



**HAL**  
open science

## Finite dimensional soluble groups

Frank O Wagner

► **To cite this version:**

| Frank O Wagner. Finite dimensional soluble groups. 2023. hal-04008765

**HAL Id: hal-04008765**

**<https://hal.science/hal-04008765>**

Preprint submitted on 28 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

# FINITE DIMENSIONAL SOLUBLE GROUPS

FRANK O. WAGNER

ABSTRACT. The results for soluble groups of finite Morley rank are generalized to the finite dimensional context. In particular, finite dimensional connected soluble non-nilpotent groups interpret a field, and have a nilpotent derived subgroup.

## INTRODUCTION

I

### 1. DIMENSION

**Definition 1.1.** A theory  $T$  is *finite-dimensional* if there is a dimension function  $\dim$  from the collection of all interpretable sets in models of  $T$  to  $\{-\infty\} \cup \omega$ , satisfying for a formula  $\varphi(x, y)$  and interpretable sets  $X$  and  $Y$ :

- *Invariance:* If  $a \equiv a'$  then  $\dim(\varphi(x, a)) = \dim(\varphi(x, a'))$ .
- *Algebraicity:*  $\dim(\emptyset) = -\infty$ , and  $\dim X = 0$  iff  $X$  is finite.
- *Union:*  $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$ .
- *Fibration:* Let  $f : X \rightarrow Y$  be an interpretable map.
  - If  $\dim(f^{-1}(y)) \geq d$  for all  $y \in Y$ , then  $\dim(X) \geq \dim(Y) + d$ .
  - If  $\dim(f^{-1}(y)) \leq d$  for all  $y \in Y$ , then  $\dim(X) \leq \dim(Y) + d$ .

Of course, fibration implies that

(†) If  $\dim(f^{-1}(y)) = d$  for all  $y \in Y$ , then  $\dim(X) = \dim(Y) + d$ .

Note that we do not suppose that the dimension is definable, i.e. that all sets  $\{y \in Y : \dim(f^{-1}(y)) = d\}$  are definable. If dimension is definable, then (†) (i.e. fibration with  $=$ ) implies the two fibration axioms (with  $\geq$  and  $\leq$ ).

**Remark 1.2.** In [6] a more general notion of dimension is defined. Our notion of finite-dimensionality would correspond to *fine finite-dimensional with lower fibration*.

Examples of finite-dimensional theories include theories of finite Lascar rank, finite SU-rank, finite  $U^b$ -rank and  $o$ -minimal theories.

---

*Date:* February 28, 2023.

*2010 Mathematics Subject Classification.* 03C45, 03C20, 03C60, 12L12, 20F24.

*Key words and phrases.* finite dimension, group, nilpotent, soluble, field interpretation.

Partially supported by ANR AAPG2019 GeoMod.

**Fact 1.3.** [6] *In a finite-dimensional theory, if  $f : G \rightarrow H$  is a definable homomorphism of definable groups, then  $\dim G = \dim H + \dim \ker f$ . In particular, if  $H \leq G$ , then  $\dim G = \dim H + \dim G/H$ . So  $H$  has infinite index in  $G$  iff  $\dim H < \dim G$ .*

It follows that we have a descending and an ascending chain condition on definable subgroups up to finite index. Note however that there is no *a priori* assumption on the existence of connected components (although we usually assume connectedness of one of the groups involved). Part of the work is then overcome the lack of connectedness elsewhere.

**Lemma 1.4.** *Let  $G$  be a connected group acting on the infinite abelian group  $A$  in a finite-dimensional theory. Suppose that  $A$  has no infinite  $G$ -invariant definable subgroup of smaller dimension, and there is  $g \in G$  with  $(g-1)A$  infinite. Then  $A$  has a  $G$ -invariant  $G$ -minimal definable subgroup of finite index.*

*Proof.* Suppose  $A^* \leq A$  is a  $G$ -invariant definable subgroup of finite index. Then  $C_G(A/A^*)$  has finite index in  $G$  and must be the whole of  $G$ . So  $(g-1)A \leq A^*$ ; it is thus contained in every  $G$ -invariant subgroup of finite index in  $A$ .

Now consider finitely many  $g_i \in G$  such that  $B = \sum_i (g_i - 1)A$  has maximal dimension possible. As  $(g-1)A$  is infinite for some  $g \in G$ , the group  $B$  is infinite; by maximality of  $\dim B$  it must be commensurable with  $gB \leq \sum_i [(gg_i - 1)A - (g-1)A]$  for any  $g \in G$ . By compactness the commensurability is uniform, and by Schlichting's Theorem there is a  $G$ -invariant definable subgroup  $B_0$  commensurable with  $B$ . Then  $\dim A = \dim B_0 = \dim B$ , so there is a maximal sum  $A_0 = \sum_j (g_j - 1)A$  of finite index in  $A$ . Then  $A_0$  is  $G$ -invariant, and contained in every  $G$ -invariant definable subgroup of finite index in  $A$ , whence  $G$ -minimal.  $\square$

In order to preserve connectedness in nilpotent groups, we shall need the following Lemma.

**Lemma 1.5.** *Let  $G$  be a finite-dimensional nilpotent connected group. Then  $G$  has a definable connected lower central series.*

*Proof.* We use induction on the dimension. If  $G$  is abelian, the statement is clear. Otherwise choose  $g \in \gamma_{n-1}G \setminus Z(G)$ , where  $n$  is the nilpotency class of  $G$ . Then  $x \mapsto [x, g]$  is a homomorphism from  $G$  to  $\gamma_n G \leq Z(G)$ ; its image  $Z$  is definable, connected and infinite. Since  $\dim(G/Z) < \dim G$  and  $\gamma_i(G)$  is the pre-image of  $\gamma_i(G/Z)$  with respect to the projection  $G \rightarrow G/Z$ , we finish by induction.  $\square$

**Lemma 1.6.** *Let  $H_1, H_2$  be normal subgroups of  $G$ . If  $G/H_1$  and  $G/H_2$  are connected-by-finite, so is  $G/(H_1 \cap H_2)$ .*

*Proof.* Let  $(G_k)_k$  be an infinite strictly descending sequence of subgroups of  $G$  containing  $H_1 \cap H_2$ . Since  $G/H_1$  is connected-by-finite, the sequence  $(G_k H_1)_k$  will eventually be stationary, and we may assume that it is constant for all  $k$ . Similarly, the

sequence  $(G_k \cap H_1)/(H_1 \cap H_2) \cong (G_k \cap H_1)H_2/H_2$  will eventually be stationary, and we may assume  $G_k \cap H_1$  is constant for all  $k$ . But then  $G_k/(G_k \cap H_1) \cong G_k H_1/H_1$  is stationary, as is  $G_k$ , a contradiction.  $\square$

**Lemma 1.7.** *Let  $H$  be a normal subgroup of  $G$  and suppose  $G/H$  and  $H$  are connected-by-finite. Then  $G$  is connected-by-finite.*

*Proof.* Let  $(G_k)_k$  be an infinite strictly descending sequence of subgroups of  $G$ . Then  $G_k H$  and  $G_k \cap H$  will eventually be stationary, and we may assume they are stationary for all  $k$ . But then  $G_k/(G_k \cap H) \cong G_k H/H$  is stationary for all  $k$ , as is  $G_k$ , a contradiction.  $\square$

**Proposition 1.8.** *Let  $G$  be a connected group acting faithfully on the abelian group  $A$  in a finite-dimensional theory. Suppose that  $C_A(G)$ , as well as  $A[n]$  are finite for all  $n < \omega$ . Then  $A$  is connected-by-finite.*

Note that since the  $n$ -torsion is finite for all  $n < \omega$ , a connected subgroup of  $A$  must be divisible (and a divisible group is connected).

*Proof.* By induction on  $\dim A$ . If  $(g-1)A$  were finite for all  $g \in G$ , then

$$\dim\{(g, a) : ga = a\} = \dim G + \dim A,$$

so  $\dim C_G(a) = \dim G$  for  $\dim A$  many  $a \in A$ , whence  $C_G(a) = G$  by connectedness of  $G$ , and  $C_A(G)$  has finite index in  $A$ , a contradiction. As in Lemma 1.4, there is a  $G$ -invariant subgroup  $B \leq A$  commensurable with a finite sum of the form  $\sum_i (g_i - 1)A$  of maximal dimension possible.

If  $\dim B = \dim A$ , then as in Lemma 1.4 there is a  $G$ -invariant subgroup  $A_0$  of finite index in  $A$  of the form  $A_0 = \sum_j (g_j - 1)A$ , which is contained in all  $G$ -invariant subgroups of finite index in  $A$ . Since  $A_0[n]$  is finite,  $nA_0$  has finite index in  $A$ , so  $A_0 \leq nA_0$  for all  $n > 0$  and  $A_0$  is divisible, whence connected.

If  $\dim B < \dim A$ , then  $B$  is connected-by-finite by induction hypothesis. Choose  $n < \omega$  minimal such that there are  $g_i \in G$  for  $i \leq n$  such that  $\sum_{i < n} (g_i - 1)A$  is commensurable with  $B$ , and put  $N = \sum_{i < n} (g_i - 1)A$ . Then  $(g_n - 1)A/((g_n - 1)A \cap N)$  is infinite and commensurable with  $B/N$ , whence connected-by-finite. Put  $K = \{a \in A : (g_n - 1)a \in N\}$ . Then  $A/K \cong (g_n - 1)A/((g_n - 1)A \cap N)$  is infinite, and connected-by-finite. Choose finitely many  $h_j \in G$  such that  $K_0 = \bigcap_j h_j K$  has minimal dimension possible. Then  $K_0$  is uniformly commensurable with all its  $G$ -conjugates; by Schlichting's Theorem there is a  $G$ -invariant  $K_1$  commensurable with  $K_0$ . By Lemma 1.6,  $A/K_0$  is connected-by-finite; since  $\dim K_1 < \dim A$  also  $K_1$  is connected-by-finite by inductive hypothesis, as is  $K_0$ ; finally  $A$  is connected-by-finite by Lemma 1.7.  $\square$

**Corollary 1.9.** *Let  $K$  be a field definable in a finite-dimensional theory. Then  $K$  does not have a connected infinite group of automorphisms.*

*Proof.* If  $C_K(G)$  is infinite, then  $[K : C_K(G)]$  is a field extension of degree  $\dim K / \dim C_K(G)$ , which is finite. Hence  $G$  embeds into  $\text{Gal}(K/C_K(G))$ , which is finite. Thus  $G$  is trivial by connectedness.

If  $C_K(G)$  is finite, we consider  $G$  as a connected group of automorphisms of  $K^\times$ . Then  $K$  is divisible-by-finite by Proposition 1.8. However, the subfield  $K^{abs}$  of absolutely algebraic elements is contained in  $C_K(G)$  by connectivity of  $G$ , and must be finite. But then  $K^{abs}$  has a cyclic extension of every prime degree  $q$ , as does  $K$ , and  $(K^\times)^q$  has index at least  $q$  in  $K^\times$ , a contradiction.  $\square$

## 2. SOLUBLE GROUPS

**Theorem 2.1.** *Let  $M$  be a connected abelian group acting faithfully on the infinite abelian group  $A$  in a finite-dimensional theory. Suppose that  $A$  has no infinite  $M$ -invariant definable subgroup of smaller dimension and  $C_A(M)$  is trivial. Then  $A$  is  $M$ -minimal, and there is a definable field  $K$  such that  $A \cong K^+$  and  $M \hookrightarrow K^\times$ .*

*Proof.* We consider the ring  $R$  of definable endomorphisms of  $A$  generated by  $M$ . Then  $R$  is commutative, and contains  $M$  as a multiplicative subgroup. Moreover, for any  $r \in R$  both  $\ker r$  and  $\text{im } r$  are definable  $M$ -invariant, whence either finite or of finite index in  $A$ . But the centralizer of any finite  $M$ -invariant subgroup  $F$  has finite index in  $M$ , and must equal  $M$  by connectedness. Since  $C_A(M)$  is trivial, so is  $F$ . But then, if  $\ker r$  has finite index,  $\text{im } r$  is finite, whence trivial, and  $r = 0$ . Thus  $\ker r = \{0\}$  for all  $r \in R \setminus \{0\}$ , and  $R$  is an integral domain.

Let  $A_0$  be the  $M$ -invariant  $M$ -minimal subgroup of finite index in  $A$  given by Lemma 1.4. Then  $(m - 1)A = A_0 = (m - 1)^2A$  for any  $m \in M \setminus \{1\}$ , since these groups are all  $M$ -invariant of finite index in  $A$  and  $M$  centralises  $A/A_0$ . Hence  $A = A_0 \oplus \ker(m - 1)$ ; as  $\ker(m - 1)$  is trivial,  $A_0 = A$ . But then the image of any  $r \in R \setminus \{0\}$  is infinite  $M$ -invariant, and must be equal to  $A$  by  $M$ -minimality, so  $R$  acts on  $A$  by automorphisms.

It follows that any  $r \in R$  is determined by the couple  $(a, ra)$  for any  $r \in R$  and non-zero  $a \in A$ . Thus  $\dim X \leq 2 \dim A$  for any type-definable subset  $X$  of  $R$ . By [7, Proposition 3.6 and Proposition 3.3] the field of fractions  $K$  of  $R$  is definable, and  $A$  is a finite-dimensional vector space over  $K$ . By minimality of  $\dim A$  it is 1-dimensional, so  $A \cong K^+$ ; clearly  $M \hookrightarrow R^\times = K^\times$ .  $\square$

**Theorem 2.2.** *Let  $G$  be a nilpotent connected group acting on an infinite abelian group  $A$ , in a dimensional theory. Then either the action is nilpotent, or there is a definable centreless 2-soluble section which naturally interprets a field.*

*Proof.* By induction on  $\dim G + \dim A$ . If  $C_A(G)$  is infinite,  $\dim(A/C_A(G)) < \dim A$  and we may use the induction hypothesis for  $G$  acting on  $A/C_A(G)$ . So we may assume that  $C_A(G)$  is finite. Then  $a^G$  is finite for any  $a \in A$  centralised by  $G$

modulo  $C_A(G)$ , so  $C_G(a)$  has finite index and  $g \in C_A(G)$  by connectedness. Thus  $C_{A/C_A(G)}(G)$  is trivial, and we may assume  $C_A(G)$  is trivial. In particular, a finite  $G$ -invariant subgroup of  $A$  must be contained in  $C_A(G)$ , and be trivial.

By Lemma 1.5 the derived subgroup  $G'$  is connected and definable, and  $\dim(G') < \dim(G)$ . By induction hypothesis either the action of  $G'$  on  $A$  is nilpotent, or a field is naturally definable. So we may assume that we are in the former case, and we consider the definable series  $A_0 = \{0\}$ ,  $A_{n+1} = \{a \in A : [a, G'] \subseteq A_n\}$  which terminates in  $A$ . Note that  $G/G'$  acts on all the quotients  $A_{n+1}/A_n$ . Let  $Q$  be such a quotient. If it is finite, it is centralized by  $G$  by connectivity. If  $C_Q(G)$  is finite, again by connectivity of  $G$  the action of  $G$  on  $Q/C_Q(G)$  is centreless; if  $Q_0 \leq Q$  is definable, infinite and  $G$ -invariant of minimal dimension possible, Theorem 2.1 yields an interpretable field  $K$  such that  $Q_0 \cong K^+$  and  $G/C_G(Q_0) \hookrightarrow K^\times$ . Finally, if  $C_Q(G)$  is infinite, we divide out and finish by induction.  $\square$

**Corollary 2.3.** *A connected soluble non-nilpotent dimensional group interprets naturally a field.*

*Proof.* If  $Z(G)$  is infinite, we divide by  $Z(G)$  and use induction on  $\dim G$ . If  $Z(G)$  is finite, then  $G/Z(G)$  is centreless by connectivity, and we may assume that  $G$  is centreless. Then for any non-trivial  $a$  in the last non-trivial derived group,  $A = Z(C_G(a^G))$  is a definable infinite abelian group. Either  $G/A$  is non-nilpotent and we finish by induction on  $\dim G$ , or  $G/A$  is nilpotent and we finish by Theorem 2.2.  $\square$

**Theorem 2.4.** *Let  $G$  be a connected group acting faithfully and without fixed points on the infinite abelian group  $A$  in a finite-dimensional theory. Suppose that  $A$  is  $G$ -minimal, and that  $G$  has a definable infinite finite-by-abelian normal subgroup  $M$ . Then  $A$  is definably a vector space of finite linear dimension over a definable field  $K$ , such that the action of  $G$  is  $K$ -linear and  $M$  has a  $G$ -invariant subgroup  $M_0$  of finite index acting by scalars (so  $M_0$  is central in  $G$  and embeds into  $K^\times$ ).*

*Proof.* Let  $\mathcal{C}$  be the  $G$ -invariant family of centralisers in  $M$  of finite tuples  $\bar{m} \in M$ , and note that they all have finite index in  $M$ . Consider an infinite definable subgroup  $B \leq A$  of minimal possible dimension and invariant under some subgroup  $N_B \in \mathcal{C}$ . Choose finitely many  $g_i \in G$  such that  $S = \sum_i g_i B \leq A$  has maximal dimension possible. Then  $\dim S + gS = \dim S = \dim gS$  for any  $g \in G$ , so the family of translates  $\{gS : g \in G\}$  is commensurable, and uniformly so by compactness. By Schlichting's Theorem there is a  $G$ -invariant group  $S_0$  commensurable with  $S$ ; by minimality of  $\dim A$  we have  $\dim A = \dim S_0 = \dim S$  and  $S$  has finite index in  $A$ . It follows that there is a maximal finite sum of  $G$ -translates of  $B$  containing  $S$ ; it must be  $G$ -invariant, whence equal to  $A$  by  $G$ -minimality. Renaming, we may assume that  $S = A = \sum_i g_i B$ .

Suppose  $C_M(B)$  has finite index in  $M$ . Then so does  $C_M(g_j B) = C_M(B)^{g_j^{-1}}$ , as well as the intersection

$$C = \bigcap_j C_M(B)^{g_j^{-1}} = C_M\left(\sum_j g_j B\right) = C_M(A) = \{1\},$$

a contradiction.

Let  $R$  be the group of endomorphisms of  $A$  generated by  $M$ , and for any subgroup  $N \in \mathcal{C}$  let  $R_N$  be the group of endomorphisms of  $A$  generated by  $N$ . Put  $I_B = \{r \in R_{N_B} : rB \text{ finite}\}$ . Now  $B$  is  $R_{N_B}$ -invariant, and for any  $r \in R$  generated using elements  $\bar{m} \in M$ ,  $rB$  is invariant under  $N_B \cap C_M(\bar{m}) \in \mathcal{C}$ . It follows by minimality that  $rB$  is either finite or has dimension  $\dim rB = \dim B$ , so  $B \cap \ker r$  is finite. In particular  $I$  is a prime ideal in  $R_{M_B}$ . Note that  $N_{gB} = N_B^{g^{-1}}$ ,  $R_{N_{gB}} = R_{N_B}^{g^{-1}}$  and  $I_{gB} = I_B^{g^{-1}}$ .

We shall call  $g$  and  $h$  equivalent if there is a subgroup  $N \in \mathcal{C}$  containing  $N_{gB} \cap N_{hB}$  such that  $I_{gB} \cap R_N = I_{hB} \cap R_N$ . This clearly is a  $G$ -invariant equivalence relation. Suppose  $(h_j : j \leq k)$  are inequivalent for some  $h_0, \dots, h_k \in G$ , and put  $N = \bigcap_{j \leq k} N_{h_j B}$ , again a group in  $\mathcal{C}$ . Then all  $I_{h_j B} \cap R_N$  are distinct prime ideals in  $R_N$ , and one of them is minimal, say  $I_{h_k B}$ . So there is  $r \in R_N \cap \bigcap_{j < k} I_{h_j B} \setminus I_{h_k B}$ . Hence  $rh_j B$  is finite for  $j < k$  and  $rh_k B$  has finite index in  $h_k B$ . Thus  $h_k B \cap \sum_{j < k} h_j B$  is finite; iterating we find that the sum  $\sum_{j \leq k} h_j B$  is almost direct. So  $(k+1) \dim B = \dim \sum_{j \leq k} h_j B \leq \dim A$ , and  $k < \dim A$ , as  $\dim B > 0$ . Thus there are only finitely many equivalence classes among the elements of  $G$ , and we may choose representatives  $(e_i : i < c)$ .

For any  $i < c$  let  $S_i$  be a sum  $\sum_{j_i} h_{j_i, i} B$  of maximal dimension, where all  $h_{j_i, i}$  are equivalent to  $g_i$ . Then as above we see that the sum  $\sum_{i < c} S_i$  is almost direct, and  $h$  is equivalent to  $g_i$  if and only if  $hB$  is almost contained in  $S_i$ . It follows that the action of  $G$  on the equivalence classes is definable, and trivial by connectivity. So there is just one equivalence class, since the action is transitive.

Recall that  $A = \sum_i g_i B$ . Take  $N \in \mathcal{C}$  containing  $\bigcap_i N_{g_i B}$  such that all  $I_{g_i B} \cap R_N$  are the same prime ideal  $I_N$  of  $R_N$ . Now any  $r \in R_N$  has finite image or image of finite index on every  $g_i B$ , whence on  $A$ . It follows that  $I_N = \{r \in R_N : rA \text{ finite}\}$ . Now for  $m, m' \in N$  we have  $m + I_N = m' + I_N$  iff  $m - m' \in I_N$  iff  $\text{im}(m - m')$  is finite iff  $\text{im}(m'^{-1}m - 1)$  is finite iff  $C_A(m'^{-1}m)$  has finite index in  $A$  iff  $mM_0 = m'M_0$ , where  $M_0 = \{m \in M : |A : C_A(m)| \text{ finite}\}$ . Note that the finiteness is definable by compactness, so  $M_0$  is definable.

Suppose almost all  $m \in N$  are equivalent modulo  $I$ . Then  $M_0$  has finite index in  $M$ . Consider the set

$$\{(a, m) \in A \times M : ma = a\}.$$

It has dimension at least  $\dim M_0 + \dim A = \dim M + \dim A$ . It follows that the definable subgroup  $A_0 = \{a \in A : Ma \text{ finite}\}$  has finite index in  $A$ , and both are  $G$ -invariant. Then  $A_0 \rtimes M_0$  has finite derived group, which is a finite  $G$ -invariant subgroup of  $A$ , whence contained in  $C_A(G) = \{1\}$ . So  $M_0$  acts trivially on  $A_0$ . But  $A_0 = A$  by  $G$ -minimality, contradicting faithfulness. Hence  $M/M_0$  is infinite.

Suppose that there is no maximal finite subgroup of  $A$  invariant under some subgroup in  $\mathcal{C}$ . Then take a countably infinite restricted sum  $S$  of such subgroups; they are invariant under a subgroup  $N_0$  of  $M$  which is a countable intersection of subgroups in  $\mathcal{C}$ . Since  $M_0$  has infinite index in  $M$ , it has infinite index in  $N \cap N_0$ , and there are  $m, m' \in N \cap N_0$  with  $m'^{-1}m \notin M_0$  with the same action on  $S$ . Hence  $\ker(m - m') \supseteq S$  is infinite, so  $\text{im}(m - m')$  is finite, contradicting  $m'^{-1}m \notin M_0$ . It follows that there is a unique maximal finite subgroup invariant under some subgroup in  $\mathcal{C}$ . By uniqueness it is  $G$ -invariant, whence centralized by  $G$  by connectedness. But  $C_A(G)$  is trivial, so there is no non-trivial finite subgroup of  $A$  invariant under some subgroup in  $\mathcal{C}$ . Thus  $R_N \setminus \{0\}$  consists of injective endomorphisms whose image has finite index in  $A$ . In particular  $I_N$  is trivial.

Choose  $a \in A \setminus \{0\}$ . For any  $r \in R_N$  the couple  $(a, ra)$  determines  $r$ , so we get a bound on  $\dim R_N$ , and the skew field of fractions  $K_N$  is definable by [7, ].

The derived subgroup  $M'$  is finite, say of size  $n$ . It is characteristic in  $M$ , whence normal in  $G$ , and must be centralized by  $G$  by connectivity. It follows that

$$[x, y^n] = [x, y]^n = 1$$

for all  $x, y \in M$ . Moreover,

$$(xy)^{2n} = x^{2n}y^{2n}[y, x]^{(2n-1)2n/2} = x^{2n}y^{2n}.$$

So the map  $x \mapsto x^{2n}$  is an endomorphism of  $M$  with characteristic abelian image; we want to show that the image  $M_0$  has finite index. So suppose otherwise. Then the kernel  $M[2n]$  is infinite, as is  $N[2n] = M[2n] \cap N$ . Since centralisers of finite tuples have finite index in  $M$ , we can find an abelian subgroup  $A$  of exponent  $2n$  and size  $> 2n$  in  $N$ . But  $A$  generates a commutative subfield of  $K_N$ , a contradiction.

It follows that  $M_0$  has finite index in  $M$ ; clearly it is  $G$ -invariant. We repeat the above argument with  $M_0$  instead of  $M$ , noting that this time  $\mathcal{C} = \{M_0\}$ . So  $R_{M_0}$  is a commutative ring of injective endomorphisms of  $A$  with finite kernels, whose field of fractions  $K_0$  is definable. Then  $G$  acts on  $K_0$  definably, and trivially by Corollary 1.9. Hence  $m = m^g$  for all  $m \in M_0$  and  $g \in G$ , and  $M_0$  is central in  $G$ . Now  $\text{im } r$  is  $G$ -invariant for any  $0 \neq r \in R_{M_0}$ , so  $\text{im } r = A$  by  $G$ -minimality. It follows that  $A$  is a  $K_0$ -vector space of finite linear dimension,  $G$  acts  $K_0$ -linearly and  $M_0$  acts  $K_0$ -scalarly.  $\square$

**Theorem 2.5.** *A finite-dimensional connected soluble group  $G$  has a nilpotent derived subgroup.*



*Proof.* Dividing out by the hypercentre, we may assume that  $G$  is centreless. In particular, it has no finite  $G$ -invariant subgroups. If  $a \neq 1$  is in the last non-trivial derived subgroup of  $G$ , then  $Z(C_G(a^G))$  is a definable  $G$ -invariant abelian subgroup of  $G$ . So we can choose a non-trivial definable  $G$ -invariant abelian subgroup  $A \leq G$  such that  $C_G(A)$  has maximal dimension possible. Replacing  $A$  by a definable subgroup, we may assume that  $\dim A$  is minimal possible. Note that  $A$  is infinite.

If  $[g, A]$  is finite for all  $g \in G$ , consider the set

$$\{(g, a) \in G \times A : ga = a\}$$

and note that it has dimension  $\dim G + \dim A$ . Hence  $\dim\{a \in A : [G, a] \text{ finite}\} = \dim A$ . But by connectivity  $[G, a]$  is finite if and only if it is trivial, contradicting the fact that  $G$  is centreless. By Lemma 1.4 we may assume that  $A$  is  $G$ -minimal. We claim that  $G' \leq C_G(A)$ . So suppose not.

If  $C_{G'}(A)$  has finite index in  $G'$ , then for every  $g \in G$  the index  $|G : C_G(g/C_G(A))|$  is finite, whence trivial by connectivity. It follows that  $G' \leq C_G(A)$ , a contradiction. Hence  $C_{G'}(A)$  has infinite index in  $G'$  and there is maximal  $k$  such that  $(G')^{(k)}/C_G(A)$  is infinite. Put  $N = C_G(A)(G')^{(k+1)}$ , a definable finite extension of  $C_G(A)$ . Then  $Z(C_{G/N}((G')^{(k)}/N))$  is an infinite definable normal abelian subgroup of  $G/N$  containing  $(G')^{(k)}/N$  (it is definable since the centralizer is the centraliser of the set of basic commutators generating  $(G')^{(k)}$ ). Its preimage  $M$  in  $G/C_G(A)$  is finite-by-abelian. By Theorem 2.4 there is a definable  $G$ -invariant subgroup  $M_0$  of finite index in  $M$  centralized by  $G$ ; moreover  $A$  is definably a finite-dimensional vector space over a definable field, the action of  $G$  is linear and the action of  $M_0$  is scalar. But  $M_0 \cap (G')^{(k)}$  consists of scalar matrices of determinant 1 and must be finite, again a contradiction. Hence  $G'$  centralizes  $A$ . We divide out by  $A$  and finish by induction on the dimension.  $\square$

## REFERENCES

- [1] Reinhold Baer. *Endlichkeitskriterien für Kommutatorgruppen*. Math. Ann. 124, 1952.
- [2] Alexandre Borovik and Ali Nesin. *Groups of finite Morley rank*, Oxford Logic Guides, vol. 26, The Clarendon Press — Oxford University Press, New York, 1994.
- [3] Nadja Hempel. *Almost group theory*, preprint (2016), hal.archives-ouvertes.fr/hal-01206954
- [4] Bernhard H. Neumann. *Groups with finite classes of conjugate elements*, Proc. London Math. Soc., 1:178–187, 1951.
- [5] Maxwell Rosenlicht. *On a result of Baer*. Proc. Amer. Math. Soc. 13(1):99–101, 1962.
- [6] Frank O. Wagner. *Simple Theories*. Kluwer Academic Publishers, 2000.
- [7] Frank O. Wagner. *Dimensional groups and fields*, J. Symb; Logic, 85(3):918-936, 2020.

UNIVERSITÉ LYON 1; CNRS; INSTITUT CAMILLE JORDAN UMR 5208, 21 AVENUE CLAUDE BERNARD, 69622 VILLEURBANNE-CEDEX, FRANCE

*Email address:* wagner@math.univ-lyon1.fr