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Exploring students' conceptions of proof in high-school and university: A proposal for collaborations in Europe

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Although there is a considerable amount of research on proof in mathematics education, very few studies explore students' conceptions of proof at the transition between secondary and tertiary level. Moreover, new didactic issues arise with proof assistants. The aims of this article are twofold: to highlight methods and theoretical backgrounds used in previous research mathematics education in order to identify undergraduate students' conceptions of proof, and to propose a questionnaire (in a large-scale survey perspective) in order to trigger collaborations in Europe on this topic.

Keywords: Proof, conception, survey, high-school and undergraduate students, proof assistants

Introduction – A rupture between secondary and tertiary education

Proof is epistemologically constitutive of mathematical activity. Fundamental at all levels of schooling and teacher training, proof contributes significantly to the learning processes of mathematical knowledge (e.g. Hanna, 1996). Large-scale surveys at the international level show that proof occupies too marginal a place in education, at all levels of schooling, with insufficient or even inadequate teacher training (e.g. Stylianides, 2016). The rupture between secondary and tertiary education is internationally pointed out by researchers in mathematics education (e.g. Gueudet, 2008; Selden et al., 2010). In this transition, there is a change in the didactical contract, but also in the relationship to the mathematical concepts and proofs which requires a more complex and formal view of objects and processes of proof with a kind of acculturation to mathematicians' practices (e.g. Dawkins & Weber, 2017; Selden, 2012). The students themselves feel like they are struggling, especially with regard to the "ways of thinking for mathematics" (Di Martino & Gregorio, 2019) and to the logic and formalism required in the activity of proving and writing proofs (e.g. Selden, 2012; Selden & Selden, 2003). Then, internationally, syntheses (those quoted for instance by Stylianides, 2016; Selden, 2012; Gueudet & Vandebrouck, 2022) converge on students' difficulties and their causes: an insufficient knowledge of concepts and mathematical theorems, the abstract mathematical notions raising conceptualisation difficulties, the expectations referring to the expert practices of mathematicians, the lack of meta-knowledge about proof, a difference in institutional culture between secondary and tertiary education, a lack of articulation between semantic, syntactic and pragmatic approaches and, then, difficulties in writing a proof. Early college students (freshmen) use inappropriate reasoning, often based on everyday logic, when faced with mathematical concepts (Selden & Selden, 2003; Selden et al., 2010; Selden, 2012). Besides, one does not really know how students validate their proofs, both when they construct them and when they read them (except for the qualitative results of Weber's research). Other factors explaining the "weak performance of students at the secondary and undergraduate levels on proof" exist such as "the disjointedness of reform efforts" (see the still relevant discussion of Harel & Fuller, 2009). In order to solve the persistent problem of students' difficulties with proof, which seems to be real in all countries, the

arrival of proof assistants in didactic research can provide new opportunities for the teaching and the learning of proof in mathematics. I will explain why.

Current issues on proof in mathematics, computer science and didactics

New technologies influence mathematical practices and create new dynamics at the interface with computer science, reflecting the reality of a work done by contemporary mathematicians in proof. Hanna and Knipping (2020) show the need to develop new approaches to proof teaching taking into account proof assistants (henceforth noted PA). PAs (Coq, LEAN, Isabelle etc.) used by mathematicians are free open-source software that mechanically verify a proof, making the logic and the formalism visible. These transform the epistemology of proof, the best-known example being the verification of the 4-color theorem. Recently used in teaching with graphical interfaces (DEDUCTION and Edukera for LEAN and Coq), they directly question mathematicians and didacticians on how proof processes should be transmitted to teaching. Thus, the question of teaching and learning proofs, problematic at the secondary-tertiary transition, joins current issues in mathematics, computer science and didactics. Isolated case studies show the interest of working at the articulation between formal proofs with PA and paper-and-pencil proofs as well as the contribution of PA for students in terms of their autonomy in controlling and structuring the proofs they produce, and understanding the mathematical statements (Kerjean et al., 2022; Avigad, 2019; Thoma & Iannone, 2022). Thus, recent syntheses on the question, based on the hindsight of Technology Enhanced Learning (TELEs), and on the nature of the feedback provided by the machine, call for a multidisciplinary structuring of research to analyse the uses and contributions of PAs in the learning of proof (Hanna et al., 2019; Balacheff & Boy de la Tour, 2019).

Research questions

Bearing in mind the above context, the questions of my current research project coincide with two of those listed by Harel and Fuller (2009, pp. 355–356): What theoretical tools seem suitable to investigate students' conceptions of proof? (Q1). What are students' current conceptions of proof? (Q2). Considering PAs, my third research questions is: How PAs do impact students' conceptions and performances in proof? (Q3). To answer to Q3, in a large-scale study, I will implement PAs with freshmen in three universities in France next year, using a didactical engineering, in the frame of a collaborative national project. My aim is to evaluate how interventions with a PA impact students' conceptions and performances. I will use two devices to answer to Q2: a questionnaire for an initial large-scale study (with open and multiple-choices questions) and interviews with pairs of students to refine the results of the questionnaire. Note that such devices aim at being reusable to evaluate the impact of PAs on students' conceptions and difficulties (before and after the didactical engineering). Designing such a questionnaire is difficult for several reasons: it aims to be large, and it requires synthesising previous results about students' conceptions of proof, while taking into account the underlying theoretical frameworks used about proof. Indeed, the nature of the questions depends on the theoretical frameworks about proof. It leads me to finely consider Q1 with a bibliographical study to identify where and how undergraduate students' conceptions of proof are explored in research. In this paper, firstly, I will present a preliminary bibliographical study (linked with Q1). Secondly, I will explain my theoretical and methodological choices to assess students' proof conceptions. I will

exemplify an exploratory questionnaire to identify proof conceptions of high-school and undergraduate students and to conduct pre- and post- diagnosis when specific interventions (with PAs for instance) are implemented (linked with Q2). Thirdly, I will propose a way to explore students' conceptions at the secondary-tertiary transition in a collaborative perspective in Europe.

Bibliographical study

To circumscribe how researchers in mathematics education have investigated the undergraduate students' conceptions about proof, I have conducted a bibliographical study aimed at identifying the nature of the mathematical domains involved, the methods and the theoretical underlying backgrounds used in such surveys. I used only one database, the Education Resources Information Center (ERIC), a free, large, online digital library of education research that is easily accessible to anyone, recognizing that a search in English will not allow us to identify existing research in other languages. My search in ERIC in February 2023 was the following: “(mathematics AND proof AND students AND undergraduate) AND (conception OR survey OR interview OR questionnaire)”. I found for “All Publication Type” and “Any Education Level”: 63 articles since 2004, 38 since 2014, 17 since 2019, 4 since 2022. 47 are journal articles, 43 are research reports, and the rest are proceedings, conference papers etc. 56 articles are in higher education, 28 in postsecondary education, and 5 in secondary education. For this article, I then focused on “Journal Articles” of the last ten years (since 2014): 30 articles remained (8 are reports or proceedings). I read the 30 articles and classified them with 8 characteristics: name of the journal, year, country of the study, research questions and methods, population analysed in the study, number of participants, theoretical tools and mathematical contents. Regarding my research questions, I only kept articles that deal with a specific population (students in university or at the secondary-tertiary transition) and whose research questions focus on proof and proving processes. The reading of the articles led me to define five exclusion criteria¹ and to eliminate 14 articles, leaving a corpus of 16 articles (Table 1). 10 papers of the 16 retained referred to the USA, 1 from Algeria, 2 from Indonesia, 1 from Iran, 1 from South Africa (Lesotho). Europe is poorly represented with only 1 article from Turkey.

Most of the papers used interviews with students, always after a preliminary survey, questionnaire, assessment, or analysis of students' writings or readings proofs. Only one paper conducted a complete large-scale survey with early undergraduate students (Stylianou et al., 2015). The methods and the theoretical background clearly depend on the research questions and on the mathematical contents. Most of the mathematical fields involved in the 16 selected papers are very specific to university (functions and continuity, differentiability, integrability, Cauchy generalized mean value theorem; limits of sequences; infinite series; linear algebra; holomorphic functions; proof by contradiction). Some of the papers explore more transversal domains such as arithmetic, geometry and algebra. The following list of the emblematic theoretical backgrounds used in the selected papers reflects a wide variety of approaches: Tall (2008)'s three worlds, Harel (1998)'s intellectual need, Mejia-Ramos et al. (2017)'s assessment model, Harel and Sowder (1998)'s proof schemes, Weber (2008)'s

¹ 5 excluded articles deal with resources (syllabi, workbooks etc.) or teaching methods, task design, etc.; 3 focus on mathematical concepts but not on proof; 2 focus on mathematicians; 3 focus on teachers and 1 was off topic.

classification on proof validation, Balacheff' (1988ab)'s typology of arguments. To date, in the last 10 years, no systematic large-scale diagnostic survey has been conducted in Europe on students' conceptions of proof, particularly at the transition from secondary to tertiary education.

Bearing in mind my aim to design a large-scale survey questionnaire, I have focused on Stylianou et al. (2015), but also on the theoretical backgrounds used in the other articles (snowball method). When tracing their common references, some authors return regularly: L. Healy and C. Hoyles, G. Harel and L. Sowder, and K. Weber (alone or with others). Moreover, one of the underlying backgrounds shared by Stylianou et al. (2015) and Healy and Hoyles (1998) is that of Balacheff (1988ab). I have therefore structured my questionnaire using these frameworks and some of the types of questions used in the large-scale study by Healy and Hoyles (1998) reused in Stylianou et al. (2015) and analysed with the typology of Balacheff (1998ab), itself based on the Lakatosian epistemology.

Table 1: References of the 16 selected papers

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| <p>Azrou, N. & Khelladi, A. (2019). Why do students write poor proof texts? A case study in undergraduates' proof writing. <i>ESM</i>, 102(2), 257–274.</p> <p>Derrick, J. & Cavey, L. (2021). High school students' understanding of proof. <i>Mathematics Teacher: Learning and Teaching PK-12</i>, 114(3), 212–218.</p> <p>Erickson, S.A. & Lockwood, E. (2021). Investigating combinatorial provers' reasoning about multiplication. <i>IJRUME</i>, 7(1), 77–106.</p> <p>Faizah, S., Nusantara, T., Sudirman, Rahardi, R. (2022). Constructing students' thinking process through assimilation and accommodation framework. <i>Mathematics Teaching Research Journal</i>, 14(1), 253–269.</p> <p>Huda, N., Subanji, Nusantar, T., Susiswo, Sutawidjaja, A., Rahardjo, S. (2016). University students' metacognitive failures in mathematical proving investigated based on the framework of assimilation and accommodation. <i>Educational research and reviews</i>, 11(12), 1119–1128.</p> <p>Kolahdouz, F., Radmehr, F., Alamolhodaei, H. (2020). Exploring students' proof comprehension of the Cauchy generalized mean value theorem. <i>Teaching Maths and Its Applications</i>, 39(3), 213–235.</p> <p>Mejia-Ramos, J.P., Lew, K., de la Torre, J., Weber, K. (2017). Developing and validating proof comprehension tests in undergraduate mathematics. <i>RME</i>, 19(2), 130–146.</p> <p>Melhuish, K., Larsen, S., Cook, S. (2019). When students prove a theorem without explicitly using a necessary condition: Digging into a subtle problem from pPractice. <i>IJRUME</i>, 5(2), 205–227.</p> <p>Moru, E. K., Nchejane, J., Ramollo, M., Rammea, L. (2017). University undergraduate science students' validation and comprehension of written proof in the context of infinite series. <i>African Journal of Research in Mathematics, Science and Technology Education</i>, 21(3), 256–270.</p> <p>Rabin, J. M. & Quarfoot, D. (2022). Sources of students' difficulties with proof by contradiction. <i>IJRUME</i>, 8(3), 521–549.</p> <p>Sevimli, E. (2018). Undergraduates' propositional knowledge and proof schemes regarding differentiability and integrability concepts. <i>IJMEST</i>, 49(7), 1052–1068.</p> <p>Stewart, S. & Thomas, M. O. J. (2019). Student perspectives on proof in linear algebra. <i>ZDM</i>, 51(7), 1069–1082.</p> <p>Stylianou, D.A., Blanton, M.L., Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. <i>IJRUME</i>, 1(1), 91–134.</p> <p>Weber, K. (2015). Effective proof reading strategies for comprehending mathematical proofs. <i>IJRUME</i>, 1(3), 289–314.</p> <p>Weber, K. & Mejia-Ramos, J.P. (2014). Mathematics majors' beliefs about proof reading. <i>IJMEST</i>, 45(1), 89–103.</p> <p>Zazkis, D. & Villanueva, M. (2016). Student conceptions of what it means to base a proof on an informal argument. <i>IJRUME</i>, 2(3), 318–337.</p> |
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Design of a questionnaire on students' conceptions of proof

My research questions guided the design of the questionnaire, considering both proof as a product and proving as a process: How do students construct proofs? What are students' judgements of given

proofs? What are students' global perceptions of proof? My aim is to build a questionnaire which will be usable throughout high-school, college, and pre-service teacher training, mainly with specialist students in mathematics, for several snapshots: to identify students' conceptions and to evaluate the development of these conceptions (also after interventions, for instance with PAs). I have taken into account several criteria: selecting mathematical contents where the obstacle of mastering the involved concepts is minimized; choosing mathematical problems outside of formalism (so that it is not an obstacle) and outside of curricula (to avoid obstacles or ready-made results and processes). So far, the questionnaire is currently being administered in universities in France and Belgium in a pilot testing phase (with freshmen and pre-service secondary teachers training). It is available online (<https://hal.science/hal-03987587>). Five sections structure the questionnaire: Part I – Identification, enunciation and use of mathematical properties; Part II – Writing proofs; Part III – View of proofs and evaluation of proof; Part IV – General conception of proof; Part V – Respondent information.

Theoretical backgrounds and aspects addressed by the questionnaire

Built upon different theoretical backgrounds and results previously quoted (mainly the significant students' difficulties even with deductive short proofs), my questionnaire explores “basic” components of deductive proofs through the above-described aspects.

In Part I, the students are asked to state a known formal statement (of their choice) and to write a formal statement starting from a representation. They also have to use a given theorem (*Thales intercept theorem*) in a specific configuration: I want to evaluate their abilities to check the hypotheses of a given theorem by themselves and to specify the instances of this theorem, as well as their abilities with formal statements (to correlate them with students' proof writings in Part II).

Table 2: An example of Part II of the questionnaire (formal proofs to be written by the students)

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| <p>One defines an imaginary formal system with objects and rules to work with these objects. One wants to prove theorems in this formal system. Our starting points are as follows:</p> <p>Definition: a « TOY » is a sequence of an integer (strictly positive) number of symbols, consisting only of dots and circles, marked with \bullet and \circ. For example, $\circ\bullet\circ\bullet\bullet$ is a TOY.</p> <p>To start building objects, we assume that: A1: $\circ\bullet$ is valid and A2: $\bullet\circ$ is valid. Three rules can be used:</p> <p>R1 : For any TOY w and v, if wv and vw are valid then w is valid</p> <p>R2 : For any TOY w and v, if w and v are valid then $w\bullet v$ is valid</p> <p>R3 : For any TOY w and v, if $wv\bullet$ is valid then $w\circ$ is valid</p> <p>Prove the following theorems: Thm A: \bullet is valid; Thm B: $\bullet\bullet$ is valid; Thm C: $\bullet\bullet\circ$ is valid.</p> |
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In Part II, the students are asked to write the three following proofs: a familiar one with several deductive steps in calculus (familiar because learnt at the beginning of high-school in France: “for any a, b positive real numbers, prove that $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$ ”); a unfamiliar one in arithmetic (but with familiar contents: “prove that the sum of two consecutive odd numbers is divisible by 4”); and a unfamiliar one in a formal, local imaginary axiomatic with non-representational symbols to see how the students make the statements work (see below). The latter is inspired by Monks quoted in Reid and Knipping (2010, p.141). The symbols do not represent anything, and the rules only concern syntax (assessing students' conceptions of this type of proof seems particularly interesting from the perspective of implementing PAs), see Table 2 above.

In Part III, the students are asked to evaluate proofs. The design of the proof proposals follows previous large-scale surveys (Stylianou et al., 2015 with undergraduate students and Healy and Hoyles, 1998 with younger students) and Balacheff (1988ab)'s characterisation of arguments (mainly naïve empiricism, generic argument and intellectual proof). The main difference lies in the choice of the mathematical domains. Four correct mathematical statements are given (both in familiar and unfamiliar fields: arithmetic², geometry³, graph theory⁴, combinatorial geometry⁵). For each of them, there are four proof proposals, presented in a variety of forms and arguments (following Balacheff 1988a's typology), as in Table 3 for the statement in combinatorial geometry. Some of the proof proposals are false or incomplete from a mathematical point of view. The students are asked to tick their answer(s) in multiple choices questions (adjustments of Healy and Hoyles 1998's and Stylianou et al. 2015's questions): Which answer is nearest to what you would have done? Which answer would you choose to explain the "solution" to your classmates? Which answer is the most "thorough" for you? What is/are the correct proof(s) for you? What is/are the incomplete proof(s) in your opinion?

In Part IV, the questions explore the students' global conception of proof (the way they think about proof) with open questions leading to kinds of proofs, functions of proofs, declarative students' tools to write/read/evaluate proofs, and word clouds.

Table 3: An example of Part III – Proofs in combinatorial geometry to be evaluated by the students

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| <p>Answer A</p> <p>Let n be the number of vertices of a convex polygon. A vertex can be joined to all vertices except itself and its two neighbours, i.e. $(n-3)$ diagonals per vertex. The same applies to all n vertices, so there are $n(n-3)$ diagonals. But a diagonal joins two vertices X and Y: it is therefore counted twice (once for X and once for Y). The number of diagonals in a polygon with n vertices is therefore necessarily $n(n-3)/2$.</p> <p>Answer B</p> <p>Suppose that there exists a polynomial function $f(n)$ that calculates the number of the diagonals of a polygon. Suppose it is a first-degree polynomial function of the form: $f(n)=an+b$ where a and b are real numbers. Using special cases (the square, the pentagon, the heptagon) to find values for a and b, I see that there are no values for a and b that work. Therefore, I assume that $f(n)$ is a second-degree polynomial function with the following form: $f(n)=an^2+bn+c$ where a, b and c are real numbers. Using the same method as above to find the values of a, b and c, I obtain $f(n)=n^2/2-3n/2$.</p> <p>Answer C</p> <p>For example, if I take a five-sided polygon: two diagonals start from a vertex. The same applies to each vertex. So, there are 10 diagonals, but I have counted them twice: there are 10 divided by 2, so, there are 5. I conclude that there are $n(n-3)/2$ diagonals.</p> <p>Answer D</p> <p>There are as many diagonals as edges: this property is true for the 5-sided polygon. Then, check for a polygon with a large number of edges and conclude that the number of diagonals is equal to n (where n is the number of edges).</p> |
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² "The product of three consecutive numbers is a multiple of 6." I made adjustments of Healy and Hoyles (1998).

³ "The sum of the angles of a triangle is a straight angle."

⁴ "The sum of the degrees of the vertices of any (finite) graph is even." Definitions of graph, vertex and degree are given as well as an example.

⁵ "Let n be the number of vertices of a convex polygon (n is a strictly positive integer). Calculate the number of its diagonals and prove the formula." I made adjustments of Balacheff (1988b).

Proposals for European collaborations

I contend that all questions of my questionnaire, mathematical contents (familiar for several years or unfamiliar but accessible with discrete concepts) and direct proofs were chosen in order to have a transferable questionnaire, for high-school, university and teacher training. Then, it is a tool to engage discussions to conduct an initial large-scale survey about the development of students' conceptions and difficulties with proof and proving from high-school to university, and also in pre-service teacher training, in Europe. Currently in a pilot testing phase, it is available in open access (in French and English). The questionnaire is designed with mathematical contents that are not specific to university. Only one proof depends on the current French high-school curricula: it can therefore be easily adapted to another country. One can hypothesise that students' conceptions are not very different across Europe. Moreover, the underlying didactic frameworks – shared by our research community – and the chosen mathematical concepts and problems follow those of Balacheff (1988ab) and the design of Healy and Hoyles (1998) partially redesigned by Stylianou et al. (2015), both in a large-scale study. Besides, a collaborative work may lead researchers to develop additional diagnostic tools to evaluate the impact of the use of specific environments (such as PAs) or interventions on students' conceptions of proof in high-school and university. This is an ambitious goal which will be enriched by working together on – *a priori* – shared educational problematics.

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