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Exploring students' conceptions of proof in high-school and university

A proposal for collaborations in Europe

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Although there is a considerable amount of research on proof in mathematics education, very few studies explore students' conceptions of proof at the transition between secondary and tertiary level. Moreover, new didactic issues arise with proof assistants. The aims of this article are twofold: to highlight methods and theoretical backgrounds used in previous research mathematics education in order to identify undergraduate students' conceptions of proof, and to propose a questionnaire (in a large-scale survey perspective) in order to trigger collaborations in Europe on this topic.

Keywords: Proof, Conception, Survey, High-School and Undergraduate Students, Proof Assistants.

Introduction – A rupture between secondary and tertiary education

Proof is epistemologically constitutive of mathematical activity. Fundamental at all levels of schooling and teacher training, proof contributes significantly to the learning processes of mathematical knowledge (e.g. Hanna, 1996). Large-scale surveys at the international level show that proof occupies too marginal a place in education, at all levels of schooling, with insufficient or even inadequate teacher training (Stylianides, 2016). The rupture between secondary and tertiary education is internationally pointed out by researchers in mathematics education (e.g. Alcock & Simpson, 2004; Gueudet, 2008; Selden et al., 2010). In this transition, there is a change in the didactical contract, but also in the relationship to the mathematical concepts and proofs which requires a more complex and formal view of objects and processes of proof with a kind of acculturation to mathematicians' practices (e.g. Dawkins & Weber, 2017; Selden, 2012). The students themselves feel like they are struggling, especially with regard to the "ways of thinking for mathematics" (Di Martino & Gregorio, 2019) and to the logic and formalism required in the activity of proving and writing proofs (e.g. Gueudet & Vandebrouck, 2022; Selden, 2012; Selden & Selden, 2003). Then, internationally, syntheses (those quoted for instance by Stylianides 2016; Selden, 2012; Gueudet & Vandebrouck, 2022) converge on students' difficulties and their causes: an insufficient knowledge of concepts and mathematical theorems (e.g. Dreyfus, 1999), the abstract mathematical notions raising conceptualisation difficulties, the expectations referring to the expert practices of mathematicians, the lack of meta-knowledge about proof (e.g. Morselli, 2007 quoted in Azrou, 2019), a difference in institutional culture between secondary and tertiary education, a lack of articulation between semantic, syntactic and pragmatic approaches (e.g. Deloustal-Jorrand et al., 2020) and then difficulties in writing a proof. Early college students use inappropriate reasoning, often based on everyday logic, when faced with mathematical concepts (Selden & Selden, 2003; Selden et al., 2010; Selden, 2012; Gueudet & Vandebrouck, 2022). Besides, one does not really know how students validate their proofs, both when they construct them and when they read them (except for the qualitative results of Weber's research). Other factors explaining the "weak performance of students at the secondary and undergraduate levels on proof" exist such as "the disjointedness of reform efforts" (see the still relevant discussion of Harel & Fuller, 2009).

Current issues on proof in mathematics, computer science and didactics

New technologies influence mathematical practices and create new dynamics at the interface with computer science, reflecting the reality of a work done by contemporary mathematicians in proof. Hanna and Knipping (2020) show the need to develop new approaches to proof teaching taking into account proof assistants (henceforth noted PA). PAs (Coq, LÉVN, Isabelle etc.) used by mathematicians are free open-source software that mechanically verify a proof, making the logic and the formalism visible. These transform the epistemology of proof, the best-known examples being the verification of the 4-color theorem or of Kepler's conjecture. Recently used in teaching with graphical interfaces (DÉVDUCTION & Edukera for LÉVN and Coq) (e.g. Kerjean et al., 2022), they directly question mathematicians and didacticians on how proof processes should be transmitted to teaching (Durand-Guerrier et al., 2019). Thus, the question of teaching and learning proofs, problematic at the secondary-tertiary transition, joins current issues in mathematics, computer science and didactics. Isolated case studies show the interest of working at the articulation between formal proofs with PA and paper-and-pencil proofs as well as the contribution of PA for students in terms of their autonomy in controlling the proofs they produce, structuring the proofs, and understanding the mathematical statements (Kerjean et al., 2022; Avigad, 2019; Thoma & Iannone, 2022). Thus, recent syntheses on the question, based on the hindsight of Technology Enhanced Learning (TEs), in particular on the nature of the feedback provided by the machine, call for a multidisciplinary structuring of research to analyse the uses and contributions of PAs in the learning of proof (Hanna et al., 2019; Balacheff & Boy de la Tour, 2019).

Research questions

I have been designing a study to explore and impact high-school and undergraduate students' conceptions of proof. I am using three devices in France: a questionnaire for a large-scale study (with open and multiple-choices questions), interviews with pairs of students to refine the results of the questionnaire, and interventions with PA to try to impact students' conceptions and performances. My research questions coincide with two of those listed by Harel & Fuller (2009, p. 355-356): What are students' current conceptions of proof? What theoretical tools seem suitable to investigate students' conceptions of proof? My main interest lies in the secondary-tertiary transition, but I am also interested in teacher training. I will propose an exploratory questionnaire to identify proof conceptions of high-school and undergraduate students and to conduct pre- and post- diagnosis when specific interventions (with PA for instance) are implemented. In this context, the aims of this paper are as follows:

- to conduct a preliminary bibliographical study in order to identify where and how undergraduate students' conceptions of proof are explored in research,
- and to propose a way to explore such conceptions in a collaborative perspective with a questionnaire (usable in a large-scale study).

Firstly, I will present a bibliographical study with a focus on the beginning of university, and, secondly, my theoretical and methodological choices to assess students' proof conceptions and to gain insights into students' understanding of the nature of proof. I will conclude with proposals for collaborations.

Bibliographical study

In order to circumscribe how researchers in mathematics education have investigated the undergraduate students' conceptions about proof, I have conducted a bibliographical study aimed at identifying the nature of the mathematical domains involved, the methods and the theoretical underlying background used in such surveys. I have only used one database, the Education Resources Information Center (ERIC), a large online digital library of education research. With the following search in ERIC in February 2023 “(mathematics AND proof AND students AND undergraduate) AND (conception OR survey OR interview OR questionnaire)”, I found for “All Publication Type” and “Any Education Level”, the results summarised in Table 1.

PUBLICATION DATE		PUBLICATION TYPE		EDUCATION LEVEL	
In 2023	0	Journal Articles	47	Higher Education	56
Since 2022	4	Reports - Research	43	Postsecondary Education	28
Since 2019 (last 5 years)	17	Collected Works - Proceedings	8	Elementary Secondary Education	7
Since 2014 (last 10 years)	38	Dissertations/Theses - ...	8	Education	
Since 2004 (last 20 years)	63	Reports - Descriptive	5	Secondary Education	5
		Speeches/Meeting Papers	5	High Schools	2
		Reports - Evaluative	4	Early Childhood Education	1
		Books	1	Elementary Education	1
		Collected Works - General	1	Junior High Schools	1
		Tests/Questionnaires	1		

Table 1: Results of the search in ERIC (Feb. 2023) to identify publications exploring undergraduate students' conceptions of proof

For this article, I have then made a focus on “Journal Articles” these last ten years (Since 2014)¹: 30 articles remained. I have read the 30 articles and classified them with 8 criteria: name of the journal, year, country of the study, method, population analysed in the study, number of participants, theoretical tools and mathematical contents. The reading of the articles led me to define 5 exclusion criteria (Table 2). This process eliminated 14 articles, leaving a corpus of 16 articles. 10 papers of the 16 retained referred to the USA. Europe is poorly represented with only 1 article (Table 3).

The majority of the papers used interviews with students, always after a preliminary survey, questionnaire, assessment, or analysis of students' writings or readings proofs. Only one paper conducted a complete large-scale survey with early undergraduate students (Stylianou et al., 2015).

The methods and the theoretical background clearly depend on the research questions and on the mathematical contents. Most of the mathematical fields involved in the 16 selected papers are very specific to university (functions and continuity, differentiability, integrability, Cauchy generalized mean value theorem; limits of sequences; infinite series; linear algebra; holomorphic functions; proof by contradiction). Some of the papers explore more transversal domains such as arithmetic, geometry and algebra. The following list of the theoretical backgrounds used in the selected papers reflects a wide variety of approaches: Tall's three worlds and Harel's intellectual need, metacognition, local and holistic understanding of proof, Mejia and Ramos's assessment model inspired by those of Yang and Lin, Harel and Sowder's proof schemes, Weber's classification on proof validation, semantic and

¹ I could present the results of the entire bibliographical study during the Congress.

syntactic perspective, cognitive unity, Balacheff's typology of arguments. To date, in the last 10 years, no systematic large-scale diagnostic survey has been conducted in Europe on students' conceptions of proof, particularly at the transition from secondary to tertiary education.

Exclusion criteria	Number of excluded articles
Focus on resources (syllabi, workbooks etc.), teaching methods, task design, etc.	5
Focus on mathematical concepts but not on proof	3
Focus on mathematicians	2
Focus on teachers	3
Off topic	1
Total	14

Table 2: Exclusion criteria

Countries	
Algeria	1
Indonesia	2
Iran	1
Lesotho (South Africa)	1
Turkey	1
USA	10
Total	16

Table 3: Countries of the 16 selected papers

Design of a questionnaire on students' conceptions of proof

My research questions guided the design of the questionnaire, taking into account both proof as a product and proving as a process: How do students construct proofs? What are students' judgements of given proofs? What are students' global perceptions of proof? My aim is to build a questionnaire which will be usable throughout high-school, college, and pre-service teacher training for several snapshots: to identify students' conceptions and to evaluate the development of these conceptions in particular after classrooms' interventions (for instance with PA). Such a questionnaire requires several criteria: selecting mathematical contents where the obstacle of mastering the involved concepts is minimized; choosing mathematical problems outside of formalism (so that it is not an obstacle) and outside of curricula (to avoid obstacles or ready-made results and processes); placing global questions about proof at the end of the questionnaire, after students have worked on proofs.

The sections of the questionnaire

So far, the questionnaire has been written in French² and is currently being administered in universities in France and Belgium in a pilot testing phase. Five sections structure the questionnaire: Part I – Identification, enunciation and use of mathematical properties; Part II – Writing proofs; Part III – View of proofs and evaluation of proof; Part IV – General conception of proof; Part V – Respondent information (gender, school/university level, etc.).

² Available online: <https://hal.science/hal-03987587>

Theoretical backgrounds and aspects addressed by the questionnaire

Built upon different theoretical backgrounds and results previously quoted (mainly the significant students' difficulties even with deductive short proofs), my questionnaire explores "basic" components of deductive proofs through the following aspects:

In Part I, the students are asked to state a known formal statement (of their choice) and to write a formal statement starting from a representation (a representation used in Nelsen's visual proofs). They also have to use a given theorem in a specific configuration: I want to evaluate their abilities to check the hypotheses of a given theorem by themselves and to specify the instances of this theorem, as well as their abilities with formal statements (to correlate them with students' proof writings in Part II).

In Part II, the students are asked to write three proofs: a familiar one with several deductive steps in calculus (familiar because learnt at the beginning of high-school)³, a unfamiliar one in arithmetic (but with familiar contents such as divisible and odd numbers), a unfamiliar one in a formal, local imaginary axiomatic with non-representational symbols to see how the students make the statements work. The latter is inspired by Monks quoted in Reid and Knipping (2010, p.141): the symbols do not represent anything, and the rules only concerne syntax (assessing students' conceptions of this type of proof seems particularly interesting from the perspective of implementing PA).

In Part III, the students are asked to evaluate proofs. The design of the proof proposals follows previous large-scale surveys (Stylianou et al., 2015 with undergraduate students and Healy & Hoyles, 1998 with younger students) and Balacheff (1988a)'s characterisation of arguments (mainly naïve empiricism, generic argument and intellectual proof). The main difference lies in the choice of the mathematical domains. Four correct mathematical statements are given (both in familiar and unfamiliar fields: arithmetic, geometry, graph theory, combinatorial geometry). One comes from Healy & Hoyles (1998) (arithmetic) and one from Balacheff (1988b) (combinatorial geometry), both with adjustments. One is a classical adaptation of the sum of the angles of a triangle and one is in graph theory. For each of them, there are four proof proposals, presented in a variety of forms and arguments (following Balacheff 1988a's typology). Some of the proof proposals are false or incomplete from a mathematical point of view. The students are asked to tick their answer(s) in the following multiple choices questions (Table 4).

In Part IV, the questions explore the students' global conception of proof (the way they think about proof) with open questions leading to kinds of proofs, functions of proofs, declarative students' tools to write/read/evaluate proofs, and word clouds.

I contend that all questions, mathematical contents (familiar for several years or unfamiliar but accessible with discrete concepts) and direct proofs were chosen in order to have a transferable questionnaire (for different levels of high-school, university and teacher training). The main results (qualitative and quantitative) of the pilot testing phase of the questionnaire in France and Belgium will be presented at the Congress.

³ This statement to be proved can be changed depending on the country.

	Answer A	Answer B	Answer C	Answer D
Which answer is nearest to what you would have done?	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Which answer would you choose to explain the "solution" to your classmates?	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Which answer is the most "rigorous" for you?	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
What is/are the correct proof(s) for you?	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
What is/are the incomplete proof(s) in your opinion?	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

Table 4: Multiple-choices questions of Part III of the questionnaire

This article is an invitation for European collaborations on two levels: a large-scale survey about the development of students' difficulties with proof and proving from high-school to university and pre-service teacher training and an exploration of the potential of proof assistants.

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