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ON THE GLOBAL WELL-POSEDNESS OF THE CALOGERO–SUTHERLAND DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

RANA BADREDDINE

ABSTRACT. We consider the Calogero–Sutherland derivative nonlinear Schrödinger equation in the focusing (with sign $+$) and defocusing case (with sign $-$)

$$i\partial_t u + \partial_x^2 u \pm \frac{2}{i} \partial_x \Pi(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T},$$

where Π is the Szegő projector $\Pi(\sum_{n \in \mathbb{Z}} \hat{u}(n) e^{inx}) = \sum_{n \geq 0} \hat{u}(n) e^{inx}$. Thanks to a Lax pair formulation, we derive the *explicit solution* to this equation. Furthermore, we prove the *global well-posedness* for this L^2 -critical equation in all the Hardy Sobolev spaces $H_+^s(\mathbb{T})$, $s \geq 0$, with small L^2 -initial data in the focusing case, and for arbitrarily L^2 -data in the defocusing case. In addition, we establish the relative compactness of the trajectories in all $H_+^s(\mathbb{T})$, $s \geq 0$.

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1. INTRODUCTION

This paper aims to prove the global well-posedness for the Calogero–Sutherland derivative nonlinear Schrödinger equations on the torus ($x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$) :

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm 2D_+(|u|^2)u = 0, \\ u(t=0, x) = u_0, \quad x \in \mathbb{T}, \end{cases} \quad (\text{CS})$$

for small L^2 -initial data u_0 in the focusing case (with sign $+$), and for arbitrarily L^2 -initial data in the defocusing case (with sign $-$). The operator D_+ in the nonlinear term of (CS) denotes $D\Pi$, where $D = -i\partial_x$, and Π is the Szegő projector acting on $L^2(\mathbb{T})$ as

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) := \sum_{n \geq 0} \widehat{u}(n) e^{inx}, \quad (1.1)$$

with value onto the Hardy space

$$L_+^2(\mathbb{T}) := \{u \in L^2(\mathbb{T}) \mid \widehat{u}(n) = 0, \forall n \in \mathbb{Z}_{\leq -1}\} \equiv \Pi(L^2(\mathbb{T})). \quad (1.2)$$

We equip $L_+^2(\mathbb{T})$ with the standard inner product of $L^2(\mathbb{T})$, $\langle u \mid v \rangle = \int_0^{2\pi} u \bar{v} \frac{dx}{2\pi}$. Our interest focuses on studying this equation with an unknown function u taken in the Hardy space of the torus, with a certain regularity. Thus, we denote by $H_+^s(\mathbb{T})$, the subspace of the Sobolev space $H^s(\mathbb{T})$, defined as

$$H_+^s(\mathbb{T}) := H^s(\mathbb{T}) \cap L_+^2(\mathbb{T}), \quad s \geq 0, \quad (1.3)$$

and equipped with the Sobolev norm

$$\|u\|_{H^s} = \|\langle D \rangle^s u\|_{L^2}, \quad \langle D \rangle^s = (1 + |D|^2)^{s/2}.$$

In Physics, this dynamical (CS)-equation, also called as the Calogero–Moser equation in the real line, is derived from the classical Calogero–Sutherland–Moser system (or Toda system) introduced in the end sixties–early seventies [Ca69, Su71]. This physical model corresponds to a N -body problem describing the pairwise interactions of N identical particles on a circle, with an inverse sin-square potential (trigonometric-type potential) in the periodic case. Abanov–Bettelheim–Wiegmann show in [ABW09] that taking the thermodynamic limit of such a model, and applying a change of variables leads to the (CS)-equation. This latter equation can also be obtained as a limit of the *intermediate nonlinear Schrödinger equation* introduced by Pelinovsky [Pe95],

$$i\partial_t u = \partial_x u^2 + (i - T)\partial_x(|u|^2)u, \quad (\text{INS})$$

where T is the integral operator

$$Tu(t, x) = \frac{1}{2\delta} \text{p.v.} \int_{-\infty}^{+\infty} \coth\left(\frac{\pi(x-y)}{2\delta}\right) u(t, y) dy,$$

by taking $\delta \rightarrow \infty$. The complex function u in (INS) represents the envelope of the fluid, and δ denotes its total depth. By passing to the limit $\delta \rightarrow \infty$, one obtains [Pe95] the same equation as (INS) but with the Hilbert transform

$$Hu(t, x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{u(t, y)}{x - y} dy,$$

instead of T . And since the Szegő projector $\Pi_{\mathbb{R}} = \frac{1}{2}(\text{Id} + iH)$, then the (CS)–equation can also be interpreted as a model describing the interfacial wave packets in a deep stratified fluid.

It turns out that the Calogero–Sutherland DNLS equation is completely integrable. Thus, what does the word “integrability” mean? In line with the different perspectives developed by various schools, a number of definitions have been raised. If the word “integrable system” means for some researchers the existence of action–angle variables, a coordinate system in which the equation is completely solvable by quadratures, others would say that it refers to the existence of a Lax operator associated with the equation, and satisfying the isospectral property¹. However, a common facet of all these definitions is the presence of infinitely independent *integrals of motion*, or what we can also call *conservation laws*. Naturally, this infinite number of conservation laws plays a crucial role in proving some global well–posedness results.

In our case, Gérard–Lenzmann derived in [GL22, Lemma 5.1], for u sufficient regular, a Lax operator so that the focusing Calogero–Sutherland DNLS equation (CS)⁺ enjoys a Lax pair formulation on the real line \mathbb{R} . i.e., for any $u \in H_+^s(\mathbb{R})$ with s sufficiently large, there exist two operators (L_u, B_u) such that the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u], \quad [B_u, L_u] := B_u L_u - L_u B_u, \quad (1.4)$$

is satisfied with

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2. \quad (1.5)$$

The operator T_u is the Toeplitz operator of symbol u , and is defined for any $u \in L^\infty$ by

$$T_u f = \Pi(uf), \quad \forall f \in L_+^2, \quad (1.6)$$

where Π is the Szegő projector given in (1.1). In what follows, we check that this Lax equation holds true on the torus \mathbb{T} by retrieving the same Lax operators (L_u, B_u) as on the real line. And, as expected, through this Lax formalism, we

¹ See Remark 2.1.

derive infinite conservation laws $\langle (L_u + \lambda)^s 1 | 1 \rangle$, $\lambda \gg 0$, $s \geq 0$, in order to control the growth of the Sobolev norms $\|u(t)\|_{\dot{H}^s}$ uniformly for all $t \in \mathbb{R}$.²

Observe, the Calogero–Sutherland DNLS equation is invariant under the scaling

$$u(t, x) \mapsto \lambda^{1/2} u(\lambda^2 t, \lambda x), \quad \lambda \in \mathbb{R}, (t, x) \in I \times \mathbb{R}. \quad (1.7)$$

This suggests the L^2 –criticality of (CS) on \mathbb{R} as well as on \mathbb{T} . In [GL22, Theorem 2.1], the local well–posedness of the (CS) equation was achieved in $H_+^s(\mathbb{R})$ for $s > \frac{1}{2}$ by following the analysis of [deMP10]. In particular, for $s > \frac{3}{2}$, Gérard–Lenzmann [GL22, Proposition 2.1] used iterative schemes of Kato’s type and energy estimates to derive the local well–posedness in $H_+^s(\mathbb{R})$ [Sa79]. On \mathbb{T} , the same proof of iterative schemes holds, and we deduce the local well–posedness in $H_+^s(\mathbb{T})$ for $s > \frac{3}{2}$. Therefore, we denote by $\mathcal{S}^+(t)$ the flow of the focusing Calogero–Sutherland DNLS equation (CS)⁺ and by $\mathcal{S}^-(t)$ the flow of the defocusing equation (CS)[−] : for all $s > \frac{3}{2}$, $t \in I_{\max}$,

$$\begin{aligned} \mathcal{S}^\pm(t) : H_+^s(\mathbb{T}) &\longrightarrow H_+^s(\mathbb{T}) \\ u_0 &\longmapsto u(t) \end{aligned}, \quad (1.8)$$

where I_{\max} denotes the maximal interval of the existence of the solution.

1.1. Main results.

Some notation. In the sequel, we denote for any nonnegative integer a , by $\mathbb{N}_{\geq a}$ the subset of \mathbb{Z} given by $\{k \in \mathbb{Z} \mid k \geq a\}$. Moreover, we denote by $\mathcal{B}_{L_+^2}(r)$ the open ball of $L_+^2(\mathbb{T})$ centered at the origin, with radius $r > 0$.

The goal of the paper is to prove the global well–posedness of the L^2 –critical equation (CS) in all $H_+^s(\mathbb{T})$, $s \geq 0$. As a starting point, we state the results for the more challenging equation, the focusing Calogero–Sutherland DNLS equation

$$i\partial_t u + \partial_x^2 u + 2D_+(|u|^2)u = 0, \quad (\text{CS}^+) \quad (1.9)$$

then, we present the results for the defocusing case³

$$i\partial_t u + \partial_x^2 u - 2D_+(|u|^2)u = 0. \quad (\text{CS}^-) \quad (1.10)$$

Theorem 1.1. *For all $s > \frac{3}{2}$, the Calogero–Sutherland DNLS focusing equation (CS⁺) is globally well–posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a priori*

² In particular, one can see that the usual conservation laws : the average $\langle 1 | u \rangle$, and the L^2 norm $\|u\|_{L^2}$ are conserved for $s = 1$ and 2 , since by definition of $L_u = D - T_u T_u$ we have $L_u 1 = -\langle 1 | u \rangle u$.

³ We refer to the introduction of Weinstein [We15] for a mathematical and physical meaning of the terms focusing and defocusing for any dispersive equation.

bound holds,

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C ,$$

where $C = C(u_0, s) > 0$ is a positive constant.

Remark 1.1. The restriction of smallness on the L^2 -norm of the initial data, namely $\|u_0\|_{L^2} < 1$, appears after applying a sharp inequality (Lemma 2.7) in order to control the growth of the Sobolev norms $\|u(t)\|_{\dot{H}^s}$, $s \geq 0$ by the conservation laws. More details for an eventual way to avoid this condition are presented in Section 6, but so far it is still an open problem.

As a second step, we focus on the main point of this paper : how the flow $\mathcal{S}^+(t)$ defined globally on $H_+^2(\mathbb{T})$ for $u_0 \in \mathcal{B}_{L_+^2}(1)$, can be extended to less regularity spaces for instance $L_+^2(\mathbb{T})$? Recall, as noted in (1.7), the Calogero–Sutherland DNLS equation is L^2 -critical. Based on the previous Theorem, and under the notation $u^\varepsilon(t) = \mathcal{S}^+(t)u_0^\varepsilon$, $\varepsilon > 0$, we state the following result.

Theorem 1.2. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists a unique potential $u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T}))$ such that, for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, we have for all $T > 0$,*

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Moreover, the L^2 -norm of the limit potential u is conserved in time :

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}. \quad (1.9)$$

As a consequence, Theorem 1.2 leads to the global well-posedness of the (CS⁺) problem in $L_+^2(\mathbb{T})$ in the following sense : There exists a unique continuous extension of the flow defined on $H_+^2(\mathbb{T})$, to $L_+^2(\mathbb{T})$, generating a unique continuous map

$$u_0 \in \mathcal{B}_{L_+^2}(1) \longmapsto u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T})).$$

The key ingredient of the proof is to obtain $H^{\frac{1}{2}}$ bounds (inequality (3.20)) on the eigenfunctions of the Lax operator L_{u^ε} , which also constitute an orthonormal basis of $L_+^2(\mathbb{T})$. Therefore, we deduce the strong convergence of these eigenfunctions in L^2 . Finally, using Parseval's identity, we infer (1.9).

We also need to emphasize the important aspect of the uniqueness of the limit potential $u(t)$, obtained independently of the choice of the sequence (u_0^ε) that approximates $u_0 \in L_+^2(\mathbb{T})$. For this purpose, we derive in Proposition 2.5, **an explicit formula of the solution** of the focusing (CS⁺) equation. Thus, for any initial data u_0 , the solution of the (CS⁺) focusing equation is given by

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 \mid 1 \rangle, \quad z \in \mathbb{D} := \{|z| < 1\}, \quad (1.10)$$

where S^* denotes the adjoint of the Shift operator $S : h \mapsto zh$ in $L_+^2(\mathbb{T})$, and L_{u_0} is the Lax operator at $t = 0$. We underline two important facts about (1.10) :

- I. First, this inversion dynamical formula defined inside the open unit disc consists an explicit solution for the nonlinear PDE (CS⁺). This is not the first time that an explicit solution occurs while dealing with nonlinear integrable PDEs. Indeed, Gérard–Grellier derived in [GG15] an explicit solution for the Szegő equation, and recently Gérard also prove in [Ger22] that the Benjamin–Ono equation has an explicit solution on \mathbb{R} and on \mathbb{T} . The common point to all these dynamical explicit formulas is that they all rely closely on the structure of the Lax operators induced by these equations.
- II. Beyond the fact that we have an explicit solution, this formula stresses out that the dynamics of the (CS⁺) equation are encoded by the Lax operator L_{u_0} , suggesting thus, that the so-called *actions–angles variables* must be related to the spectral elements of the Lax operators L_u .

In view of Theorem 1.2, we state the third result.

Corollary 1.3. *For all $0 \leq s \leq \frac{3}{2}$, the Calogero–Sutherland DNLS focusing equation (CS⁺) is globally well-posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C ,$$

where $C = C(u_0, s) > 0$ is a positive constant.

Remark 1.2. There is a subtlety hidden in the worlds of “globally well-posed” in the last statement. In fact, it is important to distinguish here the two different aspects of global well-posedness. First, we have the classical definition of GWP used in Theorem 1.1: for any $u_0 \in H_+^s$ there exists a unique solution u defined on \mathbb{R} with value in H_+^s , such that u depends continuously on the initial data u_0 as a map $u_0 \in H_+^s \mapsto u \in \mathcal{C}(\mathbb{R}, H_+^s)$. The second definition is the one described in Theorem 1.2 in the sense : we suppose that the equation is defined at least in the distribution sense, then we extend the flow defined on high regularity spaces to low regularity spaces through continuous extension.



In this corollary, the global well-posedness is in the sense used in Theorem 1.2. This will become clearer once the proof is established (see Section 4). We also expect that, following arguments in [deMP10], one can go down for the global well-posedness in the classical sense to $H_+^s(\mathbb{T})$ with $s > \frac{1}{2}$.

Beyond the global well-posedness results on the Cauchy Problem of (CS⁺), we are interested in some qualitative properties about the flow $\mathcal{S}^+(t)$ of this equation.

Theorem 1.4. *Given an initial data $u_0 \in \mathcal{B}_{L^2_+}(1) \cap H^s_+(\mathbb{T})$, $s \geq 0$, the orbit of the solution $\{\mathcal{S}^+(t)u_0; t \in \mathbb{R}\}$ is relatively compact in $H^s_+(\mathbb{T})$.*

The defocusing equation (CS⁻). Moving now to the defocusing case of the Calogero–Sutherland DNLS equation, this latter equation enjoys also a Lax pair structure : for any $u(t) \in H^s_+$, with s large enough, there exist two operators

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2,$$

satisfying the Lax equation

$$\frac{d\tilde{L}_u}{dt} = [\tilde{B}_u, \tilde{L}_u].$$

Therefore, using the same methods as on the focusing case, we prove that the conservation laws $\langle \tilde{L}_u^s 1 \mid 1 \rangle$, $s \geq 0$, controls uniformly the growth of the Sobolev norms without requiring any additional condition on the initial data. As a consequence, we obtain similar results in the defocusing case as in the focusing case, *regardless of how large the initial data is in L^2* . To summarize, we have the following.

Theorem 1.5. *The Calogero–Sutherland DNLS defocusing equation (CS⁻) is globally well-posed in $H^s_+(\mathbb{T})$ for any $s \geq 0$ in the sense of Remark 1.2. In addition, for all $u_0 \in H^s_+(\mathbb{T})$,*

$$u(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2it\tilde{L}_{u_0}} S^*)^{-1} u_0 \mid 1 \right\rangle,$$

is the solution to the (CS⁻)-defocusing equation. Furthermore, the trajectories

$$\{\mathcal{S}^-(t)u_0; t \in \mathbb{R}\}$$

are relatively compact in $H^s_+(\mathbb{T})$.

Other related equations. As explained in [GL22], the Calogero–Sutherland DNLS equation (CS) can be seen as mass critical version of the Benjamin–Ono equation. We refer to [GK21, GKT20] for a deep study of this latter equation on the torus. Of course, the Calogero–Sutherland DNLS equation (CS) is also considered as part of the nonlinear Schrödinger’s family. Several authors have been interested in different types of NLS-equations over the years. Some of these

equations are classified and presented in [Bo99]. Maybe the most closely related to the (CS)–equation are :

(i) **Cubic NLS equation.**

$$i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, \quad (\text{NLS-cubic})$$

which is considered as one of the simplest PDE enjoying complete integrable properties. Zakharov–Shabat have studied this equation in [ZS72] using inverse scattering method. Moreover, global well-posedness results in $L^2(\mathbb{T})$ are presented in Bourgain [Bo93] after he introduced the $X^{s,b}$ –spaces. His proof relies on establishing $L^4(\mathbb{T})$ –Strichartz estimates and using L^2 –conservation norm. Actually, this result of L^2 –well-posedness is known to be sharp, and it is illustrated by various types of ill-posedness results below the regularity $L^2(\mathbb{T})$. Indeed, Burq–Gérard–Tzvetkov proved in [BGT02] that the flow map of (NLS-cubic) fails to be uniformly continuous for Sobolev regularity below L^2 . Christ–Colliander–Tao [CCT03] and Molinet [Mo09] showed the discontinuity of the map solution in $H^s(\mathbb{T})$ for $s < 0$.

For a deep study of (NLS-cubic) using integrable tools, Birkhoff normal form, and some applications, we refer to Kappeler–Lohrmann–Topalov–Zung [KL+17], Grébert–Kappeler [GK14] and Kappeler–Schaad–Topalov [KST17]. For a study on the line \mathbb{R} , we cite [HKV20]. More references are also provided in [OS12].

(ii) **DNLS equation.**

$$i\partial_t u + \partial_x^2 u + \pm i\partial_x (|u|^2 u) = 0, \quad (\text{DNLS})$$

which is also an integrable equation enjoying infinite conservation laws [KN78]. Using the I-method, Win proved in [Wi10] the global well-posed of (DNLS)–equation in $H^s(\mathbb{T})$, $s > \frac{1}{2}$ for small data in $L^2(\mathbb{T})$. More recently, Klaus–Schipa [KS22] presented law regularity a priori estimates of $\|u\|_{H^s}$ for $0 < s < \frac{1}{2}$ upon small L^2 –norm, where $u \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{S}(\mathbb{T}))$ and $\mathcal{S}(\mathbb{T})$ denotes the Schwartz space. Actually, they proved the a priori estimates

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{B_{r,2}^s} \lesssim \|u(0)\|_{B_{r,2}^s}$$

in any Besov space $B_{r,2}^s$, with $r \in [1, \infty]$ and $0 < s < \frac{1}{2}$. For a study on the line \mathbb{R} , we cite [JL+20, BP22, BLP21, KNV21, HKV21, HKNV22].

1.2. Outline of the paper.

The paper is organized as follows.

In *Section 2*, we discuss some properties about the Lax operators of the Calogero–Sutherland DNLS focusing equation (CS⁺). We derive the explicit formula of the solution of (CS⁺) in the first subsection 2.1. Then, we prove in the

second subsection 2.2, the global well-posedness of the (CS^+) problem in $H_+^s(\mathbb{T})$ for any $s > \frac{3}{2}$.

In Section 3, we extend the flow $\mathcal{S}^+(t)$ of (CS^+) continuously from $H_+^2(\mathbb{T})$ to $L_+^2(\mathbb{T}) \equiv H_+^0(\mathbb{T})$. To this end, we use an approximation method, and we characterize in the first subsection 3.1.1 the limit potential $u(t)$ for all $t \in \mathbb{R}$. Then, in the second subsection 3.2, we make sure that the lack of compactness in $L_+^2(\mathbb{T})$ do not occur while passing to the limit from $H_+^2(\mathbb{T})$ to $L_+^2(\mathbb{T})$. In the same subsection, we derive an orthonormal basis of $L_+^2(\mathbb{T})$ where the coordinates of the solution $u(t)$ have nice evolution in this basis. This evolution suggests that the so-called “Birkhoff coordinates” are the coordinates of $u(t)$ in this basis.

After that, we deal in Section 4 with the problem of global well-posedness of (CS^+) in $H_+^s(\mathbb{T})$ for $0 < s \leq \frac{3}{2}$. Moreover, we address the property of relative compactness of the orbits of (CS^+) in $H_+^s(\mathbb{T})$, $s \geq 0$.

Moving to Section 5, we present the Lax pair for the defocusing Calogero–Sutherland DNLS equation (CS^-) and we state the analogous results of (CS^+) in the case of (CS^-) .

Finally, in Section 6, we discuss some remarks and open problems related to this equation.

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2. THE LAX PAIR STRUCTURE

As noted in the introduction, we first check that the Lax pair defined in (1.5) holds the same in the context of the torus \mathbb{T} as on the real line \mathbb{R} , even though on the real line \mathbb{R} , a complex function f is decomposed as

$$f = \Pi f + \overline{\Pi f}, \quad \widehat{\Pi f}(\xi) = \mathbf{1}_{\xi > 0} \widehat{f}(\xi), \quad \xi \in \mathbb{R},$$

while on the torus \mathbb{T} ,

$$f = \Pi f + \overline{\Pi f} - \langle f | 1 \rangle, \quad \Pi \left(\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx} \right) := \sum_{n \in \mathbb{N}_{\geq 0}} \widehat{f}(n) e^{inx},$$

Proposition 2.1 (The Lax pair). *For any $s > \frac{3}{2}$, let $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$ be a solution of the focusing equation (CS^+) . Then, there exist two operators*

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$$

satisfying the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u],$$

where T_u is the Toeplitz operator defined in (1.6).

Proof. Let $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, be a solution of (CS^+) equation. On the one hand, we have by definition of L_u and for all $h \in H_+^1(\mathbb{T})$,

$$\begin{aligned} \frac{dL_u(h)}{dt} &= -T_{\partial_x u} T_{\bar{u}}(h) - T_u T_{\partial_x \bar{u}}(h) \\ &= -T_{i\partial_x^2 u + 2u\partial_x \Pi(|u|^2)} T_{\bar{u}} h - T_u T_{-i\partial_x^2 \bar{u} + 2\bar{u}\partial_x \overline{\Pi(|u|^2)}} h \end{aligned}$$

Therefore, since u belongs to the Hardy space,

$$\frac{dL_u}{dt} = i[T_u T_{\partial_x^2 \bar{u}} - T_{\partial_x^2 u} T_{\bar{u}}] - 2u \left[\partial_x \Pi(|u|^2) \Pi(\bar{u} \cdot) + \Pi(\partial_x \overline{\Pi(|u|^2)} \bar{u} \cdot) \right]. \quad (2.1)$$

On the other hand, expanding the commutator $[B_u, L_u](h) = B_u L_u h - L_u B_u h$, we obtain

$$\begin{aligned} &T_u T_{\partial_x \bar{u}} D h - T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}} h - T_{\partial_x u} T_{\bar{u}} D h + T_{\partial_x u} T_{\bar{u}} T_u T_{\bar{u}} h + i(T_u T_{\bar{u}})^2 D h \\ &- D T_u T_{\partial_x \bar{u}} h + T_u T_{\bar{u}} T_u T_{\partial_x \bar{u}} h + D T_{\partial_x u} T_{\bar{u}} h - T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} h - i D (T_u T_{\bar{u}})^2 h, \end{aligned}$$

where by the Leibniz rule, $D T_u h = -i T_{\partial_x u} h + T_u D h$, so that

$$\begin{aligned} D T_u T_{\partial_x \bar{u}} &= T_u T_{\partial_x \bar{u}} D - i T_{\partial_x u} T_{\partial_x \bar{u}} - i T_u T_{\partial_x^2 \bar{u}}, \\ D T_{\partial_x u} T_{\bar{u}} &= T_{\partial_x u} T_{\bar{u}} D - i T_{\partial_x^2 u} T_{\bar{u}} - i T_{\partial_x u} T_{\partial_x \bar{u}}, \\ D (T_u T_{\bar{u}})^2 &= -i(T_{\partial_x u} T_{\bar{u}} T_u T_{\bar{u}} + T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}} + T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} + T_u T_{\bar{u}} T_u T_{\partial_x \bar{u}}) + (T_u T_{\bar{u}})^2 D. \end{aligned}$$

As a consequence,

$$\begin{aligned} [B_u, L_u] &= i T_u T_{\partial_x^2 \bar{u}} - i T_{\partial_x^2 u} T_{\bar{u}} - 2(T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} + T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}}) \\ &= i[T_u T_{\partial_x^2 \bar{u}} - T_{\partial_x^2 u} T_{\bar{u}}] - 2u \Pi(\partial_x |u|^2 \Pi(\bar{u} \cdot)). \end{aligned} \quad (2.2)$$

Comparing (2.1) and (2.2), it appears that all that remains to be proved is

$$\left[\partial_x \Pi(|u|^2) \Pi(\bar{u} h) + \Pi(\partial_x \overline{\Pi(|u|^2)} \bar{u} h) \right] = \Pi(\partial_x |u|^2 \Pi(\bar{u} h)), \quad h \in H_+^1(\mathbb{T}). \quad (2.3)$$

In fact, any complex function $f \in L^2(\mathbb{T})$ can be decomposed as

$$f = \Pi f + \overline{\Pi f} - \langle f | 1 \rangle.$$

In particular, for $f = \bar{u} h$, we have $\Pi(\partial_x \overline{\Pi(|u|^2)} \bar{u} h)$ equal to

$$\Pi(\partial_x \overline{\Pi(|u|^2)} \Pi(\bar{u} h)) + \Pi(\partial_x \overline{\Pi(|u|^2)} \overline{\Pi(\bar{u} h)}) - \langle \bar{u} h | 1 \rangle \Pi(\partial_x \overline{\Pi(|u|^2)}),$$

where the last two terms vanishes, since Π is an orthogonal projector into the Hardy space. Therefore, the left-hand side of (2.3) coincides with

$$\Pi(\partial_x \Pi(|u|^2) \Pi(\bar{u} h)) + \Pi(\partial_x \overline{\Pi(|u|^2)} \Pi(\bar{u} h)),$$

which is equal to $\Pi(\partial_x |u|^2 \Pi(\bar{u} h))$ since $\langle \partial_x(|u|^2) | 1 \rangle = 0$. \square

2.1. The explicit formula of the solution. Using this Lax pair structure, we derive in this subsection the explicit formula, solution of the focusing Calogero–Sutherland DNLS equation (CS⁺). To this end, we also need the shift operator introduced in the following paragraph.

Some Preliminaries. We recall one of the most important operator on Hardy's space, *the shift operator*, defined on $L_+^2(\mathbb{T})$ as the isometric map

$$S: h \in L_+^2(\mathbb{T}) \mapsto e^{ix}h \in L_+^2(\mathbb{T}).$$

Its adjoint in $L_+^2(\mathbb{T})$ is given by

$$S^*: h \in L_+^2(\mathbb{T}) \mapsto S^*h = T_{e^{-ix}}h = \Pi(e^{-ix}h) \in L_+^2(\mathbb{T}).$$

In particular, we have

$$S^*S = \text{Id}, \quad SS^* = \text{Id} - \langle \cdot | 1 \rangle 1, \quad (2.4)$$

leading to the fact that the shift map S is injective but not surjective. Pointing out that the Hardy space can be defined with different approaches, for instance,

$$\mathbb{H}_2(\mathbb{D}) := \left\{ u \in \text{Hol}(\mathbb{D}); \sup_{0 \leq r < 1} \int_0^{2\pi} |u(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\},$$

which is equivalent via the isometric isomorphism

$$u(z) = \sum_{k \geq 0} \hat{u}(k) z^k \mapsto u^*(x) := \sum_{k \geq 0} \hat{u}(k) e^{ikx},$$

to the Hardy space $L_+^2(\mathbb{T})$ defined in (1.2), then one could read the shift operator acting as multiplication by z . In what follows, we use indifferently u and the boundary function u^* , by making a slight abuse of notation and denoting both by u .

Coming back to the problem, we need some commutator identities to obtain the explicit formula. This is the purpose of the next Lemma.

Lemma 2.2. *Let $u \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, then*

$$\begin{aligned} [S^*, L_u] &= S^* - \langle \cdot | u \rangle S^* u, \\ [S^*, B_u] &= i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right). \end{aligned} \quad (2.5)$$

Proof. The first identity is a direct consequence of proving

$$L_u S = S L_u + S - \langle \cdot | S^* u \rangle u, \quad (2.6)$$

and taking the adjoint of all these operators in $L_+^2(\mathbb{T})$. Recall $L_u = D - T_u T_{\bar{u}}$. On the one hand, we have by the Leibniz rule $D(Sh) = S(\text{Id} + D)h$, for all $h \in H_+^1(\mathbb{T})$. On the other hand, observe for all $f \in L^2(\mathbb{T})$,

$$\Pi(Sf) = S\Pi(f) + \langle Sf | 1 \rangle.$$

In particular, for $f = h\bar{u}$, we infer

$$T_{\bar{u}}(Sh) = ST_{\bar{u}}h + \langle Sh | u \rangle. \quad (2.7)$$

Hence, taking into consideration that the operators S and T_u commute, we deduce identity (2.6).

Now, to prove the second point of (2.5) we use the first point. Recall that $B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$, and by (2.7) we have $[T_{\bar{u}}, S] = \langle \cdot | S^* u \rangle$, in other words, $[S^*, T_u] = \langle \cdot | 1 \rangle S^* u$. Thus, after noting that S^* and $T_{\bar{u}}$ commute, we deduce

$$\begin{aligned} [S^*, T_u T_{\partial_x \bar{u}}] &= \langle \cdot | \partial_x u \rangle S^* u, \\ [S^*, T_{\partial_x u} T_{\bar{u}}] &= \langle \cdot | u \rangle S^* \partial_x u, \\ [S^*, (T_u T_{\bar{u}})^2] &= \langle \cdot | T_u T_{\bar{u}} u \rangle S^* u + T_u T_{\bar{u}} (\langle \cdot | u \rangle S^* u). \end{aligned}$$

As a result,

$$[S^*, B_u] = \langle \cdot | \partial_x u \rangle S^* u - \langle \cdot | u \rangle S^* \partial_x u + i \langle \cdot | T_u T_{\bar{u}} u \rangle S^* u + iT_u T_{\bar{u}} (\langle \cdot | u \rangle S^* u).$$

Using the adjoint Leibniz rule $S^* D = (D + \text{Id})S^*$ and since $L_u = D - T_u T_{\bar{u}}$, we infer

$$\begin{aligned} [S^*, B_u] &= -i \langle \cdot | L_u u \rangle S^* u - i L_u (\langle \cdot | u \rangle S^* u) - i \langle \cdot | u \rangle S^* u \\ &= -i (\langle \cdot | u \rangle S^* u) L_u - i (L_u + \text{Id}) (\langle \cdot | u \rangle S^* u). \end{aligned}$$

We conclude by the first identity of (2.5) that $-\langle \cdot | u \rangle S^* u = S^* L_u - L_u S^* - S^*$ and hence

$$[S^*, B_u] = i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right).$$

□

Proposition 2.3. *Let $u(t) \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$. The Lax operator $(L_{u(t)}, H_+^1(\mathbb{T}))$ is a self-adjoint operator with a discrete spectrum bounded from below. Moreover, $B_{u(t)}$ is a skew-symmetric bounded operator on $L_+^2(\mathbb{T})$.*

Proof. The proof is a direct consequence of Kato–Rellich’s theorem. Indeed, the differential operator $(D, H_+^1(\mathbb{T}))$ is a positive self-adjoint operator on the Hardy space $L_+^2(\mathbb{T})$. In addition, $T_u T_{\bar{u}}$ is relatively bounded with respect to D , since for all $h \in H_+^1(\mathbb{T})$,

$$\|T_u T_{\bar{u}} h\|_{L^2} \leq \|u\|_{L^\infty}^2 \|h\|_{L^2} \leq \varepsilon \|Dh\|_{L^2(\mathbb{T})} + \|u\|_{L^\infty}^2 \|h\|_{L^2(\mathbb{T})}, \quad 0 \leq \varepsilon < 1.$$

Furthermore, the spectrum of L_u is discrete since the resolvent of L_u is compact by the Rellich–Kondrachov theorem. And it is bounded from below as the operator L_u is a semi-bounded operator. Besides, one can easily observe by definition of $B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$, that this operator is a skew-symmetric operator.

□

In view of the previous proposition, we denote by $(\lambda_n(u))_{n \geq 0}$ the eigenvalues of L_u ordered by increasing modulus, and taking into account their multiplicity

$$\lambda_0(u) \leq \lambda_1(u) \leq \lambda_2(u) \leq \dots \leq \lambda_n(u) \leq \dots$$

Remark 2.1. As discovered in the modern theory of integrable systems [GG+67] and reformulated by [Lax68], the eigenvalues of a Lax operator correspond to the integrals of motion of the associated equation. In fact, any Lax operator satisfies the *isospectral property*, namely, there exists a one parameter family of unitary operators $U(t)$ such that $U(t)^{-1}L_{u(t,\cdot)}U(t)$ is independent of t . That is,

$$U(t)^{-1}L_{u(t)}U(t) = L_{u_0}. \quad (2.8)$$

In addition, this family of unitary operators $U(t)$ is solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt}U(t) = B_{u(t,\cdot)}U(t) \\ U(0) = \text{Id}. \end{cases} \quad (2.9)$$

Therefore, by (2.8) the eigenvalues $(\lambda_n(u))$ of L_u are all conserved along the flow of (CS⁺). That means, $\lambda_n(u(t)) = \lambda_n(u_0)$ for all $t \in \mathbb{R}$.

The following lemma provides a rewrite of the Calogero–Sutherland DNLS equation focusing on (CS⁺) in terms of the Lax operators L_u and B_u . This will certainly be useful during the proof of the dynamical explicit formula.

Lemma 2.4. *Given $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, a solution of (CS⁺) equation, then*

$$\partial_t u = B_u u - iL_u^2 u.$$

Proof. By definition of $B_u := T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$,

$$\begin{aligned} \partial_t u - B_u u &= i\partial_x^2 u + 2iD_+(|u|^2)u - T_u T_{\partial_x \bar{u}} u + T_{\partial_x u} T_{\bar{u}} u - i(T_u T_{\bar{u}})^2 u \\ &= -i \left[D^2 u - 2u.D\Pi(|u|^2) + u.\Pi(D\bar{u}.u) - Du.\Pi(|u|^2) + (T_u T_{\bar{u}})^2 u \right]. \end{aligned}$$

Applying Leibniz's rule on $D(u.\Pi(|u|^2))$, we infer

$$\partial_t u - B_u u = -i \left[D^2 u - D \left[\Pi(|u|^2).u \right] + u.\Pi(D\bar{u}.u) - u.D\Pi(|u|^2) + (T_u T_{\bar{u}})^2 u \right].$$

Again, using Leibniz's rule on the term $D\Pi(|u|^2)$,

$$\partial_t u - B_u u = -i \left[D^2 u - DT_u T_{\bar{u}} u - u.\Pi(\bar{u}.Du) + (T_u T_{\bar{u}})^2 u \right] = -iL_u^2 u.$$

□

Following [Ger22] and [GG15], we derive the explicit formula for the solution of the Calogero–Sutherland DNLS focusing equation.

Proposition 2.5 (The explicit formula). *Given $u_0 \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, the solution of the focusing Calogero–Sutherland DNLS equation (CS⁺) is given by*

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 \mid 1 \rangle, \quad \forall z \in \mathbb{D}.$$

Proof. Since $u(t, \cdot) \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, for all $t \in [-T, T]$, then for all $z \in \mathbb{D}$

$$u(t, z) = \sum_{k=0}^{\infty} \widehat{u}(t, k) z^k = \sum_{k=0}^{\infty} \langle u(t) \mid S^k 1 \rangle z^k = \sum_{k=0}^{\infty} \langle (S^*)^k u(t) \mid 1 \rangle z^k,$$

where by the Neumann series of

$$\sum_{k=0}^{\infty} (zS^*)^k = (\text{Id} - zS^*)^{-1},$$

we infer

$$u(t, z) = \langle (\text{Id} - zS^*)^{-1} u(t) \mid 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (2.10)$$

Recall by Remark 2.1, there exists a unitary operator $U(t)$ solution of the Cauchy problem (2.9). Hence,

$$\begin{aligned} u(t, z) &= \langle U(t)^* (\text{Id} - zS^*)^{-1} u(t) \mid U(t)^* 1 \rangle \\ &= \langle (\text{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* u(t) \mid U(t)^* 1 \rangle. \end{aligned} \quad (2.11)$$

Since $\partial_t U(t) = B_{u(t)} U(t)$ by (2.9), and as the operator B_u is a skew-adjoint operator (Proposition 2.3), we find

- $\frac{d}{dt}[U(t)^* 1] = -U(t)^* B_{u(t)} 1 = -iU(t)^* L_{u(t)}^2 1.$
- $\frac{d}{dt}[U(t)^* u(t)] = -U(t)^* B_{u(t)} u(t) + U(t)^* \partial_t u(t) = -iU(t)^* L_{u(t)}^2 u(t)$ by Lemma 2.4.
- $\frac{d}{dt}[U(t)^* S^* U(t)] = -U(t)^* B_{u(t)} S^* U(t) + U(t)^* S^* B_{u(t)} U(t) = U(t)^* [S^*, B_{u(t)}] U(t).$

where the third point is equal to

$$\frac{d}{dt}[U(t)^* S^* U(t)] = iU(t)^* \left(S^* L_{u(t)}^2 - (L_{u(t)} + \text{Id})^2 S^* \right) U(t)$$

by Lemma 2.2. Therefore, using the identity $U(t)^* L_{u(t)} = L_{u_0} U(t)^*$ of (2.8), we deduce

- $\frac{d}{dt}[U(t)^* 1] = -iL_{u_0}^2 [U(t)^* 1].$
- $\frac{d}{dt}[U(t)^* u(t)] = -iL_{u_0}^2 [U(t)^* u(t)].$
- $\frac{d}{dt}[U(t)^* S^* U(t)] = i \left([U(t)^* S^* U(t)] L_{u_0}^2 - (L_{u_0} + \text{Id})^2 [U(t)^* S^* U(t)] \right)$

As a consequence,

$$U(t)^* 1 = e^{-itL_{u_0}^2} 1, \quad U(t)^* u(t) = e^{-itL_{u_0}^2} u(t), \quad (2.12)$$

and

$$U(t)^* S^* U(t) = e^{-it(L_{u_0} + \text{Id})^2} S^* e^{itL_{u_0}^2}. \quad (2.13)$$

Combining (2.11), (2.12) and (2.13), the claimed formula follows. \square

2.2. Global well-posedness of (CS^+) in $H_+^s(\mathbb{T})$, $s > \frac{3}{2}$. To prove the global well-posedness of (CS^+) , we need to derive some conservation laws and energy estimates.

Lemma 2.6 (Conservation laws). *Let $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $r > \frac{3}{2}$. For all $\lambda \gg 0$, the family $\{\mathcal{H}_s(u) := \langle (L_u + \lambda)^s u | u \rangle ; 0 \leq s \leq 2r\}$ is conserved by the flow of (CS) .*

Remark 2.2. Using complex interpolation method [Ta81, Chapter I. 4.], one can observe as demonstrate in Proposition 2.8, that the $\mathcal{H}_s(u) \leq C \|u\|_{H^{\frac{s}{2}}}^2$.

Proof. Given $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $r > \frac{3}{2}$, we consider the unitary operator $U(t)$ defined in Remark 2.1. Then, by (2.12), we know that $U(t)^* u(t) = e^{-itL_{u_0}^2} u(t)$. And, since L_u is a self-adjoint operator by Proposition 2.3, we infer by (2.8),

$$U(t)^*(L_{u(t)} + \lambda)^s U(t) = (L_{u_0} + \lambda)^s.$$

Therefore, for all $0 \leq s \leq 2r$,

$$\begin{aligned} \mathcal{H}_s(u(t)) &= \langle U(t)^*(L_{u(t)} + \lambda)^s u(t) | U(t)^* u(t) \rangle \\ &= \langle (L_{u_0} + \lambda)^s U(t)^* u(t) | U(t)^* u(t) \rangle \\ &= \langle (L_{u_0} + \lambda)^s e^{-itL_{u_0}^2} u(t) | e^{-itL_{u_0}^2} u(t) \rangle \end{aligned}$$

As a consequence, $\mathcal{H}_s(u(t)) = \mathcal{H}_s(u_0)$ as $(L_{u_0} + \lambda)^s$ and $e^{-itL_{u_0}^2}$ commute. \square

Remark 2.3. Using the identity $U(t)^* 1 = e^{-itL_{u_0}^2} 1$ of (2.12) and repeating the same proof of Lemma 2.6, one can also deduce for $\lambda \gg 0$, that the quantities $\langle (L_u + \lambda)^q 1 | u \rangle$ and $\langle (L_u + \lambda)^p 1 | 1 \rangle$ are conserved by the flow. Another way to show this, is to observe by definition of $L_u = D - T_u T_{\bar{u}}$ we have $L_u 1 = -\langle 1 | u \rangle u$ and the average $\langle u | 1 \rangle$ is conserved along the evolution, since

$$\partial_t \langle u | 1 \rangle = i \langle \partial_x^2 u | 1 \rangle + 2i \langle D\Pi(|u|^2) | \bar{u} \rangle = 0.$$

To prove the energy estimates and for future requests, we need the following lemma.

Lemma 2.7. *Let $h \in H_+^{\frac{1}{2}}(\mathbb{T})$, $u \in L_+^2(\mathbb{T})$,*

$$\|T_{\bar{u}} h\|_{L^2(\mathbb{T})}^2 \leq \left(\langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2 \right) \|u\|_{L^2(\mathbb{T})}^2. \quad (2.14)$$

Proof. By Parseval's identity,

$$\|T_{\bar{u}} h\|_{L^2(\mathbb{T})}^2 = \sum_{n \geq 0} |\widehat{T_{\bar{u}} h}(n)|^2,$$

where

$$\widehat{T_{\bar{u}} h}(n) = \widehat{\Pi(h\bar{u})}(n) = \sum_{p \geq 0} \widehat{h}(n+p) \overline{\widehat{u}(p)}. \quad (2.15)$$

Applying Cauchy–Schwarz’s inequality, we infer

$$\|T_{\bar{u}}h\|_{L^2(\mathbb{T})}^2 \leq \|u\|_{L^2(\mathbb{T})}^2 \sum_{p \geq 0} \sum_{n \geq 0} |\widehat{h}(n+p)|^2.$$

Set $k = n + p$, then

$$\sum_{p \geq 0} \sum_{n \geq 0} |\widehat{h}(n+p)|^2 = \sum_{k \geq 0} (k+1) |\widehat{h}(k)|^2 = \langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2.$$

□

Remark 2.4. (1) Recall that the embedding $H^{\frac{1}{2}}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ fails to be true. However, taking the potential u as an element of the Hardy space $L_+^2(\mathbb{T})$, improved the estimate from $\|T_{\bar{u}}h\|_{L^2} \leq \|h\|_{L^\infty} \|u\|_{L^2}$ to (2.14).

(2) From (2.15), one could see that, for all $h \in H_+^{\frac{1}{2}}(\mathbb{T})$, the Hilbert–Schmidt norm of the antilinear operator $u \in L_+^2(\mathbb{T}) \mapsto T_{\bar{u}}h$ is given by

$$\|\Pi(\cdot h)\|_{HS}^2 = \sum_{p \geq 0} \sum_{n \geq 0} |\widehat{h}(n+p)|^2 = \sum_{k \geq 0} (k+1) |\widehat{h}(k)|^2 = \langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2.$$

In particular, we have $u \mapsto T_{\bar{u}}h$ is a compact antilinear operator in $L_+^2(\mathbb{T})$.

(3) The inequality (2.14) of Lemma 2.7 is a sharp inequality, since its proof relies on a simple application of the Cauchy–Schwarz inequality. In particular, if $h = u$, inequality (2.14) is an equality, if and only if

$$u(z) = \frac{c}{1 - qz}, \quad |q| < 1, \quad c \in \mathbb{C}.$$

Indeed,⁴ Cauchy–Schwarz’s inequality applied to (2.15) is an equality, if and only if for all $n, p \geq 0$, there exists $c_n \in \mathbb{C}$ such that

$$\widehat{u}(n+p) = c_n \widehat{u}(p). \quad (2.16)$$

Hence, if $\widehat{u}(1) \neq 0$ and $\widehat{u}(0) \neq 0$,

$$c_n = \frac{\widehat{u}(n+1)}{\widehat{u}(1)} = \frac{\widehat{u}(n)}{\widehat{u}(0)},$$

or in other words,

$$\widehat{u}(n) = \left(\frac{\widehat{u}(1)}{\widehat{u}(0)} \right)^n \widehat{u}(0), \quad \forall n \in \mathbb{N}.$$

Therefore, the sequence $(\widehat{u}(n))$ is a geometric progression with common ratio $q := \frac{\widehat{u}(1)}{\widehat{u}(0)}$, where $0 < |q| < 1$ since $\sum_{n=0}^{\infty} |\widehat{u}(n)|^2 < +\infty$. Hence,

$$u(z) = \sum_{n=0}^{\infty} \widehat{u}(n) z^n = \frac{\widehat{u}(0)}{1 - qz}.$$

Now, if $\widehat{u}(1) = 0$ or $\widehat{u}(0) = 0$ then by (2.16) we infer for $p = 0$ or 1 , $u = \widehat{u}(0)$.

⁴ Following arguments used in [GG08, Lemma 1.]

We recall that $\mathcal{B}_{L_+^2}(r)$ denotes the open ball of $L_+^2(\mathbb{T})$ centered at the origin, with radius $r > 0$.

Proposition 2.8. *Let $u_0 \in \mathcal{B}_{L_+^2}(1) \cap H_+^r(\mathbb{T})$, $r > \frac{3}{2}$. Then, for all $s \geq 0$, $\lambda \gg 0$, there exists $C = C(u_0, s) > 0$ independent of t , such that for every $f \in H_+^s(\mathbb{T})$,*

$$\frac{1}{C} \|f\|_{H^s} \leq \|(L_{u(t)} + \lambda)^s f\|_{L^2} \leq C \|f\|_{H^s} \quad (2.17)$$

Remark 2.5. The condition $r > \frac{3}{2}$ can be omitted once we prove in Section 4 that the flow $u_0 \in \mathcal{B}_{L_+^2}(1) \cap H_+^r(\mathbb{T}) \mapsto u(t) \in H_+^r(\mathbb{T})$ exists for all $r \geq 0$.

Proof. The proof is done by induction on every interval of length $1/2$.

Step 1 : $s \in [0, \frac{1}{2}]$. Let $s = \frac{1}{2}$, we have by definition of $L_u = D - T_u T_{\bar{u}}$,

$$\begin{aligned} \|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 &= \langle (L_u + \lambda) f \mid f \rangle \\ &\geq \|f\|_{\dot{H}^{\frac{1}{2}}}^2 - \|T_{\bar{u}} f\|_{L^2}^2 + \lambda \|f\|_{L^2}^2. \end{aligned} \quad (2.18)$$

Hence, applying the sharp inequality of Lemma 2.7,

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq (1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - \|u\|_{L^2}^2) \|f\|_{L^2}^2. \quad (2.19)$$

Thus, we infer since $\|u\|_{L^2} = \|u_0\|_{L^2} < 1$,

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq C \|f\|_{\dot{H}^{\frac{1}{2}}}^2,$$

where $C > 0$ a positive constant independent of t . On the other hand, using the definition of L_u , it is easy to see that

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2} \leq C \|f\|_{\dot{H}^{\frac{1}{2}}}.$$

Therefore, by complex interpolation [Ta81, Chapter I. 4], we deduce that inequality (2.17) holds true for all $s \in [0, \frac{1}{2}]$.

Step 2 : Uniform bounds on $\|u(t)\|_{L^p}$, $p \in [2, \infty)$. By step 1, and using the conservation laws of Lemma 2.6, we infer for $f = u$,

$$\frac{1}{C} \|u(t)\|_{H^s} \leq \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2} = \|(L_{u_0} + \lambda)^s u_0\|_{L^2} \leq C \|u_0\|_{H^s},$$

for all $s \in [0, \frac{1}{2}]$. Therefore, $\sup_{t \in \mathbb{R}} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} < \infty$, and thus by Sobolev embedding

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{L^p} < \infty, \quad \forall p \in [2, \infty).$$

Step 3 : $s \in [\frac{1}{2}, 1]$. Let $s = 1$, by Step 1,

$$\begin{aligned}
\|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})}^2 &= \|L_{u(t)}f\|_{L^2}^2 + \lambda \langle (L_{u(t)} + \lambda)f | f \rangle + \lambda \langle f | L_{u(t)}f \rangle \\
&\geq \|L_{u(t)}f\|_{L^2}^2 + \lambda \left((1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - \|u\|_{L^2}^2) \|f\|_{L^2}^2 \right) \\
&\quad + \lambda \left((1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 - \|u\|_{L^2}^2 \|f\|_{L^2}^2 \right) \\
&= \|Df\|_{L^2}^2 + \|uT_{\bar{u}}f\|_{L^2}^2 - 2\operatorname{Re} \langle Df | uT_{\bar{u}}f \rangle \\
&\quad + 2\lambda(1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + \lambda(\lambda - 2\|u\|_{L^2}^2) \|f\|_{L^2}^2
\end{aligned}$$

Using Young's inequality, we deduce

$$\begin{aligned}
\|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})}^2 &\geq (1 - \varepsilon) \|Df\|_{L^2}^2 + (1 - C_\varepsilon) \|uT_{\bar{u}}f\|_{L^2}^2 \\
&\quad + 2\lambda(1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - 2\|u\|_{L^2}^2) \|f\|_{L^2}^2.
\end{aligned}$$

Now, applying Cauchy-Schwarz's inequality on $\|uT_{\bar{u}}f\|_{L^2}$ and since $\|u\|_{L^8}$ and $\|u\|_{L^4}$ are uniformly bounded by Step 2., we infer

$$\|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})}^2 \geq C \|f\|_{H^1}^2.$$

On the other hand, we have by definition of L_u ,

$$\|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})} \leq C \|f\|_{H^1}.$$

We conclude by complex interpolation as in Step 1. that inequality (2.17) holds true for all $s \in [\frac{1}{2}, 1]$. Then, by induction on all the intervals $s \in [\frac{n}{2}, \frac{n}{2} + \frac{1}{2}]$, $n \in \mathbb{N}$, we have that inequality (2.17) is true for all $s \geq 0$. \square

Theorem. 1.1. *For all $s > \frac{3}{2}$, the Calogero-Sutherland DNLS focusing equation (CS⁺) is globally well-posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C,$$

where $C = C(u_0, s) > 0$ is a positive constant.

Proof. Let $u_0 \in H_+^s(\mathbb{T})$. Recall by (1.8), there exists a unique solution $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, satisfying $u(0, \cdot) = u_0$. In addition, by the previous proposition we infer for $\|u_0\|_{L^2} < 1$ and for all $s > \frac{3}{2}$,

$$\frac{1}{C(u_0, s)} \|u(t)\|_{H^s} \leq \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2} = \|(L_{u_0} + \lambda)^s u_0\|_{L^2} \leq C(u_0, s).$$

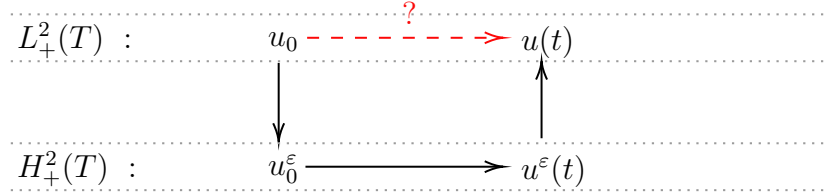
This allows us to conclude. \square

3. EXTENSION OF THE FLOW OF (CS^+) TO $L_+^2(\mathbb{T})$

In this section, we establish our main result, which states that, for $\|u_0\|_{L^2} < 1$, the flow of (CS^+)

$$\begin{array}{ccc} \mathcal{S}^+(t) : H_+^2(\mathbb{T}) & \longrightarrow & H_+^2(\mathbb{T}) \\ u_0 & \longmapsto & u(t) \end{array}, \quad (3.1)$$

defined globally on $H_+^2(\mathbb{T})$ via Theorem 1.1, can be extended continuously to the critical regularity $L_+^2(\mathbb{T})$. Our strategy is the following. Starting from $u_0 \in L_+^2(\mathbb{T})$, $\|u_0\|_{L^2} < 1$, we approximate u_0 by a sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$. After that, we consider the time evolution of (u_0^ε) , i.e. the potentials $(u^\varepsilon(t))$ defined as $u^\varepsilon(t) := \mathcal{S}^+(t)u_0^\varepsilon$ for all $t \in \mathbb{R}$. Our goal is to prove that (u^ε) has a unique limit u in $\mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T}))$ independent of the approximate sequence (u_0^ε) .



Note that, due to the presence of the nonlinear term $D\Pi_+(|u|^2)u$ in the equation, it may seem intriguing to say that there exists a solution with L^2 -regularity. Nevertheless, the equation is still well-defined in the distribution sense. As a matter of fact, one can find multiple examples of nonlinear PDE with this scenario. Take, for instance, the family of PDEs

$$i\partial_t u = \lambda u + |u|^q u, \quad \lambda \in \mathbb{R}, q > 1.$$

For an initial value $u_0 \in L^2$, the Cauchy problem associated with these equations—defined in the distribution sense—has a unique solution in L^2

$$u(t) = u_0 e^{-i(|u_0|^q + \lambda)t} \in L^2,$$

even though, at first sight, the nonlinear term $|u|^q u$ is not well-defined in L^2 .

In our approach, we use the approximation method described above, and the limit potential $u(t)$ obtained by this approximation method shall be called “solution” to the Cauchy problem (CS^+) . This solution will be uniquely well-characterized, continuous in time, inducing a global continuous flow on $L_+^2(\mathbb{T})$. In addition, it satisfies the L^2 invariant mass (i.e. $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ for all $t \in \mathbb{R}$).

3.1. Uniqueness of the limit and weak convergence in $L_+^2(\mathbb{T})$.

By passing to the limit as $\varepsilon \rightarrow 0$, it is necessary to prove first that the limit potential $u(t)$ is uniquely well-characterized for all $t \in \mathbb{R}$, and is independent of the choice of the sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ that approximate $u_0 \in L_+^2(\mathbb{T})$. The key point

is to use the explicit formula of the solution of the focusing Calogero–Sutherland DNLS equation. Thus, for all $(u^\varepsilon(t)) \subseteq H_+^2(\mathbb{T})$, we have by Proposition 2.5,

$$u^\varepsilon(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2itL_{u_0^\varepsilon}} S^*)^{-1} u_0^\varepsilon \mid 1 \right\rangle, \quad \forall z \in \mathbb{D}. \quad (3.2)$$

Our goal in this subsection is to pass to the limit in this formula. Therefore, we need at a first stage to give a meaning to the operator L_{u_0} with $u_0 \in L_+^2(\mathbb{T})$. To handle this, we recall in a few lines the work of Gérard–Lenzmann [GL22, Appendix A] who defined the operator L_u with $u \in L_+^2(\mathbb{R})$ via the standard theory of quadratic form. This new operator will coincide with the former Lax operator $L_u = D - T_u T_{\bar{u}}$ if $u \in H_+^2(\mathbb{R})$. The same proof presented in [GL22, Appendix A], works out on the torus \mathbb{T} . We recall the main points of the proof :

- (i) For $u \in L_+^2(\mathbb{T})$ and $f, g \in H_+^{\frac{1}{2}}(\mathbb{T})$, we consider the quadratic form

$$\mathcal{Q}_u(f, g) = \langle D^{1/2} f \mid D^{1/2} g \rangle - \langle T_{\bar{u}} f \mid T_{\bar{u}} g \rangle.$$

- (ii) Using the following inequality : for all $\eta > 0$, $\exists N_\eta \in \mathbb{N}_{\geq 0}$,

$$\|T_{\bar{u}} h\|_{L^2(\mathbb{T})}^2 < 2\eta^2 \left(\langle Dh \mid h \rangle + \|h\|_{L^2(\mathbb{T})}^2 \right) + 2N_\eta^2 \|u\|_{L^2}^2 \|h\|_{L^2(\mathbb{T})}^2, \quad (3.3)$$

–which can be obtained by decomposing u in high and low frequency

$$u(x) = u_N(x) + R_N(u, x), \quad \begin{cases} u_N(x) := \sum_{n \geq 0}^N \hat{u}(n) e^{inx} \\ R_N(u, x) := \sum_{n \geq N+1} \hat{u}(n) e^{inx} \end{cases},$$

and using Lemma 2.7– we infer

$$\mathcal{Q}_u(f, f) \geq (1 - 2\eta^2) \|f\|_{H^{1/2}}^2 - 2(N_\eta^2 \|u\|_{L^2}^2 + \eta^2) \|f\|_{L^2(\mathbb{T})}^2. \quad (3.4)$$

- (iii) Therefore, choosing η small enough, we introduce the positive definite quadratic form

$$\tilde{\mathcal{Q}}_u(f, g) := \mathcal{Q}_u(f, g) + K \langle f \mid g \rangle, \quad K := K(u) > 0, \quad f, g \in H_+^{\frac{1}{2}}(\mathbb{T}),$$

which define a new inner product on $H_+^{\frac{1}{2}}(\mathbb{T})$.

- (iv) Using the theory of quadratic forms (see [RS72]), we define for $u \in L_+^2(\mathbb{T})$,

$$\text{Dom}(L_u) = \left\{ h \in H_+^{\frac{1}{2}}(\mathbb{T}); \exists C > 0, |\tilde{\mathcal{Q}}_u(h, g)| \leq C \|g\|_{L^2(\mathbb{T})}, \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}) \right\}, \quad (3.5)$$

and for any $f \in \text{Dom}(L_u)$

$$\langle L_u(f) \mid g \rangle = \mathcal{Q}_u(f, g), \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}), \quad (3.6)$$

and one shows that this new operator L_u is a self-adjoint operator with a dense domain in $H_+^{\frac{1}{2}}(\mathbb{T})$.

3.1.1. *Spectral properties of L_{u_0} for $u_0 \in L_+^2(\mathbb{T})$.* Now that the operator L_{u_0} has been introduced for $u_0 \in L_+^2(\mathbb{T})$, one can examine some of its spectral properties. As noted above, the operator L_{u_0} is a self-adjoint operator with compact resolvent then it has discrete spectrum. In addition, the quadratic form \mathcal{Q}_{u_0} of L_{u_0} is bounded from below. Hence,

$$\sigma(L_{u_0}) := \{\lambda_0(u_0) \leq \dots \leq \lambda_n(u_0) \leq \dots\}, \quad \lambda_0 > -\infty. \quad (3.7)$$

To characterize this spectrum, we use the following proposition.

Proposition 3.1. *For every $n \in \mathbb{N}_{\geq 0}$, the map $u \in L_+^2(\mathbb{T}) \mapsto \lambda_n(u)$ is Lipschitz continuous on every bounded subsets of $L_+^2(\mathbb{T})$.*

Proof. Let $u \in L_+^2(\mathbb{T})$. By the min-max principle,

$$\lambda_n(u) = \max_{\substack{F \subseteq L_+^2 \\ \dim F \leq n}} \min \left\{ \mathcal{Q}_u(h, h); h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}), \|h\|_{L^2} = 1 \right\}.$$

For any subspace F of $L_+^2(\mathbb{T})$ of dimension n , let $h \in H_+^{\frac{1}{2}}(\mathbb{T}) \cap F^\perp$ and let $v \in L_+^2(\mathbb{T})$,

$$\begin{aligned} |\langle L_u h | h \rangle - \langle L_v h | h \rangle| &= \left| \|T_{\bar{v}} h\|_{L^2(\mathbb{T})}^2 - \|T_{\bar{u}} h\|_{L^2(\mathbb{T})}^2 \right| \\ &\leq \|T_{(\bar{u}-\bar{v})} h\|_{L^2(\mathbb{T})} \left(\|T_{\bar{v}} h\|_{L^2(\mathbb{T})} + \|T_{\bar{u}} h\|_{L^2(\mathbb{T})} \right) \end{aligned}$$

Applying the inequality of Lemma 2.7, we infer

$$\langle L_v h | h \rangle \leq \langle L_u h | h \rangle + C \|u - v\|_{L^2(\mathbb{T})} \left(\|u\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})} \right) \|h\|_{H^{\frac{1}{2}}}^3. \quad (3.8)$$

From $\text{Dom}(L_u) \cap F^\perp$, we select

$$h \in \bigoplus_{k=0}^n \ker(L_u - \lambda_k(u) \text{Id}) \cap \text{span} \{e^{ikx}; k = 0, \dots, n\}, \quad \|h\|_{L^2(\mathbb{T})} = 1.$$

Note that the intersection of all these sets is not empty (by checking dimensions). Therefore, $h = \sum_{k=0}^m c_k g_k(u)$, $m \geq n$, where the $(g_k(u))_{k=0}^m$ denotes any orthonormal basis of $\bigoplus_{k=0}^n \ker(L_u - \lambda_k(u) \text{Id})$, thus

$$\langle L_u h | h \rangle \leq \sum_{k=0}^n |c_k|^2 \lambda_k(u) \leq \lambda_n(u). \quad (3.9)$$

Moreover, since $h \in \text{span} \{e^{ikx}; k = 0, \dots, n\}$ then

$$\|h\|_{H^{\frac{1}{2}}}^2 = \langle (1 + D)h | h \rangle \leq 1 + n. \quad (3.10)$$

Combining (3.8), (3.9) and (3.10), we infer

$$\lambda_n(v) \leq \lambda_n(u) + C(1 + n)^{3/2} \|u - v\|_{L^2(\mathbb{T})} \left(\|u\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})} \right).$$

□

Corollary 3.2 (Characterization of the spectrum of L_{u_0}). *Let $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$. Then the spectrum of L_{u_0} , is given by*

$$\sigma(L_{u_0}) = \left\{ \lim_{\varepsilon \rightarrow 0} \lambda_n(u_0^\varepsilon) \mid \lambda_n(u_0^\varepsilon) \in \sigma(L_{u_0^\varepsilon}), n \in \mathbb{N}_{\geq 0} \right\}.$$

Proof. In light of the previous proposition, the result follows directly. \square

Proposition 3.3. *Let $u_0 \in L_+^2(\mathbb{T})$ and $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$. Then $L_{u_0^\varepsilon} \rightarrow L_{u_0}$ in the strong resolvent sense as $\varepsilon \rightarrow 0$.*

Proof. For $\lambda \ll 0$, let $\phi_\lambda^\varepsilon := (L_{u_0^\varepsilon} - \lambda)^{-1} h$, $h \in L_+^2(\mathbb{T})$. Since $(L_{u_0^\varepsilon} - \lambda)\phi_\lambda^\varepsilon = h$, then taking the inner product with ϕ_λ^ε ,

$$\mathcal{Q}_{u_0^\varepsilon}(\phi_\lambda^\varepsilon, \phi_\lambda^\varepsilon) - \lambda \|\phi_\lambda^\varepsilon\|_{L^2}^2 = \langle h \mid \phi_\lambda^\varepsilon \rangle. \quad (3.11)$$

Notice that since $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$ then the integer N_η in (3.3) is uniform for all $\varepsilon > 0$, and thus by (3.4) we deduce for $\eta = \frac{1}{2}$,

$$\mathcal{Q}_{u_0^\varepsilon}(\phi_\lambda^\varepsilon, \phi_\lambda^\varepsilon) - \lambda \|\phi_\lambda^\varepsilon\|_{L^2}^2 \geq \frac{1}{2} \|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2 - 2(N^2 \|u_0^\varepsilon\|_{L^2} + \frac{1}{4} + \lambda) \|\phi_\lambda^\varepsilon\|_{L^2}^2,$$

which leads, for $\lambda \ll 0$, to

$$\mathcal{Q}_{u_0^\varepsilon}(\phi_\lambda^\varepsilon, \phi_\lambda^\varepsilon) - \lambda \|\phi_\lambda^\varepsilon\|_{L^2}^2 \geq \frac{1}{2} \|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2,$$

That is, by (3.11),

$$\langle h \mid \phi_\lambda^\varepsilon \rangle \geq \frac{1}{2} \|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2. \quad (3.12)$$

Besides, in view of Corollary 3.2 and by (3.7), we have for all $\varepsilon > 0$,

$$\|\phi_\lambda^\varepsilon\|_{L^2} \leq \sup_n \frac{1}{|\lambda_n(u_0^\varepsilon) - \lambda|} \|h\|_{L^2} \leq C(\lambda) \|h\|_{L^2}. \quad (3.13)$$

Therefore, applying Cauchy–Schwarz’s inequality to (3.12), we deduce by (3.13),

$$\|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2 \leq C(\lambda) \|h\|_{L^2}^2, \quad \forall \varepsilon > 0.$$

Thus, up to a subsequence, there exists $\phi_\lambda \in L_+^2$ such that

$$\phi_\lambda^\varepsilon \rightharpoonup \phi_\lambda \text{ in } H_+^2(\mathbb{T}) \quad \text{and} \quad \phi_\lambda^\varepsilon \rightarrow \phi_\lambda \text{ in } L_+^2(\mathbb{T}).$$

It remains to show that $\phi_\lambda = (L_{u_0} - \lambda)^{-1} h$. Indeed, for any $g \in H_+^{\frac{1}{2}}(\mathbb{T})$, we have by definition of ϕ_λ^ε , $\langle (L_{u_0^\varepsilon} - \lambda) \phi_\lambda^\varepsilon \mid g \rangle = \langle h \mid g \rangle$. Namely,

$$\begin{aligned} \langle h \mid g \rangle &= \mathcal{Q}_{u_0^\varepsilon}(\phi_\lambda^\varepsilon, g) - \lambda \langle \phi_\lambda^\varepsilon \mid g \rangle \\ &:= \langle D^{\frac{1}{2}} \phi_\lambda^\varepsilon \mid D^{\frac{1}{2}} g \rangle - \langle T_{\bar{u}_0^\varepsilon} \phi_\lambda^\varepsilon \mid T_{\bar{u}_0^\varepsilon} g \rangle - \lambda \langle \phi_\lambda^\varepsilon \mid g \rangle. \end{aligned} \quad (3.14)$$

Since

$$\begin{cases} T_{\bar{u}_0^\varepsilon} g \longrightarrow T_{\bar{u}_0} g \\ T_{\bar{u}_0^\varepsilon} \phi_\lambda^\varepsilon \rightharpoonup T_{\bar{u}_0} \phi_\lambda \\ D^{\frac{1}{2}} \phi_\lambda^\varepsilon \rightharpoonup D^{\frac{1}{2}} \phi_\lambda \end{cases}$$

in $L_+^2(\mathbb{T})$ as $\varepsilon \rightarrow 0$, then passing to the limit in (3.14), we infer for all $g \in H_+^{\frac{1}{2}}(\mathbb{T})$,

$$\langle h | g \rangle = \langle D^{\frac{1}{2}} \phi_\lambda | D^{\frac{1}{2}} g \rangle - \langle T_{\bar{u}_0^\varepsilon}^\varepsilon \phi_\lambda^\varepsilon | T_{\bar{u}_0^\varepsilon}^\varepsilon \rangle - \lambda \langle \phi_\lambda | g \rangle =: \mathcal{Q}(\phi_\lambda, g) - \lambda \langle \phi_\lambda | g \rangle.$$

That is, $\phi_\lambda \in \text{Dom}(L_{u_0})$ and $\phi_\lambda = (L_{u_0} - \lambda)^{-1} h$. Therefore, $L_{u_0^\varepsilon} \rightarrow L_{u_0}$ in the strong resolvent sense as $\varepsilon \rightarrow 0$. \square

3.1.2. Characterization of the limit $u(t)$.

Proposition 3.4 (Uniqueness of the limit potential $u(t)$). *Let $u_0 \in L_+^2(\mathbb{T})$. There exists a unique potential $u(t) \in L_+^2(\mathbb{T})$,*

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}, \quad (3.15)$$

such that, for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ with $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, we have

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ in } L_+^2(\mathbb{T}), \quad \forall t \in \mathbb{R}.$$

Proof. By the conservation of the L^2 -norm (Lemma 2.6), we have for all $t \in \mathbb{R}$,

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \lesssim \|u_0\|_{L^2}, \quad \forall \varepsilon \ll 1.$$

Then, $\forall t \in \mathbb{R}$, $\exists u_t^* \in L_+^2(\mathbb{T})$ such that

$$u^\varepsilon(t) \rightharpoonup u_t^* \text{ in } L_+^2(\mathbb{T}), \quad \text{and} \quad \|u_t^*\|_{L^2} \lesssim \|u_0\|_{L^2}. \quad (3.16)$$

In order to guarantee that the limit u_t^* is indeed $u(t)$, for a well-defined potential u , we use the explicit formula. Indeed, recall by Proposition 2.5,

$$u^\varepsilon(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0^\varepsilon}} S^*)^{-1} u_0^\varepsilon | 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (3.17)$$

In addition, by the last Proposition, we have $L_{u_0^\varepsilon} \rightarrow L_{u_0}$ as $\varepsilon \rightarrow 0$ in the strong resolvent sense, since $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$. Thus, $f(L_{u_0^\varepsilon}) \rightarrow f(L_{u_0})$ strongly for all bounded continuous functions f [deO09, Proposition 10.1.9]. In particular for all $t \in \mathbb{R}$, we have for $f(x) = e^{-2ixt}$,

$$e^{-2itL_{u_0^\varepsilon}} \longrightarrow e^{-2itL_{u_0}}$$

in the strong operator topology as $\varepsilon \rightarrow 0$. Therefore, passing to the limit in (3.17), we deduce

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}.$$

By uniqueness of the limit, we conclude that $u_t^*(z) = u(t, z)$, with $u(t) \in L_+^2(\mathbb{T})$ thanks to the inequality $\|u(t)\|_{L^2} \lesssim \|u_0\|_{L^2}$ of (3.16). \square

3.2. Strong convergence in $L_+^2(\mathbb{T})$ and conservation of the L^2 -mass. In the light of the previous subsection, it remains to prove that

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

in order to infer the strong convergence of $u^\varepsilon(t) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$, and induce a proof to Theorem 1.2. Indeed, as $\varepsilon \rightarrow 0$,

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \longrightarrow \|u_0\|_{L^2}.$$

The main idea is to use Parseval's identity on $u(t)$, where $u(t)$ is written in a suitable evolving L_+^2 -basis (f_n^t) and satisfying

$$|\langle u(t) | f_n^t \rangle| = |\langle u_0 | f_n^0 \rangle|, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (3.18)$$

Definition 3.5 (An orthonormal basis of $L_+^2(\mathbb{T})$). For all $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{C}(\mathbb{R}, H_+^2(\mathbb{T}))$. We denote by $(f_n^{\varepsilon,t})$ the evolving orthonormal basis of $L_+^2(\mathbb{T})$ along the curve $t \mapsto u^\varepsilon(t)$ and satisfying the Cauchy problem

$$\begin{cases} \partial_t f_n^{\varepsilon,t} = B_{u^\varepsilon(t)} f_n^{\varepsilon,t} \\ f_n^{\varepsilon,t}|_{t=0} = f_n^{\varepsilon,0} \end{cases},$$

for all n , where $(f_n^{\varepsilon,0})$ is the orthonormal basis of $L_+^2(\mathbb{T})$ constituted from the eigenfunctions of the self-adjoint Lax operator $L_{u_0^\varepsilon}$, and $B_{u^\varepsilon(t)}$ is the skew-adjoint operator defined in (1.5).

Remark 3.1.

- (1) Since for all $\varepsilon > 0$, $B_{u^\varepsilon(t)}$ is a skew-adjoint bounded operator (cf. Proposition 2.3) then the orthogonality of the $(f_n^{\varepsilon,t})$ is conserved in time. Indeed, for all $t \in \mathbb{R}$,

$$\partial_t \langle f_n^{\varepsilon,t} | f_m^{\varepsilon,t} \rangle = \langle B_u f_n^{\varepsilon,t} | f_m^{\varepsilon,t} \rangle + \langle f_n^{\varepsilon,t} | B_u f_m^{\varepsilon,t} \rangle = 0,$$

- (2) By [Ku06, Lemma 4.1], such orthonormal basis is formed by the eigenfunctions of the Lax operator $L_{u^\varepsilon(t)}$. Typically, we have for all $\varepsilon > 0$, for all $t \in \mathbb{R}$,

$$L_{u^\varepsilon(t)} f_n^{\varepsilon,t} = \lambda_n(u_0^\varepsilon) f_n^{\varepsilon,t}.$$

With this choice of $L_+^2(\mathbb{T})$ -basis, we have a nice description of the evolution of the coordinates of $u^\varepsilon(t)$. This is the aim of the next Lemma.

Lemma 3.6. *For all $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{C}(\mathbb{R}, H_+^2(\mathbb{T}))$. Under the same notation of Definition 3.5, we have for any $n \in \mathbb{N}_{\geq 0}$,*

$$\langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle = \langle u_0^\varepsilon | f_n^{\varepsilon,0} \rangle e^{-i\lambda_n(u_0^\varepsilon)^2 t}. \quad (3.19)$$

Proof. By Lemma 2.4, and since $L_{u^\varepsilon(t)}$ and $B_{u^\varepsilon(t)}$ are respectively self-adjoint and skew-adjoint operators

$$\begin{aligned}\partial_t \langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle &= \langle B_{u^\varepsilon(t)} u^\varepsilon(t) - i L_{u^\varepsilon(t)}^2 u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle + \langle u^\varepsilon(t) | B_{u^\varepsilon(t)} f_n^{\varepsilon,t} \rangle \\ &= -i \lambda_n^2(u_0^\varepsilon) \langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle ,\end{aligned}$$

which leads to the statement. \square

CONSEQUENCE. From the previous lemma, we infer for all $\varepsilon > 0$, $n \in \mathbb{N}_{\geq 0}$,

$$|\langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle| = |\langle u_0^\varepsilon | f_n^{\varepsilon,0} \rangle|.$$

At this stage, we want to take $\varepsilon \rightarrow 0$ in this identity in order to deduce (3.18). However, one first might ask two questions :

- I. Does the orthonormal basis $(f_n^{\varepsilon,0})$ constituted from the eigenfunctions of the self-adjoint Lax operator $L_{u_0^\varepsilon}$ remains an orthonormal basis of $L_+^2(\mathbb{T})$ under the limit $\varepsilon \rightarrow 0$?
- II. Suppose that the answer to the former question is affirmative, and denote by (f_n^0) this orthonormal basis limit. Based on Definition 3.5, could we construct a time-evolving orthonormal basis, coinciding at $t = 0$ with (f_n^0) , and inducing a nice evolution as in Lemma 3.6 of the coordinates of u in this basis? A priori, the operator B_u defined in (1.5) is not well-defined for $u \in \mathcal{C}_t[L_+^2(\mathbb{T})]_x$. Therefore, we should find another way to circumvent this problem.

The following first proposition aims to answer question I. and to characterize the eigenfunctions of L_{u_0} for $u_0 \in L_+^2(\mathbb{T})$, by finding a uniform bound on the growth of the Sobolev norm $\|f_n^{\varepsilon,0}\|_{H^{\frac{1}{2}}}$. For the second question II., we avoid the problem of defining (f_n^t) via Definition 3.5 by using the same strategy done in the previous subsection, that is, we characterize the limit f_n^t , for all $t \in \mathbb{R}$. Therefore, we should derive an explicit formula of $f_n^{\varepsilon,t}$, for all $\varepsilon > 0$ in order to pass to the limit. Unfortunately, we won't directly obtain that the limit (f_n^t) forms an orthonormal basis of $L_+^2(\mathbb{T})$. However, it shall be an orthonormal family in $L_+^2(\mathbb{T})$, which will be sufficient to conclude.

Proposition 3.7. *Given $u_0 \in L_+^2(\mathbb{T})$. There exists a sequence $(f_n^0) \subseteq \text{Dom}(L_{u_0})$, such that for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, we have up to a subsequence*

$$\lim_{\varepsilon \rightarrow 0} \|f_n^{\varepsilon,0} - f_n^0\|_{L^2} = 0, \quad \forall n \in \mathbb{N}_{\geq 0}.$$

In addition, for all n ,

$$L_{u_0} f_n^0 = \lambda_n(u_0) f_n^0.$$

Proof. By definition of $L_{u_0^\varepsilon} = D - T_{u_0^\varepsilon} T_{\bar{u}_0^\varepsilon}$, and since $L_{u_0^\varepsilon} f_n^{\varepsilon,0} = \lambda_n(u_0^\varepsilon) f_n^{\varepsilon,0}$, it follows

$$\lambda_n(u_0^\varepsilon) + \|T_{\bar{u}_0^\varepsilon} f_n^{\varepsilon,0}\|_{L^2}^2 = \|f_n^{\varepsilon,0}\|_{\dot{H}_+^{\frac{1}{2}}}^2, \quad \forall n \geq 0.$$

Note that as $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, and by applying inequality (3.3), we infer that $\exists N \geq 0$ independent of ε , such that

$$\lambda_n(u_0^\varepsilon) + \frac{1}{2} \|f_n^{\varepsilon,0}\|_{\dot{H}_+^{\frac{1}{2}}}^2 + \left(2N^2 \|u_0^\varepsilon\|_{L^2} + \frac{1}{2}\right) \|f_n^{\varepsilon,0}\|_{L^2}^2 > \|f_n^{\varepsilon,0}\|_{\dot{H}_+^{\frac{1}{2}}}^2.$$

Hence, by Proposition 3.1 and since $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$,

$$\|f_n^{\varepsilon,0}\|_{\dot{H}_+^{1/2}}^2 \lesssim \lambda_n(u_0). \quad (3.20)$$

Therefore, up to a subsequence, $\exists (f_n^0)$ such that, as $\varepsilon \rightarrow 0$,

$$f_n^{\varepsilon,0} \rightharpoonup f_n^0 \text{ in } H_+^{\frac{1}{2}}(\mathbb{T}) \quad \text{and} \quad f_n^{\varepsilon,0} \rightarrow f_n^0 \text{ in } L_+^2(\mathbb{T}). \quad (3.21)$$

At present, for the second part of the proof we show that the (f_n^0) are eigenfunctions of L_{u_0} . Note that by Lemma 2.7, one can directly check that $(f_n^0) \subseteq \text{Dom}(L_{u_0})$ where $\text{Dom}(L_{u_0})$ was defined in (3.5). Besides, by definition of $L_{u_0^\varepsilon}$, we have for all $g \in H_+^{\frac{1}{2}}(\mathbb{T})$,

$$\langle D^{\frac{1}{2}} f_n^{\varepsilon,0} | D^{\frac{1}{2}} g \rangle - \langle T_{\bar{u}_0^\varepsilon} f_n^{\varepsilon,0} | T_{\bar{u}_0^\varepsilon} g \rangle = \lambda_n(u_0^\varepsilon) \langle f_n^{\varepsilon,0} | g \rangle, \quad (3.22)$$

where by Lemma 2.7 $T_{\bar{u}_0^\varepsilon} g \rightarrow T_{\bar{u}_0} g$ in $L_+^2(\mathbb{T})$, by Proposition 3.1 $\lambda_n(u_0^\varepsilon) \rightarrow \lambda_n(u_0)$, and by (3.21) $T_{\bar{u}_0^\varepsilon} f_n^{\varepsilon,0} \rightharpoonup T_{\bar{u}_0} f_n^0$. Hence, passing to the limit in (3.22), we infer

$$\langle L_{u_0} f_n^0 | g \rangle = \lambda_n(u_0) \langle f_n^0 | g \rangle, \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}),$$

leading to $L_{u_0} f_n^0 = \lambda_n(u_0) f_n^0$ for all $n \geq 0$, where $(\lambda_n(u_0))$ denotes all the spectrum of L_{u_0} by Corollary 3.2. □

In the sequel, thanks to Corollary 3.2 and Proposition 3.7, we denote by (f_n^0) the orthonormal basis of $L_+^2(\mathbb{T})$ made up of the eigenfunctions of L_{u_0} obtained in the previous proposition. The following lemma aims to give an explicit formula of the $(f_n^{\varepsilon,t})$ defined in Definition 3.5 in order to characterize their limits.

Lemma 3.8 (The explicit formula of $f_n^{\varepsilon,t}$). *Under the same notation of Definition 3.5, we have for all $\varepsilon > 0$, $t \in \mathbb{R}$,*

$$f_n^{\varepsilon,t}(z) = \left\langle \left(\text{Id} - z e^{-it(L_{u_0^\varepsilon} + \text{Id})^2} S^* e^{itL_{u_0^\varepsilon}^2} \right)^{-1} f_n^{\varepsilon,0} | e^{-itL_{u_0^\varepsilon}^2} 1 \right\rangle, \quad \forall z \in \mathbb{D}. \quad (3.23)$$

Proof. Like the proof of Proposition 2.5, we have

$$f_n^{\varepsilon,t}(z) = \langle (\text{Id} - z S^*)^{-1} f_n^{\varepsilon,t} | 1 \rangle, \quad \forall z \in \mathbb{D}.$$

Using the unitary operator $U(t)$ introduced in Remark 2.1, we deduce

$$\begin{aligned} f_n^{\varepsilon,t}(z) &= \langle U(t)^* (\text{Id} - zS^*)^{-1} f_n^{\varepsilon,t} \mid U(t)^* 1 \rangle \\ &= \langle (\text{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* f_n^{\varepsilon,t} \mid U(t)^* 1 \rangle \\ &= \langle (\text{Id} - zU(t)^* S^* U(t))^{-1} f_n^{\varepsilon,0} \mid U(t)^* 1 \rangle. \end{aligned} \quad (3.24)$$

By the same computation of (2.12) and (2.13), the explicit formula of $f_n^{\varepsilon,t}$ follows. \square

Proposition 3.9. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. Under the same notation of Definition 3.5, there exists an orthonormal family (f_n^t) of $L_+^2(\mathbb{T})$, such that for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, we have up to a subsequence,*

$$\|f_n^{\varepsilon,t} - f_n^t\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof. This proof is similar to the one done in Proposition 3.7. However, it presents two main differences. We will discuss these later in the upcoming remark. Now, coming back to the proof, recall by Proposition 3.4, there exists a unique $u(t) \in L_+^2(\mathbb{T})$ such that for any $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$ we have $u^\varepsilon(t) \rightharpoonup u(t)$ in $L_+^2(\mathbb{T})$ as $\varepsilon \rightarrow 0$. Therefore, by definition of $L_{u^\varepsilon(t)} = D - T_{u^\varepsilon(t)} T_{\bar{u}^\varepsilon(t)}$, and since $L_{u^\varepsilon(t)} f_n^{\varepsilon,t} = \lambda_n(u_0^\varepsilon) f_n^{\varepsilon,t}$ by the second point of Remark 3.1,

$$\lambda_n(u_0^\varepsilon) + \|T_{\bar{u}^\varepsilon(t)} f_n^{\varepsilon,t}\|_{L^2}^2 = \|f_n^{\varepsilon,t}\|_{\dot{H}^{\frac{1}{2}}}^2, \quad \forall n \geq 0.$$

Thus, applying Lemma 2.7,

$$(1 - \|u^\varepsilon(t)\|_{L^2}) \|f_n^{\varepsilon,t}\|_{\dot{H}^{\frac{1}{2}}} \leq \|u^\varepsilon(t)\|_{L^2} + \lambda_n(u_0^\varepsilon), \quad \forall n \geq 0.$$

Taking ε small enough to guarantee $\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} < 1$, we deduce by Proposition 3.1,

$$\|f_n^{\varepsilon,t}\|_{\dot{H}^{1/2}}^2 \lesssim \lambda_n(u_0), \quad \forall n \geq 0. \quad (3.25)$$

Hence, up to a subsequence,

$$f_n^\varepsilon(t) \rightharpoonup f_{n,t}^* \text{ in } H_+^{\frac{1}{2}}(\mathbb{T}), \quad f_n^\varepsilon(t) \rightarrow f_{n,t}^* \text{ in } L_+^2(\mathbb{T}).$$

It remains to show that $f_{n,t}^*$ is well characterized for all t . Using the explicit formula of Lemma 3.8,

$$f_n^{\varepsilon,t}(z) = \left\langle \left(\text{Id} - z e^{-it(L_{u_0^\varepsilon} + \text{Id})^2} S^* e^{itL_{u_0^\varepsilon}^2} \right)^{-1} f_n^{\varepsilon,0} \mid e^{-itL_{u_0^\varepsilon}^2} 1 \right\rangle, \quad \forall z \in \mathbb{D},$$

and applying the same arguments presented in the proof of Proposition 3.4 to characterize $u(t)$, one can conclude that there exists

$$f_n^t(z) = \left\langle \left(\text{Id} - z e^{-it(L_{u_0} + \text{Id})^2} S^* e^{itL_{u_0}^2} \right)^{-1} f_n^0 \mid e^{-itL_{u_0}^2} 1 \right\rangle, \quad \forall z \in \mathbb{D}, \quad (3.26)$$

where (f_n^0) denotes the eigenfunctions of L_{u_0} obtained in Proposition 3.7. Therefore, the limit $f_{n,t}^* = f_n^t$ for all t . Finally, observe that since the $(f_n^{\varepsilon,t})$ is an orthonormal

basis of $L_+^2(\mathbb{T})$ and since $f_n^{\varepsilon,t} \rightarrow f_n^t$ in $L_+^2(\mathbb{T})$, then (f_n^t) forms an orthonormal family in $L_+^2(\mathbb{T})$. \square

Remark 3.2. There are two main differences between the proof of Proposition 3.7 and Proposition 3.9:

- (i) First, note that in the last proof, we cannot control the growth of the Sobolev norm $\|f_n^{\varepsilon,t}\|_{H^{\frac{1}{2}}}$ uniformly for all t by using the inequality (3.3), since the integer N_η in (3.3) is not uniform for all $t \in \mathbb{R}$. As an alternative, we rely on Lemma 2.7. Consequently, the condition of $\|u^\varepsilon(t)\|_{L^2} < 1$ for ε small enough, is crucial here in order to conclude.
- (ii) Second in the previous proof, we had to give a meaning to the limit $f_{n,t}^*$ by characterizing this limit for all $t \in \mathbb{R}$.

A common feature about these two proofs is to obtain a uniform bounds on the growth of the Sobolev norm $H_+^{\frac{1}{2}}(\mathbb{T})$ of the eigenfunctions $f_n^{\varepsilon,0}$ and $f_n^{\varepsilon,t}$ to be able to conclude.

In view of Proposition 3.1, Lemma 3.6, Proposition 3.7 and Proposition 3.9, we infer the following lemma.

Lemma 3.10. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists an orthonormal family (f_n^t) of $L_+^2(\mathbb{T})$ such that for all $n \geq 0$,*

$$\langle u(t) | f_n^t \rangle = \langle u_0 | f_n^0 \rangle e^{-it\lambda_n^2(u_0)}, \quad \forall t \in \mathbb{R}. \quad (3.27)$$

We are, at this stage, in a position to prove Theorem 1.2.

Theorem. 1.2. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists a unique potential $u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T}))$ such that, for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, we have for all $T > 0$,*

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

In addition,

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (3.28)$$

Moreover, the L^2 -norm of the limit potential u is conserved in time :

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

Proof. Let $(t^\varepsilon) \subseteq \mathbb{R}$ such that $t^\varepsilon \rightarrow t$ as $\varepsilon \rightarrow 0$. Since $\|u_0^\varepsilon - u_0\|_{L^2} \rightarrow 0$, then

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \lesssim \|u_0\|_{L^2}.$$

Hence, for any $t^\varepsilon \rightarrow t$, there exists $u_t^* \in L_+^2(\mathbb{T})$ such that up to a subsequence, $u^\varepsilon(t^\varepsilon) \rightharpoonup u_t^*$ in $L_+^2(\mathbb{T})$ and

$$\|u_t^*\|_{L^2} \leq \liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(t^\varepsilon)\|_{L^2} = \liminf_{\varepsilon \rightarrow 0} \|u_0^\varepsilon\|_{L^2} = \|u_0\|_{L^2}. \quad (3.29)$$

Our goal is to show that $u^\varepsilon(t^\varepsilon)$ converges strongly in $L_+^2(\mathbb{T})$. As a first step, we stress out that the weak limit potential u_t^* is well characterized for all t , and is equal to a unique limit $u(t)$. For that, we repeat the same proof of Proposition 3.4 by exchanging t into t^ε with $t^\varepsilon \rightarrow t$, and obtain $u(t)$ defined in equation (3.15). Therefore, $u^\varepsilon(t^\varepsilon) \rightharpoonup u(t)$ in $L_+^2(\mathbb{T})$, with

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad (3.30)$$

by (3.29). As a second step, we prove that this weak convergence in $L_+^2(\mathbb{T})$ is actually a strong convergence. This can be achieved by checking

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2(\mathbb{T})} \rightarrow \|u(t)\|_{L^2(\mathbb{T})}, \quad \varepsilon \rightarrow 0.$$

In fact, it is actually sufficient to prove that $\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$ since

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \rightarrow \|u_0\|_{L^2}, \quad \text{as } \varepsilon \rightarrow 0. \quad (3.31)$$

Thanks to (3.30), we already have $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$. Now, to prove $\|u(t)\|_{L^2} \geq \|u_0\|_{L^2}$, we use Lemma 3.10 to infer the existence of an orthonormal family (f_n^t) of $L_+^2(\mathbb{T})$ such that

$$\sum_{n=0}^{\infty} |\langle u(t) | f_n^t \rangle|^2 = \sum_{n=0}^{\infty} |\langle u_0 | f_n^0 \rangle|^2 = \|u_0\|_{L^2}^2.$$

Hence, by Bessel's inequality

$$\|u(t)\|_{L^2} \geq \|u_0\|_{L^2}.$$

As a conclusion, we have proved for any $t^\varepsilon \rightarrow t$, $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$. This means, $u \in \mathcal{C}(R, L_+^2(\mathbb{T}))$ and for all $T > 0$,

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

□

In view of the last Theorem, we denote through on, $u(t)$ the solution of (CS⁺) in $L_+^2(\mathbb{T})$ starting from an initial datum u_0 that lies inside the open ball $\mathcal{B}_{L_+^2}(1)$ of $L_+^2(\mathbb{T})$.

Corollary 3.11. *The spectrum $\sigma(L_{u(t)})$ is invariant under the flow of (CS⁺).*

Proof. Let $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $\|u_0^\varepsilon - u_0\|_{L^2} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since, for all $n \in \mathbb{N}_{\geq 0}$,

$$\lambda_n(u_0^\varepsilon) = \lambda_n(u^\varepsilon(t)), \quad \forall t \in \mathbb{R},$$

then by passing to the limit, we infer by Proposition 3.1 and Theorem 1.2 that the spectrum of $L_{u(t)}$ is conserved in time.

□

Corollary 3.12. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists an orthonormal basis (f_n^t) of $L_+^2(\mathbb{T})$ constituted from the eigenfunction of $L_{u(t)}$, such that for all $n \in \mathbb{N}_{\geq 0}$,*

$$\langle u(t) | f_n^t \rangle = \langle u_0 | f_n^0 \rangle e^{-it\lambda_n^2(u_0)}, \quad \forall t \in \mathbb{R}. \quad (3.32)$$

Proof. Taking into account Lemma 3.10, we only need to prove that the orthonormal family (f_n^t) found in Proposition 3.9 as

$$f_n^{\varepsilon,t} \rightharpoonup f_n^t \text{ in } H_+^{\frac{1}{2}}(\mathbb{T}), \quad \text{and} \quad f_n^{\varepsilon,t} \rightarrow f_n^t \text{ in } L_+^2(\mathbb{T}), \quad (3.33)$$

is actually an orthonormal basis of $L_+^2(\mathbb{T})$. On the one hand, using (3.33) and since $u^\varepsilon(t) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$, one can directly prove that $L_{u^\varepsilon(t)} f_n^{\varepsilon,t} \rightharpoonup L_{u(t)} f_n^t$ in $L_+^2(\mathbb{T})$. On the other hand, using Proposition 3.1, we infer that taking $\varepsilon \rightarrow 0$ in

$$\langle L_{u^\varepsilon(t)} f_n^{\varepsilon,t} | g \rangle = \lambda_n(u^\varepsilon(t)) \langle f_n^{\varepsilon,t} | g \rangle, \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}),$$

leads to

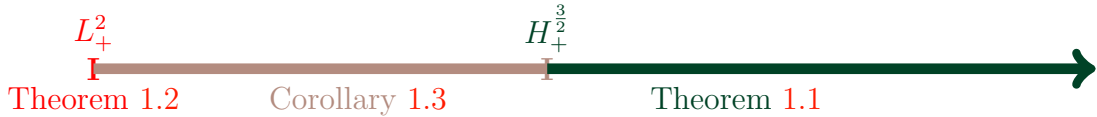
$$\langle L_{u(t)} f_n^t | g \rangle = \lambda_n(u(t)) \langle f_n^t | g \rangle, \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}).$$

As a result, the (f_n^t) describes all the eigenfunctions of the self-adjoint operator $L_{u(t)}$, thanks to Corollary 3.11 and Corollary 3.2. Hence, they form an orthonormal basis of $L_+^2(\mathbb{T})$. \square

Remark 3.3. The nice evolution in (3.32) of such coordinates suggests that the so-called ‘‘Birkhoff coordinates’’ of (CS^+) are the $(\langle u(t) | f_n^t \rangle)$.

4. PROOF OF COROLLARY 1.3 AND THEOREM 1.4

To summarize, we have proved the global well-posedness of (CS^+) -equation in $H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, and for $s = 0$ which correspond to $H_+^0(\mathbb{T}) \equiv L_+^2(\mathbb{T})$. The following corollary aims to prove the global well-posedness for $0 < s \leq \frac{3}{2}$.



Corollary. 1.3. *For all $0 \leq s \leq \frac{3}{2}$, the Calogero-Sutherland DNLS focusing equation (CS^+) is globally well-posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C,$$

where $C = C(u_0, s) > 0$ is a positive constant.

Proof. For $s = 0$, we infer by Theorem 1.2 the global well-posedness of the problem in $L_+^2(\mathbb{T})$ in the sense of continuous extension of the flow from $H_+^2(\mathbb{T})$ to $L_+^2(\mathbb{T})$.

For $0 < s \leq \frac{3}{2}$, let $u_0 \in H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. We consider $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $u_0^\varepsilon \rightarrow u_0$ in $H_+^s(\mathbb{T})$. Then by Proposition 2.8,

$$\frac{1}{C} \|u^\varepsilon(t^\varepsilon)\|_{H^s} \leq \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon)\|_{L^2} = \|(L_{u_0^\varepsilon} + \lambda)^s u_0^\varepsilon\|_{L^2} \leq C \|u_0^\varepsilon\|_{H^s},$$

thanks to Lemma 2.6. Therefore, for all $\varepsilon > 0$,

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \leq C \|u_0\|_{H^s}. \quad (4.1)$$

Hence, as $t^\varepsilon \rightarrow t$, we have $u^\varepsilon(t^\varepsilon) \rightharpoonup u(t)$ in $H_+^s(\mathbb{T})$, where u is a well characterized function for all t obtained as in Proposition 3.4. In particular, we infer $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$ with

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}, \quad (4.2)$$

by (4.1). As of now, to deduce the strong convergence in $H_+^s(\mathbb{T})$ we use Proposition 2.8. Thus, for all $\varepsilon > 0$,

$$\begin{aligned} \|u^\varepsilon(t^\varepsilon) - u(t)\|_{H^s}^2 &\lesssim \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s (u^\varepsilon(t^\varepsilon) - u(t))\|_{L^2}^2 \\ &= \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon)\|_{L^2}^2 + \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u(t)\|_{L^2}^2 \\ &\quad - 2\operatorname{Re} \langle (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon) | (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u(t) \rangle \end{aligned} \quad (4.3)$$

Recall that $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$ which leads by Proposition 3.3 to $L_{u^\varepsilon(t^\varepsilon)} \rightarrow L_{u(t)}$ in the strong resolvent sense. Thus, by functional calculus, (see the following lemma– Lemma 4.1) we infer

$$\begin{cases} (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u(t) \rightarrow (L_{u(t)} + \lambda)^s u(t) \text{ in } L_+^2(\mathbb{T}), \\ (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon) \rightharpoonup (L_{u(t)} + \lambda)^s u(t) \text{ in } L_+^2(\mathbb{T}), \end{cases}$$

as $\varepsilon \rightarrow 0$. In addition, for $\varepsilon > 0$, recall by Lemma 2.6,

$$\|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon)\|_{L^2}^2 = \|(L_{u_0^\varepsilon} + \lambda)^s u_0^\varepsilon\|_{L^2}^2. \quad (4.4)$$

Therefore, passing to the limit in (4.3), and since $u_0^\varepsilon \rightarrow u_0$ in $H_+^s(\mathbb{T})$ combined with Lemma 4.1 and Proposition 2.8, we deduce

$$\|u^\varepsilon(t^\varepsilon) - u(t)\|_{H^s}^2 \lesssim \|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 - \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2}^2.$$

At this stage, it remains to show that the right-hand side of the previous inequality is vanishing. Indeed, by Corollary 3.11,

$$\begin{aligned} \langle (L_{u(t)} + \lambda)^s u(t) | u(t) \rangle &= \sum_{n \geq 0} (\lambda_n(u(t)) + \lambda)^s |\langle u(t) | f_n^t \rangle|^2 \\ &= \sum_{n \geq 0} (\lambda_n(u_0) + \lambda)^s |\langle u_0 | f_n^0 \rangle|^2 = \langle (L_{u_0} + \lambda)^s u_0 | u_0 \rangle, \end{aligned}$$

where (f_n^t) are the orthonormal basis obtained in Corollary 3.12 since $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$. As a result, $\|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 = \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2}^2$, and as $\varepsilon \rightarrow 0$,

$$\|u^\varepsilon(t^\varepsilon) - u(t)\|_{H^s}^2 \longrightarrow 0.$$

Hence, $u \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T}))$ such that (4.2) is satisfied, and for all $T > 0$,

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{H^s} \rightarrow 0.$$

□

Remark 4.1. (1) The inequality (4.2) in the proof implies that the flow

$$u_0 \in H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1) \mapsto u \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T})),$$

defined in the previous corollary, is continuous.

- (2) As observed in the proof, one has for any $u \in \mathcal{C}(\mathbb{R}, H_+^r(\mathbb{T}))$, $r \geq 0$ such that $\|u\|_{L^2} < 1$, the family $\{\mathcal{H}_s(u) := \langle (L_u + \lambda)^s u \mid u \rangle ; 0 \leq s \leq 2r\}$, $\lambda \gg 0$, is conserved along the flow. This means, that the conservation laws of Lemma 2.6 are extended to less regular u . Consequently, as noted in Remark 2.5, Proposition 2.8 holds for any $u_0 \in \mathcal{B}_{L_+^2}(1) \cap H_+^r(\mathbb{T})$, for all $r \geq 0$.

To conclude the proof of Corollary 4, we need to prove the following functional analysis result.

Lemma 4.1. *Let (A_ε) be a sequence of positive self-adjoint operators in L^2 . Suppose that $A_\varepsilon \rightarrow A$ in the strong resolvent sense as $\varepsilon \rightarrow 0$, and for all $s \geq 0$,*

$$\text{Dom}(A_\varepsilon^s) = \text{Dom}(A^s) = H^s, \quad \varepsilon > 0.$$

Moreover, assume that for all $u \in H^s$, the $(A_\varepsilon^s u)$ are uniformly bounded with respect to $\varepsilon > 0$ in the following sense $\|A_\varepsilon^s u\| \leq C\|u\|_{H^s}$. Then, for all $s \geq 0$,

$$A_\varepsilon^s u \longrightarrow A^s u \text{ in } \mathcal{H}, \quad \varepsilon \rightarrow 0. \quad (4.5)$$

Proof. For all $R > 0$, let $\chi_R \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that $\chi_R \equiv 1$ on $[0, R]$ and $\text{supp}(\chi_R) \subseteq [0, 2R]$. Note that, for all $s \geq 0$, the subset $\{\chi_R(A)u ; u \in H^s, R > 0\}$ is dense in H^s . Then, since the $(A_\varepsilon^s u)$ are uniformly bounded with respect to ε , it is sufficient to prove for all $R > 0$,

$$A_\varepsilon^s \chi_R(A)u \longrightarrow A^s \chi_R(A)u, \quad \varepsilon \rightarrow 0,$$

to obtain (4.5). Toward this end, let $R > 0$, and write for any $s \geq 0$, for all $\varepsilon > 0$,

$$A_\varepsilon^s \chi_R(A)u = A_\varepsilon^s \chi_{\tilde{R}}(A_\varepsilon) \chi_R(A)u + A_\varepsilon^s (1 - \chi_{\tilde{R}}(A_\varepsilon)) \chi_R(A)u, \quad (4.6)$$

where $\chi_{\tilde{R}} \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that $\chi_{\tilde{R}} \equiv 1$ on $[0, \tilde{R}]$ and $\text{supp}(\chi_{\tilde{R}}) \subseteq [0, 2\tilde{R}]$, $\tilde{R} \geq 2R$. Notice that,

$$\begin{aligned} \|A_\varepsilon^s (1 - \chi_{\tilde{R}}(A_\varepsilon)) \chi_R(A)u\| &= \|A_\varepsilon^{-s} (1 - \chi_{\tilde{R}}(A_\varepsilon)) A_\varepsilon^{2s} \chi_R(A)u\| \\ &\leq \frac{C^2}{(2\tilde{R})^s} \|u\|_{H^s}. \end{aligned}$$

Therefore, for all $\eta > 0$, there exists $\tilde{R} \gg 0$, such that for all $\varepsilon > 0$,

$$\|A_\varepsilon^s (1 - \chi_{\tilde{R}}(A_\varepsilon)) \chi_R(A) u\| < \eta,$$

and so, by (4.6),

$$\|A_\varepsilon^s \chi_R(A) u - A_\varepsilon^s \chi_{\tilde{R}}(A_\varepsilon) \chi_R(A) u\| < \eta, \quad \forall \varepsilon > 0. \quad (4.7)$$

Besides, recall that $A_\varepsilon \rightarrow A$ in the strong resolvent sense as $\varepsilon \rightarrow 0$. Hence, by [deO09, Proposition 10.1.9], $f(A_\varepsilon) \rightarrow f(A)$ as $\varepsilon \rightarrow 0$, in the operator norm for all continuous bounded f . In particular, for $f(x) = x^s \chi_{\tilde{R}}(x)$, we have

$$A_\varepsilon^s \chi_{\tilde{R}}(A_\varepsilon) \chi_R(A) u \longrightarrow A^s \chi_R(A) u, \quad \varepsilon \rightarrow 0. \quad (4.8)$$

Thus, combining (4.7) and (4.8), we infer for all $\tilde{\eta} > 0$, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon > \varepsilon_0$,

$$\|A_\varepsilon^s \chi_R(A) u - A^s \chi_R(A) u\| < \tilde{\eta}.$$

□

Beyond the global well-posedness results of the (CS⁺) Cauchy's Problem, we are interested in some qualitative properties about the flow $\mathcal{S}^+(t)$ of this equation. Therefore, we prove that all weak limit points of the orbit are actually strong limit points.

Theorem. 1.4 . *Given an initial data $u_0 \in \mathcal{B}_{L^2_+}(1) \cap H^s_+(\mathbb{T})$, $s \geq 0$, the orbit of the solution $\{\mathcal{S}^+(t)u_0; t \in \mathbb{R}\}$ is relatively compact in $H^s_+(\mathbb{T})$.*

Proof. Let $(t_n) \subseteq \mathbb{R}$ such that $t_n \rightarrow \infty$.

Step 1 : $s = 0$. By Theorem 1.2,

$$\|u(t_n)\|_{L^2} = \|u_0\|_{L^2}. \quad (4.9)$$

Then, $\exists \tilde{u} \in L^2_+(\mathbb{T})$ such that, up to a subsequence,

$$u(t_n) \rightharpoonup \tilde{u} \text{ in } L^2_+(\mathbb{T}) \quad \text{and} \quad \|\tilde{u}\|_{L^2} \leq \|u_0\|_{L^2}.$$

In order to obtain the strong convergence $u(t_n) \rightarrow \tilde{u}$ in $L^2_+(\mathbb{T})$, all it remains is to show that $\|u(t_n)\|_{L^2} \rightarrow \|\tilde{u}\|_{L^2}$, or by (4.9),

$$\|u_0\|_{L^2} = \|\tilde{u}\|_{L^2}.$$

Observe that we already have $\|u_0\|_{L^2} \geq \|\tilde{u}\|_{L^2}$. Now, to prove $\|u_0\|_{L^2} \leq \|\tilde{u}\|_{L^2}$, recall by Corollary 3.12,

$$\langle u(t_n) | f_m^{t_n} \rangle = \langle u_0 | f_m^0 \rangle e^{-it_n \lambda_m^2(u_0)}, \quad (4.10)$$

where $(f_m^{t_n})$ is the orthonormal basis of $L^2_+(\mathbb{T})$ constituted of the eigenfunctions of $L_{u(t_n)}$. The idea is to pass to the limit as $t_n \rightarrow \infty$ in the above identity and conclude by using Bessel's identity. First, we have $u(t_n) \rightharpoonup \tilde{u}$ in $L^2_+(\mathbb{T})$. Second,

notice that the $(f_m^{t_n})$ converges strongly in $L_+^2(\mathbb{T})$ as $t_n \rightarrow \infty$ to an orthonormal family denoted by (g_m) . Indeed, by definition of $L_{u(t_n)}$,

$$\lambda_m(u_0) + \|T_{u(t_n)}^{-1} f_m^{t_n}\|_{L^2}^2 = \|f_m^{t_n}\|_{\dot{H}^{\frac{1}{2}}}^2,$$

leading to $\|f_m^{t_n}\|_{\dot{H}^{\frac{1}{2}}} \lesssim \lambda_m(u_0)$ for all m , thanks to Lemma 2.7. Hence, by Rellich–Kondrachov’s Theorem, $f_m(t_n) \rightarrow g_m$ in $L_+^2(\mathbb{T})$ as $t_n \rightarrow \infty$. Third, using Cantor diagonalization procedure, one can extract a subsequence $t_n \rightarrow \theta_m \bmod(\frac{2\pi}{\lambda_m^2})$, as the circle is compact. Hence, by passing to the limit in (4.10), we obtain

$$\langle \tilde{u} | g_m \rangle = \langle u_0 | f_m^0 \rangle e^{-i\lambda_m^2(u_0)\theta_m},$$

As a result, using Bessel’s inequality, we conclude

$$\|\tilde{u}\|_{L^2}^2 \geq \sum_{n=0}^{\infty} |\langle \tilde{u} | g_n \rangle|^2 = \sum_{n=0}^{\infty} |\langle u_0 | f_n^0 \rangle|^2 = \|u_0\|_{L^2}^2,$$

Consequently, $\|u(t_n)\|_{L^2} \rightarrow \|\tilde{u}\|_{L^2}$ and thus $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$.

Step 2 : $s > 0$. By inequality (4.2) of the proof of Corollary 1.3, we have $\|u(t_n)\|_{H^s} \lesssim \|u_0\|_{H^s}$ leading to

$$u(t_n) \rightharpoonup \tilde{u} \text{ in } H_+^s(\mathbb{T}) \quad \text{and} \quad \|\tilde{u}\|_{H^s} \lesssim \|u_0\|_{H^s}.$$

In particular, $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$. Then, in view of Remark 2.5,

$$\begin{aligned} \|u(t_n) - \tilde{u}\|_{H^s}^2 &\lesssim \|(L_{u(t_n)} + A \text{Id})^s (u(t_n) - \tilde{u})\|_{L^2}^2 \\ &= \|(L_{u(t_n)} + A \text{Id})^s u(t_n)\|_{L^2}^2 + \|(L_{u(t_n)} + A \text{Id})^s \tilde{u}\|_{L^2}^2, \\ &\quad - 2\text{Re} \langle (L_{u(t_n)} + \lambda)^s u(t_n) | (L_{u(t_n)} + \lambda)^s \tilde{u} \rangle, \end{aligned} \tag{4.11}$$

where by Remark 2,

$$\|(L_{u(t_n)} + A \text{Id})^s u(t_n)\|_{L^2}^2 = \|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2.$$

Besides, as $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$, then $L_{u(t_n)} \rightarrow L_{\tilde{u}}$ in the strong resolvent sense thanks to Proposition 3.3. Hence, by functional calculus (see Lemma 4.1), we infer

$$\begin{cases} (L_{u(t_n)} + \lambda)^s \tilde{u} \rightarrow (L_{\tilde{u}} + \lambda)^s \tilde{u} \text{ in } L_+^2(\mathbb{T}), \\ (L_{u(t_n)} + \lambda)^s u(t_n) \rightharpoonup (L_{\tilde{u}} + \lambda)^s \tilde{u} \text{ in } L_+^2(\mathbb{T}), \end{cases}$$

as $n \rightarrow \infty$. Therefore, by passing to the limit in (4.11), we deduce

$$\|u(t_n) - \tilde{u}\|_{H^s}^2 \lesssim \|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 - \|(L_{\tilde{u}} + \lambda)^s \tilde{u}\|_{L^2}^2,$$

where the right-hand side vanishes. Indeed, by Corollary 3.11 and Proposition 3.1, $\lambda_n(\tilde{u}) = \lambda_n(u_0)$. Hence,

$$\begin{aligned} \langle (L_{\tilde{u}} + \lambda)^{2s} \tilde{u} | \tilde{u} \rangle &= \sum_{n \geq 0} (\lambda_n(u_0) + \lambda)^{2s} |\langle \tilde{u} | g_n \rangle|^2 = \sum_{n \geq 0} (\lambda_n(u_0) + \lambda)^{2s} |\langle u_0 | f_n^0 \rangle|^2 \\ &= \langle (L_{u_0} + \lambda)^{2s} u_0 | u_0 \rangle, \end{aligned}$$

where (g_n) is the orthonormal family of $L_+^2(\mathbb{T})$ found in Step 1. Nevertheless, since $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$, one could show as in Corollary 3.12 that this orthonormal family is indeed an orthonormal basis of $L_+^2(\mathbb{T})$ by proving that the (g_n) constitutes all the eigenfunction of the self-adjoint operator $L_{\tilde{u}}$. As a consequence,

$$\|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 = \|(L_{\tilde{u}} + \lambda)^s \tilde{u}\|_{L^2}^2,$$

and thus $\|u(t_n) - \tilde{u}\|_{H^s}^2 \rightarrow 0$, as $n \rightarrow \infty$. \square

5. THE CALOGERO–SUTHERLAND DNLS DEFOCUSING EQUATION (CS[−])

In this section, we consider the defocusing equation of (CS)

$$i\partial_t u + \partial_x^2 u - 2D_+(|u|^2)u = 0. \quad (\text{CS}^-)$$

Note that by adapting the argument of [GL22, Proposition 2.1] to the defocusing equation, one can infer the local well-posedness of the (CS[−]) problem in $H_+^s(\mathbb{T})$ for $s > \frac{3}{2}$. And we expect that one can go down to $s > \frac{1}{2}$ by following [deMP10].

Below are a series of lemmas, propositions, and theorems that can be proved similarly to their analogs in the focusing case. Again, the integrable methods are the main ingredients to conclude. The first proposition is to announce that the defocusing equation of (CS[−]) enjoys also a Lax pair formalism.

Proposition 5.1 (Lax pair for (CS[−])). *Let $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, be a solution of (CS[−]). There exist two operators*

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$$

satisfying the Lax equation

$$\frac{d\tilde{L}_u}{dt} = [\tilde{B}_u, \tilde{L}_u].$$

Lemma 5.2. *Given $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $r > \frac{3}{2}$, a solution of (CS[−]) equation, then*

$$\partial_t u = \tilde{B}_u u - i\tilde{L}_u^2 u.$$

As a consequence, the quantities $\tilde{\mathcal{H}}_s(u) := \langle \tilde{L}_u^s u \mid u \rangle$ are conserved by the flow $\mathcal{S}^-(t)$ of (CS[−]) for all $0 \leq s \leq 2r$.

Remark 5.1. Expanding the conservation laws $\tilde{\mathcal{H}}_k(u)$ for all $k \in \mathbb{N}_{\geq 0}$, we have

$$\begin{aligned}\tilde{\mathcal{H}}_0(u) &= \langle u(t) \mid u(t) \rangle = \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \\ \tilde{\mathcal{H}}_1(u) &= \langle \tilde{L}_{u(t)} u(t) \mid u(t) \rangle = \|u(t)\|_{\dot{H}^{1/2}}^2 + \|T_{\tilde{u}(t)} u(t)\|_{L^2}^2 \geq \|u(t)\|_{\dot{H}^{1/2}}^2 \\ \tilde{\mathcal{H}}_2(u) &= \|\tilde{L}_{u(t)} u(t)\|_{L^2}^2 \geq \|Du(t)\|_{L^2}^2 - \|T_{u(t)} T_{\tilde{u}(t)} u(t)\|_{L^2}^2 \geq \|u(t)\|_{\dot{H}^1}^2 - C(\|u\|_{\dot{H}^{1/2}}) \\ &\vdots\end{aligned}$$

Unlike the focusing case, here we deduce the uniform control of the growth of Sobolev norms of the solution u by the conservation laws, without requiring any additional condition of smallness on the initial data u_0 . Therefore, Proposition 2.8 holds in the defocusing case for all $u_0 \in H_+^r(\mathbb{T})$, $r > \frac{3}{2}$.

As a result, we state the following theorem which is the analog of Theorem 1.1 but for equation (CS⁻).

Theorem 5.3. *For all $s > \frac{3}{2}$, let $u_0 \in H_+^s(\mathbb{T})$. There exists a unique global solution $u \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T}))$ of the defocusing equation (CS⁻), satisfying at $t = 0$, $u(0, \cdot) = u_0$. Furthermore, for all $s > \frac{3}{2}$,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C,$$

where $C = C(u_0, s) > 0$ is a positive constant.

As for the focusing case, the defocusing Calogero–Sutherland DNLS has an explicit solution.

Lemma 5.4. *Let $u_0 \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$ then the solution of the defocusing Calogero–Sutherland DNLS equation (CS⁻) is given by*

$$u(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2it\tilde{L}_{u_0}} S^*)^{-1} u_0 \mid 1 \right\rangle.$$

In particular, using this explicit formula, we extend the flow $\mathcal{S}^-(t)$ continuously from $H_+^2(\mathbb{T})$ to $H_+^s(\mathbb{T})$, for $0 \leq s \leq \frac{3}{2}$. Therefore, we have :

Theorem. 1.5 . *The Calogero–Sutherland DNLS defocusing equation (CS⁻) is globally well-posed in $H_+^s(\mathbb{T})$ for any $s \geq 0$, in the sense of Remark 1.2. In addition, for all $u_0 \in H_+^s(\mathbb{T})$,*

$$u(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2it\tilde{L}_{u_0}} S^*)^{-1} u_0 \mid 1 \right\rangle,$$

is solution of the (CS⁻)-defocusing equation. Furthermore, the trajectories

$$\{\mathcal{S}^-(t)u_0; t \in \mathbb{R}\}$$

are relatively compact in $H_+^s(\mathbb{T})$.

6. FINAL REMARKS AND OPEN PROBLEMS

Let us briefly discuss here some remarks related to the previous sections.

1. One interesting feature about the focusing Calogero–Sutherland DNLS equation is that it admits a rich dynamic in comparison to the defocusing equation. For instance, as we shall see [Ba], the focusing equation has a wider collection of traveling wave solutions.

2. The problem of global well-posedness of the focusing Calogero–Sutherland DNLS equation (CS^+) without restriction on the initial data is wide open. Nevertheless, we expect that the explicit solution (equation (3.28))

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 \mid 1 \rangle, \quad (6.1)$$

is a key ingredient to answer this question. Indeed, writing for all $t \in \mathbb{R}$, Σ_t the operator $S e^{2itL_{u_0}} e^{it}$, we have by (6.1)

$$u(t, z) = \sum_{n \geq 0} \langle u_0 \mid \Sigma_t^n 1 \rangle z^n. \quad (6.2)$$

Observe that, if u_0 belongs to the space \mathcal{J} generated by the *orthonormal family* $\{\Sigma_t^n 1, n \geq 0\}$, then using Parseval's identity on (6.2), we infer

$$\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2}, \quad t \in \mathbb{R}, \quad (6.3)$$

leading to say that the set $\{u(t), t \in \mathbb{R}\}$ is relatively compact in $L^2_+(\mathbb{T})$. Hence, the integer N_η set out in inequality (3.3) is now independent of t , and thus applying inequality (3.3) to (2.18), we obtain for all $\eta > 0$,

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq (1 - 2\eta^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - 2\eta^2 - 2N_\eta^2 \|u_0\|_{L^2}) \|f\|_{L^2}^2,$$

instead of having inequality (2.19) :

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq (1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - \|u\|_{L^2}^2) \|f\|_{L^2}^2.$$

Therefore, using (3.3), we control the growth of all the Sobolev norm $\|u(t)\|_{H^s}$ for all $s \geq 0$, and we infer the global well-posedness of the focusing (CS^+) in all $H^s_+(\mathbb{T})$, $s > \frac{3}{2}$ for arbitrary initial data. In addition, by the same manner, we deduce also $H^{\frac{1}{2}}$ -bounds on the eigenfunctions $(f_n^{\varepsilon, t})$ –inequality (3.25)– implying that the flow $\mathcal{S}^+(t)$ can be extended to $L^2_+(\mathbb{T})$, for arbitrary initial data. Besides, if u_0 does not belong to \mathcal{J} , then we expect blow-up results in finite time T .

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