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# Infinite-dimensional internal models for nonlinear robust output regulation

Daniele Astolfi

**Abstract**— We consider the problem of robust asymptotic output regulation of nonlinear systems in the presence of generic unstructured nonlinear model uncertainties. Inspired by recent results establishing that this problem cannot be solved by a finite dimensional smooth regulator, we propose a new paradigm based on the use of linear infinite-dimensional internal models. As a special case, we consider the problem of output regulation in the presence of quasi-periodic signals, e.g. signals composed by a sum of periodic signals with incommensurable periods.

**Index Terms**— Output regulation, nonlinear systems, infinite-dimensional regulators, repetitive control.

## I. INTRODUCTION

The problem of robust output regulation consists in designing a dynamical regulator able to reject disturbances while tracking desired references, despite the presence of model uncertainties. For linear systems, such a problem has been solved in the 70's in the seminal works [1], [2] with the introduction of the *internal model principle*, claiming that the asymptotic regulation property can be robust only if the regulator contains a copy of the dynamical system generating the disturbances or references. Such a property is verified for linear systems and parametric uncertainties in the plant. In the recent set of contributions [3], [4] it is however shown that smooth finite dimensional regulators are generically not robust to unstructured (nonlinear) model uncertainties. This result is a change of paradigm in the theory of robust nonlinear output regulation that modifies the perspectives of internal-model based regulators and the asymptotic properties that they can achieve. From a practical point of view, the set of works [3], [4] push towards the designs of regulators able to guarantee robust practical/approximate output regulation. However, at the same time, they leave the door open to the more theoretical question whether asymptotic regulation can be achieved with regulators that fall out of the class of smooth finite dimensional systems. As a consequence, either one investigates the use of non-smooth regulators, such as sliding-mode controllers (see, e.g., [5] and references therein), or the use of infinite-dimensional regulators, such as repetitive-control approaches (see, e.g. [6], [7] and references therein).

In this note, we follow this second idea and we try to develop a general framework for the use of linear infinite-dimensional internal models for the problem of robust output regulation. We focus in this work on strongly minimum-phase nonlinear systems. We describe the internal-model with the linear operator theory over Hilbert spaces [8] providing very generic conditions for the design of asymptotic regulators. Such conditions are inspired by the seminal works [9], [10] in which the steady-state control input is supposed to satisfy a nonlinear regression law allowing for a finite-dimensional high-gain observer based nonlinear regulator. We generalize such an idea by

supposing that the steady-state control input can be generated by a skew-adjoint linear operator [11, Chapter 3.7], i.e. an operator with a spectrum lying on the imaginary axes. In other words, the operator is conservative and it is able to generate a large variety of signals comprising  $C^1$  periodic and quasi-periodic signals. As a side result, we recover and extend in this work the recent results [6], [12] derived in the specific context of repetitive control.

We highlight that one of the motivations of studying the use of infinite-dimensional regulators is the possibility of obtaining finite-dimensional smooth regulators with approximate regulation properties which are robust with respect to generic model uncertainties. For instance, in the context of periodic signals, it is known that approximating the exact infinite-dimensional operator with a Fourier truncation (i.e. implementing a finite number of linear oscillators) allows to obtain practical regulation with asymptotic properties on the  $L_2$  norm of the output [13]–[15]. In particular, by increasing the dimension of the regulator (i.e. by taking a model which is more and more close to the exact infinite-dimensional one), the  $L_2$  norm is reduced and harmonic cancellation is guaranteed [14, Theorem 1]. In contrast, it is not easy to derive similar properties for the approximation of non-smooth regulators which presents also less flexibility in terms of system structures compared to smooth regulators. For instance, with linear internal models, very generic classes of input-affine systems can be handled, e.g. [15], [16].

The rest of the paper is organized as follows. In Section II we present some preliminary notions on Hilbert spaces, the problem statement and the main result of this paper. In Section III we provide an explicit design of an infinite dimensional internal model regulator in the case of quasi-periodic signals. The proof of the main result is given in Section IV. In Section V, we draw the conclusions and perspectives of this work.

**Notation.** Set  $\mathbb{R}_{\geq 0} = [0, \infty)$ . For any  $n \in \mathbb{N}$ , we denote by  $|\cdot|$  the Euclidean norm in  $\mathbb{R}^n$ . A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{K}_\infty$  if  $\alpha$  is continuous, increasing,  $\alpha(0) = 0$  and  $\lim_{x \rightarrow +\infty} \alpha(x) = +\infty$ . We compactly denote  $\alpha \in \mathcal{K}_\infty$ . For a function  $w : (t, x) \in \mathbb{R}_{\geq 0} \times [0, 1] \mapsto w(t, x) \in \mathbb{R}^n$ , the notation  $w_t$  (resp.  $w_x$ ) denotes the partial derivative of  $w$  with respect to the variable  $t$  (resp. with respect to the variable  $x$ ). We keep the notation for the weak and the strong definition of partial derivatives. When a function  $w$  depends only on the variable of the time  $t$  (resp. space  $x$ ), we denote its derivative  $\dot{w}$  (resp.  $w'$ ). The functional space  $L^2(0, 1)$  denotes the set of function  $f$  such that  $\int_0^1 f(x)^2 dx < +\infty$ . We denote with  $C^k(X, Y)$  the set of  $C^k$  functions from  $X$  to  $Y$ .

## II. MAIN RESULT

### A. Preliminaries on Linear Operators

Given a Hilbert space  $\mathcal{H}$ , we denote with  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  the inner product on  $\mathcal{H}$  and with  $\|\cdot\|_{\mathcal{H}}$  its induced norm. Given two Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , the space  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  denotes the space of bounded linear operators from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ , and  $\mathcal{L}(\mathcal{H}_1) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_1)$ . Given an operator  $\mathcal{A}$  we denote with  $D(\mathcal{A})$  its domain. We denote with  $\mathcal{A}^*$

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the adjoint operator of  $\mathcal{A}$ . We recall now the following definitions that will be used throughout the paper.

**Definition 1** A family  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  of operators in  $\mathfrak{L}(\mathcal{H})$  is a strongly continuous semigroup on  $\mathcal{H}$  if  $\mathbb{T}_0 = I$ ,  $\mathbb{T}_{t+\tau} = \mathbb{T}_t \mathbb{T}_\tau$  for every  $t, \tau \geq 0$ , and  $\lim_{t \rightarrow 0^+} \mathbb{T}_t v = v$  for all  $v \in \mathcal{H}$ .

**Definition 2** Let  $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$  be densely defined in  $\mathcal{H}$ .  $\mathcal{A}$  is called a skew-adjoint operator if  $\langle \mathcal{A}w, v \rangle = -\langle w, \mathcal{A}v \rangle$  for all  $w, v \in D(\mathcal{A})$ .

**Definition 3** Let  $\mathcal{H}$  and  $\mathcal{Y}$  be two real Hilbert spaces,  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  be the generator of a strongly continuous semigroup  $(\mathbb{T}(t))_{t \geq 0}$  and  $\mathcal{C} \in \mathfrak{L}(D(\mathcal{A}), L^2([0, \infty); \mathcal{Y}))$  be defined by  $(\psi v_0)(t) = \mathcal{C} \mathbb{T}(t) v_0$  for all  $v_0 \in D(\mathcal{A})$  and all  $t \geq 0$ . Then, the pair  $(\mathcal{A}, \mathcal{C})$  is approximately observable in infinite-time if and only if  $\text{Ker } \psi = \{0\}$ .

We refer to Definition 2.1.1, Definition 2.2.1, Section 3.7 and Definition 6.1.1 in [11] for the definitions of strongly  $C_0$  semigroup, bounded operators, skew-symmetric operators and approximate observability in infinite-time. Furthermore, we recall that skew-adjoint operators are generator of a  $C^0$  semi-groups, see, e.g., [11, Theorem 3.8.6].

## B. Problem Statement

In this article we consider the *global robust output regulation problem* (see, e.g. [17]) for systems that can be rewritten, under suitable change of coordinates, in the form

$$\begin{aligned} \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + u \end{aligned}$$

in which  $z \in \mathbb{R}^n$  is the state of the zero-dynamics,  $e \in \mathbb{R}$  is the measured output to be regulated to zero and  $u \in \mathbb{R}$  is the control input. The signal  $w \in \mathbb{R}^{n_x}$  represents references to be tracked or disturbances to be rejected. We refer for instance to [6], [9], [10], [14], [17]–[19] and references therein for examples of control problems that can be recast in such a form. Note that in this article we suppose that the system has unitary relative-degree between the input  $u$  and the regulated output  $e$ . The case of higher relative degree can be easily dealt with by adopting a high-gain observer as in [12], [14], [20]. In typical output regulation problems, see, e.g. [9], [10], [17], the exosignal  $w$  is supposed to be generated by an autonomous system of the form

$$\dot{w} = s(w)$$

with  $s$  being a neutrally stable dynamics. However, as remarked in [4], when dealing with robustness to model uncertainties, such a representation is not very useful since model uncertainties don't allow to correctly characterize the properties of the overall steady-state behaviour and distinguish between the structural properties of  $f, q$  and  $s$ . As a consequence, in what follows, we replace the signal  $w$  by a generic function of time  $t$ , i.e. we consider time-varying systems of the form

$$\dot{z} = f(t, z, e) \quad (1a)$$

$$\dot{e} = q(t, z, e) + u. \quad (1b)$$

We suppose that the functions  $f, q$  are globally Lipschitz with a Lipschitz constant which is uniform in  $t$ . Furthermore, we stress that the function  $q$  is non-zero when  $e = 0$ . As anticipated above, we are interested in this work in the design of a dynamical smooth feedback controller regulating  $e$  to zero with a global domain of attraction. To this end, we suppose that the zero-dynamics of system (1) is globally

convergent (see, e.g. [21]). Although such an assumption may seem rather strong it allows to simplify the derivations in the proof of the main result in which an interconnection between a nonlinear ODE and a linear PDE is addressed. We conjecture that is should be possible to relax such assumptions by restricting the attention to a semi-global framework in which solutions starts from a known compact set. However, this would require mathematical developments that would unnecessarily over-complicate the exposition of the developments of this work. To this end, we prefer to state the following strong assumption not to blur up the main message of this work.

**Assumption 1** The zero-dynamics (1a) is (uniformly in time) convergent and ISS with respect to  $e$ . In particular, there exist a unique bounded solution  $\bar{z} \in C(\mathbb{R}; \mathbb{R}^n)$  satisfying for all  $t \in \mathbb{R}$

$$\dot{\bar{z}} = f(t, \bar{z}, 0),$$

a positive definite function  $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , functions  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  and real numbers  $\alpha, \gamma > 0$  satisfying

$$\underline{\alpha}(|z - \bar{z}|) \leq V(t, z, \bar{z}) \leq \bar{\alpha}(|z - \bar{z}|),$$

$$\dot{V} := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial z} f(t, z, e) + \frac{\partial V}{\partial \bar{z}} f(t, \bar{z}, 0) \leq -\alpha|z - \bar{z}|^2 + \gamma|e|^2$$

for any  $(t, z, e) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ .

We remark that Assumption 1 can be verified for instance under the well-known Demidovich condition (see, e.g., [20], [22])

$$P \frac{\partial f}{\partial z}(t, z, e) + \frac{\partial f}{\partial z}(t, z, e)^\top P \leq -Q.$$

It follows from previous assumption that the steady-state that we aim at stabilizing is  $(\bar{z}, 0)$ . This steady-state can be made invariant if the control input is able to generate the corresponding *friend*, i.e. the signal  $\varphi(t) = -q(t, \bar{z}(t), 0)$  that makes the origin of the  $e$ -dynamics invariant. In the context of output regulation, it is typically supposed that  $\varphi$  satisfies a (possibly nonlinear) regression law [9], [10] or a mild detectability condition [18]. Both assumptions allow to design a finite-dimensional nonlinear internal-model based regulator of the form

$$\dot{\eta} = \phi(\eta, e), \quad u = -\kappa e + \psi(\eta)$$

which is however shown not to be robust in the presence of unstructured  $C^1$  model uncertainties [3], [4]. Indeed, combining the set of works [3], [4] it can be shown that in the context of smooth regulators, only infinite-dimensional linear systems can be generically robust w.r.t. unstructured  $C^1$  model uncertainties in  $f, q$  and provide asymptotic regulation. In the context of periodic functions  $f, q$ , a solution based on a repetitive control approach was given in [6]. In this work, we generalize this idea by supposing that the friend  $\varphi$  can be generated by an infinite-dimensional skew-adjoint operator. As detailed later in Section III, such an assumption allows, for instance, to consider the class of signals that are periodic or quasi-periodic.

**Assumption 2** There exist a skew-adjoint operator  $\mathcal{S} : D(\mathcal{S}) \subset \mathcal{H} \rightarrow \mathcal{H}$ , with  $D(\mathcal{S})$  dense in  $\mathcal{H}$  and a (possibly unbounded) operator  $\mathcal{E} \in \mathfrak{L}(D(\mathcal{S}), \mathbb{R})$  and for any initial time  $t_0 \in \mathbb{R}$  there exist an initial condition  $v_0 \in D(\mathcal{S})$  such that the function  $\varphi(t) := -q(t, \bar{z}(t), 0)$  satisfies for all  $t \geq t_0$

$$\begin{aligned} \dot{v}(t) &= \mathcal{S}v(t), & v(0) &= v_0, \\ \mathcal{E}v(t) &= \varphi(t). \end{aligned}$$

### C. Feedback Design

Based on previous assumptions, we consider now the following linear infinite-dimensional regulator

$$\begin{aligned} \dot{\eta} &= \mathcal{S}\eta + \mathcal{G}e \\ u &= -\kappa e + \mu \mathcal{M}^*(\eta - \mathcal{M}e) \end{aligned} \quad (2)$$

where the  $\mathcal{S} : D(\mathcal{S}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is defined as in Assumption 2,  $\mathcal{G} : \mathcal{L}(\mathbb{R}, \mathcal{H})$  is a bounded operator to be chosen so that the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  is approximately observable in infinite time (see Definition 3) and  $\mathcal{M} : \mathbb{R} \rightarrow \mathcal{H}$  is a bounded linear operator selected as the solution to the (infinite-dimensional) Sylvester equation

$$-\kappa \mathcal{M} = \mathcal{S}\mathcal{M} + \mathcal{G}, \quad (3)$$

and  $\mathcal{M}^* : \mathcal{H} \rightarrow \mathbb{R}$  is its adjoint operator. Note that since  $\mathcal{S}$  is a skew-symmetric operator, for any  $\kappa > 0$  the spectrum of  $-\kappa$  and  $\mathcal{S}$  are disjoint and therefore the solution to (3) is unique, see, e.g. [23, Lemma 22]. Also,  $\mathcal{M}$  takes values in  $\mathcal{H}$  as remarked in [24]. The design of the regulator (2) is inspired by the forwarding technique developed for the stabilization of cascade systems and successfully used in many contexts, e.g. [6], [14]–[16], [24]–[26]. See, in particular, [24], [25] for the relevant case of ODE-PDE interconnections. We can state therefore the main result of this work.

**Theorem 1** *Consider system (1) and suppose Assumptions 1-2 hold. Consider the internal-model based regulator (2) and suppose that the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  is approximately observable according to Definition 3. Then, there exists a  $\kappa^* > 0$  such that, for any  $\kappa > \kappa^*$  and any  $\mu > 0$ , the following statements hold.*

- For any initial condition  $(t_0, z_0, e_0, \eta_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times D(\mathcal{S})$  the closed-loop system (1), (2) admits a unique strong<sup>1</sup> solution  $(z, e, \eta)$  in  $C^1([t_0, +\infty); \mathbb{R}^n \times \mathbb{R} \times \mathcal{H}) \cup C^0([t_0, +\infty); \mathbb{R}^n \times \mathbb{R} \times D(\mathcal{S}))$ , which is bounded for all  $t \geq t_0$ , namely there exists  $\delta > 0$  such that  $|z(t)| + |e(t)| + \|\eta(t)\|_{D(\mathcal{S})} \leq \delta$  for all  $t \geq t_0$ .
- For any initial condition  $(t_0, z_0, e_0, \eta_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times D(\mathcal{S})$ , solutions of the closed-loop system (1), (2) satisfy  $\lim_{t \rightarrow \infty} e(t) = 0$ .

*Proof:* See Section IV. ■

Previous theorem establishes that the regulator (2) solves the global output regulation problem for systems of the form (1). Once the class of signals to be generated is fixed, i.e. once the operator  $\mathcal{S}$  in Assumption 2 is selected, the regulator (2) is only parametrized by the high-gain parameter  $\kappa$  that has to be selected large enough in order to guarantee incremental stability properties of the  $(z, e)$ -dynamics in (1). In Assumptions 2 we immerse the signal  $\varphi$  characterizing the friend into an infinite-dimensional generator, but the explicit knowledge of the functions  $f, q$  is not needed. As a consequence, the design is robust to any variations of such functions as long as  $f, q$  are varied within this class of signals that can be generated by the exosystem  $\mathcal{S}$ . For instance, in the case of periodic functions  $f, q$ , only the knowledge of the period is required [6]. This situation can occur, for instance, when considering functions  $f(w, z, e)$ ,  $q(w, z, e)$  of a  $T$ -periodic exosignal  $w$ . In this case, only the Lipschitz constants of  $f, q$  and the period  $T$  are needed in order to design the regulator (2). Note that with the proposed framework, we also recover the case of signals generated by linear exosystems and polynomial nonlinearities, e.g. [17].

In the next section we investigate Assumption 2 in the context of quasi-periodic signals providing an explicit design for the operators  $\mathcal{S}, \mathcal{G}, \mathcal{M}$  in (2).

<sup>1</sup>Actually one can also prove existence of weak solutions when asking  $\eta_0 \in \mathcal{H}$ . Since the PDE is part of the regulator, we focus in this case only to initial conditions guaranteeing existence of strong solutions.

### III. INTERNAL MODELS FOR QUASI-PERIODIC SIGNALS

Suppose that the signal  $\varphi$  in Assumption 2 is quasi-periodic and can be written as

$$\varphi(t) = \sum_{i=1}^N \varphi_i(t), \quad \varphi_i(t + T_i) = \varphi_i(t), \quad \forall t \geq 0, \quad (4)$$

namely as a sum of  $N$  signals in which each  $\varphi_i$  is  $T_i$ -periodic. Moreover, we suppose that the periods  $T_i$  are incommensurable real numbers, namely  $\frac{T_i}{T_j}$  is an irrational number for any pair of  $i, j, i \neq j$ . For  $N = 1$ , we fall in the context of repetitive control, e.g. [6], [7]. We recall that a generator of  $\tau$ -periodic signals can be obtained by using a transport equation [6, Section 2.1] as follows:

$$\begin{aligned} \partial_t \phi(t, x) &= -\frac{1}{\tau} \partial_x \phi(t, x), \quad \forall (t, x) \in \mathbb{R} \times [0, 1], \\ \phi(t, 1) &= \phi(t, 0), \quad \forall t \in \mathbb{R}. \end{aligned}$$

As a consequence, Assumption 2 is verified in the context of quasi-periodic signals by selecting the operators  $\mathcal{S}, \mathcal{E}$  as

$$\mathcal{S}v := -\Lambda \partial_x v, \quad \mathcal{E}v := \mathbf{1}v(t, 1), \quad (5)$$

with  $D(\mathcal{S}) = \{v \in \mathcal{H} : v(0) = v(1)\}$  with  $\mathcal{H} := L^2([0, 1]; \mathbb{R}^N)$ ,

$$\Lambda := \text{diag} \left( \frac{1}{T_1}, \dots, \frac{1}{T_N} \right) \quad (6)$$

and with  $\mathbf{1}$  being a row vector in which any element is equal to 1, namely  $\mathbf{1} := (1, \dots, 1)$ . In other words, for any continuous function  $\varphi$  in (4) and any  $t_0 \in \mathbb{R}$ , there exists an initial condition  $v(t_0) = v_0$  to (5) such that  $\varphi(t) = \mathcal{E}v(t)$  for all  $t \geq t_0$ . Note that in this case  $\mathcal{E}$  is an unbounded operator.

Then, the operators  $\mathcal{S}, \mathcal{G}$  and  $\mathcal{M}$  of internal model unit (2) can be taken as

$$\mathcal{S}\eta := -\Lambda \partial_x \eta, \quad \mathcal{G}e := B(x)e, \quad \mathcal{M}e := \int_0^1 M(x)e dx. \quad (7)$$

with  $\Lambda$  defined in (6),  $B : [0, 1] \rightarrow \mathbb{R}^N$  defined as

$$B(x) := \text{col}(\beta_1 \exp(T_1 x), \dots, \beta_N \exp(T_N x)),$$

and  $M : [0, 1] \rightarrow \mathbb{R}^N$  defined as the solution of the following two-boundary value problem

$$\begin{aligned} \Lambda M'(x) &= \kappa M(x) + B(x), \\ M(0) &= M(1), \end{aligned}$$

corresponding to the Sylvester equation (3). Note that  $M$  is a column vector and its  $i$ -th component can be explicitly computed as

$$\begin{aligned} M_i(x) &= \begin{cases} \exp(\kappa T_i x) \bar{M}_i + \beta_i \frac{\exp(T_i x) - \exp(\kappa T_i x)}{T_i(1-\kappa)}, & \kappa \neq 1, \\ \exp(\kappa T_i x) \bar{M}_i + \beta_i \exp(\kappa T_i x), & \kappa = 1, \end{cases} \\ \bar{M}_i &= \begin{cases} \frac{\beta_i}{T_i(1-\kappa)} \frac{\exp(T_i) - \exp(\kappa T_i)}{1 - \exp(\kappa T_i)}, & \kappa \neq 1, \\ \frac{\exp(\kappa T_i)}{1 - \exp(\kappa T_i)} \beta_i, & \kappa = 1. \end{cases} \end{aligned} \quad (8)$$

Moreover, the adjoint  $\mathcal{M}^*$  of  $\mathcal{M}$  is defined as

$$\mathcal{M}^* \eta = \int_0^1 M(x)^\top \eta(t, x) dx.$$

With this notation, the internal-model regulator (2) takes the more explicit form

$$\begin{aligned} \partial_t \eta(t, x) &= -\Lambda \partial_x \eta(t, x) + B(x)e(t) \\ \eta(t, 1) &= \eta(t, 0) \\ u(t) &= -\kappa e(t) + \mu \int_0^1 M(x)^\top (\eta(t, x) - M(x)e(t)) dx, \end{aligned} \quad (9)$$

defined on  $(t, x) \in \mathbb{R} \times [0, 1]$ , and with state variable  $\eta$  taking initial condition in  $L^2([0, 1]; \mathbb{R}^N)$ . We remark that the operator  $\mathcal{G}$  so defined (7) is bounded, allowing to guarantee ISS properties with respect to  $e$ . With such a choice, one can deal with systems possessing a relative degree larger than one and add a high-gain observer (estimating the derivatives of the output  $e$ ) so that to obtain a pure output feedback design, similarly to [12]. This is not the case if one consider an internal-model unit of the form

$$\begin{aligned}\partial_t \eta(t, x) &= -\Lambda \partial_x \eta(t, x) \\ \eta(t, 1) &= \eta(t, 0) + \mathbf{1}^\top e(t)\end{aligned}$$

as done instead in [6] in the context of repetitive control.

In the next lemma, we show that the pair  $\mathcal{S}, \mathcal{G}$  so defined satisfies the conditions of Theorem 1.

**Lemma 1** *Let  $\beta_i \neq 0$  for all  $i \in [1, \dots, N]$  and let  $(T_1, \dots, T_N)$ , with  $T_i \neq 0$  for all  $i \in [1, \dots, N]$  be a set of incommensurable scalars, namely  $\frac{T_i}{T_j}$  is an irrational number for any pair of  $i, j \in [1, \dots, N]$ ,  $i \neq j$ . Then, the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  defined in (7) is approximately observable in infinite time.*

*Proof:* In order to prove the desired result it suffices to check that the adjoint system of (9), namely the system

$$\begin{aligned}\partial_t \xi(t, x) + \Lambda \partial_x \xi(t, x) &= 0 \\ \xi(t, 0) &= \xi(t, 1) \\ y(t) &= \int_0^1 B^\top(x) \xi(t, x) dx\end{aligned}\tag{10}$$

is approximately observable from the output  $y$ . To this end, we adapt similar arguments used in [25, Theorem 2]. In particular, we suppose that  $y$  is constantly equal to zero. If we are able to show that the only initial solution satisfying such a property is  $\xi(0, x) = 0$  then the statement holds. By using the dynamics of  $\xi$  and its boundary conditions defined in (10), the output is computed, by doing an integration by parts, as

$$\begin{aligned}y(t) &= \int_0^1 B^\top(x) \xi(t, x) dx = \int_0^1 \sum_{i=1}^N \beta_i e^{T_i x} \xi_i(t, x) dx \\ &= \sum_{i=1}^N \frac{\beta_i}{T_i} \left[ e^{T_i x} \xi_i(t, x) \right]_0^1 - \sum_{i=1}^N \frac{\beta_i}{T_i} \int_0^1 e^{T_i x} \partial_x \xi_i(t, x) dx \\ &= \sum_{i=1}^N \frac{\beta_i}{T_i} (e^{T_i} - 1) \xi_i(t, 0) + \frac{d}{dt} \sum_{i=1}^N \int_0^1 \beta_i e^{T_i x} \xi_i(t, x) dx \\ &= \sum_{i=1}^N c_i \xi_i(t, 0) + \frac{d}{dt} y(t)\end{aligned}$$

with  $c_i = \frac{\beta_i}{T_i} (e^{T_i} - 1)$ . Since  $y(t) = 0$  for all  $t \in \mathbb{R}$  we have also  $\dot{y}(t) = 0$  for all  $t \in \mathbb{R}$ . This implies that  $\sum_{i=1}^N c_i \xi_i(t, 0) = 0$  for all  $t \in \mathbb{R}$ . Using the fact that each component  $\xi_i$  generates a periodic signal of period  $T_i$  and that all the periods  $T_i$  are incommensurable, and moreover  $c_i \neq 0$  for all  $i = 1, \dots, N$ , we conclude that necessarily the previous condition implies  $\xi_i(t, 0) = 0$  for all  $t \in \mathbb{R}$ , see [27, Theorem 2]. Then  $\xi$  satisfies for all  $t \in \mathbb{R}$ .

$$\begin{aligned}\partial_t \xi(t, x) + \Lambda \partial_x \xi(t, x) &= 0 \\ \xi(t, 0) &= \xi(t, 1) = 0.\end{aligned}$$

Now let us define

$$E_1 = \int_0^1 \xi(t, x)^\top \xi(t, x) dx, \quad E_2 = \int_0^1 e^{-x} \xi(t, x)^\top \xi(t, x) dx,$$

which are candidate Lyapunov functions equivalent to the usual norm in  $L^2(0, 1)$ . On the one hand, the derivative of  $E_1$  along solutions

satisfies

$$\begin{aligned}\dot{E}_1 &= -2 \sum_{i=1}^N \int_0^1 \frac{1}{T_i} \xi_i(t, x) \partial_x \xi_i(t, x) dx \\ &= -2 \sum_{i=1}^N \frac{1}{T_i} \left[ \xi_i(t, x)^2 \right]_0^1 = 0,\end{aligned}$$

which implies  $E_1(\xi(t, x)) = E_1(\xi(0, x))$ , namely the energy is constant. On the other hand, the derivative of  $E_2$  satisfies

$$\begin{aligned}\dot{E}_2 &= -2 \sum_{i=1}^N \int_0^1 \frac{1}{T_i} e^{-x} \xi_i(t, x) \partial_x \xi_i(t, x) dx \\ &= 2 \sum_{i=1}^N \frac{1}{T_i} \left[ e^{-x} \xi_i(t, x)^2 \right]_0^1 - \int_0^1 \frac{1}{T_i} e^{-x} \xi_i(t, x)^2 dx \\ &\leq -\varepsilon \int_0^1 e^{-x} \xi(t, x)^\top \xi(t, x) dx = -\varepsilon E_2\end{aligned}$$

for some  $\varepsilon > 0$ , implying  $E_2(\xi(t, x)) \leq e^{-\varepsilon(t-t_0)} E_2(\xi(0, x))$ . Combining  $E_1$  and  $E_2$  we obtain that the energy is both preserved and also converging to zero. This can happen only if  $\xi(0, x) = 0$  in  $L^2(0, 1)$ , concluding the proof. ■

We remark that a possible manner to implement the regulator (9) is to use a spectral decomposition and implement a finite-dimensional regulator

$$\dot{\eta} = S\eta + Ge\tag{11}$$

with  $S$  being a matrix containing only imaginary eigenvalues at frequencies  $\frac{2\pi}{T_i}$  and a desired number of higher harmonics  $k \frac{2\pi}{T_i}$ , and  $(S, G)$  and controllable pair. In this case, practical robust regulation can be achieved, as shown in [3, Theorem 3]. In the simple context of periodic signals (i.e.  $N = 1$ ), remarkable results have been obtained see, e.g. [6], [13]–[15]. In particular, one can show that once the high-gain parameter  $\kappa$  is fixed, one can reduce the  $L_2$  norm of the asymptotic steady-state regulation error by increasing the dimension of the regulator (11) and in particular by augmenting the number of linear oscillators included in the matrix  $S$ . In other words, by improving the approximation of the finite-dimensional realization (11) with respect to the infinite-dimensional exact regulator (7), see, [14].

## IV. PROOF OF MAIN RESULTS

### A. Preliminaries lemmas

In this section we state a certain number of preliminary results which are needed in order to prove Theorem 1.

**Lemma 2** *Suppose the the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  is approximately observable in infinite time. Then, the pair  $(\mathcal{S}, \mathcal{M}^*)$  defined in (3) is also approximately observable in infinite time.*

*Proof:* The proof follows by direct application of Proposition 1 of [24] since the all conditions of such a proposition are satisfied. In particular, because the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  is approximately observable, the operator  $\mathcal{S}$  is skew-adjoint, and the matrix

$$\begin{pmatrix} -\kappa + \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

is full rank for any  $\lambda$  in the spectrum of  $\mathcal{S}$ , lying on the imaginary axes. ■

**Lemma 3** *Suppose the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  is approximately observable in infinite time. Then, for any  $\mu > 0$ , the operator*

$$\mathcal{F} = \begin{pmatrix} -\kappa - \mu \mathcal{M}^* \mathcal{M} & \mathcal{M}^* \\ \mathcal{G} & \mathcal{S} \end{pmatrix}$$

generates a strongly stable  $C_0$  semigroup.

*Proof:* The proof is a direct application of Theorems 1, 2 and Proposition 1 in [24] and it is therefore just sketched. First, consider the system

$$\begin{cases} \dot{e} &= -\kappa e + \mu \mathcal{M}^*(\eta - \mathcal{M}e) \\ \dot{\eta} &= \mathcal{S}\eta + \mathcal{G}e \end{cases}$$

and apply the following change of coordinates

$$\eta \mapsto \tilde{\eta} := \eta - \mathcal{M}e$$

with  $\mathcal{M}$  defined by the Sylvester equation (14), to obtain a system in the triangular form

$$\dot{\chi} = \tilde{\mathcal{F}}\chi, \quad \tilde{\mathcal{F}} := \begin{pmatrix} -\kappa & \mu \mathcal{M}^* \\ 0 & (\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*) \end{pmatrix},$$

where  $\chi := (e, \tilde{\eta})$ . Note that since the operator  $\mathcal{S}$  is skew-adjoint, the operator  $(\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*)$  is dissipative and moreover it generates a  $C^0$  semigroup of contraction since  $\mathcal{M} \mathcal{M}^*$  is a bounded operator, see [11, Theorem 2.11.2]. Furthermore, in view of the previous lemma, the pair  $(\mathcal{S}, \mathcal{M}^*)$  is approximately observable. As a consequence, using LaSalle's Invariance Principle for infinite-dimensional systems (see, e.g., Theorem 3.1 in [28]) and the lower triangular structure of  $\tilde{\mathcal{F}}$ , one can conclude that the origin of  $\chi$  is globally asymptotically stable, i.e., the operator  $\tilde{\mathcal{F}}$  generates a strongly stable semigroup. As the change of coordinates is linear and globally defined, we conclude that also  $\mathcal{F}$  generates a strongly stable semigroup. Technical details follow from [24]. ■

**Lemma 4** *Suppose the pair  $(\mathcal{S}^*, \mathcal{G}^*)$  is approximately observable in infinite time. Then, for any  $\mu \neq 0$  and for any  $t_0 \in \mathbb{R}$  there exists  $\tilde{\eta}_0 \in D(\mathcal{S})$  such that, for all  $t \geq t_0$  the following holds*

$$\begin{aligned} \dot{\eta}(t) &= \mathcal{S}\eta(t), & \eta(0) &= \tilde{\eta}_0, \\ \mu \mathcal{M}^* \eta(t) &= \varphi(t). \end{aligned} \quad (12)$$

*Proof:* With in mind Assumption 2, consider the following system

$$\begin{cases} \dot{v} &= \mathcal{S}v \\ \dot{e} &= -\kappa e - \mathcal{E}v + \mu \mathcal{M}^*(\eta - \mathcal{M}e) \\ \dot{\eta} &= \mathcal{S}\eta + \mathcal{G}e \end{cases} \quad (13)$$

and let us study the existence of a solution  $\Pi_e, \Pi_\eta$  to the corresponding infinite-dimensional Sylvester equation

$$\begin{aligned} \Pi_e \mathcal{S} &= -(\kappa + \mu \mathcal{M}^* \mathcal{M}) \Pi_e + \mu \mathcal{M}^* \Pi_\eta - \mathcal{E} \\ \Pi_\eta \mathcal{S} &= \mathcal{S} \Pi_\eta + \mathcal{G} \Pi_e \end{aligned} \quad (14)$$

characterizing the steady-state of (13). Since the operator  $\mathcal{F}$  defined in Lemma 3 generates a strongly stable semigroup, the Sylvester equation (14) is well defined and the existence of a solution  $\Pi_e, \Pi_\eta$  is ensured, see, e.g. [29, Lemma 1]. Furthermore, it is possible to verify that the conditions of Theorem 6 in [29] are verified. Therefore, by Theorem 5 in [29], one can conclude from the second equation of (14) that  $\Pi_e = 0$ , and from the first equation of (14) that  $\mathcal{M}^* \Pi_\eta - \mathcal{E} = 0$ . Selecting  $\tilde{\eta}_0 = \Pi_\eta v_0$  with  $v_0$  given by Assumption 2 yields the desired result since  $\mu \mathcal{M}^* \Pi_\eta v(t) = \mathcal{E}v(t) = \varphi(t)$  for all  $t \geq t_0$ . ■

## B. Proof of Theorem 1

Consider the closed-loop system (1), (2). Since  $\mathcal{S}$  is a skew-adjoint operator, then it is a generator of a  $C^0$  semi-group  $\mathbb{T}$ , see, e.g., [11, Theorem 3.8.6]. As a consequence, the solution to (2) can be explicitly denoted as

$$u(t) = -(\kappa + \mu \mathcal{M}^* \mathcal{M})e(t) + \mu \int_{t_0}^t \mathcal{M}^* \mathbb{T}_{t-s} \mathcal{G}e(s) ds$$

for any  $t \geq t_0$  and the closed-loop system (1), (2) can be written as an integro-differential nonlinear equation

$$\begin{aligned} \dot{z}(t) &= f(t, z, e) \\ \dot{e}(t) &= q(t, z, e) - (\kappa + \mathcal{M}^* \mathcal{M})e(t) + \mu \int_{t_0}^t \mathcal{M}^* \mathbb{T}_{t-s} \mathcal{G}e(s) ds \end{aligned} \quad (15)$$

for all  $t \geq t_0$ . Since  $\mathcal{M}^*, \mathcal{G}$  are bounded operators, the closed-loop operator  $\mathcal{M}^* \mathbb{T}_{t-s} \mathcal{G}$  takes value from  $\mathbb{R}$  to  $\mathbb{R}$ . As a consequence, since all the functions are Lipschitz, we can use standard existence of solution theorems of integro-differential equations (see, e.g. [30, Theorem 1.1.1]) to show that system (15) and therefore equivalently system (1), (2) admits the existence of a unique solution for a non-empty interval of time  $[t_0, \tau)$ , with  $\tau$  its maximal time of definition. In the following, since the two representations are equivalent, we keep the original one (1), (2) to show the desired convergence/stability properties.

Now, given any initial time  $t_0 \in \mathbb{R}$ , consider the solution  $(\bar{z}(t), 0, \bar{\eta}(t))$  with  $\bar{\eta}$  being the solution to (12) initialized at  $\bar{\eta}_0$  such that  $\mu \mathcal{M}^* \bar{\eta}(t) \equiv \varphi(t)$  for all  $t \geq t_0$  given by Assumption 2. It can be verified that the solution  $(\bar{z}(t), 0, \bar{\eta}(t))$  so defined is a fixed point of the closed-loop system (1), (2). To ease the computation in the following, consider the change of coordinates  $\eta \rightarrow \tilde{\eta} := \eta - \bar{\eta}$ . The closed-loop system (1), (2) reads

$$\begin{aligned} \dot{z} &= f(t, z, e) \\ \dot{e} &= \Delta(t, z, e) - \kappa e + \mu \mathcal{M}^*(\tilde{\eta} - \mathcal{M}e) \\ \dot{\tilde{\eta}} &= \mathcal{S}\tilde{\eta} + \mathcal{G}e \end{aligned} \quad (16)$$

with  $\Delta(t, z, e) = q(t, z, e) - q(t, \bar{z}, 0)$ . Then, define the following Lyapunov functional

$$W(t, z, e, \eta) := cV(t, z, \bar{z}) + e^2 + \mu \|\tilde{\eta} - \mathcal{M}e\|_{\mathcal{H}}^2, \quad (17)$$

with  $\|\cdot\|_{\mathcal{H}}$  being the standard induced norm in  $\mathcal{H}$  and  $\bar{\eta}$ . Then, recalling Assumption 1, we compute

$$\begin{aligned} \dot{W} &\leq -c\alpha|z - \bar{z}|^2 + c\gamma|e|^2 + 2e(\Delta - \kappa e) + 2\mu e \mathcal{M}^*(\tilde{\eta} - \mathcal{M}e) \\ &\quad + \mu \langle \tilde{\eta} - \mathcal{M}e, (\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*)(\tilde{\eta} - \mathcal{M}e) - \mathcal{M}\Delta \rangle_{\mathcal{H}} \\ &\quad + \mu \langle (\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*)(\tilde{\eta} - \mathcal{M}\bar{e}) - \mathcal{M}\Delta, \tilde{\eta} - \mathcal{M}\bar{e} \rangle_{\mathcal{H}} \end{aligned}$$

where we used the Sylvester equation (14). Since  $\mathcal{S}$  is a skew symmetric operator, we have  $\langle \tilde{\eta}, \mathcal{S}\tilde{\eta} \rangle_{\mathcal{H}} + \langle \mathcal{S}\tilde{\eta}, \tilde{\eta} \rangle_{\mathcal{H}} = 0$ . Therefore,

$$\begin{aligned} &\mu \langle \tilde{\eta} - \mathcal{M}e, (\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*)(\tilde{\eta} - \mathcal{M}e) \rangle_{\mathcal{H}} \\ &\quad + \mu \langle (\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*)(\tilde{\eta} - \mathcal{M}e), \tilde{\eta} - \mathcal{M}e \rangle_{\mathcal{H}} \\ &\leq -2\mu^2 |\mathcal{M}^*(\tilde{\eta} - \mathcal{M}e)|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\mu \langle -\mathcal{M}\Delta, \tilde{\eta} - \mathcal{M}e \rangle_{\mathcal{H}} + \mu \langle \tilde{\eta} - \mathcal{M}e, -\mathcal{M}\Delta \rangle_{\mathcal{H}} \\ &\leq 2|\Delta|^2 + \frac{1}{2}\mu^2 |\mathcal{M}^*(\tilde{\eta} - \mathcal{M}e)|^2. \end{aligned}$$

Since  $q$  is globally Lipschitz (uniformly in time) in its arguments, so is  $\Delta$ . We denote with  $L$  its Lipschitz constant obtaining

$$\begin{aligned} 2|e\Delta| &\leq 2|e|L(|e| + |z - \bar{z}|) \leq 3L|e|^2 + L|z - \bar{z}|^2, \\ 2|\Delta|^2 &\leq 4L^2|e|^2 + 4L^2|z - \bar{z}|^2. \end{aligned}$$

Combining together all previous inequalities we obtain

$$\begin{aligned} \dot{W} &\leq -(c\alpha - 4L^2 - L)|z - \bar{z}|^2 - (2\kappa - c\gamma - 4L^2 - 3L - 1)|e|^2 \\ &\quad - \frac{1}{2}\mu^2 |\mathcal{M}^*(\tilde{\eta} - \mathcal{M}e)|^2 \end{aligned}$$

showing that the closed-loop system is globally dissipative selecting  $\kappa > \kappa^*$  with

$$c = \frac{1}{\alpha}(4L^2 + L) + \varepsilon, \quad \kappa^* = \frac{1}{2}(c\gamma + 4L^2 + 3L + 1),$$

for some arbitrarily small  $\varepsilon > 0$ . In particular, from the previous inequality we obtain

$$\dot{W} \leq -\varepsilon(|z - \bar{z}|^2 + |e|^2 + |\mathcal{M}^*(\eta - \bar{\eta} - \mathcal{M}e)|^2) \leq -\varepsilon|e|^2$$

for some  $\varepsilon > 0$ . Standard arguments can be used to show that the maximal time of existence of solutions is  $\tau = \infty$  and uniformly bounded. Details are omitted for space reasons but one can follow the same lines in the proof of [14, Proposition 4]. Hence, by integrating on  $[t_0, +\infty)$ , we obtain  $\int_{t_0}^{+\infty} |e(t)|^2 dt < +\infty$ . Since all functions are globally Lipschitz, we obtain that the functions  $t \mapsto e(t)$  is also globally Lipschitz and therefore uniformly continuous. As a consequence, by applying Barbalat's lemma, we obtain that  $\lim_{t \rightarrow +\infty} |e(t)| = 0$  concluding the proof. Note that one can also show, following the arguments described in the Appendix of [6], that the solution  $(\bar{z}, 0, \bar{\eta})$  is uniformly (in time) asymptotically stable.

## V. CONCLUSIONS

Built on recent results claiming that finite-dimensional smooth regulators cannot guarantee asymptotic regulation robustly to model uncertainties [3], [4], we investigate the use of infinite-dimensional regulators under a mild assumption on the class of signals to be compensated in steady-state. These include asymptotic regulation problems in the presence of quasi-periodic signals containing an infinite number of spectral components.

Although the proposed regulators are not implementable because of their infinite-dimensional nature, they represent a starting point for the design of large scale regulators able to achieve practical regulation in a non-high-gain feedback paradigm. In other words, we believe that the practical objective of approximate regulation can be obtained not by increasing a high-gain parameter but by appropriately discretizing (spatially and or temporally) the infinite-dimensional regulator and by tuning its dimension. For instance, in the context of periodic signals, it is shown that a spectral decomposition based on Fourier series allows to reduce the asymptotic  $L^2$  norm of the regulated output without need of increasing the high-gain parameter [14], but by increasing the numbers of eigenvalues on the imaginary axe, namely the dimension of the internal-model regulator. A detailed study will be also dedicated to the case of quasi-periodic signals to better detail the preliminary results obtained in [3, Theorem 3] so that to characterize the dimension of the resulting finite-dimensional regulator needed to guarantee desired asymptotic bounds on the regulated output.

Finally, as a future works, we would like also to study the use of infinite-dimensional systems for systems possessing a more general form, such as input-affine nonlinear systems [16], [26].

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