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# ON THE GROWTH RATE OF POWERS OF A STRONGLY KREISS BOUNDED OPERATOR ON SOME $L^p$ SPACE

LORIS ARNOLD AND CHRISTOPHE CUNY

ABSTRACT. Let  $T$  be a strongly Kreiss bounded linear operator on  $L^p$ . We obtain a somewhat optimal control on the rate of growth of the norms of the powers. The proof makes use of Fourier multipliers, in particular of Littlewood-Paley inequalities on arbitrary intervals as initiated by Rubio de Francia and developed by Kislyakov and Parilov.

## 1. INTRODUCTION

Let  $T$  be a bounded operator on a Banach space  $\mathcal{X}$ . We study the growth rate of  $\|T^n\|$  when  $\mathcal{X} = L^p(\mu)$ , with  $\mu$   $\sigma$ -finite, under the so-called Strong Kreiss condition. Let us describe the necessary background to state our main result.

In [8], Kreiss introduced the following resolvent condition.

$$(1) \quad \|R(\lambda, T)\| \leq \frac{C}{|\lambda| - 1} \quad |\lambda| > 1,$$

where  $R(\lambda, T) = (\lambda I - T)^{-1}$  and proved that, on a *finite dimensional* Banach space that condition is equivalent to power-boundedness of  $T$ , i.e.  $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$ .

Lubich and Nevanlinna [9], proved that (1) implies that  $\|T^N\| = \mathcal{O}(N)$  and, by a result of Shields [14], this is optimal.

Later, McCarthy [10] considered the following strengthening of (1)

$$(2) \quad \|R(\lambda, T)^k\| \leq \frac{C}{(|\lambda| - 1)^k} \quad |\lambda| > 1, k \in \mathbb{N},$$

known as the strong Kreiss condition (or iterated Kreiss condition).

Lubich and Nevanlinna [9] proved that (2) implies that  $\|T^N\| = \mathcal{O}(\sqrt{N})$  and they proved that this estimate is best possible for general Banach spaces.

Nevanlinna [12] proved that an operator  $T$  satisfies the strong Kreiss condition if and only if there exists  $L > 0$  such that

$$(3) \quad \|e^{zT}\| \leq Le^{|z|} \quad \forall z \in \mathbb{C}.$$

When  $\mathcal{X}$  is a Hilbert space, Cohen et al. [5] proved that (3) implies that  $\|T^N\| = \mathcal{O}((\log N)^\kappa)$  and that this logarithmic control is best possible under (2) in the Hilbert setting.

They also obtained results in the Hilbert setting under (1), which were generalized to  $L^p$  spaces (and more generally, Banach spaces with non trivial type and/or cotype) by the second author [4].

In [4], the situation of strongly Kreiss bounded operators (i.e. operators satisfying (3)) on some  $L^p$  space was left open and this is the purpose of the present work to study that situation. We obtain the following.

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**Theorem 1.1.** *Let  $T$  be a strongly Kreiss bounded operator on  $L^p(\mu)$ ,  $1 < p < \infty$ . There exists  $C, \kappa > 0$  such that for every  $N \in \mathbb{N}$ , setting  $\tau_p := \left| \frac{1}{2} - \frac{1}{p} \right|$ ,*

$$(4) \quad \|T^N\| \leq CN^{\tau_p} \log^\kappa(N+1).$$

**Remark.** When  $p = 2$  we recover the optimal result from [5].

Our bound is somewhat optimal, as it follows from the next proposition, based on an example of Lubich and Nevanlinna [9].

**Proposition 1.2.** *For every  $1 \leq p \leq \infty$ , there exists a strongly Kreiss bounded operator on  $\ell^p(\mathbb{Z})$  and some constant  $C_p \geq 1$  such that for every  $N \in \mathbb{N}$ ,  $N^{|1/2-1/p|}/C_p \leq \|T^N\| \leq C_p N^{|1/2-1/p|}$ .*

Actually, we get a single operator acting simultaneously on all  $\ell^p(\mathbb{Z})$ ,  $1 \leq p \leq \infty$ , with the above properties. If  $S$  is the right shift, then  $T = q(S)$  for some Möbius transformation of the unit disk.

Let us also mention that, see Proposition 4.2, when  $T$  is a *positive* strongly Kreiss bounded operator on  $L^p(\mu)$ , then it is possible to improve (4), when  $p \in [1, 4/5]$ .

## 2. AUXILLIARY RESULTS

Through the paper we will denote by  $\mathbb{T}$  the complex numbers of modulus one and by  $\lambda$  the Haar measure on  $\mathbb{T}$ .

Given a bounded interval  $I \subset \mathbb{Z}$ , we define an operator  $M_I$  by setting

$$M_I f := \sum_{i \in I} c_i(f) \gamma^i \quad \forall f \in L^1(\mathbb{T}),$$

where  $f(\gamma) = \sum_{n \in \mathbb{Z}} c_n(f) \gamma^n$ .

Recall the definition of the weak  $L^1$ -norm on a measure space

$$\|h\|_{1, \infty, \mu} := \sup_{t > 0} t \mu(|g| \geq t).$$

**Proposition 2.1.** *Let  $1 < p < \infty$ . Let  $p' = \min(2, p)$ . There exists  $D_p > 0$  such that for every finite collection  $(I_\ell)_{1 \leq \ell \leq L}$  of disjoint intervals of integers,*

$$(5) \quad \left\| \left( \sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \leq D_p L^{1/p'-1/2} \|f\|_{L^p(\mathbb{T})}, \quad \forall f \in L^p(\mathbb{T}).$$

*Furthermore, there exists  $D_{1, \infty} > 0$  such that for every finite collection  $(I_\ell)_{1 \leq \ell \leq L}$  of (not necessarily disjoint) intervals of integers,*

$$(6) \quad \left\| \left( \sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^{1, \infty}(\mathbb{T})} \leq D_{1, \infty} L^{1/2} \|f\|_{L^1(\mathbb{T})}, \quad \forall f \in L^1(\mathbb{T}).$$

**Proof.** For  $p \geq 2$ , (5) is the so called Littlewood-Paley-Rubio de Francia inequality, see [13]. Actually, Rubio de Francia proved the result on the real line. The result for the torus (with extensions) appears in Kislyakov-Parilov [7].

Inequality (6) appears in the middle of page 6419 of [7], see also Exercise 4.6.1 (a) page 337 of [6] for a version on the real line.

Then (5) for  $1 < p < 2$  follows by Marcinkiewicz interpolation, see Theorem 1.3.2 of [6] page 31.

**Corollary 2.2.** *Let  $1 < p < \infty$ . Let  $p'' = \max(2, p)$ . There exists  $C_p > 0$  such that for every finite collection  $(I_\ell)_{1 \leq \ell \leq L}$  of disjoint and consecutive intervals of integers such with  $\cup_{\ell=1}^L I_\ell =: I \subset [-N, N]$ ,  $N \in \mathbb{N}$  and every  $f \in L^p(\mathbb{T})$ ,*

$$(7) \quad \begin{aligned} \|M_I f\|_{L^p(\mathbb{T})} &\leq C_p L^{1/2-1/p''} \left\| \left( \sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \\ &\leq C_p L^{1/2-1/p''} \left( \sum_{\ell=1}^L \|M_{I_\ell} f\|_{L^p(\mathbb{T})}^{p'} \right)^{1/p'}. \end{aligned}$$

**Proof.** Clearly, it is enough to assume that  $f$  is a trigonometric polynomial supported in  $I$ . Let  $g \in L^q$ , with  $q = p/(p-1)$ . Notice that  $q' = \min(p/(p-1), 2) = p''/(p''-1)$  and that  $1/q' - 1/2 = 1/2 - 1/p''$ . We have, by orthogonality, Cauchy-Schwarz inequality and Hölder inequality,

$$\begin{aligned} \left| \int_{\mathbb{T}} f \bar{g} d\lambda \right| &= \left| \int_{\mathbb{T}} \sum_{\ell=1}^L M_{I_\ell} f M_{-I_\ell} \bar{g} d\lambda \right| \\ &\leq \left\| \left( \sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \left\| \left( \sum_{\ell=1}^L |M_{-I_\ell} \bar{g}|^2 \right)^{1/2} \right\|_{L^q(\mathbb{T})} \\ &\leq C_p L^{1/2-1/p''} \left\| \left( \sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \|\bar{g}\|_{L^q(\mathbb{T})}, \end{aligned}$$

where we used Proposition 2.1. Then (7) follows by taking the supremum over  $g \in L^q(\mathbb{T})$ , with  $\|g\|_{q,\lambda} = 1$ . The last estimate follows by using that  $x \mapsto x^{p/2}$  is subadditive when  $p \leq 2$  and Minkowski's inequality in  $L^{p/2}(\mathbb{T})$  when  $p \geq 2$ .  $\square$

We deduce the following.

**Corollary 2.3.** *Let  $1 < p < \infty$ . There exists  $C_p > 0$  such that for each  $N \in \mathbb{N}$  and each  $(x_1, \dots, x_{N^2}) \in (L^p(\mu))^{N^2}$ ,*

$$(8) \quad \left\| \sum_{k=1}^{N^2} \gamma^k x_k \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq C N^{1/2-1/p''} \left( \sum_{n=0}^{N-1} \left\| \sum_{k=n^2+1}^{(n+1)^2} \gamma^k x_k \right\|_{L^p(\mathbb{T}, L^p(\mu))}^{p'} \right)^{1/p'}.$$

*Proof.* Let  $N \in \mathbb{N}$  and  $I_n := [n^2 + 1, (n+1)^2)$  for  $n \in \{0, \dots, N-1\}$ . By Corollary 2.2 for each  $(a_1, \dots, a_{N^2})$  and  $f \in L^p(\mathbb{T})$ ,

$$\left\| \sum_{k=1}^{N^2} a_k \gamma^k \right\|_{L^p(\mathbb{T})} \leq 2C_p N^{1/2-1/p''} \left( \sum_{\ell=0}^{N-1} \left\| \sum_{k=n^2+1}^{(n+1)^2} a_k \gamma^k \right\|_{L^p(\mathbb{T})}^{p'} \right)^{1/p'}$$

If  $1 \leq p \leq 2$ , (8) follows Fubini's Theorem. If  $p > 2$  (8) follows from Fubini's Theorem combined with Minkowski's inequality in  $L^{p/2}(\mu)$ .  $\square$

We conclude this section by a Lemma we will use several times in this paper.

**Lemma 2.4.** *There exists  $C > 1$  such that for every  $N \in \mathbb{N}$  and every integer  $K \in [2 - 2\sqrt{N}, 0]$ ,*

$$(9) \quad \frac{e^N}{C\sqrt{N}} \leq \frac{N^{N+K}}{(N+K)!} \leq \frac{C e^N}{\sqrt{N}}$$

$$(10) \quad \sum_{N+2-2\sqrt{N} \leq n \leq N} \left| \left( \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^{-1} - \left( \sum_{n+1-\sqrt{N} \leq k \leq n+1} \frac{N^k}{k!} \right)^{-1} \right| \leq C e^{-N}.$$

In particular the sequences  $\left( e^N \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^{-1} \Big|_{N+2-2\sqrt{N} \leq n \leq N}$  have bounded variations uniformly bounded with respect to  $N$ .

**Proof.** The upper bound of (9) follows from Lemma 3.4 of [5] and the lower bound may be proved similarly. Then, (10) follows from the fact that, for  $N+2-2\sqrt{N} \leq n \leq N$ , writing  $m := \lfloor n+1-\sqrt{N} \rfloor$ , we have

$$\begin{aligned} e^{-N} \left| \left( \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^{-1} - \left( \sum_{n+1-\sqrt{N} \leq k \leq n+1} \frac{N^k}{k!} \right)^{-1} \right| &\leq e^{-N} \frac{\frac{N^m}{m!} + \frac{N^n}{n!}}{\left( \sum_{n+1-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^2} \\ &\leq \frac{\tilde{C}}{\sqrt{N}}, \end{aligned}$$

where we used (9). □

### 3. PROOF OF THEOREM 1.1 AND OF PROPOSITION 1.2

The proof of Theorem 1.1 makes use of Fourier multipliers, see pages 11 and 12 of [4] for a brief description of Fourier multipliers in UMD Banach spaces (in particular in  $L^p$  spaces). Actually, we only make use of *real valued* Fourier multipliers in our proofs and then use Fubini to obtain results for  $L^p$ -valued Fourier multipliers.

We shall use in the sequel the terminology of Riesz theorem and Stechkin theorem, already used in [4]. More precisely, let  $X$  be an UMD space and  $1 < p < \infty$ . We refer as *Riesz theorem* the following: it exists  $C_p > 0$  such that for each interval  $I \subset \mathbb{Z}$  we have

$$\forall (c_i)_{i \in \mathbb{Z}} \subset X, \quad \left\| \sum_{i \in I} \gamma^i c_i \right\|_{L^p(\mathbb{T}, X)} \leq C_p \left\| \sum_{i \in \mathbb{Z}} \gamma^i c_i \right\|_{L^p(\mathbb{T}, X)}.$$

And we refer as *Stechkin theorem* the following: if  $(a_n)_{n \in \mathbb{Z}}$  is a bounded monotone sequence of real numbers, then it exists  $D_p$  such that,

$$\forall (c_n)_{n \in \mathbb{Z}} \subset X, \quad \left\| \sum_{n \in \mathbb{Z}} a_n \gamma^n c_n \right\|_{L^p(\mathbb{T}, X)} \leq D_p \left\| \sum_{i \in \mathbb{Z}} \gamma^n c_n \right\|_{L^p(\mathbb{T}, X)}.$$

**Lemma 3.1.** *Let  $T$  be a bounded operator on  $L^p(\mu)$ ,  $1 < p < \infty$ . Assume that there exists  $D > 0$  and  $\alpha > 0$ , such that for every  $n \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,*

$$\left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DN^\alpha \|x\|_{L^p(\mu)}.$$

*Then, there exists  $E_p > 0$  independent of  $D$ ,  $\alpha$  and  $\beta$  such that, setting  $\delta_p := \frac{1}{2} \left( \frac{2}{p'} - \frac{1}{p} \right)$ , for every  $N \in \mathbb{N}$  and every  $x \in L^p(\mu)$ , we have*

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DE_p N^{\alpha + \delta_p}.$$

**Proof.** By Corollary 2.3 and the assumption, for every  $M \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,

$$\begin{aligned} \left\| \sum_{1 \leq n \leq M^2} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))}^{p'} &\leq (CM^{1/2-1/p''})^{p'} \left( \sum_{n=0}^{M-1} \left\| \sum_{k=n^2+1}^{(n+1)^2} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))}^{p'} \right) \\ &\leq 2^{p'\beta} (CD)^{p'} M^{p'(1/2-1/p'')/2} \sum_{n=0}^{M-1} (n+1)^{2p'\alpha} \\ &\leq 2^{p'\beta} (CD)^{p'} M^{p'(1/2-1/p'')+2p'\alpha+1}. \end{aligned}$$

Let  $N \in \mathbb{N}$  and  $M := \lceil \sqrt{N} \rceil + 1$ . By the Riesz theorem, using that  $M^2 \leq 4N$  and  $M \leq N$ , we infer that

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq \left\| \sum_{1 \leq n \leq M^2} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq 2^\beta CD (4N)^{(1/2-1/p'')/2+\alpha+1/2p'},$$

and the result follows.  $\square$

**Lemma 3.2.** *Let  $T$  be a strongly Kreiss bounded operator on  $L^p(\mu)$ ,  $1 < p < \infty$ . Assume that there exists a function  $f : \mathbb{R}_+ \rightarrow (1, \infty)$ ,  $D > 0$  such that for every  $n \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,*

$$(11) \quad \left\| \sum_{n=1}^N \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq Df(N) \|x\|_{L^p(\mu)}.$$

Then, there exists  $E_p > 0$  independent of  $D$  and  $f$  such that for every  $N \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,

$$\left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DE_p f(4\sqrt{N}) \|x\|_{L^p(\mu)}.$$

**Proof.** The proof is similar to the one of Lemma 4.7 of [5].

Let  $x \in L^p(\mu)$  and  $M_{N,\gamma} = \sum_{n=1}^{4\sqrt{N}} \gamma^n T^n$ . Firstly since  $T$  is strongly Kreiss bounded, using assumption (11), we have

$$(12) \quad \left\| e^{\gamma NT} M_{N,\gamma} x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq Ce^N \|M_{N,\gamma} x\|_{L^p(\mathbb{T}, L^p(\mu))} \leq CDe^N f(4\sqrt{N}).$$

Furthermore, we have

$$e^{\gamma NT} M_{N,\gamma} x = \sum_{1 \leq n \leq 4\sqrt{N}} \gamma^n T^n x \sum_{0 \leq k \leq n} \frac{N^k}{k!} + \sum_{n \geq 4\sqrt{N}} \gamma^n T^n x \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!}.$$

For every  $N$  large enough (such that  $N + 2 - 2\sqrt{N} \geq 4\sqrt{N}$ ), using the Riesz theorem, we have

$$\left\| e^{\gamma NT} M_{N,\gamma} x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \geq C_p \left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right\|_{L^p(\mathbb{T}, L^p(\mu))}.$$

By Lemma 2.4 (in particular the property of bounded variation) and the Steckin theorem,

$$(13) \quad \left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right\|_{L^p(\mathbb{T}, L^p(\mu))} \geq De^N \left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))}.$$

Combining (12) and (13), we get the desired result.  $\square$

Combining Lemma 3.1 and 3.2, we easily derive the following.

**Corollary 3.3.** *Let  $T$  be a strongly Kreiss bounded operator on  $L^p(\mu)$ ,  $1 < p < \infty$ . Assume that there exists  $D > 0$  and  $\alpha > 0$ ,  $\beta \geq 0$ , such that for every  $n \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,*

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DN^\alpha \|x\|_{L^p(\mu)}.$$

Then, there exists  $E_p$ , independent of  $\alpha$  and  $\beta$  such that for every  $N \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DE_p N^{\alpha/2 + \delta_p} \|x\|_{L^p(\mu)}.$$

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Since  $T$  is strongly Kreiss bounded, it is known (see pages 1-2 of [5]) that there exists  $C > 0$  such that for every  $N \in \mathbb{N}$  and every  $x \in L^p(\mu)$ ,

$$\left\| \sum_{n=1}^N \gamma^n T^n x \right\|_{L^p(\mu)} \leq CN \|x\|_{L^p(\mu)} \quad \forall \gamma \in \mathbb{T}.$$

Applying inductively Corollary 3.3 we see that for every  $N \in \mathbb{N}$ ,  $K \in \mathbb{N}_0 = \{0, 1, \dots\}$  and every  $x \in L^p(\mu)$ ,

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq CE_p^K N^{2\delta_p + (\alpha - \delta_p)2^{-K}}.$$

Without loss of generality, we may and do assume that  $N \geq 3$ . Let  $K \geq 0$  be the integer such that  $2^K \leq \frac{\log N}{\log(\log N)} \leq 2^{K+1}$ . Then we have  $K \leq \log(\log N)/\log 2$  and  $E_p^K \leq \exp(\log E_p \log(\log N)/\log 2) = \log(N)^{\log E_p / \log 2}$ . Moreover, there exists  $D > 0$  such that  $N^{(\alpha - \delta_p)2^{-K}} \leq e^{D \log \log N} = \log(N)^D$ .

Combining those estimates, we infer that, there exists  $C > 0$  and  $\kappa > 0$  such that for every  $N \in \mathbb{N}$ ,

$$(14) \quad \left\| \sum_{1 \leq n \leq N} \gamma^n T^n \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq CN^{2\delta_p} (\log(N+1))^\kappa \|x\|_{L^p(\mu)}.$$

Using that  $T^*$  is strongly Kreiss bounded on  $L^q(\mu)$ ,  $q = p/(p-1)$ , we obtain a similar estimate for  $T^*$ .

Hence, applying once more Lemmas 3.1 and 3.2, we see that there exists  $C, \kappa > 0$  (large that the previous ones) such that for every  $x \in L^p(\mu)$  and  $x^* \in L^q(\mu)$ ,

$$\begin{aligned} (1 + 2[\sqrt{N}]) |\langle x^*, T^{N+1} x \rangle| &= \left| \int_{\mathbb{T}} \left\langle \sum_{1 \leq n \leq 1+2\sqrt{N}} \gamma^n T^{*n}, \sum_{1 \leq m \leq 1+2\sqrt{N}} \bar{\gamma}^m T^{N+1-m} x \right\rangle d\lambda \right| \\ &\leq \left\| \sum_{1 \leq n \leq 1+2\sqrt{N}} \gamma^n T^{*n} x^* \right\|_{L^q(\mathbb{T}, L^q(\mu))} \left\| \sum_{N-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \\ &\leq CN^{\delta_p + \delta_q} (\log(N+1))^\kappa \\ &= CN^{1/\min(p,q)} (\log(N+1))^\kappa, \end{aligned}$$

and the result follows by taking the supremum over  $x^*$  with  $\|x^*\|_{L^q(\mu)} = 1$ .  $\square$

Let us now prove Proposition 1.2. Let  $0 < a < 1$  and define  $q_a(z) := \frac{z-a}{1-az}$ . Let  $S$  be the right shift, acting on  $\ell^p(\mathbb{Z})$ ,  $1 \leq p \leq \infty$ . Set  $T := q_a(S)$ .

It has been proved in [9] that  $T$  is strongly Kreiss bounded on  $\ell^\infty(\mathbb{Z})$ . Their proof works equally to prove that  $T$  is strongly Kreiss bounded on  $\ell^p(\mathbb{Z})$  for every  $1 \leq p \leq \infty$ .

Moreover, as noticed in [9], it follows from the book by Brenner, Thomée and Wahlbin [3] that

$$N^{|1/2-1/p|}/C_p \leq \|T^N\| \leq C_p N^{|1/2-1/p|},$$

for some  $C_p > 0$ .

In [9], they refer to Theorem 3.1 page 102 of [3], while the theorem is actually concerned with operators acting on  $L^p(\mathbb{R}, \text{Leb})$ . Now, using techniques from [3], pages 19-29 (in particular the bounds (4.3) page 21), it is possible to adapt the proof of Theorem 3.1 page 102 to our setting.

#### 4. SOME PARTICULAR STRONGLY KREISS BOUNDED OPERATORS

We consider here the case of *positive* strongly Kreiss bounded operators on  $L^p(\mu)$  (we do not require anymore  $\mu$  to be  $\sigma$ -finite) or *absolutely* strongly Kreiss bounded operators (on any Banach space). The basic idea used to obtain (4) (in  $L^p$ -spaces) has been already used by the first author [1] where Kreiss bounded operators were considered.

Let begin by a general result.

**Proposition 4.1.** *Let  $1 \leq p \leq 2$  and  $T$  be bounded operator on  $X$  satisfying that there exists  $C > 0$  and  $\alpha > 0$  such that for each  $N \in \mathbb{N}$ ,*

$$(15) \quad \|T^N\| \leq CN^\alpha$$

and for every  $x \in X$

$$(16) \quad \sum_{N-2\sqrt{N} \leq n \leq N} \|T^n x\| \leq CN^{p/2} \|x\|.$$

. Then,  $\|T^N\| = \mathcal{O}(N^{1/q}(\log N)^\kappa)$ , for some  $\kappa > 0$ , where  $q = p/(p-1)$ .

*Proof.* We start with the following observation. For every  $x \in X$  and  $x^* \in X^*$ ,

$$(17) \quad (1 + 2\sqrt{N})|\langle x^*, T^{N+1}x \rangle|^p = \sum_{1 \leq n \leq 1+2\sqrt{N}} |\langle T^{*n}x^*, T^{N+1-n}x \rangle|_{X^*, X}^p \\ \leq \|x^*\|^p \max_{1 \leq n \leq 1+2\sqrt{N}} \|T^{*n}\|^p \sum_{N-2\sqrt{N} \leq n \leq N} \|T^n x\|^p.$$

Taking the supremum over  $x, x^*$  of norm 1, using that  $\|T^{*n}\| = \|T^n\|$  and assumption (16), we infer that, for every  $N \in \mathbb{N}$

$$\|T^N\| \leq N^{-1/2p} \max_{1 \leq n \leq 3\sqrt{N}} \|T^n\| \sup_{\|x\| \leq 1} \left( \sum_{N-2\sqrt{N} \leq n \leq N} \|T^n x\|^p \right)^{1/p} \\ \leq CN^{1/2q} \max_{1 \leq n \leq 3\sqrt{N}} \|T^n\|.$$

Combining above estimate with (15), we get that

$$\|T^N\| \leq C^2 3^\alpha DN^{\alpha/2+1/2q}.$$

Iterating the above we get that for every integers  $N, K \in \mathbb{N}$ ,

$$\|T^N\| \leq 3^{K\alpha} C^{K+1} N^{2^{-K}\alpha+(1-2^{-K})/q},$$

and we conclude as in the proof of Theorem 1.1.  $\square$

From Proposition 4.1 we deduce a better bound than in Theorem 1.1 for a strongly Kreiss bounded positive operator on  $L^p(\mu)$  when  $p \in [1, 4/3) \cup (4, +\infty)$  (the cases where  $p = 1$  and  $p = \infty$  are discussed below).



**Proposition 4.2.** *Let  $T$  be a positive operator that is strongly Kreiss bounded on  $L^p(\mu)$ ,  $1 \leq p < \infty$ . Then,  $\|T^N\| = \mathcal{O}(N^{1/\bar{p}}(\log N)^\kappa)$ , for some  $\kappa > 0$ , where  $\bar{p} = \max(p, p/(p-1))$ .*

The proof is straightforward using following Lemma and the fact that every strongly Kreiss bounded operator satisfies (15) for  $\alpha = 1/2$ .

**Lemma 4.3.** *Let  $1 \leq p < \infty$ . Any positive strong Kreiss bounded operator  $T$  on  $L^p(\mu)$  satisfies (16) for every  $x \in L^p(\mu)$ .*

*Proof.* Using the fact that  $\|\cdot\|_{\ell^p} \leq \|\cdot\|_{\ell^1}$  and Lemma 2.4, we see that there exists  $C > 0$  such that for every  $N \in \mathbb{N}$  and every  $x \in L^p(\mu)$  with  $x \geq 0$ ,

$$(18) \quad \left( \sum_{n \geq 0} \frac{N^n T^n x}{n!} \right)^p \geq \sum_{n \geq 0} \left( \frac{N^n T^n x}{n!} \right)^p \geq \frac{C^p e^{pN}}{\sqrt{N^p}} \sum_{N-2\sqrt{N} \leq n \leq N} (T^n x)^p.$$

Integrating with respect to  $\mu$  and using Strong Kreiss boundedness, we infer  $T$  satisfies (16) for every  $x \in L^p(\mu)$  with  $x \geq 0$ . By linearity (16) it remains true for every  $x \in L^p(\mu)$ .  $\square$

We turn now to the special case of absolutely strongly Kreiss bounded operator. Let  $T$  a bounded operator on  $X$ . We say that  $T$  is absolutely strongly Kreiss bounded if there exists  $C > 0$  such that

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \|T^n x\| \leq C e^r \|x\|, \quad r > 0, x \in X.$$

Such operator is clearly strongly Kreiss bounded and using Lemma 2.4, for every  $x \in X$ , it satisfies (16) with  $p = 1$ . Then we can apply Proposition 4.1 to state that  $\|T^N\|$  has logarithmic bound.

**Proposition 4.4.** *Let  $T$  be an absolutely strongly Kreiss bounded operator on  $X$ . Then, there exists  $\kappa > 0$  such that*

$$(19) \quad \|T^N\| = \mathcal{O}(\log(N)^\kappa).$$

**Remark 4.5.** When  $X = L^p(\mu)$ ,  $1 \leq p < \infty$  the bound (19) is sharp. Indeed according [5, remark 1 page 16], for any  $\kappa > 0$  there exists an absolutely strongly Kreiss bounded  $T$  on  $L^p(\mu)$  such that  $\|T^N\| \asymp (\log(N)^\kappa)$ .

We now discuss the case of positive strong Kreiss operator on  $(AL)$  and  $(AM)$ -spaces. We refer to [11], Section 2, for more details. A Banach lattice  $X$  is an  $(AL)$ -space if the norm is additive on the positive cone on  $X$ , that is

$$(20) \quad \forall x, y \in X_+, \|x + y\| = \|x\| + \|y\|,$$

that is, the norm is additive on the positive cone of  $X$ . A Banach lattice  $X$  is an  $(AM)$ -space if the norm on  $X$  satisfies

$$\forall x, y \in X_+, \|\sup(x, y)\| = \sup(\|x\|, \|y\|).$$

If  $X$  is an  $(AM)$ -space, then  $X^*$  is an  $(AL)$ -space. It is known that an  $(AL)$ -space is isometrically isomorphic to some space  $L^1$  and that an  $(AM)$ -space with is isometrically isomorphic respectively to some  $C(K)$  where  $K$  is a compact space. We are now ready for giving a proof to the following statement.

**Proposition 4.6.** *Let  $T$  be a positive strongly Kreiss bounded operator on an  $(AL)$ -space or an  $(AM)$ -space. Then it exists  $\kappa > 0$  such that*

$$(21) \quad \|T^N\| = \mathcal{O}(\log(N)^\kappa).$$

**Remark.** By the proposition, since  $L^\infty(\mu)$  is an  $(AM)$ -space, we see that Proposition 4.2 remains true for  $p = \infty$ .

*Proof.* If  $T$  is positive strong Kreiss bounded operator on an (AL)-space, then using (20) it is straightforward that  $T$  is absolute strong Kreiss bounded and then we can conclude with Proposition 4.1.

If  $T$  is a positive strongly Kreiss bounded operator on an (AM)-space, then  $T^*$  is a positive strongly Kreiss bounded on an (AL)-space, and then we conclude by above.  $\square$

**Remark 4.7.** In view of our last section, we can ask whether the bound of Proposition 4.2 can be improved to obtain a logarithmic bound. More precisely for a positive strongly Kreiss bounded operator  $T$  on  $L^p(\mu)$  with  $1 < p < \infty$ , is there exist  $\kappa > 0$  such that  $\|T^N\| = \mathcal{O}(\log(N)^\kappa)$  ?

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