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ON THE GROWTH RATE OF POWERS OF A STRONGLY KREISS BOUNDED OPERATOR ON SOME L^p SPACE

LORIS ARNOLD AND CHRISTOPHE CUNY

ABSTRACT. Let T be a strongly Kreiss bounded linear operator on L^p . We obtain a somewhat optimal control on the rate of growth of the norms of the powers. The proof makes use of Fourier multipliers, in particular of Littlewood-Paley inequalities on arbitrary intervals as initiated by Rubio de Francia and developed by Kislyakov and Parilov.

1. INTRODUCTION

Let T be a bounded operator on a Banach space \mathcal{X} . We study the growth rate of $\|T^n\|$ when $\mathcal{X} = L^p(\mu)$, with μ σ -finite, under the so-called Strong Kreiss condition. Let us describe the necessary background to state our main result.

In [8], Kreiss introduced the following resolvent condition.

$$(1) \quad \|R(\lambda, T)\| \leq \frac{C}{|\lambda| - 1} \quad |\lambda| > 1,$$

where $R(\lambda, T) = (\lambda I - T)^{-1}$ and proved that, on a *finite dimensional* Banach space that condition is equivalent to power-boundedness of T , i.e. $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$.

Lubich and Nevanlinna [9], proved that (1) implies that $\|T^N\| = \mathcal{O}(N)$ and, by a result of Shields [14], this is optimal.

Later, McCarthy [10] considered the following strengthening of (1)

$$(2) \quad \|R(\lambda, T)^k\| \leq \frac{C}{(|\lambda| - 1)^k} \quad |\lambda| > 1, k \in \mathbb{N},$$

known as the strong Kreiss condition (or iterated Kreiss condition).

Lubich and Nevanlinna [9] proved that (2) implies that $\|T^N\| = \mathcal{O}(\sqrt{N})$ and they proved that this estimate is best possible for general Banach spaces.

Nevanlinna [12] proved that an operator T satisfies the strong Kreiss condition if and only if there exists $L > 0$ such that

$$(3) \quad \|e^{zT}\| \leq Le^{|z|} \quad \forall z \in \mathbb{C}.$$

When \mathcal{X} is a Hilbert space, Cohen et al. [5] proved that (3) implies that $\|T^N\| = \mathcal{O}((\log N)^\kappa)$ and that this logarithmic control is best possible under (2) in the Hilbert setting.

They also obtained results in the Hilbert setting under (1), which were generalized to L^p spaces (and more generally, Banach spaces with non trivial type and/or cotype) by the second author [4].

In [4], the situation of strongly Kreiss bounded operators (i.e. operators satisfying (3)) on some L^p space was left open and this is the purpose of the present work to study that situation. We obtain the following.

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Theorem 1.1. *Let T be a strongly Kreiss bounded operator on $L^p(\mu)$, $1 < p < \infty$. There exists $C, \kappa > 0$ such that for every $N \in \mathbb{N}$, setting $\tau_p := \left| \frac{1}{2} - \frac{1}{p} \right|$,*

$$(4) \quad \|T^N\| \leq CN^{\tau_p} \log^\kappa(N+1).$$

Remark. When $p = 2$ we recover the optimal result from [5].

Our bound is somewhat optimal, as it follows from the next proposition, based on an example of Lubich and Nevanlinna [9].

Proposition 1.2. *For every $1 \leq p \leq \infty$, there exists a strongly Kreiss bounded operator on $\ell^p(\mathbb{Z})$ and some constant $C_p \geq 1$ such that for every $N \in \mathbb{N}$, $N^{|1/2-1/p|}/C_p \leq \|T^N\| \leq C_p N^{|1/2-1/p|}$.*

Actually, we get a single operator acting simultaneously on all $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$, with the above properties. If S is the right shift, then $T = q(S)$ for some Möbius transformation of the unit disk.

Let us also mention that, see Proposition 4.2, when T is a *positive* strongly Kreiss bounded operator on $L^p(\mu)$, then it is possible to improve (4), when $p \in [1, 4/5]$.

2. AUXILLIARY RESULTS

Through the paper we will denote by \mathbb{T} the complex numbers of modulus one and by λ the Haar measure on \mathbb{T} .

Given a bounded interval $I \subset \mathbb{Z}$, we define an operator M_I by setting

$$M_I f := \sum_{i \in I} c_i(f) \gamma^i \quad \forall f \in L^1(\mathbb{T}),$$

where $f(\gamma) = \sum_{n \in \mathbb{Z}} c_n(f) \gamma^n$.

Recall the definition of the weak L^1 -norm on a measure space

$$\|h\|_{1, \infty, \mu} := \sup_{t > 0} t \mu(|g| \geq t).$$

Proposition 2.1. *Let $1 < p < \infty$. Let $p' = \min(2, p)$. There exists $D_p > 0$ such that for every finite collection $(I_\ell)_{1 \leq \ell \leq L}$ of disjoint intervals of integers,*

$$(5) \quad \left\| \left(\sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \leq D_p L^{1/p'-1/2} \|f\|_{L^p(\mathbb{T})}, \quad \forall f \in L^p(\mathbb{T}).$$

Furthermore, there exists $D_{1, \infty} > 0$ such that for every finite collection $(I_\ell)_{1 \leq \ell \leq L}$ of (not necessarily disjoint) intervals of integers,

$$(6) \quad \left\| \left(\sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^{1, \infty}(\mathbb{T})} \leq D_{1, \infty} L^{1/2} \|f\|_{L^1(\mathbb{T})}, \quad \forall f \in L^1(\mathbb{T}).$$

Proof. For $p \geq 2$, (5) is the so called Littlewood-Paley-Rubio de Francia inequality, see [13]. Actually, Rubio de Francia proved the result on the real line. The result for the torus (with extensions) appears in Kislyakov-Parilov [7].

Inequality (6) appears in the middle of page 6419 of [7], see also Exercise 4.6.1 (a) page 337 of [6] for a version on the real line.

Then (5) for $1 < p < 2$ follows by Marcinkiewicz interpolation, see Theorem 1.3.2 of [6] page 31.

Corollary 2.2. *Let $1 < p < \infty$. Let $p'' = \max(2, p)$. There exists $C_p > 0$ such that for every finite collection $(I_\ell)_{1 \leq \ell \leq L}$ of disjoint and consecutive intervals of integers such with $\cup_{\ell=1}^L I_\ell =: I \subset [-N, N]$, $N \in \mathbb{N}$ and every $f \in L^p(\mathbb{T})$,*

$$(7) \quad \begin{aligned} \|M_I f\|_{L^p(\mathbb{T})} &\leq C_p L^{1/2-1/p''} \left\| \left(\sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \\ &\leq C_p L^{1/2-1/p''} \left(\sum_{\ell=1}^L \|M_{I_\ell} f\|_{L^p(\mathbb{T})}^{p'} \right)^{1/p'}. \end{aligned}$$

Proof. Clearly, it is enough to assume that f is a trigonometric polynomial supported in I . Let $g \in L^q$, with $q = p/(p-1)$. Notice that $q' = \min(p/(p-1), 2) = p''/(p''-1)$ and that $1/q' - 1/2 = 1/2 - 1/p''$. We have, by orthogonality, Cauchy-Schwarz inequality and Hölder inequality,

$$\begin{aligned} \left| \int_{\mathbb{T}} f \bar{g} d\lambda \right| &= \left| \int_{\mathbb{T}} \sum_{\ell=1}^L M_{I_\ell} f M_{-I_\ell} \bar{g} d\lambda \right| \\ &\leq \left\| \left(\sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \left\| \left(\sum_{\ell=1}^L |M_{-I_\ell} \bar{g}|^2 \right)^{1/2} \right\|_{L^q(\mathbb{T})} \\ &\leq C_p L^{1/2-1/p''} \left\| \left(\sum_{\ell=1}^L |M_{I_\ell} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \|\bar{g}\|_{L^q(\mathbb{T})}, \end{aligned}$$

where we used Proposition 2.1. Then (7) follows by taking the supremum over $g \in L^q(\mathbb{T})$, with $\|g\|_{q,\lambda} = 1$. The last estimate follows by using that $x \mapsto x^{p/2}$ is subadditive when $p \leq 2$ and Minkowski's inequality in $L^{p/2}(\mathbb{T})$ when $p \geq 2$. \square

We deduce the following.

Corollary 2.3. *Let $1 < p < \infty$. There exists $C_p > 0$ such that for each $N \in \mathbb{N}$ and each $(x_1, \dots, x_{N^2}) \in (L^p(\mu))^{N^2}$,*

$$(8) \quad \left\| \sum_{k=1}^{N^2} \gamma^k x_k \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq C N^{1/2-1/p''} \left(\sum_{n=0}^{N-1} \left\| \sum_{k=n^2+1}^{(n+1)^2} \gamma^k x_k \right\|_{L^p(\mathbb{T}, L^p(\mu))}^{p'} \right)^{1/p'}$$

Proof. Let $N \in \mathbb{N}$ and $I_n := [n^2 + 1, (n+1)^2)$ for $n \in \{0, \dots, N-1\}$. By Corollary 2.2 for each (a_1, \dots, a_{N^2}) and $f \in L^p(\mathbb{T})$,

$$\left\| \sum_{k=1}^{N^2} a_k \gamma^k \right\|_{L^p(\mathbb{T})} \leq 2C_p N^{1/2-1/p''} \left(\sum_{\ell=0}^{N-1} \left\| \sum_{k=n^2+1}^{(n+1)^2} a_k \gamma^k \right\|_{L^p(\mathbb{T})}^{p'} \right)^{1/p'}$$

If $1 \leq p \leq 2$, (8) follows Fubini's Theorem. If $p > 2$ (8) follows from Fubini's Theorem combined with Minkowski's inequality in $L^{p/2}(\mu)$. \square

We conclude this section by a Lemma we will use several times in this paper.

Lemma 2.4. *There exists $C > 1$ such that for every $N \in \mathbb{N}$ and every integer $K \in [2 - 2\sqrt{N}, 0]$,*

$$(9) \quad \frac{e^N}{C\sqrt{N}} \leq \frac{N^{N+K}}{(N+K)!} \leq \frac{C e^N}{\sqrt{N}}$$

$$(10) \quad \sum_{N+2-2\sqrt{N} \leq n \leq N} \left| \left(\sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^{-1} - \left(\sum_{n+1-\sqrt{N} \leq k \leq n+1} \frac{N^k}{k!} \right)^{-1} \right| \leq C e^{-N}.$$

In particular the sequences $\left(e^N \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^{-1} \Big|_{N+2-2\sqrt{N} \leq n \leq N}$ have bounded variations uniformly bounded with respect to N .

Proof. The upper bound of (9) follows from Lemma 3.4 of [5] and the lower bound may be proved similarly. Then, (10) follows from the fact that, for $N+2-2\sqrt{N} \leq n \leq N$, writing $m := [n+1-\sqrt{N}]$, we have

$$\begin{aligned} e^{-N} \left| \left(\sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^{-1} - \left(\sum_{n+1-\sqrt{N} \leq k \leq n+1} \frac{N^k}{k!} \right)^{-1} \right| &\leq e^{-N} \frac{\frac{N^m}{m!} + \frac{N^n}{n!}}{\left(\sum_{n+1-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right)^2} \\ &\leq \frac{\tilde{C}}{\sqrt{N}}, \end{aligned}$$

where we used (9). □

3. PROOF OF THEOREM 1.1 AND OF PROPOSITION 1.2

The proof of Theorem 1.1 makes use of Fourier multipliers, see pages 11 and 12 of [4] for a brief description of Fourier multipliers in UMD Banach spaces (in particular in L^p spaces). Actually, we only make use of *real valued* Fourier multipliers in our proofs and then use Fubini to obtain results for L^p -valued Fourier multipliers.

We shall use in the sequel the terminology of Riesz theorem and Stechkin theorem, already used in [4]. More precisely, let X be an UMD space and $1 < p < \infty$. We refer as *Riesz theorem* the following: it exists $C_p > 0$ such that for each interval $I \subset \mathbb{Z}$ we have

$$\forall (c_i)_{i \in \mathbb{Z}} \subset X, \quad \left\| \sum_{i \in I} \gamma^i c_i \right\|_{L^p(\mathbb{T}, X)} \leq C_p \left\| \sum_{i \in \mathbb{Z}} \gamma^i c_i \right\|_{L^p(\mathbb{T}, X)}.$$

And we refer as *Stechkin theorem* the following: if $(a_n)_{n \in \mathbb{Z}}$ is a bounded monotone sequence of real numbers, then it exists D_p such that,

$$\forall (c_n)_{n \in \mathbb{Z}} \subset X, \quad \left\| \sum_{n \in \mathbb{Z}} a_n \gamma^n c_n \right\|_{L^p(\mathbb{T}, X)} \leq D_p \left\| \sum_{i \in \mathbb{Z}} \gamma^n c_n \right\|_{L^p(\mathbb{T}, X)}.$$

Lemma 3.1. *Let T be a bounded operator on $L^p(\mu)$, $1 < p < \infty$. Assume that there exists $D > 0$ and $\alpha > 0$, such that for every $n \in \mathbb{N}$ and every $x \in L^p(\mu)$,*

$$\left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DN^\alpha \|x\|_{L^p(\mu)}.$$

Then, there exists $E_p > 0$ independent of D , α and β such that, setting $\delta_p := \frac{1}{2} \left(\frac{2}{p'} - \frac{1}{p} \right)$, for every $N \in \mathbb{N}$ and every $x \in L^p(\mu)$, we have

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DE_p N^{\alpha + \delta_p}.$$

Proof. By Corollary 2.3 and the assumption, for every $M \in \mathbb{N}$ and every $x \in L^p(\mu)$,

$$\begin{aligned} \left\| \sum_{1 \leq n \leq M^2} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))}^{p'} &\leq (CM^{1/2-1/p''})^{p'} \left(\sum_{n=0}^{M-1} \left\| \sum_{k=n^2+1}^{(n+1)^2} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))}^{p'} \right) \\ &\leq 2^{p'\beta} (CD)^{p'} M^{p'(1/2-1/p'')/2} \sum_{n=0}^{M-1} (n+1)^{2p'\alpha} \\ &\leq 2^{p'\beta} (CD)^{p'} M^{p'(1/2-1/p'')+2p'\alpha+1}. \end{aligned}$$

Let $N \in \mathbb{N}$ and $M := \lceil \sqrt{N} \rceil + 1$. By the Riesz theorem, using that $M^2 \leq 4N$ and $M \leq N$, we infer that

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq \left\| \sum_{1 \leq n \leq M^2} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq 2^\beta CD (4N)^{(1/2-1/p'')/2+\alpha+1/2p'},$$

and the result follows. \square

Lemma 3.2. *Let T be a strongly Kreiss bounded operator on $L^p(\mu)$, $1 < p < \infty$. Assume that there exists a function $f : \mathbb{R}_+ \rightarrow (1, \infty)$, $D > 0$ such that for every $n \in \mathbb{N}$ and every $x \in L^p(\mu)$,*

$$(11) \quad \left\| \sum_{n=1}^N \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq Df(N) \|x\|_{L^p(\mu)}.$$

Then, there exists $E_p > 0$ independent of D and f such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mu)$,

$$\left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DE_p f(4\sqrt{N}) \|x\|_{L^p(\mu)}.$$

Proof. The proof is similar to the one of Lemma 4.7 of [5].

Let $x \in L^p(\mu)$ and $M_{N,\gamma} = \sum_{n=1}^{4\sqrt{N}} \gamma^n T^n$. Firstly since T is strongly Kreiss bounded, using assumption (11), we have

$$(12) \quad \left\| e^{\gamma NT} M_{N,\gamma} x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq Ce^N \|M_{N,\gamma} x\|_{L^p(\mathbb{T}, L^p(\mu))} \leq CDe^N f(4\sqrt{N}).$$

Furthermore, we have

$$e^{\gamma NT} M_{N,\gamma} x = \sum_{1 \leq n \leq 4\sqrt{N}} \gamma^n T^n x \sum_{0 \leq k \leq n} \frac{N^k}{k!} + \sum_{n \geq 4\sqrt{N}} \gamma^n T^n x \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!}.$$

For every N large enough (such that $N + 2 - 2\sqrt{N} \geq 4\sqrt{N}$), using the Riesz theorem, we have

$$\left\| e^{\gamma NT} M_{N,\gamma} x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \geq C_p \left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right\|_{L^p(\mathbb{T}, L^p(\mu))}.$$

By Lemma 2.4 (in particular the property of bounded variation) and the Steckin theorem,

$$(13) \quad \left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \sum_{n-\sqrt{N} \leq k \leq n} \frac{N^k}{k!} \right\|_{L^p(\mathbb{T}, L^p(\mu))} \geq De^N \left\| \sum_{N+2-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))}.$$

Combining (12) and (13), we get the desired result. \square

Combining Lemma 3.1 and 3.2, we easily derive the following.

Corollary 3.3. *Let T be a strongly Kreiss bounded operator on $L^p(\mu)$, $1 < p < \infty$. Assume that there exists $D > 0$ and $\alpha > 0$, $\beta \geq 0$, such that for every $n \in \mathbb{N}$ and every $x \in L^p(\mu)$,*

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DN^\alpha \|x\|_{L^p(\mu)}.$$

Then, there exists E_p , independent of α and β such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mu)$,

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq DE_p N^{\alpha/2 + \delta_p} \|x\|_{L^p(\mu)}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since T is strongly Kreiss bounded, it is known (see pages 1-2 of [5]) that there exists $C > 0$ such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mu)$,

$$\left\| \sum_{n=1}^N \gamma^n T^n x \right\|_{L^p(\mu)} \leq CN \|x\|_{L^p(\mu)} \quad \forall \gamma \in \mathbb{T}.$$

Applying inductively Corollary 3.3 we see that for every $N \in \mathbb{N}$, $K \in \mathbb{N}_0 = \{0, 1, \dots\}$ and every $x \in L^p(\mu)$,

$$\left\| \sum_{1 \leq n \leq N} \gamma^n T^n \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq CE_p^K N^{2\delta_p + (\alpha - \delta_p)2^{-K}}.$$

Without loss of generality, we may and do assume that $N \geq 3$. Let $K \geq 0$ be the integer such that $2^K \leq \frac{\log N}{\log(\log N)} \leq 2^{K+1}$. Then we have $K \leq \log(\log N)/\log 2$ and $E_p^K \leq \exp(\log E_p \log(\log N)/\log 2) = \log(N)^{\log E_p / \log 2}$. Moreover, there exists $D > 0$ such that $N^{(\alpha - \delta_p)2^{-K}} \leq e^{D \log \log N} = \log(N)^D$.

Combining those estimates, we infer that, there exists $C > 0$ and $\kappa > 0$ such that for every $N \in \mathbb{N}$,

$$(14) \quad \left\| \sum_{1 \leq n \leq N} \gamma^n T^n \right\|_{L^p(\mathbb{T}, L^p(\mu))} \leq CN^{2\delta_p} (\log(N+1))^\kappa \|x\|_{L^p(\mu)}.$$

Using that T^* is strongly Kreiss bounded on $L^q(\mu)$, $q = p/(p-1)$, we obtain a similar estimate for T^* .

Hence, applying once more Lemmas 3.1 and 3.2, we see that there exists $C, \kappa > 0$ (large that the previous ones) such that for every $x \in L^p(\mu)$ and $x^* \in L^q(\mu)$,

$$\begin{aligned} (1 + 2[\sqrt{N}]) |\langle x^*, T^{N+1} x \rangle| &= \left| \int_{\mathbb{T}} \left\langle \sum_{1 \leq n \leq 1+2\sqrt{N}} \gamma^n T^{*n}, \sum_{1 \leq m \leq 1+2\sqrt{N}} \bar{\gamma}^m T^{N+1-m} x \right\rangle d\lambda \right| \\ &\leq \left\| \sum_{1 \leq n \leq 1+2\sqrt{N}} \gamma^n T^{*n} x^* \right\|_{L^q(\mathbb{T}, L^q(\mu))} \left\| \sum_{N-2\sqrt{N} \leq n \leq N} \gamma^n T^n x \right\|_{L^p(\mathbb{T}, L^p(\mu))} \\ &\leq CN^{\delta_p + \delta_q} (\log(N+1))^\kappa \\ &= CN^{1/\min(p,q)} (\log(N+1))^\kappa, \end{aligned}$$

and the result follows by taking the supremum over x^* with $\|x^*\|_{L^q(\mu)} = 1$. \square

Let us now prove Proposition 1.2. Let $0 < a < 1$ and define $q_a(z) := \frac{z-a}{1-az}$. Let S be the right shift, acting on $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$. Set $T := q_a(S)$.

It has been proved in [9] that T is strongly Kreiss bounded on $\ell^\infty(\mathbb{Z})$. Their proof works equally to prove that T is strongly Kreiss bounded on $\ell^p(\mathbb{Z})$ for every $1 \leq p \leq \infty$.

Moreover, as noticed in [9], it follows from the book by Brenner, Thomée and Wahlbin [3] that

$$N^{|1/2-1/p|}/C_p \leq \|T^N\| \leq C_p N^{|1/2-1/p|},$$

for some $C_p > 0$.

In [9], they refer to Theorem 3.1 page 102 of [3], while the theorem is actually concerned with operators acting on $L^p(\mathbb{R}, \text{Leb})$. Now, using techniques from [3], pages 19-29 (in particular the bounds (4.3) page 21), it is possible to adapt the proof of Theorem 3.1 page 102 to our setting.

4. SOME PARTICULAR STRONGLY KREISS BOUNDED OPERATORS

We consider here the case of *positive* strongly Kreiss bounded operators on $L^p(\mu)$ (we do not require anymore μ to be σ -finite) or *absolutely* strongly Kreiss bounded operators (on any Banach space). The basic idea used to obtain (4) (in L^p -spaces) has been already used by the first author [1] where Kreiss bounded operators were considered.

Let begin by a general result.

Proposition 4.1. *Let $1 \leq p \leq 2$ and T be bounded operator on X satisfying that there exists $C > 0$ and $\alpha > 0$ such that for each $N \in \mathbb{N}$,*

$$(15) \quad \|T^N\| \leq CN^\alpha$$

and for every $x \in X$

$$(16) \quad \sum_{N-2\sqrt{N} \leq n \leq N} \|T^n x\| \leq CN^{p/2} \|x\|.$$

. Then, $\|T^N\| = \mathcal{O}(N^{1/q}(\log N)^\kappa)$, for some $\kappa > 0$, where $q = p/(p-1)$.

Proof. We start with the following observation. For every $x \in X$ and $x^* \in X^*$,

$$(17) \quad (1 + 2\sqrt{N})|\langle x^*, T^{N+1}x \rangle|^p = \sum_{1 \leq n \leq 1+2\sqrt{N}} |\langle T^{*n}x^*, T^{N+1-n}x \rangle|_{X^*, X}^p \\ \leq \|x^*\|^p \max_{1 \leq n \leq 1+2\sqrt{N}} \|T^{*n}\|^p \sum_{N-2\sqrt{N} \leq n \leq N} \|T^n x\|^p.$$

Taking the supremum over x, x^* of norm 1, using that $\|T^{*n}\| = \|T^n\|$ and assumption (16), we infer that, for every $N \in \mathbb{N}$

$$\|T^N\| \leq N^{-1/2p} \max_{1 \leq n \leq 3\sqrt{N}} \|T^n\| \sup_{\|x\| \leq 1} \left(\sum_{N-2\sqrt{N} \leq n \leq N} \|T^n x\|^p \right)^{1/p} \\ \leq CN^{1/2q} \max_{1 \leq n \leq 3\sqrt{N}} \|T^n\|.$$

Combining above estimate with (15), we get that

$$\|T^N\| \leq C^2 3^\alpha DN^{\alpha/2+1/2q}.$$

Iterating the above we get that for every integers $N, K \in \mathbb{N}$,

$$\|T^N\| \leq 3^{K\alpha} C^{K+1} N^{2^{-K}\alpha+(1-2^{-K})/q},$$

and we conclude as in the proof of Theorem 1.1. \square

From Proposition 4.1 we deduce a better bound than in Theorem 1.1 for a strongly Kreiss bounded positive operator on $L^p(\mu)$ when $p \in [1, 4/3) \cup (4, +\infty)$ (the cases where $p = 1$ and $p = \infty$ are discussed below).

Proposition 4.2. *Let T be a positive operator that is strongly Kreiss bounded on $L^p(\mu)$, $1 \leq p < \infty$. Then, $\|T^N\| = \mathcal{O}(N^{1/\bar{p}}(\log N)^\kappa)$, for some $\kappa > 0$, where $\bar{p} = \max(p, p/(p-1))$.*

The proof is straightforward using following Lemma and the fact that every strongly Kreiss bounded operator satisfies (15) for $\alpha = 1/2$.

Lemma 4.3. *Let $1 \leq p < \infty$. Any positive strong Kreiss bounded operator T on $L^p(\mu)$ satisfies (16) for every $x \in L^p(\mu)$.*

Proof. Using the fact that $\|\cdot\|_{\ell^p} \leq \|\cdot\|_{\ell^1}$ and Lemma 2.4, we see that there exists $C > 0$ such that for every $N \in \mathbb{N}$ and every $x \in L^p(\mu)$ with $x \geq 0$,

$$(18) \quad \left(\sum_{n \geq 0} \frac{N^n T^n x}{n!} \right)^p \geq \sum_{n \geq 0} \left(\frac{N^n T^n x}{n!} \right)^p \geq \frac{C^p e^{pN}}{\sqrt{N^p}} \sum_{N-2\sqrt{N} \leq n \leq N} (T^n x)^p.$$

Integrating with respect to μ and using Strong Kreiss boundedness, we infer T satisfies (16) for every $x \in L^p(\mu)$ with $x \geq 0$. By linearity (16) it remains true for every $x \in L^p(\mu)$. \square

We turn now to the special case of absolutely strongly Kreiss bounded operator. Let T a bounded operator on X . We say that T is absolutely strongly Kreiss bounded if there exists $C > 0$ such that

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \|T^n x\| \leq C e^r \|x\|, \quad r > 0, x \in X.$$

Such operator is clearly strongly Kreiss bounded and using Lemma 2.4, for every $x \in X$, it satisfies (16) with $p = 1$. Then we can apply Proposition 4.1 to state that $\|T^N\|$ has logarithmic bound.

Proposition 4.4. *Let T be an absolutely strongly Kreiss bounded operator on X . Then, there exists $\kappa > 0$ such that*

$$(19) \quad \|T^N\| = \mathcal{O}(\log(N)^\kappa).$$

Remark 4.5. When $X = L^p(\mu)$, $1 \leq p < \infty$ the bound (19) is sharp. Indeed according [5, remark 1 page 16], for any $\kappa > 0$ there exists an absolutely strongly Kreiss bounded T on $L^p(\mu)$ such that $\|T^N\| \asymp (\log(N)^\kappa)$.

We now discuss the case of positive strong Kreiss operator on (AL) and (AM) -spaces. We refer to [11], Section 2, for more details. A Banach lattice X is an (AL) -space if the norm is additive on the positive cone on X , that is

$$(20) \quad \forall x, y \in X_+, \|x + y\| = \|x\| + \|y\|,$$

that is, the norm is additive on the positive cone of X . A Banach lattice X is an (AM) -space if the norm on X satisfies

$$\forall x, y \in X_+, \|\sup(x, y)\| = \sup(\|x\|, \|y\|).$$

If X is an (AM) -space, then X^* is an (AL) -space. It is known that an (AL) -space is isometrically isomorphic to some space L^1 and that an (AM) -space with is isometrically isomorphic respectively to some $C(K)$ where K is a compact space. We are now ready for giving a proof to the following statement.

Proposition 4.6. *Let T be a positive strongly Kreiss bounded operator on an (AL) -space or an (AM) -space. Then it exists $\kappa > 0$ such that*

$$(21) \quad \|T^N\| = \mathcal{O}(\log(N)^\kappa).$$

Remark. By the proposition, since $L^\infty(\mu)$ is an (AM) -space, we see that Proposition 4.2 remains true for $p = \infty$.

Proof. If T is positive strong Kreiss bounded operator on an (AL)-space, then using (20) it is straightforward that T is absolute strong Kreiss bounded and then we can conclude with Proposition 4.1.

If T is a positive strongly Kreiss bounded operator on an (AM)-space, then T^* is a positive strongly Kreiss bounded on an (AL)-space, and then we conclude by above. \square

Remark 4.7. In view of our last section, we can ask whether the bound of Proposition 4.2 can be improved to obtain a logarithmic bound. More precisely for a positive strongly Kreiss bounded operator T on $L^p(\mu)$ with $1 < p < \infty$, is there exist $\kappa > 0$ such that $\|T^N\| = \mathcal{O}(\log(N)^\kappa)$?

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