



**HAL**  
open science

# Developments for MCRB Computation in Multipath Scenarios

Corentin Lubeigt, Lorenzo Ortega, Jordi Vilà-Valls, Eric Chaumette

► **To cite this version:**

Corentin Lubeigt, Lorenzo Ortega, Jordi Vilà-Valls, Eric Chaumette. Developments for MCRB Computation in Multipath Scenarios. 2022. hal-04006396

**HAL Id: hal-04006396**

**<https://hal.science/hal-04006396>**

Preprint submitted on 27 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Technical Note

## Developments for MCRB Computation in Multipath Scenarios

Corentin Lubeigt, Lorenzo Ortega, Jordi Vilà-Valls and Eric Chaumette

September 15, 2022

### 1 Derivation of Convolution Terms Using the Fourier Transform Properties

Let  $s(t)$  and  $r(t)$  be band-limited signals of bandwidth smaller than  $B$  sampled at frequency  $F_s$  over  $N = N_2 - N_1$  samples. The aim of this note is to provide details on the evaluation of the following integral terms:

$$w_1 = \int_{\mathbb{R}} s(t - \tau)r(t - \tau_k)^* e^{-j2\pi f_c(b(t-\tau)-b_k(t-\tau_k))} dt, \quad (1)$$

$$w_2 = \int_{\mathbb{R}} (t - \tau)s(t - \tau)r(t - \tau_k)^* e^{-j2\pi f_c(b(t-\tau)-b_k(t-\tau_k))} dt, \quad (2)$$

$$w_3 = \int_{\mathbb{R}} (t - \tau)^2 s(t - \tau)r(t - \tau_k)^* e^{-j2\pi f_c(b(t-\tau)-b_k(t-\tau_k))} dt, \quad (3)$$

$$w_4 = \int_{\mathbb{R}} s^{(1)}(t - \tau)r(t - \tau_k)^* e^{-j2\pi f_c(b(t-\tau)-b_k(t-\tau_k))} dt, \quad (4)$$

$$w_5 = \int_{\mathbb{R}} (t - \tau)s^{(1)}(t - \tau)r(t - \tau_k)^* e^{-j2\pi f_c(b(t-\tau)-b_k(t-\tau_k))} dt, \quad (5)$$

$$w_6 = \int_{\mathbb{R}} s^{(2)}(t - \tau)r(t - \tau_k)^* e^{-j2\pi f_c(b(t-\tau)-b_k(t-\tau_k))} dt, \quad (6)$$

where  $\tau$  is a time delay,  $(1 - b)$  is a dilatation term due to Doppler effect,  $f_c$  is carrier frequency and the superscript  $^{(i)}$  refers to the  $i$ -th time derivative of signal  $s(t)$ .

#### 1.1 Prior Considerations

First the Fourier transform of a set of functions are to be evaluated. Remembering that the signal is band-limited of band  $B \leq F_s$ , one has:

$$s(t) \Leftrightarrow \text{FT} \{s(t)\} (f) \triangleq S(f) = \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n T_s} \right) 1_{[-\frac{F_s}{2}, \frac{F_s}{2}]} \quad (7)$$

In order to tackle the issue that may come from the spectral shift due to Doppler effect. One simply needs to take  $F_s$  large enough so that  $\frac{F_s}{2} \geq \frac{B}{2} + f_c \max\{|b|, |b_k|, |b - b_k|\}$ .

A first expression is a simple application of the frequency shift relation when using the Fourier transform of a signal multiplied by a complex time-varying exponential.

$$s(t)e^{j2\pi f_c b t} \Leftrightarrow \text{FT} \{s(t)e^{j2\pi f_c b t}\} (f) \triangleq S(f - f_c b) \quad (8)$$

Then, let  $s_1$  be defined by  $s_1(t; b) = s(t)e^{j2\pi f_c b t}$ , it is known that

$$t s_1(t; b) \Leftrightarrow \frac{j}{2\pi} \frac{d}{df} \underbrace{(\text{FT} \{s_1(t; b)\} (f))}_{(8)} \quad (9)$$

therefore

$$t s(t)e^{j2\pi f_c b t} \Leftrightarrow \frac{j}{2\pi} \frac{d}{df} (S(f - f_c b)) \quad (10)$$

Similarly

$$t^2 s(t)e^{j2\pi f_c b t} \Leftrightarrow \left(\frac{j}{2\pi}\right)^2 \frac{d^2}{df^2} (S(f - f_c b)) \quad (11)$$

Besides, with the superscript <sup>(1)</sup> referring to the first time derivative,

$$\begin{aligned} s_1^{(1)}(t; b) &\triangleq \frac{d}{dt}(s_1(t; b)) = s^{(1)}(t)e^{j2\pi f_c b t} + (j2\pi f_c b)s_1(t; b) \\ \Leftrightarrow s^{(1)}(t)e^{j2\pi f_c b t} &= s_1^{(1)}(t; b) - (j2\pi f_c b)s_1(t; b) \end{aligned}$$

Then, knowing the Fourier transform of the  $k$ -th time derivative of a function

$$\text{FT} \{s^{(k)}(t)\} (f) \triangleq (j2\pi f)^k S(f) \quad (12)$$

one directly gets

$$s^{(1)}(t)e^{j2\pi f_c b t} \Leftrightarrow j2\pi (f - f_c b) S(f - f_c b) \quad (13)$$

Now, if  $s_2$  is defined as  $s_2(t; b) = t s(t)e^{j2\pi f_c b t}$ ,

$$\begin{aligned} s_2^{(1)}(t; b) &= s_1(t; b) + t s^{(1)}(t)e^{j2\pi f_c b t} + (j2\pi f_c b)s_2(t; b) \\ \Leftrightarrow t s^{(1)}(t)e^{j2\pi f_c b t} &= \underbrace{-s_1(t; b)}_{(8)} + \underbrace{s_2^{(1)}(t; b)}_{(12)} - (j2\pi f_c b) \underbrace{s_2(t; b)}_{(10)} \end{aligned}$$

therefore,

$$t s^{(1)}(t)e^{j2\pi f_c b t} \Leftrightarrow -S(f - f_c b) - (f - f_c b) \frac{d}{df} (S(f - f_c b)) \quad (14)$$

Finally, by taking again  $s_1$  as  $s_1(t; b) = s(t)e^{j2\pi f_c b t}$ ,

$$s_1^{(2)}(t; b) = s^{(2)}(t)e^{j2\pi f_c b t} + 2(j2\pi f_c b)s^{(1)}(t)e^{j2\pi f_c b t} + (j2\pi f_c b)^2 s_1(t; b) \quad (15)$$

$$\Leftrightarrow s^{(2)}(t)e^{j2\pi f_c b t} = \underbrace{s_1^{(2)}(t; b)}_{(12)} - \underbrace{(j4\pi f_c b) s^{(1)}(t)e^{j2\pi f_c b t}}_{(13)} + 4\pi^2 (f_c b)^2 \underbrace{s_1(t; b)}_{(8)} \quad (16)$$

one obtains,

$$s^{(2)}(t)e^{j2\pi f_c b t} \Leftrightarrow (j2\pi f)^2 S(f - f_c b) + 8\pi^2 f_c b (f - f_c b) S(f - f_c b) + 4\pi^2 (f_c b)^2 S(f - f_c b) \quad (17)$$

$$\Leftrightarrow -4\pi^2 (f - f_c b)^2 S(f - f_c b) \quad (18)$$

## 1.2 Evaluation of the Integrals

### 1.2.1 Integral $w_1$

$$\begin{aligned} w_1 &= \int_{\mathbb{R}} s(t - \tau) r(t - \tau_k)^* e^{-j\omega_c(b(t-\tau) - b_k(t-\tau_k))} dt \\ &= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} s(u) r(u - \Delta\tau_k)^* e^{j\omega_c \Delta b_k u} du \end{aligned}$$

with  $\Delta\tau_k \triangleq \tau_k - \tau$  and  $\Delta b_k \triangleq b_k - b$ . Then, using the Fourier transform properties over the hermitian product,

$$\begin{aligned} w_1 e^{j\omega_c \Delta b_k u} &= \int_{\mathbb{R}} \underbrace{s(u) e^{j\omega_c \Delta b_k u}}_{(8)} (r(u - \Delta\tau_k))^* du \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} S(f - f_c \Delta b_k) (R(f) e^{-j2\pi f \Delta\tau_k})^* df \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f - f_c \Delta b_k)nT_s} \right) e^{j2\pi f \Delta\tau_k} \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* df \\ &= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* df \\ &= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \mathbf{s}^T \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) df \\ &= \frac{1}{F_s} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} df \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\ &= \frac{1}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \end{aligned}$$

Hence

$$\boxed{w_1 = \frac{1}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} e^{-j\omega_c b_k \Delta\tau_k}} \quad (19)$$

with

$$\boldsymbol{\nu}(f) = \left( \dots e^{j2\pi f n} \dots \right)_{N_1 \leq n \leq N_2}^T \quad (20)$$

$$\mathbf{U}(p) = \text{diag} \left( \dots e^{-j2\pi p n} \dots \right)_{N_1 \leq n \leq N_2} \quad (21)$$

$$\mathbf{V}^{\Delta,0}(q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} df \quad (22)$$

and

$$\begin{aligned}
[\mathbf{V}^{\Delta,0}(q)]_{k,l} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k-l-q)} \mathbf{d}f \\
&= \left[ \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{\sin(\pi(k-l-q))}{\pi(k-l-q)} = \text{sinc}(k-l-q)
\end{aligned} \tag{23}$$

### 1.2.2 Integral $w_2$

$$\begin{aligned}
w_2 &= \int_{\mathbb{R}} (t-\tau)s(t-\tau)r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))} \mathbf{d}t \\
&= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} us(u)r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u} \mathbf{d}u
\end{aligned}$$

Therefore,

$$\begin{aligned}
w_2 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{us(u)e^{j\omega_c \Delta b_k u}}_{(10)} (r(u-\Delta\tau_k))^* \mathbf{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( \frac{j}{2\pi} \frac{\mathbf{d}}{\mathbf{d}f} (S(f-f_c \Delta b_k)) \right) (R(f)e^{-j2\pi f \Delta\tau_k})^* \mathbf{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( \frac{1}{F_s} \frac{j}{2\pi} (-j2\pi T_s) \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi(f-f_c \Delta b_k)nT_s} \right) e^{j2\pi f \Delta\tau_k} \\
&\quad \times \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* \mathbf{d}f \\
&= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* \mathbf{d}f \\
&= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \mathbf{s}^T \mathbf{D} \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) \mathbf{d}f \\
&= \frac{1}{F_s^2} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathbf{d}f \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} \\
&= \frac{1}{F_s^2} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s}
\end{aligned}$$

Hence

$$\boxed{w_2 = \frac{1}{F_s^2} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} e^{-j\omega_c b_k \Delta\tau_k}} \tag{24}$$

with  $\mathbf{U}$  and  $\mathbf{V}^{\Delta,0}$ , defined in (21) and (22), respectively, and with

$$\mathbf{D} = \text{diag} \left( \dots \quad n \quad \dots \right)_{N_1 \leq n \leq N_2} \tag{25}$$

### 1.2.3 Integral $w_3$

$$\begin{aligned} w_3 &= \int_{\mathbb{R}} (t - \tau)^2 s(t - \tau) r(t - \tau_k)^* e^{-j\omega_c(b(t-\tau) - b_k(t-\tau_k))} dt \\ &= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} u^2 s(u) r(u - \Delta\tau_k)^* e^{j\omega_c \Delta b_k u} du \end{aligned}$$

Therefore,

$$\begin{aligned} w_3 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{u^2 s(u) e^{j\omega_c \Delta b_k u}}_{(11)} (r(u - \Delta\tau_k))^* du \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( \left( \frac{j}{2\pi} \right)^2 \frac{d^2}{df^2} (S(f - f_c \Delta b_k)) \right) (R(f) e^{-j2\pi f \Delta\tau_k})^* df \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( \frac{1}{F_s} \left( \frac{j}{2\pi} \right)^2 (-j2\pi T_s)^2 \sum_{n=N_1}^{N_2} s(nT_s) n^2 e^{-j2\pi(f - f_c \Delta b_k) n T_s} \right) e^{j2\pi f \Delta\tau_k} \\ &\quad \times \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* df \\ &= \frac{1}{F_s^3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{n=N_1}^{N_2} s(nT_s) n^2 e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k n}{F_s}} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* df \\ &= \frac{1}{F_s^3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \mathbf{s}^T \mathbf{D}^2 \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) df \\ &= \frac{1}{F_s^3} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} df \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D}^2 \mathbf{s} \\ &= \frac{1}{F_s^3} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D}^2 \mathbf{s} \end{aligned}$$

Hence

$$\boxed{w_3 = \frac{1}{F_s^3} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D}^2 \mathbf{s} e^{-j\omega_c b_k \Delta\tau_k}} \quad (26)$$

with  $\mathbf{U}$ ,  $\mathbf{V}^{\Delta,0}$  and  $\mathbf{D}$  defined in (21), (22) and (25) respectively.

### 1.2.4 Integral $w_4$

$$\begin{aligned} w_4 &= \int_{\mathbb{R}} s^{(1)}(t - \tau) r(t - \tau_k)^* e^{-j\omega_c(b(t-\tau) - b_k(t-\tau_k))} dt \\ &= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} s^{(1)}(u) r(u - \Delta\tau_k)^* e^{j\omega_c \Delta b_k u} du \end{aligned}$$

Therefore,

$$\begin{aligned}
w_4 e^{j\omega_c b_k \Delta \tau_k} &= \int_{\mathbb{R}} \underbrace{s^{(1)}(u) e^{j\omega_c \Delta b_k u}}_{(13)} (r(u - \Delta \tau_k))^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} (j2\pi(f - f_c \Delta b_k) S(f - f_c \Delta b_k)) (R(f) e^{-j2\pi f \Delta \tau_k})^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( j2\pi(f - f_c \Delta b_k) \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f - f_c \Delta b_k)nT_s} \right) e^{j2\pi f \Delta \tau_k} \\
&\quad \times \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( j2\pi(f F_s - f_c \Delta b_k) \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \\
&\quad \times \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( j2\pi(f F_s - f_c \Delta b_k) \mathbf{s}^T \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta \tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) \mathrm{d}f \\
&= \mathbf{r}^H \left( j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&\quad - \frac{j2\pi f_c \Delta b_k}{F_s} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&= \mathbf{r}^H \mathbf{V}^{\Delta,1} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} - \frac{j2\pi f_c \Delta b_k}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s}
\end{aligned}$$

Hence

$$\boxed{
\begin{aligned}
w_4 &= \left( \mathbf{r}^H \mathbf{V}^{\Delta,1} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} - \frac{j\omega_c \Delta b_k}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta \tau_k}{T_s} \right) \right) \\
&\quad \times \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} e^{-j\omega_c b_k \Delta \tau_k}
\end{aligned}
} \tag{27}$$

with  $\mathbf{U}$  and  $\mathbf{V}^{\Delta,0}$  defined in (21) and (22) and

$$\mathbf{V}^{\Delta,1}(q) = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} \mathrm{d}f \tag{28}$$

and

$$\begin{aligned}
[\mathbf{V}^{\Delta,1}(q)]_{k,l} &= j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi f(k-l-q)} \mathbf{d}f \\
&= j2\pi \left( \left[ \frac{f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathbf{d}f \right) \\
&= \frac{j2\pi}{j2\pi(k-l-q)} \left( \left[ \frac{1}{2} e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right) e^{-j\pi(k-l-q)} \right] - \left[ \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \right) \\
&= \frac{1}{k-l-q} (\cos(\pi(k-l-q)) - \text{sinc}(k-l-q)) \tag{29}
\end{aligned}$$

### 1.2.5 Integral $w_5$

$$\begin{aligned}
w_5 &= \int_{\mathbb{R}} (t-\tau) s^{(1)}(t-\tau) r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))} \mathbf{d}t \\
&= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} u s^{(1)}(u) r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u} \mathbf{d}u
\end{aligned}$$



Therefore,

$$\begin{aligned}
w_5 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{u s^{(1)}(u) e^{j\omega_c \Delta b_k u}}_{(14)} (r(u - \Delta\tau_k))^* du \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( -S(f - f_c \Delta b_k) - (f - f_c \Delta b_k) \frac{d}{df} (S(f - f_c \Delta b_k)) \right) (R(f) e^{-j2\pi f \Delta\tau_k})^* df \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( - \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f - f_c \Delta b_k) n T_s} \right) \right. \\
&\quad \left. - (f - f_c \Delta b_k) \left( \frac{1}{F_s} (-j2\pi T_s) \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi(f - f_c \Delta b_k) n T_s} \right) \right) \\
&\quad \times e^{j2\pi f \Delta\tau_k} \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* df \\
&= -\frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* df \\
&\quad + \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f \left( \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* df \\
&\quad - j2\pi \frac{f_c \Delta b_k}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left( \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* df \\
&= -\frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \mathbf{s}^T \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) df \\
&\quad + \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f \left( \mathbf{s}^T \mathbf{D} \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) df \\
&\quad - j2\pi \frac{f_c \Delta b_k}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \mathbf{s}^T \mathbf{D} \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) df \\
&= -\frac{1}{F_s} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta\tau_k}{T_s}} df \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&\quad + \frac{1}{F_s} \mathbf{r}^H \left( j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta\tau_k}{T_s}} df \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} \\
&\quad - j2\pi \frac{f_c \Delta b_k}{F_s^2} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta\tau_k}{T_s}} df \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} \\
&= -\frac{1}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} + \frac{1}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,1} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} \\
&\quad - j2\pi \frac{f_c \Delta b_k}{F_s^2} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s}
\end{aligned}$$

Hence

$$w_5 = \left( -\frac{1}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} + \frac{1}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,1} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} \right. \\ \left. - j \frac{\omega_c \Delta b_k}{F_s^2} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta\tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{D} \mathbf{s} \right) e^{-j\omega_c b_k \Delta\tau_k} \quad (30)$$

with  $\mathbf{U}$ ,  $\mathbf{V}^{\Delta,0}$ ,  $\mathbf{V}^{\Delta,1}$  and  $\mathbf{D}$  defined in (21), (22), (28) and (25).

### 1.2.6 Integral $w_6$

$$w_6 = \int_{\mathbb{R}} s^{(2)}(t - \tau) r(t - \tau_k)^* e^{-j\omega_c (b(t-\tau) - b_k(t-\tau_k))} dt \\ = e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} s^{(2)}(u) r(u - \Delta\tau_k)^* e^{j\omega_c \Delta b_k u} du$$

$$\begin{aligned}
w_6 e^{j\omega_c b_k \Delta \tau_k} &= \int_{\mathbb{R}} \underbrace{s^{(2)}(u) e^{j\omega_c \Delta b_k u} (r(u - \Delta \tau_k))^*}_{(18)} \mathbf{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} (-4\pi^2(f - f_c \Delta b_k)^2 S(f - f_c \Delta b_k)) (R(f) e^{-j2\pi f \Delta \tau_k})^* \mathbf{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left( (-4\pi^2 f^2 + 8\pi^2 f f_c \Delta b_k - 4\pi^2 (f_c \Delta b_k)^2) \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f - f_c \Delta b_k)nT_s} \right) \right) \\
&\quad \times e^{j2\pi f \Delta \tau_k} \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* \mathbf{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( (-4\pi^2 (f F_s)^2 + 8\pi^2 (f F_s) f_c \Delta b_k - 4\pi^2 (f_c \Delta b_k)^2) \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k n}{F_s}} \right) \right) \\
&\quad \times e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \left( \frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* \mathbf{d}f F_s \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \left( -F_s (4\pi^2 f^2) - j4\pi f_c \Delta b_k (j2\pi f) - 4\pi^2 \frac{(f_c \Delta b_k)^2}{F_s} \right) \left( \mathbf{s}^T \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \boldsymbol{\nu}(f)^* \right) \right) \\
&\quad \times e^{j2\pi f \frac{\Delta \tau_k}{T_s}} (\mathbf{r}^H \boldsymbol{\nu}(f)) \mathbf{d}f \\
&= -F_s \mathbf{r}^H \left( 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \mathbf{d}f \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&\quad - j4\pi f_c \Delta b_k \mathbf{r}^H \left( j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \mathbf{d}f \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&\quad - 4\pi^2 \frac{(f_c \Delta b_k)^2}{F_s} \mathbf{r}^H \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta \tau_k}{T_s}} \mathbf{d}f \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&= -F_s \mathbf{r}^H \mathbf{V}^{\Delta,2} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} - j4\pi f_c \Delta b_k \mathbf{r}^H \mathbf{V}^{\Delta,1} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \\
&\quad - 4\pi^2 \frac{(f_c \Delta b_k)^2}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s}
\end{aligned}$$

Hence

$$\boxed{
\begin{aligned}
w_6 &= \left( -F_s \mathbf{r}^H \mathbf{V}^{\Delta,2} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} - j2\omega_c \Delta b_k \mathbf{r}^H \mathbf{V}^{\Delta,1} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \right. \\
&\quad \left. - \frac{(\omega_c \Delta b_k)^2}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0} \left( -\frac{\Delta \tau_k}{T_s} \right) \mathbf{U} \left( -\frac{f_c \Delta b_k}{F_s} \right) \mathbf{s} \right) e^{-j\omega_c b_k \Delta \tau_k}
\end{aligned}
} \tag{31}$$

with  $\mathbf{U}$ ,  $\mathbf{V}^{\Delta,0}$  and  $\mathbf{V}^{\Delta,1}$  defined in (21), (22) and (28) and

$$\mathbf{V}^{\Delta,2}(q) = 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} \mathbf{d}f \tag{32}$$

and

$$\begin{aligned}
[\mathbf{V}^{\Delta,2}(q)]_{k,l} &= 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 e^{j2\pi f(k-l-q)} \mathbf{d}f \\
&= 4\pi^2 \left( \left[ \frac{f^2 e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{2f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathbf{d}f \right) \\
&= \frac{4\pi^2}{j2\pi(k-l-q)} \frac{1}{4} [e^{j\pi(k-l-q)} - e^{-j\pi(k-l-q)}] \\
&\quad - \frac{8\pi^2}{j2\pi(k-l-q)} \left( \left[ \frac{f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathbf{d}f \right) \\
&= \pi^2 \text{sinc}(k-l-q) - \frac{8\pi^2}{(j2\pi(k-l-q))^2} \\
&\quad \times \left( \left[ \frac{1}{2} e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right) e^{-j\pi(k-l-q)} \right] - \left[ \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \right) \\
&= \pi^2 \text{sinc}(k-l-q) + \frac{8\pi^2}{4\pi^2(k-l-q)^2} \\
&\quad \times (\cos(\pi(k-l-q)) - \text{sinc}(k-l-q)) \\
&= \pi^2 \text{sinc}(k-l-q) + 2 \frac{\cos(\pi(k-l-q)) - \text{sinc}(k-l-q)}{(k-l-q)^2} \tag{33}
\end{aligned}$$

### 1.3 Matrix Properties

Based on the definitions of matrices  $\mathbf{V}^{\Delta,0}$ ,  $\mathbf{V}^{\Delta,1}$ ,  $\mathbf{V}^{\Delta,2}$  and  $\mathbf{U}$ , one can do the following remarks:

- $(\mathbf{V}^{\Delta,0}(q))^H = \mathbf{V}^{\Delta,0}(-q)$ ,
- $(\mathbf{V}^{\Delta,1}(q))^H = -\mathbf{V}^{\Delta,1}(-q)$ ,
- $(\mathbf{V}^{\Delta,2}(q))^H = \mathbf{V}^{\Delta,2}(-q)$ ,
- $(\mathbf{U}(p))^H = \mathbf{U}(-p)$ .

## 2 Kullback-Leibler Divergence and Mispesified Maximum Likelihood Estimator

### 2.1 Signal Models

In this section two signal models are considered. One, referred to as the true model considers the reception of two sources embedded in an additive white Gaussian noise:

$$\mathbf{x} = \alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_N), \tag{34}$$

with, for  $i \in \{0, 1\}$ ,  $\alpha_i = \rho_i e^{j\phi_i}$  and, for  $n \in [N_1, N_2]$ ,

$$\begin{aligned}\mathbf{x}^T &= (\dots, x(nT_s), \dots), \\ \mathbf{a}_i^T &= \mathbf{a}(\boldsymbol{\eta}_i)^T = (\dots, a(nT_s; \boldsymbol{\eta}_i), \dots), \\ \mathbf{w}^T &= (\dots, w(nT_s), \dots).\end{aligned}$$

$\boldsymbol{\eta}_i$  is a vector of unknown parameters (delay and Doppler parameters for instance) that parameterized the signal of interest  $\mathbf{a}$  and  $\boldsymbol{\theta}_i^T = [\boldsymbol{\eta}_i^T, \rho_i, \phi_i]$  is the concatenated vector of parameters the  $i$ -th contributor. Consequently, the true data model PDF, noted  $p_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\theta}_1)$ , is written as:

$$p_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\theta}_1) = \mathcal{CN}(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1, \sigma_n^2 \mathbf{I}_N). \quad (35)$$

The second signal model, referred to as pseudotrue model considers the reception of a single source with an additive white Gaussian noise:

$$\mathbf{x} = \alpha_{pt} \mathbf{a}_{pt} + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_N), \quad (36)$$

with,  $\alpha_{pt} = \rho_{pt} e^{j\phi_{pt}}$  and, for  $n \in [N_1, N_2]$ ,

$$\mathbf{a}_{pt}^T = \mathbf{a}(\boldsymbol{\eta}_{pt})^T = (\dots, a(nT_s; \boldsymbol{\eta}_{pt}), \dots),$$

and the subscript  $pt$  refers to *pseudotrue* which will necessarily depend on the true values  $\boldsymbol{\theta}_0$  and  $\boldsymbol{\theta}_1$ . Then, the misspecified data model PDF, noted  $f_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_{pt})$  is written as:

$$f_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_{pt}) = \mathcal{CN}(\alpha_{pt} \mathbf{a}_{pt}, \sigma_n^2 \mathbf{I}_N) \quad (37)$$

where  $\boldsymbol{\theta}_{pt} = [\boldsymbol{\eta}_{pt}, \rho_{pt}, \phi_{pt}]^T$  is the vector of pseudotrue parameters.

## 2.2 Kullback-Leibler Divergence Computation

In the case of a misspecified scenario, one uses a the single source model (37) whereas the true model is a dual source model (35). In this case, the pseudotrue parameters  $\boldsymbol{\theta}_{pt}$  are the values of the parameters  $\boldsymbol{\theta} = [\boldsymbol{\eta}^T, \rho, \phi]^T$  that minimise the Kullback-Leibler Divergence (KLD) between the true and misspecified distribution models.

$$D(p_{\mathbf{x}}||f_{\mathbf{x}}) = E_p \{ \ln(p_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\theta}_0, \boldsymbol{\theta}_1)) - \ln(f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\theta})) \} \quad (38)$$

$$\boldsymbol{\theta}_{pt} = \arg \min_{\boldsymbol{\theta}} \{ D(p_{\mathbf{x}}||f_{\mathbf{x}}) \} = \arg \min_{\boldsymbol{\theta}} \{ E_p \{ -\ln(f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\theta})) \} \}, \quad (39)$$

where  $E_p\{\cdot\}$  is the expectation with respect to the true model's pdf, and

$$-\ln(f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\theta})) = -N \ln(\pi) - 2N \ln(\sigma_n) + \frac{1}{\sigma_n^2} \|\mathbf{x} - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2. \quad (40)$$

The last term of (40) can be expanded as follows:

$$\|\mathbf{x} - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2 = \|\mathbf{x} - (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1) + (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1) - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2 \quad (41)$$

$$\begin{aligned}&= \|\mathbf{x} - (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1)\|^2 + \|\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2 \\ &+ (\mathbf{x} - (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1))^H (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta})) \\ &+ (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta}))^H (\mathbf{x} - (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1))\end{aligned} \quad (42)$$

The expectation of the first term of (42) is the noise covariance, which cannot be minimized and the expectation of the last two terms of (42) are null. Consequently, to minimize the expectation of (40) w.r.t. the argument  $\boldsymbol{\theta}$ , the equation can be simplified as,

$$\arg \min_{\boldsymbol{\theta}} \{E_p \{-\ln(f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\theta}))\}\} = \arg \min_{\boldsymbol{\theta}} \{\|\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2\}. \quad (43)$$

Let  $\mathbf{P}_{\mathbf{a}}$  be the orthogonal projector and  $\mathbf{P}_{\mathbf{a}}^{\perp} = \mathbf{I}_N - \mathbf{P}_{\mathbf{a}}$  with  $\mathbf{P}_{\mathbf{a}} = \frac{\mathbf{a}(\boldsymbol{\eta})\mathbf{a}(\boldsymbol{\eta})^H}{\|\mathbf{a}(\boldsymbol{\eta})\|^2}$ , which leads to

$$\begin{aligned} \|\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2 &= \|(\mathbf{P}_{\mathbf{a}} + \mathbf{P}_{\mathbf{a}}^{\perp})(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta}))\|^2 \\ &= \|\mathbf{P}_{\mathbf{a}}(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta}))\|^2 + \|\mathbf{P}_{\mathbf{a}}^{\perp}(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta}))\|^2 \\ &= \left\| \mathbf{a}(\boldsymbol{\eta}) \left( \frac{\mathbf{a}(\boldsymbol{\eta})^H}{\|\mathbf{a}(\boldsymbol{\eta})\|^2} (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1) - \alpha \right) \right\|^2 + \|\mathbf{P}_{\mathbf{a}}^{\perp}(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1)\|^2, \end{aligned}$$

then the parameters that minimize the KLD are,

$$\boldsymbol{\theta}_{pt} = \arg \min_{\boldsymbol{\theta}} \{\|\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 - \alpha \mathbf{a}(\boldsymbol{\eta})\|^2\} \Leftrightarrow \begin{cases} \boldsymbol{\eta}_{pt} = \arg \max_{\boldsymbol{\eta}} \left\{ \|\mathbf{P}_{\mathbf{a}}^{\perp}(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1)\|^2 \right\} \\ \alpha_{pt} = \frac{\mathbf{a}(\boldsymbol{\eta}_{pt})^H}{\|\mathbf{a}(\boldsymbol{\eta}_{pt})\|^2} (\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1) \end{cases} \quad (44)$$

with  $\alpha_{pt} = \rho_{pt} e^{j\Phi_{pt}}$ . This result shows that minimizing the KLD between the true and the misspecified distribution is equivalent to performing misspecified maximum likelihood estimation.