



**HAL**  
open science

## Global exponential set-point regulation for linear operator semigroups with input saturation

Daniele Astolfi, Swann Marx, Vincent Andrieu, Christophe Prieur

### ► To cite this version:

Daniele Astolfi, Swann Marx, Vincent Andrieu, Christophe Prieur. Global exponential set-point regulation for linear operator semigroups with input saturation. CDC 2022 - 61st IEEE Conference on Decision and Control, IEEE, Dec 2022, Cancun, Mexico. pp.7358 - 7363, 10.1109/CDC51059.2022.9993082 . hal-04005700

**HAL Id: hal-04005700**

**<https://hal.science/hal-04005700v1>**

Submitted on 27 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Global exponential set-point regulation for linear operator semigroups with input saturation

Daniele Astolfi<sup>1</sup>, Swann Marx<sup>2</sup>, Vincent Andrieu<sup>1</sup> and Christophe Prieur<sup>3</sup>

**Abstract**—In this paper the problem of set-point regulation of linear operator semigroups is considered, in presence of saturation of the control input. It is shown how global exponential stability can still be obtained by adding an anti-windup loop to the integral action. As first contribution towards the solution of this regulation problem, we give some new results related to the characterization of global exponential stability of the equilibrium of a subclass of semi-linear dynamical systems.

## I. INTRODUCTION

Recent years have witnessed a growing interest in the problem of feedback control of dynamical systems modeled by partial differential equation or more generally with state evolving in an infinite-dimensional space. Among the problems that have been studied, the output regulation (as considered e.g., in [1]) holds a great place because of its importance in applications. Hence, many results have been obtained in this field and in this context, integral actions are essential to obtain set-point regulation (see [2] or more recently [3]–[7]). However, when applying these regulation methods in the case of saturated controls if the global regulation can be preserved, the global exponential regulation can be lost (see, e.g., [8] in the context of finite-dimensional systems). In this note, inspired by so-called anti-windup methods (see, e.g., [9]–[11]), we provide an explicit solution to the regulation problem, by an appropriate choice of an integral action, while obtaining the global exponential stabilization of the equilibrium.

Before solving this regulation problem, we give some new results concerning global exponential stability conditions for nonlinear infinite-dimensional systems composed of a linear operator to which is added a globally Lipschitz and bounded function. These type of systems are a subclass of semi-linear systems. The proposed

sufficient conditions and necessary conditions extend the corpus of existing results for such a class of systems, e.g. [12]–[18].

The rest of the article is organized in two main parts. In Section II, we characterize the property of global exponential stability for a class of semi-linear systems. These results are then applied in Section III to the particular problem of set-point regulation in case of saturated control input. Conclusions are drawn in Section IV.

**Notation:** Given two Hilbert spaces  $\mathcal{X}_1, \mathcal{X}_2$ ,  $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$  denotes the class of linear bounded operators from  $\mathcal{X}_1$  to  $\mathcal{X}_2$ , and we define  $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$ . Given a real number  $L > 0$ , the saturation function  $\text{sat}_L : \mathbb{R} \rightarrow \mathbb{R}$  and the dead-zone function  $\text{dz}_L : \mathbb{R} \rightarrow \mathbb{R}$  are defined as

$$\text{sat}_L(s) := \begin{cases} L & \text{if } s \geq L, \\ s & \text{if } |s| \leq L, \\ -L & \text{if } s \leq -L, \end{cases} \quad (1)$$

$$\text{dz}_L(s) := s - \text{sat}_L(s). \quad (2)$$

## II. GLOBAL EXPONENTIAL STABILITY CONDITIONS FOR SOME SEMI-LINEAR SYSTEMS

### A. Preliminaries

Let  $\mathcal{X}$  be a Hilbert space equipped with a scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ , from which one can deduce a norm  $\|\cdot\|_{\mathcal{X}}$ . We consider in this section semi-linear systems of the form

$$\frac{d}{dt}x = \mathcal{F}x + \mathcal{G}\psi_L(\mathcal{K}x) := \mathcal{F}_{\psi}(x) \quad (3)$$

in which  $\mathcal{F} : D(\mathcal{F}) \subset \mathcal{X} \rightarrow \mathcal{X}$  is a linear operator which is an infinitesimal generator of a strongly continuous semigroup denoted  $(e^{t\mathcal{F}})_{t \geq 0}$ ,  $\mathcal{G} : \mathbb{R} \rightarrow \mathcal{X}$  and  $\mathcal{K} : \mathcal{X} \rightarrow \mathbb{R}$  are bounded linear operators, and  $\psi_L : \mathbb{R} \rightarrow \mathbb{R}$  is a function satisfying the following assumption.

**Definition 1.** Given a positive real number  $L$ ,  $\psi_L : \mathbb{R} \rightarrow \mathbb{R}$  is a generalized saturation function if it satisfies the following properties

- 1)  $\psi_L(0) = 0$ .
- 2)  $|\psi_L(s') - \psi_L(s'')| \leq |s' - s''|$  for any  $s', s'' \in \mathbb{R}$ .
- 3)  $\sup_{s \in \mathbb{R}} |\psi_L(s)| \leq L$ .

In particular,  $\psi_L$  is a continuous globally bounded function with Lipschitz constant equal to 1. For instance,

<sup>1</sup> D. Astolfi and V. Andrieu are with Univ Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne, France. [astolfi@univ-lyon1.fr](mailto:astolfi@univ-lyon1.fr).

<sup>2</sup> S. Marx is with LS2N, École Centrale de Nantes & CNRS UMR 6004, F-44000 Nantes, France. [swann.marx@ls2n.fr](mailto:swann.marx@ls2n.fr).

<sup>3</sup> C. Prieur is with Univ. Grenoble Alpes, CNRS, Grenoble-INP, GIPSA-lab, F-38000 Grenoble, France. [christophe.prieur@gipsa-lab.fr](mailto:christophe.prieur@gipsa-lab.fr). His work has been partially supported by MIAI@Grenoble Alpes (ANR-19-P3IA-0003)

This research was partially supported by the French Grant ANR ODISSE (ANR-19-CE48-0004-01), and by MIAI@Grenoble Alpes (ANR-19-P3IA-0003).

the saturation function defined in (1) satisfies all properties of Assumption 1. We stress that the Lipschitz constant is selected equal to 1 to ease the presentation of the following section, but we could remove the knowledge assumption on the value of the Lipschitz constant.

Next, we study the solution to system (3). The generalized saturation function being globally Lipschitz, applying [12, Theorem 11.1.5], the following well-posedness result follows. Note that we refer the reader to [12, Definition 11.1.3] and [12, Definition 11.1.2] for the definitions of mild and classical solutions, respectively.

**Proposition 1.** *For any  $x_0 \in \mathcal{X}$  (resp.  $x_0 \in D(\mathcal{F})$ ), there exists a unique mild (resp. classical) solution  $x \in C(\mathbb{R}_+; \mathcal{X})$  (resp.  $x \in C^1(\mathbb{R}_+; \mathcal{X}) \cap C(\mathbb{R}_+; D(\mathcal{F}))$ ), that we will denote by  $x(t) := e^{t\mathcal{F}\psi}(x_0)$ , to (3).*

In this first section, our aim is to study the exponential stability of the origin of (3). This notion of stability is defined as follows.

**Definition 2.** *The origin is said to be globally exponentially stable for system (3) if there exist positive real numbers  $c$  and  $\lambda$  such that the following inequality holds*

$$\|e^{t\mathcal{F}\psi}(x_0)\|_{\mathcal{X}} \leq ce^{-\lambda t} \|x_0\|_{\mathcal{X}} \quad (4)$$

for all  $(x_0, t)$  in  $\mathcal{X} \times \mathbb{R}_+$ .

It has to be noticed that in this property it is required that the positive real number  $c$  and  $\lambda$  are uniform. In the following part of this section a necessary condition and two sufficient conditions are given to obtain this global exponential stability property.

#### B. A necessary condition for global exponential stability

We start by stating the following necessary condition. Even though the result is probably known in the finite-dimensional context, we haven't found any proof of it. Note that this condition is in the same spirit of some other results related to feedback stabilization with saturated control, e.g. [10, Theorem 2.2, Chapter 2.4.4] or [9, Theorem 1], although these results are based on the existence of a quadratic Lyapunov function (differently from the proof below which is based on a trajectory analysis).

**Proposition 2** (Necessary Conditions). *The origin of system (3) is globally exponentially stable only if the origin of  $\frac{d}{dt}x = \mathcal{F}x$  is exponentially stable.*

**Proof.** Assume that (4) holds for some positive real numbers  $c$  and  $\lambda$ . By definition of mild solutions [12, Definition 11.1.3], note that, for all  $x$  in  $\mathcal{X}$  and for all  $t \geq 0$

$$e^{t\mathcal{F}\psi}(x) = e^{t\mathcal{F}}x + \int_0^t e^{(t-s)\mathcal{F}}\mathcal{G}\psi_L(\mathcal{K}e^{s\mathcal{F}\psi}x) ds. \quad (5)$$

Let  $v \in \mathcal{X}$  satisfying  $\|v\|_{\mathcal{X}} = 1$ . Note that, by definition of the norm, for all  $\mu > 0$  and all  $t \geq 0$

$$\begin{aligned} \|e^{t\mathcal{F}}v\|_{\mathcal{X}} &= \frac{1}{\mu} \|e^{t\mathcal{F}}\mu v\|_{\mathcal{X}}, \\ &\leq \frac{1}{\mu} \|e^{t\mathcal{F}\psi}(\mu v)\|_{\mathcal{X}} + \frac{1}{\mu} \|e^{t\mathcal{F}\psi}(\mu v) - e^{t\mathcal{F}}\mu v\|_{\mathcal{X}}. \end{aligned}$$

With inequality (4), this implies by Definition 2 and equation (5) that for all  $\mu > 0$  and all  $t \geq 0$

$$\begin{aligned} \|e^{t\mathcal{F}}v\|_{\mathcal{X}} &\leq ce^{-\lambda t} \|v\|_{\mathcal{X}} \\ &\quad + \frac{1}{\mu} \left\| \int_0^t e^{(t-s)\mathcal{F}}\mathcal{G}\psi_L(e^{s\mathcal{F}\psi}\mu v) ds \right\|_{\mathcal{X}}, \end{aligned}$$

which gives, using the fact that  $\|v\|_{\mathcal{X}} = 1$ , the Cauchy Schwarz inequality and item 3) of Definition 1

$$\|e^{t\mathcal{F}}v\|_{\mathcal{X}} \leq ce^{-\lambda t} + \frac{L}{\mu} \int_0^t \|e^{(t-s)\mathcal{F}}\|_{\mathcal{L}(\mathcal{X})} \|\mathcal{G}\|_{\mathcal{L}(\mathbb{R}, \mathcal{X})} ds.$$

Letting  $\mu$  going to infinity yields, for all  $t \geq 0$

$$\|e^{t\mathcal{F}}v\|_{\mathcal{X}} \leq ce^{-\lambda t}, \quad (6)$$

for all  $v$  in  $\mathcal{X}$  such that  $\|v\|_{\mathcal{X}} = 1$ . By linearity of the semigroup  $(e^{t\mathcal{F}})_{t \geq 0}$  it holds, for all  $w$  in  $\mathcal{X}$ ,  $\frac{1}{\|w\|_{\mathcal{X}}} \|e^{t\mathcal{F}}w\|_{\mathcal{X}} = \|e^{t\mathcal{F}}\frac{w}{\|w\|_{\mathcal{X}}}\|_{\mathcal{X}}$  and using (6) with  $v = \frac{w}{\|w\|_{\mathcal{X}}}$ , we deduce that  $\mathcal{F}$  is an infinitesimal generator of an exponential strongly continuous semigroup.  $\square$

#### C. Sufficient conditions for global exponential stability

Sufficient conditions exist to prove the asymptotic stability from separate properties on the linear operator and on the semilinear map. See, e.g., [14, Theorem 4] (see also [19], [20]) for exponentially stable linear operators and dissipative semilinear maps; [14, Theorem 5] for an other sufficient condition based on the comparison of the decay rate of the linear operator and the Lipschitz constant of the semilinear map; [13] for conditions based on the circle-criterion for Lur'e type systems. In our context, we give sufficient conditions based on Lyapunov consideration. Before presenting the next result, we provide the definition of coercive Lyapunov functionals for linear systems, that will be used all along the sequel (see [18, Definition 2.11] for the definition of coercive Lyapunov functionals for more general systems).

**Definition 3.** *Let  $\mathcal{F} : D(\mathcal{F}) \subset \mathcal{X} \rightarrow \mathcal{X}$  be a linear operator generating a strongly continuous semigroup. A functional  $V(x) := \langle \mathcal{P}x, x \rangle_{\mathcal{X}}$ , with  $\mathcal{P} \in \mathcal{L}(\mathcal{X})$  a self-adjoint operator, is said to be a coercive Lyapunov functional for  $\mathcal{F}$  if there exist positive constants  $\alpha$  and  $\lambda$  satisfying*

$$\alpha \|x\|_{\mathcal{X}}^2 \leq V(x), \quad \forall x \in \mathcal{X}, \quad (7)$$

$$\langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}x \rangle_{\mathcal{X}} \leq -\lambda \|x\|_{\mathcal{X}}^2, \quad \forall x \in D(\mathcal{F}). \quad (8)$$

Inequality (7) and the assumption  $\mathcal{P} \in \mathcal{L}(\mathcal{X})$  imply in particular that  $\sqrt{V(\cdot)}$  is equivalent to the usual norm  $\|\cdot\|_{\mathcal{X}}$ . Note that the existence of non-coercive Lyapunov functional (possibly implying a non-equivalence between  $\sqrt{V(\cdot)}$  and  $\|\cdot\|_{\mathcal{X}}$ ) is a particular feature of infinite-dimensional systems, as discussed in [21].

In the following, two sufficient conditions for the exponential stability of the origin of systems of the form (3) are stated. The first condition is based on a ‘‘small-gain control property’’ which extends the result [8, Theorem 2.4] presented in the case of finite-dimensional linear systems in the particular case in which the function  $\psi_L$  is a saturation function. It is a trivial extension of the fact that exponential stability is robust to *small* Lipschitz perturbations in the dynamics when there exists a coercive Lyapunov functional.

**Proposition 3.** *Consider system (3) with the function  $\psi_L$  satisfying Definition 1. Suppose that there exists a coercive Lyapunov functional  $V$  for  $\mathcal{F}$  given by, for all  $x \in \mathcal{X}$ ,  $V(x) := \langle \mathcal{P}x, x \rangle_{\mathcal{X}}$  for some  $\mathcal{P} \in \mathcal{L}(\mathcal{X})$ . Then, for all bounded linear operator  $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$  satisfying  $\|\mathcal{K}\|_{\mathcal{L}(\mathcal{X}, \mathbb{R})} < \frac{\lambda}{2\|\mathcal{P}\mathcal{G}\|_{\mathcal{X}}}$  the origin of (3) is globally exponentially stable.*

**Proof.** We consider classical solutions to (3), i.e. we consider initial condition  $x_0$  in  $D(\mathcal{F})$  and we will prove that the result of Proposition 3 for such solutions. One can deduce the same result for any initial condition  $x_0 \in \mathcal{X}$  by applying a standard density argument (see e.g., [17, Lemma 1]).

Consider the Lyapunov functional  $V(x) := \langle \mathcal{P}x, x \rangle_{\mathcal{X}}$  of the statement. Its derivative along classical solutions to (3) yields, for all  $x$  in  $D(\mathcal{F})$

$$\begin{aligned} \frac{d}{dt}V(x) &= \langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}x \rangle_{\mathcal{X}} \\ &\quad + \langle \mathcal{G}\psi_L(\mathcal{K}x), \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{G}\psi_L(\mathcal{K}x) \rangle_{\mathcal{X}}, \\ &\leq -\lambda\|x\|_{\mathcal{X}}^2 + 2\|x\|_{\mathcal{X}}\|\mathcal{P}\mathcal{G}\|_{\mathcal{X}}|\psi_L(\mathcal{K}x)|, \end{aligned} \quad (9)$$

where in the second step we used the property (8). Let  $\varepsilon > 0$  be such that  $\varepsilon < \frac{\lambda}{2\|\mathcal{P}\mathcal{G}\|_{\mathcal{X}}} - \|\mathcal{K}\|_{\mathcal{L}(\mathcal{X}, \mathbb{R})}$ . Such  $\varepsilon > 0$  exists by assumption on the bound of  $\mathcal{K}$  in Proposition 3. Using Definition 1 (point 1. and 2.) and the definition of  $\varepsilon$ , one obtains that, for all  $x$  in  $\mathcal{X}$ ,  $|\psi_L(\mathcal{K}x)| \leq |\mathcal{K}x| \leq \frac{\lambda}{2\|\mathcal{P}\mathcal{G}\|_{\mathcal{X}}}\|x\|_{\mathcal{X}} - \varepsilon\|x\|_{\mathcal{X}}$ . We deduce with (9)

$$\frac{d}{dt}V(x) \leq -2\varepsilon\|\mathcal{P}\mathcal{G}\|_{\mathcal{X}}\|x\|_{\mathcal{X}}^2 \quad \forall x \in D(\mathcal{F}).$$

Using the coercivity of  $V$  and the boundedness of the linear operator  $\mathcal{P}$ , we obtain the desired result. Indeed,  $\mathcal{P}$  being coercive,  $\sqrt{V(\cdot)}$  is equivalent to the usual norm of  $\mathcal{X}$ . Using this equivalence together with the Grönwall’s inequality, one can deduce that the origin of (3) is globally exponentially stable (in the sense of Definition 2).  $\square$

The second sufficient condition is based on the existence

of a common coercive Lyapunov functional between the operators  $\mathcal{F}$  and  $\mathcal{F} + \mathcal{G}\mathcal{K}$ . For the finite-dimensional case in which the function  $\psi_L$  is a saturation function this result is well known, see, e.g. [11].

**Proposition 4.** *Consider system (3) with the function  $\psi_L$  satisfying Definition 1. Suppose there exists a common coercive Lyapunov functional  $V$  for  $\mathcal{F}$  and  $\mathcal{F} + \mathcal{G}\mathcal{K}$ , namely there exists a functional  $V$  defined as  $V(x) := \langle \mathcal{P}x, x \rangle_{\mathcal{X}}$ , for all  $x$  in  $\mathcal{X}$ , satisfying (7) and*

$$\begin{aligned} \langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}x \rangle_{\mathcal{X}} &\leq -\lambda\|x\|_{\mathcal{X}}^2 \\ \langle (\mathcal{F} + \mathcal{G}\mathcal{K})x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, (\mathcal{F} + \mathcal{G}\mathcal{K})x \rangle_{\mathcal{X}} &\leq -\lambda\|x\|_{\mathcal{X}}^2 \end{aligned} \quad (10)$$

for all  $x \in D(\mathcal{F})$ , for some  $\lambda > 0$ . Then the origin of (3) is globally exponentially stable.

**Proof.** Similarly to the proof of Proposition 3, we consider here classical solutions to (3), i.e. we consider initial condition  $x_0$  in  $D(\mathcal{F})$ , and we will prove that the result of Proposition 4 for any initial condition  $x_0$  in  $D(\mathcal{F})$ . One can deduce the same result for any initial condition  $x_0 \in \mathcal{X}$  by applying a standard density argument (see e.g., [17, Lemma 1]).

Define  $\mathcal{F}_{\mathcal{K}} := \mathcal{F} + \mathcal{G}\mathcal{K}$  and let  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined as  $\alpha(0) = 0$  and

$$\alpha(s) := 1 - \frac{\psi_L(s)}{s}, \quad \forall s \neq 0,$$

Note that with Points 1 and 2 of the definition 1, it yields that  $\alpha(s) \in [0, 1]$  for all  $s$ . With such a notation, the system (3) can be rewritten as

$$\frac{d}{dt}x = \alpha(\mathcal{K}x)\mathcal{F}x + (1 - \alpha(\mathcal{K}x))\mathcal{F}_{\mathcal{K}}x. \quad (11)$$

Now consider the coercive Lyapunov functional given in the statement  $V(x) := \langle \mathcal{P}x, x \rangle_{\mathcal{X}}$  with  $\mathcal{P} \in \mathcal{L}(\mathcal{X})$  satisfying (7) and (10). By compactly writing  $\alpha = \alpha(\mathcal{K}x)$ , we compute its derivative along classical solutions to (3), i.e. equivalently (11). Using (10), we obtain, for all  $x$  in  $D(\mathcal{F})$ ,

$$\begin{aligned} \frac{d}{dt}V(x) &= \alpha\langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} + \alpha\langle \mathcal{P}x, \mathcal{F}x \rangle_{\mathcal{X}} \\ &\quad + (1 - \alpha)\langle \mathcal{F}_{\mathcal{K}}x, \mathcal{P}x \rangle_{\mathcal{X}} + (1 - \alpha)\langle \mathcal{P}x, \mathcal{F}_{\mathcal{K}}x \rangle_{\mathcal{X}} \\ &\leq -\alpha\lambda\|x\|_{\mathcal{X}}^2 - (1 - \alpha)\lambda\|x\|_{\mathcal{X}}^2 \leq -\lambda\|x\|_{\mathcal{X}}^2. \end{aligned}$$

The proof concludes by following the same arguments used in the proof of Proposition 3.  $\square$

### III. SET-POINT REGULATION

#### A. Problem statement and assumptions

In this section we consider a class of controlled linear systems with saturated control. A slightly modified notation is used. In particular, we consider  $\mathcal{H}$  as a Hilbert

space equipped with a norm  $\|\cdot\|_{\mathcal{H}}$  and a scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ . We consider systems which are described as

$$\frac{d}{dt}\xi = \mathcal{A}\xi + \mathcal{B}\text{sat}_L(u) + w, \quad y = \mathcal{C}\xi, \quad (12)$$

in which  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is an infinitesimal generator of a  $C_0$ -semigroup denoted  $t \mapsto e^{t\mathcal{A}}$ , the operator  $\mathcal{B} : \mathbb{R} \rightarrow \mathcal{H}$  is linear and bounded, the operator  $\mathcal{C}$  is in  ${}^1\mathcal{L}(D(\mathcal{A}), \mathbb{R})$ ,  $u$  in  $\mathbb{R}$  is the control input,  $y$  in  $\mathbb{R}$  is the measured output,  $w$  in  $\mathcal{H}$  is an unknown constant disturbance vector, and the saturation function  $\text{sat}_L$  is defined as in (1) for some  $L > 0$ . Given a constant reference  $y_{\text{ref}} \in \mathbb{R}$ , we are interested in the design of a feedback law for system (12) so that there exists a unique equilibrium which is exponentially stable and

$$\lim_{t \rightarrow \infty} y(t) = y_{\text{ref}}, \quad (13)$$

by means of an integral action. Note that the regulation problem could be defined without requiring existence and uniqueness of equilibrium leading to weaker assumptions. It is however beyond the scope of this paper.

To this end, we follow the set of conditions established below (similarly to [7]). Comments are postponed at the end of the section.

**Assumption 1.** *The operator  $\mathcal{A}$  generates a strongly continuous semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$ , and moreover there exist a coercive and self-adjoint operator  $\mathcal{Q} \in \mathcal{L}(X)$  and a positive constant  $\nu$  such that for all  $\xi \in D(\mathcal{A})$*

$$\langle \mathcal{A}\xi, \mathcal{Q}\xi \rangle_{\mathcal{H}} + \langle \mathcal{Q}\xi, \mathcal{A}\xi \rangle_{\mathcal{H}} \leq -2\nu\|\xi\|_{\mathcal{H}}^2. \quad (14)$$

Then, following [2], we assume (up to a reparametrization) the following.

**Assumption 2.** *The condition  $\mathcal{C}\mathcal{A}^{-1}\mathcal{B} = 1$  holds.*

Following classical finite-dimensional strategies, we extend the system (12) with an integral action [22] and an anti-windup mechanism [9]–[11]. In particular, we propose the following regulator

$$\frac{d}{dt}z = \mathcal{C}\xi - y_{\text{ref}} - \mu \text{dz}_L(u), \quad u = kz, \quad (15)$$

with  $z$  in  $\mathbb{R}$ , the function  $\text{dz}_L$  defined as in (2), and  $\mu, k$  positive parameters to be selected so that the solutions to the closed-loop system (12), (15) converge exponentially to an equilibrium on which the regulation objective (13) is attained.

We complete this section by introducing the following compact form. In particular, recalling the definition of the deadzone function in (2), the extended closed-loop system (12), (15) reads as

$$x = \begin{pmatrix} \xi \\ z \end{pmatrix}, \quad \frac{d}{dt}x = \mathcal{F}x + \mathcal{G}\text{sat}_L(\mathcal{K}x) + \Gamma, \quad (16)$$

<sup>1</sup>Since  $\mathcal{A}$  is an infinitesimal generator of a  $C_0$ -semigroup, its domain  $D(\mathcal{A})$  is a Hilbert space equipped with the graph norm  $\|\xi\|_{D(\mathcal{A})} = \|\xi\|_{\mathcal{H}} + \|\mathcal{A}\xi\|_{\mathcal{H}}$ .

with  $\mathcal{F}, \mathcal{G}, \mathcal{L}$  defined as

$$\mathcal{F} := \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{C} & -\mu k \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} \mathcal{B} \\ \mu \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} 0 & k \end{pmatrix}, \quad (17)$$

and  $\Gamma = (w, y_{\text{ref}})^\top$ . The state space is defined by  $\mathcal{X} = \mathcal{H} \times \mathbb{R}$  and it is equipped with the norm  $\|x\|_{\mathcal{X}} = \|(\xi, z)\|_{\mathcal{X}} = \sqrt{\langle \xi, \xi \rangle_{\mathcal{H}} + |z|^2}$ , with the associated scalar product  $\langle (\xi_1, z_1), (\xi_2, z_2) \rangle_{\mathcal{X}} = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}} + z_1 z_2$ . Moreover, we define the domain of the operator  $\mathcal{F}$  given by  $D(\mathcal{F}) = \{(\xi, z) \in \mathcal{H} \times \mathbb{R} \mid \mathcal{A}\xi \in \mathcal{H}\}$ , which is equipped with the usual graph norm.

**Remark 1.** *Assumption 1 establishes that the semigroup generated by  $\mathcal{A}$  is exponentially stable. This assumption is necessary in finite dimension in view of Proposition 2 and the lower triangular structure of the operator  $\mathcal{F}$ . Note that exponential stability could be obtained without assuming coercivity of the operator  $\mathcal{Q}$ . However, it is not straightforward to obtain exponential stability for the closed loop system that we study, because it is nonlinear due to the saturation operator. Following [21], it may be possible to remove this coercivity assumption. It is left for future research. Assumption 2 could be also relaxed as  $\mathcal{C}\mathcal{A}^{-1}\mathcal{B} \neq 0$ , which is necessary for the design of an integral action feedback design (see, e.g., [22] for the finite-dimensional case and [7] for the infinite-dimensional case). Here we select its value set to 1 to ease the rest of the presentation.*

## B. Main results

The next result ensures that there exists a unique equilibrium point to (16). This existence result relies mainly on Assumption 2. In addition to this existence, we provide a condition ensuring that, if the perturbation  $w$  and the reference  $y_{\text{ref}}$  are sufficiently small, then at the equilibrium the regulation objective  $\mathcal{C}\xi^\circ = y_{\text{ref}}$  is satisfied, despite the perturbation  $w$ .

**Lemma 1.** *There exists a unique equilibrium point  $(\xi^\circ, z^\circ) \in D(\mathcal{F})$  to (16). In other words, there exists a unique solution  $(\xi^\circ, z^\circ) \in D(\mathcal{F})$  to the following set of equations:*

$$\begin{aligned} 0 &= \mathcal{A}\xi^\circ + \mathcal{B}\text{sat}_L(kz^\circ) + w \\ 0 &= \mathcal{C}\xi^\circ - y_{\text{ref}} - \mu \text{dz}_L(kz^\circ). \end{aligned} \quad (18)$$

Moreover, let  $\Omega \subset \mathcal{H} \times \mathbb{R}$  be defined as

$$\Omega = \{(w, y_{\text{ref}}) \in \mathcal{H} \times \mathbb{R} : |\mathcal{C}\mathcal{A}^{-1}w + y_{\text{ref}}| \leq L\}.$$

Then, for all  $(w, y_{\text{ref}}) \in \Omega$ , one has  $\mathcal{C}\xi^\circ = y_{\text{ref}}$ .

**Proof.** The first line of (18) can be equivalently written as:

$$\xi^\circ = -\mathcal{A}^{-1}\mathcal{B}\text{sat}_L(kz^\circ) - \mathcal{A}^{-1}w. \quad (19)$$

Plugging this identity in the second line of (18) leads to  $-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\text{sat}_L(kz^\circ) - y_{\text{ref}} - \mathcal{C}\mathcal{A}^{-1}w - \mu \text{dz}_L(kz^\circ) = 0$ .

Using Assumption 2 and the definition of the dead-zone function, one finds the equivalent equation

$$f(kz^\circ) = -y_{\text{ref}} - \mathcal{C}\mathcal{A}^{-1}w, \quad (20)$$

with  $f$  defined as

$$f : s \in \mathbb{R} \mapsto \mu s - (\mu - 1) \text{sat}_L(s) \in \mathbb{R}.$$

By definition of the saturation function,  $f$  can be equivalently rewritten as:

$$f : s \mapsto f(s) = \begin{cases} \mu s - (\mu - 1)L & \text{if } s \geq L, \\ s & \text{if } |s| \leq L, \\ \mu s + (\mu - 1)L & \text{if } s \leq -L. \end{cases} \quad (21)$$

Note that once one is able to prove that there exists a unique  $z^\circ \in \mathbb{R}$  solving (20), the result of Lemma 1 follows, thanks to (19). If  $\mu = 1$ , the result follows easily, and we have even an explicit solution given by  $kz^\circ = -y_{\text{ref}} - \mathcal{C}\mathcal{A}^{-1}w$ . Suppose then that  $\mu \neq 1$ . Note that  $f$  is continuous. Moreover,  $\lim_{s \rightarrow +\infty} f(s) = +\infty$  and  $\lim_{s \rightarrow -\infty} f(s) = -\infty$ . Then, according to the intermediate value theorem, and since  $f$  is increasing, for every  $\tilde{s} \in \mathbb{R}$ , there exists a unique  $s \in \mathbb{R}$  such that  $f(s) = \tilde{s}$ . Setting  $s = kz^\circ$  and  $\tilde{s} = -y_{\text{ref}} - \mathcal{C}w$ , there exists a unique solution to (18).

Now, we prove the second part of Lemma 1. The definition of the set  $\Omega$  and the condition (20) gives  $|f(kz^\circ)| \leq L$ . As a consequence, in view of the definition of the function  $f$  given in (21), we obtain  $f(kz^\circ) = kz^\circ$  and  $|kz^\circ| \leq L$ . Furthermore, using the definitions of  $\text{sat}_L$  and  $\text{dz}_L$  in (1), (2), the second line of equation (18) reads  $0 = \mathcal{C}\xi^\circ - y_{\text{ref}}$  concluding the proof.  $\square$

Our aim now is to transform (16) into a system in the form (3). To do so, we introduce the change of coordinates

$$\xi \mapsto \tilde{\xi} := \xi - \xi^\circ, \quad z \mapsto \tilde{z} := z - z^\circ,$$

where the pair  $(\xi^\circ, z^\circ)$  is given by Lemma 1. The system (16) reads in these new coordinates:

$$\begin{aligned} \frac{d}{dt} \tilde{\xi} &= \mathcal{A}\tilde{\xi} + \mathcal{B}[\text{sat}_L(k\tilde{z} + kz^\circ) - \text{sat}_L(kz^\circ)] \\ \frac{d}{dt} \tilde{z} &= \mathcal{C}\tilde{\xi} - \mu[\text{dz}_L(k\tilde{z} + kz^\circ) - \text{dz}_L(z^\circ)] \end{aligned}$$

or equivalently, selecting  $\tilde{x} := (\tilde{\xi}, \tilde{z})$ ,

$$\frac{d}{dt} \tilde{x} = \mathcal{F}\tilde{x} + \mathcal{G}\psi_L(\mathcal{K}\tilde{x}), \quad (22)$$

where  $\psi_L(\mathcal{K}\tilde{x}) := \text{sat}_L(\mathcal{K}(\tilde{x} + x^\circ)) - \text{sat}_L(\mathcal{K}x^\circ)$ . System (22) is in the form (3) and moreover the function  $\psi_L$  so defined satisfies Definition 1 (picking  $2L$  instead of  $L$ ). Well-posedness of system (22) follows from direct application of Proposition 1.

Finally, we can show that an appropriate selection of the parameters  $\mu, k$  allows to apply Proposition 4

in order to show that the origin of system (22) is globally exponentially stable. To this end, we need to introduce a new operator that will be denoted as  $\mathcal{M}$ . In particular, since  $\mathcal{A}$  generates a strongly continuous semigroup which origin is exponentially stable, it admits an inverse,  $\mathcal{A}^{-1}$  which belongs to  $\mathcal{L}(\mathcal{H}, D(\mathcal{A}))$  (see e.g., [23, Theo. 2.1]). As a consequence,  $\mathcal{C}$  being  $\mathcal{A}$ -bounded, the operator  $\mathcal{M} = \mathcal{C}\mathcal{A}^{-1}$  belongs to  $\mathcal{L}(\mathcal{H}, \mathbb{R})$ . Note that such an operator  $\mathcal{M}$  is solution to the following equation

$$\mathcal{M}\mathcal{A}\xi = \mathcal{C}\xi \quad \forall \xi \in D(\mathcal{A}). \quad (23)$$

**Proposition 5.** *Suppose Assumption 1 and 2 hold. For any  $\mu > 0$  and  $k \in (0, k^*)$  with  $k^*$  given by*

$$k^* = \min \left\{ \frac{\nu}{\mu \|\mathcal{M}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})}^2}, \frac{\nu}{\|(\mathcal{Q} + \mathcal{M}^*\mathcal{M})\mathcal{B}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})}^2} \right\}, \quad (24)$$

*the origin of (22) is globally exponentially stable.*

**Proof.** The main idea of the proof of this proposition consists in showing the existence of a common Lyapunov functional for the operators  $\mathcal{F}$  and for  $\mathcal{F} + \mathcal{G}\mathcal{K}$ , which is the context of Proposition 4. To this end, let us define the operator  $\mathcal{P}$  in  $\mathcal{L}(\mathcal{X})$  as

$$\mathcal{P}(\xi, z) = \begin{pmatrix} \mathcal{Q} + \mathcal{M}^*\mathcal{M} & -\mathcal{M}^* \\ \star & 1 \end{pmatrix} \begin{pmatrix} \xi \\ z \end{pmatrix}, \quad (25)$$

with  $\mathcal{Q}$  given by Assumption 1 and  $\mathcal{M}$  defined by (23). We compute now, for all  $x = (\xi, z)$  in  $D(\mathcal{F})$ ,

$$\begin{aligned} \langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} &= \langle \mathcal{P}\mathcal{F}x, x \rangle_{\mathcal{X}}, \\ &= \left\langle \begin{pmatrix} \mathcal{Q}\mathcal{A} & \mathcal{M}^*\mu k \\ \star & -\mu k \end{pmatrix} x, x \right\rangle_{\mathcal{X}}, \end{aligned} \quad (26)$$

where we used the fact that  $\mathcal{M}^*\mathcal{M}\mathcal{A}\xi - \mathcal{M}^*\mathcal{C}\xi = 0$  in view of (23). Then, using (14), inequality (26) gives

$$\langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}x \rangle_{\mathcal{X}} \leq - \left( \frac{\|\xi\|_{\mathcal{H}}}{|z|} \right)^\top S_1 \left( \frac{\|\xi\|_{\mathcal{H}}}{|z|} \right)$$

with

$$S_1 = 2 \begin{pmatrix} \nu I & -\mu k \|\mathcal{M}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})} \\ \star & \mu k \end{pmatrix}.$$

Finally, it can be verified that for any  $k \in (0, k^*)$ , with  $k^*$  smaller than the first value in (24), there exists  $\varepsilon_1 > 0$  such that  $S_1 \geq \varepsilon_1 I$  from which we obtain

$$\langle \mathcal{F}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}x \rangle_{\mathcal{X}} \leq -\varepsilon_1 \|x\|_{\mathcal{X}}^2. \quad (27)$$

Then, denoting

$$\mathcal{F}_{\mathcal{K}} = \mathcal{F} + \mathcal{G}\mathcal{K} = \begin{pmatrix} \mathcal{A} & k\mathcal{B} \\ \mathcal{C} & 0 \end{pmatrix}$$

and recalling that  $\mathcal{M}\mathcal{B} = 1$  by Assumption 2, we compute for all  $x = (\xi, z)$  in  $D(\mathcal{F})$

$$\begin{aligned} \langle \mathcal{F}_{\mathcal{K}}x, \mathcal{P}x \rangle_{\mathcal{X}} &= \langle \mathcal{P}\mathcal{F}_{\mathcal{K}}x, x \rangle_{\mathcal{X}}, \\ &= \left\langle \begin{pmatrix} \mathcal{Q}\mathcal{A} & k(\mathcal{Q} + \mathcal{M}^*\mathcal{M})\mathcal{B} \\ \star & -k \end{pmatrix} x, x \right\rangle_{\mathcal{X}}. \end{aligned}$$

With Assumption 1, it implies

$$\langle \mathcal{F}_{\mathcal{K}}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}_{\mathcal{K}}x \rangle_{\mathcal{X}} \leq - \left( \frac{\|\xi\|_{\mathcal{H}}}{|z|} \right)^{\top} S_2 \left( \frac{\|\xi\|_{\mathcal{H}}}{|z|} \right)^{\top}$$

with

$$S_2 = 2 \begin{pmatrix} -\nu I & k\|(\mathcal{Q} + \mathcal{M}^* \mathcal{M})\mathcal{B}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})} \\ \star & -k \end{pmatrix}.$$

As in the previous case, it can be verified that for any  $k \in (0, k^*)$ , with  $k^*$  smaller than the second value in (24), there exists  $\varepsilon_2 > 0$  such that  $S_2 \geq \varepsilon_2 I$ . Hence, we obtain  $\langle \mathcal{F}_{\mathcal{K}}x, \mathcal{P}x \rangle_{\mathcal{X}} + \langle \mathcal{P}x, \mathcal{F}_{\mathcal{K}}x \rangle_{\mathcal{X}} \leq -\varepsilon_2 \|x\|_{\mathcal{X}}^2$ . Combining the last inequality with (27), we conclude that  $\mathcal{P}$  is a common coercive Lyapunov functional for the operators  $\mathcal{F}$  and  $\mathcal{F} + \mathcal{G}\mathcal{K}$ . As a consequence, the conditions (10) of Proposition 4 are satisfied and the origin of system (22) is globally exponentially stable.  $\square$

**Remark 2.** From condition (24) we obtain that if  $\mu \in \left(0, \frac{\|(\mathcal{Q} + \mathcal{M}^* \mathcal{M})\mathcal{B}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})}^2}{\|\mathcal{M}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})}^2}\right)$ , then  $k^*$  is independent of  $\mu$ , allowing for a larger interval for the choice of  $k$ .

Finally, in view of the previous result, we state the following result which combines the result of Proposition 5 with the properties of the equilibria of the closed-loop dynamics established in Lemma 1. In summary, global exponential set-point regulation is obtained.

**Corollary 1.** Consider the controller (15) and select  $\mu, k$  according to Proposition 5. For any  $w, y_{\text{ref}} \in \Omega$ , with  $\Omega$  defined in Lemma 1, the closed-loop system (12), (15) admits a unique equilibrium  $(\xi^\circ, z^\circ)$  which is globally exponentially stable and satisfies  $y = C\xi^\circ = y_{\text{ref}}$ .

#### IV. CONCLUSION

In this paper, we have first given some necessary and sufficient conditions for the global exponential stability of some subclass of semi-linear abstract Cauchy problems. This class of system is composed of a linear generator to which is added a nonlinear function which is globally Lipschitz and globally bounded. This has allowed us to tackle the set-point regulation problem with saturated control input. Indeed, employing an anti-windup design strategy, we have shown how to obtain global exponential set point regulation for infinite-dimensional linear systems. The case of boundary control is currently under investigation.

#### REFERENCES

- [1] L. Paunonen and S. Pohjolainen, "Internal model theory for distributed parameter systems," *SIAM Journal on Control and Optimization*, vol. 48, no. 7, pp. 4753–4775, 2010.
- [2] S. Pohjolainen, "Robust multivariable PI-controller for infinite dimensional systems," *IEEE Trans. on Autom. Contr.*, vol. 27, no. 1, pp. 17–30, 1982.
- [3] J.-M. Coron and A. Hayat, "PI controllers for 1-D nonlinear transport equation," *IEEE Trans. on Autom. Contr.*, vol. 64, no. 11, pp. 4570–4582, 2019.
- [4] N. Vanspranghe and L. Brivadis, "Output regulation of infinite-dimensional nonlinear systems: a forwarding approach for contraction semigroups," *arXiv:2201.10146*, 2022.
- [5] H. Lhachemi and C. Prieur, "Finite-dimensional observer-based PI regulation control of a reaction-diffusion equation," *IEEE Trans. on Autom. Contr.*, to appear, 2022.
- [6] I. Balogoun, S. Marx, and D. Astolfi, "ISS Lyapunov strictification via observer design and integral action control for a Korteweg-de-Vries equation," *arXiv:2107.09541*, 2021.
- [7] A. Terrand-Jeanne, V. Andrieu, V. D. S. Martins, and C.-Z. Xu, "Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems," *IEEE Trans. on Autom. Contr.*, vol. 65, no. 11, pp. 4481–4492, 2019.
- [8] E. D. Sontag, "An algebraic approach to bounded controllability of linear systems," *International Journal of Control*, vol. 39, no. 1, pp. 181–188, 1984.
- [9] G. Grimm, J. Hatfield, I. Postlethwaite, A. R. Teel, M. C. Turner, and L. Zaccarian, "Antiwindup for stable linear systems with input saturation: an LMI-based synthesis," *IEEE Trans. on Autom. Contr.*, vol. 48, no. 9, pp. 1509–1525, 2003.
- [10] S. Tarbouriech, G. Garcia, J. M. Gomes da Silva Jr, and I. Queinnec, *Stability and stabilization of linear systems with saturating actuators*. Springer, 2011.
- [11] B. G. Romanchuk, "Some comments on anti-windup synthesis using LMIs," *International Journal of Robust and Nonlinear Control*, vol. 9, no. 10, pp. 717–734, 1999.
- [12] R. Curtain and H. Zwart, *Introduction to infinite-dimensional systems theory: a state-space approach*. Springer, 2020, vol. 71.
- [13] B. Jayawardhana, H. Logemann, and E. P. Ryan, "Infinite-dimensional feedback systems: the circle criterion and input-to-state stability," *Communications in Information & Systems*, vol. 8, no. 4, pp. 413–444, 2008.
- [14] I. Aksikas, "Asymptotic behaviour of contraction non-autonomous semi-flows in a Banach space: Application to first-order hyperbolic PDEs," *Automatica*, vol. 65, pp. 140–146, 2016.
- [15] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Springer, 2012.
- [16] S. Marx, Y. Chitour, and C. Prieur, "Stability analysis of dissipative systems subject to nonlinear damping via Lyapunov techniques," *IEEE Trans. on Autom. Contr.*, vol. 65, no. 5, pp. 2139–2146, 2019.
- [17] S. Marx, V. Andrieu, and C. Prieur, "Cone-bounded feedback laws for  $m$ -dissipative operators on Hilbert spaces," *Mathematics of Control, Signals, and Systems*, vol. 29, no. 4, pp. 1–32, 2017.
- [18] A. Mironchenko and C. Prieur, "Input-to-state stability of infinite-dimensional systems: recent results and open questions," *SIAM Review*, vol. 62, no. 3, pp. 529–614, 2020.
- [19] M. Dus, F. Ferrante, and C. Prieur, "On  $L^\infty$  stabilization of diagonal semilinear hyperbolic systems by saturated boundary control," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 26, p. 23, 2020.
- [20] A. Hayat, "Global exponential stability and Input-to-State Stability of semilinear hyperbolic systems for the  $L^2$  norm," *Systems & Control Letters*, vol. 148, p. 104848, 2021.
- [21] A. Mironchenko and F. Wirth, "Non-coercive Lyapunov functions for infinite-dimensional systems," *Journal of Differential Equations*, vol. 266, no. 11, pp. 7038–7072, 2019.
- [22] D. Astolfi and L. Praly, "Integral action in output feedback for multi-input multi-output nonlinear systems," *IEEE Trans. on Autom. Contr.*, vol. 62, no. 4, pp. 1559–1574, 2017.
- [23] R. Chill, D. Seifert, and Y. Tomilov, "Semi-uniform stability of operator semigroups and energy decay of damped waves," *Philosophical Trans. of the Royal Society A*, vol. 378, no. 2185, p. 20190614, 2020.