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The Domino problem is undecidable on every rhombus subshift

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Abstract

We extend the classical Domino problem to any tiling of rhombus-shaped tiles. For any subshift X of edge-to-edge rhombus tilings, such as the Penrose subshift, we prove that the associated X -Domino problem is Π_1^0 -hard and therefore undecidable. It is Π_1^0 -complete when the subshift X is given by a computable sequence of forbidden patterns.

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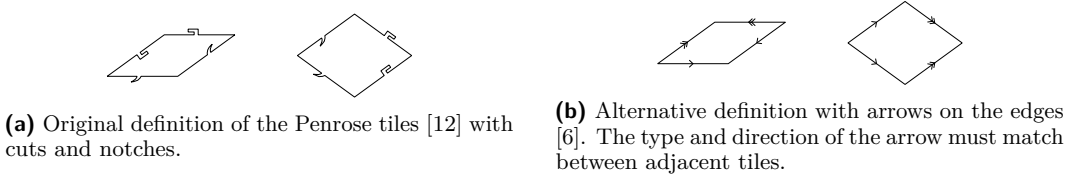
1 Introduction

Tilings come, roughly speaking, in two families. Geometrical tilings are coverings of a space, usually the euclidean plane, by geometrical tiles without overlap; the constraints come from the geometry of the tiles. A famous example of geometrical tilings is the Penrose tilings [12]. Symbolic tilings are colourings of a discrete structure, usually $\mathbb{Z}^d$ for some d , whose constraints are given by forbidden patterns.

Both families have received a lot of attention for their dynamical and combinatorial properties. The specificities of geometrical tilings are their symmetries and their links with mathematical cristallography [3]. Symbolic tilings, on the other hand, have more links to computability and decidability theory. The seminal example is the Domino problem: given a set of colours and a set of forbidden patterns, is there a colouring of $\mathbb{Z}^2$ that satisfies those constraints? The proof by Berger [5] that this problem is undecidable shaped the whole domain of research.

There has been much work to extend Domino problems to structures that extend the classical symbolic results on $\mathbb{Z}^2$. An active area of research considers Domino problem on groups in order to relate properties of the group and of the tiling spaces; see [1] for a survey. Other considered extensions are Domino problems in self-similar structures to understand the limit between dimension 1 and 2 [4] or Domino problems inside a $\mathbb{Z}^2$ -subshift to understand the effect of dynamical restrictions [2].

Coming back to geometrical tilings, complex examples such as the Penrose tilings were originally defined with jigsaw-type tiles with indentations on their edges (see Fig. 1). It can be restated as simple polygon tiles with symbols on their edges [6] with the condition that symbols must match. In essence these tilings are both symbolic and geometrical.



■ **Figure 1** Definitions of Penrose tiles: adding symbols on the edges of the tiles allow for simple rhombus shapes.

Similarly, given a set of shapes, we define symbolic tiles on those shapes by adding colours on the edges and study the induced symbolic-geometrical tilings. Let us consider the set of symbolic-geometrical tiles $\mathbf{T}$ of Fig. 2. Since the shapes are Penrose rhombuses, there are two natural questions regarding this tileset:

1. is there an infinite valid tiling of $\mathbb{R}^2$ with tiles in $\mathbf{T}$ up to translation?
2. is there an infinite valid tiling of $\mathbb{R}^2$ with tiles in $\mathbf{T}$ up to translation that projects to a geometrical Penrose tiling (that is, when removing the colours from the tiles)?

It is not hard to see that the first question is at least as hard as (and is in fact equivalent to) the classical Domino problem, which corresponds to the case where the input tiles are all the same rhombus. This motivates us to study the second question, where the geometrical (Penrose) subshift forces the use of the different rhombuses.

These are instances of the Domino problem in geometrical subshifts, which is the object of the present article. For any set of rhombuses $\mathcal{T}$ and a geometrical subshift X on these rhombuses, the X -Domino problem is defined as follows: given as input a set of tiles, that is, rhombuses from $\mathcal{T}$ with a colour on each edge, decide whether it is possible to tile the plane in such a way that:

1. tiles with a common edge have the same colour along the shared edge, and
2. the geometrical tiling (when colours are erased) is valid for X .

Our main result is the following:

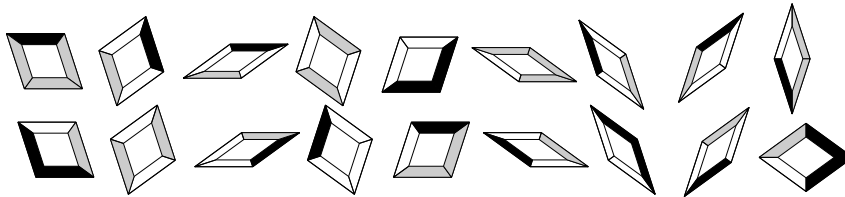
► **Theorem 1.** *Let X be a geometrical subshift given by a computable list of forbidden patterns. X -Domino is many-one equivalent to the classical Domino problem on $\mathbb{Z}^2$, that is, co-computably enumerable-complete, and thus undecidable.*

2 Definitions

2.1 Geometrical tiling spaces

► **Definition 2** (Shapes and patches). *We call shape a geometrical rhombus given as a pair of vectors $(\vec{u}, \vec{v})$ and a position p .*

We call shapeset a finite set of shapes considered up-to-translation, see Fig. 3a.



■ **Figure 2** An example of symbolic-geometrical tiles on Penrose rhombuses.

We call patch an edge-to-edge simply connected finite set of shapes, i.e., any two tiles are either disjoint, share a single common vertex or a full common edge, and there is no hole in the patch. We call support of a patch the union of its shapes. We call pattern a patch up to translation.

Note that shapes are not taken up to rotation.

► **Definition 3** (Tilings, full shift and subshifts). *Given a shapese $\mathcal{T}$, we call $\mathcal{T}$ -tiling an edge-to-edge covering of the euclidean plane without overlap by translates of the shapes in $\mathcal{T}$: see Fig. 3b.*

We call full shift on $\mathcal{T}$, denoted by $X_{\mathcal{T}}$, the set of all $\mathcal{T}$ -tilings.

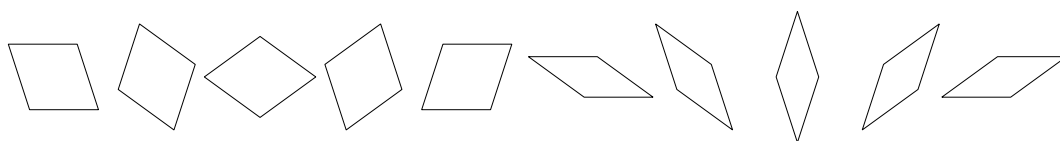
We call subshift of $X_{\mathcal{T}}$ any subset X of $X_{\mathcal{T}}$ that is invariant by translation and closed for the tiling topology [13].

Edge-to-edge rhombus tilings with finitely many shapes up to translation have *Finite Local Complexity* (FLC): that is, for any compact $C \subseteq \mathbb{R}^2$, there are finitely many patterns whose support is included in C . The FLC hypothesis appears a lot in the study of geometrical tilings and Delone sets; see [3]. In particular, FLC ensures that the tiling space shares most properties with standard $\mathbb{Z}^d$ tiling spaces, such as being compact for the usual tiling topology [13, 10].

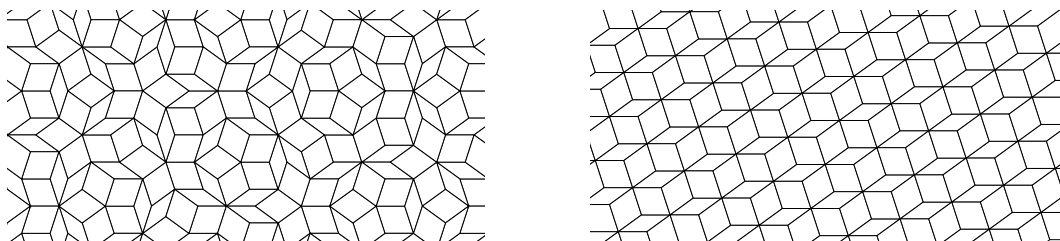
A subshift X can always be characterized by a countable (possibly infinite) set of forbidden patterns, that we denote as a sequence $\mathcal{F} := (f_n)_{n \in \mathbb{N}}$. In other words, X is the set of all tilings where no pattern in $\mathcal{F}$ appears. When $\mathcal{F}$ is computable, we say that the subshift X is *effective*.

We say that a pattern (or patch) has *minimal radius* r when its support contains a disk of radius r centered on a vertex of the pattern, and when removing any shape from the patch would break that property.

A sequence $\mathcal{F}$ of forbidden patterns being fixed, we call locally-allowed patterns $\mathcal{A}(X)$ the set of finite patterns where no forbidden pattern appears, and rank- r locally-allowed patterns $\mathcal{A}_r(X)$ the set of patterns of minimal radius r that do not contain any of the first r forbidden patterns. Note that when X is an FLC subshift $\mathcal{A}_r(X)$ is finite for all r : indeed, there exists a constant d (maximum diameter of the shapes) such that the support of any minimal radius r pattern is included in an $r + d$ disk.

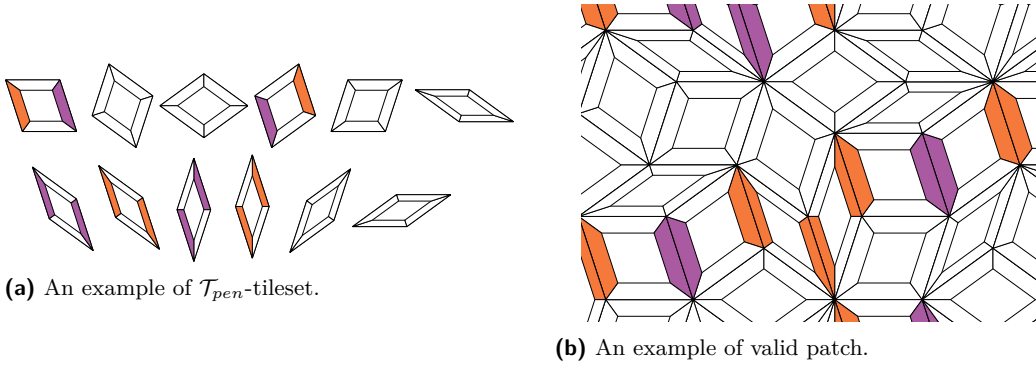


(a) An example of shapese: the set $\mathcal{T}_{pen}$ of Penrose rhombuses up to translation.



(b) Two examples of geometrical $\mathcal{T}_{pen}$ -tilings: on the left a geometrical Penrose tiling and on the right a "cube"-tiling.

■ **Figure 3** Shapes and geometrical tilings.



■ **Figure 4** Rhombus wang tiles.

Note that patterns in $\mathcal{A}(X)$ may not be globally allowed in X (appear in no infinite tiling in X). They may even appear in no tiling of the full shift $X_{\mathcal{T}}$ if they contain a geometrical impossibility. Such patterns are called *deceptions* [7].

The interest of locally allowed patterns is that, as seen below in Lemma 15, they are computable from the list of forbidden patterns, whereas it is not the case for globally allowed patterns.

2.2 Symbolic-geometrical tiling spaces

► **Definition 4** (Symbols, tiles and tilesets). *A tile is a shape endowed with a colour on each edge, as seen in Fig. 4(a).*

Formally, given a finite set C whose elements are called colours and a rhombus shape r , we call r -Wang tile or simply r -tile a quintuple (r, a_0, a_1, a_2, a_3) with $a_i \in C$. Formally, with $r = (\vec{u}, \vec{v}, p)$, the side $(p, p + \vec{u})$ has colour a_0 , the side $(p + \vec{u}, p + \vec{u} + \vec{v})$ has colour a_1 and so on.

Given a shapset $\mathcal{T}$, we call $\mathcal{T}$ -tile a r -tile for some $r \in \mathcal{T}$. We call $\mathcal{T}$ -tileset a finite set of $\mathcal{T}$ -tiles, considered up to translation, such that each shape has at least a tile.

► **Definition 5** (Colour erasing operator π). *We define the colour erasing operator π by:*

- *for any r -tile t , $\pi(t) := r$*
- *for a tiling x (or finite patch of tiles), $\pi(x) := \{\pi(t), t \in x\}$*
- *for a set of tilings X , $\pi(X) := \{\pi(x), x \in X\}$*

► **Definition 6** (Tiling). *Given a finite set of colours C and a $\mathcal{T}$ -tileset $\mathbf{T}$, we call $\mathbf{T}$ -tiling a tiling x such that $\pi(x) \in X_{\mathcal{T}}$ and such that any two tiles in x that share an edge have the same colour on their shared edge. See Fig. 4(b).*

We denote by $X_{\mathbf{T}}$ the subshift of all $\mathbf{T}$ -tilings.

A symbolic-geometrical subshift X is given by a set of shapes (geometrical constraints), a set of forbidden patterns (geometrical subshift) and colourings on the tiles (symbolic constraints). Even when the geometrical subshift is a full shift, geometrical and symbolic constraints can interact in interesting ways. For example, there is a choice of tiles on the Penrose rhombuses such that all valid tilings correspond to a geometrical Penrose tiling after erasing colours (in particular, no valid tiling use a single shape); see Appendix A.

The definitions of minimal radius r patterns and locally allowed patterns extend naturally to symbolic-geometrical tilings.

2.3 Computability and decidability

► **Definition 7.** A decision problem is a function $A : \text{dom}(A) \rightarrow \{0, 1\}$, where $\text{dom}(A)$ is called the input domain of A .

► **Definition 8.** A decision problem A is said to be decidable when there exists an algorithm (or Turing machine) that, given as input any $x \in \text{dom}(A)$, terminates and outputs $A(x)$.

A weaker notion of computability for decision problems is the following:

► **Definition 9** (co-computably enumerable, Π_1^0). A decision problem A is called co-computably enumerable, also known as co-recursively enumerable, when there exists a total computable function $f : \text{dom}(A) \times \mathbb{N} \rightarrow \{0, 1\}$ such that:

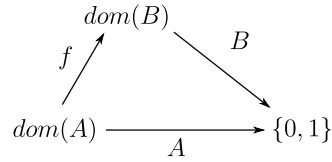
$$\forall x \in \text{dom}(A), \quad A(x) \Leftrightarrow \forall n \in \mathbb{N}, f(x, n)$$

Alternatively, a decision problem A is co-computably enumerable if there is an algorithm that, on input x , terminates if and only if $A(x)$ is false.

We denote by Π_1^0 the class of co-computably enumerable problems.

Π_1^0 is a class of the arithmetical hierarchy; see [9, 11].

► **Definition 10** (Many-one reductions). Given two decision problems A and B , we say that A many-one reduces to B , and write $A \leq_m B$, when there exists a total computable function $f : \text{dom}(A) \rightarrow \text{dom}(B)$ such that $A = B \circ f$.



■ **Figure 5** The diagram of a many one-reduction, $A \leq_m B$ when f is computable and the diagram commutes.

► **Definition 11** (Π_1^0 -hardness and Π_1^0 -completeness). A problem A is called Π_1^0 -hard if $B \leq_m A$ for any problem B in Π_1^0 .

A problem A is called Π_1^0 -complete when it is both in Π_1^0 and Π_1^0 -hard.

Notice that Π_1^0 -hard problems are undecidable. The canonical example of a Π_1^0 -complete problem is the co-halting problem, that is the problem of deciding whether a Turing Machine does not terminate in finite time.

Many-one reductions are a restrictive case of Turing reductions that are appropriate to study classes of decision problems such as Π_1^0 , as the following Lemma shows:

► **Lemma 12** (Π_1^0 -hardness). Given two problems A and B such that $A \leq_m B$,

- if B is Π_1^0 , then A is Π_1^0 .
- if A is Π_1^0 -hard, then B is Π_1^0 -hard.

2.4 Domino problems

In this paper, our goal is to prove the Π_1^0 -hardness of a generalisation of the classical Domino problem, which is known to be Π_1^0 -complete.

The classical Domino problem asks, given as input a finite set of Wang tiles, *i.e.*, square tiles with a colour on each edge, whether there exists an infinite valid tiling with these tiles: see Fig. 6.

► **Theorem 13** (Berger66 [5]). *The classical Domino problem is Π_1^0 -complete.*

Berger's paper provides a many-one reduction to the co-halting problem. We extend this classical problem to rhombus-shaped Wang tiles.

► **Definition 14** (X -Domino). *Given a subshift X on $\mathcal{T}$, the Domino problem on X is defined as:*

Input A finite set $\mathbf{T}$ of $\mathcal{T}$ -tiles

Output Is there a $\mathbf{T}$ -tiling $x \in X_{\mathbf{T}}$ such that $\pi(x) \in X$?

The classical Domino problem is $X_{\{\square\}}$ -Domino, that is, the domino problem on the full shift with a single shape (usually a square shape, but any single rhombus works). *Penrose-Domino* would be, given a set of tiles on Penrose rhombuses as in Fig. 4 looking for a Penrose tiling with matching edges.

3 Complexity of the Domino problem on rhombus subshifts

3.1 The Domino problem on effective rhombus subshifts is Π_1^0

► **Lemma 15.** *Let $\mathcal{T}$ be a finite set of rhombus shapes. The following problem is computable:*

Input an integer n , a finite list of forbidden patterns $\mathcal{F}$, and a tileset $\mathbf{T}$ of $\mathcal{T}$ -tiles

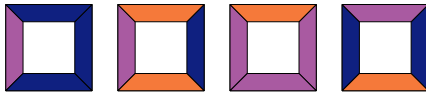
Output the list of all minimal radius n patterns of $\mathbf{T}$ -tiles that avoid all patterns in $\mathcal{F}$

Proof. The algorithm is as follows:

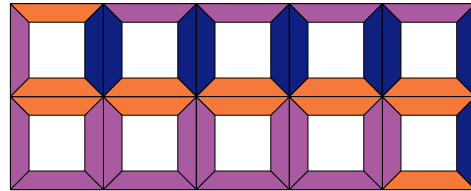
1. By combinatorial exploration, try all possibilities and list all $\mathbf{T}$ -patterns with minimal radius n .
2. Eliminate all listed patterns x such that the colour-erased pattern $\pi(x)$ contains some pattern in $\mathcal{F}$.
3. Output the remaining patterns.

In Point 1, remember that the set of edge-to-edge tilings on a fixed finite set of rhombus tiles have finite local complexity, so this process terminates in finite time. ◀

► **Proposition 16** (X -Domino $\in \Pi_1^0$). *For any shapese set $\mathcal{T}$ and any subshift X on $\mathcal{T}$ defined by a computable enumeration of forbidden patterns $\mathcal{F}$, X -Domino is co-computably enumerable.*



(a) A tileset of 4 tiles.



(b) A valid patch with these tiles.

■ **Figure 6** Tiles and tilings: a patch or tiling is called *valid* when any two adjacent tiles have the same colour on their shared edge.

Note that, if X is not effective, then X -Domino is Π_1^0 when provided with an enumeration of $\mathcal{F}$ as oracle.

This is essentially the same proof as for the classical Domino problem.

Proof. The following problem, called disk-tiling- X , is decidable:

Input A finite set $\mathbf{T}$ of $\mathcal{T}$ -tiles and an integer n

Output Is there a valid (finite) patch x with tiles in $\mathbf{T}$ such that $\pi(x)$ is a rank n locally allowed pattern of X , *i.e.*, $\pi(x) \in \mathcal{A}_n(X)$?

Simply compute the first n forbidden patterns $\mathcal{F}_n$, which is possible because X is effective, then apply Lemma 15 on $(n, \mathcal{F}_n, \mathbf{T})$. For any input tileset $\mathbf{T}$, both the geometrical subshift X and the symbolic full shift $X_{\mathbf{T}}$ have Finite Local Complexity so they are compact [13], and

$$X\text{-Domino}(\mathbf{T}) \Leftrightarrow \forall n \in \mathbb{N}, \text{disk-tiling-}X(\mathbf{T}, n).$$

Remark that $\text{disk-tiling-}X(\mathbf{T}, n) = 1$ when there exists a rank n locally allowed pattern, that is, a pattern of minimal radius n that avoids the first n forbidden patterns $\mathcal{F}$ of X with tileset $\mathbf{T}$. If $\forall n \in \mathbb{N}$, $\text{disk-tiling-}X(\mathbf{T}, n)$, there exists a sequence $(p_n)_{n \in \mathbb{N}}$ with $\pi(p_n) \in \mathcal{A}_n(X)$. Since the radius of the patches tends to infinity, by compactity there exists a limit tiling x to which a subsequence converges. Now remark that $\pi(x) \in X$ because it avoids all forbidden patterns in $\mathcal{F}$. Indeed, for any k , the k th forbidden pattern does not appear in any $\pi(p_n)$ for $n \geq k$, so it does not appear in $\pi(x)$. Since $\text{disk-tiling-}X$ is computable, we indeed have $\text{domino-}X \in \Pi_1^0$. $\blacktriangleleft$

If X is a full shift on some shapeset $\mathcal{T}$, it is easy to see that the corresponding Domino problem is Π_1^0 -hard by reduction to the classical version: given a finite set $\mathbf{T}$ of square Wang tiles, choose an arbitrary shape in $\mathcal{T}$ and colour it like $\mathbf{T}$, and colour every other shape with four new different *fresh* colours (so that any valid tiling may only use the first shape). In the rest of the paper, we extend this idea to work on an arbitrary subshift X .

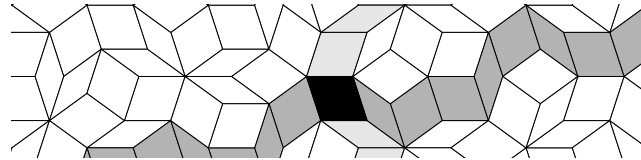
3.2 The Domino problem on shape uniformly recurrent subshifts is Π_1^0 -hard

The key concept is the concept of *chains* of rhombuses [8] (also called *ribbons* [14]).

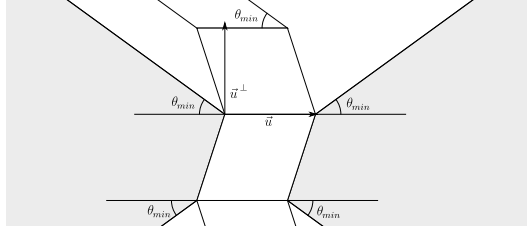
► **Definition 17** (Chains of rhombuses). *We call chain of rhombuses a bi-infinite sequence of rhombuses that share an edge direction; see Figure 7.*

A chain of rhombuses is characterized by its normal vector $\vec{v}$: the direction of the common edge.

► **Lemma 18** (Occurrences of a rhombus, [8]). *In an edge-to-edge rhombus tiling, rhombuses of edge directions $\vec{u}$ and $\vec{v}$ correspond exactly to the intersections of two chains of normal vectors $\vec{u}$ and $\vec{v}$. Moreover, two chains can cross at most once. See Fig. 7.*



■ **Figure 7** In an edge-to-edge rhombus tiling, a rhombus (in black) is at the intersection of two chains of rhombuses (in shades of grey).



■ **Figure 8** A rhombus r and the chain c . The chain c is outside the grey cones left and right of the rhombus r .

As a consequence, two chains of same normal vector cannot cross, otherwise there would be an impossible flat rhombus at the intersection. Such chains are therefore called parallel.

► **Lemma 19** (Uniform monotonicity). *Given a finite shapese set $\mathcal{T}$, let $\theta_{\min}$ be the smallest angle in a rhombus of $\mathcal{T}$. For any $\mathcal{T}$ -tiling x , for any rhombus r appearing in a chain c of normal vector $\vec{u}$, the chain c is outside the cone centered in r and of half-angle $\theta_{\min}$ along $\vec{u}$; see Fig. 8.*

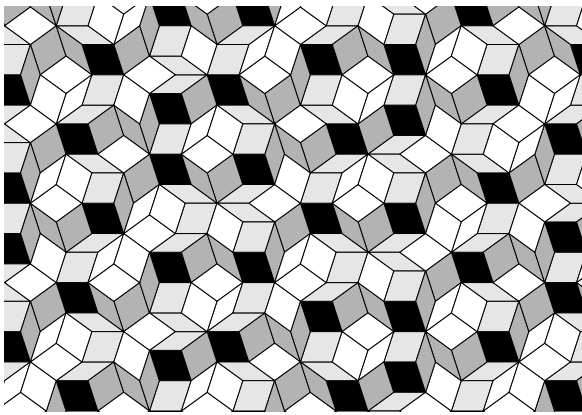
Overall, an edge-to-edge rhombus tiling can be decomposed as d sets of parallel chains of rhombuses where d is the number of edge directions. Given an edge direction $\vec{u}$, the $\vec{u}$ chains can be indexed by either $\mathbb{Z}$, $\mathbb{N}$, $-\mathbb{N}$ or a finite integer interval in such a way that, starting from any position and moving along $\vec{u}$ one crosses the $\vec{u}$ chains in increasing order.

► **Definition 20** (Shape uniform recurrence). *Given a rhombus shape r and a tiling x we say that r is uniformly recurrent in x , or that x is r -uniformly recurrent, if r appears in any disk of radius R in x for some R .*

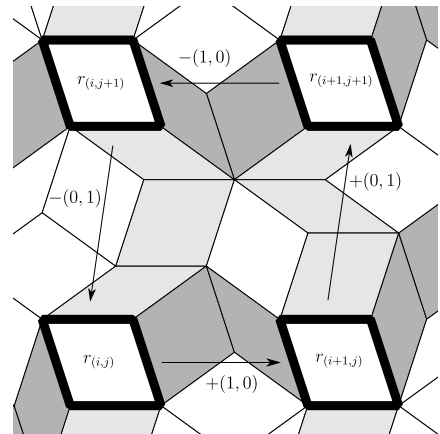
A tiling x is called shape uniformly recurrent when it is r -uniformly-recurrent for every shape r that appears in x .

A subshift X is called shape uniformly recurrent when, for every shape r that appears in some tiling $y \in X$, every tiling $x \in X$ is r -uniformly-recurrent.

Note that this is much weaker than the usual uniform recurrence, which holds for every pattern instead of a single shape.



(a) Occurrences of r (in black) in a Penrose tiling. The chains that link the occurrences of r are highlighted in medium grey and light grey.



(b) Indexing the occurrences of r .

■ **Figure 9** Occurrences of a uniformly recurrent rhombus r in a tiling.

► **Lemma 21.** *Let x be a edge-to-edge rhombus tiling and r a shape that is uniformly recurrent in x . The occurrences of r in x can be indexed by coordinates in $\mathbb{Z}^2$ such that two consecutive occurrences of r along a chain have adjacent $\mathbb{Z}^2$ coordinates; see Fig. 9b.*

Proof. Let x be a tiling in X and r a uniformly recurrent shape in x . Denote by $\vec{u}$ and $\vec{v}$ the two edge directions of the rhombus r . As explained in Lemma 18, the occurrences of r are exactly the intersections of a $\vec{u}$ chain and a $\vec{v}$ chain.

As seen above, the $\vec{u}$ chains can be indexed by either $\mathbb{Z}$, $\mathbb{N}$, $-\mathbb{N}$ or a finite integer interval. Since r is uniformly recurrent, only the case of $\mathbb{Z}$ is possible. Indeed by uniform recurrence of r , there exists R such that any disk of radius R in x contains an occurrence of r , and in particular intersects a $\vec{u}$ chain. So, starting from any position, one finds arbitrarily many $\vec{u}$ chains in both the $\vec{u}$ and $-\vec{u}$ directions. The same holds for $\vec{v}$.

Denote $r_{(i,j)}$ the occurrence of r at the intersection of the i th $\vec{u}$ chain and the j th $\vec{v}$ chain. By definition of the indexing of chains, we see on Fig. 9b that starting from occurrence $r_{(i,j)}$ and going along a $\vec{u}$ chain, the next occurrence of r is $r_{(i+1,j)}$. ◀

► **Proposition 22.** *Let X be a non-empty subshift of edge-to-edge rhombus tilings that is shape uniformly recurrent. X -Domino is Π_1^0 -hard.*

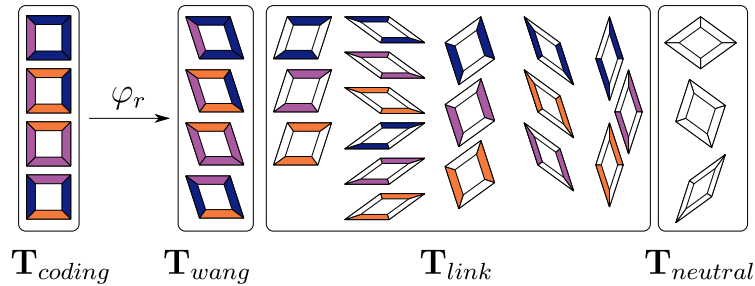
Proof. We proceed by many-one reduction to the classical Domino problem which is known to be Π_1^0 -complete [5].

Let $\mathcal{T}$ be the shapeset on which X is defined, and choose an arbitrary shape $r \in \mathcal{T}$. Define the reduction φ_r as follows. We are given as input a finite set of square tiles $\mathbf{T}_{wang}$ on the set of colours S_{wang} . Define a tileset $\mathbf{T}_{rhombus} = \varphi_r(\mathbf{T}_{wang})$ on colours $S_{rhombus} := S_{wang} \cup \{\text{blank}\}$, where blank is a fresh colour, as $\mathbf{T}_{rhombus} := \mathbf{T}_{coding} \cup \mathbf{T}_{link} \cup \mathbf{T}_{neutral}$, with:

1. $\mathbf{T}_{coding}$ is a copy of $\mathbf{T}_{wang}$ on the shape r . Formally, for each tile $(a_0, a_1, a_2, a_3) \in \mathbf{T}_{wang}$, $\mathbf{T}_{coding}$ contains a tile (r, a_0, a_1, a_2, a_3) . See Fig. 10.
2. $\mathbf{T}_{link}$ is a set of rhombus tiles that link occurrences of r and transmit the colours. Formally, for each shape $r' \neq r$ such that r and r' share exactly one edge direction, say $\vec{u}$, and for each colour $a \in S_{wang}$ $\mathbf{T}_{link}$ contains a tile of shape r' with colour a on both edges along $\vec{u}$ edges and colour blank on other edges. See Fig. 10.
3. $\mathbf{T}_{neutral}$ completes the tileset with blank colour. Formally, for each shape r' that shares no edge direction with r , $\mathbf{T}_{neutral}$ contains one tile of shape r' with blank colour on each edge. See Fig. 10.

We prove that $\mathbf{T}_{wang}$ admits a valid tiling of $\mathbb{Z}^2$ if and only if $\varphi_r(\mathbf{T}_{wang})$ admits a valid tiling x such that $\pi(x) \in X$.

Assume that $\mathbf{T}_{wang}$ admits a valid x_{wang} tiling of $\mathbb{Z}^2$. Let us pick $x \in X$. By hypothesis, r is uniformly recurrent in X . We colour x as follows (see Fig. 11):



■ **Figure 10** The reduction φ_r .

Coding tiles: index occurrences of r as $\{r_{(i,j)} \mid (i,j) \in \mathbb{Z}^2\}$ as explained in Lemma 21. Copy the tiles from x_{wang} to the occurrences of r , i.e., if at position (i,j) in x_{wang} there is a (a_0, a_1, a_2, a_3) tile, then colour $r_{(i,j)}$ as the coding tile $(r_{(i,j)}, a_0, a_1, a_2, a_3)$.

Linker tiles: by construction, the north colour of $r_{(i,j)}$ is equal to the south colour of $r_{(i,j+1)}$ so linkers of that colour can be put along that portion of chain, and similarly for east-west links.

Neutral tiles: remaining tiles share no edge direction with r , so they must be neutral tiles. The converse also holds: if there exists a valid tiling x on $\varphi_r(\mathbf{T}_{wang})$, since r is uniformly recurrent its occurrences can be indexed by $\mathbb{Z}^2$ (Lemma 21). By construction of the linker tiles, colours of the coding tiles $r_{(i,j)}$ correspond to a valid $\mathbf{T}_{wang}$ tiling in $\mathbb{Z}^2$. ◀

3.3 The Domino problem on any rhombus subshift is Π_1^0 -hard

In this section, for any subshift X , we build a subshift X' that has a uniformly recurrent shape and such that $X'\text{-Domino} \preceq_m X\text{-Domino}$.

► **Definition 23** (Restriction). *Given any rhombus subshift X on shapset $\mathcal{T}$, and a subset $\mathcal{T}' \subset \mathcal{T}$, define $\rho_{\mathcal{T}'}(X)$ as the restriction of X to the configurations that contain only shapes in $\mathcal{T}'$, that is, $\rho_{\mathcal{T}'}(X) := X \cap X_{\mathcal{T}'}$.*

► **Lemma 24.** *Let X be a nonempty subshift on a shapset $\mathcal{T}$. For any $r \in \mathcal{T}$, either r is uniformly recurrent in every $x \in X$, or the restriction $\rho_{\mathcal{T} \setminus \{r\}}(X)$ is nonempty.*

Proof. This proof, once again, comes from finite local complexity and therefore compactness of edge-to-edge rhombus tilings.

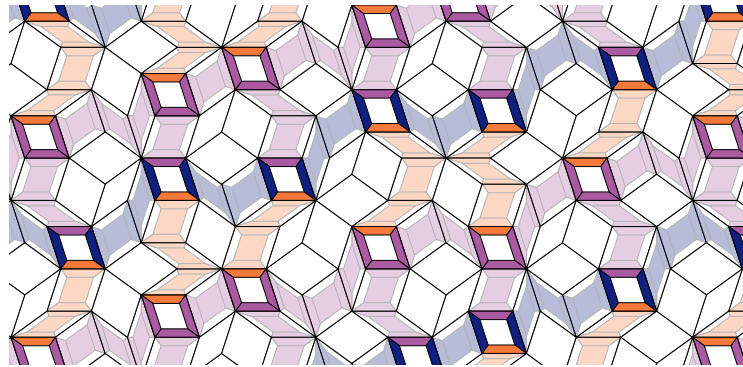
If r is not uniformly recurrent in every $x \in X$, by definition there exist arbitrarily large patterns in X that do not contain r . By compactness there exists an infinite tiling in X containing no r . Hence $\rho_{\mathcal{T} \setminus \{r\}}(X)$ is non-empty. ◀

► **Lemma 25.** *Let X be a nonempty subshift on a finite set of rhombuses $\mathcal{T}$. There is a subset $\mathcal{T}' \subseteq \mathcal{T}$ such that $\rho_{\mathcal{T}'}(X) \neq \emptyset$, and there exists $r \in \mathcal{T}'$ which is uniformly recurrent in every configuration $x \in \rho_{\mathcal{T}'}(X)$.*

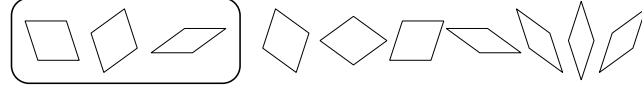
Proof. We prove this by induction on the number of shapes, i.e., on $\#\mathcal{T}$.

If $\mathcal{T} = \{r\}$, take $\mathcal{T}' = \mathcal{T}$ and r is uniformly recurrent in any tiling $x \in X$.

Assume the result holds for at most n shapes and $\#\mathcal{T} = n + 1$. Pick some $r \in \mathcal{T}$: by Lemma 24 either r is uniformly recurrent in all tilings $x \in X$ or $\rho_{\mathcal{T} \setminus \{r\}}(X)$ is non-empty.



■ **Figure 11** A valid tiling x with tileset $\varphi_r(\mathbf{T}_{wang})$. Link tiles and neutral tiles have been visually modified to improve readability.



■ **Figure 12** A set of shapes $\mathcal{T}$, and in the left box the subset $\mathcal{T}'$.

In the first case, we conclude with $\mathcal{T}' = \mathcal{T}$. In the second case, we apply the induction hypothesis on $\rho_{\mathcal{T} \setminus \{r\}}(X)$, obtaining some $\mathcal{T}' \subseteq \mathcal{T} \setminus \{r\}$ and a tile $r' \in \mathcal{T}'$ such that r' is uniformly recurrent in $\rho_{\mathcal{T}'}(X)$. ◀

► **Theorem 26.** *Let X be a non-empty subshift of edge-to-edge rhombus tilings. X -Domino is Π_1^0 -hard.*

This result, with Proposition 16, implies that X -Domino is Π_1^0 -complete for effective subshifts X .

Proof. By Lemma 25 there exists a subset of the shapset $\mathcal{T}' \subset \mathcal{T}$ such that the subshift $X' := \rho_{\mathcal{T}'}(X)$ has a uniformly recurrent tile t .

By Proposition 22, X' -Domino is Π_1^0 -hard. We now show that X' -Domino $\preceq_m$ X -Domino so that X -Domino is also Π_1^0 -hard.

The many-one reduction $\varphi_{\mathcal{T}'}$ from X' -Domino to X -Domino is defined as follows: given a $\mathcal{T}'$ -wang tiling $\mathbf{T}'$ we define $\varphi_{\mathcal{T}'}(\mathbf{T}') := \mathbf{T}' \cup \mathbf{T}_{\text{fresh}}$ where $\mathbf{T}_{\text{fresh}}$ contains a tile with a *fresh* colour on each edge for each shape in $\mathcal{T} \setminus \mathcal{T}'$; see Figure 13.

Remark that there is no reason that choosing the suitable $\mathcal{T}'$ given $\mathcal{T}$ and X (as an enumeration of forbidden patterns) can be done in a computable manner. It only matters that, for a fixed $\mathcal{T}'$, $\varphi_{\mathcal{T}'}$ is computable, which is easily seen with the above definition.

This reduction $\varphi_{\mathcal{T}'}$ is well defined as we have

$$X'\text{-Domino}(\mathbf{T}') \Leftrightarrow X\text{-Domino}(\varphi_{\mathcal{T}'}(\mathbf{T}')).$$

The implication holds because $X' \subset X$ and $\mathbf{T}' \subset \varphi_{\mathcal{T}'}(\mathbf{T}')$, so a $\mathbf{T}'$ -tiling that projects by erasing colours in X' is also a $\varphi_{\mathcal{T}'}(\mathbf{T}')$ -tiling that projects in X .

The converse holds because the tiles in $\mathbf{T}_{\text{fresh}}$ cannot appear in $\varphi_{\mathcal{T}'}(\mathbf{T}')$ -tilings because they have a fresh colour on each side so no tile can be placed next to it. So a $\varphi_{\mathcal{T}'}(\mathbf{T}')$ -tiling x is actually a $\mathbf{T}'$ -tiling. Since x contains only tiles in $\mathbf{T}'$, $\pi(x)$ contains only shapes in $\mathcal{T}'$ so $\pi(x) \in \rho_{\mathcal{T}'}(X) = X'$. ◀

► **Remark 27 (Fresh colours).** Remark that, if we remove the condition that a tiling on shapset $\mathcal{T}$ must contain at least a tile for each shape $r \in \mathcal{T}$, we can take $\varphi(\mathbf{T}') := \mathbf{T}'$. In essence the fact of taking fresh colours simulates that.

► **Remark 28 (Restriction).** We could consider, instead of taking X' as a restriction $\rho_{\mathcal{T}'}(X)$, taking an arbitrary minimal subshift $X'' \subset X$ (where all patterns are uniformly recurrent). However, there is no clear reduction from X'' -Domino to X -Domino.



■ **Figure 13** The reduction from a $\mathcal{T}'$ -tiling to a $\mathcal{T}$ -tiling : in the left box the original $\mathcal{T}'$ -tiling, on the right the tiles with fresh colours $\mathbf{T}_{\text{fresh}}$.

► **Remark 29** (Beyond Π_1^0). If X is not given by a computable enumeration of forbidden patterns $\mathcal{F}$, but $\mathcal{F}$ is given as an oracle, we remarked earlier that X -Domino is Π_1^0 relative to the oracle $\mathcal{F}$. However, we have only proved that X -Domino is Π_1^0 -hard, but not relative to $\mathcal{F}$. In particular we have not proved that X -Domino is Π_1^0 -complete relative to the oracle $\mathcal{F}$.

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A Rhombus Wang tiles to define Penrose tilings

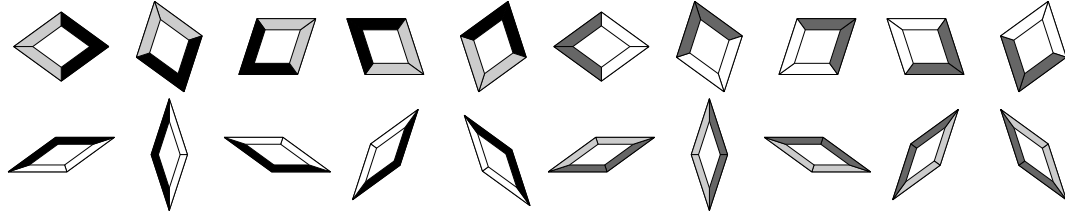
Rhombus Penrose tilings were originally defined as jigsaw-type tiles with indentations on their edges [12] which then reformulated as rhombus tiles with arrows as labels on their edges [6]. With the arrow labels, two adjacent tiles must have the same type and direction of arrow on their shared edge. In both the original and modern definitions, the tiles are defined up to isometry.

These arrow labels are not strictly speaking the same as putting a single colour on each edge as for Wang tiles. However, if we define the tileset up to translation and not up to isometry we can translate these arrow labels as single colours, see Fig. 14.



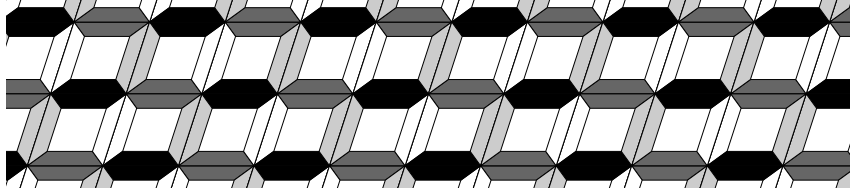
■ **Figure 14** Given an edge direction, each type of arrow is translated into two colours. By rotating this figure by multiples of $\frac{2\pi}{5}$, one obtains the arrow-colour translation for all directions.

This process gives us a tileset shown in Fig. 15 that defines the subshift of Penrose tilings. This tileset contains 20 tiles up to translation, which can be considered as less elegant as the 2 tiles up to isometry of the classical definition. Note however that if we use Wang-type



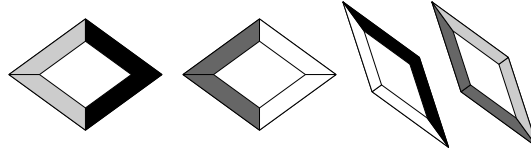
■ **Figure 15** A rhombus Wang tileset definition of Penrose tilings.

colours on the edges, there exists no aperiodic up-to-isometry tileset, indeed a single tile up-to-isometry always tiles the plane in a periodic way (see Fig. 16). This implies that if we want to define Penrose tilings with Wang rhombus tiles, we cannot use an up-to-isometry or up-to-rotation tileset.



■ **Figure 16** Given a rhombus Wang tile with different colours on all four edges, there exists a periodic tiling of the plane with only this tile up to translation and rotation (of angle π).

However, if we consider the Penrose tileset of Fig. 15 up to $\frac{2\pi}{5}$ rotations we have a reduced tileset with 4 tiles shown in Figure 17. Note that, in this definition, tiles can only be rotated by multiples of $\frac{2\pi}{5}$ so the counterexample of Fig. 16 does not apply as a rotation of angle π is necessary for this counterexample.



■ **Figure 17** A rhombus Wang tileset up to translations and $\frac{2\pi}{5}$ rotations that defines Penrose tilings.