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Error estimates for Gaussian beams at a fold caustic

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Abstract

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1 Introduction

We consider the particular case of a fold caustic in two dimensions generated by the index of refraction $n(x,y)^2 = 1-x$. To be precise, we consider the out-going solution for the Helmholtz equation with a compactly supported source that concentrates on $x = 0$,

$$\Delta u + k^2(1-x)u = ik\delta(x)f(y),$$

where we take

$$f(y) = \frac{1}{\xi_0}A_0(y)e^{i\eta_0 y}, \qquad \xi_0 = \sqrt{1-\eta_0^2}.$$

This represents an incoming plane wave in the direction θ with amplitude envelope $A_0(y)$, where $\eta_0 = \cos\theta$.

picture of solution

background on Gaussian beams and error estimates

2 Exact solution

We let $\hat{u}(x,\eta)$ be the k-scaled Fourier transform in y of the exact solution, i.e.

$$\hat{u}(x,\eta) = \sqrt{\frac{k}{2\pi}}\int e^{-i\eta k y}u(x,y)dy.$$

For $x > 0$ we can derive an exact expression written in terms of x, η and k using Airy functions,

$$\hat{u}(x,\eta) = \bar{\alpha}\frac{\mathrm{Ai}(k^{\frac{2}{3}}(x+\eta^2-1))}{\mathrm{Ai}(\alpha k^{\frac{2}{3}}(1-\eta^2))}\hat{u}_+(0,\eta). \quad (1)$$

In this expression, one notices that the denominator never vanish because all the roots of the Airy function are on th e negative real axis.

what is α and $\hat{u}_+$?

3 Gaussian beam approximation

The approximation is made up of a superposition of Gaussian beams,

$$u_{GB}(x,y) = \sqrt{\frac{k}{2\pi}}\int v(x,y;z)dz.$$

where v is the Gaussian beam starting at $y = z$,

$$v(\boldsymbol{x};z) = A(s;z)\exp\Big(ik\big[S(s;z) + (\boldsymbol{x}-\gamma(s;z))\cdot\boldsymbol{p}(s;z) + \frac{1}{2}(\boldsymbol{x}-\gamma(s;z))^T M(s;z)(\boldsymbol{x}-\gamma(s;z))\big]\Big),$$

where $\boldsymbol{x} = (x,y)$. The beam parameters are the central ray position γ, the beam direction $\boldsymbol{p}$ the shape M, the initial phase S and the amplitude A. They all satisfy known ODEs in s with initial data determined by z. In this particular case the ODEs can be solved exactly,

$$\begin{aligned}
\gamma(s;z) &= (x(s;z), y(s;z) = (2s\xi_0 - s^2,\ z + 2s\eta_0)\\
\boldsymbol{p}(s;z) &= (\xi(s;z),\eta(s;z))) = (\xi_0 - s, \eta_0),\\
S(s;z) &= \eta_0 z + 2s - 2s^2\xi_0 + \frac{2}{3}s^3,\\
M(s;z) &= (I + 2sM_0)^{-1}M_0,\\
A(s;z) &= \frac{\tilde{A}_0(z)}{\sqrt{1+2si-s^2\beta}}.
\end{aligned}$$

what is M_0, β and $\tilde{A}_0(z)$?

For the evaluation of the beam we connect s to a particular y-coordinate by

$$s(y;z) = \frac{y-z}{2\eta_0}.$$

Then we can derive the following expression for the k-scaled Fourier transform of u_{GB},

$$\hat{u}_{GB}(x,\eta) = 2\eta_0\hat{\tilde{A}}_0(\eta-\eta_0)\sqrt{\frac{k}{2\pi}}e^{ik(-2\eta_0\xi_0(\eta-\eta_0)+2\xi_0^3/3)} \times \int \frac{e^{ik\phi(x,\theta,\eta-\eta_0)}}{\sqrt{1+2(\xi_0+\theta)i-(\xi_0+\theta)^2\beta}}d\theta, \quad (2)$$

where

$$\phi(x,\theta,\eta) = -\frac{1}{3}\theta^3+\theta(\xi_0^2-x-2\eta_0\eta)+\frac{1}{2}m_{11}(\xi_0+\theta)(x-\xi_0^2+\theta^2)^2$$

and m_{11}?

4 Error estimate

We want to compute the max error at a fixed x and use the error in the k-scaled Fourier transform to estimate this,

$$\sup_y |u_{GB}(x,y)-u(x,y)| \leq \sqrt{\frac{k}{2\pi}}\int |\hat{u}_{GB}(x,\eta)-\hat{u}(x,\eta)|d\eta.$$

Away from the caustic where $0 \leq x < \xi_0^2$ we obtain the expected $O(k^{-1})$ error by using standard stationary phase estimates of the integral in the expression for $\hat{u}_{GB}$ in (2) and asymptotic estimates for large arguments of Airy functions in the expression for $\hat{u}$. in (1).

The more interesting case is at the caustic where $x = \xi_0^2$. Then the asymptotic estimates above fail and a more precise analysis is needed. We use the Malgrange preparation theorem to change variables in (2). The critical values of the phase ϕ for η small are needed. Observing that $\partial^3_{\theta^3}(\xi_0^2,0,0) < 0$, let $\theta(x)$ solve (locally) $\partial^2_{\theta^2}(x,\theta(x),\eta) = 0$. One has $\theta(x) = (x-\xi_0^2)k(x)$, the (complex) critical points are $\theta_\pm(x,\eta) = \theta(x) \pm \delta_\pm\sqrt{\partial_\theta\phi}$ for $\Re\partial_\theta\phi > 0$, hence, for η small, two critical values arise:

$$\begin{aligned}\phi_\pm(x,\eta) &= \phi_0(x) \pm \tfrac{2}{3}(\tilde{\rho}(x,\eta))^{\frac{3}{2}}, \text{ with}\\ \tilde{\rho}(x,\eta) &= \partial_\theta\phi(x,\theta(x),\eta)r(x,\eta)\\ &= (\xi_0^2 - x + 2\eta\eta_0 + O((x-\xi_0^2)^2)r.\end{aligned}$$

Deduce then

$$\hat{u}_{GB}(x,\eta) = 2\eta_0\hat{\tilde{A}}_0(\eta-\eta_0)\sqrt{2\pi}k^{\frac{1}{6}}\times \quad (3)$$

$$[\sigma_0 Ai + ik^{-\frac{1}{3}}\sigma_1 Ai'](-k^{\frac{2}{3}}\tilde{\rho}(x,\eta)) \quad (4)$$

$$\times\, e^{ik(-2\eta_0\xi_0(\eta-\eta_0)+2\xi_0^3/3+\phi_0(x,\eta))},$$

Comparing (3) and (1), using the principal term of $\tilde{\rho}$, $\phi_0, \sigma_0, \sigma_1$, the max error is of the order $O(k^{-5/6})$, which is demonstrated in numerical computations. Since the size of the solution itself grows as $O(k^{1/6})$ at the caustic, the relative error is then still $O(k^{-1})$.

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