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► **To cite this version:**

Olivier Lafitte, Olof Runborg. Error estimates for Gaussian beams at a fold caustic. Waves 2019, TU Wien, Aug 2019, Wien, Austria. hal-04004714

HAL Id: hal-04004714

<https://hal.science/hal-04004714>

Submitted on 25 Feb 2023

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Error estimates for Gaussian beams at a fold caustic

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Abstract

Keywords: high frequency approximations, Gaussian beams, caustics

1 Introduction

We consider the particular case of a fold caustic in two dimensions generated by the index of refraction $n(x, y)^2 = 1 - x$. To be precise, we consider the out-going solution for the Helmholtz equation with a compactly supported source that concentrates on $x = 0$,

$$\Delta u + k^2(1 - x)u = ik\delta(x)f(y),$$

where we take

$$f(y) = \frac{1}{\xi_0} A_0(y) e^{i\eta_0 y}, \quad \xi_0 = \sqrt{1 - \eta_0^2}.$$

This represents an incoming plane wave in the direction θ with amplitude envelope $A_0(y)$, where $\eta_0 = \cos \theta$.

picture of solution

background on Gaussian beams and error estimates

2 Exact solution

We let $\hat{u}(x, \eta)$ be the k -scaled Fourier transform in y of the exact solution, i.e.

$$\hat{u}(x, \eta) = \sqrt{\frac{k}{2\pi}} \int e^{-i\eta ky} u(x, y) dy.$$

For $x > 0$ we can derive an exact expression written in terms of x, η and k using Airy functions,

$$\hat{u}(x, \eta) = \bar{\alpha} \frac{\text{Ai}(k^{\frac{2}{3}}(x + \eta^2 - 1))}{\text{Ai}(\alpha k^{\frac{2}{3}}(1 - \eta^2))} \hat{u}_+(0, \eta). \quad (1)$$

In this expression, one notices that the denominator never vanishes because all the roots of the Airy function are on the negative real axis.

what is α and \hat{u}_+ ?

3 Gaussian beam approximation

The approximation is made up of a superposition of Gaussian beams,

$$u_{GB}(x, y) = \sqrt{\frac{k}{2\pi}} \int v(x, y; z) dz.$$

where v is the Gaussian beam starting at $y = z$,

$$v(\mathbf{x}; z) = A(s; z) \exp\left(ik\left[S(s; z) + (\mathbf{x} - \gamma(s; z)) \cdot \mathbf{p}(s; z) + \frac{1}{2}(\mathbf{x} - \gamma(s; z))^T M(s; z)(\mathbf{x} - \gamma(s; z))\right]\right),$$

where $\mathbf{x} = (x, y)$. The beam parameters are the central ray position γ , the beam direction \mathbf{p} the shape M , the initial phase S and the amplitude A . They all satisfy known ODEs in s with initial data determined by z . In this particular case the ODEs can be solved exactly,

$$\gamma(s; z) = (x(s; z), y(s; z)) = (2s\xi_0 - s^2, z + 2s\eta_0)$$

$$\mathbf{p}(s; z) = (\xi(s; z), \eta(s; z)) = (\xi_0 - s, \eta_0),$$

$$S(s; z) = \eta_0 z + 2s - 2s^2\xi_0 + \frac{2}{3}s^3,$$

$$M(s; z) = (I + 2sM_0)^{-1}M_0,$$

$$A(s; z) = \frac{\tilde{A}_0(z)}{\sqrt{1 + 2si - s^2\beta}}.$$

what is M_0, β and $\tilde{A}_0(z)$?

For the evaluation of the beam we connect s to a particular y -coordinate by

$$s(y; z) = \frac{y - z}{2\eta_0}.$$

Then we can derive the following expression for the k -scaled Fourier transform of u_{GB} ,

$$\hat{u}_{GB}(x, \eta) = 2\eta_0 \hat{A}_0(\eta - \eta_0) \sqrt{\frac{k}{2\pi}} e^{ik(-2\eta_0\xi_0(\eta - \eta_0) + 2\xi_0^3/3)} \quad (2)$$

$$\times \int \frac{e^{ik\phi(x, \theta, \eta - \eta_0)}}{\sqrt{1 + 2(\xi_0 + \theta)i - (\xi_0 + \theta)^2\beta}} d\theta,$$

where

$$\phi(x, \theta, \eta) = -\frac{1}{3}\theta^3 + \theta(\xi_0^2 - x - 2\eta_0\eta) + \frac{1}{2}m_{11}(\xi_0 + \theta)(x - \xi_0^2 + \theta^2)$$

and m_{11} ?

4 Error estimate

We want to compute the max error at a fixed x and use the error in the k -scaled Fourier transform to estimate this,

$$\sup_y |u_{GB}(x, y) - u(x, y)| \leq \sqrt{\frac{k}{2\pi}} \int |\hat{u}_{GB}(x, \eta) - \hat{u}(x, \eta)| d\eta.$$

Away from the caustic where $0 \leq x < \xi_0^2$ we obtain the expected $O(k^{-1})$ error by using standard stationary phase estimates of the integral in the expression for \hat{u}_{GB} in (2) and asymptotic estimates for large arguments of Airy functions in the expression for \hat{u} . in (1).

The more interesting case is at the caustic where $x = \xi_0^2$. Then the asymptotic estimates above fail and a more precise analysis is needed. We use the Malgrange preparation theorem to change variables in (2). The critical values of the phase ϕ for η small are needed. Observing that $\partial_{\theta^3}^3(\xi_0^2, 0, 0) < 0$, let $\theta(x)$ solve (locally) $\partial_{\theta^2}^2(x, \theta(x), \eta) = 0$. One has $\theta(x) = (x - \xi_0^2)k(x)$, the (complex) critical points are $\theta_{\pm}(x, \eta) = \theta(x) \pm \delta_{\pm}\sqrt{\partial_{\theta}\phi}$ for $\Re\partial_{\theta}\phi > 0$, hence, for η small, two critical values arise:

$$\begin{aligned} \phi_{\pm}(x, \eta) &= \phi_0(x) \pm \frac{2}{3}(\tilde{\rho}(x, \eta))^{\frac{3}{2}}, \text{ with} \\ \tilde{\rho}(x, \eta) &= \partial_{\theta}\phi(x, \theta(x), \eta)r(x, \eta) \\ &= (\xi_0^2 - x + 2\eta_0\eta + O((x - \xi_0^2)^2))r. \end{aligned}$$

Deduce then

$$\hat{u}_{GB}(x, \eta) = 2\eta_0 \hat{A}_0(\eta - \eta_0) \sqrt{2\pi} k^{\frac{1}{6}} \times \quad (3)$$

$$\begin{aligned} &[\sigma_0 Ai + ik^{-\frac{1}{3}}\sigma_1 Ai'](-k^{\frac{2}{3}}\tilde{\rho}(x, \eta)) \quad (4) \\ &\times e^{ik(-2\eta_0\xi_0(\eta - \eta_0) + 2\xi_0^3/3 + \phi_0(x, \eta))}, \end{aligned}$$

Comparing (3) and (1), using the principal term of $\tilde{\rho}$, $\phi_0, \sigma_0, \sigma_1$, the max error is of the order $O(k^{-5/6})$, which is demonstrated in numerical computations. Since the size of the solution itself grows as $O(k^{1/6})$ at the caustic, the relative error is then still $O(k^{-1})$.

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