

High frequency analysis of the Dirichlet to Neumann operator for an infinite cylinder (and an infinite elliptic cylinder) coated with dielectric material

Olivier Lafitte

IRL CRM-CNRS, Université de Montréal, Canada and LAGA, Université Paris 13,
Sorbonne Paris Cité, Villetaneuse, France

Waves 2022, ENSTAParistech, Palaiseau, July 29th, 2022

Abstract result: the Dirichlet to Neumann operator

Case of a coated cylinder

The case of an elliptic coated cylinder

Problem studied

Define $\Omega = \mathcal{O} \times \mathbf{R}$, \mathcal{O} bounded convex openset in \mathbf{R}^2 . Let \mathcal{O}^e bounded convex openset in \mathbf{R}^2 such that $\mathcal{O} \subset \mathcal{O}^e$ and there exists $\alpha_0 > 0$ such that $\text{dist}(\mathcal{O}, (\mathcal{O}^e)^c) \geq \alpha_0$.

Assume that \mathcal{O} is a perfectly conducting body, and that \mathcal{O}_e is filled with a dielectric material of dielectric constants ϵ, μ satisfying $\Im \epsilon \mu < 0$, $\Re \epsilon \mu > 0$, fixed (independent of ω).

Our aim is to replace the Helmholtz equation in $\mathcal{O} \cup (\mathcal{O}_e - \mathcal{O})$ by a boundary condition on $\partial \mathcal{O}_e$.

Abstract classical result

Theorem

The problem

$$\begin{cases} (\Delta + \omega^2 \epsilon \mu)u = 0, & (\mathcal{O}^e - \mathcal{O}) \times \mathbf{R}, \\ u|_{\partial \mathcal{O} \times \mathbf{R}} = 0, \\ u|_{\partial \mathcal{O}^e \times \mathbf{R}} = u_0 \end{cases} \quad (1)$$

has a unique solution $U(u_0)$ in $H^1((\mathcal{O}^e - \mathcal{O}) \times \mathbf{R})$ for each $u_0 \in H^{\frac{1}{2}}(\partial \mathcal{O}^e \times \mathbf{R})$.

The application $u_0 \rightarrow \partial_n U(u_0)$, ∂_n being the normal derivative on $\partial \mathcal{O}^e \times \mathbf{R}$ is well defined from $H^{\frac{1}{2}}(\partial \mathcal{O}^e \times \mathbf{R})$ to $H^{-\frac{1}{2}}(\partial \mathcal{O}^e \times \mathbf{R})$. It is called the Dirichlet to Neumann operator (DtN).

Known expressions of the Dirichlet to Neumann operator

- If $(\mathcal{O}^e - \mathcal{O}) \times \mathbf{R}$ is replaced by an infinite slab, the DtN operator is a Fourier multiplier, of symbol

$$i\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}, \Re\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} > 0.$$

As $\epsilon\mu \notin \mathbf{R}$, $\Re\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2 - k_3^2} > 0$ never vanish.

- Microlocal approx. (Waves 1993) of the DtN (impedance) operator¹ is a Fourier multiplier of order 1, of symbol²

$$i\omega \frac{\nu(x^0, \eta)}{\tan \omega l \nu(x^0, \eta)}, \text{ where}$$

$$\nu(x^0, \eta) = \sqrt{\epsilon\mu - g_{11}\eta_1^2 - 2g_{12}\eta_1\eta_2 - g_{22}\eta_2^2}$$

¹ Ω bounded C^∞ convex in \mathbf{R}^3 + layer l of dielectric material

²the unit vector $(\eta_1, \eta_2) \in T_{(x_1^0, x_2^0)}^*(\partial\Omega)$, the matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} (x_1^0, x_2^0)$

is the matrix of the metric induced by the Euclidean metric on the boundary at (x_1^0, x_2^0) , under $lw \simeq c_0 > 0$.

Known expressions of the Dirichlet to Neumann operator

- If $(\mathcal{O}^e - \mathcal{O}) \times \mathbf{R}$ is replaced by an infinite slab, the DtN operator is a Fourier multiplier, of symbol

$$i\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}, \Re\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} > 0.$$

As $\epsilon\mu \notin \mathbf{R}$, $\Re\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2 - k_3^2} > 0$ never vanish.

- Microlocal approx. (Waves 1993) of the DtN (impedance) operator¹ is a Fourier multiplier of order 1, of symbol²

$$i\omega \frac{\nu(x^0, \eta)}{\tan \omega l \nu(x^0, \eta)}, \text{ where}$$

$$\nu(x^0, \eta) = \sqrt{\epsilon\mu - g_{11}\eta_1^2 - 2g_{12}\eta_1\eta_2 - g_{22}\eta_2^2}$$

¹ Ω bounded C^∞ convex in \mathbf{R}^3 + layer l of dielectric material

²the unit vector $(\eta_1, \eta_2) \in T_{(x_1^0, x_2^0)}^*(\partial\Omega)$, the matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} (x_1^0, x_2^0)$ is the matrix of the metric induced by the Euclidean metric on the boundary at (x_1^0, x_2^0) , under $l\omega \simeq c_0 > 0$.

Known expressions of the Dirichlet to Neumann operator

- If $(\mathcal{O}^e - \mathcal{O}) \times \mathbf{R}$ is replaced by an infinite slab, the DtN operator is a Fourier multiplier, of symbol

$$i\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}, \Re\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} > 0.$$

As $\epsilon\mu \notin \mathbf{R}$, $\Re\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2 - k_3^2} > 0$ never vanish.

- Microlocal approx. (Waves 1993) of the DtN (impedance) operator¹ is a Fourier multiplier of order 1, of symbol²

$$i\omega \frac{\nu(x^0, \eta)}{\tan \omega l \nu(x^0, \eta)}, \text{ where}$$

$$\nu(x^0, \eta) = \sqrt{\epsilon\mu - g_{11}\eta_1^2 - 2g_{12}\eta_1\eta_2 - g_{22}\eta_2^2}$$

¹ Ω bounded C^∞ convex in \mathbf{R}^3 + layer l of dielectric material

²the unit vector $(\eta_1, \eta_2) \in T_{(x_1^0, x_2^0)}^*(\partial\Omega)$, the matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} (x_1^0, x_2^0)$

is the matrix of the metric induced by the Euclidean metric on the boundary at (x_1^0, x_2^0) , under $l\omega \simeq c_0 > 0$.

The infinite cylinder (classical approach)

Assume $\mathcal{O} = B(0, r_0)$, $\mathcal{O}^e = B(0, R)$, $R > r_0$. Fourier transform in z (k). Equation

$$\begin{cases} (\Delta + \epsilon\mu\omega^2 - k^2)\hat{u}(\cdot, k) = 0, r_0 < r < R, \\ \hat{u}(\cdot, k)|_{r=r_0} = 0, \hat{u}(\cdot, k)|_{r=R} = \hat{u}_0(\cdot, k). \end{cases}$$

Fourier series expansion:

$$\hat{u}(x, y, k) = \sum_{n=0}^{+\infty} e^{in\theta} v_n(r, k),$$

where v_n solves the Bessel equation:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_n}{dr} \right) + \left(\epsilon\mu\omega^2 - k^2 - \frac{n^2}{r^2} \right) v_n = 0.$$

Dirichlet condition $v_n(r_0) = 0$.

Necessarily, there exists A and B (depending of ω, k) such that³

$$v_n(r) = AJ_n(k_3r) + BY_n(k_3r), k_3 = \sqrt{\epsilon\mu\omega^2 - k^2}, \Im k_3 > 0.$$

$$\Rightarrow \exists \tilde{A}, v_n(r) = \tilde{A}(J_n(k_3r)Y_n(k_3r_0) - Y_n(k_3r)J_n(k_3r_0)).$$

$$\tilde{A}(J_n(k_3R)Y_n(k_3r_0) - Y_n(k_3R)J_n(k_3r_0)) = \hat{u}_0(n, k):$$

$$v_n(r) = \frac{J_n(k_3r)Y_n(k_3r_0) - Y_n(k_3r)J_n(k_3r_0)}{J_n(k_3R)Y_n(k_3r_0) - Y_n(k_3R)J_n(k_3r_0)} \hat{u}_0(n, k). \quad (2)$$

Proposition

There exists a constant C , independent of (k, ω) , such that for all n, k , $|v_n(r)| \leq C|\hat{u}_0(n, k)|$.

³ $\max_{[r_0, R]} |AJ_n(k_3r) + BY_n(k_3r)| \leq C(k^2 + \omega^2)^t$, C independent of (k, ω)

High frequency analysis of the Dirichlet to Neumann multiplier: the usual approach

Expression:

$$\begin{aligned} v'_n(R) &= k_3 \frac{J'_n(k_3 R) Y_n(k_3 r_0) - Y'_n(k_3 R) J_n(k_3 r_0)}{J_n(k_3 R) Y_n(k_3 r_0) - Y_n(k_3 R) J_n(k_3 r_0)} \hat{u}_0(n, k), \\ v_n(R) &= \frac{J_n(k_3 R) Y_n(k_3 r_0) - Y_n(k_3 R) J_n(k_3 r_0)}{J_n(k_3 R) Y_n(k_3 r_0) - Y_n(k_3 R) J_n(k_3 r_0)} \hat{u}_0(n, k), \end{aligned} \quad \text{implies}$$

$$v'_n(R) = k_3 \frac{J'_n(k_3 R) Y_n(k_3 r_0) - Y'_n(k_3 R) J_n(k_3 r_0)}{J_n(k_3 R) Y_n(k_3 r_0) - Y_n(k_3 R) J_n(k_3 r_0)} v_n(R). \quad (3)$$

DtN: $u \in H^{\frac{1}{2}}(\partial\mathcal{O}^e)$, hence $u \in L^2(0, 2\pi)$, $u(\theta) = \sum_{n=-\infty}^{+\infty} u_n e^{in\theta}$,
 $\sum (1+n^2)^{\frac{1}{4}} |u_n|^2 < +\infty \Rightarrow$

$$DtN(u) = \sum_{n=-\infty}^{+\infty} DtN_n u_n e^{in\theta}.$$

Expression for each mode in the usual asymptotics

Use $H_n^{(1)}(k_3 r) = J_n(k_3 r) + iY_n(k_3 r)$. Analysis with

$$H_n^{(1)}(k_3 r) = \frac{2}{\pi k_3 r} [P^*(n, k_3 r) + iQ^*(n, k_3 r)] e^{i(k_3 r - (\frac{1}{2}n + \frac{\pi}{4}))}$$

$$(H^{(1)})'_n(k_3 r) = \frac{2}{\pi k_3 r} [iR^*(n, k_3 r) - S^*(n, k_3 r)] e^{i(k_3 r - (\frac{1}{2}n + \frac{\pi}{4}))}$$

$$P^* = 1 + O((k_3 r)^{-2}), Q^* = \frac{4n^2 - 1}{8k_3 r} (1 + O((k_3 r)^{-2})),$$

$$R^* = 1 + O((k_3 r)^{-2}), S^* = \frac{4n^2 + 3}{8k_3 r} (1 + O((k_3 r)^{-2})).$$

$$\text{Estimate: } \frac{v'_n(R)}{v_n(R)} = ik_3 (1 + i\frac{4}{8k_3 R}) (1 + O((k_3 R)^{-1})).$$

It is obtained with $|e^{i(k_3 R - (\frac{1}{2}n + \frac{\pi}{4}) - k_3 r_0 + (\frac{1}{2}n + \frac{\pi}{4}))}| = e^{-\Im k_3 (R - r_0)}$
(selecting one term).

$$DtN(n): ik_3 - \frac{1}{2R} + \dots$$

Unsatisfactory because the microlocal analysis would lead to

$$i\sqrt{\epsilon\mu\omega^2 - k^2 - \frac{n^2}{R^2}}. \quad (4)$$

(approximation $i(k_3 - \frac{n^2}{2k_3 R^2})$). How come?

Expression for each mode in the usual asymptotics

Use $H_n^{(1)}(k_3 r) = J_n(k_3 r) + iY_n(k_3 r)$. Analysis with

$$H_n^{(1)}(k_3 r) = \frac{2}{\pi k_3 r} [P^*(n, k_3 r) + iQ^*(n, k_3 r)] e^{i(k_3 r - (\frac{1}{2}n + \frac{\pi}{4}))}$$

$$(H^{(1)})'_n(k_3 r) = \frac{2}{\pi k_3 r} [iR^*(n, k_3 r) - S^*(n, k_3 r)] e^{i(k_3 r - (\frac{1}{2}n + \frac{\pi}{4}))}$$

$$P^* = 1 + O((k_3 r)^{-2}), Q^* = \frac{4n^2 - 1}{8k_3 r} (1 + O((k_3 r)^{-2})),$$

$$R^* = 1 + O((k_3 r)^{-2}), S^* = \frac{4n^2 + 3}{8k_3 r} (1 + O((k_3 r)^{-2})).$$

$$\text{Estimate: } \frac{v'_n(R)}{v_n(R)} = ik_3 (1 + i\frac{4}{8k_3 R}) (1 + O((k_3 R)^{-1})).$$

It is obtained with $|e^{i(k_3 R - (\frac{1}{2}n + \frac{\pi}{4}) - k_3 r_0 + (\frac{1}{2}n + \frac{\pi}{4}))}| = e^{-\Im k_3 (R - r_0)}$
(selecting one term).

$$DtN(n): ik_3 - \frac{1}{2R} + \dots$$

Unsatisfactory because the microlocal analysis would lead to

$$i\sqrt{\epsilon\mu\omega^2 - k^2 - \frac{n^2}{R^2}}. \quad (4)$$

(approximation $i(k_3 - \frac{n^2}{2k_3 R^2})$). How come?

Expression for each mode in the usual asymptotics

Use $H_n^{(1)}(k_3 r) = J_n(k_3 r) + iY_n(k_3 r)$. Analysis with

$$H_n^{(1)}(k_3 r) = \frac{2}{\pi k_3 r} [P^*(n, k_3 r) + iQ^*(n, k_3 r)] e^{i(k_3 r - (\frac{1}{2}n + \frac{\pi}{4}))}$$

$$(H^{(1)})'_n(k_3 r) = \frac{2}{\pi k_3 r} [iR^*(n, k_3 r) - S^*(n, k_3 r)] e^{i(k_3 r - (\frac{1}{2}n + \frac{\pi}{4}))}$$

$$P^* = 1 + O((k_3 r)^{-2}), Q^* = \frac{4n^2 - 1}{8k_3 r} (1 + O((k_3 r)^{-2})),$$

$$R^* = 1 + O((k_3 r)^{-2}), S^* = \frac{4n^2 + 3}{8k_3 r} (1 + O((k_3 r)^{-2})).$$

$$\text{Estimate: } \frac{v'_n(R)}{v_n(R)} = ik_3 (1 + i\frac{4}{8k_3 R}) (1 + O((k_3 R)^{-1})).$$

It is obtained with $|e^{i(k_3 R - (\frac{1}{2}n + \frac{\pi}{4}) - k_3 r_0 + (\frac{1}{2}n + \frac{\pi}{4}))}| = e^{-\Im k_3 (R - r_0)}$
(selecting one term).

$$DtN(n): ik_3 - \frac{1}{2R} + \dots$$

Unsatisfactory because the microlocal analysis would lead to

$$i\sqrt{\epsilon\mu\omega^2 - k^2 - \frac{n^2}{R^2}}. \quad (4)$$

(approximation $i(k_3 - \frac{n^2}{2k_3 R^2})$). How come?

Bessel function toolbox (I)

Assumption: $\frac{n}{R}$ is a high frequency parameter.

First case: 'hyperbolic' regime ($k_3 R > n$ if $\epsilon\mu$ real, $\omega^2 \epsilon\mu - k^2 > \frac{n^2}{R^2}$). introduce β such that $k_3 R = \frac{n}{\cos \beta}$.

Hyperbolic regime ($\Psi(n, \beta) = n(\tan \beta - \beta)$):

$$\begin{cases} H_n^{(1)}\left(\frac{n}{\cos \beta}\right) = \sqrt{\frac{2}{\pi n \tan \beta}} (L - iM) e^{i\Psi(n, \beta)}, \\ (H_n^{(1)})'\left(\frac{n}{\cos \beta}\right) = \sqrt{\frac{\sin 2\beta}{\pi n}} (iN - O) e^{i\Psi(n, \beta)}, \\ L = 1 + O(n^{-2}), n(\tan \beta)M = \frac{1}{8} + \frac{5}{24} \tan^{-2} \beta + O(n^{-2}), \\ N = 1 + O(n^{-2}), n(\tan \beta)O = \frac{3}{8} + \frac{7}{24} \tan^{-2} \beta + O(n^{-2}). \end{cases}$$

Bessel function toolbox (II)

Second expression 'elliptic' regime $k_3 R > n \epsilon \mu$ real. Define z through $k_3 r = nz$.

$$H_n^{(1)}(nz) = 2e^{-i\frac{\pi}{3}} \left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} \left[n^{-\frac{1}{3}} Ai(jn^{\frac{2}{3}}\zeta)A - n^{-\frac{5}{3}} j\zeta^{-\frac{1}{2}} B Ai'(jn^{\frac{2}{3}}\zeta)\right],$$

$$A = 1 + O(n^{-2}), \quad B = U_1 + O(n^{-2}), \quad \zeta = \left(\frac{3}{2} \int_z^1 \frac{\sqrt{1-t^2}}{t} dt\right)^{\frac{2}{3}}$$

Equivalent:

$$Ai(jn^{\frac{2}{3}}\zeta) \simeq (jn^{\frac{2}{3}}\zeta)^{-\frac{1}{4}} e^{-\frac{2}{3}(jn^{\frac{2}{3}}\zeta)^{\frac{3}{2}}} = j^{-\frac{1}{4}} n^{-\frac{1}{6}} \zeta^{-\frac{1}{4}} e^n \int_z^1 \frac{\sqrt{1-t^2}}{t} dt.$$

Eikonal equation (hyperbolic regime)

$$k_3 = \frac{d}{dr} \left(\frac{n}{\cos \beta} \right) = \frac{n}{\cos^2 \beta} \sin \beta \frac{d\beta}{dr} \Rightarrow \frac{d\Psi}{dr} = n(1 + \tan^2 \beta - 1) \frac{d\beta}{dr}:$$

$$\left(\frac{d\Psi}{dr} \right)^2 = n^2 \tan^4 \beta \left(\frac{k_3 \cos^2 \beta}{n \sin \beta} \right)^2 = k_3^2 \sin^2 \beta = k_3^2 \left(1 - \frac{n^2}{k_3^2 r^2} \right)$$

$$\left(\frac{d\Psi}{dr} \right)^2 = \epsilon \mu \omega^2 - k^2 - \frac{n^2}{r^2}. \quad (5)$$

One then recovers the complete eikonal equation with this phase depending of β .

In the elliptic regime ($nz = k_3 r$):

$$\tilde{\Psi}(z) = -\frac{2}{3} in \zeta^{\frac{3}{2}} \rightarrow \frac{d\tilde{\Psi}}{dr} = -in \frac{\sqrt{1-z^2}}{z} \frac{dz}{dr} \rightarrow \left(\frac{d\tilde{\Psi}}{dr} \right)^2 = \left(1 - \frac{1}{z^2} \right) \left(n \frac{dz}{dr} \right)^2.$$

Eikonal equation (hyperbolic regime)

$$k_3 = \frac{d}{dr} \left(\frac{n}{\cos \beta} \right) = \frac{n}{\cos^2 \beta} \sin \beta \frac{d\beta}{dr} \Rightarrow \frac{d\Psi}{dr} = n(1 + \tan^2 \beta - 1) \frac{d\beta}{dr}:$$

$$\left(\frac{d\Psi}{dr} \right)^2 = n^2 \tan^4 \beta \left(\frac{k_3 \cos^2 \beta}{n \sin \beta} \right)^2 = k_3^2 \sin^2 \beta = k_3^2 \left(1 - \frac{n^2}{k_3^2 r^2} \right)$$

$$\left(\frac{d\Psi}{dr} \right)^2 = \epsilon \mu \omega^2 - k^2 - \frac{n^2}{r^2}. \quad (5)$$

One then recovers the complete eikonal equation with this phase depending of β .

In the elliptic regime ($nz = k_3 r$):

$$\tilde{\Psi}(z) = -\frac{2}{3} in \zeta^{\frac{3}{2}} \rightarrow \frac{d\tilde{\Psi}}{dr} = -in \frac{\sqrt{1-z^2}}{z} \frac{dz}{dr} \rightarrow \left(\frac{d\tilde{\Psi}}{dr} \right)^2 = \left(1 - \frac{1}{z^2} \right) \left(n \frac{dz}{dr} \right)^2.$$

Eikonal equation (hyperbolic regime)

$$k_3 = \frac{d}{dr} \left(\frac{n}{\cos \beta} \right) = \frac{n}{\cos^2 \beta} \sin \beta \frac{d\beta}{dr} \Rightarrow \frac{d\Psi}{dr} = n(1 + \tan^2 \beta - 1) \frac{d\beta}{dr}:$$

$$\left(\frac{d\Psi}{dr} \right)^2 = n^2 \tan^4 \beta \left(\frac{k_3 \cos^2 \beta}{n \sin \beta} \right)^2 = k_3^2 \sin^2 \beta = k_3^2 \left(1 - \frac{n^2}{k_3^2 r^2} \right)$$

$$\left(\frac{d\Psi}{dr} \right)^2 = \epsilon \mu \omega^2 - k^2 - \frac{n^2}{r^2}. \quad (5)$$

One then recovers the complete eikonal equation with this phase depending of β .

In the elliptic regime ($nz = k_3 r$):

$$\tilde{\Psi}(z) = -\frac{2}{3} in \zeta^{\frac{3}{2}} \rightarrow \frac{d\tilde{\Psi}}{dr} = -in \frac{\sqrt{1-z^2}}{z} \frac{dz}{dr} \rightarrow \left(\frac{d\tilde{\Psi}}{dr} \right)^2 = \left(1 - \frac{1}{z^2} \right) \left(n \frac{dz}{dr} \right)^2.$$

DtN

Consider only one term (between $H^{(1)}$ and $H^{(2)}$), use the expansion

$$\begin{aligned} k_3 \frac{\sqrt{\frac{\sin 2\beta}{\pi n}} (iN - O)}{\sqrt{\frac{2}{\pi n \tan \beta}} (L - iM)} &= ik_3 \sqrt{\frac{2 \sin^2 \beta}{2}} \frac{N + iO}{L - iM} = i \frac{d\Psi}{dr} \frac{N + iO}{L - iM} \\ &= ik_3 \sin \beta \left(1 + i \frac{1}{n \tan \beta} \left(\frac{4}{8} + \tan^{-2} \beta \frac{5+7}{24} \right) \right) \end{aligned}$$

$$DtN_n = i \sqrt{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} - \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} + \dots \quad (6)$$

In the elliptic regime, an interesting quantity is

$q_* := n^{\frac{1}{3}} \zeta_*^{\frac{1}{2}} \frac{Ai}{Ai'}(n^{\frac{2}{3}} \zeta_*)$. It satisfies $q_* \simeq 1$. The quantity DtN_n writes

$$-k_3 \frac{2}{z_*} \sqrt{\frac{1-z_*^2}{4\zeta_*}} \frac{Ai'}{Ai}(n^{\frac{2}{3}} \zeta_*) \frac{D_* + C_*(q_* n)^{-1}}{A_* + B_* q_* n^{-1}}, \text{ that is}$$

$$DtN_n = \sqrt{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \dots \quad (7)$$

DtN

Consider only one term (between $H^{(1)}$ and $H^{(2)}$), use the expansion

$$k_3 \frac{\sqrt{\frac{\sin 2\beta}{\pi n}} (iN - O)}{\sqrt{\frac{2}{\pi n \tan \beta}} (L - iM)} = ik_3 \sqrt{\frac{2 \sin^2 \beta}{2}} \frac{N + iO}{L - iM} = i \frac{d\Psi}{dr} \frac{N + iO}{L - iM}$$

$$= ik_3 \sin \beta \left(1 + i \frac{1}{n \tan \beta} \left(\frac{4}{8} + \tan^{-2} \beta \frac{5+7}{24} \right) \right)$$

$$DtN_n = i \sqrt{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} - \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} + \dots \quad (6)$$

In the elliptic regime, an interesting quantity is

$q_* := n^{\frac{1}{3}} \zeta_*^{\frac{1}{2}} \frac{Ai}{Ai'}(n^{\frac{2}{3}} \zeta_*)$. It satisfies $q_* \simeq 1$. The quantity DtN_n writes

$$-k_3 \frac{2}{z_*} \sqrt{\frac{1-z_*^2}{4\zeta_*}} \frac{Ai'}{Ai}(n^{\frac{2}{3}} \zeta_*) \frac{D_* + C_*(q_* n)^{-1}}{A_* + B_* q_* n^{-1}}, \text{ that is}$$

$$DtN_n = \sqrt{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \dots \quad (7)$$

DtN

Consider only one term (between $H^{(1)}$ and $H^{(2)}$), use the expansion

$$k_3 \frac{\sqrt{\frac{\sin 2\beta}{\pi n}} (iN - O)}{\sqrt{\frac{2}{\pi n \tan \beta}} (L - iM)} = ik_3 \sqrt{\frac{2 \sin^2 \beta}{2}} \frac{N + iO}{L - iM} = i \frac{d\Psi}{dr} \frac{N + iO}{L - iM}$$

$$= ik_3 \sin \beta \left(1 + i \frac{1}{n \tan \beta} \left(\frac{4}{8} + \tan^{-2} \beta \frac{5+7}{24} \right) \right)$$

$$DtN_n = i \sqrt{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} - \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} + \dots \quad (6)$$

In the elliptic regime, an interesting quantity is

$q_* := n^{\frac{1}{3}} \zeta_*^{\frac{1}{2}} \frac{Ai}{Ai'}(n^{\frac{2}{3}} \zeta_*)$. It satisfies $q_* \simeq 1$. The quantity DtN_n writes

$$-k_3 \frac{2}{z_*} \sqrt{\frac{1-z_*^2}{4\zeta_*}} \frac{Ai'}{Ai}(n^{\frac{2}{3}} \zeta_*) \frac{D_* + C_*(q_* n)^{-1}}{A_* + B_* q_* n^{-1}}, \text{ that is}$$

$$DtN_n = \sqrt{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \dots \quad (7)$$

DtN

Consider only one term (between $H^{(1)}$ and $H^{(2)}$), use the expansion

$$k_3 \frac{\sqrt{\frac{\sin 2\beta}{\pi n}} (iN - O)}{\sqrt{\frac{2}{\pi n \tan \beta}} (L - iM)} = ik_3 \sqrt{\frac{2 \sin^2 \beta}{2}} \frac{N + iO}{L - iM} = i \frac{d\Psi}{dr} \frac{N + iO}{L - iM}$$

$$= ik_3 \sin \beta \left(1 + i \frac{1}{n \tan \beta} \left(\frac{4}{8} + \tan^{-2} \beta \frac{5+7}{24} \right) \right)$$

$$DtN_n = i \sqrt{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} - \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{\epsilon \mu \omega^2 - k^2 - \frac{n^2}{R^2}} + \dots \quad (6)$$

In the elliptic regime, an interesting quantity is

$q_* := n^{\frac{1}{3}} \zeta_*^{\frac{1}{2}} \frac{Ai}{Ai'}(n^{\frac{2}{3}} \zeta_*)$. It satisfies $q_* \simeq 1$. The quantity DtN_n writes

$$-k_3 \frac{2}{z_*} \sqrt{\frac{1-z_*^2}{4\zeta_*}} \frac{Ai'}{Ai}(n^{\frac{2}{3}} \zeta_*) \frac{D_* + C_*(q_* n)^{-1}}{A_* + B_* q_* n^{-1}}, \text{ that is}$$

$$DtN_n = \sqrt{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \frac{1}{2R} \frac{\epsilon \mu \omega^2 - k^2}{k^2 + \frac{n^2}{R^2} - \epsilon \mu \omega^2} + \dots \quad (7)$$

Helmholtz equation in elliptic coordinates

$$\mathcal{O} = \{\rho(\cosh u \cos v, \sinh u \sin v), u \in [0, u_0], v \in [0, 2\pi)\},$$

$$\mathcal{O}^e = \{\rho(\cosh u \cos v, \sinh u \sin v), u \in [0, u_1], v \in [0, 2\pi)\}$$

Helmholtz operator

$$\Delta = \frac{1}{\rho^2(\cosh^2 u - \cos^2 v)} (\partial_{u^2}^2 + \partial_{v^2}^2) + \frac{\partial^2}{\partial z^2}.$$

Separation of variables $u(x, y, z) = e^{ikz} F(u)G(v)$

$$F''(u) + \left(\frac{k_3^2}{2}\rho^2 \cosh^2 u - a\right)F = 0, u_0 \leq u \leq u_1,$$

(Modified Mathieu)

$$G''(v) - \left(\frac{k_3^2}{2}\rho^2 \cos^2 v - a\right)G = 0, 0 \leq v \leq 2\pi.$$

(Mathieu)

Bloch theory

Periodic solutions of the Mathieu equation \Rightarrow Floquet modes.

Yields:

$a_n(k_3\rho)$ (even solutions), $b_n(k_3\rho)$ (odd solutions).

Obtained through the fact that, for any solution U of Mathieu equation, there exist $\nu(a, k_3\rho)$ and $P_{a, k_3\rho}$ periodic such that $U(v) = e^{\pm i\nu(a, k_3\rho)v} P_{a, k_3\rho}(\pm v)$.

The sequence of values $a_n(k_3\rho)$ solves $\nu(a, k_3\rho) = n$ (and the opposite sign for b), even in the complex case.

Solutions denoted by $ce_n(v)$ and $se_n(v)$. Generalization of cosine and sine.

Quantization relation:

$$\int_0^\pi \sqrt{a - \frac{k_3^2 \rho^2}{2} \cos 2v} dv + O(h^2) = \pi h n.$$

Bloch theory

Periodic solutions of the Mathieu equation \Rightarrow Floquet modes.

Yields:

$a_n(k_3\rho)$ (even solutions), $b_n(k_3\rho)$ (odd solutions).

Obtained through the fact that, for any solution U of Mathieu equation, there exist $\nu(a, k_3\rho)$ and $P_{a, k_3\rho}$ periodic such that $U(v) = e^{\pm i\nu(a, k_3\rho)v} P_{a, k_3\rho}(\pm v)$.

The sequence of values $a_n(k_3\rho)$ solves $\nu(a, k_3\rho) = n$ (and the opposite sign for b), even in the complex case.

Solutions denoted by $ce_n(v)$ and $se_n(v)$. Generalization of cosine and sine.

Quantization relation:

$$\int_0^\pi \sqrt{a - \frac{k_3^2 \rho^2}{2} \cos 2v} dv + O(h^2) = \pi h n.$$

Bloch theory

Periodic solutions of the Mathieu equation \Rightarrow Floquet modes.

Yields:

$a_n(k_3\rho)$ (even solutions), $b_n(k_3\rho)$ (odd solutions).

Obtained through the fact that, for any solution U of Mathieu equation, there exist $\nu(a, k_3\rho)$ and $P_{a, k_3\rho}$ periodic such that $U(v) = e^{\pm i\nu(a, k_3\rho)v} P_{a, k_3\rho}(\pm v)$.

The sequence of values $a_n(k_3\rho)$ solves $\nu(a, k_3\rho) = n$ (and the opposite sign for b), even in the complex case.

Solutions denoted by $ce_n(v)$ and $se_n(v)$. Generalization of cosine and sine.

Quantization relation:

$$\int_0^\pi \sqrt{a - \frac{k_3^2 \rho^2}{2} \cos 2v} dv + O(h^2) = \pi h n.$$

Bloch theory

Periodic solutions of the Mathieu equation \Rightarrow Floquet modes.

Yields:

$a_n(k_3\rho)$ (even solutions), $b_n(k_3\rho)$ (odd solutions).

Obtained through the fact that, for any solution U of Mathieu equation, there exist $\nu(a, k_3\rho)$ and $P_{a, k_3\rho}$ periodic such that $U(v) = e^{\pm i\nu(a, k_3\rho)v} P_{a, k_3\rho}(\pm v)$.

The sequence of values $a_n(k_3\rho)$ solves $\nu(a, k_3\rho) = n$ (and the opposite sign for b), even in the complex case.

Solutions denoted by $ce_n(v)$ and $se_n(v)$. Generalization of cosine and sine.

Quantization relation:

$$\int_0^\pi \sqrt{a - \frac{k_3^2 \rho^2}{2} \cos 2v} dv + O(h^2) = \pi h n.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Modified Mathieu functions

Equation

$$F''(u) = \begin{pmatrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{pmatrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u F(u)$$

Program:

Decomposition into growing at $+\infty$ and decaying at $+\infty$ solution

F_g, F_d , (in u as well as in ω).

Solution satisfying $F(u_0) = 0$: $\exists C$ such that

$$F(u) = C(F_g(u)F_d(u_0) - F_d(u)F_g(u_0)).$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED. The DtN uses:

$$\frac{F'_g(u_1)F_d(u_0) - F'_d(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Obtention of the DtN

Normal vector:

$$\vec{n} = \frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}} (\sinh u_1 \cos v, \cosh u_1 \sin v).$$

Normal derivative involves u **AND** v .

Curvature at a point of the boundary $\partial\mathcal{O}^e$

$$\rho(s(v)) = \frac{\rho}{\cosh u_1 \sinh u_1} (\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}$$

Regimes for obtaining the values of the DtN:

- bottom of the well (near $-\frac{k_3^2 \rho^2}{2}$): classical Schrodinger well, (corresponds to n finite)
- above the maximum of the potential (elliptic regime)
- between the maximum and the minimum of the potential (hyperbolic region).

Obtention of the DtN

Normal vector:

$$\vec{n} = \frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}} (\sinh u_1 \cos v, \cosh u_1 \sin v).$$

Normal derivative involves u **AND** v .

Curvature at a point of the boundary $\partial\mathcal{O}^e$

$$\rho(s(v)) = \frac{\rho}{\cosh u_1 \sinh u_1} (\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}$$

Regimes for obtaining the values of the DtN:

- bottom of the well (near $-\frac{k_3^2 \rho^2}{2}$): classical Schrodinger well, (corresponds to n finite)
- above the maximum of the potential (elliptic regime)
- between the maximum and the minimum of the potential (hyperbolic region).

Asymptotic expressions used in the hyperbolic regime

Introduce $N = \sqrt{a_n(k_3\rho)}$, $\frac{d\Theta}{du} = \sqrt{\frac{2\rho^2 k_3^2}{a_n(k_3\rho)} \cosh^2 u - 1}$.

$F''(u) = (a_n(k_3\rho) - \frac{k_3^2 \rho^2}{2} \cosh^2 u) F(u)$ rewrites
 $F''(u) = (iN)^2 (\Theta'(u))^2 F(u)$.

$$F(u) = y(\Theta(u)) \Rightarrow \frac{d\Theta}{du} \frac{d}{d\Theta} \left(\frac{d\Theta}{du} \frac{dy}{d\Theta} \right) = (iN)^2 (\Theta'(u))^2 y(\Theta)$$

$$r(\Theta) = (\Theta'(u))^{-\frac{1}{2}} y(\Theta) \Rightarrow \frac{d^2 r}{d\Theta^2} = [(iN)^2 + \delta(\Theta)] r(\Theta) \Rightarrow r(\Theta) \simeq e^{\pm iN\Theta} (1 + O(N^{-2}))$$

Asymptotic expressions used in the hyperbolic regime

Introduce $N = \sqrt{a_n(k_3\rho)}$, $\frac{d\Theta}{du} = \sqrt{\frac{2\rho^2 k_3^2}{a_n(k_3\rho)} \cosh^2 u - 1}$.

$F''(u) = (a_n(k_3\rho) - \frac{k_3^2 \rho^2}{2} \cosh^2 u) F(u)$ rewrites
 $F''(u) = (iN)^2 (\Theta'(u))^2 F(u)$.

$$F(u) = y(\Theta(u)) \Rightarrow \frac{d\Theta}{du} \frac{d}{d\Theta} \left(\frac{d\Theta}{du} \frac{dy}{d\Theta} \right) = (iN)^2 (\Theta'(u))^2 y(\Theta)$$

$$r(\Theta) = (\Theta'(u))^{-\frac{1}{2}} y(\Theta) \Rightarrow \frac{d^2 r}{d\Theta^2} = [(iN)^2 + \delta(\Theta)] r(\Theta) \Rightarrow r(\Theta) \simeq e^{\pm iN\Theta} (1 + O(N^{-2}))$$

The mode N

Choose $\Im N\Theta(u_1) < 0$. Growing solution $e^{iN\Theta(u)}(1 + O(N^{-2}))$.

DtN in the u part:

$$\begin{aligned} \frac{d}{du} \left(\frac{e^{iN\Theta(u)}(1+O(N^{-2}))}{e^{iN\Theta(u)}(1+O(N^{-2}))} \right) &\simeq iN\Theta'(u) + O(N^{-1}) \\ &= i\omega\rho\sqrt{\frac{\epsilon\mu-\eta^2}{2} \cosh 2u_1 - \frac{a_n(k_3\rho)}{\omega^2\rho^2}} + O(N^{-1}). \end{aligned}$$

The v parts couples modes.

Concluding remarks

1. Need to consider n as a high frequency parameter to have the total high frequency analysis of the impedance operator
2. Never have, in the hypothesis $l\mathfrak{S}\epsilon\mu$ independent of ω the glancing regime
3. Need to include in the impedance operator the elliptic and the hyperbolic regime
4. Allows to get well defined solutions in the \mathcal{S}' sense (not all solutions can have a Fourier transform)
5. In the case of an elliptic boundary, use the Floquet values a_n, b_n for the Mathieu equation
6. The impedance (DtN) operator couples all modes modes in the case of an elliptic boundary. No longer a Fourier multiplier.