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Conditional indicators

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Abstract: In this paper, we introduce a large class of (so-called) conditional indicators, on a complete probability space with respect to a sub σ -algebra. A conditional indicator is a positive mapping, which is not necessary linear, but may share common features with the conditional expectation, such as the tower property or the projection property. Several characterizations are formulated. Beyond the definitions, we provide some non trivial examples that are used in finance and may inspire new developments in the theory of operators on Riesz spaces.

Keywords and phrases: Positive operator; conditional expectation; tower-property; projection; conditional risk-measure; stochastic basis; mathematical finance.

1. Introduction

In mathematical finance, the positive expectation operator and, more generally, the conditional expectation operator, is certainly the indicator the most used by the practitioners. It provides the best estimation $E(X)$, say today, of any future wealth or price X , modeled as a random variable, that is only revealed at some horizon date. Actually, there exists a large variety of indicators that are used in statistics, economics but, also, in finance, in order to control for example the risk of financial strategies.

The conditional expectation is the key tool when estimating the portfolio process replicating a contingent claim in a complete financial market model under the usual no-arbitrage condition, see [13], [7]. Under this condition, the price process is a martingale under the so-called risk-neutral probability measure, which is fundamental to identify the unique replicating portfolio process

from its terminal value, see [26]. This notion of martingale is generally defined with respect to the expectation operator, which is made possible by the well-known tower and projection properties. But we may find generalizations to other operators, such as in [4]. Actually, the expectation operator appears naturally in the classical theory because of the financial models themselves that are conic by definition. In particular, the no-arbitrage condition which is imposed appears to be equivalent to the existence of a risk-neutral probability measure, by virtue of the Hahn-Banach separation theorem with respect to the $\sigma(L^\infty, L^1)$ weak topologie, see for example [10] and [16] in discrete time.

Nevertheless, as soon as we consider more realistic financial markets with transaction costs, the models are not necessary conic and, worst, they are not convex if there are fixed costs, see [21]. In that case, the usual arguments derived from the standard duality of the convex analysis, see [25], can not be used. In the recent papers [6] and [12], a new approach is proposed. Not only there is no need to impose a no-arbitrage condition which is, in general, difficult to verify in practice but it is possible to compute numerically the super-hedging prices backwardly thanks to new results on random optimization, see [11]. To do so, the fundamental operator we use (called indicator in this paper as it is not linear) is the conditional essential supremum, both with its dual indicator, i.e. the conditional essential infimum, see [21], [22], [24]. Similarly to the conditional expectation operator, it satisfies the tower property and other common features. In particular, this is possible to consider martingales w.r.t. such an indicator.

In our paper, we define conditional indicators I with respect to a sub σ -algebra \mathcal{H} as mappings that map real-valued random variables X into the subset of \mathcal{H} -measurable random variables. Precisely, $I(X)$ is supposed to belong to the convex hull of the conditional support of a all possible values of X , and satisfies $I(X) = X$ if X is \mathcal{H} -measurable. In particular, I is positive i.e. $I(X) \geq 0$ if $X \geq 0$. This is the usual projection property. In finance, a σ -algebra \mathcal{H} is generally interpreted as an available information on the market. Then, a conditional indicator is an indicator whose value is updated thanks to the information \mathcal{H} . Classical indicators in finance, but also in statistics, are the quantiles, e.g. the Value At Risk [20], in the domain of risk measures for banking and insurance regulation, see [8] and [9] among others.

In Section 2, we introduce the main definitions and we give some typical examples of conditional indicators. In Section 3, we consider and characterize the conditional indicators I that are said regular, i.e. they equivalently satisfy

the property $I(X1_{\mathcal{H}}) = I(X)1_{\mathcal{H}}$ for all $H \in \mathcal{H}$. This property is observable in many examples of conditional indicators and used to define the projection property related to the tower property of Section 5. In Section 4, the dual of a conditional indicator is naturally introduced and we give an example we meet in finance. Natural questions arise such as identifying the set of all self-dual conditional indicators. In Section 6, we make a link between conditional indicators and the conditional risk measures of financial regulation, see also [14]. At last, Section 7 is devoted to the conditional expectation defined on the whole space $L^0(\mathbf{R}, \mathcal{F})$. We also provide some minimal conditions under which a conditional indicator is necessarily a conditional expectation under some absolutely continuous probability measure.

We expect that the notion of conditional indicators may be extended to the general setting of Riesz spaces. Some interesting problems are open such as characterizing the linear conditional indicators, studying the indicators satisfying the tower property and the associated notion of martingales but, also, identifying the stochastic indicators which are uniquely defined by the projection property, see a first result in that direction given by Proposition 5.4.

Actually, the notion of conditional expectation in the field of Riesz spaces and positive operators is very popular and has given rise to new developments recently, see [3] for an overview on positive operators and the papers [17] and [15] on conditional expectation, among others. Naturally, the concept of martingale has been introduced, see [17] and [15]. As mentioned above, some non linear positive operators are also needed in finance and we think that they may inspire interesting problems for the community of people working on positive operators and Riesz spaces, see for example [2].

2. Conditional indicators

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the σ -algebra \mathcal{F} is supposed to be complete. Let \mathcal{H} be a sub σ -algebra of \mathcal{F} which is also supposed to be complete. In the whole paper, we use the following notations.

Notations and conventions:

1) For any $r \in \mathbf{R}$, we adopt the conventions that $r \pm \infty = \pm \infty$, $\infty - \infty = 0$, and $\infty + \infty = \infty$ and $0 \times \pm \infty = 0$.

By virtue of our notational conventions, we deduce that $\alpha(a-b) = \alpha a - \alpha b$ for all $\alpha \in \mathbf{R}$ and $a, b \in \overline{\mathbf{R}}$.

2) For any random subset $G(\omega) \subseteq \mathbf{R}$, we denote by $\mathbb{L}^0(G, \mathcal{F})$ (resp. $\mathbb{L}^0(G, \mathcal{H})$) the set of all \mathcal{F} -measurable (resp. \mathcal{H} -measurable) random variables X such that $X(\omega) \in G(\omega)$ a.s..

3) We define the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$.

We recall the concept of conditional supremum and infimum, see [19][Section 5.3.1], [18]:

Theorem 2.1. *Let Γ be a family of \mathcal{F} -measurable random variables with values in $\overline{\mathbf{R}}$ and let \mathcal{H} be a sub σ -algebra of \mathcal{F} . There exists a unique \mathcal{H} -measurable random variable $\text{ess sup}_{\mathcal{H}} \Gamma$ such that:*

- 1) $\text{ess sup}_{\mathcal{H}} \Gamma \geq \gamma$, for all $\gamma \in \Gamma$,
- 2) If $\hat{\gamma}$ is \mathcal{H} -measurable and $\hat{\gamma} \geq \gamma$, for all $\gamma \in \Gamma$, then $\hat{\gamma} \geq \text{ess sup}_{\mathcal{H}} \Gamma$.

Note that $\text{ess sup}_{\mathcal{H}} \Gamma$ is smallest \mathcal{H} -measurable variable that dominates the family Γ . Symmetrically, we define $\text{ess inf}_{\mathcal{H}} \Gamma := -\text{ess sup}_{\mathcal{H}}(-\Gamma)$ as the largest \mathcal{H} -measurable variable that is dominated by the family Γ .

Definition 2.2. *Let \mathbb{D}_I be a subset of $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$ containing 0. We say that a mapping*

$$\begin{aligned} I_{\mathcal{H}} : \mathbb{D}_I &\longrightarrow \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H}). \\ X &\longmapsto I_{\mathcal{H}}(X) \end{aligned}$$

is a *Conditional Indicator (C.I.)* if the following properties hold:

(P1) $I_{\mathcal{H}}(X) \in \text{c-sup}_{\mathcal{H}}(X) := [\text{ess inf}_{\mathcal{H}}(X), \text{ess sup}_{\mathcal{H}}(X)]$ a.s.

(P2) $\mathbb{D}_I + \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H}) \subseteq \mathbb{D}_I$.

Remark 2.3. *For the sake of simplicity, we write I instead of $I_{\mathcal{H}}$ when \mathcal{H} is fixed without any possible confusion. Note that, for all $X \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$, $I(X) = X$, i.e. I is idempotent. We also observe that it is always possible to extend a conditional indicator to the whole set $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$. Indeed, it suffices to define for example $I(X) = \text{ess sup}_{\mathcal{H}}(X)$ for $X \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F}) \setminus \mathbb{D}_I$. In the following, the domain of definition of any conditional indicator is always denoted by \mathbb{D}_I .*

Remark 2.4. *The natural extension of Definition 2.2 to multi-varied random variables is to suppose that $I_{\mathcal{H}}(X)$ belongs a.s. to the convex hull of the*

conditional support $\text{c-supp}_{\mathcal{H}}(X)$, see the definition in [11], which is no more an interval. This possible generalization is an open problem beyond the scope of this paper.

Lemma 2.5. *Let \mathcal{H} be a sub- σ -algebra of \mathcal{F} , and let I be a C.I. w.r.t. \mathcal{H} . Then, I is a positive indicator, i.e. $I(X) \geq 0$, for all $X \in \mathbb{D}_I$ such that $X \geq 0$. In particular, if I is linear, then I is increasing.*

Proof. Let $X \in \mathbb{D}_I^+$. As $I(X) \in \text{c-supp}_{\mathcal{H}}(X)$, then $I_{\mathcal{H}}(X) \geq \text{essinf}_{\mathcal{H}}(X) \geq 0$ and the conclusion follows. \square

Definition 2.6. *Let I be a C.I. Then,*

- 1) *I is said increasing if, for all $X, Y \in \mathbb{D}_I$ such that $X \leq Y$, we have $I(X) \leq I(Y)$.*
- 2) *I is said \mathcal{H} -translation invariant if $I(X + Y_{\mathcal{H}}) = I(X) + I(Y_{\mathcal{H}})$ for all $X \in \mathbb{D}_I$ and $Y_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$ such that $X + Y_{\mathcal{H}} \in \mathbb{D}_I$.*
- 3) *I is said \mathcal{H} -positively-homogeneous if, for every $\alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}_+, \mathcal{H})$, we have $\alpha_{\mathcal{H}}\mathbb{D}_I \subset \mathbb{D}_I$ and for any $X \in \mathbb{D}_I$, $I(\alpha_{\mathcal{H}}X) = \alpha_{\mathcal{H}}I(X)$.*
- 4) *I is said \mathcal{H} -linear if, for all $\alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$, $\alpha_{\mathcal{H}}\mathbb{D}_I + \mathbb{D}_I \subset \mathbb{D}_I$, and for every $X, Y \in \mathbb{D}_I$, $I(\alpha_{\mathcal{H}}X + Y) = \alpha_{\mathcal{H}}I(X) + I(Y)$.*
- 5) *If $I(\limsup_n X_n) \geq \limsup_n I(X_n)$ (resp. $I(\liminf_n X_n) \leq \liminf_n I(X_n)$), for any sequence $(X_n)_n \in \mathbb{D}_I$ such that $\limsup_n X_n \in \mathbb{D}_I$ (resp. $\liminf_n X_n \in \mathbb{D}_I$), we say that I satisfies the upper (resp. lower) Fatou property.*
- 6) *I is said conditionally convex if, for any $\alpha_{\mathcal{H}} \in \mathbb{L}^0([0, 1], \mathcal{H})$, we have $\alpha_{\mathcal{H}}\mathbb{D}_I + (1 - \alpha_{\mathcal{H}})\mathbb{D}_I \subset \mathbb{D}_I$ and for all $X \in \mathbb{D}_I$,*

$$I(\alpha_{\mathcal{H}}X + (1 - \alpha_{\mathcal{H}})Y) \leq \alpha_{\mathcal{H}}I(X) + (1 - \alpha_{\mathcal{H}})I(Y).$$

Remark 2.7.

The conditional expectation operator $I^1(X) = \mathbb{E}(X|\mathcal{H})$ is a well known example of conditional indicator which is \mathcal{H} -linear, \mathcal{H} -translation invariant and increasing on $\mathbb{D}_{I^1} = \mathbb{L}^1(\mathbf{R}, \mathcal{F}) \cup \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$, where $\mathbb{L}^1(\mathbf{R}, \mathcal{F})$ is the set of all integrable random variables.

The conditional supremum $I^2(X) = \text{ess sup}_{\mathcal{H}}(X)$ is another exemple defined on $\mathbb{D}_{I^2} = \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$. Note that I^2 is increasing, \mathcal{H} -translation invariant, \mathcal{H} -positively-homogeneous and sub-additive.

If $I : \mathbb{D}_I \rightarrow \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$ is increasing and satisfies $I(X_{\mathcal{H}}) = X_{\mathcal{H}}$ for all $X_{\mathcal{H}} \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$, then I is a C.I.

Lemma 2.8 (lower and upper extensions of a conditional indicator). *Consider a conditional indicator I defined on some domain \mathbb{D}_I which is monotone. Suppose that E_I is a subset of \mathbb{D}_I containing $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$. Let us define:*

$$I^L(X) : = \text{ess sup}_{\mathcal{H}}\{I(Y) : Y \in E_I \text{ and } Y \leq X\}, \quad (2.1)$$

$$I^U(X) : = \text{ess inf}_{\mathcal{H}}\{I(Y) : Y \in E_I \text{ and } Y \geq X\}. \quad (2.2)$$

Then, I^L and I^U are two monotone conditional indicators defined on $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$ that coincide with I on E_I and satisfies $I^L \leq I \leq I^U$ on \mathbb{D}_I . We say that I^L and I^U are lower and upper extensions of I on E_I . If $E_I + E_I \subseteq E_I$, then I^L and I^U are respectively super and sub-additive.

Proof. First observe that, if $Y \in E_I$ is such that $Y \leq X$, then we have $I(Y) \leq \text{ess sup}_{\mathcal{H}}(Y) \leq \text{ess sup}_{\mathcal{H}}(X)$ hence, taking the essential supremum, we get that $J(X) \leq \text{ess sup}_{\mathcal{H}}(X)$. Moreover, if $X \in \mathbb{D}_I$, $Y \leq X$ implies, by assumption, that $I(Y) \leq I(X)$ hence $J(X) \leq I(X) \leq$ for $X \in \mathbb{D}_I$. On the other hand, as $\text{ess inf}_{\mathcal{H}}(X) \leq X$, we deduce that $J(X) \geq \text{ess inf}_{\mathcal{H}}(X)$. At last, if $X_{\mathcal{H}} \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$, then $X_{\mathcal{H}} \in E_I$ hence $J(X_{\mathcal{H}}) \geq X_{\mathcal{H}}$. Moreover, $J(X_{\mathcal{H}}) \leq I(X_{\mathcal{H}}) = X_{\mathcal{H}}$ so that $J(X_{\mathcal{H}}) = X_{\mathcal{H}}$. Note that, if $X \in E_I$, then $J(X) \geq I(X)$. As $J(X) \leq I(X)$, we conclude that $J(X) = I(X)$. The same types of argument hold for K . \square

Remark 2.9. *If $(I_k)_{k \in K}$ is a family of conditional indicators w.r.t. the σ -algebra \mathcal{H} , then $I_1(X) = \text{ess inf}_{k \in K} I_k(X)$ and $I_2(X) = \text{ess sup}_{k \in K} I_k(X)$ are still conditional indicators w.r.t. to \mathcal{H} on $\mathbb{D}_{I_1} = \mathbb{D}_{I_2} = \cap_{k \in K} \mathbb{D}_{I_k}$. As Lemma 2.8 proves the existence of upper and lower extensions, we then deduce the following corollary.*

Corollary 2.10. *Consider a conditional indicator I , w.r.t. the σ -algebra \mathcal{H} , defined on some domain \mathbb{D}_I , which is monotone. Suppose that E_I is a subset of \mathbb{D}_I containing $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$. There exists a (unique) smallest conditional indicator I^+ (resp. a largest conditional indicator I^-) which coincides with I on E_I and such that $I^- \leq I \leq I^+$ on \mathbb{D}_I .*

Proof. By Lemma 2.8, there exists conditional indicators J, K , defined on $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$ such that $J \leq I \leq K$ on \mathbb{D}_I . By Lemma 2.9, it suffices to define the indicator $I^-(X) = \text{ess sup}_{k \in K} J_k(X)$ where $(J_k)_{k \in K}$ is the non empty family of conditional indicators that is dominated by I on \mathbb{D}_I and coincides with I on E_I and the indicator $I^+(X) = \text{ess inf}_{k \in K} K_k(X)$ where $(K_k)_{k \in K}$ is the family of non empty conditional indicators that dominate I on \mathbb{D}_I and coincides with I on E_I . \square

3. Regularity

Definition 3.1. A subset E of $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$ is said \mathcal{H} -decomposable if, for all $H \in \mathcal{H}$, and $X, Y \in E$, we have $X1_H + Y1_{\Omega \setminus H} \in E$.

Lemma 3.2. Let \mathcal{H} be a sub σ -algebra of \mathcal{F} and let I be a C.I. w.r.t. \mathcal{H} , which is defined on an \mathcal{H} -decomposable subset \mathbb{D}_I . Then, for all $X \in \mathbb{D}_I$ and $H \in \mathcal{H}$, $I(X1_H)1_{H^c} = 0$. Therefore, we have $I(X1_H) = I(X1_H)1_H$.

Proof. Consider $X \in \mathbb{D}_I$ and $H \in \mathcal{H}$. As $I(X1_{H^c})1_H \in \text{c-supp}_{\mathcal{H}}(X1_{H^c})1_H$ and $\text{c-supp}_{\mathcal{H}}(X1_{H^c})1_H = \text{c-supp}_{\mathcal{H}}(X1_{H^c}1_H) = \{0\}$ by the properties satisfied by the essential infimum and supremum, we get $I(X1_H)1_{H^c} = 0$. Therefore, we have the equality $I(X1_H) = I(X1_H)1_H + I(X1_H)1_{H^c} = I(X1_H)1_H$. \square

Definition 3.3. Consider \mathcal{H} a sub σ -algebra of \mathcal{F} . A conditional indicator I w.r.t. \mathcal{H} is said regular if \mathbb{D}_I is \mathcal{H} -decomposable and, for all $X, Y \in \mathbb{D}_I$ and $H \in \mathcal{H}$, we have:

$$X1_H = Y1_H \Rightarrow I(X)1_H = I(Y)1_H.$$

The proof of the following lemma is trivial:

Lemma 3.4. Let I be a C.I. w.r.t. \mathcal{H} , which is defined on an \mathcal{H} -decomposable subset \mathbb{D}_I . The following statements are equivalent.

1. I is regular.
2. $I(X1_H) = I(X)1_H$, for all $X \in \mathbb{D}_I$ and $H \in \mathcal{H}$.
3. $I(X1_H + Y1_{H^c}) = I(X)1_H + I(Y)1_{H^c}$, for all $X, Y \in \mathbb{D}_I$ and $H \in \mathcal{H}$.

Proposition 3.5. Let \mathcal{H} be a sub σ -algebra of \mathcal{F} and let I be a regular and \mathcal{H} -positively-homogeneous conditional indicator w.r.t. \mathcal{H} . Then, for all $X \in \mathbb{D}_I$ and $h \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$, we have

$$I(hX) = h^+I(X) + h^-I(-X).$$

Proof. Let $X \in \mathbb{D}_I$ and $h \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$. By Propostion 3.4, we get that

$$\begin{aligned} I(hX) &= I(hX1_{\{h \geq 0\}}) + I(hX1_{\{h < 0\}}) \\ &= I(h^+X) + I(-h^-X) \\ &= h^+I(X) + h^-I(-X) \end{aligned}$$

\square

Proposition 3.6. *If a conditional indicator is conditionally convex and \mathbb{D}_I is \mathcal{H} -decomposable, then it is regular.*

Proof. Let us consider $H \in \mathcal{H}$ and $X \in \mathcal{L}^0(\mathbf{R}, \mathcal{F})$. Since $1_H \in \mathcal{L}^0([0, 1], \mathcal{H})$ and $1_{H^c} = 1 - 1_H$, the conditional convexity of I implies that

$$I(1_H X) = I(1_H X + 1_{H^c} 0) \leq 1_H I(X) + 1_{H^c} I(0) = 1_H I(X).$$

Similarly, we have

$$I(X) = I(1_H(1_H X) + 1_{H^c}(1_{H^c} X)) \leq 1_H I(1_H X) + 1_{H^c} I(1_{H^c} X).$$

By Lemma 3.2, we deduce that $1_H I(X) \leq 1_H I(1_H X) = I(1_H X)$. Therefore, $I(1_H X) = 1_H I(X)$. The conclusion follows by Lemma 3.4. \square

Lemma 3.7. *Let I be a C.I. w.r.t. the σ -algebra \mathcal{H} . If I is sub-additive and \mathbb{D}_I is \mathcal{H} -decomposable, then we have $1_H I(X) \leq I(1_H X)$ for all $H \in \mathcal{H}$ and $X \in \mathcal{L}^0(\mathbf{R}, \mathcal{F})$. Moreover, if I is additive, then I is regular.*

Proof. Consider $H \in \mathcal{H}$ and $X \in \mathcal{L}^0(\mathbf{R}, \mathcal{F})$. If I is sub-additive, then by Lemma 3.2, we have:

$$\begin{aligned} 1_H I(X) &= 1_H I(1_H X + 1_{H^c} X) \leq 1_H I(1_H X) + 1_H I(1_{H^c} X) \\ &\leq 1_H I(1_H X) = I(1_H X). \end{aligned}$$

If I is additive, then

$$\begin{aligned} 1_H I(X) &= 1_H I(1_H X + 1_{H^c} X) = 1_H I(1_H X) + 1_H I(1_{H^c} X) \\ &= 1_H I(1_H X) = I(1_H X). \end{aligned}$$

\square

Proposition 3.8. *Let I^L and I^U be the extensions of I defined by (2.1) and (2.2) in Lemma 2.8 when $E_I = \mathbb{D}_I$. Suppose that I is regular. Then, I^L and I^U are regular.*

Proof. Let $H \in \mathcal{H}$ and $X \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$. Consider any $Y \leq X 1_H$ where $Y \in \mathbb{D}_I$. Let us define $Z = Y 1_H + (\text{ess inf}_{\mathcal{H}} X) 1_{\Omega \setminus H}$. Then, $Z \leq X$ and $Z \in \mathbb{D}_I$ so that $I(Z) \leq I^L(X)$. As I is regular by assumption, we deduce that $I(Y) 1_H + (\text{ess inf}_{\mathcal{H}} X) 1_{\Omega \setminus H} \leq I^L(X)$. Therefore, $I(Y) 1_H \leq I^L(X) 1_H$. Taking the essential supremum, we deduce that $I^L(X 1_H) 1_H \leq I^L(X) 1_H$. On the other hand, for any $Y \leq X$ such that $Y \in \mathbb{D}_I$, we have $Y 1_H \leq X 1_H$

and $Y1_H \in \mathbb{D}_I$. Therefore, $J(X1_H) \geq I(Y1_H) = I(Y)1_H$. Taking the essential supremum, we deduce that $J(X1_H) \geq J(X)1_H$. With the first part of the proof, we deduce that $J(X1_H)1_H = J(X)1_H$ for any $X \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$ and $H \in \mathcal{H}$. Replacing X by $X1_H$ and H by $\Omega \setminus H$, we then deduce the equality $J(X1_H)1_{\Omega \setminus H} = J(X1_H1_{\Omega \setminus H})1_{\Omega \setminus H} = J(0)1_{\Omega \setminus H} = 0$. Therefore, we have $J(X1_H) = J(X)1_H$ hence $J(X1_H) = J(X)1_H$. The conclusion follows by Lemma 3.4. The reasoning is similar for I^U . \square

Corollary 3.9. *Consider a conditional indicator I , w.r.t. the σ -algebra \mathcal{H} , defined on some domain \mathbb{D}_I , which is monotone and regular. Suppose that E_I is a subset of \mathbb{D}_I containing $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H})$. There exists a (unique) smallest regular conditional indicator I^+ (resp. a largest regular conditional indicator I^-) which coincides with I on E_I and such that $I^- \leq I \leq I^+$ on \mathbb{D}_I .*

Proof. It suffices to repeat the proof of Corollary 2.10 by restricting the families to the regular indicators. Indeed, existence holds by Proposition 3.8. \square

Lemma 3.10. *Consider the conditional expectation $I(X) = E(X|\mathcal{H})$ for $X \in \mathbb{D}_I$ where $\mathbb{D}_I = \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H}) \cup \mathbb{L}^1(\mathbf{R}, \mathcal{F}) \cup \mathbb{L}^0(\mathbf{R}_+, \mathcal{F}) \cup \mathbb{L}^0(\mathbf{R}_-, \mathcal{F})$. Suppose that $E_I = \mathbb{D}_I$, see Corollary 3.9. Then, there exists regular extensions I^- and I^+ of the conditional expectation to the whole set $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$.*

Proof. Consider $X = X^+ - X^-$, supposed to be integrable, where we recall that $X^+ = \max(X, 0) \geq 0$ and $X^- = -\min(X, 0) \geq 0$. Then, $X^+ = \lim_n X^n$ where $X^n = X^+ \wedge n$, $n \geq 1$, is an increasing sequence of integrable random variables. Then, we get that $X^n - X^+ \in E_I = \mathbb{D}_I$ and $X^n \leq X^+$ hence we get that $I^-(X) \geq I(X^n)$. Taking the limit, we get that $I^-(X) \geq I(X)$ so that $I^-(X) = I(X)$ for all $X = X^+ - X^-$ where X^- is integrable. More generally, if $E(X^-|\mathcal{H}) < \infty$ a.s., by regularity of I^- , we get that

$$\begin{aligned} I^-(X) &= \sum_{k=0}^{\infty} I^-(X)1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}} \\ &= \sum_{k=0}^{\infty} I^-(X1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}})1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}}. \end{aligned}$$

As $X1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}}$ is of the form $X^+ - X^-$ where X^- is integrable, we

deduce by above that

$$\begin{aligned} I^-(X1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}}) &= I(X1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}}) \\ &= (E(X^+|\mathcal{H}) - E(X^-|\mathcal{H}))1_{\{k \leq E(X^-|\mathcal{H}) < k+1\}} \end{aligned}$$

and, finally $I^-(X) = E(X^+|\mathcal{H}) - E(X^-|\mathcal{H})$ for every X such that we have $E(X^-|\mathcal{H}) < \infty$. Similarly, we have $I^+(X) = E(X^+|\mathcal{H}) - E(X^-|\mathcal{H})$ for every X such that $E(X^+|\mathcal{H}) < \infty$. Note that I^- and I^+ are natural extensions of the conditional expectation with the conventions $+\infty - \mathbf{R} = \{+\infty\}$ and $\mathbf{R} - \infty = \{-\infty\}$. \square

4. Dual indicators

Definition 4.1. Let I be a C.I. on a domain \mathbb{D}_I . The dual indicator I^* of I is defined on $\mathbb{D}_{I^*} = -\mathbb{D}_I$ as $I^*(X) = -I(-X)$, $X \in \mathbb{D}_{I^*}$. If $I = I^*$, we say that I is self-dual.

Proposition 4.2. The dual I^* of a C.I. I is still a C.I. such that $(I^*)^* = I$ and we have:

- 1) If I is monotone, then I^* is monotone.
- 2) If I is \mathcal{H} -translation invariant, then I^* is \mathcal{H} -translation invariant.
- 3) If I is super-linear, then I^* is sub-linear.

Example 4.3. If $I_1(X) = E(X|\mathcal{H})$ is defined for $X \in \mathbb{D}_{I_1} = \mathbb{L}^1(\mathbf{R}, \mathcal{F})$, then $I_1^* = I_1$. The indicator $I_2(X) = \text{ess sup}_{\mathcal{H}}(X)$, $X \in \mathbb{D}_{I_2} = \mathbb{L}^0(\mathbf{R}, \mathcal{F})$, admits the dual $I_2^*(X) = \text{ess inf}_{\mathcal{H}}(X)$. Let I be any C.I.. Then, $T = \frac{1}{2}I + \frac{1}{2}I^*$, defined on $\mathbb{D}_T = \mathbb{D}_I \cap \mathbb{D}_{I^*}$, is self-dual and is still a C.I.. Reciprocally, any self-dual indicator T is of the form $T = \frac{1}{2}I + \frac{1}{2}I^*$. Indeed, it suffices to choose $I = T$.

The proof of the following is left to the readers:

Lemma 4.4. Consider the lower and upper conditional indicators $I^L = I^{L(E)}$ and $I^U = I^{U(E)}$ with respect to a subset E of the domain of definition \mathbb{D}_I of a conditional indicator I , as defined in Lemma 2.8. Then, we have the following equalities: $(I^{L(E)})^* = (I^*)^{U(-E)}$ and $(I^{U(E)})^* = (I^*)^{L(-E)}$.

Theorem 4.5. Let \mathcal{H} be a sub σ -algebra of \mathcal{F} , and let I be a C.I. w.r.t. \mathcal{H} , defined on a vector space \mathbb{D}_I of $\mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F})$ such that $I^* = I$. Then, I is additive if and only if I is \mathcal{H} -linear.

Proof. As I is additive, we deduce that I is regular by lemma 3.7, i.e. $I(X1_H) = I(X)1_H$, for all $X \in \mathbb{D}_I$ and $H \in \mathcal{H}$. Consider $X \in \mathbb{D}_I$ and

$n_H \in \mathbb{L}^0(\mathbb{N}, \mathcal{H})$. By additivity, $I(kX) = kI(X)$, for all $k \in \mathbb{N}$, so that we have:

$$\begin{aligned}
I_{\mathcal{H}}(n_H X) &= \sum_{k=0}^{\infty} 1_{\{n_H=k\}} I(n_H X) \\
&= \sum_{k=0}^{\infty} 1_{\{n_H=k\}} I_{\mathcal{H}}(n_H X 1_{\{n_H=k\}}) \\
&= \sum_{k=0}^{\infty} 1_{\{n_H=k\}} I(kX) \\
&= \sum_{k=0}^{\infty} 1_{\{n_H=k\}} kI(X) = n_H I(X).
\end{aligned}$$

Also, we get that $I(-n_H X) = n_H I(-X) = -n_H I^*(X) = -n_H I(X)$. We then deduce that $I(n_H X) = n_H I(X)$ for all $n_H \in \mathbb{L}^0(\mathbb{Z}, \mathcal{H})$. Let us consider $r_H \in \mathbb{L}^0(\mathbb{Q}, \mathcal{H})$. By Lemma 8.1, there exists $p_H, q_H \in \mathbb{L}^0(\mathbb{Z}, \mathcal{H}) \times \mathbb{L}^0(\mathbb{N}^*, \mathcal{H})$ such that $r_H = p_H/q_H$. By the first step, we get that

$$p_H I(X) = I(q_H p_H / q_H X) = q_H I(p_H / q_H X).$$

Therefore, $I(r_H X) = r_H I(X)$. Finally, consider $\alpha \in \mathbb{L}^0(\mathbb{R}, \mathcal{H})$. By lemma 8.2, there exists two sequences $(r_n)_n, (q_n)_n \in \mathbb{L}^0(\mathbb{Q}, \mathcal{H})$ such that

$$\alpha - 1/n \leq r_n \leq \alpha \leq q_n \leq \alpha + 1/n.$$

By the properties above, we deduce that I is increasing. We deduce by the second step that

$$\begin{aligned}
I(r_n X) &\leq I(\alpha X) \leq I(q_n X), \\
r_n I(X) &\leq I(\alpha X) \leq q_n I(X).
\end{aligned}$$

We conclude that $I(\alpha X) = \alpha I(X)$, as $n \rightarrow \infty$. □

5. Stochastic indicators: tower property and projection w.r.t. a filtration

Let $(\mathcal{F}_t)_{t \in [0, T]}$ be a complete filtration, i.e. a sequence of complete σ -algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t$. Consider a family $(I_t)_{t \in [0, T]}$ of adapted conditional indicators in the sense that I_t is a conditional indicator w.r.t. \mathcal{F}_t , for every $t \in [0, T]$. We say that $I = (I_t)_{t \in [0, T]}$ is a stochastic indicator.

Definition 5.1. Consider a stochastic indicator $I = (I_t)_{t \in [0, T]}$. We say that $(I_t)_{t \in [0, T]}$ satisfies the tower property if, for any $s \leq t$, $I_t(\mathbb{D}_{I_s}) \subseteq \mathbb{D}_{I_s} \subseteq \mathbb{D}_{I_t}$ and

$$I_s(I_t(X)) = I_s(X), \text{ for all } X \in \mathbb{D}_{I_s}.$$

Example 5.2. The conditional essential supremum indicator satisfies the tower property. In particular, we have

$$\text{ess sup}_{\mathcal{F}_0}(\text{ess sup}_{\mathcal{F}_t}(X)1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X1_{F_t}), \forall F_t \in \mathcal{F}_t, \forall X \in L^0(\mathbf{R}, \mathcal{F}).$$

Note that, if a stochastic indicator $I = (I_t)_{t \in [0, T]}$ satisfies the tower property, so does its dual $I^* = (I_t^*)_{t \in [0, T]}$.

Definition 5.3. Let I_0 be a conditional indicator w.r.t. \mathcal{F}_0 such that D_{I_0} is \mathcal{F}_t -decomposable for every $t \geq 0$. We say that $Z_t \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_t)$ satisfies the projection equality at time t w.r.t. I_0 if the following condition holds:

$$\mathbf{Pr} : I_0(X1_{F_t}) = I_0(Z_t1_{F_t}), \text{ for all } F_t \in \mathcal{F}_t.$$

Proposition 5.4. Suppose that the stochastic indicator $I = (I_t)_{t \in [0, T]}$ is such that I_0 is super-additive, $I_t(X)$ satisfies the projection property at given time t for some $X \in \mathbb{D}_{I_t}$ and, for every $Y \in \mathcal{L}^0(\mathbf{R}^+, \mathcal{F}_T)$, we have $I_0(Y) \leq 0$ if and only if $Y = 0$. Then, $I_t(X)$ is the unique \mathcal{F}_t -measurable random variable satisfying the projection equality (**Pr**).

Proof. Suppose that there exists $Z_t \in \mathcal{L}^0(\mathbf{R}, \mathcal{F}_t)$ such that, for all $F_t \in \mathcal{F}_t$,

$$I_0(X1_{F_t}) = I_0(Z_t1_{F_t}). \tag{5.3}$$

Let us show that $Z_t = I_t(X)$. Take $F_t = \{Z_t > I_t(X)\}$. By the projection property (**Pr**) for $I_t(X)$ and (5.3), $I_0(I_t(X)1_{F_t}) = I_0(Z_t1_{F_t})$. It follows by the super-additivity of I_0 that $I_0((Z_t - I_t(X))1_{F_t}) \leq 0$. Since $(Z_t - I_t(X))1_{F_t} \geq 0$ then $(Z_t - I_t(X))1_{F_t} = 0$ by assumption. So $1_{F_t} = 0$ and $Z_t \leq I_t(X)$ a.s.. Analogously $Z_t \geq I_t(X)$ a.s.. The conclusion follows. \square

Proposition 5.5. Suppose that the stochastic indicator $I = (I_t)_{t \in [0, T]}$ is such that I_0 is linear, $I_t(X)$ satisfies the projection property at time t for all $X \in \mathcal{L}^0(\mathbf{R}, \mathcal{F}_T)$ and, for all $Y \in \mathcal{L}^0(\mathbf{R}^+, \mathcal{F}_T)$, we have $I_0(Y) \leq 0$ if and only if $Y = 0$. Then, the following statements hold:

1. I_t is \mathcal{F}_t -linear, for all $t \in [0, T]$.
2. I_t is increasing, for all $t \in [0, T]$.

3. I_t is regular, for all $t \in [0, T]$.

4. The stochastic indicator $I = (I_t)_{t \in [0, T]}$ satisfies the tower property.

Proof. Let us show that I_t is linear. Consider $X, Y \in \mathcal{L}^0(\mathbf{R}, \mathcal{F}_T)$, $\alpha \in \mathbf{R}$ and $F_t \in \mathcal{F}_t$. By linearity and the projection property, we get that

$$\begin{aligned} I_0((\alpha I_t(X) + I_t(Y))1_{F_t}) &= \alpha I_0(I_t(X)1_{F_t}) + I_0(I_t(Y)1_{F_t}) \\ &= \alpha I_0(X1_{F_t}) + I_0(Y1_{F_t}) \\ &= I_0((\alpha X + Y)1_{F_t}). \end{aligned}$$

By Proposition 5.4, we deduce $I_t((\alpha X + Y)) = \alpha I_t(X) + I_t(Y)$. By Theorem 4.5, I_t is then \mathcal{F}_t -linear. The second statement is an immediate consequence of the first one. The third one is also direct consequence by Proposition 5.4. At last, consider $X \in \mathcal{L}^0(\mathbf{R}, \mathcal{F}_T)$ and let $F_s \in \mathcal{F}_s$ where $s \leq t$. By the projection property, we have

$$I_0(I_s(I_t(X))1_{F_s}) = I_0(I_t(X)1_{F_s}) = I_0(X1_{F_s}).$$

We conclude by Proposition 5.4 that $I_s(I_t(X)) = I_s(X)$. \square

The conditional expectation $E(X|\mathcal{F}_t)$, $X \in \mathbb{L}^1(\mathbf{R}^+, \mathcal{F}_T)$, is the unique \mathcal{F}_t -measurable random variable such that we have $E(X1_{F_t}) = E(E(X|\mathcal{F}_t)1_{F_t})$, for all $F_t \in \mathcal{F}_t$. As soon as a stochastic indicator $I = (I_t)_{t \in [0, T]}$ satisfies the tower property and is such that I_t is \mathcal{F}_t -regular, for all $t \leq T$, we have $I_0(X1_{F_t}) = I_0(I_t(X)1_{F_t}) = I_0(I_t(X)1_{F_t})$, for all $F_t \in \mathcal{F}_t$. The natural question is whether $I_t(X)$ is the unique \mathcal{F}_t -measurable random variable satisfying the projection property. Below, we study the case of the stochastic essential supremum indicator.

Theorem 5.6. *Let $X \in \mathbb{L}^0(\mathbf{R}_+, \mathcal{F}_T)$ such that $\text{ess sup}_{\mathcal{F}_0}(X) \in \mathbf{R}$, i.e. X is bounded. There exists a unique $Z_t \in \mathbb{L}^\infty(\mathbf{R}_+, \mathcal{F}_t)$ that satisfies the projection property*

$$\text{ess sup}_{\mathcal{F}_0}(Z_t1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X1_{F_t}), \forall F_t \in \mathcal{F}_t.$$

Proof. Suppose that there exists $Z_t \in \mathbb{L}^\infty(\mathbf{R}_+, \mathcal{F}_t)$ such that

$$\text{ess sup}_{\mathcal{F}_0}(Z_t1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X1_{F_t}), \forall F_t \in \mathcal{F}_t.$$

Let us show that $Z_t = \text{ess sup}_{\mathcal{F}_t}(X)$. To do so, let us denote $X_t = \text{ess sup}_{\mathcal{F}_t}(X)$. Consider, for any $\varepsilon > 0$, the set $F_t^\varepsilon = \{\text{ess sup}_{\mathcal{F}_t}(X) \leq Z_t - \varepsilon\} \in \mathcal{F}_t$. Then, $X1_{F_t^\varepsilon} \leq (Z_t - \varepsilon)1_{F_t^\varepsilon}$. Moreover, by assumption, we have

$$\text{ess sup}_{\mathcal{F}_0}(Z_t1_{F_t^\varepsilon}) = \text{ess sup}_{\mathcal{F}_0}(X1_{F_t^\varepsilon}) \leq \text{ess sup}_{\mathcal{F}_0}((Z_t - \varepsilon)1_{F_t^\varepsilon}) \leq \text{ess sup}_{\mathcal{F}_0}(Z_t1_{F_t^\varepsilon}).$$

Therefore, we have:

$$\text{ess sup}_{\mathcal{F}_0}((Z_t - \varepsilon)1_{F_t^\varepsilon}) = \text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t^\varepsilon}).$$

Suppose that $\mathbb{P}(F_t^\varepsilon) > 0$. We claim that $\text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t^\varepsilon}) \neq 0$. Otherwise, $Z_t 1_{F_t^\varepsilon} \leq 0$ and so $X \leq -\varepsilon$ on F_t^ε in contradiction with $X \geq 0$ a.s.. As $\text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t^\varepsilon}) \leq \text{ess sup}_{\mathcal{F}_0}(Z_t) \in \mathbf{R}$, then $\text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t^\varepsilon}) \in \mathbf{R} \setminus \{0\}$. By Corollary 8.4, $\varepsilon = 0$ in contradiction with the assumption that $\varepsilon > 0$. Therefore, $\mathbb{P}(F_t^\varepsilon) = 0$ hence $\mathbb{P}(F_t^{1/n}) = 0$, for any $n \geq 1$. We deduce that $\mathbb{P}(\bigcap_{n \geq 1} (\Omega \setminus F_t^{1/n})) = 1$, which means that, a.s., $aZ_t - 1/n < \text{ess sup}_{\mathcal{F}_t}(X)$ for any $n \geq 1$. As $n \rightarrow \infty$, we get that $Z_t \leq X_t$. Now consider the sets $G_t^\varepsilon = \{Z_t \leq X_t - \varepsilon\}$. Similarly, we obtain that

$$\text{ess sup}_{\mathcal{F}_0}((X_t - \varepsilon)1_{G_t^\varepsilon}) = \text{ess sup}_{\mathcal{F}_0}(X_t 1_{G_t^\varepsilon}).$$

As $\text{ess sup}_{\mathcal{F}_0}(X_t) \neq 0$ if $\mathbb{P}(G_t^\varepsilon) > 0$. We apply again Corollary 8.4 and deduce that $\varepsilon = 0$, i.e. a contradiction. We deduce that $X_t \leq Z_t$ a.s. and the conclusion follows. \square

Corollary 5.7. *Let $X \in \mathbb{L}^\infty(\mathbf{R}, \mathcal{F}_T)$ such that $X > 0$ a.s.. Then, there exists a unique $Z_t \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_t)$ such that $\text{ess sup}_{\mathcal{F}_0}(Z_t) \in \mathbf{R}$ that satisfies the projection property*

$$\text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X 1_{F_t}), \forall F_t \in \mathcal{F}_t.$$

Proof. Suppose that there exists $Z_t \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_t)$ such that

$$\text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X 1_{F_t}), \forall F_t \in \mathcal{F}_t.$$

Let us show that $Z_t \geq 0$. Consider $F_t = \{0 \geq Z_t\}$. Then, we have $\text{ess sup}_{\mathcal{F}_0}(X 1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(Z_t 1_{F_t}) \leq 0$. Therefore, $X 1_{F_t} \leq 0$ on F_t hence $\mathbb{P}(F_t) = 0$. The conclusion follows by Theorem 5.6. \square

Corollary 5.8. *Consider $X \in \mathbb{L}^\infty(\mathbf{R}_-, \mathcal{F}_T)$ (resp. s.t. $X < 0$). There exists a unique $Z_t \in \mathbb{L}^\infty(\mathbf{R}_-, \mathcal{F}_t)$ (resp. $Z_t \in \mathbb{L}^0(\mathbf{R}_-, \mathcal{F}_t)$ bounded from below) that satisfies the projection property*

$$\text{ess inf}_{\mathcal{F}_0}(Z_t 1_{F_t}) = \text{ess inf}_{\mathcal{F}_0}(X 1_{F_t}), \forall F_t \in \mathcal{F}_t.$$

The following counter-example shows that uniqueness does not hold in general for the essential supremum indicator.

Example 5.9. Consider $\Omega = \mathbf{R}$, $\mathcal{F} = \mathcal{B}(\mathbf{R})$ and \mathbb{P} the probability measure defined by its density $d\mathbb{P}/dx = \alpha/(1+x^2)$, $\alpha > 0$, w.r.t. the Lebesgue measure dx . We consider $X = 0$ and we define $Z(\omega) = -\exp(\omega)$ for all $\omega \in \Omega$.

We claim that $0 = \text{ess sup}_{\mathcal{F}_0}(Z)$. First, as $Z \leq 0$, $\text{ess sup}_{\mathcal{F}_0}(Z) \leq 0$. Secondly, as $\lim_{\omega \rightarrow -\infty} Z(\omega) = 0$, for any $\alpha < 0$, there exists $x \in \mathbf{R}$ such that for any $\omega \leq x$, $0 \geq Z(\omega) \geq \alpha$. So $0 \geq \text{ess sup}_{\mathcal{F}_0}(Z) \geq \alpha$ for any $\alpha \in \mathbf{R}^-$. Therefore, $\text{ess sup}_{\mathcal{F}_0}(Z) = 0$.

Let $A = \{Z \leq -1\}$ and $\mathcal{F}_1 = \sigma(1_A)$, i.e. $\mathcal{F}_1 = \{A, A^c, \Omega, \emptyset\}$. Let us introduce $Z_1 = \text{ess sup}_{\mathcal{F}_1}(Z)$. Note that $Z_1 1_A = \text{ess sup}_{\mathcal{F}_1}(Z 1_A) = -1_A$ and $Z_1 1_{A^c} = \text{ess sup}_{\mathcal{F}_1}(Z 1_{A^c}) = 0$, i.e. $Z_1 = -1_A$.

Let us now consider any $F_1 \in \mathcal{F}_1$. Of course, $\text{ess sup}_{\mathcal{F}_0}(X 1_{F_1}) = 0$. On the other hand, we have by the tower property and by sub-additivity:

$$\begin{aligned} 0 = \text{ess sup}_{\mathcal{F}_0}(Z) &= \text{ess sup}_{\mathcal{F}_0}(Z_1) \\ &\leq \text{ess sup}_{\mathcal{F}_0}(Z_1 1_{F_1}) + \text{ess sup}_{\mathcal{F}_0}(Z_1 1_{F_1^c}) \leq 0. \end{aligned}$$

We deduce that $\text{ess sup}_{\mathcal{F}_0}(Z_1 1_{F_1}) = \text{ess sup}_{\mathcal{F}_0}(Z_1 1_{F_1^c}) = 0$. Therefore,

$$\text{ess sup}_{\mathcal{F}_0}(Z_1 1_{F_1}) = 0 = \text{ess sup}_{\mathcal{F}_0}(X 1_{F_1}), \forall F_1 \in \mathcal{F}_1.$$

This means that the projection property for $X = 0$ is both satisfied by $\text{ess sup}_{\mathcal{F}_0}(X) = 0$ and Z_1 , i.e. uniqueness does not hold. Note that, with $\mathcal{F}_2 = \mathcal{B}(\mathbf{R})$ and $Z_2 = \text{ess sup}_{\mathcal{F}_2}(Z) = Z$, we have, for any $F_2 \in \mathcal{F}_2$, $\text{ess sup}_{\mathcal{F}_0}(X 1_{F_2}) = 0 = \text{ess sup}_{\mathcal{F}_0}(Z_2 1_{F_2})$. However, $Z_2 = Z \neq 0$.

6. Risk measures derived from conditional indicators

Let I be a C.I. w.r.t. a σ -algebra \mathcal{H} . We define the positive elements of I as the set

$$\mathbb{D}_I^+ := \{X \in \mathbb{D}_I : I(X) \geq 0\}.$$

In the setting of risk measures in finance, the elements of \mathbb{D}_I^+ are interpreted as the acceptable financial positions. We then define:

$$\mathcal{M}_I(X) := (\mathbb{D}_I^+ - X) \cap \mathbb{L}^0(\mathbf{R}, \mathcal{H}), \quad X \in \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{F}).$$

Note that $\mathcal{M}_I(X)$ may be empty and we have $\mathcal{M}_I(X + \alpha_{\mathcal{H}}) = \mathcal{M}_I(X) - \alpha_{\mathcal{H}}$ for all $\alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$. Moreover, $Y_{\mathcal{H}} \in \mathcal{M}_I(X)$ if and only if $Y_{\mathcal{H}} + X$ is acceptable. We then define:

$$\rho_I(X) := \text{ess inf } \mathcal{M}_I(X), \quad X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}), \quad (6.4)$$

with the convention $\text{ess inf } \emptyset = +\infty$. Here, we use the usual notation $\text{ess inf } \Gamma$ without mentioning the σ -algebra when this one is shared with the elements of Γ , i.e. $\mathcal{H} = \mathcal{F}$ in the definition. We denote by $\text{Dom } \rho_I$ the set of all $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$, such that $\mathcal{M}_I(X) \neq \emptyset$.

The proof of the following lemma, being simple, is left to the readers.

Lemma 6.1. *Let I be a conditional operator w.r.t. \mathcal{H} . We have the following properties:*

1. *If I is \mathcal{H} -positively homogeneous, then $\alpha_{\mathcal{H}}X \in \text{Dom } \rho_I$ for every X in $\text{Dom } \rho_I$ and $\alpha_{\mathcal{H}} \in \mathcal{L}^0(\mathbf{R}_+, \mathcal{H})$.*
2. *If I is super-additive, then $X_1 + X_2 \in \text{Dom } \rho_I$ for any $X_1, X_2 \in \text{Dom } \rho_I$.*
3. *If I is non decreasing, then $X_1 \in \text{Dom } \rho_I$ for any $X_1 \geq X_2$ such that $X_2 \in \text{Dom } \rho_I$.*
4. *If \mathbb{D}_I^+ is \mathcal{H} -convex (e.g. if I is \mathcal{H} -convex) then $\text{Dom } \rho_I$ is \mathcal{H} -convex.*

We now recall the definition of a risk measure, see [14] for example.

Definition 6.2. *Let \mathbb{D} be a subset of $\mathbb{L}^0(\mathbf{R}, \mathcal{F})$ containing 0 and such that $\mathbb{D} + \mathbb{L}^0(\mathbf{R}, \mathcal{H}) \subset \mathbb{D}$. We say that a mapping*

$$\begin{aligned} \rho_{\mathcal{H}} : \mathbb{D} &\longrightarrow \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H}). \\ X &\longmapsto \rho_{\mathcal{H}}(X) \end{aligned}$$

is an \mathcal{H} -conditional risk measure if the following properties hold:

- (P1) *Normalization: $\rho_{\mathcal{H}}(0) = 0$.*
- (P2) *Monotonicity: $\rho_{\mathcal{H}}(X_1) \leq \rho_{\mathcal{H}}(X_2)$, for any $X_1, X_2 \in \mathbb{D}$ such that we have $X_1 \geq X_2$.*
- (P3) *Cash invariance: for all $X \in \mathbb{D}$ and $\alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$, we have the equality $\rho_{\mathcal{H}}(X + \alpha_{\mathcal{H}}) = \rho_{\mathcal{H}}(X) - \alpha_{\mathcal{H}}$.*

Recall that we say that $\mathbb{D} \subseteq \mathbb{L}^0(\mathbf{R}, \mathcal{F})$ is \mathcal{H} -convex if, for all $X_1, X_2 \in \mathbb{D}$ and $\alpha_{\mathcal{H}} \in \mathbb{L}^0([0, 1], \mathcal{H})$, we have $\alpha_{\mathcal{H}}X_1 + (1 - \alpha_{\mathcal{H}})X_2 \in \mathbb{D}$. When $\alpha_{\mathcal{H}}X \in \mathbb{D}$ for all $X \in \mathbb{D}$ and $\alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}_+, \mathcal{H})$, we say that \mathbb{D} is positively homogenous.

Definition 6.3. *An \mathcal{H} -conditional risk measure ρ on \mathbb{D}_{ρ} is said:*

1. conditionally convex if \mathbb{D}_ρ is \mathcal{H} -convex and, for all $X_1, X_2 \in \mathbb{D}_\rho$ and $\alpha_{\mathcal{H}} \in \mathbb{L}^0([0, 1], \mathcal{H})$, we have:

$$\rho(\alpha_{\mathcal{H}}X_1 + (1 - \alpha_{\mathcal{H}})X_2) \leq \alpha_{\mathcal{H}}\rho(X_1) + (1 - \alpha_{\mathcal{H}})\rho(X_2).$$

2. conditionally positively homogeneous if \mathbb{D} is positively homogenous and, for all $X \in \mathbb{D}_\rho$ and $\alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}_+, \mathcal{H})$, $\rho(\alpha_{\mathcal{H}}X) = \alpha_{\mathcal{H}}\rho(X)$.

A conditional convex risk measure which is positively homogeneous is called a conditional coherent risk measure.

The proof of the following lemma is standard:

Proposition 6.4. *Let I be a non decreasing conditional operator. Consider the mapping ρ_I defined by (6.4) and the associated domain $\text{Dom } \rho_I$. We have the following properties:*

1. The mapping ρ_I is a conditional risk measure on $\text{Dom } \rho_I$.
2. If \mathbb{D}_I^+ is \mathcal{H} -convex (for example if I is \mathcal{H} -convex), then ρ_I is \mathcal{H} -convex.
3. If I is \mathcal{H} -positively homogeneous, then ρ_I is \mathcal{H} -positively homogeneous.
4. If I is super-additive, then ρ_I is sub-additive.

Lemma 6.5. *Let us consider a C.I. I and let us define $\rho(X) = I(-X)$. Then, ρ is a conditional risk-measure if and only if I is increasing and \mathcal{H} -translation invariant.*

Lemma 6.6. *Let us consider a C.I. I and let us define $\rho(X) = -I(X)$. Then, ρ is a conditional risk-measure if and only if I is increasing and \mathcal{H} -translation invariant.*

Under some conditions, we may show that a risk-measure admits a dual representation at least on $L^1(\mathbf{R}, \mathcal{F}_T)$, see [14] and the recent result on $L^0(\mathbf{R}, \mathcal{F}_T)$ in [23]. The open question is whether a conditional indicator may have such a characterization, at least if it is convex.

7. The conditional expectation indicator

The conditional expectation knowing $\mathcal{H} \subseteq \mathcal{F}$ is defined on $\mathbb{L}^0(\mathbf{R}, \mathcal{F})$ with the conventions introduced in the beginning of the paper by:

$$\mathbb{E}(X|\mathcal{H}) := \mathbb{E}(X^+|\mathcal{H}) - \mathbb{E}(X^-|\mathcal{H}).$$

Lemma 7.1. *The mapping $I(X) = \mathbb{E}(X|\mathcal{H})$ is a conditional indicator on $X \in \mathbb{D}_I = \mathbb{L}^0(\mathbf{R}, \mathcal{F})$. Moreover, if $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$, we have:*

$$1_H I(X) = I(X1_H), \text{ for all } H \in \mathcal{H}, \quad (7.5)$$

$$I(X + \alpha_{\mathcal{H}}) = I(X) + \alpha_{\mathcal{H}}, \text{ on the set } \{\mathbb{E}(X^+|\mathcal{H}) \neq +\infty \text{ or } \mathbb{E}(X^-|\mathcal{H}) \neq +\infty\}, \quad (7.6)$$

$$I(\alpha_{\mathcal{H}}X) = \alpha_{\mathcal{H}}I(X), \text{ for all } \alpha_{\mathcal{H}} \in \mathbb{L}^0(\mathbf{R}, \mathcal{H}). \quad (7.7)$$

Proof. Note that $(X1_H)^+ = 1_H X^+$ and $(X1_H)^- = 1_H X^-$ so that (7.5) holds by the property satisfied by the usual conditional expectation defined on the non negative random variables. Therefore, we may show the next properties on each subset of a \mathcal{H} -measurable partition of Ω . Actually, the equality (7.6) is a particular case of Lemma 7.2 we show below.

To show (7.7), it suffices to consider the cases $\alpha_{\mathcal{H}} \geq 0$ and $\alpha_{\mathcal{H}} < 0$. When, $\alpha_{\mathcal{H}} < 0$, we have $(\alpha_{\mathcal{H}}X)^+ = |\alpha_{\mathcal{H}}|X^-$ and $(\alpha_{\mathcal{H}}X)^- = |\alpha_{\mathcal{H}}|X^+$. We deduce that $\mathbb{E}(\alpha_{\mathcal{H}}X|\mathcal{H}) = |\alpha_{\mathcal{H}}|\mathbb{E}(X^-|\mathcal{H}) - |\alpha_{\mathcal{H}}|\mathbb{E}(X^+|\mathcal{H})$ and we conclude that $\mathbb{E}(X|\mathcal{H}) = -\alpha_{\mathcal{H}}\mathbb{E}(X^-|\mathcal{H}) + \alpha_{\mathcal{H}}\mathbb{E}(X^+|\mathcal{H}) = \alpha_{\mathcal{H}}\mathbb{E}(X|\mathcal{H})$.

At last, $X \leq \alpha_{\mathcal{H}} = \text{ess sup}_{\mathcal{H}}(X)$ and it is clear that $\mathbb{E}(X|\mathcal{H}) \leq \alpha_{\mathcal{H}}$ when $\alpha_{\mathcal{H}} = +\infty$. Otherwise, as $\mathbb{E}(X^+|\mathcal{H}) \leq (\text{ess sup}_{\mathcal{H}}(X))^+$, $\alpha_{\mathcal{H}} \neq +\infty$ implies that $\mathbb{E}(X^+|\mathcal{H}) \neq +\infty$ hence (7.6) applies. So, $\mathbb{E}(X - \alpha_{\mathcal{H}}|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}) - \alpha_{\mathcal{H}}$. As, $X - \alpha_{\mathcal{H}} \leq 0$, we get that $\mathbb{E}(X - \alpha_{\mathcal{H}}|\mathcal{H}) \leq 0$ hence $\mathbb{E}(X|\mathcal{H}) - \alpha_{\mathcal{H}} \leq 0$ and, finally, $\mathbb{E}(X|\mathcal{H}) \leq \alpha_{\mathcal{H}}$. Indeed, it suffices to observe that $\mathbb{E}(X|\mathcal{H}) \neq +\infty$.

As $X \geq \alpha_{\mathcal{H}} = \text{ess inf}_{\mathcal{H}}(X)$, we conclude similarly and the conclusion follows. \square

Lemma 7.2. *Let us consider the operator $I(X) = \mathbb{E}(X|\mathcal{H})$ on $\mathbb{D}_L = \mathbb{L}^0(\mathbf{R}, \mathcal{F})$. If $X, Y \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$, then we have $I(X + Y) = I(X) + I(Y)$ on the set $F = \bigcup_{i \in [1,5]} F_i \in \mathcal{H}$ where*

$$\begin{aligned} F_1 &= \{\mathbb{E}(|X||\mathcal{H}), \mathbb{E}(|Y||\mathcal{H}) \in \mathbf{R}\}, \\ F_2 &= \{\mathbb{E}(X^+|\mathcal{H}) = +\infty, (\mathbb{E}(X^-|\mathcal{H}), \mathbb{E}(Y^-|\mathcal{H})) \in \mathbf{R}^2\}, \\ F_3 &= \{\mathbb{E}(X^-|\mathcal{H}) = +\infty, (\mathbb{E}(X^+|\mathcal{H}), \mathbb{E}(Y^+|\mathcal{H})) \in \mathbf{R}^2\}, \\ F_4 &= \{\mathbb{E}(Y^+|\mathcal{H}) = +\infty, (\mathbb{E}(X^-|\mathcal{H}), \mathbb{E}(Y^-|\mathcal{H})) \in \mathbf{R}^2\}, \\ F_5 &= \{\mathbb{E}(Y^-|\mathcal{H}) = +\infty, (\mathbb{E}(X^+|\mathcal{H}), \mathbb{E}(Y^+|\mathcal{H})) \in \mathbf{R}^2\}. \end{aligned}$$

Proof. In the following, we shall use the convexity and positive homogeneity of the mappings $x \mapsto x^+ = \max(x, 0)$ and $x \mapsto x^- = \max(-x, 0)$. This

implies that $(x + y)^+ \leq x^+ + y^+$ and $(x + y)^- \leq x^- + y^-$ for all $x, y \in \mathbf{R}$. Consider $X, Y \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$. We then have:

$$\begin{aligned} \mathbb{E}((X + Y)^+ | \mathcal{H}) &\leq \mathbb{E}(X^+ | \mathcal{H}) + \mathbb{E}(Y^+ | \mathcal{H}), \\ \mathbb{E}((X + Y)^- | \mathcal{H}) &\leq \mathbb{E}(X^- | \mathcal{H}) + \mathbb{E}(Y^- | \mathcal{H}), \\ \mathbb{E}(X^+ | \mathcal{H}) &\leq \mathbb{E}((X + Y)^+ | \mathcal{H}) + \mathbb{E}(Y^- | \mathcal{H}), \\ \mathbb{E}(X^- | \mathcal{H}) &\leq \mathbb{E}((X + Y)^- | \mathcal{H}) + \mathbb{E}(Y^+ | \mathcal{H}). \end{aligned} \tag{7.8}$$

Note that the inequalities above are obvious as soon as one of the terms in the r.h.s. is $+\infty$. Otherwise, we may argue as if the random variables were integrable.

1st case: On the set F_1 , X and Y are integrable so the result holds by linearity of the conditional expectation for integrable random variables.

2nd case: On the set F_2 , by (7.8), we have $\mathbb{E}((X + Y)^- | \mathcal{H}) \in \mathbf{R}$ and $\mathbb{E}((X + Y)^+ | \mathcal{H}) = +\infty$. Therefore, $L(X + Y) = L(X) + L(Y) = +\infty$ and the equality holds.

3rd case: On the set F_3 , by (7.8), we have $\mathbb{E}((X + Y)^- | \mathcal{H}) = +\infty$ and $\mathbb{E}((X + Y)^+ | \mathcal{H}) \in \mathbf{R}$. Therefore, $L(X + Y) = L(X) + L(Y) = -\infty$.

By symmetry, the same conclusion holds on the subsets F_4 and F_5 . \square

Proposition 7.3. *Let \mathcal{H} be a sub σ -algebra of \mathcal{F} and let I be a C.I. w.r.t. \mathcal{H} such that $\mathbb{D}_I = L^1(\mathbf{R}, \mathcal{F})$ and*

- 1) $I^* = I$
- 2) I is sub-additive (respectively super-additive).
- 3) $E(|I(X)|) \leq E(|X|)$.

Then $I = \mathbb{E}(\cdot | \mathcal{H})$ on $L^1(\mathbf{R}, \mathcal{F})$.

Proof. By Proposition 4.5, I is a linear indicator. As $1 \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$, $I(1) = 1$. Moreover, I is contractive by assumption. We conclude by Douglas Theorem, see [1], that $I = \mathbb{E}(\cdot | G)$ where G is the σ -algebra generated by the fixed points of I . It is clear that $G = \mathcal{H}$ hence $I = \mathbb{E}(\cdot | \mathcal{H})$. \square

Theorem 7.4. *Suppose that I is a conditional indicator defined on the domain $\mathbb{D}_I = L^1(\mathbf{R}, \mathcal{F})$ and satisfies the following properties:*

- 1) I is self-dual.
- 2) $I(X + Y) = I(X) + I(Y)$ for all $X, Y \in L^1(\mathbf{R}, \mathcal{F})$.

3) I satisfies the Fatou property i.e., for any sequence $(X_n)_n$ of $\mathbb{L}^1(\mathbf{R}, \mathcal{F})$, we have $I(\liminf_n X_n) \leq \liminf I(X_n)$.

Then, there exists a probability measure $\mu \ll \mathbb{P}$, with $\rho = d\mu/d\mathbb{P} \in \mathbb{L}^1(\mathbf{R}_+, \mathcal{F})$ such that $I(X) = E_\mu(X|\mathcal{H}) = \mathbb{E}(\rho X|\mathcal{H})$, for all $X \in \mathbb{L}^1(\mathbf{R}, \mathcal{H})$.

Proof. Since L is sub-additive and self-dual, we deduce by Lemma 4.5 that L is \mathcal{H} -linear. Moreover, L is increasing by Lemma 2.5. Let us define the mapping $\mu(A) = \mathbb{E}(I(1_A))$, for any $A \in \mathcal{F}$. Let us prove that μ is a probability measure.

As $1_A \in [0, 1]$, then $I(1_A) \in [0, 1]$ a.s. hence $\mu(A) \in [0, 1]$. Moreover, $I(1_\Omega) = I(1) = 1$. So $\mu(\Omega) = 1$. Also $\mu(\emptyset) = \mathbb{E}(I(0)) = 0$. Consider a partition $(A_n)_{n \in \mathbb{N}}$ of Ω . We have

$$\sum_{n=0}^{\infty} 1_{A_n} = \lim_{N \rightarrow \infty} \sum_{n=0}^N 1_{A_n} \geq \sum_{n=0}^N 1_{A_n}.$$

Therefore,

$$I\left(\sum_{n=0}^{\infty} 1_{A_n}\right) \geq \lim_{N \rightarrow \infty} I\left(\sum_{n=0}^N 1_{A_n}\right) = \lim_{N \rightarrow \infty} \sum_{n=0}^N I(1_{A_n}).$$

We deduce that $\mu(\sum_{n=0}^{\infty} 1_{A_n}) \geq \sum_{n=0}^{\infty} \mu(1_{A_n})$. Moreover, by the Fatou property, we have

$$I\left(\lim_{N \rightarrow \infty} \uparrow \sum_{n=0}^N 1_{A_n}\right) \leq \lim_{N \rightarrow \infty} \uparrow I\left(\sum_{n=0}^N 1_{A_n}\right) = \lim_{N \rightarrow \infty} \sum_{n=0}^N I(1_{A_n}).$$

So, $\mu(\sum_{n=0}^{\infty} 1_{A_n}) \leq \sum_{n=0}^{\infty} \mu(1_{A_n})$. We conclude that μ is a probability measure. Note that, if $\mathbb{P}(N) = 0$, then $I(1_N) = I(0) = 0$, i.e. $\mu(N) = 0$. Therefore μ is absolutely continuous w.r.t. P . Let $\rho = d\mu/dP$ be the Radon-Nikodym derivative. We aim to show that $I = \mathbb{E}_\mu[\cdot|\mathcal{H}]$.

Consider $A \in \mathcal{H}$. In one hand, $\mu(A) = \mathbb{E}(I(1_A)) = \mathbb{E}(1_A)$ by definition of μ and I . In the other hand, $\mu(A) = \mathbb{E}(\rho 1_A)$ as $\rho = d\mu/d\mathbb{P}$. Therefore, $\mathbb{E}(\rho 1_A) = \mathbb{E}(1_A)$ for any $A \in \mathcal{H}$ hence $\mathbb{E}(\rho|\mathcal{H}) = 1$. Moreover, for any $A \in \mathcal{F}$, as $I(1_A)$ is \mathcal{H} -measurable, we get that

$$\mathbb{E}_\mu(I(1_A)) = \mathbb{E}(\rho I(1_A)) = \mathbb{E}(I(1_A)) = \mu(A) = \mathbb{E}_\mu(1_A).$$

So, for any $A \in \mathcal{F}$ and $B \in \mathcal{H}$, we have:

$$\mathbb{E}_\mu(1_A 1_B) = \mathbb{E}_\mu(I(1_A 1_B)) = \mathbb{E}_\mu(1_B I(1_A)).$$

This implies that $\mathbb{E}_\mu[1_A | \mathcal{H}] = I(1_A)$, for all $A \in \mathcal{F}$.

Consider now $X \in L^0(\mathbf{R}_+, \mathcal{F})$. We use the standard arguments, i.e. we have $X = \lim_n \uparrow X_n$ where $X^n = \sum_{i=0}^n \alpha_i^n 1_{A_i^n}$, $(A_i^n)_i$ is a partition of Ω in \mathcal{F} and $\alpha_i^n \in \mathbf{R}$. By the Fatou property, $I(X) \leq \lim_n \uparrow \sum_{i=0}^n \alpha_i^n I(1_{A_i^n})$. On the other hand, $X \geq \sum_{i=0}^n \alpha_i^n 1_{A_i^n}$ implies that $I(X) \geq \sum_{i=0}^n \alpha_i^n I(1_{A_i^n})$ for any n . Therefore, $I(X) \geq \lim_n \uparrow \sum_{i=0}^n \alpha_i^n I(1_{A_i^n})$. We deduce that:

$$I(X) = \lim_n \uparrow \sum_{i=0}^n \alpha_i^n I(1_{A_i^n}) = \lim_n \uparrow \sum_{i=0}^n \alpha_i^n \mathbb{E}_\mu[1_{A_i^n} | \mathcal{H}] = \mathbb{E}_\mu[X | \mathcal{H}].$$

Finally, for any $X \in L^1(\mathbf{R}, \mathcal{F})$, we have:

$$I(X) = I(X^+) - I(X^-) = \mathbb{E}_\mu[X^+ | \mathcal{H}] - \mathbb{E}_\mu[X^- | \mathcal{H}] = \mathbb{E}_\mu[X | \mathcal{H}].$$

Since $\mathbb{E}(\rho | H) = 1$, we finally deduce that $I(X) = \mathbb{E}(\rho X | \mathcal{H})$. \square

In the following, we construct linear indicators that are not conditional expectations.

Counter-example The following is standard. Consider the space $\Omega = \mathbb{N}$ of all non negative integers endowed with the σ -algebra \mathcal{F} of all subsets of \mathbb{N} . The probability measure is defined as $P(A) = \sum_{n=0}^\infty 2^{-n-1} \delta_n(A)$ where δ_n is the Dirac measure at point n . Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ be the trivial sub σ -algebra. Each random variable w.r.t. (Ω, \mathcal{F}, P) is identifiable with the sequence $(X(n))_{n \in \mathbb{N}}$ and we have X_k converges a.s. to X when $k \rightarrow \infty$ if and only if $X_k(n) \rightarrow X(n)$, for all $n \in \mathbb{N}$. The L^∞ norm is $\|X\|_\infty = \sup_n |X(n)|$ and we have $\text{ess sup}_{\mathcal{F}_0} = \sup_n X(n)$ and $\text{ess inf}_{\mathcal{F}_0} = \inf_n X(n)$. In the following, we consider the set \mathbb{D} of all $X \in L^0(\mathbf{R}, \mathcal{F})$ such that $\lim_n X(n)$ exists in \mathbf{R} . We define the linear positive operator $T(X) = \lim_n X(n)$ on the domain \mathbb{D} . Note that $X \in L^0(\mathbf{R}, \mathcal{F}_0)$ if and only if X is a constant sequence so that $X \in \mathbb{D}$. In particular, $T(X) = X$ for any $X \in L^0(\mathbf{R}, \mathcal{F}_0)$ and T is a C.I. w.r.t. \mathcal{F}_0 . As $|T(X)| \leq \|X\|_\infty$ for all $X \in L^\infty(\mathbf{R}, \mathcal{F})$, the Hahn-Banach theorem states the existence of a (continuous) linear mapping \bar{T} defined on the whole space $L^\infty(\mathbf{R}, \mathcal{F}) \supset \mathbb{D}$ such that $\bar{T} = T$ on \mathbb{D} , see [5][Chapter 2].

For every $A \in \mathcal{F}$, consider the random variable

$$X^A(\omega) = \frac{\text{card}(A \cap [0, n])}{n+1},$$

where card designates the number of elements that contains a subset. We have $X^A \in L^\infty[0, 1], \mathcal{F}$. Let us define $m(A) = \bar{T}(X^A)$. We may show that $m(\emptyset) = 0$, $m(\Omega) = 1$ and $m(\sum_{i=1}^n A_i) = \sum_{i=1}^n m(A_i)$ if $(A_i)_{i=1}^n$ is a finite partition.

Moreover, suppose that \bar{T} is an expectation on $L^\infty(\mathbf{R}, \mathcal{F})$, i.e. there exists an integrable random variable $Y \in L^1(\mathbf{R}_+, \mathcal{F})$ such that $\bar{T}(X) = E(YX)$ for any $X \in L^\infty(\mathbf{R}, \mathcal{F})$. In that case, if $(A_i)_{i=1}^n$ is an infinite partition, we get that

$$1 = m(\Omega) = \sum_{i=1}^{\infty} m(A_i).$$

With $A_i = \{i\}$, $i \geq 0$, we get that $\hat{T}(X^{A_i}) = T(X^{A_i}) = 0$, i.e. $m(A_i) = 0$ for all $i \in \mathbb{N}$. This is in contradiction with the equality above.

Another example is to consider $\mathcal{F}_1 = \{\emptyset, \Omega, I, I^c\}$ where $I^c = \Omega \setminus I$ and $I = 2\mathbb{N} + 1$. Let us introduce the indicator:

$$\begin{aligned} T_1(X) &= \hat{T}(\tilde{X})1_I + E(X|\mathcal{F}_1)1_{I^c}, \quad X \in L^\infty(\mathbf{R}, \mathcal{F}), \\ \tilde{X}(n) &= X(n)1_I(n) + X(n+1)1_{I^c}(n). \end{aligned}$$

We observe that T_1 is linear, $T_1(X)$ is \mathcal{F}_1 -measurable for any $X \in L^\infty(\mathbf{R}, \mathcal{F})$, i.e. $T_1(X)$ is constant on I and I^c respectively and, moreover, $T_1(X) = X$ if $X \in L^\infty(\mathbf{R}, \mathcal{F})$. Therefore, we deduce by monotony that T_1 is a linear conditional indicator. By the same arguments, we then prove that T_1 is not a conditional expectation of the form $T_1(X) = E(XY|\mathcal{F}_1)$, $X \in L^\infty(\mathbf{R}, \mathcal{F})$, for some $Y \in L^\infty(\mathbf{R}, \mathcal{F})$. Indeed, otherwise, we get the equality $1 = 0$ on the non null set I .

8. Appendix

Lemma 8.1. *For $r_H \in \mathbb{L}^1(\mathbf{Q}, \mathcal{H})$. There exist $p_H, q_H \in \mathbb{L}^1(\mathbb{Z}, \mathcal{H}) \times \mathbb{L}^1(\mathbb{N}^*, \mathcal{H})$ such that $r_H = p_H/q_H$.*

Proof. Consider the random set $\Gamma(\omega) = \{(p, q) \in \mathbb{Z} \times \mathbb{N}^* : r_H q = p\}$. We observe that its graph $Graph\Gamma = \{(\omega, p, q) : (p, q) \in \Gamma(\omega)\}$ is a measurable set of $\mathcal{H} \times \mathcal{B}(\mathbb{Z}) \times \mathcal{B}(\mathbb{N}^*)$, σ -algebra product of \mathcal{H} and the Borel σ -algebras of \mathbb{Z} and \mathbb{N}^* respectively and $\Gamma(\omega)$ is non empty. Indeed $f : \omega \mapsto r_H(\omega)q - p$ is measurable. Therefore, we conclude by a measurable selection argument, see [19][Section A.4], that there exists a measurable selector (p_H, q_H) of Γ . \square

Lemma 8.2. For any $\alpha \in \mathbb{L}^1(\mathbf{R}, \mathcal{H})$, there exist $(r_n)_n, (q_n)_n \in \mathbb{L}^1(\mathbf{Q}, \mathcal{H})$ such that

$$\alpha - 1/n \leq r_n \leq \alpha \leq q_n \leq \alpha + 1/n.$$

Proof. Consider $\Gamma(\omega) = \{(r_n, q_n) \in \mathbb{Q}^2 : \alpha - 1/n \leq r_n \leq \alpha \leq q_n \leq \alpha + 1/n\}$. It is non empty a.s. and its graph is a measurable set of $\mathcal{H} \times \mathcal{B}(\mathbb{Z}) \times \mathcal{B}(\mathbb{N}^*)$. We then conclude by a measurable argument, see [19][Section A.4]. \square

Lemma 8.3. Consider $F_t \in \mathcal{F}_t$ such that $\mathbb{P}(F_t) > 0$ and $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_T)$ such that $X = X1_{F_t}$ and $\text{ess sup}_{\mathcal{F}_0}(X) \neq 0$. If, for some $\varepsilon \in \mathbf{R}_+$, we have

$$\text{ess sup}_{\mathcal{F}_0}(X - \varepsilon 1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X), \quad (8.9)$$

then $\varepsilon = 0$.

Proof. A first case is when $1_{F_t} = 1$. In that case $\text{ess sup}_{\mathcal{F}_0}(X) - \varepsilon = \text{ess sup}_{\mathcal{F}_0}(X)$, thus $\varepsilon = 0$. Suppose that $1_{F_t} < 1$, i.e. $\mathbb{P}(\Omega \setminus F_t) > 0$. As $\text{ess sup}_{\mathcal{F}_0}(X) \geq X$ a.s., we deduce that $\text{ess sup}_{\mathcal{F}_0}(X) \geq 0$ on $\Omega \setminus F_t \neq \emptyset$ hence $\text{ess sup}_{\mathcal{F}_0}(X) \geq 0$. Note that (8.9) is equivalent to $\alpha_0 := \text{ess sup}_{\mathcal{F}_0}(X + \varepsilon 1_{\Omega \setminus F_t}) = \text{ess sup}_{\mathcal{F}_0}(X) + \varepsilon$. We observe that $\alpha_0 \geq \text{ess sup}_{\mathcal{F}_0}(X) \vee \varepsilon$ and, also, $\text{ess sup}_{\mathcal{F}_0}(X) \vee \varepsilon \geq X + \varepsilon 1_{\Omega \setminus F_t}$ on $\Omega \setminus F_t$.

On the other hand, on the set F_t , we also have

$$\alpha_0 \geq \text{ess sup}_{\mathcal{F}_0}(X) \vee \varepsilon \geq X = X + \varepsilon 1_{F_t^c}.$$

Therefore, a.s. we have $\alpha_0 \geq \text{ess sup}_{\mathcal{F}_0}(X) \vee \varepsilon \geq \text{ess sup}_{\mathcal{F}_0}(X + \varepsilon 1_{\Omega \setminus F_t})$. We deduce that $\text{ess sup}_{\mathcal{F}_0}(X) \vee \varepsilon = \alpha_0$. This implies that (8.9) is equivalent to

$$\text{ess sup}_{\mathcal{F}_0}(X) \vee \varepsilon = \text{ess sup}_{\mathcal{F}_0}(X) + \varepsilon.$$

If $\text{ess sup}_{\mathcal{F}_0}(X) \geq \varepsilon$, then $\text{ess sup}_{\mathcal{F}_0}(X) = \text{ess sup}_{\mathcal{F}_0}(X) + \varepsilon$ and thus $\varepsilon = 0$. If $\text{ess sup}_{\mathcal{F}_0}(X) < \varepsilon$, we get that $\text{ess sup}_{\mathcal{F}_0}(X) = 0$ in contradiction with the assumption. \square

Corollary 8.4. Consider $F_t \in \mathcal{F}_t$ such that $\mathbb{P}(F_t) > 0$ and $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_T)$ such that $\text{ess sup}_{\mathcal{F}_0}(X1_{F_t}) \neq 0$. If, for some $\varepsilon \geq 0$,

$$\text{ess sup}_{\mathcal{F}_0}((X - \varepsilon)1_{F_t}) = \text{ess sup}_{\mathcal{F}_0}(X1_{F_t}),$$

then $\varepsilon = 0$.

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