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# Reasoning about Cyclic Space: Axiomatic and Computational Aspects

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**Abstract.** In this paper we propose models of the axioms for linear and cyclic orders. First, we describe explicitly the relations between linear and cyclic models, from a logical point of view. The second part of the paper is concerned with qualitative constraints: we study the cyclic point algebra. This formalism is based on ternary relations which allow to express cyclic orientations. We give some results of complexity about the consistency problem in this formalism. The last part of the paper is devoted to conceptual spaces. The notion of a conceptual space is related to the complexity properties of temporal and spatial qualitative formalisms, including the cyclic point algebra.

## 1 Introduction

Much attention in the domain of qualitative temporal and spatial reasoning has been devoted to the study of spaces which are ultimately based on some version of a Euclidean space: Allen's calculus [1] is the qualitative study of pairs of points in the 1-D Euclidean space, the real line; the Cardinal Direction calculus [8,20], the  $n$ -point calculus [4], the rectangle calculus [3], the  $n$ -block calculus [5], the line segments calculus [22], refer to entities in (Cartesian products of the real line), which is an unbounded, dense linear ordering.

There are however good reasons for considering spaces which are not based on linear orderings. The set of directions around a reference points has a cyclic, rather than a linear structure. Schlieder's concepts of orientation and panoramas [25,26] are examples of proposals for reasoning about cyclic situations. Such is Röhrig's theory CycOrd [24] and work of Sogo *et al.* [28]. More recently, Cohn and Isli [12] have considered points on a circle and the ternary relations between them, obtaining substantial results about the complexity of the corresponding calculi. Finally, the binary relations between intervals on a circle have been considered [7]. If we think of the particular field of applications to reasoning about geographical or cartographic entities, it is clear that many applications may need to consider cycles such as parallels or meridians on the Earth's surface.

When studying spaces with a cyclic structure, it seems quite reasonable not to consider the cyclic case as a *tabula rasa*. After all, from a topological point of view, a circle is easily derivable from a segment (or a line) by identifying the end-points. Conversely, cutting a circle makes it into a line. This is the intuition behind the work presented in this paper: The idea is to exploit, as much as possible, the relationships between linear

and cyclic models. Technically, this will also involve the relationships between binary relations between points on a line, which are enough to characterize the “qualitative” relation between them, and ternary relations between three points on a circle, which are necessary for the analogous characterization.

The structure of the paper is as follows: Firstly, we describe explicitly the relations between linear and cyclic models, from a logical point of view. The main result is that, in the same way as there is basically one countable model of an unbounded, dense and linear ordering (Cantor’s theorem), a similar result obtains for suitable well chosen axioms involving one “betweenness” ternary relation between points on a circle (the main condition here is to have at least two points, plus density). Then, we consider six possible ternary relations between three points on a circle, which are jointly exhaustive and pairwise disjoint (JEPD) relations, and develop a qualitative calculus, and examine the problem of determining consistency for the corresponding constraint networks. We describe various subsets where we can prove either tractability or NP-completeness. Finally, in a last section, we consider the problem of extending the known complexity results for the linear calculi to the cyclic cases. Although the initial results (about Allen’s algebra) were first proved using logical tools, it appears that most of them can also be expressed in geometric and topological terms, which can be also understood as particular cases of conceptual spaces introduced by Gärdenfors. We present the basic notions of the framework of conceptual spaces, in relation to the characterization of tractable subclasses. We then speculate on the possibility of extending the geometric and topological characterizations of tractable classes to the cyclic case.

## 2 Models of Axioms for Linear and Cyclic Orders

This section is devoted to the semantical analysis of the relationship between linear orders and cyclic orders.

### 2.1 Linear Orders

A linear order is a structure of the form  $\mathcal{M} = (\mathcal{T}, <)$  where  $\mathcal{T}$  is a set of points and  $<$  is a binary relation on  $\mathcal{T}$  subject to the following universal conditions for all  $x, y, z \in \mathcal{T}$ :

- Not  $x < x$ ;
- If  $x < y$  and  $y < z$  then  $x < z$ ;
- Either  $x = y$  or  $x < y$  or  $y < x$ .

A linear order  $\mathcal{M} = (\mathcal{T}, <)$  is dense if it satisfies the following principle:

- For all  $x, y \in \mathcal{T}$ , if  $x < y$  then there is  $z \in \mathcal{T}$  such that  $x < z$  and  $z < y$ .

A linear order  $\mathcal{M} = (\mathcal{T}, <)$  is unbounded if it satisfies the following principles:

- For all  $x \in \mathcal{T}$ , there is  $y \in \mathcal{T}$  such that  $y < x$ ;
- For all  $x \in \mathcal{T}$ , there is  $y \in \mathcal{T}$  such that  $x < y$ .

Before turning to cyclic orders and to a detailed investigation of their relationship with linear orders, let us remind the reader of the following result.

**Proposition 1.** *Let  $\mathcal{M}_1 = (\mathcal{T}_1, <_1)$  and  $\mathcal{M}_2 = (\mathcal{T}_2, <_2)$  be two countable linear orders. If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are dense and unbounded then they are isomorphic.*

**Proof.** By Cantor's well known zig-zag argument.  $\square$

In technical terminology, the set of all dense and unbounded linear orders is countably categorical. Hence as far as countable structures are concerned, there is only one dense and unbounded linear order, the structure  $(\mathbf{Q}, <)$  of the rational numbers.

## 2.2 Cyclic Orders

A cyclic order is a structure of the form  $\mathcal{M} = (\mathcal{T}, \prec)$  where  $\mathcal{T}$  is a nonempty set of points and  $\prec$  is a ternary relation on  $\mathcal{T}$  subject to the following universal conditions for all  $x, y, z, t \in \mathcal{T}$ :

- Not  $\prec(x, y, y)$ ;
- If  $\prec(x, y, z)$  and  $\prec(x, z, t)$  then  $\prec(x, y, t)$ ;
- If  $x \neq y$  and  $x \neq z$  then either  $y = z$  or  $\prec(x, y, z)$  or  $\prec(x, z, y)$ ;
- $\prec(x, y, z)$  iff  $\prec(y, z, x)$  iff  $\prec(z, x, y)$ .

A cyclic order  $\mathcal{M} = (\mathcal{T}, \prec)$  is standard if it satisfies the following principles:

- For all  $x, y \in \mathcal{T}$ , if  $x \neq y$  then there is  $z \in \mathcal{T}$  such that  $\prec(x, z, y)$ ;
- For all  $x, y \in \mathcal{T}$ , if  $x \neq y$  then there is  $z \in \mathcal{T}$  such that  $\prec(x, y, z)$ .

Referring to propositions 1, 4 and 6, we easily obtain a proof of the following result.

**Proposition 2.** *Let  $\mathcal{M}_1 = (\mathcal{T}_1, \prec_1)$  and  $\mathcal{M}_2 = (\mathcal{T}_2, \prec_2)$  be two countable cyclic orders. If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are standard then they are isomorphic.*

**Proof.** Assume that  $\mathcal{M}_1 = (\mathcal{T}_1, \prec_1)$  and  $\mathcal{M}_2 = (\mathcal{T}_2, \prec_2)$  are standard. Let  $\infty_1$  be a point such that  $\infty_1 \in \mathcal{T}_1$  and  $\infty_2$  be a point such that  $\infty_2 \in \mathcal{T}_2$ . Let  $\mathcal{M}'_1 = (\mathcal{T}'_1, <'_1)$  be the linear order on  $\mathcal{M}_1 = (\mathcal{T}_1, \preceq_1)$  and  $\infty_1$  and  $\mathcal{M}'_2 = (\mathcal{T}'_2, <'_2)$  be the linear order on  $\mathcal{M}_2 = (\mathcal{T}_2, \preceq_2)$  and  $\infty_2$ . By proposition 4,  $\mathcal{M}'_1 = (\mathcal{T}'_1, <'_1)$  and  $\mathcal{M}'_2 = (\mathcal{T}'_2, <'_2)$  are dense and unbounded. By proposition 1,  $\mathcal{M}'_1 = (\mathcal{T}'_1, <'_1)$  and  $\mathcal{M}'_2 = (\mathcal{T}'_2, <'_2)$  are isomorphic. Let  $\infty'_1$  be a point such that  $\infty'_1 \notin \mathcal{T}'_1$  and  $\infty'_2$  be a point such that  $\infty'_2 \notin \mathcal{T}'_2$ . Let  $\mathcal{M}''_1 = (\mathcal{T}''_1, \prec''_1)$  be the cyclic order on  $\mathcal{M}'_1 = (\mathcal{T}'_1, <'_1)$  and  $\infty'_1$  and  $\mathcal{M}''_2 = (\mathcal{T}''_2, \prec''_2)$  be the cyclic order on  $\mathcal{M}'_2 = (\mathcal{T}'_2, <'_2)$  and  $\infty'_2$ . We see without difficulty that  $\mathcal{M}''_1 = (\mathcal{T}''_1, \prec''_1)$  and  $\mathcal{M}''_2 = (\mathcal{T}''_2, \prec''_2)$  are isomorphic. By proposition 6,  $\mathcal{M}_1 = (\mathcal{T}_1, \prec_1)$  and  $\mathcal{M}''_1 = (\mathcal{T}''_1, \prec''_1)$  are isomorphic and  $\mathcal{M}_2 = (\mathcal{T}_2, \prec_2)$  and  $\mathcal{M}''_2 = (\mathcal{T}''_2, \prec''_2)$  are isomorphic. Hence  $\mathcal{M}_1 = (\mathcal{T}_1, \prec_1)$  and  $\mathcal{M}_2 = (\mathcal{T}_2, \prec_2)$  are isomorphic.  $\square$

In this way, the set of all standard cyclic orders is countably categorical. Consequently as far as countable structures are concerned, there is only one standard cyclic order, the structure  $(\mathbf{Q} \cup \{\infty\}, \prec)$  obtained from the structure  $(\mathbf{Q}, <)$  of the rational numbers by the construction of section 2.3.

## 2.3 From Linear Orders to Cyclic Orders

Let  $\mathcal{M} = (\mathcal{T}, <)$  be a linear order and  $\infty$  be a point such that  $\infty \notin \mathcal{T}$ . Let  $\mathcal{M}' = (\mathcal{T}', \prec')$  be the structure where  $\mathcal{T}' = \mathcal{T} \cup \{\infty\}$  and  $\prec'$  is the ternary relation on  $\mathcal{T}'$  defined as follows for all  $x, y, z \in \mathcal{T}'$ :

- $\prec'$   $(x, y, z)$  iff either  $(x, y, z \in \mathcal{T}$  and  $x < y < z)$  or  $(x, y, z \in \mathcal{T}$  and  $y < z < x)$  or  $(x, y, z \in \mathcal{T}$  and  $z < x < y)$  or  $(x = \infty, y, z \in \mathcal{T}$  and  $y < z)$  or  $(y = \infty, x, z \in \mathcal{T}$  and  $z < x)$  or  $(z = \infty, x, y \in \mathcal{T}$  and  $x < y)$ .

$\mathcal{M}' = (\mathcal{T}', \prec')$  is called cyclic order on  $\mathcal{M} = (\mathcal{T}, <)$  and  $\infty$ . The reader may easily verify the following result.

**Proposition 3.** *Let  $\mathcal{M} = (\mathcal{T}, <)$  be a linear order and  $\infty$  be a point such that  $\infty \notin \mathcal{T}$ . The cyclic order  $\mathcal{M}' = (\mathcal{T}', \prec')$  on  $\mathcal{M} = (\mathcal{T}, <)$  and  $\infty$  is a cyclic order. Moreover, if  $\mathcal{M} = (\mathcal{T}, <)$  is dense and unbounded then  $\mathcal{M}' = (\mathcal{T}', \prec')$  is standard.*

## 2.4 From Cyclic Orders to Linear Orders

Let  $\mathcal{M} = (\mathcal{T}, \prec)$  be a cyclic order and  $\infty$  be a point such that  $\infty \in \mathcal{T}$ . Let  $\mathcal{M}' = (\mathcal{T}', \prec')$  be the structure where  $\mathcal{T}' = \mathcal{T} \setminus \{\infty\}$  and  $\prec'$  is the binary relation on  $\mathcal{T}'$  defined as follows for all  $x, y \in \mathcal{T}'$ :

- $x \prec' y$  iff  $\prec(x, y, \infty)$ .

$\mathcal{M}' = (\mathcal{T}', \prec')$  is called linear order on  $\mathcal{M} = (\mathcal{T}, \prec)$  and  $\infty$ . The reader may easily verify the following result.

**Proposition 4.** *Let  $\mathcal{M} = (\mathcal{T}, \prec)$  be a cyclic order and  $\infty$  be a point such that  $\infty \in \mathcal{T}$ . The linear order  $\mathcal{M}' = (\mathcal{T}', \prec')$  on  $\mathcal{M} = (\mathcal{T}, \prec)$  and  $\infty$  is a linear order. Moreover, if  $\mathcal{M} = (\mathcal{T}, \prec)$  is standard then  $\mathcal{M}' = (\mathcal{T}', \prec')$  is dense and unbounded.*

## 2.5 An Equivalence Result

We now prove that the two constructions defined above yield an equivalence result between the two classes of models. Let  $\mathcal{M} = (\mathcal{T}, <)$  be a countable linear order,  $\infty$  be a point such that  $\infty \notin \mathcal{T}$ ,  $\mathcal{M}' = (\mathcal{T}', \prec')$  be the cyclic order on  $\mathcal{M} = (\mathcal{T}, <)$  and  $\infty$ ; let  $\infty'$  be a point such that  $\infty' \in \mathcal{T}'$  and  $\mathcal{M}'' = (\mathcal{T}'', \prec'')$  be the linear order on  $\mathcal{M}' = (\mathcal{T}', \prec')$  and  $\infty'$ . Let us assume that  $\mathcal{M} = (\mathcal{T}, <)$  is dense and unbounded. If  $\infty = \infty'$  then it is easily seen that the function  $f$  with domain  $\mathcal{T}$  and range  $\mathcal{T}''$  defined as follows for all  $x \in \mathcal{T}$ :

- $f(x) = x$ ;

is an isomorphism from  $\mathcal{M} = (\mathcal{T}, <)$  to  $\mathcal{M}'' = (\mathcal{T}'', \prec'')$ . If  $\infty \neq \infty'$  then the submodels  $\mathcal{M}^- = (\mathcal{T}^-, \prec^-)$  and  $\mathcal{M}^+ = (\mathcal{T}^+, \prec^+)$  where  $\mathcal{T}^- = \{x: x < \infty'\}$  and  $\mathcal{T}^+ = \{x: \infty' < x\}$  are dense and unbounded. By proposition 1,  $\mathcal{M}^- = (\mathcal{T}^-, \prec^-)$  and  $\mathcal{M}^+ = (\mathcal{T}^+, \prec^+)$  are isomorphic. Hence there is a function  $f^\mp$  with domain  $\mathcal{T}^-$  and range  $\mathcal{T}^+$  such that  $f^\mp$  is an isomorphism from  $\mathcal{M}^- = (\mathcal{T}^-, \prec^-)$  to  $\mathcal{M}^+ = (\mathcal{T}^+, \prec^+)$  and there is a function  $f^\pm$  with domain  $\mathcal{T}^+$  and range  $\mathcal{T}^-$  such that  $f^\pm$  is an isomorphism from  $\mathcal{M}^+ = (\mathcal{T}^+, \prec^+)$  to  $\mathcal{M}^- = (\mathcal{T}^-, \prec^-)$ . Therefore it is easily seen that the function  $f$  with domain  $\mathcal{T}$  and range  $\mathcal{T}''$  defined as follows for all  $x \in \mathcal{T}$ :

- If  $x = \infty'$  then  $f(x) = \infty$ ;
- If  $x < \infty'$  then  $f(x) = f^\mp(x)$ ;

- If  $\infty' < x$  then  $f(x) = f^\pm(x)$ ;

is an isomorphism from  $\mathcal{M} = (\mathcal{T}, <)$  to  $\mathcal{M}'' = (\mathcal{T}'', <'')$ . Hence the following result.

**Proposition 5.** *Let  $\mathcal{M} = (\mathcal{T}, <)$  be a countable linear order,  $\infty$  be a point such that  $\infty \notin \mathcal{T}$ ,  $\mathcal{M}' = (\mathcal{T}', <')$  be the cyclic order on  $\mathcal{M} = (\mathcal{T}, <)$  and  $\infty$ ; let  $\infty'$  be a point such that  $\infty' \in \mathcal{T}'$  and  $\mathcal{M}'' = (\mathcal{T}'', <'')$  be the linear order on  $\mathcal{M}' = (\mathcal{T}', <')$  and  $\infty'$ . If  $\mathcal{M} = (\mathcal{T}, <)$  is dense and unbounded then  $\mathcal{M} = (\mathcal{T}, <)$  and  $\mathcal{M}'' = (\mathcal{T}'', <'')$  are isomorphic.*

Let  $\mathcal{M} = (\mathcal{T}, <)$  be a cyclic order,  $\infty$  be a point such that  $\infty \in \mathcal{T}$ ,  $\mathcal{M}' = (\mathcal{T}', <')$  be the linear order on  $\mathcal{M} = (\mathcal{T}, <)$  and  $\infty, \infty'$  be a point such that  $\infty' \notin \mathcal{T}'$  and  $\mathcal{M}'' = (\mathcal{T}'', <'')$  be the cyclic order on  $\mathcal{M}' = (\mathcal{T}', <')$  and  $\infty'$ . It is easily seen that the function  $f$  with domain  $\mathcal{T}$  and range  $\mathcal{T}''$  defined as follows for all  $x \in \mathcal{T}$ :

- If  $x \neq \infty$  then  $f(x) = x$ ;
- If  $x = \infty$  then  $f(x) = \infty'$ ;

is an isomorphism from  $\mathcal{M} = (\mathcal{T}, <)$  to  $\mathcal{M}'' = (\mathcal{T}'', <'')$ . Hence the following result.

**Proposition 6.** *Let  $\mathcal{M} = (\mathcal{T}, <)$  be a cyclic order,  $\infty$  be a point such that  $\infty \in \mathcal{T}$ ,  $\mathcal{M}' = (\mathcal{T}', <')$  be the linear order on  $\mathcal{M} = (\mathcal{T}, <)$  and  $\infty, \infty'$  be a point such that  $\infty' \notin \mathcal{T}'$  and  $\mathcal{M}'' = (\mathcal{T}'', <'')$  be the cyclic order on  $\mathcal{M}' = (\mathcal{T}', <')$  and  $\infty'$ .  $\mathcal{M} = (\mathcal{T}, <)$  and  $\mathcal{M}'' = (\mathcal{T}'', <'')$  are isomorphic.*

## 2.6 Elimination of Quantifiers

Let  $\mathcal{L}_l$  be the first-order language consisting of the binary predicate  $<$  and the binary predicate  $=$ . The theory  $\Delta_l$  of dense and unbounded linear orders has 6 axioms:

- $(\forall x)(x \not< x)$ ;
- $(\forall xyz)(x < y \wedge y < z \rightarrow x < z)$ ;
- $(\forall xy)(x = y \vee x < y \vee y < x)$ ;
- $(\forall xy)(x < y \rightarrow (\exists z)(x < z \wedge z < y))$ ;
- $(\forall x)(\exists y)(y < x)$ ;
- $(\forall x)(\exists y)(x < y)$ .

By the Löwenheim-Skolem theorem, every  $\Delta_l$ -consistent sentence  $\phi$  in  $\mathcal{L}_l$  has a countable model. By proposition 1, this model is isomorphic to the structure  $(\mathbf{Q}, <)$  of the rational numbers. Hence for every sentence  $\phi$  in  $\mathcal{L}_l$ , either  $\phi$  is a consequence of  $\Delta_l$  or  $\neg\phi$  is a consequence of  $\Delta_l$ . Consequently the set of consequences of  $\Delta_l$  is maximal consistent and  $\Delta_l$  is a complete theory. The method of elimination of quantifiers applies to  $\Delta_l$  and gives a way of deciding whether or not a sentence  $\phi$  in  $\mathcal{L}_l$  is a consequence of  $\Delta_l$ , see Langford [14]. It consists of proving the following result.

**Proposition 7.** *For every formula  $\phi$  in  $\mathcal{L}_l$  with free variables in  $\{x, y_1, \dots, y_I\}$ , the formula  $(\exists x)\phi$  is  $\Delta_l$ -equivalent to a Boolean combination of atomic formulas in  $\mathcal{L}_l$  with free variables in  $\{y_1, \dots, y_I\}$ .*

Our aim is to define a similar method for the theory  $\Delta_c$  of standard cyclic orders. Let  $\mathcal{L}_c$  be the first-order language consisting of the ternary predicate  $\prec$  and the binary predicate  $=$ . The theory  $\Delta_c$  of standard cyclic orders has 6 axioms:

- $(\forall xy)(\not\prec (x, y, y))$ ;
- $(\forall xyz)(\prec (x, y, z) \wedge \prec (x, z, t) \rightarrow \prec (x, y, t))$ ;
- $(\forall xyz)(x \neq y \wedge x \neq z \rightarrow y = z \vee \prec (x, y, z) \vee \prec (x, z, y))$ ;
- $(\forall xyz)(\prec (x, y, z) \leftrightarrow (y, z, x) \leftrightarrow (z, x, y))$ ;
- $(\forall xy)(x \neq y \rightarrow (\exists z) \prec (x, z, y))$ ;
- $(\forall xy)(x \neq y \rightarrow (\exists z) \prec (x, z, y))$ .

Following the line of reasoning suggested above within the framework of dense linear orders, the reader may easily verify the following results:

- Every  $\Delta_c$ -consistent sentence  $\phi$  in  $\mathcal{L}_c$  has a countable model;
- This model is isomorphic to the structure  $(\mathbf{Q} \cup \{\infty\}, \prec)$  obtained from the structure  $(\mathbf{Q}, <)$  of the rational numbers by the construction of section 2.3;
- For every sentence  $\phi$  in  $\mathcal{L}_c$ , either  $\phi$  is a consequence of  $\Delta_c$  or  $\neg\phi$  is a consequence of  $\Delta_c$ ;
- The set of consequences of  $\Delta_c$  is maximal consistent;
- $\Delta_c$  is a complete theory.

We now come to the method of elimination of quantifiers applied to the theory  $\Delta_c$ . For our purpose, it suffices to prove that for every conjunction  $\phi$  of the form  $\prec (x, y_1, z_1) \wedge \dots \wedge \prec (x, y_I, z_I) \wedge \not\prec (x, t_1, u_1) \wedge \dots \wedge \not\prec (x, t_J, u_J) \wedge x = v_1 \wedge \dots \wedge x = v_K \wedge x \neq w_1 \wedge \dots \wedge x \neq w_L$ , the formula  $(\exists x)\phi$  is  $\Delta_c$ -equivalent to a Boolean combination of atomic formulas with free variables in  $\{y_1, \dots, y_I, z_1, \dots, z_I, t_1, \dots, t_J, u_1, \dots, u_J, v_1, \dots, v_K, w_1, \dots, w_L\}$ . Firstly, it is easy to show that for every  $i, i' \in \{1, \dots, I\}$ , the formula  $\prec (x, y_i, z_i) \wedge \prec (x, y_{i'}, z_{i'})$  is  $\Delta_c$ -equivalent to a disjunction of formulas of the form  $\prec (x, y, z) \wedge \phi''$  where  $y, z \in \{y_i, y_{i'}, z_i, z_{i'}\}$  and  $\phi''$  is a Boolean combination of atomic formulas in  $\mathcal{L}_c$  with free variables in  $\{y_i, y_{i'}, z_i, z_{i'}\}$ . Hence we may consider that  $I = 0$  or  $I = 1$ . Secondly, we observe that for every  $j \in \{1, \dots, J\}$ , the formulas  $\not\prec (x, t_j, u_j)$  and  $x = t_j \vee x = u_j \vee t_j = u_j \vee \prec (x, u_j, t_j)$  are  $\Delta_c$ -equivalent. Consequently we may consider that  $J = 0$ . Thirdly, it should be clear that if  $K \geq 1$  then the formula  $(\exists x)\phi$  is  $\Delta_c$ -equivalent to  $\prec (v_1, y_1, z_1) \wedge \dots \wedge \prec (v_1, y_I, z_I) \wedge \not\prec (v_1, t_1, u_1) \wedge \dots \wedge \not\prec (v_1, t_J, u_J) \wedge v_1 = v_2 \wedge \dots \wedge v_1 = v_K \wedge v_1 \neq w_1 \wedge \dots \wedge v_1 \neq w_L$ . Therefore let us assume that  $K = 0$ . Fourthly, the reader may check that for every  $l, l' \in \{1, \dots, L\}$ , the formulas  $x \neq w_l \wedge x \neq w_{l'}$  and  $(w_l = w_{l'} \wedge x \neq w_l) \vee (w_l = w_{l'} \wedge x \neq w_{l'}) \vee \prec (x, w_l, w_{l'}) \vee \prec (x, w_{l'}, w_l)$  are  $\Delta_c$ -equivalent. Thus we may consider that  $L = 0$  or  $L = 1$ . Since:

- the formulas  $(\exists x)(\prec (x, y_1, z_1) \wedge x \neq w_1)$  and  $y_1 \neq z_1$  are  $\Delta_c$ -equivalent;
- the formulas  $(\exists x)(\prec (x, y_1, z_1))$  and  $y_1 \neq z_1$  are  $\Delta_c$ -equivalent;
- the formulas  $(\exists x)(x \neq w_1)$  and  $\top$  are  $\Delta_c$ -equivalent;

our proof of the following result is complete.

**Proposition 8.** *For every formula  $\phi$  in  $\mathcal{L}_c$  with free variables in  $\{x, y_1, \dots, y_I\}$ , the formula  $(\exists x)\phi$  is  $\Delta_c$ -equivalent to a Boolean combination of atomic formulas in  $\mathcal{L}_c$  with free variables in  $\{y_1, \dots, y_I\}$ .*

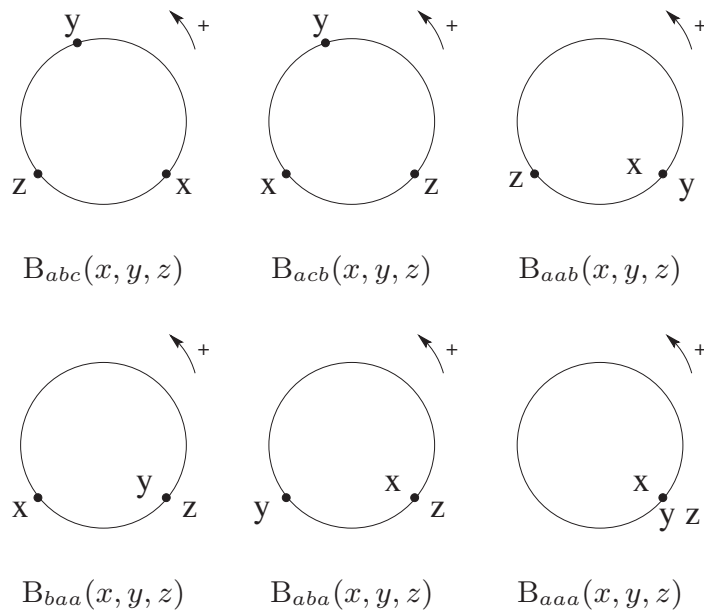
### 3 The Cyclic Point Algebra

#### 3.1 Entities and Relations

Let  $\mathcal{C}$  be an oriented circle. The entities we consider are the points of this circle; we will denote them by  $v, w, x, \text{etc.}$ , and we will call them the cyclic points. Sometimes we will use a rational number belonging to the interval  $[0, 360[$  to define a point of  $\mathcal{C}$ . Such a rational number expresses the angle from the horizontal line to the line passing through the center of  $\mathcal{C}$  and intersecting the point of  $\mathcal{C}$ . Given two points  $x, y \in \mathcal{C}$ ,  $[x, y]$  denotes the set of the points found onto  $\mathcal{C}$  by going from  $x$  to  $y$  by following the orientation of  $\mathcal{C}$ . The atomic relations considered between points of  $\mathcal{C}$  are the six ternary relations defined in the following way :

$$\begin{aligned} B_{abc} &= \{(x, y, z) \in \mathcal{C}^3 : x \neq y, x \neq z, y \neq z \text{ and } y \in [x, z]\}, \\ B_{acb} &= \{(x, y, z) \in \mathcal{C}^3 : x \neq y, x \neq z, y \neq z \text{ and } z \in [x, y]\}, \\ B_{aab} &= \{(x, x, y) \in \mathcal{C}^3 : x \neq y\}, \\ B_{baa} &= \{(y, x, x) \in \mathcal{C}^3 : x \neq y\}, \\ B_{aba} &= \{(x, y, x) \in \mathcal{C}^3 : x \neq y\}, \\ B_{aaa} &= \{(x, x, x) \in \mathcal{C}^3\}. \end{aligned}$$

These six relations are illustrated in Fig. 3.1. We denote the set of these atomic relations by  $\mathcal{B}_{\mathcal{C}}$  and in the sequel we will use  $a, b, c, \text{etc}$  to designate them. We note that these atomic relations are complete and mutually exclusive, *i.e.* three cyclic points satisfy one, and only one, atomic relation of this set of qualitative relations.  $B_{abc}$  and  $B_{acb}$  correspond to both atomic relations satisfied in the cases where the three points are distinct points.  $B_{aab}$ ,  $B_{baa}$  and  $B_{aba}$  are concerned with the cases where two of the three points are the same. The atomic relation  $B_{aaa}$  corresponds to the case in which the three points are equal. From the atomic relations of  $\mathcal{B}_{\mathcal{C}}$  we define the set of the complex relations of



**Fig. 1.** Atomic relations of  $\mathcal{B}_{\mathcal{C}}$ .



the cyclic point algebra by taking the subsets of  $\mathcal{B}_C$ , *i.e.*  $2^{\mathcal{B}_C}$ . In the sequel we will say relation for complex relation.  $\alpha, \beta, \gamma$ , *etc.*, will denote the relations. We have a set of  $2^6 = 64$  relations with two particular relations : the empty relation  $\{\}$  (also denoted by  $\emptyset$ ) and the total relation  $\{B_{aaa}, B_{aab}, B_{baa}, B_{aba}, B_{abc}, B_{acb}\}$  (also improperly denoted by  $2^{\mathcal{B}_C}$ ). Given a relation  $\alpha \in 2^{\mathcal{B}_C}$  and three cyclic points  $x, y, z$ , we have  $\alpha(x, y, z)$  if, and only if, there exists  $a \in \alpha$  such that  $a(x, y, z)$ . Such a relation  $\alpha$  can be seen as the disjunction of its atomic relations. With relations of  $\mathcal{B}_C$  we can represent incomplete information about relative positions of cyclic points.

### 3.2 Basic Operations

The binary operations of intersection  $\cap$ , union  $\cup$  and the unary operation of complement are defined onto  $2^{\mathcal{B}_C}$  in the following way :

$$\begin{aligned} \forall x, y, z \in \mathcal{C}, \forall \alpha, \beta \in 2^{\mathcal{B}_C}, \\ (\alpha \cup \beta)(x, y, z) &\leftrightarrow \alpha(x, y, z) \text{ or } \beta(x, y, z), \\ (\alpha \cap \beta)(x, y, z) &\leftrightarrow \alpha(x, y, z) \text{ and } \beta(x, y, z), \\ \bar{\alpha}(x, y, z) &\leftrightarrow \text{not } \alpha(x, y, z). \end{aligned}$$

These operations can be seen as the usual set operations since we have :

$$\begin{aligned} \forall a \in \mathcal{B}_C, \forall \alpha, \beta \in 2^{\mathcal{B}_C}, \\ a \in (\alpha \cup \beta) &\leftrightarrow a \in \alpha \text{ or } a \in \beta, \\ a \in (\alpha \cap \beta) &\leftrightarrow a \in \alpha \text{ and } a \in \beta, \\ a \in \bar{\alpha} &\leftrightarrow a \notin \alpha. \end{aligned}$$

We also define the unary operations of permutation (denoted by  $\overset{\circlearrowleft}{\rightarrow}$ ) and of rotation (denoted by  $\overset{\circlearrowright}{\rightarrow}$ ) in the following way:

$$\begin{aligned} \text{Let } a \in \mathcal{B}_C, \\ \forall x, y, z \quad a^{\overset{\circlearrowleft}{\rightarrow}}(x, z, y) &\leftrightarrow a(x, y, z), \\ \forall x, y, z \quad a^{\overset{\circlearrowright}{\rightarrow}}(y, z, x) &\leftrightarrow a(x, y, z). \end{aligned}$$

**Table 1.** Permutations and rotations of the atomic relations of  $\mathcal{B}_C$ .

$a$	$B_{aaa}$	$B_{aab}$	$B_{baa}$	$B_{aba}$	$B_{abc}$	$B_{acb}$
$a^{\overset{\circlearrowleft}{\rightarrow}}$	$B_{aaa}$	$B_{aba}$	$B_{baa}$	$B_{aab}$	$B_{acb}$	$B_{abc}$
$a^{\overset{\circlearrowright}{\rightarrow}}$	$B_{aaa}$	$B_{aba}$	$B_{aab}$	$B_{baa}$	$B_{abc}$	$B_{acb}$

Table 1 gives the permutation and the rotation of each atomic relation. We extend these operations to the relations of  $2^{\mathcal{B}_C}$ . The permutation (resp. the rotation) of  $\alpha \in 2^{\mathcal{B}_C}$ , denoted by  $\alpha^{\overset{\circlearrowleft}{\rightarrow}}$  (resp. by  $\alpha^{\overset{\circlearrowright}{\rightarrow}}$ ), is the union of the permutations (resp. the rotations) of its atomic relations. Given four cyclic points  $w, x, y, z$ , from the atomic relation  $a$  satisfied by  $w, x, y$  and the atomic relation  $b$  satisfied by  $x, y, z$ , we can deduce the possible atomic relations of  $\mathcal{B}_C$  satisfied by  $w, x, z$ . This set of atomic relations is given by the binary

operation of composition, which is denoted by  $\circ$ . More formally,  $a \circ b$  is the relation of  $2^{\mathcal{B}_c}$  defined by:

$$a \circ b = \{c \in \mathcal{B}_c : \exists w, x, y, z \ a(w, x, y) \ \& \ b(x, y, z) \ \& \ c(w, x, z)\}.$$

The composition of two relations  $\alpha$  and  $\beta$  of  $2^{\mathcal{B}_c}$  is defined by  $\alpha \circ \beta = \bigcup_{a \in \alpha, b \in \beta} a \circ b$ . In the sequel we will suppose that the unary operations have priority. Knowing the relations  $\alpha$  and  $\beta$  satisfied respectively by three cyclic points  $w, x, y$  and the three points  $x, y, z$ , with the operation of composition we can deduce the possible atomic relations satisfied by  $w, x, z$ . Let us note that with the operations of composition, permutation and rotation we can also find the set of atomic relations which can be satisfied by  $w, y, z$ . This set of atomic relations corresponds to the relation  $\alpha^{\curvearrowright} \circ \beta^{\curvearrowright}$ .

**Table 2.** Table of composition for the atomic relations of  $\mathcal{B}_c$ .

$\circ \curvearrowright$	$B_{aaa}$	$B_{aab}$	$B_{baa}$	$B_{aba}$	$B_{abc}$	$B_{acb}$
$B_{aaa}$	$\{B_{aaa}\}$	$\{B_{aab}\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$B_{aab}$	$\emptyset$	$\emptyset$	$\{B_{aab}\}$	$\{B_{aaa}\}$	$\{B_{aab}\}$	$\{B_{aab}\}$
$B_{baa}$	$\{B_{baa}\}$	$\{B_{aba}, B_{abc}, B_{acb}\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$B_{aba}$	$\emptyset$	$\emptyset$	$\{B_{aba}\}$	$\{B_{baa}\}$	$\{B_{acb}\}$	$\{B_{abc}\}$
$B_{abc}$	$\emptyset$	$\emptyset$	$\{B_{abc}\}$	$\{B_{baa}\}$	$\{B_{aba}, B_{abc}, B_{acb}\}$	$\{B_{abc}\}$
$B_{acb}$	$\emptyset$	$\emptyset$	$\{B_{acb}\}$	$\{B_{baa}\}$	$\{B_{acb}\}$	$\{B_{aba}, B_{abc}, B_{acb}\}$

### 3.3 Constraint Networks of Cyclic Points

To represent spatial information about cyclic points we use particular ternary constraint networks which we call constraint networks of cyclic points, CNCP in short. Each variable of a CNCP represents a cyclic point and each ternary constraint is defined by a relation of  $2^{\mathcal{B}_c}$ . This relation represents all allowed relative positions satisfied by the three points represented by the three concerned variables. More formally, a CNCP is defined in the following way.

**Definition 9.** A CNCP is a pair  $\mathcal{N} = (V, C)$ , where:

- $V$  is a nonempty set of  $n$  ordered variables  $\{V_1, \dots, V_n\}$  representing  $n$  cyclic points.
- $D$  is a mapping from  $(V \times V \times V)$  to  $2^{\mathcal{B}_c}$  defining the constraints onto  $V$ . In the sequel, usually we will denote  $C(V_i, V_j, V_k)$  by  $C_{ijk}$  or  $C_{v_i v_j v_k}$ .

With no loss of generality, we may suppose that each CNCP satisfies the following properties:

- (a) for all  $i, j, k \in \{1, \dots, n\}$ ,  $C_{ijk} = C_{kij}^{\curvearrowright} = C_{ikj}^{\curvearrowright}$ .
- (b) For all  $i, j \in \{1, \dots, n\}$ ,  $C_{ijj} \subseteq \{B_{aab}, B_{aaa}\}$ ,
- (c) For all  $i \in \{1, \dots, n\}$ ,  $C_{iii} \subseteq \{B_{aaa}\}$ .

Intuitively, the requirement (a) stipulates that constraints between three variables must be coherent. Consequently, giving the constraints  $C_{ijk}$  for all  $i, j, k \in \{1, \dots, n\}$  with  $i \leq j \leq k$  is sufficient to define a CNCP. Concerning conditions (b) and (c), let us note that for convenience reasons we allow the empty relation for  $C_{iij}$  and  $C_{iii}$ . The second and third conditions contain necessary atomic relations allowing one or many equalities of cyclic points.

Given a CNCP, an important issue is to determine its consistency, *i.e.* whether there exists a set of cyclic points satisfying its constraints. More formally, given a CNCP  $\mathcal{N} = (V, C)$ , we have the following definitions:

- An instantiation  $m$  of  $\mathcal{N}$  is a function from  $V$  to  $\mathcal{C}$  associating with each variable  $V_i \in V$  a cyclic point  $m(V_i)$  (with  $i \in \{1, \dots, |V|\}$ ). In what follows,  $m(V_i)$  will be sometimes denoted by  $m_i$ , or  $m_{V_i}$ .  $m_{ijk}$  (equally denoted by  $m_{V_i V_j V_k}$  or  $m(V_i, V_j, V_k)$ ) is the atomic relation of  $\mathcal{B}_{\mathcal{C}}$  satisfied by the points  $m_i, m_j$  and  $m_k$  with  $i, j, k \in \{1, \dots, n\}$  ( $n = |V|$ ).
- An instantiation  $m$  of  $\mathcal{N}$  is consistent iff for all  $i, j, k \in \{1, \dots, n\}$ ,  $m_{ijk} \subseteq C_{ijk}$  ( $n = |V|$ ). We will say that  $m$  is a solution of  $\mathcal{N}$ .
- A partial instantiation  $m$  of  $\mathcal{N}$  is a mapping from  $V'$  to  $\mathcal{C}$ , with  $V' \subseteq V$ , which associates with each variable of  $V'$  a cyclic point.  $m$  is a partial solution of  $\mathcal{N}$  iff the points associated with the variables of  $V'$  satisfy the ternary constraints uniquely concerning by the variables of  $V'$ .
- $\mathcal{N}$  is consistent iff it admits a consistent instantiation.

To solve the consistency problem of a qualitative binary constraint network, the path-consistency method is usually used. This method consists of obtaining a constraint network equivalent to the initial network (a network with exactly the same solutions) by deleting some atomic relations which do not participate in any solution. This method uses the operations of composition, inverse and intersection. The obtained network is 3-consistent, *i.e.* we can always extend a partial solution concerning two variables to a partial solution concerning a third variable in addition to the first ones. In a similar way, we can use the operations of composition, intersection, rotation and permutation to remove impossible atomic relations in the constraints of a CNPC. In particular we can apply the following operations onto a CNPC  $\mathcal{N} = (V, C)$ :

$$\begin{aligned} C_{ijk} &\leftarrow C_{ijk} \cap (C_{ijl} \circ C_{jlk}), \\ C_{jki} &\leftarrow C_{ijk}^{\curvearrowright}, C_{kij} \leftarrow C_{jki}^{\curvearrowright}, \\ C_{ikj} &\leftarrow C_{ijk}^{\leftrightarrow}, C_{jik} \leftarrow C_{jki}^{\leftrightarrow}, C_{kji} \leftarrow C_{kij}^{\leftrightarrow}. \end{aligned}$$

onto all each 4-tuple  $i, j, k, l \in \{1, \dots, |V|\}$  until a fixed point is reached. This method is accomplished in polynomial time, we will call it the composition closure method. The obtained CNCP admits the same solutions as the initial CNCP. Moreover this constraint network is closed for composition: we say that a CNCP  $\mathcal{N} = (V, C)$  is closed for the operation of composition iff

$$\forall i, j, k, l \in \{1, \dots, |V|\}, C_{ijk} \subseteq C_{ijl} \circ C_{jlk}.$$

To close this subsection let us note that for all  $i, j, k, l \in \{1, \dots, |V|\}$ , if  $C_{ijk} \subseteq C_{ijl} \circ C_{jlk}$  then the inclusion  $C_{jki} \subseteq C_{jkl} \circ C_{kli}$  and the inclusion  $C_{ikj} \subseteq C_{ikl} \circ C_{klj}$  are not always satisfied.

### 3.4 Tractability Results about CNCP

In the sequel we will prove that the consistency problem for the CNCP, which we will denote C-CNCP, is NP-complete in the general case. This subsection is devoted to the definitions of tractable cases and intractable cases. These cases will be parametrized by two nonempty sets  $B$  and  $T$ , where  $B$  is the union of  $\{\emptyset\}$  and zero or several sets among the following ones :  $\{\{B_{aaa}\}\}$ ,  $\{\{B_{aab}\}, \{B_{aba}\}, \{B_{baa}\}\}$ ,  $\{\{B_{aaa}, B_{aab}\}, \{B_{aaa}, B_{aba}\}, \{B_{aaa}, B_{baa}\}\}$ , and where  $T$  is a subset of  $2^{\mathcal{B}^c}$  containing the empty relation.  $B$  contains the possible constraints between two distinct variables, whereas  $T$  contains the possible constraints between three distinct variables. In other words,  $B$  corresponds to the relations  $C_{ijj}$  and  $T$  to the relations  $C_{ijk}$ , with  $i, j, k$  three distinct integers.  $T$  must be closed for the operations of permutation and composition. CNCP( $B, T$ ) will denote the set of the CNCP  $\mathcal{N} = (V, C)$  such that for all  $i, j \in \{1, \dots, n\}$  (with  $n = |V|$ ) if  $i \neq j$  then  $C_{ijj} \in B$  and for all  $i, j, k \in \{1, \dots, n\}$  if  $i, j, k$  are pairwise distinct then  $C_{ijk} \in T$ . C-CNCP( $B, T$ ) will denote the consistency problem restricted to the networks of CNCP( $B, T$ ).

**Table 3.** The sets  $B_0, B_1, B_2$  and  $B_3$ .

	$\emptyset$	$\{B_{aaa}\}$	$\{B_{aab}\}, \{B_{aba}\}, \{B_{baa}\}$	$\{B_{aaa}, B_{aab}\}, \{B_{aaa}, B_{aba}\}, \{B_{aaa}, B_{baa}\}$
$B_0$	• •			
$B_1$	•		•	
$B_2$	•		•	•
$B_3$	• •		•	•
$B_4$	• •	•		

Let us note that we have eight possible different sets  $B$ . The sets considered for  $B$  in the sequel are given in Table 3 and those for  $T$  are defined in Table 5. Firstly we define tractable cases and secondly, we give some intractable cases. Let us start our study with an easy case.

**Proposition 10.** *C-CNCP( $B_0, T$ ) is a polynomial problem.*

**Proof.** Let  $\mathcal{N} = (V, C) \in \text{CNCP}(B_0, T)$ .  $\mathcal{N}$  is a consistent network iff for all  $i, j, k \in \{1, \dots, |V|\}$ ,  $B_{aaa} \in C_{ijk}$ . This test can be established in time  $O(n^3)$  with  $n = |V|$ .  $\square$

**Proposition 11.** *Let  $B$  be a set of relations such that  $B_1 \subseteq B$  and let  $T$  be a set closed for the operation of intersection (in addition to the closure for the operations of permutation and rotation). Let  $T' = (T \setminus B_3) \cup \{\emptyset\}$  and  $B' = (B \setminus B_0) \cup \{\emptyset\}$ . C-CNCP( $B, T$ ) is a polynomial problem (resp. a NP-complete problem) iff C-CNCP( $B', T'$ ) is a polynomial problem (resp. a NP-complete problem).*

**Proof.** Trivially, since  $B' \subseteq B$  and  $T' \subseteq T$  if C-CNCP( $B, T$ ) is a polynomial problem then C-CNCP( $B', T'$ ) is also a polynomial problem and if C-CNCP( $B', T'$ ) is a NP-complete complete then C-CNCP( $B, T$ ) is equally a NP-complete problem. Now, let us define a polynomial transformation from C-CNCP( $B, T$ ) to C-CNCP( $B', T'$ ). Let

$\mathcal{N} = (V, C)$  be a CNCP belonging to  $\text{CNCP}(B, T)$ . From  $\mathcal{N}$  we define a CNCP  $\mathcal{N}''$  by making the following steps.

**Step 1.** We initialize  $\mathcal{N}' = (V', C')$  by  $\mathcal{N}$ .

**Step 2.** We define the binary graph  $G = (S, E)$  in the following way : for each variable  $V'_i \in V'$  there is an associated node  $s_i \in S$  and  $(s_i, s_j)$  belongs to the set of edges  $E$  iff  $C'_{ijk} \subseteq \{B_{aaa}, B_{aab}\}$  for some  $k \in \{1, \dots, |V'|\}$ . Let  $C_1, \dots, C_p$  be the strongly connected components of  $G$ .

**Step 3.** We define the CNCP  $\mathcal{N}'' = (V'', C'')$  by:

- with each component  $C_i$  is associated a variable  $V''_i$  for each  $i \in \{1, \dots, p\}$ .
- $C''_{ijk} = \bigcap_{s_r \in C_i, s_s \in C_j, s_t \in C_k} C'_{rst}$  for each  $i, j, k \in \{1, \dots, p\}$ .

**Step 4.** If  $\mathcal{N}'$  and  $\mathcal{N}''$  are identical then we stop. If not we set  $\mathcal{N}'$  to  $\mathcal{N}''$  and we go back to Step 2.

From the construction of  $\mathcal{N}''$  we note that  $\mathcal{N}''$  is consistent iff  $\mathcal{N}$  is consistent. Since  $T$  is closed for the operation of intersection we know that  $C''_{ijk}$  belongs to  $T$  for all distinct integers  $i, j, k$ . Since the relations of  $B$  contain at most two atomic relations and  $B_1 \subseteq B$  it follows that  $C''_{ijj} \in B \cup \{B_{aaa}\}$ . Moreover, by construction  $C''_{ijk}$  cannot belong to the set  $B_3 \setminus \{\emptyset\}$  and  $C''_{ijj}$  cannot be  $\{B_{aaa}\}$ . It follows that  $\mathcal{N}'' \in \text{CNCP}(B', T')$ . The construction of  $\mathcal{N}''$  is realized in polynomial time. It follows that if  $\text{C-CNCP}(B', T')$  is a polynomial problem then  $\text{C-CNCP}(B, T)$  is also a polynomial problem and if  $\text{C-CNCP}(B, T)$  is a NP-complete problem then  $\text{C-CNCP}(B', T')$  is also a NP-complete problem.  $\square$

**Lemma 12.** *The composition closure method solves  $\text{C-CNCP}(B_2, \{\{B_{abc}\}, \{B_{acb}\}, \emptyset\})$ .*

**Proof.** Let  $\mathcal{N}' = (V, C') \in \text{C-CNCP}(B_2, \{\{B_{abc}\}, \{B_{acb}\}, \emptyset\})$  and let  $\mathcal{N} = (V, C)$  be the network obtained by applying the composition closure method on  $\mathcal{N}'$ . Let us show that  $\mathcal{N}' \in \text{CNCP}(B_1, \{\{B_{abc}\}, \{B_{acb}\}, \emptyset\})$ . Let  $i, j \in \{1, \dots, n\}$  with  $n = |V|$  and  $i \neq j$ . We will suppose that  $n > 3$  since the case  $n \leq 3$  is trivial. We now prove that  $C_{ijj} \neq \{B_{aaa}\}$  and  $C_{ijj} \neq \{B_{aaa}, B_{aab}\}$ . Let us suppose the contrary; since  $\mathcal{N}$  is closed for composition it follows that  $C_{ijj} \subseteq C_{iik} \circ C_{ikj}$  for all  $k \in \{1, \dots, n\}$ . In the case where  $k$  is distinct from  $i$  and  $j$  we have  $C_{ijk} = \{B_{abc}\}$  or  $C_{ijk} = \{B_{acb}\}$ . It follows that  $C_{ijj} \subseteq \{B_{aaa}, B_{aab}\} \circ \{B_{acb}, B_{abc}\}$ . Consequently,  $C_{ijj} \subseteq \{B_{aab}\}$ , which is a contradiction. We can conclude that  $\mathcal{N}' \in \text{CNCP}(B_1, \{\{B_{abc}\}, \{B_{acb}\}\})$ .

Let us suppose that  $\mathcal{N}$  does not contain the empty relation. We are going to construct a consistent instantiation  $m$  for  $\mathcal{N}$  in which two distinct variables are associated with two distinct cyclic points. It is always possible to instantiate the first three variables. Now, let us suppose that we have a partial solution  $m_1, \dots, m_{q-1}$  with  $q > 3$  and  $q \leq n$ , such that  $m_i \neq m_j$  for all  $i, j \in \{1, \dots, q-1\}$ . Let us show that we can extend this partial solution to the variable  $V_q$  by a cyclic point different from the other cyclic points used. Firstly, we renumber the variables  $V_1, \dots, V_{q-1}$  such that  $m_{i(i+1)(i+2)} = B_{abc}$  for each  $i \in \{1, \dots, q-3\}$  and  $m_{(q-2)(q-1)1} = B_{abc}$ . Let  $l \in \{1, \dots, q-1\}$  be such that  $C_{lq(l \bmod (q-1)+1)} = \{B_{abc}\}$ . Let us show the existence of  $l$ . Let us suppose that  $l$  does not exist. Hence we have for each  $l \in \{1, \dots, q-2\}$ ,  $C_{lq(l+1)} = \{B_{acb}\}$  and  $C_{(q-1)q1} = \{B_{acb}\}$ . Since  $\mathcal{N}$  is closed for the operation of composition we have

$C_{1q3} \subseteq C_{1q2} \circ C_{q23}$ . Consequently,  $C_{1q3} \subseteq \{B_{acb}\} \circ \{B_{abc}\}$ . Since this last composition equals  $\{B_{acb}\}$  we can deduce that  $C_{1q3} = \{B_{acb}\}$ . By propagation and by using the more general fact that  $C_{1qi} \subseteq C_{1q(i-1)} \circ C_{q(i-1)i}$  for each  $i \in \{3, \dots, q-1\}$  we obtain the following equality:  $C_{1q(q-1)} = \{B_{acb}\}$ , hence by permutation we obtain  $C_{1(q-1)q} = \{B_{abc}\}$ . By rotation we have  $C_{(q-1)q1} = \{B_{abc}\}$ , which is a contradiction. We can conclude that there exists an integer  $l$  satisfying the given conditions. By defining  $m_q$  by a cyclic point such that  $m_{lq(l \bmod (q-1)+1)}$  is  $B_{abc}$  (i.e. any intermediate cyclic point between  $m_l$  and  $m_{l \bmod (q-1)+1}$  by following the circle orientation we extend  $m$  to a partial instantiation such that the valuations are pairwise distinct. Let us show that  $m$  is always a partial solution. In order to do that, let us suppose that there exists  $i, j \in \{1, \dots, q-1\}$  such that  $m_{ijq} \notin C_{ijq}$ .  $i, j, q$  must be pairwise distinct. With no loss of generality we may suppose that  $i < j$ ; then three cases are possible:

- $i < j \leq l$ . We have  $m_{ijq} = B_{abc}$ , consequently  $C_{ijq} = \{B_{acb}\}$ . If  $j = l$  then  $C_{ilq} = \{B_{acb}\}$ . In the contrary case, since  $C_{liq} \subseteq C_{lij} \circ C_{ijq}$ , it follows that  $C_{liq} \subseteq \{B_{abc}\} \circ \{B_{acb}\}$  and hence  $C_{liq} = \{B_{abc}\}$ . Consequently  $C_{ilq} = \{B_{acb}\}$ . If  $i = 1$  and  $l = q-1$  then  $C_{lqi} = \{B_{abc}\}$  and hence  $C_{ilq} = \{B_{abc}\}$ , which is a contradiction. In the contrary case  $C_{ilq} \subseteq C_{il(l \bmod (q-1)+1)} \circ C_{l(l \bmod (q-1)+1)q}$  with  $C_{il(l \bmod (q-1)+1)} \circ C_{l(l \bmod (q-1)+1)q} = \{B_{abc}\} \circ \{B_{acb}\} = \{B_{abc}\}$ . Consequently  $C_{ilq} = \{B_{abc}\}$ , which is a contradiction.
- $l \leq i < j$ .
  - Let us consider the case where  $j = l+1$ . It follows that  $i = l$ . Hence  $m_{ijq} = B_{acb}$  and by consequence  $C_{ijq} = \{B_{abc}\}$ . This is a contradiction since  $C_{l(l+1)q} = \{B_{abc}\}$ .
  - Let us consider the case where  $i = l+1$ . It follows that  $m_{ijq} = B_{abc}$  and then  $C_{ijq} = \{B_{acb}\}$ . It follows that  $C_{(l+1)jq} = \{B_{acb}\}$ . We know that  $C_{j(l+1)l} \subseteq C_{j(l+1)q} \circ C_{(l+1)ql}$ . Hence,  $C_{j(l+1)l} \subseteq \{B_{abc}\} \circ \{B_{acb}\}$  and  $C_{j(l+1)l} = \{B_{abc}\}$ . This is a contradiction since  $m_{j(l+1)l} = B_{acb}$ .
  - Let us consider the case where  $i \neq l+1$  and  $j \neq l+1$ . By propagating the fact that  $C_{q(l+1)(q-m-2)} \subseteq C_{q(l+1)(q-m-1)} \circ C_{(l+1)(q-m-1)(q-m-2)}$  for  $m \in \{0, \dots, q-2-j\}$ . We obtain  $C_{q(l+1)(q-m-2)} \subseteq \{B_{abc}\} \circ \{B_{acb}\}$  and hence  $C_{q(l+1)(q-m-2)} \subseteq \{B_{abc}\}$ , for  $m \in \{1, \dots, q-2-j\}$ . It follows that  $C_{q(l+1)j} = \{B_{abc}\}$ . Since  $C_{qji} \subseteq C_{qj(l+1)} \circ C_{j(l+1)i}$ . It results that  $C_{qji} \subseteq \{B_{abc}\} \circ \{B_{acb}\}$  and thus  $C_{qji} = \{B_{abc}\}$ , which is a contradiction.
- $i < l < j$ . Consequently we have  $m_{ijq} = B_{acb}$  and then  $C_{ijq} = \{B_{abc}\}$ . Since  $C_{i(l+1)q} \subseteq C_{i(l+1)l} \circ C_{(l+1)lq}$ . we obtain  $C_{i(l+1)q} \subseteq \{B_{acb}\} \circ \{B_{abc}\}$  and thus  $C_{i(l+1)q} = \{B_{acb}\}$ . If  $j = l+1$  then we get a contradiction. Let us suppose that  $j \neq (l+1)$ . As  $C_{qij} \subseteq C_{qi(l+1)} \circ C_{i(l+1)j}$  it follows that  $C_{qij} \subseteq \{B_{acb}\} \circ \{B_{abc}\}$  and thus  $C_{qij} = \{B_{acb}\}$ . By rotation we have  $C_{ijq} = \{B_{acb}\}$ , which is a contradiction.

□

**Proposition 13.** Let  $T_0$  be the set  $B_3 \cup \{\{B_{abc}\}, \{B_{acb}\}, \emptyset\}$ .  $C\text{-CNCP}(B_3, T_0)$  is a polynomial problem.

**Proof.** From Lemma 12 it follows that  $C\text{-CNCP}(B_2, \{\{B_{abc}\}, \{B_{acb}\}, \emptyset\})$  is a polynomial problem. From Proposition 11 we can conclude that  $C\text{-CNCP}(B_2 \cup B_0, \{\{B_{abc}\},$

$\{B_{acb}\} \cup B_3$ ), i.e.  $C\text{-CNCP}(B_3, T_0)$ , is also a polynomial problem since  $T_0$  is closed for the operation of intersection.  $\square$

**Proposition 14.** *Let  $T_1$  be a set composed of all relations minus the relations containing both atomic relations  $\{B_{abc}\}$  and  $\{B_{acb}\}$ .  $C\text{-CNCP}(B_4, T_1)$  is a polynomial problem.*

**Proof.** The set  $T_1$  is closed for the operation of intersection and  $B_1$  is a subset of  $B_4$ , from Proposition 11 it follows that  $C\text{-CNCP}(B_4, T_1)$  is a polynomial problem if, and only if,  $C\text{-CNCP}(B_1, T_1 \setminus B_3)$  is a polynomial problem. Let  $\mathcal{N} = (V, C) \in \text{CNCP}(B_1, T_1 \setminus B_3)$  and a solution  $m$  de  $\mathcal{N}$ . For all  $i, j, k \in \{1, \dots, n\}$  pairwise distinct, we have  $m_{ijk} \in \{B_{abc}, B_{acb}\}$  (because of the possible constraints allowed by the set  $B_1$ ). It follows that  $\mathcal{N} = (V, C)$  admits the same solutions that the  $\text{CNCP}(V, C')$  defined by  $C'_{ijk} = C_{ijk} \setminus \{B_{aaa}, B_{aba}, B_{baa}, B_{aab}\}$  if  $i, j, k$  are pairwise distinct integers,  $C'_{ijk} = C_{ijk}$  else, for  $i, j, k \in \{1, \dots, n\}$ .  $(V, C') \in \text{CNCP}(B_1, \{B_{abc}, B_{acb}\})$ . As  $C\text{-CNCP}(B_1, \{B_{abc}, B_{acb}\})$  is a polynomial problem (Lemma 12) we can conclude that  $C\text{-CNCP}(B_1, T_1)$  is also a polynomial problem.  $\square$

**Lemma 15.** *Let  $T$  be the set  $\{\{B_{abc}\}, \{B_{acb}\}, \{B_{aaa}\}, \{B_{abc}, B_{aaa}\}, \{B_{acb}, B_{aaa}\}, \emptyset\}$ .  $C\text{-CNCP}(B_3, T)$  is a polynomial problem.*

**Proof.** Let be  $\mathcal{N}' = (V, C') \in \text{RCPC}(B_3, T)$ . By applying the method of composition closure on  $\mathcal{N}'$  we obtain  $\mathcal{N} = (V, C)$  belonging also to  $C\text{-CNCP}(B_3, T)$ . Let us suppose that  $\mathcal{N}$  does not contain the empty constraint. Let us show that  $\mathcal{N}$  is consistent. Let  $i, j, k, l$  be four pairwise distinct integers belonging to the set  $\{1, \dots, n\}$  with  $n = |V|$ . Let us suppose that  $C_{ijk}$  contains the atomic relation  $B_{aaa}$ . Since that  $C_{ijk} \subseteq C_{ijl} \circ C_{jlk}$  and  $C_{jik} \subseteq C_{jil} \circ C_{ilk}$  we can deduce that  $B_{aaa} \in C_{ijl}, B_{aaa} \in C_{jlk}, B_{aaa} \in C_{jil}$  and  $B_{aaa} \in C_{ilk}$ . Consequently, a constraint on three distinct variables contains the atomic relation  $B_{aaa}$  if, and only if, every constraint on three distinct variables contains the atomic relation  $B_{aaa}$ . Let us suppose that for two distinct integers  $i, j \in \{1, \dots, n\}$ ,  $C_{ijj}$  does not contain the atomic relation  $B_{aaa}$ , i.e.  $C_{ijj} = \{B_{aab}\}$ . Let  $k \in \{1, \dots, n\}$  different from  $i$  and  $j$ . Let  $k$  be an integer different from  $i$  and  $j$ . We have  $C_{ijk} \subseteq C_{ijj} \circ C_{jik}$ . Hence  $C_{ijk} \subseteq \{B_{aba}\} \circ \{B_{aaa}, B_{abc}, B_{acb}\}$  and hence  $C_{ijk} \subseteq \{B_{abc}, B_{acb}\}$ . It follows that  $C_{ijk}$  does not contain the atomic relation  $B_{aaa}$ . We can conclude that either all constraints of  $\mathcal{N}$  contain the atomic relation  $B_{aaa}$ , in which case  $\mathcal{N}$  is consistent, or that  $\mathcal{N}$  belongs  $\text{RCPC}(B_3, T_0)$  and consequently deciding consistency is polynomial (Proposition 13).  $\square$

**Proposition 16.** *Let  $T_3$  be the set composed of the relations of  $B_3$  and the relations of the set  $\{\{B_{abc}\}, \{B_{acb}\}, \{B_{abc}, B_{aaa}\}, \{B_{acb}, B_{aaa}\}\}$ .  $C\text{-CNCP}(B_3, T_3)$  is a polynomial problem.*

**Proof.**  $T_3$  is closed for the operation of intersection and moreover  $B_0 \subseteq B_3$ . It follows that  $C\text{-CNCP}(B_3, T_3)$  is a polynomial problem iff  $C\text{-CNCP}(B_2, (T_1 \setminus B_3) \cup \{\emptyset\})$  is a polynomial problem. This is actually the case by Lemma 15.  $\square$

**Proposition 17.** Let  $T_2$  be the set composed of the relations of  $B_3$  and all the relations including the relation  $\{B_{abc}, B_{acb}\}$ .  $C\text{-CNCP}(B_3, T_2)$  is a polynomial problem.

**Proof.**  $T_2$  is a set closed for the operation of intersection, moreover  $B_0 \subseteq B_3$ . It follows that  $C\text{-CNCP}(B_3, T_2)$  is a polynomial problem iff  $C\text{-CNCP}(B_2, (T_2 \setminus B_3) \cup \{\emptyset\})$  is a polynomial problem. Let us show that  $C\text{-CNCP}(B_2, T_2 \setminus B_3)$  is a polynomial problem. Let  $\mathcal{N} \in \text{CNCP}(B_2, T_2 \setminus B_3)$ . By giving to the variables pairwise distinct values we obtain a solution for  $\mathcal{N}$  since  $\{B_{abc}, B_{acb}\} \subseteq C_{ijk}$  for all  $i, j, k$  pairwise distinct integers and no constraint belonging to  $B_2$  implies the equality of two variables.  $\square$

**Proposition 18.**  $C\text{-CNCP}(B_3, T_4)$  is a polynomial problem.

**Proof.** Let  $\mathcal{N}$  be a CNCP belonging to  $\text{CNCP}(B_3, T_4)$ . Each consistent instantiation  $m$  of  $\mathcal{N}$  uses only one or two distinct cyclic points. Consequently, we can reduce the domain of the variables to two distinct cyclic points  $u$  and  $v$ . From this fact we can define a polynomial transformation from  $\text{CNCP}(B_3, T_4)$  to the 2-SAT problem in the following way. Let  $\mathcal{N} = (V, C)$  be a CNCP belonging to  $\text{CNCP}(B_3, T_4)$ . We note  $|V|$  by  $n$ . Let  $L = \{l_1, \dots, l_n\}$  be a set of literals. For each  $i \in \{1, \dots, n\}$ , the variable  $V_i$  is associated with the literal  $l_i$ . Intuitively,  $l_i$  is true will correspond to the assignation  $u$  to  $V_i$  and  $l_i$  is false will correspond to the assignation  $v$  to  $V_i$ . We define a set of clauses  $\mathcal{C}$  over  $L$  by adding to  $\mathcal{C}$  a set of clauses  $s_{ijk}$  for each  $i, j, k \in \{1, \dots, n\}$  in accordance with the values of  $C_{ijk}$ . This translation is given by Table 4.

**Table 4.** Translation of the cyclic constraints with clauses

the constraint $C_{ijk}$	the clause $c_{ijk}$
$\{B_{aaa}\}$	$(l_i \vee \bar{l}_j) \wedge (l_i \vee \bar{l}_k) \wedge (l_j \vee \bar{l}_i) \wedge (l_j \vee \bar{l}_k) \wedge (l_k \vee \bar{l}_i) \wedge (l_k \vee \bar{l}_j)$
$\{B_{aab}\}$	$(l_i \vee \bar{l}_j) \wedge (l_i \vee l_k) \wedge (\bar{l}_i \vee l_j) \wedge (l_j \vee l_k) \wedge (\bar{l}_i \vee \bar{l}_k) \wedge (\bar{l}_j \vee \bar{l}_k)$
$\{B_{aba}\}$	$c_{ikj}$
$\{B_{baa}\}$	$c_{jki}$
$\{B_{aaa}, B_{aab}\}$	$(\bar{l}_i \vee l_j) \wedge (l_i \vee \bar{l}_j)$
$\{B_{aaa}, B_{aba}\}$	$c_{ikj}$
$\{B_{aaa}, B_{baa}\}$	$c_{kji}$
$\{B_{aaa}, B_{aab}, B_{aba}, B_{baa}\}$	$\emptyset$

Given  $m$  a solution of  $\mathcal{N} = (V, C)$  we define the following interpretation  $\mathcal{I}$  onto  $\mathcal{L}$ : for each  $l_i \in L$ ,  $\mathcal{I}(l_i) = \text{true}$  iff  $m(V_i) = u$ . The reader can verify that  $\mathcal{I}$  is a model of all sets of clauses  $s_{ijk}$ . Now, from a model  $\mathcal{I}$  of all sets of clauses  $s_{ijk}$ , a solution of  $\mathcal{N}$  can be obtained by taking  $m(V_i) = u$  iff  $l_i \in L$ ,  $\mathcal{I}(l_i) = \text{true}$ , else  $m(V_i) = v$ , for each  $V_i \in V$ . It can be verified that  $m$  is a solution of  $\mathcal{N}$ .  $\square$

Let us note that the relations  $\{B_{aaa}, B_{aab}, B_{aba}\}$ ,  $\{B_{aaa}, B_{aba}, B_{baa}\}$  and  $\{B_{aaa}, B_{aab}, B_{aba}\}$  can be expressed with clauses containing three literals and cannot be expressed with clauses containing only two literals. Despite this fact, we cannot assert that adding these relations implies NP-completeness.

**Lemma 19.** Let  $T$  be a set containing both relations  $\{B_{abc}\}$ ,  $\{B_{acb}\}$  and any relation  $\alpha$  including  $\{B_{abc}, B_{acb}\}$ .  $C\text{-CNCP}(B_1, T)$  is a NP-complete problem.



**Proof.** We give a polynomial transformation from the *cyclic ordering* problem – a NP-complete referenced in [10] – to C-CNCP( $B_1, T$ ). The *cyclic ordering* problem is as follows: given a set  $A$  and a set  $Tr$  of 3-tuples  $(e, f, g)$  with  $e, f, g$  three distinct elements belonging to  $A$ , answer the following question: is there a one-to-one function  $\mathcal{F}$  which associates an integer belonging to  $\{1, \dots, |A|\}$  with each element of  $A$  such that, for each 3-tuple  $(e, d, f) \in Tr$  we have  $\mathcal{F}(e) < \mathcal{F}(f) < \mathcal{F}(g)$  or  $\mathcal{F}(g) < \mathcal{F}(e) < \mathcal{F}(f)$  or  $\mathcal{F}(f) < \mathcal{F}(g) < \mathcal{F}(e)$ .

Let  $(A, Tr)$  be an instance of the *cyclic ordering* problem. Let  $\mathcal{N} = (V, C)$  be a CNCP defined in the following way:

- $V$  is a set of  $|A|$  variables. To each of these variables corresponds an element belonging to  $A$  (to two distinct variables corresponds two distinct elements). We denote by  $V_e$  the variable associated with the element  $e \in A$ .
- The constraints of  $C$  are defined as follows: given  $e, f, g \in A$  which are pairwise distinct, if  $(e, f, g)$  or  $(f, g, e)$  or  $(g, e, f) \in Tr$  then  $C(V_e, V_f, V_g) = \{B_{abc}\}$ , else if  $(f, e, g)$  or  $(e, g, f)$  or  $(g, f, e) \in Tr$  then  $C(V_e, V_f, V_g) = \{B_{acb}\}$ , else  $C(V_e, V_f, V_g) = \alpha$ .  $C(V_e, V_e, V_f) = \{B_{aab}\}$  and  $C(V_e, V_e, V_e) = \{B_{aaa}\}$

Let us prove that there exists a one-to-one mapping  $\mathcal{F}$  solution of  $(A, T)$  if, and only if, there exists a solution  $m$  of  $\mathcal{N}$ .

- Let  $\mathcal{F}$  be a solution of  $(A, T)$ . Let  $P$  denote a cyclic point and  $O$  the center of the circle. We define an instantiation  $m$  of  $\mathcal{N}$  by for each  $e \in A$ :  $m(V_e)$  is the cyclic point such that the angle  $(OP, Om(V_e))$  equals  $(\mathcal{F}(e) - 1) \times (360/|A|)$  degrees. Then  $m$  is a solution of  $\mathcal{N}$ . Indeed, we note that if a 3-tuple  $(e, f, g)$  belongs to  $Tr$  then  $\mathcal{F}(e) < \mathcal{F}(f) < \mathcal{F}(g)$  or  $\mathcal{F}(g) < \mathcal{F}(e) < \mathcal{F}(f)$  or  $\mathcal{F}(f) < \mathcal{F}(g) < \mathcal{F}(e)$ . Consequently, by moving along the circle starting from  $P$  we first meet  $m(V_e)$ , then  $m(V_f)$  and finally  $m(V_g)$  or  $m(V_g)$ , then  $m(V_e)$  and finally  $m(V_f)$  or  $m(V_f)$ , then  $m(V_g)$  and finally  $m(V_e)$ . It follows that  $B_{abc}(m(V_e), m(V_f), m(V_g))$ .
- Let us consider a consistent instantiation  $m$  of  $\mathcal{N}$ . Let us define the one-to-one function  $\mathcal{F}$  from  $A$  to  $\{1, \dots, |A|\}$  in the following way. Because of the constraints in  $C$  the cyclic points associated with the variables are pairwise distinct. Let  $P$  be a cyclic point. Let  $e \in A$ ;  $\mathcal{F}(e)$  is the cardinal of the set  $\{v \in V : m(v) \in [P, m(V_e)]\}$ .  $\mathcal{F}$  is a one-to-one function, moreover  $\mathcal{F}$  is a solution for  $(A, T)$ . Indeed, let us suppose that  $(e, f, g) \in T$ ; it follows that  $C(V_e, V_f, V_g) = \{B_{abc}\}$ . Consequently, by moving along the circle starting from  $P$  we first meet  $m(V_e)$ , then  $m(V_f)$  and finally  $m(V_g)$ , or  $m(V_f)$ , then  $m(V_g)$  and finally  $m(V_e)$  or  $m(V_g)$ , then  $m(V_e)$  and finally  $m(V_f)$ . It follows that  $\mathcal{F}(e) < \mathcal{F}(f) < \mathcal{F}(g)$  or  $\mathcal{F}(g) < \mathcal{F}(e) < \mathcal{F}(f)$  or  $\mathcal{F}(f) < \mathcal{F}(g) < \mathcal{F}(e)$ . Hence  $\mathcal{F}$  is a solution.

□

From the previous proposition we can deduce that in the general case C-CNCP( $B_2, 2^{Bc}$ ) is a NP-complete problem.

## 4 Complexity and Conceptual Spaces

For a whole family of temporal and spatial calculi based on linear orderings, the complexity properties are closely related to geometrical and topological properties of the

**Table 5.** A listing of  $T_i$  sets.

Relations \ Sets	$T_0$	$T_1$	$T_2$	$T_3$	$T_4$	Relations \ Sets	$T_0$	$T_1$	$T_2$	$T_3$	$T_4$
$\emptyset$	•	•	•	•	•	$\{B_{aaa}, B_{baa}, B_{abc}\}$		•			
$\{B_{aaa}\}$	•	•	•	•	•	$\{B_{aaa}, B_{baa}, B_{acb}\}$		•			
$\{B_{aab}\}$	•	•	•	•	•	$\{B_{aaa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aba}\}$	•	•	•	•	•	$\{B_{aab}, B_{aba}, B_{baa}\}$		•			
$\{B_{baa}\}$	•	•	•	•	•	$\{B_{aab}, B_{aba}, B_{abc}\}$		•			
$\{B_{abc}\}$	•	•		•		$\{B_{aab}, B_{aba}, B_{acb}\}$		•			
$\{B_{acb}\}$	•	•		•		$\{B_{aab}, B_{baa}, B_{abc}\}$		•			
$\{B_{aaa}, B_{aab}\}$	•	•	•	•	•	$\{B_{aab}, B_{baa}, B_{acb}\}$		•			
$\{B_{aaa}, B_{aba}\}$	•	•	•	•	•	$\{B_{aab}, B_{abc}, B_{acb}\}$			•		
$\{B_{aaa}, B_{baa}\}$	•	•	•	•	•	$\{B_{aba}, B_{baa}, B_{abc}\}$		•			
$\{B_{aaa}, B_{abc}\}$		•		•		$\{B_{aba}, B_{baa}, B_{acb}\}$		•			
$\{B_{aaa}, B_{acb}\}$		•		•		$\{B_{aba}, B_{abc}, B_{acb}\}$			•		
$\{B_{aab}, B_{aba}\}$		•			•	$\{B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aab}, B_{baa}\}$		•			•	$\{B_{aaa}, B_{aab}, B_{aba}, B_{baa}\}$		•			•
$\{B_{aab}, B_{abc}\}$		•			•	$\{B_{aaa}, B_{aab}, B_{aba}, B_{abc}\}$		•			
$\{B_{aab}, B_{acb}\}$		•				$\{B_{aaa}, B_{aab}, B_{aba}, B_{acb}\}$		•			
$\{B_{aba}, B_{baa}\}$		•			•	$\{B_{aaa}, B_{aba}, B_{baa}, B_{abc}\}$		•			
$\{B_{aba}, B_{abc}\}$		•				$\{B_{aaa}, B_{aba}, B_{baa}, B_{acb}\}$		•			
$\{B_{aba}, B_{acb}\}$		•				$\{B_{aaa}, B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{baa}, B_{abc}\}$		•				$\{B_{aab}, B_{aba}, B_{baa}, B_{abc}\}$		•			
$\{B_{baa}, B_{acb}\}$		•				$\{B_{aab}, B_{aba}, B_{baa}, B_{acb}\}$		•			
$\{B_{abc}, B_{acb}\}$			•			$\{B_{aab}, B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aaa}, B_{aab}, B_{aba}\}$	•					$\{B_{aba}, B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aaa}, B_{aab}, B_{baa}\}$	•					$\{B_{aaa}, B_{aab}, B_{aba}, B_{baa}, B_{abc}\}$		•			
$\{B_{aaa}, B_{aab}, B_{abc}\}$	•					$\{B_{aaa}, B_{aab}, B_{aba}, B_{baa}, B_{acb}\}$		•			
$\{B_{aaa}, B_{aab}, B_{acb}\}$	•					$\{B_{aaa}, B_{aba}, B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aaa}, B_{aba}, B_{baa}\}$	•					$\{B_{aab}, B_{aba}, B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aaa}, B_{aba}, B_{abc}\}$	•					$\{B_{aaa}, B_{aab}, B_{aba}, B_{baa}, B_{abc}, B_{acb}\}$			•		
$\{B_{aaa}, B_{aba}, B_{acb}\}$	•										

relations. In this section, we give a quick survey of the results of that kind, relating them to the concept of *conceptual space* introduced by Gärdenfors [9]. We then examine the question of interpreting the preceding results about points on a circle in that context.

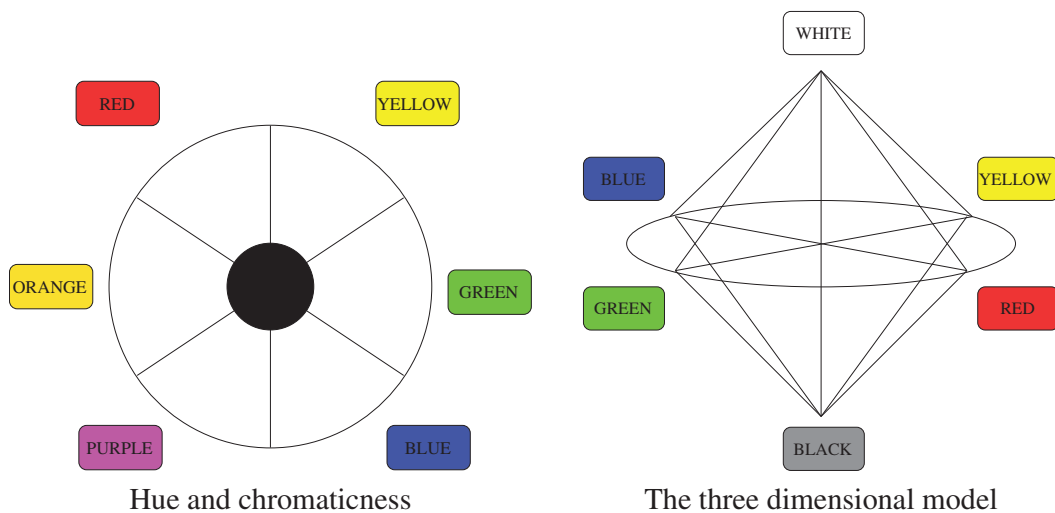
#### 4.1 Conceptual Spaces

Conceptual spaces are based on domains. A typical example of a domain is the *color domain* as represented by the Swedish natural color system (NCS) [11] which is a perceptual model of color perception. It describes the phenomenal structure of colors, that is, colors as we perceive them, using three dimensions: *hue*, *chromaticness* (or saturation), and *brightness*.

The first dimension, hue, is represented by the color circle. Colors lying opposite to each other are complementary colors: for example, green is complementary to red, orange to blue.

The second dimension, chromaticness, ranges from zero color intensity to increasingly greater intensities. It is modelled by a segment. Hence hue and chromaticness taken together are modelled by a disk, where colors can be distinguished on the periphery, and become more and more blurred as one comes closer to the center.

The third dimension is brightness which varies from white to black, and is consequently also represented by a segment. Brightness and chromaticness do not vary independently: variations in chromaticness decrease in range when brightness approaches black or white. Hence, for a given hue, the space of possible pairs (chromaticness, brightness) describes a triangle.



**Fig. 2.** The NCS model of colors.

Globally, then the model is called the NCS color spindle [27]. Gärdenfors gives a detailed discussion of the use of the model for explaining linguistic phenomena (such as the use of color terms), based on the assumption that terms referring to “natural” properties, that is in particular properties which can be named, correspond to *convex* subsets of the model.

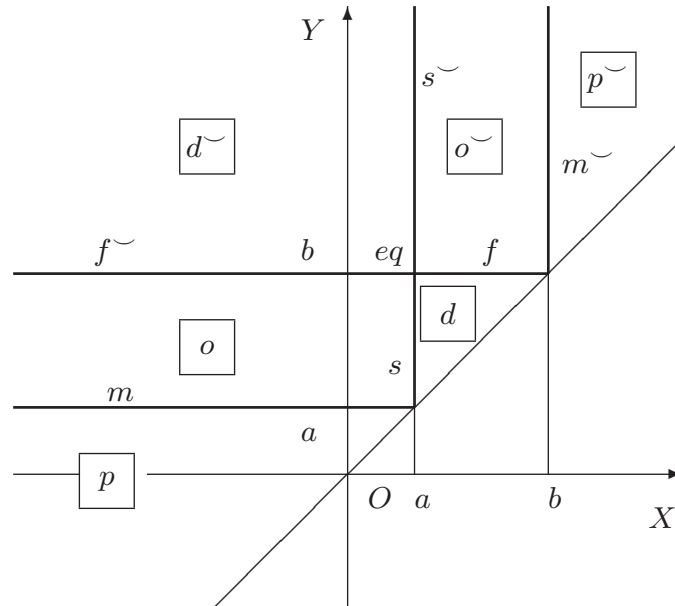
The color model example is only a particular instance of the general hypothesis about natural properties: they should correspond to convex regions in some suitable conceptual model. The interested reader should refer to [9].

The NCS model is an example of a *phenomenal* conceptual space. Other conceptual spaces are theoretical conceptual spaces: For instance, the conceptual model of space in Newtonian physics is a 3-dimensional Euclidean space, time being an independent (in Gärdenfors’ terminology, separable) dimension. By contrast, the temporal dimension in relativistic physics is an integral dimension of the 4-dimensional Minkowski space.

## 4.2 The Conceptual Space of Allen’s Relations

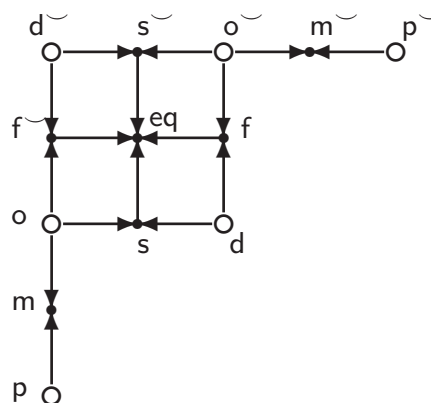
As a first, and typical case of a conceptual space in the domain of qualitative temporal reasoning, we consider the case of the Allen calculus [1]. Since an interval on the real line is characterized by a (strictly) increasing pair of real numbers, a model of the set

of all intervals is the open half-plane delimited by the first bisector in the  $(X, Y)$ -plane. This half-plane is defined by the equation  $Y > X$ . Given a fixed interval  $(a, b)$ , with  $a < b$ , the basic Allen relations correspond to 13 regions in the half-plane, as shown in Fig. 3.

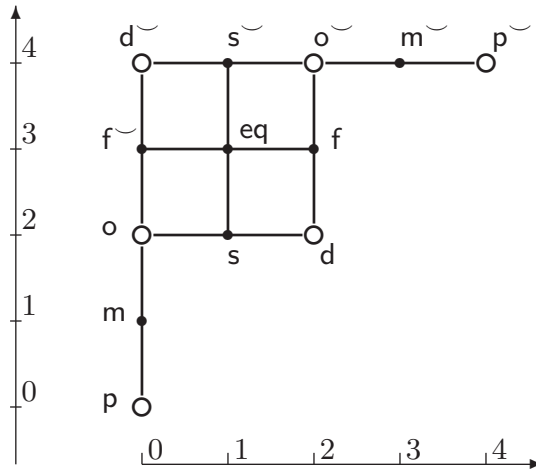


**Fig. 3.** The conceptual space of atomic relations in Allen's algebra

The conceptual space of Allen's relations has a much richer structure than the mere algebra. In particular, each relation has a dimension (the dimension of the corresponding region, which corresponds to the number of degrees of freedom of the relation). The incidence structure of the set of regions is a graph whose vertices correspond to the atomic relations, where there is an arc from  $r_1$  to  $r_2$  if  $r_2$  belongs to the boundary of  $r_1$ . This incidence structure can be deduced from the conceptual space, cf. Fig. 4. It contains enough topological information to encode the closure properties of the relations. In particular, the closure of any relation can be read from this graph.



**Fig. 4.** The incidence graph of Allen's relations



**Fig. 5.** The lattice of Allen's relations

Closely related to the incidence structure is the lattice of atomic relations represented in Fig. 5, which summarizes the order properties of the relations.

A basic problem in studying the complexity of reasoning with Allen's relations is the problem of determining whether a given constraint network is consistent. The general class of networks using any disjunction of atomic relations is known to be NP-complete. It is a remarkable fact that tractable subclasses of relations can be characterized in geometrical terms, as shown in [17] and subsequent papers [18,20].

Basic relations (as regions in the half-plane) are convex relations. In fact, they have a stronger property: they are also *saturated* (with respect to projections on the axes), in the sense that for such a region  $R$ ,  $R = pr_1(R) \times pr_2(R)$ , where  $pr_1$  and  $pr_2$  are the  $X$  and  $Y$  projections respectively.

*Convex* relations are those unions of atomic relations which are both convex and saturated. In the lattice representation, this is equivalent to relations which are intervals in the lattice. More generally, *pre-convex* relations are those relations whose topological closure is a convex relation. Although they are neither convex nor saturated in general, they differ from the smallest convex closure by only "small" pieces, in the sense that the difference contains only relations whose dimension is strictly smaller. An argument based on this fact, together with the known fact that convex relations are tractable implies that the class of pre-convex relations is tractable [18]. In fact, it is the unique maximal tractable subclass containing all atomic relations [19].

Those results also can be obtained with purely syntactic methods: pre-convex relations coincide with ORD-Horn relations in the sense of [23], and their tractability is a consequence of the properties of Horn theories.

### 4.3 Conceptual Spaces and Complexity in Calculi Based on Linear Orderings

Allen's calculus fits into a larger family of calculi based on linear orderings:

- Generalized interval calculi [15,16,20] which consider finite strictly increasing sequences of points (in this context, Allen's calculus is the particular case of two points);

- The  $n$ -point calculus [4] where the basic objects are points in a  $n$ -D Euclidean space (the time point calculus is the case where  $n = 1$ ). The case where  $n = 2$  has been considered in [20] under the name of Cardinal Direction Calculus.
- The  $n$ -block calculus [5,2], whose basic objects are blocks (products of intervals) in a  $n$ -D Euclidean space (Allen's calculus is the case where  $n = 1$ ).

Conceptual spaces are easily derived using the same method as for Allen's calculus in each particular case:

- In the case of generalized interval calculi, the conceptual space associated to  $(p, q)$ -relations, that is, relations from one  $p$ -interval to one  $q$ -interval ( $p, q \geq 1$ ) is defined as follows:

Consider in the Euclidean  $q$ -space the cone  $C_q$  defined by:

$$X_1 < X_2 < \dots < X_q$$

Fix a point  $(a_1, \dots, a_q)$  in  $C_q$

The set of atomic  $(p, q)$ -relations  $\Pi_{p,q}$  is the set of non-decreasing sequences of length  $p$  of integers between 0 and  $2q$ , where no odd integer occurs more than once. Each  $(p, q)$ -relation is associated with a region in  $C_q$ , and, globally, these regions constitute a partition of  $C_q$ .

The explicit definition of the region associated to a given  $(p, q)$ -relation  $(b_1, \dots, b_p)$  is intricate but straightforward: if  $b_i = 2n_i + 1$ , then we consider the equation  $X_i = a_{n_i}$ ; if  $b_i = 2n_i$ , then we consider the inequations  $a_{n_i} < X_i < a_{n_i+1}$ . The region is defined by the conjunctions of equations and inequations associated to each  $b_i$ , for  $1 \leq i \leq p$ .

Clearly, since it is defined by a Cartesian product of points or open intervals, (the region associated to ) each atomic relation is convex and saturated.

Quite analogously to the case of Allen's relations, *convex* relations can be defined, e.g. as intervals in the lattice  $\Pi_{p,q}$ . They are convex and saturated. *Pre-convex* relations are those relations which have a convex topological closure. The same kinds of considerations as in Allen's case show that pre-convex relations, or more precisely the subclass of pre-convex relations, called strongly pre-convex relations [6] is tractable.

- For the  $n$ -point calculus, the conceptual space associated to basic relations of the  $n$ -point calculus (which are sequences of length  $n$  of point relations) is a partition of the Euclidean  $n$ -space:

Consider any point  $a_1, \dots, a_n$  in the  $n$ -space.

A basic relation is a  $n$ -tuple  $b_1, \dots, b_n$ , where  $b_i \in \{<, =, >\}$ .

The region associated to such a basic relation is defined by the conjunction of the equations or inequations:

$$X_i = a_i \text{ if } b_i \text{ is } =, X_i < a_i \text{ if } b_i \text{ is } <, \text{ and } X_i > a_i \text{ if } b_i \text{ is } >.$$

Clearly again, these regions are convex and saturated. Convex, pre-convex, and strongly pre-convex relations can be defined, and tractability results obtained for strongly pre-convex relations [4].

- Finally, for the  $n$ -block calculus, the basic relations are sequences of Allen’s relations. The corresponding conceptual space is a product of copies of the space for Allen’s relations, and again, similar results obtain for pre-convex relations.

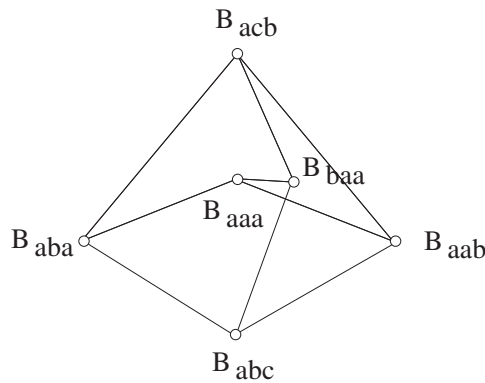
It must be mentioned, moreover, that in all three classes of calculi, strongly pre-convex relations coincide with ORD-Horn relations, which gives an independent motivation for their tractability, and constitutes a nice point of agreement between geometrically and syntactically motivated notions [6].

#### 4.4 Points on a Circle

For all calculi based on linear orderings by taking sequences or products, the geometric structure of the basic relations, as represented in the corresponding conceptual space and in the lattice and incidence graph representations, are closely related to tractability properties. In line with the general considerations in Gärdenfors’ framework, convexity and the stronger property of convexity plus saturation, play a crucial role.

The sad fact is that this does not seem to be the case any longer if we consider relations in the cyclic case. For the ternary relations between points considered in this paper, the incidence graph of the basic relations is easily obtained: starting from the relation where all three points coincide, that is, relation  $B_{aaa}$ , one gets, by separating either  $x$ ,  $y$ , or  $z$ , one of the three relations  $B_{baa}$ ,  $B_{aba}$ ,  $B_{aab}$ . Going further and separating the remaining two points leads either to  $B_{abc}$  or to  $B_{acb}$ . Hence we get the graph in Fig. 4.4.

However, it is not at all clear how the (partial) complexity results we have obtained in this paper relate to geometric properties of this graph. This negative phenomenon may be related to the fact that, for the *binary* qualitative relations between intervals on a circle, path-consistent atomic networks may be inconsistent [7], or, in other terms, that weak-representations in the sense of [15,21] may well be inconsistent.



**Fig. 6.** The incidence graph of the cyclic point relations.

## 5 Conclusion

In a first part we described the relations between linear and cyclic models. Then, we considered six possible ternary relations between three points on a circle, which are jointly exhaustive and pairwise disjoint (JEPD) relations, and developed a qualitative calculus called the cyclic point algebra. We examined the consistency problem of the cyclic point networks. We have characterized several tractable and untractable cases for this problem.

The continuation of this work will be the complete characterization of all the tractable cases in the cyclic point algebra. Because of the small size of the set of relations of the cyclic point algebra, this goal seems to be reasonable.

Another perspective will be considering cyclic arcs instead of cyclic points. The relations considered will be those characterized by Balbiani and Osmani in [7]. A first task will consist in defining an axiom system for these relations. To this end, we can use the axiom system of the cyclic orders (see [13] for a similar work). Concerning the constraint aspects, our study of cyclic point networks can be certainly used to characterize new tractable cases for the consistency problem of the cyclic arc networks.

We presented the basic notions of the framework of conceptual spaces, in relation to the characterization of tractable subclasses for formalisms such as the Interval Algebra. An open question is: is there a geometric and topological characterization of tractable classes in the cyclic cases? It appears that finding a suitable conceptual space is more difficult (less natural) than in the linear case.

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