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# The discontinuous Galerkin approximation of the grad-div and curl-curl operators in first-order form is involution-preserving and spectrally correct\*

Alexandre Ern<sup>†</sup>      Jean-Luc Guermond<sup>‡</sup>

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## Abstract

The discontinuous Galerkin approximation of the grad-div and curl-curl problems formulated in conservative first-order form is investigated. It is shown that the approximation is spectrally correct, thereby confirming numerical observations made by various authors in the literature. This result hinges on the existence of discrete involutions which are formulated as discrete orthogonality properties. The involutions are crucial to establish discrete versions of weak Poincaré—Steklov inequalities that hold at the continuous level.

**Keywords.** Grad-div problem, curl-curl problem, first-order system, Friedrichs system, involution, spectrum approximation

**MSC.** 65M60, 65M12, 65N30, 35L02, 35L05, 35Q61

## 1 Introduction

Many conservation equations generate involutions, e.g., the elastodynamics equations, Maxwell’s equations, the magnetohydrodynamics equations, the wave equation, etc. For instance, on domains with trivial topology, if one considers the wave equation (which is a linearized version of the compressible Euler equations), the involution on the velocity,  $\mathbf{v}$ , is  $\nabla \times \mathbf{v} = \mathbf{0}$ , and if one considers Maxwell’s equations, the involutions on the electric and magnetic fields  $\mathbf{E}$ ,  $\mathbf{B}$ , are Gauss laws, i.e.,  $\nabla \cdot \mathbf{E} = \mathbf{0}$  (in the absence of free charges) and  $\nabla \cdot \mathbf{B} = \mathbf{0}$ . An algebraic characterization of involutions for general conservation equations is given in Boillat [6]. Involutions are important to prove compactness, entropy inequalities, and to establish well-posedness. For instance, the magnetohydrodynamics equations satisfy an entropy inequality only if the divergence of the magnetic field is zero; see, e.g., Godunov [18], Barth [4, §2.2]. The question addressed in this paper is whether using the discontinuous Galerkin (dG) method to approximate in space conservation equations endowed with involutions generates discrete involutions that are strong enough to establish compactness. Since the problem is very difficult for generic nonlinear conservation equations, we restrict ourselves

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to the grad-div and curl-curl operators written in first-order conservation form as they are good representatives of the problems encountered with the wave equation and Maxwell's equations. For both operators, the involutions are formulated as orthogonality properties (see, e.g., Hiptmair [21, §4.1] for handling Gauss' law involutions). We restrict ourselves to the investigation of the spectral properties of the dG approximation, leaving the study of other discretization methods to future work.

The spectral properties of the dG method for approximating the curl-curl problem written in second-order form (wherein one of the two fields is eliminated) have been established in Buffa and Perugia [8, Prop. 7.3] (this result is prefigured in Houston et al. [22, Cor. 3.6] for constant coefficients). It is shown in [8] that the dG approximation is spectrally correct, i.e., the point spectrum of the approximation does not contain spurious eigenvalues. Whether this is also the case when the equations are written in first-order form has not yet been established at the time of this writing and to the best of the authors' knowledge. The dG approximation of the time-dependent Maxwell's equations written in first-order conservation form has been investigated in Hesthaven and Warburton [19]. Using energy arguments, it is shown in Theorem 4.2 therein that the approximation is convergent, and an observation is made in Theorem 4.3 regarding the involution properties of the scheme. This observation is, however, not sufficient to establish that the associated eigenvalue problem is spectrally correct. The eigenvalue problem is numerically investigated in Alvarez et al. [1], Cohen and Duruflé [12, §3.1], Hesthaven and Warburton [20], and the authors observe that the eigenvalue problem is pollution free provided the numerical flux is dissipative, i.e., includes penalty terms on the jump of the tangential components of both fields. Furthermore, spectral correctness has been established for a combination of conforming and dG approximations of Maxwell's equations by Campos Pinto and Sonnendrücker [9]. It is also observed therein that the full dG approximation is compatible with Gauss laws.

In the paper, we prove that indeed the dG method yields approximations of the first-order form of the grad-div and curl-curl operators that are spectrally correct. In particular, we show that it is essential to invoke discrete counterparts of the involutions associated with the continuous problem to establish this result. The discrete involutions, which are formulated as (topology-blind) discrete orthogonality properties, are crucial to establish discrete (weak) Poincaré–Steklov inequalities. These inequalities are, in turn, pivotal to gain full  $L^2$ -stability of the dG approximation. The discrete Poincaré–Steklov inequalities usually available in the literature involve the  $L^2$ -norm of the gradient, curl or divergence when using conforming spaces (see, e.g., Hiptmair [21, Thm. 4.7] or Monk and Demkowicz [24, Cor. 3.2]) or reconstructions thereof when using dG approximations (see Buffa and Perugia [8, Lem. 7.6]). The route followed here consists instead of extending weak Poincaré–Steklov inequalities that hold at the continuous level and which involve a dual norm of the gradient, curl or divergence, thus leading to sharper inequalities.

The material is organized as follows. We present in §2 the continuous operators whose spectrum we want to approximate with the dG method (see Definitions 2.12 and 2.16). The involutions mentioned above are formalized as orthogonality properties (see Remarks 2.1 and 2.5). The finite element setting used in the approximation is introduced in §3. Lemma 3.2, which establishes discrete Poincaré–Steklov inequalities hinging on discrete involutions, is the key result of this section. As in the continuous setting, the discrete involutions are formalized as orthogonality properties. The dG approximation of the grad-div and curl-curl operators is analyzed in §4 and §5, respectively. The main results of these sections are Theorems 4.11 and 5.10. Owing to standard spectral approximation results (see, e.g., Bramble and Osborn [7, Lem. 2.2], Osborn [25, Thm. 3&4], Boffi [5, Prop. 7.4]), these theorems prove the spectral correctness of the dG approximation. For completeness, some standard results on Helmholtz decompositions are collected in Appendix A. Although everything that is said in the paper could be written using the unified formalism of finite element exterior calculus (see Arnold et al. [3]), we prefer to use the formalism of vector calculus

to be more explicit even though the structure of the grad-div and curl-curl operators is similar. Finally, we observe that the main compactness result invoked for both operators is the consequence of the smoothness regularity of the solution in fractional-order Sobolev spaces with regularity index  $s > \frac{1}{2}$  (see Lemma 2.13 and 2.17). This regularity is known in the literature to hold in the case of homogeneous materials; the case of heterogeneous materials is left to future work.

## 2 Continuous setting

Let  $D$  be an open, bounded, Lipschitz polyhedron of  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ , with unit outward normal vector  $\mathbf{n}_D$ . Additional topological assumptions on  $D$  are collected in §A.1. We implicitly assume that  $d = 3$  whenever working with the curl operator. To be dimensionally consistent, we introduce a length scale associated with  $D$ ,  $\ell_D$  (it could be for instance the diameter of  $D$ ).

### 2.1 Functional spaces

We use standard notation for Lebesgue and Sobolev spaces. We use boldface fonts for  $\mathbb{R}^d$ -valued vectors, vector fields, and functional spaces composed of such fields. The spaces  $L^2(D)$  and  $\mathbf{L}^2(D)$  are composed of Lebesgue integrable scalar-valued functions and vector fields that are square integrable, respectively. The canonical inner products in these two spaces are denoted  $(\cdot, \cdot)_{L^2(D)}$  and  $(\cdot, \cdot)_{\mathbf{L}^2(D)}$ , respectively. Depending on the context, the symbol  $^\perp$  denotes the orthogonality in  $L^2(D)$  or  $\mathbf{L}^2(D)$ . We recall the following Hilbert Sobolev spaces:

$$H^1(D) := H(\mathbf{grad}; D) := \{p \in L^2(D) \mid \nabla p \in \mathbf{L}^2(D)\}, \quad (1a)$$

$$\mathbf{H}(\mathbf{curl}; D) := \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{v} \in \mathbf{L}^2(D)\}, \quad (1b)$$

$$\mathbf{H}(\mathbf{div}; D) := \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \cdot \mathbf{v} \in L^2(D)\}. \quad (1c)$$

These spaces are equipped with their natural graph norms

$$\|p\|_{H^1(D)} := \|p\|_{L^2(D)} + \ell_D \|\nabla p\|_{\mathbf{L}^2(D)}, \quad (2a)$$

$$\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl}; D)} := \|\mathbf{v}\|_{\mathbf{L}^2(D)} + \ell_D \|\nabla \times \mathbf{v}\|_{\mathbf{L}^2(D)}, \quad (2b)$$

$$\|\mathbf{v}\|_{\mathbf{H}(\mathbf{div}; D)} := \|\mathbf{v}\|_{\mathbf{L}^2(D)} + \ell_D \|\nabla \cdot \mathbf{v}\|_{L^2(D)}. \quad (2c)$$

We also consider the following closed subspaces:

$$H_0^1(D) := H_0(\mathbf{grad}; D) := \{p \in H^1(D) \mid \gamma_{\partial D}^g(p) = 0\}, \quad (3a)$$

$$\mathbf{H}_0(\mathbf{curl}; D) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; D) \mid \gamma_{\partial D}^c(\mathbf{v}) = \mathbf{0}\}, \quad (3b)$$

$$\mathbf{H}_0(\mathbf{div}; D) := \{\mathbf{v} \in \mathbf{H}(\mathbf{div}; D) \mid \gamma_{\partial D}^d(\mathbf{v}) = 0\}, \quad (3c)$$

where  $\gamma_{\partial D}^g : H^1(D) \rightarrow H^{\frac{1}{2}}(\partial D)$  is the extension by density of the usual trace operator such that  $\gamma_{\partial D}^g(p) = p|_{\partial D}$  for every smooth function  $p \in H^s(D)$ ,  $s > \frac{1}{2}$ , and the tangential and normal trace operators  $\gamma_{\partial D}^c : \mathbf{H}(\mathbf{curl}; D) \rightarrow \mathbf{H}^{-\frac{1}{2}}(\partial D)$  and  $\gamma_{\partial D}^d : \mathbf{H}(\mathbf{div}; D) \rightarrow H^{-\frac{1}{2}}(\partial D)$  are the extensions by density of the tangent and normal trace operators such that  $\gamma_{\partial D}^c(\mathbf{v}) = \mathbf{v}|_{\partial D} \times \mathbf{n}_D$  and  $\gamma_{\partial D}^d(\mathbf{v}) = \mathbf{v}|_{\partial D} \cdot \mathbf{n}_D$  for every smooth field  $\mathbf{v} \in \mathbf{H}^s(D)$ ,  $s > \frac{1}{2}$  (see, e.g., [16, §4.3]). We are also going to make

use of the following closed subspaces:

$$\mathbb{P}_0 := H(\mathbf{grad} = \mathbf{0}; D) := \{p \in H^1(D) \mid \nabla p = \mathbf{0}\}, \quad (4a)$$

$$\mathbf{H}(\mathbf{curl} = \mathbf{0}; D) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; D) \mid \nabla \times \mathbf{v} = \mathbf{0}\}, \quad (4b)$$

$$\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) := \{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \nabla \times \mathbf{v} = \mathbf{0}\}, \quad (4c)$$

$$\mathbf{H}(\text{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}(\text{div}; D) \mid \nabla \cdot \mathbf{v} = 0\}, \quad (4d)$$

$$\mathbf{H}_0(\text{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}_0(\text{div}; D) \mid \nabla \cdot \mathbf{v} = 0\}. \quad (4e)$$

Finally, we recall the following result (see Amrouche et al. [2, Prop. 7]) which is a consequence of the elliptic regularity theory: There exists  $s \in (\frac{1}{2}, 1]$  so that for all  $\mathbf{e} \in \{\mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}_0(\text{div}; D), \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}(\text{div}; D)\}$ ,

$$\|\mathbf{e}\|_{\mathbf{H}^s(D)} \leq C_D (\|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl}; D)} + \|\mathbf{e}\|_{\mathbf{H}(\text{div}; D)}). \quad (5)$$

Here and in what follows,  $C_D$  denotes a generic constant whose value can change at each occurrence as long as it only depends on  $D$ .

## 2.2 Heuristics for the involutions

We introduce in this section the differential operators associated with the grad-div and curl-curl operators when written either as eigenvalue problems or as first-order conservation equations. In particular, we identify the associated involutions and interpret them as orthogonality properties.

### 2.2.1 Grad-div eigenvalue problem

Given a scaling factor  $\mathfrak{c} > 0$  (see Remark 2.3), we consider the following eigenvalue problem: Find  $\lambda \in \mathbb{C}$  and a nonzero pair  $(\mathbf{v}, p) \in \mathbf{H}_0(\text{div}; D) \times H^1(D)$  so that

$$\nabla p = \lambda \mathbf{v}, \quad \mathfrak{c}^2 \nabla_0 \cdot \mathbf{v} = \lambda p, \quad (6)$$

with the operators  $\nabla : H^1(D) \ni q \mapsto \nabla q \in \mathbf{L}^2(D)$  and  $\nabla_0 \cdot : \mathbf{H}_0(\text{div}; D) \ni \mathbf{w} \mapsto \nabla \cdot \mathbf{w} \in L^2(D)$ . Notice that  $-\nabla$  and  $\nabla_0 \cdot$  are adjoint to each other since  $(q, \nabla_0 \cdot \mathbf{w})_{L^2(D)} = -(\nabla q, \mathbf{w})_{L^2(D)}$  for all  $q \in H^1(D)$  and all  $\mathbf{w} \in \mathbf{H}_0(\text{div}; D)$ . Moreover, we will see in §2.3 that both operators have a closed range.

We are only interested in the case  $\lambda \neq 0$ . The assumption  $\lambda \neq 0$  implies that have  $p = \nabla_0 \cdot (\lambda^{-1} \mathfrak{c}^2 \mathbf{v})$  and  $\mathbf{v} = \nabla (\lambda^{-1} p)$ . This means that

$$p \in \text{im}(\nabla_0 \cdot) = \ker(\nabla)^\perp = \mathbb{P}_0^\perp, \quad (7a)$$

$$\mathbf{v} \in \text{im}(\nabla) = \ker(\nabla_0 \cdot)^\perp = \mathbf{H}_0(\text{div} = 0; D)^\perp, \quad (7b)$$

where we recall that the symbol “im” means “image of” or “range of” and the symbol “ker” means “kernel of” or “nullspace of”. (Notice that  $p \in \mathbb{P}_0^\perp$  simply means that  $(p, 1)_{L^2(D)} = 0$ , i.e.,  $p$  has zero mean-value over  $D$ .) We call *involutions* of the eigenvalue problem (6) the properties  $p \in \mathbb{P}_0^\perp$  and  $\mathbf{v} \in \mathbf{H}_0(\text{div} = 0; D)^\perp$ . One objective of the paper is to prove that the dG approximation of (6) preserves discrete versions of the involutions (7), and that the approximation of the spectrum is pollution free.

**Remark 2.1** (Involutions and topology of  $D$ ). *The involution property (7b) implies that  $\nabla \times \mathbf{v} = \mathbf{0}$  since  $\mathbf{H}_0(\text{div} = 0; D)^\perp = \nabla H^1(D)$  (see (68c)). However, the statement  $\nabla \times \mathbf{v} = \mathbf{0}$  does not tell the full story when the topology of  $D$  is not trivial. Indeed, the decomposition  $\mathbf{H}(\mathbf{curl} = \mathbf{0}; D) =$*

$\nabla H^1(D) \oplus \mathbf{K}_T(D)$  (see (67a)) implies that  $\mathbf{H}_0(\operatorname{div} = 0; D)^\perp \subset \mathbf{H}(\operatorname{curl} = \mathbf{0}; D)$  with equality iff  $D$  is simply connected. Thus,  $\nabla \times \mathbf{v} = \mathbf{0}$  is the correct description of the involution  $\mathbf{v} \in \mathbf{H}_0(\operatorname{div} = 0; D)^\perp$  only if  $D$  is simply connected.

**Remark 2.2** (Time domain). Consider the wave equation in the time domain:

$$\partial_t \mathbf{v} + \nabla p = \mathbf{0}, \quad \partial_t p + \mathbf{c}^2 \nabla_0 \cdot \mathbf{v} = 0, \quad (8)$$

with initial conditions  $p(\cdot, 0) = p^0(\cdot)$ ,  $\mathbf{v}(\cdot, 0) = \mathbf{v}^0(\cdot)$  and boundary condition  $\gamma_{\partial D}^d(\mathbf{v}) = 0$ . Arguing as above, we observe that if  $p^0 \in \mathbb{P}_0^\perp$ ,  $\mathbf{v}^0 \in \mathbf{H}_0(\operatorname{div} = 0; D)^\perp$ , then the involutions  $p(\cdot, t) \in \mathbb{P}_0^\perp$ ,  $\mathbf{v}(\cdot, t) \in \mathbf{H}_0(\operatorname{div} = 0; D)^\perp$  hold at all times. That the dG method satisfies discrete counterparts of these involutions guarantees that the (semi-discrete) system behaves properly over long times.

**Remark 2.3** (Scaling factor). The scaling factor  $\mathbf{c}$  is introduced to remind us that the field  $\mathbf{v}$  and the function  $p$  can have different units. Here,  $\mathbf{c}$  has the same units as the ratio of  $p$  to  $\|\mathbf{v}\|$ . In applications, one often thinks of  $\mathbf{c}$  as a wave speed. With this scaling, the eigenvalue  $\lambda$  scales as a frequency. The reader can assume that  $\mathbf{c} = 1$  without losing anything essential in what follows.

**Remark 2.4** (Other BCs). The problems (6) and (8) can be equipped with other boundary conditions. For instance, one can enforce  $\gamma_{\partial D}^e(p) = 0$  instead of  $\gamma_{\partial D}^d(\mathbf{v}) = 0$ . In this case, the involutions are  $p \in \{0\}^\perp$  and  $\mathbf{v} \in \mathbf{H}(\operatorname{div} = 0; D)^\perp$ . (Notice that the involution  $p \in \{0\}^\perp$  is trivial.) The present analysis extends to this situation; see Remarks 2.10 and 4.1.

## 2.2.2 Curl-curl eigenvalue problem

Given a scaling factor  $\mathbf{c} > 0$ , we consider the following eigenvalue problem: Find  $\lambda \in \mathbb{C}$  and a nonzero pair  $(\mathbf{B}, \mathbf{E}) \in \mathbf{H}_0(\operatorname{curl}; D) \times \mathbf{H}(\operatorname{curl}; D)$  so that

$$-\nabla \times \mathbf{E} = \lambda \mathbf{B}, \quad \mathbf{c}^2 \nabla_0 \times \mathbf{B} = \lambda \mathbf{E}, \quad (9)$$

with the operators  $\nabla \times : \mathbf{H}(\operatorname{curl}; D) \ni \mathbf{e} \mapsto \nabla \times \mathbf{e} \in \mathbf{L}^2(D)$  and  $\nabla_0 \times : \mathbf{H}_0(\operatorname{curl}; D) \ni \mathbf{b} \mapsto \nabla \times \mathbf{b} \in \mathbf{L}^2(D)$ . These operators are adjoint to each other since we have  $(\mathbf{e}, \nabla_0 \times \mathbf{b})_{\mathbf{L}^2(D)} = (\mathbf{b}, \nabla \times \mathbf{e})_{\mathbf{L}^2(D)}$  for all  $\mathbf{e} \in \mathbf{H}(\operatorname{curl}; D)$  and all  $\mathbf{b} \in \mathbf{H}_0(\operatorname{curl}; D)$ . Moreover, we will see in §2.3 that both operators have a closed range.

We are only interested in the case  $\lambda \neq 0$ . The assumption  $\lambda \neq 0$  implies that  $\mathbf{E} = \nabla_0 \times (\lambda^{-1} \mathbf{c}^2 \mathbf{B})$  and  $\mathbf{B} = -\nabla \times (\lambda^{-1} \mathbf{E})$ . This means that

$$\mathbf{E} \in \operatorname{im}(\nabla_0 \times) = \ker(\nabla \times)^\perp = \mathbf{H}(\operatorname{curl} = \mathbf{0}; D)^\perp, \quad (10a)$$

$$\mathbf{B} \in \operatorname{im}(\nabla \times) = \ker(\nabla_0 \times)^\perp = \mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D)^\perp. \quad (10b)$$

We call *involutions* of the eigenvalue problem (9) the fact that  $\mathbf{E} \in \mathbf{H}(\operatorname{curl} = \mathbf{0}; D)^\perp$  and  $\mathbf{B} \in \mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D)^\perp$ . One objective of the paper is to prove that the dG approximation of (9) preserves discrete versions of the involutions (10) and that the approximation of the spectrum is pollution free.

**Remark 2.5** (Involutions and topology of  $D$ ). Owing to (67a) and (68c), we have  $\mathbf{H}(\operatorname{curl} = \mathbf{0}; D)^\perp \subset \mathbf{H}_0(\operatorname{div} = 0; D)$ , and equality holds iff  $D$  is simply connected. Thus, the statement  $\nabla \cdot \mathbf{E} = 0$  and  $\gamma_{\partial D}^d(\mathbf{E}) = 0$  tells the full story on the involution satisfied by  $\mathbf{E}$  only if  $D$  is simply connected. Similarly, owing to (67b) and (68d), we have  $\mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D)^\perp \subset \mathbf{H}(\operatorname{div} = 0; D)$ , and equality holds iff  $\partial D$  is connected. Thus, the statement  $\nabla \cdot \mathbf{B} = 0$  tells the full story on the involution satisfied by  $\mathbf{B}$  only if  $\partial D$  is connected.

**Remark 2.6** (Time domain). *Consider Maxwell's equations in the time domain:*

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0}, \quad \partial_t \mathbf{E} - \mathbf{c}^2 \nabla_0 \times \mathbf{B} = \mathbf{0}, \quad (11)$$

with initial conditions  $\mathbf{B}(\cdot, 0) = \mathbf{B}^0(\cdot)$ ,  $\mathbf{E}(\cdot, 0) = \mathbf{E}^0(\cdot)$  and boundary condition  $\gamma_{\partial D}^{\circ}(\mathbf{B}) = \mathbf{0}$ . Assuming  $\mathbf{B}^0 \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ ,  $\mathbf{E}^0 \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ , then the involutions  $\mathbf{B}(\cdot, t) \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ ,  $\mathbf{E}(\cdot, t) \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$  hold at all times. That the dG method satisfies discrete counterparts of these involutions guarantees that the (semi-discrete) system behaves properly over long times.

### 2.3 Weak Poincaré–Steklov inequalities

In this section, we identify important properties of the differential operators involved in the eigenvalue problems (6) and (9) and in the time-evolution problems (8) and (11). All these results are consequences of well-known Helmholtz decompositions which we recall in Appendix A. Recall also that we defined  $H^1(D) := H(\mathbf{grad}; D)$  and  $\mathbb{P}_0 := H(\mathbf{grad} = \mathbf{0}; D)$ , and that orthogonality is meant in  $L^2$  and  $\mathbf{L}^2$  depending on the context. We consider the following subspaces:

$$X^g := H(\mathbf{grad}; D) \cap H(\mathbf{grad} = \mathbf{0}; D)^\perp = H^1(D) \cap \mathbb{P}_0^\perp, \quad (12a)$$

$$\mathbf{X}_0^d := \mathbf{H}_0(\mathbf{div}; D) \cap \mathbf{H}_0(\mathbf{div} = 0; D)^\perp, \quad (12b)$$

$$\mathbf{X}^c := \mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (12c)$$

$$\mathbf{X}_0^c := \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp. \quad (12d)$$

Notice that these spaces are indeed closed in  $H^1(D)$ ,  $\mathbf{H}(\mathbf{div}; D)$ ,  $\mathbf{H}(\mathbf{curl}; D)$  and  $\mathbf{H}_0(\mathbf{curl}; D)$ , respectively, and are therefore Hilbert spaces when equipped with the inherited inner products.

**Lemma 2.7** (Isomorphisms). *The following operators are isomorphisms:*

$$\nabla : X^g \rightarrow \mathbf{H}_0(\mathbf{div} = 0; D)^\perp, \quad \nabla_0 \cdot : \mathbf{X}_0^d \rightarrow H(\mathbf{grad} = \mathbf{0}; D)^\perp = \mathbb{P}_0^\perp. \quad (13a)$$

$$\nabla \times : \mathbf{X}^c \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \nabla_0 \times : \mathbf{X}_0^c \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp. \quad (13b)$$

*Proof.* (1) These operators are well defined since  $\nabla(H^1(D)) \subset \mathbf{H}_0(\mathbf{div} = 0; D)^\perp$ ,  $\nabla_0 \cdot (\mathbf{H}_0(\mathbf{div}; D)) \subset \mathbb{P}_0^\perp$ ,  $\nabla \times (\mathbf{H}(\mathbf{curl}; D)) \subset \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ ,  $\nabla_0 \times (\mathbf{H}_0(\mathbf{curl}; D)) \subset \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ . They are also bounded.

(2) Injectivity. The argument is straightforward. For instance, let  $p \in X^g = H(\mathbf{grad}; D) \cap H(\mathbf{grad} = \mathbf{0}; D)^\perp$  be such that  $\nabla p = \mathbf{0}$ . Then  $p \in H(\mathbf{grad} = \mathbf{0}; D) \cap H(\mathbf{grad} = \mathbf{0}; D)^\perp = \{0\}$ . A similar argument shows that the other operators are injective as well.

(3) Surjectivity. The Helmholtz decompositions (68c) and (68b) show that the operators  $\nabla$  and  $\nabla_0 \cdot$  are surjective. The surjectivity of the  $\nabla \times$  operator follows from the Helmholtz decomposition (70a) and  $\nabla(H_\Gamma^1(D)) \subset \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$ . Similarly, the surjectivity of the  $\nabla_0 \times$  operator follows from the Helmholtz decomposition (70b) and  $\nabla_\Sigma(H_\Sigma^1(D)) \subset \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)$  (see Amrouche et al. [2, Lem. 3.11, p. 840]). □

**Remark 2.8** (Literature). *Referring to Appendix A for the notation and to Dautray and Lions [13, Tab. I, p. 314], we have  $\mathbf{H}^\Gamma(\mathbf{div} = 0; D) = \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$  and  $\mathbf{H}_0^\Sigma(\mathbf{div} = 0; D) = \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ . Hence, (13b) is a topology-blind restatement of Theorems 3.2 and 3.17 in Amrouche et al. [2].*

Lemma 2.7 implies that the images of the operators  $\nabla$ ,  $\nabla_0 \cdot$ ,  $\nabla \times$ ,  $\nabla_0 \times$  are closed. Therefore, there is  $C_D > 0$  so that  $C_D \|p\|_{H^1(D)} \leq \ell_D \|\nabla p\|_{L^2(D)}$  for all  $p \in X^g$ . Hence, we can equip  $X^g$  with the norm  $\|p\|_{X^g} := \ell_D \|\nabla p\|_{L^2(D)}$  for all  $p \in X^g$ . By using a similar argument, we equip  $\mathbf{X}_0^d$ ,  $\mathbf{X}_0^c$ ,  $\mathbf{X}^c$  with the norms  $\|\mathbf{v}\|_{\mathbf{X}_0^d} := \ell_D \|\nabla_0 \cdot \mathbf{v}\|_{L^2(D)}$ ,  $\|\mathbf{b}\|_{\mathbf{X}_0^c} := \ell_D \|\nabla_0 \times \mathbf{b}\|_{L^2(D)}$ ,  $\|\mathbf{e}\|_{\mathbf{X}^c} := \ell_D \|\nabla \times \mathbf{e}\|_{L^2(D)}$ , respectively. For all  $p \in H(\mathbf{grad} = \mathbf{0}; D)^\perp = \mathbb{P}_0^\perp$ , all  $\mathbf{v} \in \mathbf{H}_0(\mathbf{div} = \mathbf{0}; D)^\perp$ , all  $\mathbf{b} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ , and all  $\mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ , we set

$$\|\nabla p\|_{(\mathbf{X}_0^d)'} := \sup_{\mathbf{v} \in \mathbf{X}_0^d} \frac{|(p, \nabla_0 \cdot \mathbf{v})_{L^2(D)}|}{\ell_D \|\nabla_0 \cdot \mathbf{v}\|_{L^2(D)}}, \quad \|\nabla \cdot \mathbf{v}\|_{(X^g)'} := \sup_{p \in X^g} \frac{|(\mathbf{v}, \nabla p)_{L^2(D)}|}{\ell_D \|\nabla p\|_{L^2(D)}}, \quad (14a)$$

$$\|\nabla \times \mathbf{b}\|_{(\mathbf{X}^c)'} := \sup_{\mathbf{e} \in \mathbf{X}^c} \frac{|(\mathbf{b}, \nabla \times \mathbf{e})_{L^2(D)}|}{\ell_D \|\nabla \times \mathbf{e}\|_{L^2(D)}}, \quad \|\nabla \times \mathbf{e}\|_{(\mathbf{X}_0^c)'} := \sup_{\mathbf{b} \in \mathbf{X}_0^c} \frac{|(\mathbf{e}, \nabla_0 \times \mathbf{b})_{L^2(D)}|}{\ell_D \|\nabla_0 \times \mathbf{b}\|_{L^2(D)}}. \quad (14b)$$

**Corollary 2.9** (Weak Poincaré–Steklov inequalities). *The following holds:*

$$\|p\|_{L^2(D)} \leq \ell_D \|\nabla p\|_{(\mathbf{X}_0^d)'}, \quad \forall p \in H(\mathbf{grad} = \mathbf{0}; D)^\perp = \mathbb{P}_0^\perp, \quad (15a)$$

$$\|\mathbf{v}\|_{L^2(D)} \leq \ell_D \|\nabla_0 \cdot \mathbf{v}\|_{(X^g)'}, \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{div} = \mathbf{0}; D)^\perp, \quad (15b)$$

$$\|\mathbf{b}\|_{L^2(D)} \leq \ell_D \|\nabla_0 \times \mathbf{b}\|_{(\mathbf{X}^c)'}, \quad \forall \mathbf{b} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (15c)$$

$$\|\mathbf{e}\|_{L^2(D)} \leq \ell_D \|\nabla \times \mathbf{e}\|_{(\mathbf{X}_0^c)'}, \quad \forall \mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp. \quad (15d)$$

*Proof.* Owing to (13a) in Lemma 2.7, for all  $q \in \mathbb{P}_0^\perp$ , there exists a unique  $\mathbf{v}(q) \in \mathbf{X}_0^d$  so that  $\nabla_0 \cdot \mathbf{v}(q) = q$ . Let  $p \in \mathbb{P}_0^\perp$ ,  $p \neq 0$ . We have

$$\begin{aligned} \|p\|_{L^2(D)} &= \sup_{q \in \mathbb{P}_0^\perp} \frac{|(p, q)_{L^2(D)}|}{\|q\|_{L^2(D)}} = \sup_{q \in \mathbb{P}_0^\perp} \frac{|(p, \nabla_0 \cdot \mathbf{v}(q))_{L^2(D)}|}{\|\nabla_0 \cdot \mathbf{v}(q)\|_{L^2(D)}} \\ &= \sup_{\mathbf{v} \in \mathbf{X}_0^d} \frac{|(p, \nabla_0 \cdot \mathbf{v})_{L^2(D)}|}{\|\nabla_0 \cdot \mathbf{v}\|_{L^2(D)}} = \ell_D \|\nabla p\|_{(\mathbf{X}_0^d)'}. \end{aligned}$$

The proof of the other inequalities is similar.  $\square$

**Remark 2.10** (Other BCs). *For the grad-div problem, one can also consider the spaces  $X_0^g := H_0(\mathbf{grad}; D) \cap H_0(\mathbf{grad} = \mathbf{0}; D)^\perp = H_0^1(D) \cap \{0\}^\perp = H_0^1(D)$  and  $\mathbf{X}^d := \mathbf{H}(\mathbf{div}; D) \cap \mathbf{H}(\mathbf{div} = \mathbf{0}; D)^\perp$ . Then, the operators  $\nabla_0 : X_0^g \rightarrow \mathbf{H}(\mathbf{div} = \mathbf{0}; D)^\perp$  and  $\nabla \cdot : \mathbf{X}^d \rightarrow H_0(\mathbf{grad} = \mathbf{0}; D)^\perp = L^2(D)$  are isomorphisms owing, in particular, to the Helmholtz decompositions (68d) and (68a). Moreover, equipping  $X_0^g$  and  $\mathbf{X}^d$  with the norms  $\|p\|_{X_0^g} := \ell_D \|\nabla_0 p\|_{L^2(D)}$  and  $\|\mathbf{v}\|_{\mathbf{X}^d} := \ell_D \|\nabla \cdot \mathbf{v}\|_{L^2(D)}$ , respectively, the following weak Poincaré–Steklov inequalities hold true for all  $(p, \mathbf{v}) \in X_0^g \times \mathbf{X}^g$ ,*

$$\|p\|_{L^2(D)} \leq \ell_D \|\nabla_0 p\|_{(\mathbf{X}^d)'}, \quad \|\mathbf{v}\|_{L^2(D)} \leq \ell_D \|\nabla \cdot \mathbf{v}\|_{(X_0^g)'},$$

with dual norms defined as in (14). (Notice that  $\|\cdot\|_{(X_0^g)'} = \|\cdot\|_{H^{-1}(D)}$ .)

## 2.4 Eigenvalue problems

We are now ready to give a precise definition of the operators involved in the eigenvalue problems (6) and (9). The main difficulty we address is to get rid of the eigenspace associated with the 0



eigenvalue. For this purpose, we consider the following  $L^2$ - or  $\mathbf{L}^2$ -orthogonal projections:

$$\Pi^{\mathfrak{s}} : L^2(D) \rightarrow H(\mathbf{grad} = \mathbf{0}; D) = \mathbb{P}_0 = \ker(\nabla) = \text{im}(\nabla_0 \cdot)^\perp, \quad (16a)$$

$$\mathbf{\Pi}_0^{\mathfrak{d}} : \mathbf{L}^2(D) \rightarrow \mathbf{H}_0(\text{div} = 0; D) = \ker(\nabla_0 \cdot) = \text{im}(\nabla)^\perp, \quad (16b)$$

$$\mathbf{\Pi}_0^{\mathfrak{c}} : \mathbf{L}^2(D) \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) = \ker(\nabla_0 \times) = \text{im}(\nabla \times)^\perp, \quad (16c)$$

$$\mathbf{\Pi}^{\mathfrak{c}} : \mathbf{L}^2(D) \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D) = \ker(\nabla \times) = \text{im}(\nabla_0 \times)^\perp. \quad (16d)$$

Recall that  $\ell_D$  is a length scale associated with  $D$  and that  $\mathfrak{c}$  is a scaling factor typically representing a wave speed; in what follows, we use the time scale  $\tau_D := \mathfrak{c}^{-1} \ell_D$ .

### 2.4.1 Grad-div problem

Let us first address the grad-div operator.

**theorem 2.11** (Well-posedness). *For all  $(\mathbf{f}, g) \in \mathbf{L}^2(D) \times L^2(D) =: L^{\mathfrak{d}}$ , there exists a unique pair  $(\mathbf{v}, p) \in \mathbf{H}_0(\text{div}; D) \times H^1(D)$  such that*

$$\tau_D^{-1} \mathbf{\Pi}_0^{\mathfrak{d}}(\mathbf{v}) + \nabla p = (\mathbf{I} - \mathbf{\Pi}_0^{\mathfrak{d}})(\mathbf{f}), \quad (17a)$$

$$\tau_D^{-1} \Pi^{\mathfrak{s}}(p) + \mathfrak{c}^2 \nabla_0 \cdot \mathbf{v} = (I - \Pi^{\mathfrak{s}})(g). \quad (17b)$$

*This pair is in  $\mathbf{X}_0^{\mathfrak{d}} \times X^{\mathfrak{s}}$  and continuously depends on  $(\mathbf{f}, g)$ , i.e.,  $\|\mathbf{v}\|_{\mathbf{X}_0^{\mathfrak{d}}} = \ell_D \mathfrak{c}^{-2} \|(I - \Pi^{\mathfrak{s}})(g)\|_{L^2(D)}$ ,  $\|p\|_{X^{\mathfrak{s}}} = \ell_D \|(\mathbf{I} - \mathbf{\Pi}_0^{\mathfrak{d}})(\mathbf{f})\|_{L^2(D)}$ .*

*Proof.* Lemma 2.7 implies that there exists a unique pair  $(\mathbf{v}, p) \in \mathbf{X}_0^{\mathfrak{d}} \times X^{\mathfrak{s}}$  verifying  $\nabla p = (\mathbf{I} - \mathbf{\Pi}_0^{\mathfrak{d}})(\mathbf{f})$  and  $\nabla \cdot \mathbf{v} = (I - \Pi^{\mathfrak{s}})(g)$ . Since  $\Pi^{\mathfrak{s}}(q) = 0$  for all  $q \in X^{\mathfrak{s}}$  and  $\mathbf{\Pi}_0^{\mathfrak{d}}(\mathbf{w}) = \mathbf{0}$  for all  $\mathbf{w} \in \mathbf{X}_0^{\mathfrak{d}}$ , the pair  $(\mathbf{v}, p)$  solves (17). This proves the existence. Let now  $(\mathbf{v}, p) \in \mathbf{H}_0(\text{div}; D) \times H^1(D)$  be a solution to the homogeneous problem (17). Taking the inner product of (17a) with  $\mathbf{\Pi}_0^{\mathfrak{d}}(\mathbf{v})$ , we conclude that  $\mathbf{\Pi}_0^{\mathfrak{d}}(\mathbf{v}) = \mathbf{0}$ . This, in turn, implies that  $\mathbf{v} \in \mathbf{X}_0^{\mathfrak{d}}$ . Similarly, taking the inner product of (17b) with  $\Pi^{\mathfrak{s}}(p)$ , we conclude that  $p \in X^{\mathfrak{s}}$ . Finally,  $\nabla_0 \cdot \mathbf{v} = 0$  and  $\mathbf{v} \in \mathbf{X}_0^{\mathfrak{d}}$  imply that  $\mathbf{v} = \mathbf{0}$ , and  $\nabla p = \mathbf{0}$  and  $p \in X^{\mathfrak{s}}$  imply that  $p = 0$ . This proves the uniqueness. The boundedness assertion is a consequence of the definition of the norms  $\|\cdot\|_{\mathbf{X}_0^{\mathfrak{d}}}$  and  $\|\cdot\|_{X^{\mathfrak{s}}}$ .  $\square$

**Definition 2.12.** *We define the operator  $T : L^{\mathfrak{d}} \rightarrow L^{\mathfrak{d}}$  so that for all  $(\mathbf{f}, g) \in L^{\mathfrak{d}}$ , the pair  $(\mathbf{v}, p) := T(\mathbf{f}, g)$  solves (17).*

**Lemma 2.13** (Compactness). (i) *There exists  $s \in (\frac{1}{2}, 1]$  so that, for all  $(\mathbf{v}, p) \in \mathbf{X}_0^{\mathfrak{d}} \times X^{\mathfrak{s}}$ ,*

$$\|\mathbf{v}\|_{\mathbf{H}^s(D)} \leq C_D \ell_D \|\nabla_0 \cdot \mathbf{v}\|_{L^2(D)}, \quad \|p\|_{H^1(D)} \leq C_D \ell_D \|\nabla p\|_{L^2(D)}. \quad (18)$$

(ii) *The operator  $T : L^{\mathfrak{d}} \rightarrow L^{\mathfrak{d}}$  is compact.*

*Proof.* The decomposition (68c) implies that  $\mathbf{v} \in \nabla(H^1(D))$  for all  $\mathbf{v} \in \mathbf{X}_0^{\mathfrak{d}}$ ; hence,  $\nabla \times \mathbf{v} = \mathbf{0}$ . Then, the inequality (5) implies that, for all  $\mathbf{v} \in \mathbf{X}_0^{\mathfrak{d}}$ ,

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{H}^s(D)} &\leq C_D (\|\mathbf{v}\|_{\mathbf{H}(\text{div}; D)} + \ell_D \|\nabla \times \mathbf{v}\|_{L^2(D)}) \\ &= C_D \|\mathbf{v}\|_{\mathbf{H}(\text{div}; D)} \leq C'_D \|\mathbf{v}\|_{\mathbf{X}_0^{\mathfrak{d}}} = C'_D \ell_D \|\nabla_0 \cdot \mathbf{v}\|_{L^2(D)}. \end{aligned}$$

Moreover, we have already seen that  $\|p\|_{H^1(D)} \leq C_D \ell_D \|\nabla p\|_{L^2(D)} = C_D \|p\|_{X^{\mathfrak{s}}}$  for all  $p \in X^{\mathfrak{s}}$ . This proves the first assertion. The second assertion follows from the Rellich–Kondrachov compactness theorem in fractional-order Sobolev spaces implying that the embedding  $\mathbf{X}_0^{\mathfrak{d}} \times X^{\mathfrak{s}} \rightarrow L^{\mathfrak{d}}$  is compact.  $\square$

Let  $\sigma(T)$  be the spectrum of  $T$  and  $\sigma_p(T)$  be the point spectrum of  $T$ . Since  $T$  is compact, we have  $\sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$ , i.e., the spectrum of  $T$  reduces to its point spectrum away from 0. Moreover, 0 is an accumulation point. Let  $\sigma_p(-\Delta_N^{-1})$  be the point spectrum of the operator  $-\Delta_N^{-1} : L^2(D) \cap \mathbb{P}_0^\perp \rightarrow L^2(D) \cap \mathbb{P}_0^\perp$ , where  $-\Delta(-\Delta_N^{-1}(f)) = f$  and  $\partial_n(-\Delta_N^{-1}(f))|_{\partial D} = 0$ . Recall that  $\sigma_p(-\Delta_N^{-1}) \subset \mathbb{R}_{>0}$ . Notice also that  $\alpha \in \sigma_p(-\Delta_N^{-1})$  if and only if  $i\sqrt{\alpha}$  and  $-i\sqrt{\alpha}$  are both members of  $\sigma_p(T)$ . Hence, the point spectrum of  $T$  is purely imaginary. Let us now relate the operator  $T$  to the spectral problem (6).

**Lemma 2.14.** (i) Let  $\mu \neq 0$ ,  $(\mathbf{v}, p) \in \mathbf{X}_0^d \times X^g$  be an eigenpair of  $T$ . Then  $\frac{1}{\mu}$ ,  $(\mathbf{v}, p)$  is an eigenpair of (6). (ii) Let  $\lambda \neq 0$ ,  $(\mathbf{v}, p) \in \mathbf{H}_0(\text{div}; D) \times H^1(D)$  be an eigenpair of (6). Then  $\frac{1}{\lambda}$ ,  $(\mathbf{v}, p)$  is an eigenpair of  $T$ .

*Proof.* (i) Let  $\mu \neq 0$ ,  $(\mathbf{v}, p) \in \mathbf{X}_0^d \times X^g$  be an eigenpair of  $T$ . Then  $\mathbf{\Pi}_0^d(\mu\mathbf{v}) + \nabla(\mu p) = (\mathbf{I} - \mathbf{\Pi}_0^d)(\mathbf{v})$  and  $\mathbf{\Pi}^g(\mu p) + \nabla_0 \cdot (\mu\mathbf{v}) = (\mathbf{I} - \mathbf{\Pi}^g)(p)$ . Since  $\mathbf{\Pi}_0^d(\mathbf{v}) = \mathbf{0}$  and  $\mathbf{\Pi}^g(p) = 0$ , we conclude that  $\frac{1}{\mu}$ ,  $(\mathbf{v}, p)$  is an eigenpair of (6).

(ii) Let  $\lambda \neq 0$ ,  $(\mathbf{v}, p) \in \mathbf{H}_0(\text{div}; D) \times H^1(D)$  be an eigenpair of (6). Then  $\nabla(\frac{1}{\lambda}p) = \mathbf{v}$  and  $\nabla_0 \cdot (\frac{1}{\lambda}\mathbf{c}^2\mathbf{v}) = p$ . This implies that  $\mathbf{v} \in \mathbf{H}_0(\text{div} = 0; D)^\perp$  and  $p \in \mathbb{P}_0^\perp$ . Hence,  $\mathbf{\Pi}_0^d(\mathbf{v}) = \mathbf{0}$  and  $\mathbf{\Pi}^g(p) = 0$ , i.e.,  $\frac{1}{\lambda}$ ,  $(\mathbf{v}, p) \in \mathbf{X}_0^d \times X^g$  is an eigenpair of  $T$ .  $\square$

Lemma 2.14 shows that the eigenstructure of  $T$  is the same as that of (6) for the nonzero eigenvalues. Hence, it suffices to study the spectrum of  $T$  to have full knowledge of the eigenstructure of (6). In the paper, we prove that the dG approximation of  $T$  is spectrally correct, i.e., pollution free.

## 2.4.2 Curl-curl problem

We proceed as in §2.4.1 for the analysis of the eigenvalue problem associated with the curl-curl operator. The reader is referred to Monk [23, Chap. 1] for an introduction to Maxwell's equations.

**theorem 2.15** (Well-posedness). *For all  $(\mathbf{f}, \mathbf{g}) \in \mathbf{L}^2(D) \times \mathbf{L}^2(D) =: L^c$ , there exists a unique pair  $(\mathbf{B}, \mathbf{E}) \in \mathbf{X}_0^c \times \mathbf{X}^c$  such that*

$$\tau_D^{-1} \mathbf{\Pi}_0^c(\mathbf{B}) - \nabla \times \mathbf{E} = (\mathbf{I} - \mathbf{\Pi}_0^c)(\mathbf{f}), \quad (19a)$$

$$\tau_D^{-1} \mathbf{\Pi}^c(\mathbf{E}) + \mathbf{c}^2 \nabla_0 \times \mathbf{B} = (\mathbf{I} - \mathbf{\Pi}^c)(\mathbf{g}). \quad (19b)$$

*This pair is in  $\mathbf{X}_0^c \times \mathbf{X}^c$  and continuously depends on  $(\mathbf{f}, \mathbf{g})$ , i.e., we have  $\|\mathbf{B}\|_{\mathbf{X}_0^c} = \ell_D \mathbf{c}^{-2} \|(\mathbf{I} - \mathbf{\Pi}^c)(\mathbf{g})\|_{\mathbf{L}^2(D)}$ ,  $\|\mathbf{E}\|_{\mathbf{X}^c} = \ell_D \|(\mathbf{I} - \mathbf{\Pi}_0^c)(\mathbf{f})\|_{\mathbf{L}^2(D)}$ .*

*Proof.* Similar to that of Theorem 2.11.  $\square$

**Definition 2.16.** *We define the operator  $T : L^c \rightarrow L^c$  so that for all  $(\mathbf{f}, \mathbf{g}) \in L^c$ , the pair  $(\mathbf{B}, \mathbf{E}) := T(\mathbf{f}, \mathbf{g})$  solves (19).*

**Lemma 2.17** (Compactness). (i) *There exists  $s \in (\frac{1}{2}, 1]$  so that, for all  $(\mathbf{B}, \mathbf{E}) \in \mathbf{X}_0^c \times \mathbf{X}^c$ ,*

$$\|\mathbf{B}\|_{\mathbf{H}^s(D)} \leq C_D \ell_D \|\nabla_0 \times \mathbf{B}\|_{\mathbf{L}^2(D)}, \quad \|\mathbf{E}\|_{\mathbf{H}^s(D)} \leq C_D \ell_D \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)}. \quad (20)$$

(ii) *The operator  $T : L^c \rightarrow L^c$  is compact.*

*Proof.* By definition, we have  $\mathbf{X}^c \subset \mathbf{H}(\mathbf{curl}; D)$ . Moreover, the identity (67f) implies that  $\mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp = \mathbf{H}_0^\Sigma(\text{div} = 0; D)$ . Hence,  $\mathbf{X}^c \subset \mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}(\text{div} = 0; D)$ . Then, the inequality (5) implies that, for all  $\mathbf{E} \in \mathbf{X}^c$ ,

$$\begin{aligned} \|\mathbf{E}\|_{\mathbf{H}^s(D)} &\leq C_D (\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl}; D)} + \ell_D \|\nabla \cdot \mathbf{E}\|_{L^2(D)}) \\ &= C_D \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl}; D)} \leq C'_D \|\mathbf{E}\|_{\mathbf{X}^c} = C'_D \ell_D \|\nabla \times \mathbf{E}\|_{L^2(D)}. \end{aligned}$$

A similar argument is used to prove the first inequality in (20). This proves the first assertion. The second assertion follows from the Rellich–Kondrachov compactness theorem in fractional-order Sobolev spaces implying that the embedding  $\mathbf{X}_0^c \times \mathbf{X}^c \rightarrow L^c$  is compact.  $\square$

Since  $T$  is compact, we have  $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$  and 0 is an accumulation point of  $\sigma(T)$ . Let us now relate the operator  $T$  to the spectral problem (9).

**Lemma 2.18.** (i) Let  $\mu \neq 0$ ,  $(\mathbf{B}, \mathbf{E}) \in \mathbf{X}_0^c \times \mathbf{X}^c$  be an eigenpair of  $T$ . Then  $\frac{1}{\mu}$ ,  $(\mathbf{B}, \mathbf{E})$  is an eigenpair of (9). (ii) Let  $\lambda \neq 0$ ,  $(\mathbf{B}, \mathbf{E}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$  be an eigenpair of (9). Then  $\frac{1}{\lambda}$ ,  $(\mathbf{B}, \mathbf{E})$  is an eigenpair of  $T$ .

*Proof.* Similar to that of Lemma 2.14.  $\square$

Lemma 2.18 implies that the eigenstructure of  $T$  is the same as that of the spectral problem (11). Hence, it is sufficient to study the spectrum of  $T$  to have full knowledge of the eigenstructure of (11). In the paper, we prove that the dG approximation of  $T$  is spectrally correct, i.e., pollution free.

### 3 Discrete setting

In this section, we introduce the discrete setting used in the paper. The main result of this section is Lemma 3.2 which establishes discrete counterparts of the weak Poincaré–Steklov inequalities from Corollary 2.9.

#### 3.1 Broken polynomial spaces, jumps and averages

Let  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  be a shape-regular family of affine simplicial meshes such that each mesh covers  $D$  exactly. A generic mesh cell is denoted  $K$ , its diameter  $h_K$  and its outward unit normal  $\mathbf{n}_K$ . We define  $\tilde{h}$  as the piecewise constant function on  $\mathcal{T}_h$  such that  $\tilde{h}|_K = h_K$  for all  $K \in \mathcal{T}_h$ ; we set  $h := \|\tilde{h}\|_{L^\infty(D)}$ . The set of mesh faces,  $\mathcal{F}_h$ , is split into the subset of mesh interfaces (shared the two distinct mesh cells which we denote  $K_l, K_r$ ), say  $\mathcal{F}_h^\circ$ , and the subset of mesh boundary faces (shared by one mesh cell,  $K_l$ , and the boundary,  $\partial D$ ), say  $\mathcal{F}_h^\partial$ . For every mesh face  $F \in \mathcal{F}_h$ ,  $h_F$  denotes the diameter of  $F$ . Every mesh interface  $F \in \mathcal{F}_h^\circ$  is oriented by the unit normal,  $\mathbf{n}_F$ , pointing from  $K_l$  to  $K_r$ . Every boundary face  $F \in \mathcal{F}_h^\partial$  is oriented by the unit normal  $\mathbf{n}_F := \mathbf{n}_D$ . For all  $K \in \mathcal{T}_h$ ,  $\mathcal{F}_K$  denotes the collection of the mesh faces composing the boundary of  $K$ , and we set  $\mathcal{F}_K^\circ := \mathcal{F}_K \cap \mathcal{F}_h^\circ$ .

In what follows, for positive real numbers  $A, B$ , we often abbreviate as  $A \lesssim B$  the inequality  $A \leq CB$  where  $C$  is a generic constant whose value can change at each occurrence as long as it is independent of  $h \in \mathcal{H}$  and of the functions or fields involved in the inequality.

Let  $k \geq 0$  be the polynomial degree. Let  $\mathbb{P}_{k,d}$  be the space composed of  $d$ -variate polynomials of total degree at most  $k$  and set  $\mathbf{P}_{k,d} := [\mathbb{P}_{k,d}]^d$ . Consider the scalar- and vector-valued broken

polynomial spaces

$$P_k^b(\mathcal{T}_h) := \{v_h \in L^\infty(D) \mid v_h|_K \in \mathbb{P}_{k,d}, \forall K \in \mathcal{T}_h\}, \quad (21a)$$

$$\mathbf{P}_k^b(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^\infty(D) \mid \mathbf{v}_h|_K \in \mathbf{P}_{k,d}, \forall K \in \mathcal{T}_h\}. \quad (21b)$$

We introduce the  $L^2$ - and  $\mathbf{L}^2$ -orthogonal projections

$$\Pi_h^b : L^2(D) \rightarrow P_k^b(\mathcal{T}_h), \quad \mathbf{\Pi}_h^b : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h). \quad (22)$$

For every  $p_h \in P_k^b(\mathcal{T}_h)$ ,  $\nabla_h p_h$  denotes the broken gradient of  $p_h$  (evaluated piecewise over each mesh cell). For every  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,  $\nabla_h \times \mathbf{v}_h$  and  $\nabla_h \cdot \mathbf{v}_h$  denote the broken curl and divergence of  $\mathbf{v}_h$ , respectively.

For all  $K \in \mathcal{T}_h$ , all  $F \in \mathcal{F}_K$ , all  $p_h \in P_k^b(\mathcal{T}_h)$ , and all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ , we define the local trace operators such that  $\gamma_{K,F}^g(p_h)(\mathbf{x}) := p_h|_K(\mathbf{x})$ ,  $\gamma_{K,F}^g(\mathbf{v}_h)(\mathbf{x}) := \mathbf{v}_h|_K(\mathbf{x})$ ,  $\gamma_{K,F}^c(\mathbf{v}_h) := \mathbf{v}_h|_K(\mathbf{x}) \times \mathbf{n}_F$ , and  $\gamma_{K,F}^d(\mathbf{v}_h) := \mathbf{v}_h|_K(\mathbf{x}) \cdot \mathbf{n}_F$  for a.e.  $\mathbf{x} \in F$ . Then, for all  $F \in \mathcal{F}_h^\circ$  and  $\mathbf{x} \in \{g, c, d\}$ , we define the jump and average operators such that

$$\llbracket p_h \rrbracket_F^g := \gamma_{K_l,F}^g(p_h) - \gamma_{K_r,F}^g(p_h), \quad \{\!\!\{ p_h \}\!\!\}_F^g := \frac{1}{2}(\gamma_{K_l,F}^g(p_h) + \gamma_{K_r,F}^g(p_h)), \quad (23a)$$

$$\llbracket \mathbf{v}_h \rrbracket_F^{\mathbf{x}} := \gamma_{K_l,F}^{\mathbf{x}}(\mathbf{v}_h) - \gamma_{K_r,F}^{\mathbf{x}}(\mathbf{v}_h), \quad \{\!\!\{ \mathbf{v}_h \}\!\!\}_F^{\mathbf{x}} := \frac{1}{2}(\gamma_{K_l,F}^{\mathbf{x}}(\mathbf{v}_h) + \gamma_{K_r,F}^{\mathbf{x}}(\mathbf{v}_h)). \quad (23b)$$

To allow for more compact expressions, we also set  $\llbracket p_h \rrbracket_F^g := \{\!\!\{ p_h \}\!\!\}_F^g := \gamma_{K_l,F}^g(p_h)$ ,  $\llbracket \mathbf{v}_h \rrbracket_F^{\mathbf{x}} := \{\!\!\{ \mathbf{v}_h \}\!\!\}_F^{\mathbf{x}} := \gamma_{K_l,F}^{\mathbf{x}}(\mathbf{v}_h)$  for all  $F \in \mathcal{F}_h^\partial$ . Finally, we define the jump bilinear forms such that for all  $p_h, q_h \in P_k^b(\mathcal{T}_h)$  and all  $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,

$$\begin{aligned} s_h^g(p_h, q_h) &:= \sum_{F \in \mathcal{F}_h} (\llbracket p_h \rrbracket_F^g, \llbracket q_h \rrbracket_F^g)_{L^2(F)}, & s_h^{g,\circ}(p_h, q_h) &:= \sum_{F \in \mathcal{F}_h^\circ} (\llbracket p_h \rrbracket_F^g, \llbracket q_h \rrbracket_F^g)_{L^2(F)}, \\ s_h^c(\mathbf{v}_h, \mathbf{w}_h) &:= \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^c, \llbracket \mathbf{w}_h \rrbracket_F^c)_{L^2(F)}, & s_h^{c,\circ}(\mathbf{v}_h, \mathbf{w}_h) &:= \sum_{F \in \mathcal{F}_h^\circ} (\llbracket \mathbf{v}_h \rrbracket_F^c, \llbracket \mathbf{w}_h \rrbracket_F^c)_{L^2(F)}, \\ s_h^d(\mathbf{v}_h, \mathbf{w}_h) &:= \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^d, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)}, & s_h^{d,\circ}(\mathbf{v}_h, \mathbf{w}_h) &:= \sum_{F \in \mathcal{F}_h^\circ} (\llbracket \mathbf{v}_h \rrbracket_F^d, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)}, \end{aligned}$$

and the seminorms  $|q_h|_h^g := s_h^g(q_h, q_h)^{\frac{1}{2}}$ ,  $|q_h|_h^{g,\circ} := s_h^{g,\circ}(q_h, q_h)^{\frac{1}{2}}$ ,  $|\mathbf{v}_h|_h^c := s_h^c(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}}$ ,  $|\mathbf{v}_h|_h^{c,\circ} := s_h^{c,\circ}(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}}$ ,  $|\mathbf{v}_h|_h^d := s_h^d(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}}$ , and  $|\mathbf{v}_h|_h^{d,\circ} := s_h^{d,\circ}(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}}$ .

### 3.2 Discrete Poincaré–Steklov inequalities

In this section, we prove discrete counterparts to the Poincaré–Steklov inequalities established in Corollary 2.9. We first define the (broken) polynomial subspaces

$$P_{k0}^d(\text{div} = 0; \mathcal{T}_h) := P_k^b(\mathcal{T}_h) \cap \mathbf{H}_0(\text{div} = 0; D), \quad (24a)$$

$$P_k^d(\text{div} = 0; \mathcal{T}_h) := P_k^b(\mathcal{T}_h) \cap \mathbf{H}(\text{div} = 0; D), \quad (24b)$$

$$P_{k0}^c(\text{curl} = \mathbf{0}; \mathcal{T}_h) := P_k^b(\mathcal{T}_h) \cap \mathbf{H}_0(\text{curl} = \mathbf{0}; D), \quad (24c)$$

$$P_k^c(\text{curl} = \mathbf{0}; \mathcal{T}_h) := P_k^b(\mathcal{T}_h) \cap \mathbf{H}(\text{curl} = \mathbf{0}; D), \quad (24d)$$

and we consider the following orthogonal complements:

$$X_h^g := P_k^b(\mathcal{T}_h) \cap \mathbb{P}_0^\perp, \quad (25a)$$

$$\mathbf{X}_{h0}^d := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{P}_{k0}^d(\operatorname{div} = 0; \mathcal{T}_h)^\perp, \quad (25b)$$

$$\mathbf{X}_h^d := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{P}_k^d(\operatorname{div} = 0; \mathcal{T}_h)^\perp, \quad (25c)$$

$$\mathbf{X}_{h0}^c := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{P}_{k0}^c(\operatorname{curl} = \mathbf{0}; \mathcal{T}_h)^\perp, \quad (25d)$$

$$\mathbf{X}_h^c := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{P}_k^c(\operatorname{curl} = \mathbf{0}; \mathcal{T}_h)^\perp. \quad (25e)$$

The broken polynomial subspaces  $X_h^g$ ,  $\mathbf{X}_{h0}^d$ ,  $\mathbf{X}_h^d$ ,  $\mathbf{X}_{h0}^c$ , and  $\mathbf{X}_h^c$  are meant to realize nonconforming approximations of the spaces  $X^g$ ,  $\mathbf{X}_0^d$ ,  $\mathbf{X}^d$ ,  $\mathbf{X}_0^c$ , and  $\mathbf{X}^c$ , respectively. We could have set  $P_k^g(\operatorname{grad} = \mathbf{0}; \mathcal{T}_h) := P_k^b(\mathcal{T}_h) \cap H(\operatorname{grad} = \mathbf{0}; D)$  and  $X_h^g = P_k^b(\mathcal{T}_h) \cap P_k^g(\operatorname{grad} = \mathbf{0}; \mathcal{T}_h)^\perp$ , but since  $P_k^g(\operatorname{grad} = \mathbf{0}; \mathcal{T}_h) = \mathbb{P}_0$ , this would have led to the same definition of  $X^g$  as above.

**Lemma 3.1** (Discrete Poincaré–Steklov inequality for scalars). *The following holds:*

$$\|p_h\|_{L^2(D)} \leq \ell_D \|\nabla p_h\|_{(\mathbf{X}_0^d)^\prime}, \quad \forall p_h \in X_h^g. \quad (26)$$

*Proof.* Since  $X_h^g \subset \mathbb{P}_0^\perp$ , (26) is a consequence of (15a).  $\square$

The situation is not as simple for discrete vector fields. Indeed,  $\mathbf{X}_{h0}^d$  is not a subspace of  $\mathbf{H}_0(\operatorname{div} = 0; D)^\perp$ , and  $\mathbf{X}_h^d$  is not a subspace of  $\mathbf{H}(\operatorname{div} = 0; D)^\perp$ . Similarly,  $\mathbf{X}_{h0}^c$  is not a subspace of  $\mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D)^\perp$ , and  $\mathbf{X}_h^c$  is not a subspace of  $\mathbf{H}(\operatorname{curl} = \mathbf{0}; D)^\perp$ . To establish discrete Poincaré–Steklov inequalities for fields, we use that the broken polynomial spaces contain polynomial spaces from the entire discrete de Rham sequence based on Nédélec and Raviart–Thomas polynomials.

**Lemma 3.2** (Discrete Poincaré–Steklov inequalities for fields). *The following holds:*

$$\|\mathbf{v}_h\|_{L^2(D)} \lesssim \ell_D \|\nabla \cdot \mathbf{v}_h\|_{(X^g)^\prime} + h^{\frac{1}{2}} |\mathbf{v}_h|_h^d, \quad \forall \mathbf{v}_h \in \mathbf{X}_{h0}^d, \quad (27a)$$

$$\|\mathbf{w}_h\|_{L^2(D)} \lesssim \ell_D \|\nabla \cdot \mathbf{w}_h\|_{(X_0^g)^\prime} + h^{\frac{1}{2}} |\mathbf{w}_h|_h^{d,0}, \quad \forall \mathbf{w}_h \in \mathbf{X}_h^d, \quad (27b)$$

$$\|\mathbf{b}_h\|_{L^2(D)} \lesssim \ell_D \|\nabla \times \mathbf{b}_h\|_{(X^c)^\prime} + h^{\frac{1}{2}} |\mathbf{b}_h|_h^c, \quad \forall \mathbf{b}_h \in \mathbf{X}_{h0}^c, \quad (27c)$$

$$\|\mathbf{e}_h\|_{L^2(D)} \lesssim \ell_D \|\nabla \times \mathbf{e}_h\|_{(X_0^c)^\prime} + h^{\frac{1}{2}} |\mathbf{e}_h|_h^{c,0}, \quad \forall \mathbf{e}_h \in \mathbf{X}_h^c. \quad (27d)$$

*Proof.* (1) Proof of (27a). Here, the  $\mathbf{H}_0(\operatorname{div}; D)$ -conforming space composed of piecewise Raviart–Thomas polynomials of order  $k \geq 0$ ,  $\mathbf{P}_{k0}^d(\mathcal{T}_h)$ , plays a central role. Notice that  $\mathbf{P}_{k0}^d(\mathcal{T}_h)$  is not a subspace of  $\mathbf{P}_k^b(\mathcal{T}_h)$ , but this is the case whenever a divergence-free constraint is enforced. Indeed, we have

$$\mathbf{P}_{k0}^d(\mathcal{T}_h) \cap \mathbf{H}_0(\operatorname{div} = 0; D) = \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}_0(\operatorname{div} = 0; D) =: \mathbf{P}_{k0}^d(\operatorname{div} = 0; \mathcal{T}_h). \quad (28)$$

Let  $\mathbf{v}_h \in \mathbf{X}_{h0}^d$  and define

$$\mathbf{v}_h^d := \mathcal{I}_{h0}^{d,\text{av}}(\mathbf{v}_h), \quad \boldsymbol{\xi} := \mathbf{v}_h^d - \mathbf{\Pi}_0^d(\mathbf{v}_h^d).$$

where  $\mathcal{I}_{h0}^{d,\text{av}} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_{k0}^d(\mathcal{T}_h)$  is the  $\mathbf{H}_0(\operatorname{div}; D)$ -conforming averaging operator with zero normal boundary prescription constructed in [15, §6]. Since  $\nabla \cdot (\mathbf{\Pi}_0^d(\mathbf{v}_h^d)) = 0$  (because  $\mathbf{\Pi}_0^d(\mathbf{v}_h^d) \in \mathbf{H}_0(\operatorname{div} = 0; D)$ ) and since  $\boldsymbol{\xi} \in \mathbf{H}_0(\operatorname{div} = 0; D)^\perp$  by construction, the weak Poincaré–Steklov inequality (15b) from Corollary 2.9 gives

$$\|\boldsymbol{\xi}\|_{L^2(D)} \leq \ell_D \|\nabla_0 \cdot \boldsymbol{\xi}\|_{(X^g)^\prime} = \ell_D \|\nabla_0 \cdot \mathcal{I}_{h0}^{d,\text{av}}(\mathbf{v}_h)\|_{(X^g)^\prime}. \quad (29)$$

Let  $\mathcal{J}_{h_0}^d : \mathbf{L}^2(D) \rightarrow \mathbf{P}_{k_0}^d(\mathcal{T}_h)$  and  $\mathcal{J}_h^b : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h)$  be the commuting approximation operators devised in [16, §23.3] (see also Arnold et al. [3], Christiansen [10], Christiansen and Winther [11], Schöberl [26]). Since  $\mathbf{v}_h^d \in \mathbf{P}_{k_0}^d(\mathcal{T}_h)$ , we have

$$\mathbf{v}_h^d - \mathcal{J}_{h_0}^d(\boldsymbol{\xi}) = \mathcal{J}_{h_0}^d(\mathbf{v}_h^d - \boldsymbol{\xi}) = \mathcal{J}_{h_0}^d(\boldsymbol{\Pi}_0^d(\mathbf{v}_h^d)).$$

The commuting property of  $\mathcal{J}_{h_0}^d$  implies that

$$\nabla \cdot (\mathbf{v}_h^d - \mathcal{J}_{h_0}^d(\boldsymbol{\xi})) = \nabla \cdot (\mathcal{J}_{h_0}^d(\boldsymbol{\Pi}_0^d(\mathbf{v}_h^d))) = \mathcal{J}_h^b(\nabla \cdot (\boldsymbol{\Pi}_0^d(\mathbf{v}_h^d))) = \mathcal{J}_h^b(0) = 0.$$

Hence,  $\mathbf{v}_h^d - \mathcal{J}_{h_0}^d(\boldsymbol{\xi}) \in \mathbf{P}_{k_0}^d(\mathcal{T}_h) \cap \mathbf{H}_0(\operatorname{div} = 0; D) = \mathbf{P}_{k_0}^d(\operatorname{div} = 0; \mathcal{T}_h)$  owing to (28). Using that  $\mathbf{v}_h \in \mathbf{P}_{k_0}^d(\operatorname{div} = 0; \mathcal{T}_h)^\perp$  then gives

$$\begin{aligned} \|\mathbf{v}_h\|_{\mathbf{L}^2(D)}^2 &= (\mathbf{v}_h, \mathbf{v}_h - \mathbf{v}_h^d)_{\mathbf{L}^2(D)} + (\mathbf{v}_h, \mathbf{v}_h^d - \mathcal{J}_{h_0}^d(\boldsymbol{\xi}))_{\mathbf{L}^2(D)} + (\mathbf{v}_h, \mathcal{J}_{h_0}^d(\boldsymbol{\xi}))_{\mathbf{L}^2(D)} \\ &= (\mathbf{v}_h, \mathbf{v}_h - \mathbf{v}_h^d)_{\mathbf{L}^2(D)} + (\mathbf{v}_h, \mathcal{J}_{h_0}^d(\boldsymbol{\xi}))_{\mathbf{L}^2(D)}. \end{aligned}$$

Invoking the Cauchy–Schwarz inequality and the  $\mathbf{L}^2$ -stability of  $\mathcal{J}_{h_0}^d$  yields

$$\|\mathbf{v}_h\|_{\mathbf{L}^2(D)} \lesssim \|\mathbf{v}_h - \mathbf{v}_h^d\|_{\mathbf{L}^2(D)} + \|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)}.$$

Recalling the definition of  $\mathbf{v}_h^d$  and the bound (29) on  $\boldsymbol{\xi}$  then gives

$$\|\mathbf{v}_h\|_{\mathbf{L}^2(D)} \lesssim \|\mathbf{v}_h - \mathcal{I}_{h_0}^{\operatorname{d,av}}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)} + \ell_D \|\nabla \cdot \mathcal{I}_{h_0}^{\operatorname{d,av}}(\mathbf{v}_h)\|_{(X^\varepsilon)'}$$

Adding and subtracting  $\mathbf{v}_h$  to the second term on the right-hand side, using the triangle inequality and since  $\ell_D \|\nabla \cdot \boldsymbol{\phi}\|_{(X^\varepsilon)'} \leq \|\boldsymbol{\phi}\|_{\mathbf{L}^2(D)}$  for all  $\boldsymbol{\phi} \in \mathbf{L}^2(D)$ , we obtain

$$\|\mathbf{v}_h\|_{\mathbf{L}^2(D)} \lesssim \|\mathbf{v}_h - \mathcal{I}_{h_0}^{\operatorname{d,av}}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)} + \ell_D \|\nabla \cdot \mathbf{v}_h\|_{(X^\varepsilon)'}$$

Finally, we invoke the approximation properties of  $\mathcal{I}_{h_0}^{\operatorname{d,av}}$ . For all  $K \in \mathcal{T}_h$ , we have

$$\|\mathbf{v}_h - \mathcal{I}_{h_0}^{\operatorname{d,av}}(\mathbf{v}_h)\|_{\mathbf{L}^2(K)} \lesssim h_K^{\frac{1}{2}} \left\{ \sum_{F \in \mathcal{F}_K} \|\llbracket \mathbf{v}_h \rrbracket_F^d \|_{\mathbf{L}^2(F)}^2 \right\}^{\frac{1}{2}}.$$

The assertion follows from the shape-regularity of the mesh sequence.

(2) The proof of the other inequalities proceeds similarly. Here, one considers the  $\mathbf{H}(\operatorname{div}; D)$ -conforming space composed of piecewise Raviart–Thomas polynomials of order  $k \geq 0$ ,  $\mathbf{P}_k^d(\mathcal{T}_h)$ , or the  $\mathbf{H}(\operatorname{curl}; D)$ - and  $\mathbf{H}_0(\operatorname{curl}; D)$ -conforming spaces composed of piecewise Nédélec polynomials of order  $k \geq 0$ ,  $\mathbf{P}_k^c(\mathcal{T}_h)$  and  $\mathbf{P}_{k_0}^c(\mathcal{T}_h)$ . The corresponding averaging operators and commuting approximation operators are constructed in, e.g., [16, Chap. 22-23].  $\square$

### 3.3 Discrete projection operators

The (broken) polynomial subspaces introduced in (24) naturally lead to the following  $\mathbf{L}^2$ -orthogonal projections:

$$\boldsymbol{\Pi}_{h_0}^d : \mathbf{L}^2(D) \rightarrow \mathbf{P}_{k_0}^d(\operatorname{div} = 0; \mathcal{T}_h), \quad \boldsymbol{\Pi}_h^d : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^d(\operatorname{div} = 0; \mathcal{T}_h), \quad (30a)$$

$$\boldsymbol{\Pi}_{h_0}^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_{k_0}^c(\operatorname{curl} = \mathbf{0}; \mathcal{T}_h), \quad \boldsymbol{\Pi}_h^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^c(\operatorname{curl} = \mathbf{0}; \mathcal{T}_h). \quad (30b)$$

We could also have defined the  $\mathbf{L}^2$ -orthogonal projection  $\boldsymbol{\Pi}_h^g : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^g(\operatorname{grad} = \mathbf{0}; \mathcal{T}_h)$ , but since  $\mathbf{P}_k^g(\operatorname{grad} = \mathbf{0}; \mathcal{T}_h) = \mathbb{P}_0$ ,  $\boldsymbol{\Pi}_h^g$  coincides with  $\boldsymbol{\Pi}^g$ . Let us record here some instrumental properties of the above operators to be used in the analysis of the dG approximation. Recall that the continuous projection operators are defined in (16), and that the projection operators onto the broken polynomial spaces are defined in (22).

**Lemma 3.3** (Discrete projections). *The following identities hold:*

$$\mathbf{\Pi}_{h0}^d \circ \mathbf{\Pi}_0^d = \mathbf{\Pi}_{h0}^d, \quad \mathbf{\Pi}_h^d \circ \mathbf{\Pi}^d = \mathbf{\Pi}_h^d, \quad \mathbf{\Pi}_{h0}^c \circ \mathbf{\Pi}_0^c = \mathbf{\Pi}_{h0}^c, \quad \mathbf{\Pi}_h^c \circ \mathbf{\Pi}^c = \mathbf{\Pi}_h^c, \quad (31a)$$

$$\mathbf{\Pi}_{h0}^d \circ \mathbf{\Pi}_h^b = \mathbf{\Pi}_{h0}^d, \quad \mathbf{\Pi}_h^d \circ \mathbf{\Pi}_h^b = \mathbf{\Pi}_h^d, \quad \mathbf{\Pi}_{h0}^c \circ \mathbf{\Pi}_h^b = \mathbf{\Pi}_{h0}^c, \quad \mathbf{\Pi}_h^c \circ \mathbf{\Pi}_h^b = \mathbf{\Pi}_h^c. \quad (31b)$$

*Proof.* We only prove the assertions for  $\mathbf{\Pi}_{h0}^d$ , since the proof of the other statements is similar.

(1) The identity (31a) is a consequence of  $\mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h) \subset \mathbf{H}_0(\text{div} = 0; D)$ .

(2) The identity (31b) is a consequence of  $\mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h) \subset \mathbf{P}_k^b(\mathcal{T}_h)$ .  $\square$

**Remark 3.4** ( $\mathbf{\Pi}^g$ ). *The identities in (31) also hold for  $\mathbf{\Pi}^g$  and  $\mathbf{\Pi}_h^g$ . We additionally have  $\mathbf{\Pi}^g \circ \mathbf{\Pi}_h^b = \mathbf{\Pi}_h^b \circ \mathbf{\Pi}^g = \mathbf{\Pi}^g$  because  $\mathbb{P}_0 \subset \mathbf{P}_k^b(\mathcal{T}_h)$ . Finally, if the boundary condition is  $\gamma_{\partial D}^g(p) = 0$  for the grad-div problem, we simply set  $\mathbf{\Pi}_0^g = \mathbf{\Pi}_{h0}^g = 0$ .*

## 4 Discontinuous Galerkin approximation of grad-div operator

This section deals with the analysis of the dG approximation of the grad-div operator. The main result is Theorem 4.11, which implies that the approximation is spectrally correct. For simplicity, we set the scaling coefficient to  $\mathfrak{c} := 1$ .

### 4.1 Definitions

We define the discrete space  $L_h^d := \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ . The sesquilinear form  $a_h : L_h^d \times L_h^d \rightarrow \mathbb{C}$  associated with the problem (17) is

$$\begin{aligned} a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) &:= \ell_D^{-1}(\mathbf{\Pi}_{h0}^d(\mathbf{v}_h), \mathbf{w}_h)_{L^2(D)} + \ell_D^{-1}(\mathbf{\Pi}^g(p_h), q_h)_{L^2(D)} \\ &\quad - (p_h, \nabla_h \cdot \mathbf{w}_h)_{L^2(D)} - (\mathbf{v}_h, \nabla_h q_h)_{L^2(D)} \\ &\quad + \sum_{F \in \mathcal{F}_h} (\llbracket p_h \rrbracket_F^g, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)} + \sum_{F \in \mathcal{F}_h^\circ} (\llbracket \mathbf{v}_h \rrbracket_F^d, \llbracket q_h \rrbracket_F^g)_{L^2(F)} \\ &\quad + s_h^d(\mathbf{v}_h, \mathbf{w}_h) + s_h^{g,\circ}(p_h, q_h). \end{aligned} \quad (32)$$

Integrating by parts the broken divergence and gradient operators also gives

$$\begin{aligned} a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) &= \ell_D^{-1}(\mathbf{\Pi}_{h0}^d(\mathbf{v}_h), \mathbf{w}_h)_{L^2(D)} + \ell_D^{-1}(\mathbf{\Pi}^g(p_h), q_h)_{L^2(D)} \\ &\quad + (\nabla_h p_h, \mathbf{w}_h)_{L^2(D)} + (\nabla_h \cdot \mathbf{v}_h, q_h)_{L^2(D)} \\ &\quad - \sum_{F \in \mathcal{F}_h^\circ} (\llbracket p_h \rrbracket_F^g, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)} - \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^d, \llbracket q_h \rrbracket_F^g)_{L^2(F)} \\ &\quad + s_h^d(\mathbf{v}_h, \mathbf{w}_h) + s_h^{g,\circ}(p_h, q_h). \end{aligned} \quad (33)$$

Notice that the stabilization bilinear forms  $s_h^d$  and  $s_h^{g,\circ}$  could be scaled by  $O(1)$  positive weights; for simplicity, we choose these weights to be equal to 1 here.

Recalling that  $L^d := \mathbf{L}^2(D) \times \mathbf{L}^2(D)$ , we now define the discrete counterpart of the operator  $T : L^d \rightarrow L^d$  introduced in Definition 2.12. We define  $T_h : L^d \rightarrow L_h^d \subset L^d$  so that, for all  $(\mathbf{f}, g) \in L^d$ ,  $T_h(\mathbf{f}, g) := (\mathbf{v}_h, p_h)$  is the unique pair in  $L_h^d$  so that, for all  $(\mathbf{w}_h, q_h) \in L_h^d$ ,

$$a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = ((I - \mathbf{\Pi}_{h0}^d)(\mathbf{f}), \mathbf{w}_h)_{L^2(D)} + ((I - \mathbf{\Pi}^g)(g), q_h)_{L^2(D)}. \quad (34)$$

The definition of  $T_h$  makes sense owing to the stability result established in Lemma 4.6.

Our goal is to prove that  $\lim_{h \in \mathcal{H} \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^4, L^4)} = 0$ , i.e.,  $T_h$  converges strongly to  $T$ . We achieve this goal in two steps. First we prove an inf-sup condition which establishes stability. Then we prove a consistency/boundedness result. Convergence follows by combing these two results.

**Remark 4.1** (Other BCs). *To approximate the grad-div operator equipped with the boundary condition  $\gamma_{\partial D}^g(p) = 0$  (see Remarks 2.4 and 2.10), we consider the sesquilinear form*

$$\begin{aligned} a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) &:= \ell_D^{-1}(\mathbf{\Pi}_h^d(\mathbf{v}_h), \mathbf{w}_h)_{L^2(D)} - (p_h, \nabla_h \cdot \mathbf{w}_h)_{L^2(D)} - (\mathbf{v}_h, \nabla_h q_h)_{L^2(D)} \\ &+ \sum_{F \in \mathcal{F}_h^\circ} (\llbracket p_h \rrbracket_F^g, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)} + \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^d, \llbracket q_h \rrbracket_F^g)_{L^2(F)} + s_h^{d,\circ}(\mathbf{v}_h, \mathbf{w}_h) + s_h^g(p_h, q_h). \end{aligned}$$

Notice that there is no projection operator acting on  $p_h$  as  $\Pi_{h0}^g = 0$ .

## 4.2 Discrete involutions and other comments

The projection operators  $\mathbf{\Pi}_{h0}^d$  and  $\Pi^g$  are only invoked for theoretical purposes. One does not need to construct these operators in practice when one wants to approximate the eigenvalue problem (6) or when one wants to approximate the wave equation in the time domain (8). Indeed, let us consider the following sesquilinear form:

$$\begin{aligned} \hat{a}_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) &:= -(p_h, \nabla_h \cdot \mathbf{w}_h)_{L^2(D)} - (\mathbf{v}_h, \nabla_h q_h)_{L^2(D)} \\ &+ \sum_{F \in \mathcal{F}_h} (\llbracket p_h \rrbracket_F^g, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)} + \sum_{F \in \mathcal{F}_h^\circ} (\llbracket \mathbf{v}_h \rrbracket_F^d, \llbracket q_h \rrbracket_F^g)_{L^2(F)} + s_h^{d,\circ}(\mathbf{v}_h, \mathbf{w}_h) + s_h^{g,\circ}(p_h, q_h). \end{aligned} \quad (35)$$

Notice that  $\hat{a}_h((\cdot, \cdot), (\mathbf{w}_h, q_h)) = 0$  for all  $(\mathbf{w}_h, q_h) \in \mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h) \times \mathbb{P}_0$  because every field  $\mathbf{w}_h \in \mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h)$  satisfies  $\nabla_h \cdot \mathbf{w}_h = 0$  and  $\llbracket \mathbf{w}_h \rrbracket_F^d = 0$  for all  $F \in \mathcal{F}_h$ , and every function  $q_h \in \mathbb{P}_0$  satisfies  $\nabla_h q_h = 0$  and  $\llbracket q_h \rrbracket_F^g = 0$  for all  $F \in \mathcal{F}_h^\circ$ .

**Lemma 4.2** (Eigenvalue problems for  $a_h$  and  $\hat{a}_h$ ). *Let  $\lambda \neq 0$ ,  $(\mathbf{v}_h, p_h) \in L_h^d$ . Then  $T_h(\mathbf{v}_h, p_h) = \frac{1}{\lambda}(\mathbf{v}_h, p_h)$  if and only if  $\hat{a}_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = \lambda((\mathbf{v}_h, \mathbf{w}_h)_{L^2(D)} + (p_h, q_h)_{L^2(D)})$  for all  $(\mathbf{w}_h, q_h) \in L_h^d$ .*

*Proof.* (1) Let  $(\mathbf{v}_h, p_h) \in L_h^d$  be so that  $T_h(\mathbf{v}_h, p_h) = \frac{1}{\lambda}(\mathbf{v}_h, p_h)$ . This means that  $a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = \lambda(((I - \mathbf{\Pi}_{h0}^d)(\mathbf{v}_h), \mathbf{w}_h)_{L^2(D)} + ((I - \Pi^g)(p_h), q_h)_{L^2(D)})$  for all  $(\mathbf{w}_h, q_h) \in L_h^d$ . Using the test functions  $\mathbf{w}_h = \mathbf{\Pi}_{h0}^d(\mathbf{v}_h)$  and  $q_h = \Pi^g(p_h)$ , we obtain  $\ell_D^{-1} \|\mathbf{\Pi}_{h0}^d(\mathbf{v}_h)\|_{L^2(D)}^2 + \ell_D^{-1} \|\Pi^g(p_h)\|_{L^2(D)}^2 = 0$ , which then gives

$$\mathbf{\Pi}_{h0}^d(\mathbf{v}_h) = \mathbf{0}, \quad \Pi^g(p_h) = 0. \quad (36)$$

This implies that  $\hat{a}_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = \lambda((\mathbf{v}_h, \mathbf{w}_h)_{L^2(D)} + (p_h, q_h)_{L^2(D)})$  for all  $(\mathbf{w}_h, q_h) \in L_h^d$ , whence the assertion.

(2) Assume now that  $\hat{a}_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = \lambda((\mathbf{v}_h, \mathbf{w}_h)_{L^2(D)} + (p_h, q_h)_{L^2(D)})$  for all  $(\mathbf{w}_h, q_h) \in L_h^d$ . Using the test functions  $\mathbf{w}_h = \mathbf{\Pi}_{h0}^d(\mathbf{v}_h)$  and  $q_h = \Pi^g(p_h)$ , we observe that  $\hat{a}_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = 0$ , which, in turn, implies that (36) holds true. The assertion readily follows.  $\square$

**Remark 4.3** (Discrete involutions). *The proof of Lemma 4.2 shows that the involutions enforced by  $a_h$  and  $\hat{a}_h$  are (36). Notice that the projections  $\mathbf{\Pi}_{h0}^d$  and  $\Pi^g$  are not involved in the construction of  $\hat{a}_h$ . As shown in Lemma 3.2, these involutions are essential to prove the discrete Poincaré–Steklov inequalities (27). These inequalities play a pivotal in the proof of the spectral correctness of  $T_h$ , which owing to Lemma 4.2 implies the spectral correctness of the dG approximation realized by  $\hat{a}_h$ .*



Let us now consider the approximation in time and space of the wave equation (8). For simplicity, we use the backward Euler time-stepping. Letting  $(\mathbf{v}_h^n, p_h^n) \in L_h^d$  be the approximation at time  $t^n$  and letting  $\tau$  be the time step,  $(\mathbf{v}_h^{n+1}, p_h^{n+1}) \in L_h^d$  is the unique pair that solves, for all  $(\mathbf{w}_h, q_h) \in L_h^d$ ,

$$(\mathbf{v}_h^{n+1}, \mathbf{w}_h)_{L^2(D)} + (p_h^{n+1}, q_h)_{L^2(D)} + \tau \hat{\alpha}_h((\mathbf{v}_h^{n+1}, p_h^{n+1}), (\mathbf{w}_h, q_h)) = (\mathbf{v}_h^n, \mathbf{w}_h)_{L^2(D)} + (p_h^n, q_h)_{L^2(D)}. \quad (37)$$

**Lemma 4.4** (Time involution). *Assume that the pair  $(\mathbf{v}_h^n, p_h^n)$  satisfies the involutions (36). Then the pair  $(\mathbf{v}_h^{n+1}, p_h^{n+1})$  satisfies (36) as well.*

*Proof.* Using  $\mathbf{w}_h = \mathbf{\Pi}_{h0}^d(\mathbf{v}_h^{n+1})$  and  $q_h = \Pi^g(p_h^{n+1})$  yields the assertion.  $\square$

In other words, Lemma 4.4 shows that if the initial discrete velocity  $\mathbf{v}_h^0$  is orthogonal to  $\mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h)$  and the mean of  $p_h^0$  over  $D$  is zero, then this is also the case for  $(\mathbf{v}_h^n, p_h^n)$  for all  $n \geq 0$ .

### 4.3 Stability

We equip the discrete space  $L_h^d$  with the mesh-dependent norm

$$\begin{aligned} \|(\mathbf{v}_h, p_h)\|_{b,h} &:= \ell_D^{-\frac{1}{2}} \|\mathbf{v}_h\|_{L^2(D)} + \ell_D^{-\frac{1}{2}} \|p_h\|_{L^2(D)} \\ &\quad + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p_h\|_{L^2(D)} + |\mathbf{v}_h|_h^d + |p_h|_h^{g,\circ}. \end{aligned} \quad (38)$$

Recall that the broken projections  $\mathbf{\Pi}_h^b$  and  $\Pi_h^b$  are defined in (22), and the spaces  $\mathbf{X}_0^d$  are  $X^g$  are equipped with the norms  $\|\mathbf{w}\|_{\mathbf{X}_0^d} := \ell_D \|\nabla_0 \cdot \mathbf{w}\|_{L^2(D)}$  and  $\|q\|_{X^g} := \ell_D \|\nabla q\|_{L^2(D)}$ , respectively.

**Lemma 4.5** (Stability of broken projections). *The following holds:*

$$\|(\mathbf{\Pi}_h^b(\mathbf{w}), \Pi_h^b(q))\|_{b,h} \lesssim \ell_D^{-\frac{1}{2}} (\|\mathbf{w}\|_{\mathbf{X}_0^d} + \|q\|_{X^g}), \quad \forall (\mathbf{w}, q) \in \mathbf{X}_0^d \times X^g. \quad (39)$$

*Proof.* (1) Bound on  $\mathbf{\Pi}_h^b(\mathbf{w})$ . Let  $\mathbf{w} \in \mathbf{X}_0^d$ . The  $L^2$ -stability of  $\mathbf{\Pi}_h^b$  together with the closedness of the image of  $\nabla_0 \cdot$  (see Lemma 2.7) implies that

$$\ell_D^{-\frac{1}{2}} \|\mathbf{\Pi}_h^b(\mathbf{w})\|_{L^2(D)} \leq \ell_D^{-\frac{1}{2}} \|\mathbf{w}\|_{L^2(D)} \lesssim \ell_D^{-\frac{1}{2}} \|\mathbf{w}\|_{\mathbf{X}_0^d}.$$

Moreover, we observe that, for all  $K \in \mathcal{T}_h$ ,

$$\|\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))\|_{L^2(K)}^2 = (\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}) - \mathbf{w}), \nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w})))_{L^2(K)} + (\nabla \cdot \mathbf{w}, \nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w})))_{L^2(K)}.$$

Letting  $A$  be the first term on the right-hand side, two integration by parts give

$$A = -(\mathbf{\Pi}_h^b(\mathbf{w}) - \mathbf{w}, \nabla(\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))))_{L^2(K)} + (\mathbf{\Pi}_h^b(\mathbf{w}) - \mathbf{w}, \mathbf{n}_K \nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w})))_{L^2(\partial K)}.$$

Invoking the Cauchy–Schwarz inequality, the approximation properties of  $\mathbf{\Pi}_h^b$ , the boundedness of the embedding  $\mathbf{X}_0^d \hookrightarrow \mathbf{H}^s(D)$  with  $s > \frac{1}{2}$  (see Lemma 2.13), and inverse and trace inequalities for the polynomial  $\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))$ , we obtain

$$|A| \lesssim h_K^{s-1} |\mathbf{w}|_{\mathbf{H}^s(K)} \|\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))\|_{L^2(K)}.$$

For the second-term on the right-hand side, we simply apply the Cauchy–Schwarz inequality. After simplifying the inequality by  $\|\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))\|_{L^2(K)}$ , we multiply the result by  $h_K^{\frac{1}{2}}$  and obtain

$$h_K^{\frac{1}{2}} \|\nabla \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))\|_{L^2(K)} \lesssim h_K^{s-\frac{1}{2}} |\mathbf{w}|_{\mathbf{H}^s(K)} + h_K^{\frac{1}{2}} \|\nabla \cdot \mathbf{w}\|_{L^2(K)}.$$

Summing over the mesh cells, we infer that

$$\|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot (\mathbf{\Pi}_h^b(\mathbf{w}))\|_{L^2(D)} \lesssim h^{s-\frac{1}{2}} \ell_D^{-s} \|\mathbf{w}\|_{\mathbf{X}_0^d} + h^{\frac{1}{2}} \ell_D^{-1} \|\mathbf{w}\|_{\mathbf{X}_0^d} \leq 2\ell_D^{-\frac{1}{2}} \|\mathbf{w}\|_{\mathbf{X}_0^d},$$

where the last bound follows from  $h \leq \ell_D$ . Finally, since  $\mathbf{w} \in \mathbf{H}^s(D)$  with  $s > \frac{1}{2}$ , it is legitimate to assert that  $\mathbf{w}$  has zero normal jump across every mesh interface and zero normal trace at every mesh boundary face. This implies that

$$|\mathbf{\Pi}_h^b(\mathbf{w})|_h^d = |\mathbf{\Pi}_h^b(\mathbf{w}) - \mathbf{w}|_h^d \lesssim h^{s-\frac{1}{2}} |\mathbf{w}|_{\mathbf{H}^s(D)} \lesssim h^{s-\frac{1}{2}} \ell_D^{-s} \|\mathbf{w}\|_{\mathbf{X}_0^d}.$$

Hence  $|\mathbf{\Pi}_h^b(\mathbf{w})|_h^d \lesssim \ell_D^{-\frac{1}{2}} \|\mathbf{w}\|_{\mathbf{X}_0^d}$  because  $h \leq \ell_D$ . This completes the bound on  $\mathbf{\Pi}_h^b(\mathbf{w})$ .

(2) The arguments for  $\Pi^g(q)$  are similar (and simpler) and are therefore omitted.  $\square$

We are now ready to establish our main stability result.

**Lemma 4.6** (Stability). *The following holds:*

$$\|(\mathbf{v}_h, p_h)\|_{b,h} \lesssim \sup_{(\mathbf{w}_h, q_h) \in L_h^d} \frac{|a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h))|}{\|(\mathbf{w}_h, q_h)\|_{b,h}}, \quad \forall (\mathbf{v}_h, p_h) \in L_h^d. \quad (40)$$

*Proof.* Let  $(\mathbf{v}_h, p_h) \in L_h^d$  and let  $\mathbb{S}$  denote the right-hand side of (40). We need to show that  $\|(\mathbf{v}_h, p_h)\|_{b,h} \lesssim \mathbb{S}$ .

(1) The first step of the proof is classical (see, e.g., [14]). We observe that

$$\begin{aligned} \ell_D^{-1} \|\mathbf{\Pi}_{h0}^d(\mathbf{v}_h)\|_{L^2(D)}^2 + \ell_D^{-1} \|\Pi^g(p_h)\|_{L^2(D)}^2 + (|\mathbf{v}_h|_h^d)^2 + (|p_h|_h^{g,\circ})^2 \\ = a_h((\mathbf{v}_h, p_h), (\mathbf{v}_h, p_h)) \leq \mathbb{S} \|(\mathbf{v}_h, p_h)\|_{b,h}. \end{aligned} \quad (41)$$

Moreover, using  $\mathbf{w}_h := \tilde{h} \nabla_h p_h$  and  $q_h := \tilde{h} \nabla_h \cdot \mathbf{v}_h$  in the expression (33) for  $a_h$  gives

$$\|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}_h\|_{L^2(D)}^2 + \|\tilde{h}^{\frac{1}{2}} \nabla_h q_h\|_{L^2(D)}^2 = a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) - \Delta_1,$$

with

$$\begin{aligned} \Delta_1 := \ell_D^{-1} (\mathbf{\Pi}_{h0}^d(\mathbf{v}_h), \mathbf{w}_h)_{L^2(D)} + \ell_D^{-1} (\Pi^g(p_h), q_h)_{L^2(D)} + s_h^d(\mathbf{v}_h, \mathbf{w}_h) + s_h^{g,\circ}(p_h, q_h) \\ - \sum_{F \in \mathcal{F}_h^o} ([p_h]_F^g, \{\{\mathbf{w}_h\}\}_F^d)_{L^2(F)} - \sum_{F \in \mathcal{F}_h} ([\mathbf{v}_h]_F^d, \{\{q_h\}\}_F^g)_{L^2(F)}. \end{aligned}$$

Using inverse inequalities and  $h \leq \ell_D$  shows that

$$\begin{aligned} \|(\mathbf{w}_h, q_h)\|_{b,h} &\lesssim \|\tilde{h}^{-\frac{1}{2}} \mathbf{w}_h\|_{L^2(D)} + \|\tilde{h}^{-\frac{1}{2}} q_h\|_{L^2(D)} \\ &= \|\tilde{h}^{\frac{1}{2}} \nabla_h p_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}_h\|_{L^2(D)} \leq \|(\mathbf{v}_h, p_h)\|_{b,h}. \end{aligned}$$

This proves that

$$|a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h))| \lesssim \mathbb{S} \|(\mathbf{v}_h, p_h)\|_{b,h}.$$

Moreover, invoking the Cauchy–Schwarz inequality,  $h \leq \ell_D$ , inverse inequalities, and the bound from Step (1) shows that

$$\begin{aligned} |\Delta_1| &\lesssim (\ell_D^{-1} \|\mathbf{\Pi}_{h0}^d(\mathbf{v}_h)\|_{L^2(D)}^2 + \ell_D^{-1} \|\Pi^g(p_h)\|_{L^2(D)}^2 + (|\mathbf{v}_h|_h^d)^2 + (|p_h|_h^{g,\circ})^2)^{\frac{1}{2}} \\ &\quad \times (\|\tilde{h}^{-\frac{1}{2}} \mathbf{w}_h\|_{L^2(D)} + \|\tilde{h}^{-\frac{1}{2}} q_h\|_{L^2(D)}) \leq \mathbb{S}^{\frac{1}{2}} \|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{3}{2}}. \end{aligned}$$

Putting the above bounds together gives

$$\|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p_h\|_{L^2(D)} \lesssim \mathbb{S}^{\frac{1}{2}} \|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}} \|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{3}{4}}. \quad (42)$$

Combining (41) with (42) shows that

$$\begin{aligned} \ell_D^{-\frac{1}{2}} \|\mathbf{\Pi}_{h0}^d(\mathbf{v}_h)\|_{L^2(D)} + \ell_D^{-\frac{1}{2}} \|\Pi^g(p_h)\|_{L^2(D)} + |\mathbf{v}_h|_h^d + |p_h|_h^{g,\circ} \\ + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p_h\|_{L^2(D)} \lesssim \mathbb{S}^{\frac{1}{2}} \|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}} \|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{3}{4}}. \end{aligned} \quad (43)$$

(2) In the second step of the proof, we establish that, for all  $(\mathbf{v}'_h, p'_h) \in \mathbf{X}_{h0}^d \times X_h^g$  (these spaces are defined in (25)),

$$\begin{aligned} \ell_D^{-\frac{1}{2}} (\|\mathbf{v}'_h\|_{L^2(D)} + \|p'_h\|_{L^2(D)}) \lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi \\ + (h/\ell_D)^{s-\frac{1}{2}} (|\mathbf{v}'_h|_h^d + |p'_h|_h^{g,\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}'_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p'_h\|_{L^2(D)}), \end{aligned} \quad (44)$$

with

$$\mathbb{S}^\pi := \sup_{(\mathbf{w}, q) \in \mathbf{X}_0^d \times X^g} \frac{|a((\mathbf{v}'_h, p'_h), (\mathbf{\Pi}_h^b(\mathbf{w}), \Pi_h^b(q)))|}{\|\mathbf{w}\|_{\mathbf{X}_0^d} + \|q\|_{X^g}}.$$

The proof of (44) heavily relies on the discrete Poincaré–Steklov inequalities (26) and (27a). Let  $q \in X^g$  with  $\ell_D \|\nabla q\|_{L^2(D)} =: \|q\|_{X^g} = 1$ , and set  $q_h := \Pi_h^b(q)$ . Notice that  $\Pi^g(q_h) = \Pi^g(q) = 0$  since  $\Pi^g \circ \Pi_h^b = \Pi^g$  (see Remark 3.4). Using the expression (33) for  $a_h$ , and since  $q$  has zero jumps across the mesh interfaces, we infer that

$$\begin{aligned} (\mathbf{v}'_h, \nabla q)_{L^2(D)} &= -(\nabla_h \cdot \mathbf{v}'_h, q)_{L^2(D)} + \sum_{F \in \mathcal{F}_h} ([\mathbf{v}'_h]_F^d, \llbracket q \rrbracket_F^g)_{L^2(F)} \\ &= -a_h((\mathbf{v}'_h, p'_h), (\mathbf{0}, q_h)) - \Delta_2, \end{aligned}$$

with

$$\Delta_2 := (\nabla_h \cdot \mathbf{v}'_h, q - q_h)_{L^2(D)} - \sum_{F \in \mathcal{F}_h} ([\mathbf{v}'_h]_F^d, \llbracket q - q_h \rrbracket_F^g)_{L^2(F)} + s_h^{g,\circ}(p'_h, q - q_h).$$

Invoking the Cauchy–Schwarz inequality, the approximation properties of  $\Pi_h^b$ , and the continuous embedding  $X^g \hookrightarrow H^s(D)$  with  $s > \frac{1}{2}$  gives

$$|\Delta_2| \lesssim h^{s-\frac{1}{2}} \ell_D^{-s} (|\mathbf{v}'_h|_h^d + |p'_h|_h^{g,\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}'_h\|_{L^2(D)}).$$

Similarly, let  $\mathbf{w} \in \mathbf{X}_0^d$  with  $\ell_D \|\nabla_0 \cdot \mathbf{w}\|_{L^2(D)} =: \|\mathbf{w}\|_{\mathbf{X}_0^d} = 1$  and set  $\mathbf{w}_h := \mathbf{\Pi}_h^b(\mathbf{w})$ . Notice that  $\mathbf{\Pi}_{h0}^d(\mathbf{w}_h) = \mathbf{\Pi}_{h0}^d(\mathbf{\Pi}_h^b(\mathbf{w})) = \mathbf{\Pi}_{h0}^d(\mathbf{w}) = \mathbf{\Pi}_{h0}^d(\mathbf{\Pi}_0^d(\mathbf{w})) = \mathbf{0}$  owing to Lemma 3.3. Using the expression (33) for  $a_h$ , and since  $\mathbf{w}$  has zero normal jumps across the mesh interfaces and zero normal component at the mesh boundary faces, we infer that

$$\begin{aligned} (p'_h, \nabla_0 \cdot \mathbf{w})_{L^2(D)} &= -(\nabla_h p'_h, \mathbf{w})_{L^2(D)} + \sum_{F \in \mathcal{F}_h^o} ([p'_h]_F^g, \llbracket \mathbf{w} \rrbracket_F^d)_{L^2(F)} \\ &= -a_h((\mathbf{v}'_h, p'_h), (\mathbf{w}_h, 0)) - \Delta_3, \end{aligned}$$

with

$$\Delta_3 := (\nabla_h p'_h, \mathbf{w} - \mathbf{w}_h)_{\mathbf{L}^2(D)} - \sum_{F \in \mathcal{F}_h^\circ} (\llbracket p'_h \rrbracket_F^{\mathfrak{g}}, \{\!\!\{ \mathbf{w} - \mathbf{w}_h \}\!\!\}_F^{\mathfrak{d}})_{L^2(F)} + s_h^{\mathfrak{d}}(\mathbf{v}'_h, \mathbf{w} - \mathbf{w}_h).$$

Invoking the Cauchy–Schwarz inequality, the approximation properties of  $\mathbf{\Pi}_h^{\mathfrak{b}}$ , and the continuous embedding  $\mathbf{X}_0^{\mathfrak{d}} \hookrightarrow \mathbf{H}^s(D)$  with  $s > \frac{1}{2}$  gives

$$|\Delta_3| \lesssim h^{s-\frac{1}{2}} \ell_D^{-s} (|\mathbf{v}'_h|_h^{\mathfrak{d}} + |p'_h|_h^{\mathfrak{g},\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p'_h\|_{\mathbf{L}^2(D)}).$$

Using the above identities for  $(\mathbf{v}'_h, \nabla q)_{\mathbf{L}^2(D)}$  and  $(p'_h, \nabla_0 \cdot \mathbf{w})_{L^2(D)}$  together with the above bounds on  $\Delta_2$  and  $\Delta_3$  shows that

$$\begin{aligned} \|\nabla \cdot \mathbf{v}'_h\|_{(X^{\mathfrak{g}})'} + \|\nabla p'_h\|_{(X_0^{\mathfrak{d}})'} &\lesssim \mathbb{S}^\pi \\ &+ h^{s-\frac{1}{2}} \ell_D^{-s} (|\mathbf{v}'_h|_h^{\mathfrak{d}} + |p'_h|_h^{\mathfrak{g},\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}'_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p'_h\|_{L^2(D)}). \end{aligned}$$

Finally, combining this bound with the discrete Poincaré–Steklov inequalities (26) and (27a) gives

$$\begin{aligned} &\ell_D^{-\frac{1}{2}} (\|\mathbf{v}'_h\|_{\mathbf{L}^2(D)} + \|p'_h\|_{L^2(D)}) \\ &\lesssim \ell_D^{\frac{1}{2}} (\|\nabla \cdot \mathbf{v}'_h\|_{(X^{\mathfrak{g}})'} + \|\nabla p'_h\|_{(X_0^{\mathfrak{d}})'} + (h/\ell_D)^{\frac{1}{2}} (|\mathbf{v}'_h|_h^{\mathfrak{d}} + |p'_h|_h^{\mathfrak{g},\circ})) \\ &\lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi + (h/\ell_D)^{s-\frac{1}{2}} (|\mathbf{v}'_h|_h^{\mathfrak{d}} + |p'_h|_h^{\mathfrak{g},\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}'_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p'_h\|_{L^2(D)}), \end{aligned}$$

where we used that  $h \leq \ell_D$ . This completes the proof of (44).

(3) In this last step, we prove (40). Let  $(\mathbf{v}_h, p_h) \in L_h^{\mathfrak{d}}$  and set  $\mathbf{v}'_h := \mathbf{v}_h - \mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h)$ ,  $p'_h := p_h - \Pi^{\mathfrak{g}}(p_h)$ . Notice that  $(\mathbf{v}'_h, p'_h) \in \mathbf{X}_{h0}^{\mathfrak{g}} \times X_h^{\mathfrak{g}}$ . Applying (44) and since  $\mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h) \in \mathbf{P}_{k0}^{\mathfrak{d}}(\text{div} = 0; \mathcal{T}_h)$ ,  $\Pi^{\mathfrak{g}}(p_h) \in \mathbb{P}_0$ , we infer that

$$\begin{aligned} \ell_D^{-\frac{1}{2}} (\|\mathbf{v}_h - \mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)} + \|p_h - \Pi^{\mathfrak{g}}(p_h)\|_{L^2(D)}) &\lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi \\ &+ (h/\ell_D)^{s-\frac{1}{2}} (|\mathbf{v}_h|_h^{\mathfrak{d}} + |p_h|_h^{\mathfrak{g},\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \cdot \mathbf{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h p_h\|_{L^2(D)}). \end{aligned}$$

Using  $h \leq \ell_D$  and invoking (43) gives

$$\begin{aligned} \ell_D^{-\frac{1}{2}} (\|\mathbf{v}_h - \mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)} + \|p_h - \Pi^{\mathfrak{g}}(p_h)\|_{L^2(D)}) &\lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi \\ &+ \mathbb{S}^{\frac{1}{2}} \|(\mathbf{v}_h, p_h)\|_{\mathfrak{b},h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}} \|(\mathbf{v}_h, p_h)\|_{\mathfrak{b},h}^{\frac{3}{4}}. \end{aligned}$$

Owing to Lemma 4.5, we infer that

$$\ell_D^{\frac{1}{2}} \mathbb{S}^\pi \lesssim \sup_{(\mathbf{w}_h, q_h) \in L_h^{\mathfrak{d}}} \frac{|a_h((\mathbf{v}_h - \mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h), p_h - \Pi^{\mathfrak{g}}(p_h)), (\mathbf{w}_h, q_h))|}{\|(\mathbf{w}_h, q_h)\|_{\mathfrak{b},h}}.$$

We observe that

$$\begin{aligned} a_h((\mathbf{v}_h - \mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h), p_h - \Pi^{\mathfrak{g}}(p_h)), (\mathbf{w}_h, q_h)) &= a_h((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) \\ &- \ell_D^{-1} (\mathbf{\Pi}_{h0}^{\mathfrak{d}}(\mathbf{v}_h), \mathbf{w}_h)_{\mathbf{L}^2(D)} - \ell_D^{-1} (\Pi^{\mathfrak{g}}(p_h), q_h)_{L^2(D)}. \end{aligned}$$

Bounding the last two terms on the right-hand side by using the Cauchy–Schwarz inequality, invoking the estimate (41) on  $\ell_D^{-\frac{1}{2}}(\|\mathbf{\Pi}_{h0}^d(\mathbf{v}_h)\|_{L^2(D)} + \|\Pi^g(p_h)\|_{L^2(D)})$ , and using the above bounds gives

$$\begin{aligned} \ell_D^{-\frac{1}{2}}(\|\mathbf{v}_h - \mathbf{\Pi}_{h0}^d(\mathbf{v}_h)\|_{L^2(D)} + \|p_h - \Pi^g(p_h)\|_{L^2(D)}) &\lesssim \mathbb{S} \\ &\quad + \mathbb{S}^{\frac{1}{2}}\|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}}\|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{3}{4}}. \end{aligned}$$

Invoking the triangle inequality and the estimate (41) yields

$$\ell_D^{-\frac{1}{2}}(\|\mathbf{v}_h\|_{L^2(D)} + \|p_h\|_{L^2(D)}) \lesssim \mathbb{S} + \mathbb{S}^{\frac{1}{2}}\|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}}\|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{3}{4}}.$$

Combining this bound with (43) finally gives

$$\|(\mathbf{v}_h, p_h)\|_{b,h}^2 \lesssim \mathbb{S}^2 + \mathbb{S}\|(\mathbf{v}_h, p_h)\|_{b,h} + \mathbb{S}^{\frac{1}{2}}\|(\mathbf{v}_h, p_h)\|_{b,h}^{\frac{3}{2}}.$$

The inf-sup condition (40) follows by repeated applications of Young’s inequality.  $\square$

Our introducing the quantity  $\mathbb{S}^\pi$  in the proof of Lemma 4.6 may seem a bit surprising since it is not used in the final result (40). But, the inequality (44) finds its justification in the following sharper stability estimate which will be instrumental to bound the consistency error.

**Corollary 4.7** (Sharper  $L^d$ -stability). *The following inequality holds true for all  $(\mathbf{v}'_h, p'_h) \in \mathbf{X}_{h0}^d \times X_h^g$ ,*

$$\begin{aligned} \ell_D^{-\frac{1}{2}}(\|\mathbf{v}'_h\|_{L^2(D)} + \|p'_h\|_{L^2(D)}) &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \sup_{(\mathbf{w}_h, q_h) \in L_h^d} \frac{|a_h((\mathbf{v}'_h, p'_h), (\mathbf{w}_h, q_h))|}{\|(\mathbf{w}_h, q_h)\|_{b,h}} \\ &\quad + \ell_D^{\frac{1}{2}} \sup_{(\mathbf{w}, q) \in \mathbf{X}_0^d \times X^g} \frac{|a((\mathbf{v}'_h, p'_h), (\mathbf{\Pi}_h^b(\mathbf{w}), \Pi_h^b(q)))|}{\|\mathbf{w}\|_{\mathbf{X}_0^d} + \|q\|_{X^g}}. \end{aligned} \quad (45)$$

*Proof.* Combine (40) with (44).  $\square$

#### 4.4 Consistency and boundedness

The second step of our program consists of proving a consistency/boundedness result. This is done by first considering the discrete operator  $\tilde{T}_h : L^d \rightarrow L_h^d \subset L^d$  so that, for all  $(\mathbf{f}, g) \in L^d$ ,  $\tilde{T}_h(\mathbf{f}, g) := (\tilde{\mathbf{v}}_h, \tilde{p}_h)$  is the unique pair in  $L_h^d$  so that, for all  $(\mathbf{w}_h, q_h) \in L_h^d$ ,

$$a_h((\tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}_h, q_h)) = ((\mathbf{I} - \mathbf{\Pi}_0^d)(\mathbf{f}), \mathbf{w}_h)_{L^2(D)} + ((\mathbf{I} - \Pi^g)(g), q_h)_{L^2(D)}. \quad (46)$$

The definition of  $\tilde{T}_h$  is meaningful owing to Lemma 4.6. The difference between the discrete operators  $T_h$  and  $\tilde{T}_h$  lies in the way  $\mathbf{f}$  is projected on the right-hand sides of (34) and (46) (see also Remark 4.10 below). The discrete operator  $\tilde{T}_h$  is introduced since it is easier to bound the associated consistency error. We postpone the control on  $\tilde{T}_h - T_h$  to a second step (see Lemma 4.9 below). We augment the stability norm  $\|\cdot\|_{b,h}$  by defining, for all  $s \in (\frac{1}{2}, 1]$ , the following mesh-dependent norm on  $\mathbf{H}^s(D) \times H^s(D) + L_h^d$ :

$$\begin{aligned} \|(\mathbf{w}, q)\|_{\sharp, h} &:= \|(\mathbf{w}, q)\|_{b,h} + \|\tilde{h}^{-\frac{1}{2}}\mathbf{w}\|_{L^2(D)} + \|\tilde{h}^{-\frac{1}{2}}q\|_{L^2(D)} \\ &\quad + \left\{ \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} \|\gamma_{K,F}^d(\mathbf{w})\|_{L^2(F)}^2 + \|\gamma_{K,F}^g(q)\|_{L^2(F)}^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (47)$$

**Lemma 4.8** (Consistency/boundedness). *Let  $(\mathbf{f}, g) \in L^d$ . Set  $(\mathbf{v}, p) := T(\mathbf{f}, g)$ ,  $(\tilde{\mathbf{v}}_h, \tilde{p}_h) := \tilde{T}_h(\mathbf{f}, g)$ ,  $\tilde{\mathbf{e}}_h^v := \tilde{\mathbf{v}}_h - \mathbf{\Pi}_h^b(\mathbf{v})$ ,  $\boldsymbol{\xi}^v := \mathbf{v} - \mathbf{\Pi}_h^b(\mathbf{v})$ ,  $\tilde{e}_h^p := \tilde{p}_h - \mathbf{\Pi}_h^b(p)$ , and  $\xi^p := p - \mathbf{\Pi}_h^b(p)$ . The following holds for all  $(\mathbf{w}_h, q_h) \in L_h^d$ ,*

$$|a_h((\tilde{\mathbf{e}}_h^v, \tilde{e}_h^p), (\mathbf{w}_h, q_h))| \lesssim \|(\boldsymbol{\xi}^v, \xi^p)\|_{\#,h} \|(\mathbf{w}_h, q_h)\|_{\flat,h}. \quad (48)$$

*Proof.* Let  $(\mathbf{w}_h, q_h) \in L_h^d$ . By definition of  $\tilde{T}_h$ , we obtain

$$\begin{aligned} a_h((\tilde{\mathbf{e}}_h^v, \tilde{e}_h^p), (\mathbf{w}_h, q_h)) &= ((I - \mathbf{\Pi}_0^d)(\mathbf{f}), \mathbf{w}_h)_{L^2(D)} + ((I - \mathbf{\Pi}^g)(g), q_h)_{L^2(D)} \\ &\quad - a_h((\mathbf{\Pi}_h^b(\mathbf{v}), \mathbf{\Pi}_h^b(p)), (\mathbf{w}_h, q_h)). \end{aligned}$$

Since  $\nabla p = (I - \mathbf{\Pi}_0^d)(\mathbf{f})$  and  $\nabla_0 \cdot \mathbf{v} = (I - \mathbf{\Pi}^g)(g)$  by definition of  $T$ , we infer that

$$\begin{aligned} a_h((\tilde{\mathbf{e}}_h^v, \tilde{e}_h^p), (\mathbf{w}_h, q_h)) &= (\nabla p, \mathbf{w}_h)_{L^2(D)} + (\nabla_0 \cdot \mathbf{v}, q_h)_{L^2(D)} \\ &\quad - a_h((\mathbf{\Pi}_h^b(\mathbf{v}), \mathbf{\Pi}_h^b(p)), (\mathbf{w}_h, q_h)). \end{aligned}$$

We integrate by parts the first two terms on the right-hand side (this is legitimate owing to Lemma 2.13 since  $s > \frac{1}{2}$ ). Using the expression (32) for  $a_h$ , this gives

$$\begin{aligned} a_h((\tilde{\mathbf{e}}_h^v, \tilde{e}_h^p), (\mathbf{w}_h, q_h)) &= -\ell_D^{-1}(\mathbf{\Pi}_{h0}^d(\mathbf{\Pi}_h^b(\mathbf{v})), \mathbf{w}_h)_{L^2(D)} - \ell_D^{-1}(\mathbf{\Pi}^g(\mathbf{\Pi}_h^b(p)), q_h)_{L^2(D)} \\ &\quad + (\mathbf{\Pi}_h^b(p) - p, \nabla_h \cdot \mathbf{w}_h)_{L^2(D)} + (\mathbf{\Pi}_h^b(\mathbf{v}) - \mathbf{v}, \nabla_h q_h)_{L^2(D)} \\ &\quad + \sum_{F \in \mathcal{F}_h} (\{p - \mathbf{\Pi}_h^b(p)\}_F^g, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)} + \sum_{F \in \mathcal{F}_h^\circ} (\{\mathbf{v} - \mathbf{\Pi}_h^b(\mathbf{v})\}_F^d, \llbracket q_h \rrbracket_F^g)_{L^2(F)} \\ &\quad - s_h^d(\mathbf{\Pi}_h^b(\mathbf{v}), \mathbf{w}_h) - s_h^{g,\circ}(\mathbf{\Pi}_h^b(p), q_h). \end{aligned}$$

Recall from Lemma 2.11 that  $\mathbf{v} \in \mathbf{X}_0^d$  and  $p \in X^g$ , i.e.,  $\mathbf{\Pi}_0^d(\mathbf{v}) = \mathbf{0}$  and  $\mathbf{\Pi}^g(p) = 0$ . We then observe that  $\mathbf{\Pi}_{h0}^d(\mathbf{\Pi}_h^b(\mathbf{v})) = \mathbf{\Pi}_{h0}^d(\mathbf{v}) = \mathbf{\Pi}_{h0}^d(\mathbf{\Pi}_0^d(\mathbf{v})) = \mathbf{0}$  (owing to Lemma 3.3),  $\mathbf{\Pi}^g(\mathbf{\Pi}_h^b(p)) = \mathbf{\Pi}^g(p) = 0$ ,  $\llbracket \mathbf{v} \rrbracket_F^d = 0$  for all  $F \in \mathcal{F}_h$ , and  $\llbracket p \rrbracket_F^g = 0$  for all  $F \in \mathcal{F}_h^\circ$ . Recalling the notation  $\boldsymbol{\xi}^v := \mathbf{v} - \mathbf{\Pi}_h^b(\mathbf{v})$ ,  $\xi^p := p - \mathbf{\Pi}_h^b(p)$ , we infer that

$$\begin{aligned} a_h((\tilde{\mathbf{e}}_h^v, \tilde{e}_h^p), (\mathbf{w}_h, q_h)) &= -(\xi^p, \nabla_h \cdot \mathbf{w}_h)_{L^2(D)} - (\boldsymbol{\xi}^v, \nabla_h q_h)_{L^2(D)} \\ &\quad + \sum_{F \in \mathcal{F}_h} (\{\xi^p\}_F^g, \llbracket \mathbf{w}_h \rrbracket_F^d)_{L^2(F)} + \sum_{F \in \mathcal{F}_h^\circ} (\{\boldsymbol{\xi}^v\}_F^d, \llbracket q_h \rrbracket_F^g)_{L^2(F)} \\ &\quad + s_h^d(\boldsymbol{\xi}^v, \mathbf{w}_h) + s_h^{g,\circ}(\xi^p, q_h). \end{aligned}$$

The assertion follows from the Cauchy–Schwarz inequality and  $h \leq \ell_D$ .  $\square$

The second step in the consistency error analysis is to bound the difference between the discrete operators  $T_h$  and  $\tilde{T}_h$ .

**Lemma 4.9** (Bound on  $(\tilde{T}_h - T_h)$ ). *We have  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|\tilde{T}_h - T_h\|_{\mathcal{L}(L^d; L^d)} = 0$ .*

*Proof.* Let  $(\mathbf{f}, g) \in L^d$  and let us set  $(\mathbf{v}_h, p_h) := T_h(\mathbf{f}, g)$ ,  $(\tilde{\mathbf{v}}_h, \tilde{p}_h) := \tilde{T}_h(\mathbf{f}, g)$ ,  $\boldsymbol{\eta}_h^v := \mathbf{v}_h - \tilde{\mathbf{v}}_h$ , and  $\eta_h^p := p_h - \tilde{p}_h$ . We have, for all  $(\mathbf{w}_h, q_h) \in L_h^d$ ,

$$a_h((\boldsymbol{\eta}_h^v, \eta_h^p), (\mathbf{w}_h, q_h)) = ((\mathbf{\Pi}_0^d - \mathbf{\Pi}_{h0}^d)(\mathbf{f}), \mathbf{w}_h)_{L^2(D)}. \quad (49)$$

Invoking Lemma 3.3 gives  $((\mathbf{\Pi}_0^d - \mathbf{\Pi}_{h0}^d)(\mathbf{f}), \mathbf{\Pi}_{h0}^d(\boldsymbol{\eta}_h^v))_{L^2(D)} = 0$ . Then testing (49) with  $\mathbf{w}_h = \mathbf{\Pi}_{h0}^d(\boldsymbol{\eta}_h^v)$  yields  $\mathbf{\Pi}_{h0}^d(\boldsymbol{\eta}_h^v) = 0$ . Moreover, testing (49) with  $q_h = \mathbf{\Pi}^g(\eta_h^p)$  readily gives  $\mathbf{\Pi}^g(\eta_h^p) = 0$ .

Hence,  $(\boldsymbol{\eta}_h^v, \eta_h^p) \in \mathbf{X}_{h0}^d \times X_h^g$ , so that we can invoke Corollary 4.7 to bound  $(\boldsymbol{\eta}_h^v, \eta_h^p)$ . Owing to (49), we infer that

$$\sup_{(\mathbf{w}_h, q_h) \in L_h^d} \frac{|a_h((\boldsymbol{\eta}_h^v, \eta_h^p), (\mathbf{w}_h, q_h))|}{\|(\mathbf{w}_h, q_h)\|_{b,h}} \leq \ell_D^{\frac{1}{2}} \|(\boldsymbol{\Pi}_0^d - \boldsymbol{\Pi}_{h0}^d)(\mathbf{f})\|_{L^2(D)} \leq \ell_D^{\frac{1}{2}} \|\mathbf{f}\|_{L^2(D)}.$$

Moreover, we have, for all  $(\mathbf{w}, q) \in \mathbf{X}_0^d \times X^g$ ,

$$\begin{aligned} a((\boldsymbol{\eta}_h^v, \eta_h^p), (\boldsymbol{\Pi}_h^b(\mathbf{w}), \Pi_h^b(q))) &= ((\boldsymbol{\Pi}_0^d - \boldsymbol{\Pi}_{h0}^d)(\mathbf{f}), \boldsymbol{\Pi}_h^b(\mathbf{w}))_{L^2(D)} \\ &= ((\boldsymbol{\Pi}_0^d - \boldsymbol{\Pi}_{h0}^d)(\mathbf{f}), \boldsymbol{\Pi}_h^b(\mathbf{w}) - \mathbf{w})_{L^2(D)}, \end{aligned}$$

where the last equality follows from  $\mathbf{w} \in \mathbf{H}_0(\operatorname{div} = 0; D)^\perp$  and  $(\boldsymbol{\Pi}_0^d - \boldsymbol{\Pi}_{h0}^d)(\mathbf{f}) \in \mathbf{H}_0(\operatorname{div} = 0; D)$ . Invoking the Cauchy-Schwarz inequality, the approximation properties of  $\boldsymbol{\Pi}_h^b$ , and the embedding  $\mathbf{X}_0^d \hookrightarrow \mathbf{H}^s(D)$  with  $s > \frac{1}{2}$  (see Lemma 2.13) gives

$$|a((\boldsymbol{\eta}_h^v, \eta_h^p), (\boldsymbol{\Pi}_h^b(\mathbf{w}), \Pi_h^b(q)))| \lesssim h^s \|\mathbf{f}\|_{L^2(D)} \|\mathbf{w}\|_{\mathbf{H}^s(D)} \lesssim (h/\ell_D)^s \|\mathbf{f}\|_{L^2(D)} \|\mathbf{w}\|_{\mathbf{X}_0^d}.$$

Hence, we have

$$\sup_{(\mathbf{w}, q) \in \mathbf{X}_0^d \times X^g} \frac{|a((\boldsymbol{\eta}_h^v, \eta_h^p), (\boldsymbol{\Pi}_h^b(\mathbf{w}), \Pi_h^b(q)))|}{\|\mathbf{w}\|_{\mathbf{X}_0^d} + \|q\|_{X^g}} \lesssim (h/\ell_D)^s \|\mathbf{f}\|_{L^2(D)}.$$

Putting the above two bounds together and invoking Corollary 4.7 finally gives

$$\ell_D^{-\frac{1}{2}} (\|\boldsymbol{\eta}_h^v\|_{L^2(D)} + \|\eta_h^p\|_{L^2(D)}) \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\mathbf{f}\|_{L^2(D)} + (h/\ell_D)^s \ell_D^{\frac{1}{2}} \|\mathbf{f}\|_{L^2(D)}.$$

Since  $h \leq \ell_D$ , this proves that

$$\ell_D^{-\frac{1}{2}} \|\tilde{T}_h(\mathbf{f}, g) - T_h(\mathbf{f}, g)\|_{L^d} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|(\mathbf{f}, g)\|_{L^d},$$

whence we conclude that  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|\tilde{T}_h - T_h\|_{\mathcal{L}(L^d; L^d)} = 0$ .  $\square$

**Remark 4.10** ( $T_h$  vs.  $\tilde{T}_h$ ). *Although we have  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|T - \tilde{T}_h\|_{\mathcal{L}(L^d; L^d)} = 0$  (see the proof of Theorem 4.11), this does not prove the spectral correctness of the dG approximation induced by the sesquilinear form  $\hat{a}_h$  defined in (35) because Lemma 4.2 does not hold for  $\tilde{T}_h$ . More precisely, let  $\lambda \neq 0$  and assume that  $\frac{1}{\lambda}, (\mathbf{v}_h, p_h) \in L_h^d$  is an eigenpair of  $\tilde{T}_h$ . Then  $\boldsymbol{\Pi}_{h0}^d(\mathbf{v}_h) = \mathbf{0}$  and  $\Pi^g(p_h) = 0$ , but  $\hat{a}((\mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)) = \lambda(\mathbf{v}_h, \mathbf{w}_h)_{L^2(D)} + \lambda(p_h, q_h)_{L^2(D)} - \lambda((\boldsymbol{\Pi}_0^d - \boldsymbol{\Pi}_{h0}^d)(\mathbf{v}_h), \mathbf{w}_h)_{L^2(D)}$  for all  $(\mathbf{w}_h, q_h) \in L_h^d$ . Hence  $\lambda, (\mathbf{v}_h, p_h)$  is not an eigenpair of the dG approximation associated with  $\hat{a}_h$ .*

## 4.5 Conclusion

We are now ready to state the main result of this section.

**theorem 4.11** (Convergence). *We have  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^d; L^d)} = 0$ .*

*Proof.* Since we have already established in Lemma 4.9 that  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|\tilde{T}_h - T_h\|_{\mathcal{L}(L^d; L^d)} = 0$ , it suffices to prove that  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|T - \tilde{T}_h\|_{\mathcal{L}(L^d; L^d)} = 0$  and invoke the triangle inequality. Let  $(\mathbf{f}, g) \in L^d$ . Recalling the notation introduced in Lemma 4.8, and using Lemma 4.6 and Lemma 4.8, we infer that

$$\|(\tilde{\boldsymbol{\epsilon}}_h^v, \tilde{\epsilon}_h^p)\|_{b,h} \lesssim \sup_{(\mathbf{w}_h, q_h) \in L_h^d} \frac{|a_h((\tilde{\boldsymbol{\epsilon}}_h^v, \tilde{\epsilon}_h^p), (\mathbf{w}_h, q_h))|}{\|(\mathbf{w}_h, q_h)\|_{b,h}} \lesssim \|(\boldsymbol{\xi}^v, \xi^p)\|_{\#,h}.$$

This estimate combined with the triangle inequality,  $\|\cdot\|_{b,h} \leq \|\cdot\|_{\sharp,h}$ , standard approximation properties of  $\mathbf{\Pi}_h^b$  and  $\Pi_h^b$ , and  $h \leq \ell_D$  gives

$$\begin{aligned} \|(\mathbf{v} - \tilde{\mathbf{v}}_h, p - \tilde{p}_h)\|_{b,h} &\lesssim \|(\boldsymbol{\xi}^{\mathbf{v}}, \xi^p)\|_{\sharp,h} \lesssim \left\{ \sum_{K \in \mathcal{T}_h} h_K^{2s-1} |\mathbf{v}|_{\mathbf{H}^s(K)}^2 + h_K |p|_{H^1(K)}^2 \right\}^{\frac{1}{2}} \\ &\lesssim h^{s-\frac{1}{2}} (|\mathbf{v}|_{\mathbf{H}^s(D)} + \ell_D^{1-s} |p|_{H^1(D)}). \end{aligned}$$

(Recall that we have set  $h := \max_{K \in \mathcal{T}_h} h_K$ .) Lemma 2.13 gives

$$\begin{aligned} \ell_D^s |\mathbf{v}|_{\mathbf{H}^s(D)} &\leq \|\mathbf{v}\|_{\mathbf{H}^s(D)} \lesssim \ell_D \|\nabla_0 \cdot \mathbf{v}\|_{L^2(D)} = \ell_D \|(I - \Pi^{\mathfrak{G}})(g)\|_{L^2(D)} \leq \ell_D \|g\|_{L^2(D)}, \\ |p|_{H^1(D)} &= \|\nabla p\|_{L^2(D)} = \|(I - \Pi_0^{\mathfrak{d}})(\mathbf{f})\|_{L^2(D)} \leq \|\mathbf{f}\|_{L^2(D)}. \end{aligned}$$

Hence,

$$\ell_D^{-\frac{1}{2}} \|T(\mathbf{f}, g) - \tilde{T}_h(\mathbf{f}, g)\|_{L^d} \leq \|(\mathbf{v} - \tilde{\mathbf{v}}_h, p - \tilde{p}_h)\|_{b,h} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|(\mathbf{f}, g)\|_{L^d}.$$

This proves that  $\|T - \tilde{T}_h\|_{\mathcal{L}(L^d; L^d)} \lesssim \ell_D (h/\ell_D)^{s-\frac{1}{2}}$ . The proof is complete.  $\square$

## 5 Discontinuous Galerkin approximation of curl-curl operator

This section deals with the analysis of the dG approximation of the curl-curl operator. The main result is Theorem 5.10, which implies that the approximation is spectrally correct. As most of the arguments are similar to those in §4, most details are omitted. For simplicity, we set the scaling coefficient to  $\mathfrak{c} := 1$ .

### 5.1 Definitions

We define the discrete space  $L_h^{\mathfrak{c}} := \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ . The sesquilinear form  $a_h : L_h^{\mathfrak{c}} \times L_h^{\mathfrak{c}} \rightarrow \mathbb{C}$  associated with the problem (19) is

$$\begin{aligned} a_h((\mathbf{B}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) &:= \ell_D^{-1} (\mathbf{\Pi}_{h0}^{\mathfrak{c}}(\mathbf{B}_h), \mathbf{b}_h)_{L^2(D)} + \ell_D^{-1} (\mathbf{\Pi}_h^{\mathfrak{c}}(\mathbf{E}_h), \mathbf{e}_h)_{L^2(D)} \\ &\quad - (\mathbf{E}_h, \nabla_h \times \mathbf{b}_h)_{L^2(D)} + (\mathbf{B}_h, \nabla \times_h \mathbf{e}_h)_{L^2(D)} \\ &\quad - \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{E}_h \rrbracket_F^{\mathfrak{G}}, \llbracket \mathbf{b}_h \rrbracket_F^{\mathfrak{c}})_{L^2(F)} + \sum_{F \in \mathcal{F}_h^{\circ}} (\llbracket \mathbf{B}_h \rrbracket_F^{\mathfrak{G}}, \llbracket \mathbf{e}_h \rrbracket_F^{\mathfrak{c}})_{L^2(F)} \\ &\quad + s_h^{\mathfrak{c}}(\mathbf{B}_h, \mathbf{b}_h) + s_h^{\mathfrak{c}, \circ}(\mathbf{E}_h, \mathbf{e}_h). \end{aligned} \quad (50)$$

Integrating by parts the broken curl operators gives

$$\begin{aligned} a_h((\mathbf{B}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) &= \ell_D^{-1} (\mathbf{\Pi}_{h0}^{\mathfrak{c}}(\mathbf{B}_h), \mathbf{b}_h)_{L^2(D)} + \ell_D^{-1} (\mathbf{\Pi}_h^{\mathfrak{c}}(\mathbf{E}_h), \mathbf{e}_h)_{L^2(D)} \\ &\quad - (\nabla_h \times \mathbf{E}_h, \mathbf{b}_h)_{L^2(D)} + (\nabla_h \times \mathbf{B}_h, \mathbf{e}_h)_{L^2(D)} \\ &\quad - \sum_{F \in \mathcal{F}_h^{\circ}} (\llbracket \mathbf{E}_h \rrbracket_F^{\mathfrak{c}}, \llbracket \mathbf{b}_h \rrbracket_F^{\mathfrak{G}})_{L^2(F)} + \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{B}_h \rrbracket_F^{\mathfrak{c}}, \llbracket \mathbf{e}_h \rrbracket_F^{\mathfrak{G}})_{L^2(F)} \\ &\quad + s_h^{\mathfrak{c}}(\mathbf{B}_h, \mathbf{b}_h) + s_h^{\mathfrak{c}, \circ}(\mathbf{E}_h, \mathbf{e}_h). \end{aligned} \quad (51)$$



Recalling that  $L^c := \mathbf{L}^2(D) \times \mathbf{L}^2(D)$ , we now define the discrete counterpart of the operator  $T : L^c \rightarrow L^c$  introduced in Definition 2.16. We define  $T_h : L^c \rightarrow L_h^c \subset L^c$  so that, for all  $(\mathbf{f}, \mathbf{g}) \in L^c$ ,  $T_h(\mathbf{f}, \mathbf{g}) := (\mathbf{B}_h, \mathbf{E}_h)$  is the unique pair in  $L_h^c$  so that, for all  $(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c$ ,

$$a_h((\mathbf{B}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) = ((\mathbf{I} - \mathbf{\Pi}_{h0}^c)(\mathbf{f}), \mathbf{b}_h)_{\mathbf{L}^2(D)} + ((\mathbf{I} - \mathbf{\Pi}_h^c)(\mathbf{g}), \mathbf{e}_h)_{\mathbf{L}^2(D)}. \quad (52)$$

The definition of  $T_h$  makes sense owing to the stability result established in Lemma 5.5.

Our goal is to prove that  $\lim_{h \in \mathcal{H} \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^c; L^c)} = 0$ , i.e., that  $T_h$  converges strongly to  $T$ . We achieve this goal by proceeding as in §4.

## 5.2 Discrete involutions and other comments

The projection operators  $\mathbf{\Pi}_{h0}^c$  and  $\mathbf{\Pi}_h^c$  are only invoked for theoretical purposes. One does not need to construct these operators in practice when one wants to approximate the eigenvalue problem (9) or when one wants to approximate Maxwell's equations in the time domain (11). Indeed, let us consider the following sesquilinear form:

$$\begin{aligned} \hat{a}_h((\mathbf{B}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) &:= -(\mathbf{E}_h, \nabla_h \times \mathbf{b}_h)_{\mathbf{L}^2(D)} + (\mathbf{B}_h, \nabla_h \times \mathbf{e}_h)_{\mathbf{L}^2(D)} \\ &- \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{E}_h \rrbracket_F^g, \llbracket \mathbf{b}_h \rrbracket_F^c)_{\mathbf{L}^2(F)} + \sum_{F \in \mathcal{F}_h^o} (\llbracket \mathbf{B}_h \rrbracket_F^g, \llbracket \mathbf{e}_h \rrbracket_F^c)_{\mathbf{L}^2(F)} + s_h^c(\mathbf{B}_h, \mathbf{b}_h) + s_h^{c,o}(\mathbf{E}_h, \mathbf{e}_h). \end{aligned} \quad (53)$$

Notice that  $\hat{a}_h((\cdot, \cdot), (\mathbf{b}_h, \mathbf{e}_h)) = 0$  for all  $(\mathbf{b}_h, \mathbf{e}_h) \in \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \times \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$  because every field  $\mathbf{b}_h \in \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$  satisfies  $\nabla_h \times \mathbf{b}_h = \mathbf{0}$  and  $\llbracket \mathbf{b}_h \rrbracket_F^c = \mathbf{0}$  for all  $F \in \mathcal{F}_h$  and every field  $\mathbf{e}_h \in \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$  satisfies  $\nabla_h \times \mathbf{e}_h = \mathbf{0}$  and  $\llbracket \mathbf{e}_h \rrbracket_F^c = \mathbf{0}$  for all  $F \in \mathcal{F}_h^o$ .

**Lemma 5.1** (Eigenvalue problems for  $a_h$  and  $\hat{a}_h$ ). *Let  $\lambda \neq 0$ ,  $(\mathbf{B}_h, \mathbf{E}_h) \in L_h^c$ . Then  $T_h(\mathbf{B}_h, \mathbf{E}_h) = \frac{1}{\lambda}(\mathbf{B}_h, \mathbf{E}_h)$  iff  $\hat{a}_h((\mathbf{B}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) = \lambda((\mathbf{B}_h, \mathbf{b}_h)_{\mathbf{L}^2(D)} + (\mathbf{E}_h, \mathbf{e}_h)_{\mathbf{L}^2(D)})$  for all  $(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c$ .*

*Proof.* Proceed as in the proof of Lemma 4.2.  $\square$

**Remark 5.2** (Discrete involutions). *Lemma 5.1 reveals that the involutions enforced by  $a_h$  and  $\hat{a}_h$  are*

$$\mathbf{\Pi}_{h0}^c(\mathbf{B}_h) = \mathbf{0}, \quad \mathbf{\Pi}_h^c(\mathbf{E}_h) = \mathbf{0}. \quad (54)$$

*As shown in Lemma 3.2, these involutions are essential to prove discrete Poincaré–Steklov inequalities. These inequalities play a pivotal role in the proof of the spectral correctness of  $T_h$ , which owing to Lemma 5.1 then implies the spectral correctness of the dG approximation realized by  $\hat{a}_h$ .*

Let us now consider the approximation in time and space of Maxwell's equations (11). For simplicity, we use the backward Euler time-stepping. Letting  $(\mathbf{B}_h^n, \mathbf{E}_h^n) \in L_h^c$  be the approximation at time  $t^n$  and letting  $\tau$  be the time step,  $(\mathbf{B}_h^{n+1}, \mathbf{E}_h^{n+1}) \in L_h^c$  is the unique pair that solves, for all  $(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c$ ,

$$\begin{aligned} (\mathbf{B}_h^{n+1}, \mathbf{b}_h)_{\mathbf{L}^2(D)} + (\mathbf{E}_h^{n+1}, \mathbf{e}_h)_{\mathbf{L}^2(D)} + \tau \hat{a}_h((\mathbf{B}_h^{n+1}, \mathbf{E}_h^{n+1}), (\mathbf{b}_h, \mathbf{e}_h)) = \\ (\mathbf{B}_h^n, \mathbf{b}_h)_{\mathbf{L}^2(D)} + (\mathbf{E}_h^n, \mathbf{e}_h)_{\mathbf{L}^2(D)}. \end{aligned} \quad (55)$$

**Lemma 5.3** (Time involution). *Assume that the pair  $(\mathbf{B}_h^n, \mathbf{E}_h^n)$  satisfies the involutions (54). Then the pair  $(\mathbf{B}_h^{n+1}, \mathbf{E}_h^{n+1})$  satisfies (54) as well.*

*Proof.* Using  $\mathbf{b}_h = \mathbf{\Pi}_{h0}^c(\mathbf{B}_h^{n+1})$  and  $\mathbf{e}_h = \mathbf{\Pi}_h^c(\mathbf{E}_h^{n+1})$  yields the assertion.  $\square$

### 5.3 Stability

We equip the discrete space  $L_h^c$  with the mesh-dependent norm

$$\begin{aligned} \|(\mathbf{e}_h, \mathbf{b}_h)\|_{b,h} &:= \ell_D^{-\frac{1}{2}} \|\mathbf{b}_h\|_{\mathbf{L}^2(D)} + \ell_D^{-\frac{1}{2}} \|\mathbf{e}_h\|_{\mathbf{L}^2(D)} \\ &\quad + \|\tilde{h}^{\frac{1}{2}} \nabla_h \times \mathbf{v}_h\|_{\mathbf{L}^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \times \mathbf{e}_h\|_{\mathbf{L}^2(D)} + |\mathbf{b}_h|_h^c + |\mathbf{e}_h|_h^{c,\circ}. \end{aligned} \quad (56)$$

Recall that the broken projection  $\mathbf{\Pi}_h^b$  is defined in (22), and the spaces  $\mathbf{X}_0^c$  are  $\mathbf{X}^c$  are equipped with the norms  $\|\mathbf{b}\|_{\mathbf{X}_0^c} := \ell_D \|\nabla_0 \times \mathbf{b}\|_{\mathbf{L}^2(D)}$  and  $\|\mathbf{e}\|_{\mathbf{X}^c} := \ell_D \|\nabla \times \mathbf{e}\|_{\mathbf{L}^2(D)}$ , respectively.

**Lemma 5.4** (Stability of broken projections). *The following holds:*

$$\|(\mathbf{\Pi}_h^b(\mathbf{b}), \mathbf{\Pi}_h^b(\mathbf{e}))\|_{b,h} \lesssim \ell_D^{-\frac{1}{2}} (\|\mathbf{b}\|_{\mathbf{X}_0^c} + \|\mathbf{e}\|_{\mathbf{X}^c}), \quad \forall (\mathbf{b}, \mathbf{e}) \in \mathbf{X}_0^c \times \mathbf{X}^c. \quad (57)$$

*Proof.* Similar to the proof of Lemma 4.5.  $\square$

We can now state our main stability results.

**Lemma 5.5** (Stability). *The following holds:*

$$\|(\mathbf{B}_h, \mathbf{E}_h)\|_{b,h} \lesssim \sup_{(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c} \frac{|a_h((\mathbf{B}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h))|}{\|(\mathbf{b}_h, \mathbf{e}_h)\|_{b,h}}, \quad \forall (\mathbf{B}_h, \mathbf{E}_h) \in L_h^c. \quad (58)$$

*Proof.* Proceed as in the proof of Lemma 4.6. In particular, the second step of the proof establishes that, for all  $(\mathbf{b}'_h, \mathbf{e}'_h) \in \mathbf{X}_{h0}^c \times \mathbf{X}_h^c$  (these spaces are defined in (25)),

$$\begin{aligned} \ell_D^{-\frac{1}{2}} (\|\mathbf{b}'_h\|_{\mathbf{L}^2(D)} + \|\mathbf{e}'_h\|_{\mathbf{L}^2(D)}) &\lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi \\ &\quad + (h/\ell_D)^{s-\frac{1}{2}} (|\mathbf{b}'_h|_h^c + |\mathbf{e}'_h|_h^{c,\circ} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \times \mathbf{b}'_h\|_{\mathbf{L}^2(D)} + \|\tilde{h}^{\frac{1}{2}} \nabla_h \times \mathbf{e}'_h\|_{\mathbf{L}^2(D)}), \end{aligned} \quad (59)$$

with  $\mathbb{S}^\pi := \sup_{(\mathbf{b}, \mathbf{e}) \in \mathbf{X}_0^c \times \mathbf{X}^c} \frac{|a((\mathbf{b}'_h, \mathbf{e}'_h), (\mathbf{\Pi}_h^b(\mathbf{b}), \mathbf{\Pi}_h^b(\mathbf{e})))|}{\|\mathbf{b}\|_{\mathbf{X}_0^c} + \|\mathbf{e}\|_{\mathbf{X}^c}}$ .  $\square$

**Corollary 5.6** (Sharper  $L^c$ -stability). *The following inequality holds true for all  $(\mathbf{b}'_h, \mathbf{e}'_h) \in \mathbf{X}_{h0}^c \times \mathbf{X}_h^c$ ,*

$$\begin{aligned} \ell_D^{-\frac{1}{2}} (\|\mathbf{b}'_h\|_{\mathbf{L}^2(D)} + \|\mathbf{e}'_h\|_{\mathbf{L}^2(D)}) &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \sup_{(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c} \frac{|a_h((\mathbf{b}'_h, \mathbf{e}'_h), (\mathbf{b}_h, \mathbf{e}_h))|}{\|(\mathbf{b}_h, \mathbf{e}_h)\|_{b,h}} \\ &\quad + \ell_D^{\frac{1}{2}} \sup_{(\mathbf{b}, \mathbf{e}) \in \mathbf{X}_0^c \times \mathbf{X}^c} \frac{|a((\mathbf{b}'_h, \mathbf{e}'_h), (\mathbf{\Pi}_h^b(\mathbf{b}), \mathbf{\Pi}_h^b(\mathbf{e})))|}{\|\mathbf{b}\|_{\mathbf{X}_0^c} + \|\mathbf{e}\|_{\mathbf{X}^c}}. \end{aligned} \quad (60)$$

*Proof.* Combine (58) with (59).  $\square$

### 5.4 Consistency and boundedness

To establish consistency/boundedness, we proceed as above in two steps. We first introduce the discrete operator  $\tilde{T}_h : L^c \rightarrow L_h^c \subset L^c$  so that, for all  $(\mathbf{f}, \mathbf{g}) \in L^c$ ,  $\tilde{T}_h(\mathbf{f}, \mathbf{g}) := (\tilde{\mathbf{B}}_h, \tilde{\mathbf{E}}_h)$  is the unique pair in  $L_h^c$  so that, for all  $(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c$ ,

$$a_h((\tilde{\mathbf{B}}_h, \tilde{\mathbf{E}}_h), (\mathbf{b}_h, \mathbf{e}_h)) = ((\mathbf{I} - \mathbf{\Pi}_0^c)(\mathbf{f}), \mathbf{b}_h)_{\mathbf{L}^2(D)} + ((\mathbf{I} - \mathbf{\Pi}^c)(\mathbf{g}), \mathbf{e}_h)_{\mathbf{L}^2(D)}. \quad (61)$$

The definition of  $\tilde{T}_h$  is meaningful owing to Lemma 5.5. The difference between the discrete operators  $T_h$  and  $\tilde{T}_h$  lies in the way  $\mathbf{f}$  and  $\mathbf{g}$  are projected on the right-hand sides of (52) and

(61). To bound the consistency error associated with  $\tilde{T}_h$ , we augment the stability norm  $\|\cdot\|_{b,h}$  by defining, for all  $s \in (\frac{1}{2}, 1]$ , the following mesh-dependent norm on  $\mathbf{H}^s(D) \times \mathbf{H}^s(D) + L_h^c$ :

$$\begin{aligned} \|(\mathbf{b}, \mathbf{e})\|_{\sharp,h} &:= \|(\mathbf{b}, \mathbf{e})\|_{b,h} + \|\tilde{h}^{-\frac{1}{2}} \mathbf{b}\|_{\mathbf{L}^2(D)} + \|\tilde{h}^{-\frac{1}{2}} \mathbf{e}\|_{\mathbf{L}^2(D)} \\ &\quad + \left\{ \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} \|\gamma_{K,F}^c(\mathbf{b})\|_{\mathbf{L}^2(F)}^2 + \|\gamma_{K,F}^c(\mathbf{e})\|_{\mathbf{L}^2(F)}^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (62)$$

**Lemma 5.7** (Consistency/boundedness). *For all  $(\mathbf{f}, \mathbf{g}) \in L^c$ , let  $(\mathbf{B}, \mathbf{E}) := T(\mathbf{f}, \mathbf{g})$  and  $(\tilde{\mathbf{B}}_h, \tilde{\mathbf{E}}_h) := \tilde{T}_h(\mathbf{f}, \mathbf{g})$ . Define  $\tilde{\boldsymbol{\theta}}_h^b := \tilde{\mathbf{B}}_h - \Pi_h^b(\mathbf{B})$ ,  $\boldsymbol{\xi}^b := \mathbf{B} - \Pi_h^b(\mathbf{B})$ , and  $\tilde{\boldsymbol{\theta}}_h^e = \tilde{\mathbf{E}}_h - \Pi_h^b(\mathbf{E})$ ,  $\boldsymbol{\xi}^e := \mathbf{E} - \Pi_h^b(\mathbf{E})$ . The following holds for all  $(\mathbf{b}_h, \mathbf{e}_h) \in L_h^c$ ,*

$$|a_h((\tilde{\boldsymbol{\theta}}_h^b, \tilde{\boldsymbol{\theta}}_h^e), (\mathbf{b}_h, \mathbf{e}_h))| \lesssim \|(\boldsymbol{\xi}^b, \boldsymbol{\xi}^e)\|_{\sharp,h} \|(\mathbf{b}_h, \mathbf{e}_h)\|_{b,h}. \quad (63)$$

*Proof.* Proceed as in the proof of Lemma 4.8.  $\square$

The second step in the consistency error analysis consists of bounding the difference between the discrete operators  $T_h$  and  $\tilde{T}_h$ .

**Lemma 5.8** (Bound on  $(\tilde{T}_h - T_h)$ ). *We have  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|\tilde{T}_h - T_h\|_{\mathcal{L}(L^c; L^c)} = 0$ .*

*Proof.* Proceed as in the proof of Lemma 4.9.  $\square$

**Remark 5.9** ( $T_h$  vs.  $\tilde{T}_h$ ). *The fact that  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|T - \tilde{T}_h\|_{\mathcal{L}(L^c, L^c)} = 0$  is not sufficient to prove the spectral correctness of the dG approximation using the sesquilinear form  $\hat{a}_h$  defined in (53). As for the grad-div problem (see Remark 4.10), one needs to prove that  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^c, L^c)} = 0$ .*

## 5.5 Conclusion

We are now ready to state the main result of this section.

**theorem 5.10** (Convergence). *We have  $\lim_{\mathcal{H} \ni h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L^c; L^c)} = 0$ .*

*Proof.* The proof is similar to that of Theorem 4.11.  $\square$

## A Helmholtz decompositions

We recall here standard results characterizing the kernel and the image of the gradient, curl, and divergence operators. These results are mostly drawn from Amrouche et al. [2], Dautray and Lions [13], and Girault and Raviart [17].

### A.1 Topology of $D$

Recall that we assumed that  $D$  is an open, bounded, Lipschitz polyhedron of  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ . We denote  $\Gamma_0$  the boundary of the only unbounded connected component of  $\mathbb{R}^d \setminus \bar{D}$ . If  $\partial D$  is not connected, i.e.,  $\partial D \neq \Gamma_0$ , we denote  $\{\Gamma_i\}_{i \in \{1:J\}}$  the connected components of  $\partial D$  that are different from  $\Gamma_0$  (see, e.g., [17, p. 37], [2, p. 835], [13, p. 217]). If  $D$  is not simply connected, we assume that there exist  $J$  cuts  $((d-1)$ -dimensional smooth manifolds)  $\{\Sigma_j\}_{j \in \{1:J\}}$  that make the open set  $D^\Sigma := D \setminus \bigcup_{j \in \{1:J\}} \Sigma_j$  simply connected. Additional regularity assumptions on these cuts as stated in [2, Hyp. 3.3, p. 836] are assumed to hold. For all  $q \in L^2(D)$  such that  $q|_{D^\Sigma} \in H^1(D^\Sigma)$ , we denote by  $\nabla_\Sigma q$  the broken gradient of  $q$  such that  $(\nabla_\Sigma q)(\mathbf{x}) = (\nabla q|_{D^\Sigma})(\mathbf{x})$  for a.e.  $\mathbf{x} \in D$ .

For all  $i \in \mathbb{N}$ , let  $c_i$  denote any real number. We define

$$H_{\Gamma}^1(D) := \{q \in H^1(D) \mid q|_{\Gamma_0} = 0, q|_{\Gamma_i} = c_i, \forall i \in \{1:I\}\}, \quad (64a)$$

$$H_{\Sigma}^1(D) := \{q \in L^2(D) \mid q|_{D^{\Sigma}} \in H^1(D^{\Sigma}), \llbracket q \rrbracket|_{\Sigma_j} = c_j, \forall j \in \{1:J\}\}. \quad (64b)$$

We also consider the subspaces

$$\mathbf{H}^{\Gamma}(\operatorname{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div} = 0; D) \mid \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n} \, ds = 0, \forall i \in \{1:I\}\}, \quad (65a)$$

$$\mathbf{H}_0^{\Sigma}(\operatorname{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}_0(\operatorname{div} = 0; D) \mid \int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n} \, ds = 0, \forall j \in \{1:J\}\}, \quad (65b)$$

where the integrals are understood as duality products. The following spaces characterize the topology of  $D$ :

$$\mathbf{K}_{\Gamma}(D) := \mathbf{H}_0(\operatorname{div} = 0; D) \cap \mathbf{H}(\operatorname{curl} = \mathbf{0}; D), \quad (66a)$$

$$\mathbf{K}_{\Sigma}(D) := \mathbf{H}(\operatorname{div} = 0; D) \cap \mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D). \quad (66b)$$

We have  $\dim(\mathbf{K}_{\Sigma}(D)) = I$  [2, Prop. 3.18] and  $\dim(\mathbf{K}_{\Gamma}(D)) = J$  [2, Prop. 3.14].

**theorem A.1** (Orthogonal decompositions). *The following decompositions hold and are orthogonal in  $\mathbf{L}^2(D)$ :*

$$\mathbf{H}(\operatorname{curl} = \mathbf{0}; D) = \nabla H^1(D) \overset{\perp}{\oplus} \mathbf{K}_{\Gamma}(D), \quad (67a)$$

$$\mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D) = \nabla H_0^1(D) \overset{\perp}{\oplus} \mathbf{K}_{\Sigma}(D), \quad (67b)$$

$$\mathbf{H}(\operatorname{div} = 0; D) = \mathbf{H}^{\Gamma}(\operatorname{div} = 0; D) \overset{\perp}{\oplus} \mathbf{K}_{\Sigma}(D), \quad (67c)$$

$$\mathbf{H}_0(\operatorname{div} = 0; D) = \mathbf{H}_0^{\Sigma}(\operatorname{div} = 0; D) \overset{\perp}{\oplus} \mathbf{K}_{\Gamma}(D), \quad (67d)$$

$$\mathbf{L}^2(D) = \mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D) \overset{\perp}{\oplus} \mathbf{H}^{\Gamma}(\operatorname{div} = 0; D), \quad (67e)$$

$$\mathbf{L}^2(D) = \mathbf{H}(\operatorname{curl} = \mathbf{0}; D) \overset{\perp}{\oplus} \mathbf{H}_0^{\Sigma}(\operatorname{div} = 0; D). \quad (67f)$$

*Proof.* The reader is referred, e.g., to [13, p. 314].  $\square$

## A.2 Helmholtz decompositions

We recall here Helmholtz decompositions that are useful to study the grad-div and curl-curl problems. Recall that we have  $\mathbb{P}_0 = H(\operatorname{grad} = \mathbf{0}; D)$  and  $\{0\} = H_0(\operatorname{grad} = \mathbf{0}; D)$ .

**theorem A.2** (Decompositions for grad-div problem). *The following decompositions hold and are orthogonal in  $L^2(D)$  and  $\mathbf{L}^2(D)$ , respectively:*

$$L^2(D) = \{0\} \overset{\perp}{\oplus} \nabla \cdot \mathbf{H}(\operatorname{div}; D), \quad (68a)$$

$$L^2(D) = \mathbb{P}_0 \overset{\perp}{\oplus} \nabla \cdot \mathbf{H}_0(\operatorname{div}; D), \quad (68b)$$

$$\mathbf{L}^2(D) = \mathbf{H}_0(\operatorname{div} = 0; D) \overset{\perp}{\oplus} \nabla H^1(D), \quad (68c)$$

$$\mathbf{L}^2(D) = \mathbf{H}(\operatorname{div} = 0; D) \overset{\perp}{\oplus} \nabla H_0^1(D). \quad (68d)$$

*Proof.* These results are standard and are consequences of the following Poincaré–Steklov inequalities:

$$\|q\|_{L^2(D)} \leq C_D \ell_D \|\nabla q\|_{L^2(D)}, \quad \forall q \in \{H_0^1(D), H^1(D) \cap \mathbb{P}_0^\perp\}. \quad (69)$$

□

**theorem A.3** (Decompositions for curl-curl problem). *The following decompositions hold and are orthogonal in  $L^2(D)$ :*

$$L^2(D) = \nabla H_\Gamma^1(D) \dot{\oplus} \nabla \times \mathbf{H}(\mathbf{curl}; D), \quad (70a)$$

$$L^2(D) = \nabla_\Sigma H_\Sigma^1(D) \dot{\oplus} \nabla \times \mathbf{H}_0(\mathbf{curl}; D). \quad (70b)$$

*Proof.* (1) Proof of (70a). Let  $\mathbf{u} \in L^2(D)$ . Let  $q \in H_\Gamma^1(D)$  solve  $(\nabla q, \nabla r)_{L^2(D)} = (\mathbf{u}, \nabla r)_{L^2(D)}$  for all  $r \in H_\Gamma^1(D)$ . This problem is equivalent to solving  $\Delta q = \nabla \cdot \mathbf{u}$ ,  $q|_{\Gamma_0} = 0$ ,  $q|_{\Gamma_i} = c_i$  and  $\int_{\Gamma_i} (\nabla q - \mathbf{u}) \cdot \mathbf{n} \, dx = 0$  for all  $i \in \{1:I\}$ . Let  $\mathbf{w} := \mathbf{u} - \nabla q$ . We have  $\nabla \cdot \mathbf{w} = 0$  and  $\int_{\Gamma_i} \mathbf{w} \cdot \mathbf{n} \, dx = 0$  for all  $i \in \{1:I\}$ . Then, according to [17, Thm. 3.5, p. 47] or [2, Lem. 3.12, p. 841], there exists a unique  $\boldsymbol{\psi}$  in  $\mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}_0^\Sigma(\text{div} = 0; D)$  such that  $\nabla \times \boldsymbol{\psi} = \mathbf{w}$ . This proves (70a).

(2) The decomposition (70b) is proved in [2, §3.5(ii), p. 847]. □

**Remark A.4** (Uniqueness of decomposition). *The potentials in the Helmholtz decompositions from Theorems A.2 and A.3 can be made unique by requiring them to be orthogonal to the kernel of the relevant differential operator. This gives*

$$\begin{aligned} L^2(D) &= \{0\} \dot{\oplus} \nabla \cdot (\mathbf{H}(\text{div}; D) \cap \mathbf{H}(\text{div} = 0; D)^\perp), \\ L^2(D) &= \mathbb{P}_0 \dot{\oplus} \nabla \cdot (\mathbf{H}_0(\text{div}; D) \cap \mathbf{H}_0(\text{div} = 0; D)^\perp), \\ L^2(D) &= \mathbf{H}_0(\text{div} = 0; D) \dot{\oplus} \nabla (H^1(D) \cap \mathbb{P}_0^\perp), \\ L^2(D) &= \mathbf{H}(\text{div} = 0; D) \dot{\oplus} \nabla H_0^1(D), \\ L^2(D) &= \nabla (H_\Gamma^1(D) \cap \mathbb{P}_0^\perp) \dot{\oplus} \nabla \times (\mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp), \\ L^2(D) &= \nabla_\Sigma (H_\Sigma^1(D) \cap \mathbb{P}_0^\perp) \dot{\oplus} \nabla \times (\mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp). \end{aligned}$$

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