



**HAL**  
open science

# Hyperbolic geometry and real moduli of five points on the line

Olivier de Gaay Fortman

► **To cite this version:**

Olivier de Gaay Fortman. Hyperbolic geometry and real moduli of five points on the line. 2023.  
hal-04003092

**HAL Id: hal-04003092**

**<https://hal.science/hal-04003092>**

Preprint submitted on 23 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Hyperbolic geometry and real moduli of five points on the line

Olivier de Gaay Fortman, January 9, 2023

## 1 Introduction

Let  $X \cong \mathbb{A}_{\mathbb{R}}^6$  be the affine space parametrizing homogeneous degree 5 polynomials  $F \in \mathbb{R}[x, y]$ . Let the variety  $X_0 \subset X$  parametrize polynomials with distinct roots, and  $X_s \subset X$  polynomials with roots of multiplicity at most two (i.e. stable in the sense of geometric invariant theory). The principal goal of this paper is to study the moduli space of stable *real binary quintics*

$$\mathcal{M}_s(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R}) \setminus X_s(\mathbb{R}) \supset \mathrm{GL}_2(\mathbb{R}) \setminus X_0(\mathbb{R}) =: \mathcal{M}_0(\mathbb{R}).$$

If  $P_s \subset \mathbb{P}^1(\mathbb{C})^5$  is the set 5-tuples  $(x_1, \dots, x_5)$  such that no three  $x_i \in \mathbb{P}^1(\mathbb{C})$  coincide (c.f. [MS72]), and  $P_0 \subset P_s$  the subset of 5-tuples whose coordinates are distinct, then

$$\mathcal{M}_0(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_0/\mathfrak{S}_5)(\mathbb{R}) \quad \text{and} \quad \mathcal{M}_s(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R}).$$

In other words,  $\mathcal{M}_0(\mathbb{R})$  is the space of subsets  $S \subset \mathbb{P}^1(\mathbb{C})$  of cardinality  $|S| = 5$  stable under complex conjugation modulo real projective transformations, and in  $\mathcal{M}_s(\mathbb{R})$  one or two pairs of points are allowed to collapse. For  $i = 0, 1, 2$ , we define  $\mathcal{M}_i$  to be the connected component of  $\mathcal{M}_0(\mathbb{R})$  parametrizing  $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -stable subsets  $S \subset \mathbb{P}^1(\mathbb{C})$  with  $2i$  complex and  $5 - 2i$  real points.

There is a natural period map that defines an isomorphism of analytic spaces

$$\mathcal{M}_s(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \setminus X_s(\mathbb{C}) \xrightarrow{\sim} P\Gamma \setminus \mathbb{C}H^2$$

for a certain arithmetic ball quotient  $P\Gamma \setminus \mathbb{C}H^2$  (see Theorem 2.10), and strictly stable quintics correspond to points in a hyperplane arrangement  $\mathcal{H} \subset \mathbb{C}H^2$  (see Proposition 2.11). By investigating the equivariance of this period map with respect to suitable anti-holomorphic involutions  $\alpha_i: \mathbb{C}H^2 \rightarrow \mathbb{C}H^2$ , we obtain the following real analogue:

**Theorem 1.1.** *For each  $i \in \{0, 1, 2\}$ , the period map induces an isomorphism of real analytic orbifolds*

$$\mathcal{M}_i \cong P\Gamma_i \setminus (\mathbb{R}H^2 - \mathcal{H}_i). \tag{1}$$

Here  $\mathbb{R}H^2$  is the real hyperbolic plane,  $\mathcal{H}_i$  a union of geodesic subspaces in  $\mathbb{R}H^2$  and  $P\Gamma_i$  an arithmetic lattice in  $\mathrm{PO}(2, 1)$ . Moreover, the  $P\Gamma_i$  are projective orthogonal groups attached to explicit quadratic forms over  $\mathbb{Z}[\zeta_5 + \zeta_5^{-1}]$ , see Equation (28).

In particular, Theorem 1.1 endows each component  $\mathcal{M}_i \subset \mathcal{M}_0(\mathbb{R})$  with a hyperbolic metric. Since one can deform the topological type of a  $\text{Gal}(\mathbb{C}/\mathbb{R})$ -stable five-element subset of  $\mathbb{P}^1(\mathbb{C})$  by allowing two points to collide, the compactification  $\mathcal{M}_s(\mathbb{R}) \supset \mathcal{M}_0(\mathbb{R})$  is connected. One may wonder whether the metrics on the components  $\mathcal{M}_i$  extend to a metric on the whole of  $\mathcal{M}_s(\mathbb{R})$ . If so, what does the resulting space look like at the boundary? Our main result answers these questions in the following way.

**Theorem 1.2.** *There is a complete hyperbolic metric on  $\mathcal{M}_s(\mathbb{R})$  restricting to the metrics on  $\mathcal{M}_i$  induced by (1). Let  $\overline{\mathcal{M}}_{\mathbb{R}}$  denote the resulting metric space, and define*

$$\Gamma_{3,5,10} = \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_i^2 = (\alpha_1 \alpha_2)^3 = (\alpha_1 \alpha_3)^5 = (\alpha_2 \alpha_3)^{10} = 1 \rangle. \quad (2)$$

*Then there exist open embeddings  $P\Gamma_i \setminus (\mathbb{R}H^2 - \mathcal{H}_i) \hookrightarrow \Gamma_{3,5,10} \setminus \mathbb{R}H^2$  and an isometry*

$$\overline{\mathcal{M}}_{\mathbb{R}} \cong \Gamma_{3,5,10} \setminus \mathbb{R}H^2 \quad (3)$$

*extending the orbifold isomorphisms (1) in Theorem 1.1. In particular,  $\overline{\mathcal{M}}_{\mathbb{R}}$  is isometric to the hyperbolic triangle  $\Delta_{3,5,10}$  of angles  $\pi/3, \pi/5, \pi/10$ , see Figure 1 below.*

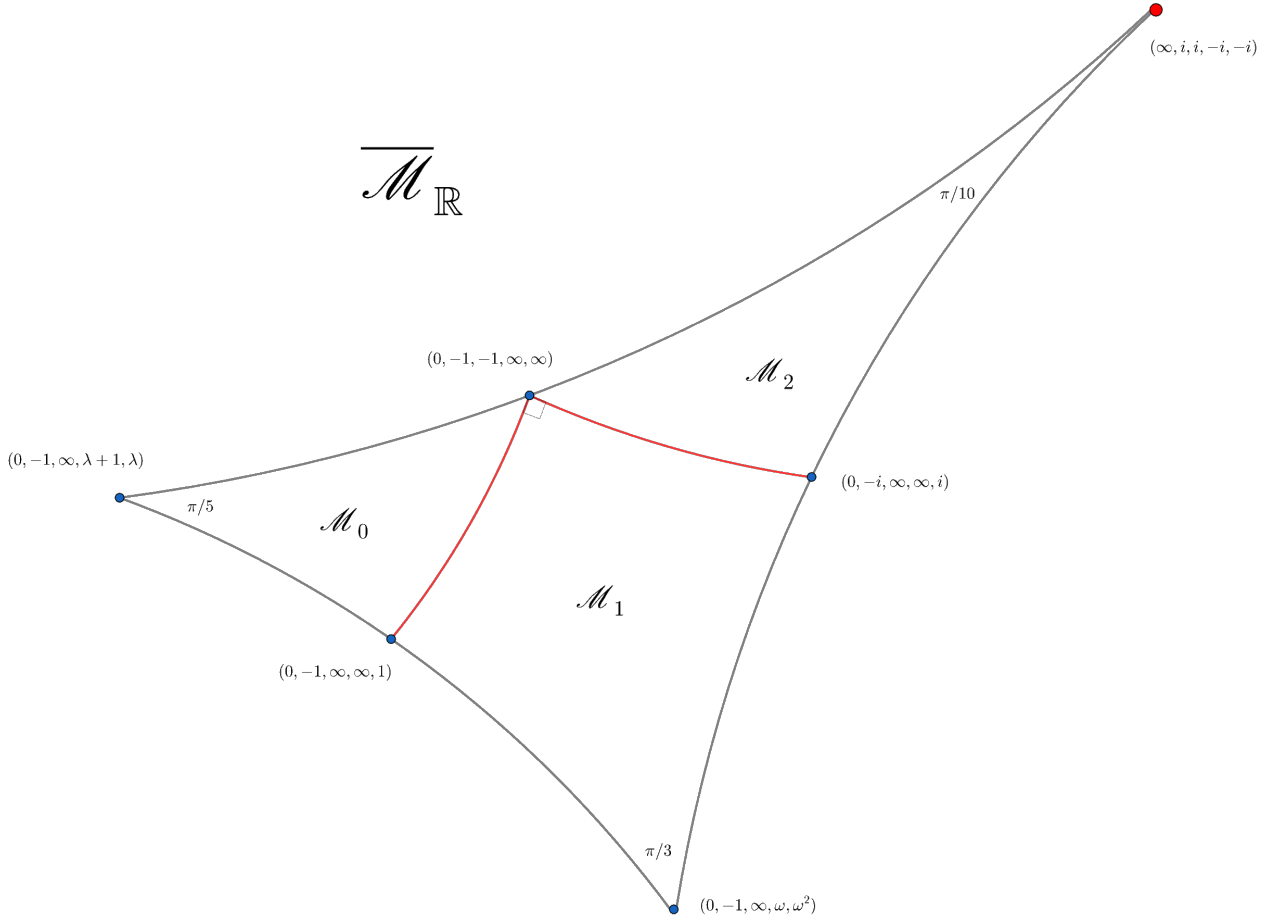


Figure 1:  $\overline{\mathcal{M}}_{\mathbb{R}}$  as the hyperbolic triangle  $\Delta_{3,5,10} \subset \mathbb{R}H^2$ . Here  $\lambda = \zeta_5 + \zeta_5^{-1}$  and  $\omega = \zeta_3$ .


Equivariant period maps arise often in real algebraic geometry as a method to obtain real uniformization of the connected components of the moduli space of smooth varieties. For instance, this works for abelian varieties [GH81], algebraic curves [SS89], K3 surfaces [Nik79] and quartic curves [HR18]. Only recently, Allcock, Carlson and Toledo have shown that in the cases of cubic surfaces [ACT10] and binary sextics [ACT06; ACT07], the real ball quotient components can be glued along the hyperplane arrangement in order to uniformize the moduli space of real stable varieties. Binary quintics provide the first new example of this phenomenon.

**Remarks 1.3.** 1. The lattice  $\Gamma_{3,5,10} \subset \mathrm{PO}(2,1)$  is *non-arithmetic*, see [Tak77].

2. The isometry (3) in Theorem 1.2 seems to provide  $\overline{\mathcal{M}}_{\mathbb{R}}$  with a hyperbolic orbifold structure. The proof of Theorem 1.2 actually goes in the other direction: we use the theory of glueing real hyperbolic orbifolds developed in [GF23] to show that the pieces on the right hand side of (1) into a complete real hyperbolic orbifold  $P\Gamma_{\mathbb{R}} \backslash \mathbb{R}H^2$ . After that, we prove that the period maps (1) glue into an isomorphism  $\overline{\mathcal{M}}_{\mathbb{R}} \cong P\Gamma_{\mathbb{R}} \backslash \mathbb{R}H^2$ , and finally, to finish the proof of Theorem 1.2, we show that  $P\Gamma_{\mathbb{R}} \cong \Gamma_{3,5,10}$ .

3. The topological space  $\mathcal{M}_s(\mathbb{R})$  underlies two orbifold structures: the natural orbifold structure of  $\mathrm{GL}_2(\mathbb{R}) \backslash X_s(\mathbb{R})$  and the structure on  $\overline{\mathcal{M}}_{\mathbb{R}}$  induced by (3). These structures only differ at one point of  $\mathcal{M}_s(\mathbb{R})$ , which is the point  $(\infty, i, i, -i, -i)$  (see Figure 1).

4. Important ingredients in the proof of Theorem 1.2 are [GF23, Theorem 3.1] and the fact that  $\mathcal{H} \subset \mathbb{C}H^2$  is an *orthogonal arrangement* in the sense of [ACT02a]. The latter holds by [GF23, Theorem 4.12 & Proposition 4.14]. Another implication of the orthogonality of  $\mathcal{H} \subset \mathbb{C}H^2$  is that neither  $\pi_1(P_0/\mathfrak{S}_5)$  nor  $\pi_1^{\mathrm{orb}}(\mathcal{M}_{\mathbb{C}})$  is a lattice in any Lie group with finitely many connected components, see [ACT02a, Theorem 1.2].

1.1 **Acknowledgements.** This project was carried out during my PhD at the ENS in Paris. I thank my advisor Olivier Benoist for his great guidance and support. Moreover, I thank Romain Branchereau, Samuel Bronstein, Nicolas Tholozan and Frans Oort for useful discussions, and I thank Nicolas Bergeron for pointing me to Takeuchi’s paper on hyperbolic triangle groups which led to Remark 1.3.1. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement N<sup>o</sup> 754362. 

## 2 Moduli of complex binary quintics

Recall from the introduction that  $X \cong \mathbb{A}_{\mathbb{R}}^6$  is the real affine space of homogeneous degree 5 polynomials  $F \in \mathbb{R}[x, y]$ ,  $X_0$  the subvariety of polynomials with distinct roots, and  $X_s \subset X$  the subvariety of polynomials with roots of multiplicity at most two, i.e. non-zero polynomials whose class in the associated projective space is stable in the sense of geometric invariant theory [MFK94] for the action of  $\mathrm{SL}_{2,\mathbb{R}}$  on it.

**Notation 2.1.** Let  $K$  be the cyclotomic field  $\mathbb{Q}(\zeta)$ , with  $\zeta = \zeta_5 = e^{2\pi i/5} \in \mathbb{C}$ . The ring of integers  $\mathcal{O}_K$  of  $K$  is  $\mathbb{Z}[\zeta]$  [Neu99, Chapter I, Proposition 10.2]. Let  $\mu_K \subset \mathcal{O}_K^*$  be the group of finite units in  $\mathcal{O}_K$ . Thus,  $\mu_K$  is cyclic of order ten, generated by  $-\zeta$ .

The goal of Section 2 is to prove that there exists a hermitian  $\mathcal{O}_K$ -lattice  $\Lambda$  of rank three, such that, if  $\Gamma = \text{Aut}(\Lambda)$ ,  $\mathbb{G}(\mathbb{C}) = \text{GL}_2(\mathbb{C})/D$ , and  $\mathcal{H} \subset \mathbb{C}H^n$  is the hyperplane arrangement defined by the norm one vectors in  $\Lambda$ , then there is an isomorphism of complex analytic spaces  $\mathcal{M}_s(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \backslash X_s(\mathbb{C}) \cong P\Gamma \backslash \mathbb{C}H^2$  restricting to an orbifold isomorphism  $\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \backslash X_0(\mathbb{C}) \cong P\Gamma \backslash (\mathbb{C}H^2 - \mathcal{H})$ .

## 2.1 The Jacobian of a quintic cover of the projective line. We begin with:

**Lemma 2.2.** *Let  $Z \subset \mathbb{P}_{\mathbb{C}}^1$  be a smooth quintic hypersurface. Let  $\mathbb{P}_{\mathbb{C}}^2 \supset C \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the quintic cover of  $\mathbb{P}^1$  ramified along  $Z$ . Then  $C$  has the following refined Hodge numbers:*

$$\begin{aligned} h^{1,0}(C)_{\zeta} &= 3, & h^{1,0}(C)_{\zeta^2} &= 2, & h^{1,0}(C)_{\zeta^3} &= 1, & h^{1,0}(C)_{\zeta^4} &= 0 \\ h^{0,1}(C)_{\zeta} &= 0, & h^{0,1}(C)_{\zeta^2} &= 1, & h^{0,1}(C)_{\zeta^3} &= 2, & h^{0,1}(C)_{\zeta^4} &= 3. \end{aligned}$$

*Proof.* This follows from the Hurwitz-Chevalley-Weil formula, see [MO13, Proposition 5.9]. Alternatively, see [CT99, Section 5].  $\square$

Fix a point  $F_0 \in X_0(\mathbb{C})$  and let

$$C = \{z^5 = F_0(x, y)\} \subset \mathbb{P}_{\mathbb{C}}^2 \quad (4)$$

be the corresponding cyclic cover of  $\mathbb{P}_{\mathbb{C}}^1$ . Let

$$\left( A = J(C) = \text{Pic}^0(C), \quad \lambda: A \rightarrow \widehat{A}, \quad \iota: \mathcal{O}_K = \mathbb{Z}[\zeta] \rightarrow \text{End}(A) \right)$$

be the Jacobian of  $C$ , viewed as a principally polarized abelian variety of dimension six equipped with an  $\mathcal{O}_K$ -action compatible with the polarization, see [GF23, Conditions 4.5]. Write  $\Lambda = H_1(A(\mathbb{C}), \mathbb{Z})$ . We have  $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = H^{-1,0} \oplus H^{0,-1}$ , the Hodge decomposition of  $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ . Define a CM-type  $\Psi \subset \text{Hom}(K, \mathbb{C})$  as follows:

$$\tau_i: K \rightarrow \mathbb{C}, \quad \tau_1(\zeta) = \zeta^3, \quad \tau_2(\zeta) = \zeta^4; \quad \Psi = \{\tau_1, \tau_2\}. \quad (5)$$

Since  $H^{-1,0} = \text{Lie}(A) = H^1(C, \mathcal{O}_C) = H^{0,1}(C)$ , Lemma 2.2 implies that

$$\dim_{\mathbb{C}} H_{\tau_1}^{-1,0} = 2, \quad \dim_{\mathbb{C}} H_{\tau_1\sigma}^{-1,0} = 1, \quad \dim_{\mathbb{C}} H_{\tau_2}^{-1,0} = 3, \quad \dim_{\mathbb{C}} H_{\tau_2\sigma}^{-1,0} = 0. \quad (6)$$

Define  $\eta = 5/(\zeta - \zeta^{-1})$ . Then  $\mathfrak{D}_K = (\eta)$ , see [GF23, Proposition 4.14]. Let  $E: \Lambda \times \Lambda \rightarrow \mathbb{Z}$  be the alternating form corresponding to the polarization  $\lambda$  of the abelian variety  $A$ . For  $a \in \mathcal{O}_K$  and  $x, y \in \Lambda$ , we have  $E(\iota(a)x, y) = E(x, \iota(a^\sigma)y)$ . Define

$$T: \Lambda \times \Lambda \rightarrow \mathfrak{D}_K^{-1}, \quad T(x, y) = \frac{1}{5} \sum_{j=0}^4 \zeta^j E(x, \iota(\zeta)^j y).$$

By [GF23, Example 4.2.2], this is the skew-hermitian form corresponding to  $E$  via [GF23, Lemma 4.1]. We obtain a hermitian form on the free  $\mathcal{O}_K$ -module  $\Lambda$  as follows:

$$\mathfrak{h} : \Lambda \times \Lambda \rightarrow \mathcal{O}_K, \quad \mathfrak{h}(x, y) = \eta T(x, y) = (\zeta - \zeta^{-1})^{-1} \sum_{j=0}^4 \zeta^j E(x, \iota(\zeta)^j y). \quad (7)$$

By Lemma [GF23, Lemma 4.1], the hermitian lattice  $(\Lambda, \mathfrak{h})$  is unimodular, because  $(\Lambda, E)$  is unimodular. For each embedding  $\varphi : K \rightarrow \mathbb{C}$ , the restriction of the hermitian form  $\varphi(\eta) \cdot E_{\mathbb{C}}(x, \bar{y})$  on  $\Lambda_{\mathbb{C}}$  to  $(\Lambda_{\mathbb{C}})_{\varphi} \subset \Lambda_{\mathbb{C}}$  coincides with  $\mathfrak{h}^{\varphi}$  by [GF23, Lemma 4.3]. Since  $\Im(\tau_i(\zeta - \zeta^{-1})) < 0$  for  $i = 1, 2$ , the signature of  $\mathfrak{h}^{\tau_i}$  on  $V_i = \Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C}$  is

$$\text{sign}(\mathfrak{h}^{\tau_i}) = \begin{cases} (h_{\tau_1}^{-1,0}, h_{\tau_1}^{0,-1}) = (2, 1) & \text{for } i = 1, \quad \text{and} \\ (h_{\tau_2}^{-1,0}, h_{\tau_2}^{0,-1}) = (3, 0) & \text{for } i = 2. \end{cases} \quad (8)$$

**2.2 The monodromy representation.** Consider the real algebraic variety  $X_0$  introduced in Section 1. Let  $D \subset \text{GL}_2(\mathbb{C})$  be the group  $D = \{\zeta^i \cdot I_2\} \subset \text{GL}_2(\mathbb{C})$  of scalar matrices  $\zeta^i \cdot I_2$ , where  $I_2 \in \text{GL}_2(\mathbb{C})$  is the identity matrix of rank two, and define

$$\mathbb{G}(\mathbb{C}) = \text{GL}_2(\mathbb{C})/D. \quad (9)$$

The group  $\mathbb{G}(\mathbb{C})$  acts from the left on  $X_0(\mathbb{C})$  in the following way: if  $F(x, y) \in \mathbb{C}[x, y]$  is a binary quintic, we may view  $F$  as a function  $\mathbb{C}^2 \rightarrow \mathbb{C}$ , and define  $g \cdot F = F(g^{-1})$  for  $g \in \mathbb{G}(\mathbb{C})$ . This gives a canonical isomorphism of complex analytic orbifolds

$$\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \backslash X_0(\mathbb{C}),$$

where  $\mathcal{M}_0$  is the moduli stack of smooth binary quintics.

Consider two families

$$\pi : \mathcal{C} \rightarrow X_0 \quad \text{and} \quad \phi : J \rightarrow X_0$$

defined as follows. We define  $\pi$  as the universal family of cyclic covers  $C \rightarrow \mathbb{P}^1$  ramified along a smooth binary quintic  $\{F = 0\} \subset \mathbb{P}^1$ . We let  $\phi$  be the relative Jacobian of  $\pi$ . By Section 2.1,  $\phi$  is an abelian scheme of relative dimension six over  $X_0$ , with  $\mathcal{O}_K$ -action of signature  $\{(2, 1), (3, 0)\}$  with respect to  $\Psi = \{\tau_1, \tau_2\}$ .

Let  $\mathbb{V} = R^1\pi_*\mathbb{Z}$  be the local system of hermitian  $\mathcal{O}_K$ -modules underlying the abelian scheme  $J/X_0$ . Attached to  $\mathbb{V}$ , we have a representation

$$\rho' : \pi_1(X_0(\mathbb{C}), F_0) \rightarrow \Gamma = \text{Aut}_{\mathcal{O}_K}(\Lambda, \mathfrak{h}),$$

whose composition with the quotient map  $\Gamma \rightarrow P\Gamma = \Gamma/\mu_K$  defines a homomorphism

$$\rho : \pi_1(X_0(\mathbb{C}), F_0) \rightarrow P\Gamma. \quad (10)$$

We shall see that  $\rho$  is surjective, see Corollary 2.4 below.

2.3 **Marked binary quintics.** For  $F \in X_0(\mathbb{C})$ , define  $Z_F$  as the hypersurface

$$Z_F = \{F = 0\} \subset \mathbb{P}_{\mathbb{C}}^1.$$

A *marking* of  $F$  is a ring isomorphism  $m : H^0(Z_F(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^5$ . To give a marking is to give a labelling of the points  $p \in Z_F(\mathbb{C})$ . Let  $\mathcal{N}_0$  be the space of marked binary quintics  $(F, m)$ ; this is a manifold, equipped with a holomorphic map

$$\mathcal{N}_0 \rightarrow X_0(\mathbb{C}). \quad (11)$$

Let  $\psi : \mathcal{Z} \rightarrow X_0(\mathbb{C})$  be the universal complex binary quintic, and consider the local system  $H = \psi_* \mathbb{Z}$  of stalk  $H_F = H^0(Z_F(\mathbb{C}), \mathbb{Z})$  for  $F \in X_0(\mathbb{C})$ . Then  $H$  corresponds to a monodromy representation

$$\tau : \pi_1(X_0(\mathbb{C}), F_0) \rightarrow \mathfrak{S}_5. \quad (12)$$

It can be shown that  $\tau$  is surjective using the results of [Bea86]. This implies that (11) is covering space, i.e. that  $\mathcal{N}_0$  is connected.

By choosing a marking  $m_0 : H^0(Z_{F_0}(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^5$  lying over our base point  $F_0 \in X_0(\mathbb{C})$  we obtain an embedding  $\pi_1(\mathcal{N}_0, m_0) \hookrightarrow \pi_1(X_0(\mathbb{C}), F_0)$  whose composition with the map  $\rho$  in (10) defines a homomorphism

$$\mu : \pi_1(\mathcal{N}_0, m_0) \rightarrow P\Gamma. \quad (13)$$

Define  $\theta = \zeta - \zeta^{-1}$  and consider the three-dimensional  $\mathbb{F}_5$  vector space  $\Lambda/\theta\Lambda$  and the quadratic space  $W := (\Lambda/\theta\Lambda, q)$ , where  $q$  is the quadratic form obtained by reducing  $\mathfrak{h}$  modulo  $\theta\Lambda$ . Define two groups  $\Gamma_\theta$  and  $P\Gamma_\theta$  as follows:

$$\Gamma_\theta = \text{Ker}(\Gamma \rightarrow \text{Aut}(W)), \quad P\Gamma_\theta = \text{Ker}(P\Gamma \rightarrow P\text{Aut}(W)) \subset \text{PU}(2, 1).$$

Remark that the composition  $\mathcal{N}_0 \rightarrow X_0(\mathbb{C}) \rightarrow X_s(\mathbb{C})$  admits an essentially unique *completion*  $\mathcal{N}_s \rightarrow X_s(\mathbb{C})$ , see [Fox57] or [DM86, §8]. Here  $\mathcal{N}_s$  a manifold and  $\mathcal{N}_s \rightarrow X_s(\mathbb{C})$  is a ramified covering space.

**Proposition 2.3.** *The image of  $\mu$  in (13) is the group  $P\Gamma_\theta$ , and the induced homomorphism  $\pi_1(X_0(\mathbb{C}), F_0)/\pi_1(\mathcal{N}_0, m_0) = \mathfrak{S}_5 \rightarrow P\Gamma/P\Gamma_\theta$  is an isomorphism. In other words, we obtain the following commutative diagram with exact rows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(\mathcal{N}_0, m_0) & \longrightarrow & \pi_1(X_0(\mathbb{C}), F_0) & \xrightarrow{\tau} & \mathfrak{S}_5 & \longrightarrow & 0 \\ & & \downarrow \mu & & \downarrow \rho & & \downarrow \gamma & & \\ 0 & \longrightarrow & P\Gamma_\theta & \longrightarrow & P\Gamma & \longrightarrow & P\text{Aut}(W) & \longrightarrow & 0. \end{array} \quad (14)$$

*Proof.* Consider the quotient

$$Q = \mathbb{G}(\mathbb{C}) \backslash \mathcal{N}_0 = \text{PGL}_2(\mathbb{C}) \backslash P_0,$$

where  $P_0 \subset \mathbb{P}^1(\mathbb{C})^5$  is the subvariety of distinct five-tuples, see Section 1. Let  $0 \in Q$  be the image of  $m_0 \in \mathcal{N}_0$ . In [DM86], Deligne and Mostow define a hermitian space

bundle  $B_Q \rightarrow Q$  over  $Q$  whose fiber over  $0 \in Q$  is  $\mathbb{C}H^2$ . Writing  $V_1 = \Lambda \otimes_{\mathcal{O}_{K,\tau_1}} \mathbb{C}$ , this gives a monodromy representation

$$\pi_1(Q, 0) \rightarrow \mathrm{PU}(V_1, \mathfrak{h}^{\tau_1}) \cong \mathrm{PU}(2, 1)$$

whose image we denote by  $\Gamma_{\mathrm{DM}}$ . Kondō has shown that in fact,  $\Gamma_{\mathrm{DM}} = P\Gamma_\theta$  [Kon07, Theorem 7.1]. Since  $\mathcal{N}_0 \rightarrow Q$  is a covering space (the action of  $\mathbb{G}(\mathbb{C})$  on  $\mathcal{N}_0$  being free) we have an embedding  $\pi_1(\mathcal{N}_0, m_0) \hookrightarrow \pi_1(Q, 0)$  whose composition with  $\pi_1(Q, 0) \rightarrow \mathrm{PU}(2, 1)$  is the map  $\mu : \pi_1(\mathcal{N}_0, m_0) \rightarrow P\Gamma \subset \mathrm{PU}(2, 1)$ .

To prove that the image of  $\mu$  is  $P\Gamma_\theta$ , it suffices to give a section of the map  $\mathcal{N}_0 \rightarrow Q$ . Indeed, such a section induces a retraction of  $\pi_1(\mathcal{N}_0, m_0) \hookrightarrow \pi_1(Q, 0)$ , so that the images of these two groups in  $\mathrm{PU}(2, 1)$  are the same. Observe that if  $\Delta \subset \mathbb{P}^1(\mathbb{C})^5$  is the union of all hyperplanes  $\{x_i = x_j\} \subset \mathbb{P}^1(\mathbb{C})^5$  for  $i \neq j$ , then

$$\begin{aligned} Q &= \mathrm{PGL}_2(\mathbb{C}) \setminus P_0 = \mathrm{PGL}_2(\mathbb{C}) \setminus (\mathbb{P}^1(\mathbb{C})^5 - \Delta) \\ &\cong \{(u_1, u_2) \in \mathbb{C}^2 : u_i \neq 0, 1 \text{ and } u_1 \neq u_2\}. \end{aligned}$$

The section  $Q \rightarrow \mathcal{N}_0$  may then be defined by sending  $(u_1, u_2)$  to the binary quintic

$$F(x, y) = x(x - y)y(x - u_1 \cdot y)(x - u_2 \cdot y) \in X_0(\mathbb{C}),$$

marked by the labelling of its roots  $\{[0 : 1], [1 : 1], [1 : 0], [u_1 : 1], [u_2 : 1]\} \subset \mathbb{P}^1(\mathbb{C})$ .

It remains to prove that the homomorphism  $\gamma : \mathfrak{S}_5 \rightarrow P\Gamma/P\Gamma_\theta$  appearing on the right in (14) is an isomorphism. We use Theorem 4.1, proven by Shimura in [Shi64], which says that

$$(\Lambda, \mathfrak{h}) \cong \left( \mathcal{O}_K^3, \mathrm{diag} \left( \frac{1 - \sqrt{5}}{2}, 1, 1 \right) \right).$$

It follows that  $P\Gamma/P\Gamma_\theta = P\mathrm{Aut}(W) \cong \mathrm{PO}_3(\mathbb{F}_5) \cong \mathfrak{S}_5$ . Next, consider the manifold  $\mathcal{N}_s$ . Remark that  $\mathfrak{S}_5$  embeds into  $\mathrm{Aut}(\mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s)$ . Moreover, there is a natural isomorphism

$$p : \mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s \cong P\Gamma_\theta \setminus \mathbb{C}H^2 \quad (\text{see [DM86; Kon07]}).$$

Compare also Equation (16) below. The two compositions

$$\mathfrak{S}_5 \subset \mathrm{Aut}(\mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s) \cong \mathrm{Aut}(P\Gamma_\theta \setminus \mathbb{C}H^2) \quad \text{and} \quad \mathfrak{S}_5 \rightarrow P\Gamma/P\Gamma_\theta \subset \mathrm{Aut}(P\Gamma_\theta \setminus \mathbb{C}H^2)$$

agree, because of the equivariance of  $p$  with respect to  $\gamma$ . Thus,  $\gamma$  is injective.  $\square$

**Corollary 2.4.** *The monodromy representation  $\rho$  in (10) is surjective.*  $\square$

**2.4 Framed binary quintics.** By a *framing* of a point  $F \in X_0(\mathbb{C})$  we mean a projective equivalence class  $[f]$ , where

$$f : \mathbb{V}_F = H^1(C_F(\mathbb{C}), \mathbb{Z}) \rightarrow \Lambda$$

is an  $\mathcal{O}_K$ -linear isometry: two such isometries are in the same class if and only if they differ by an element in  $\mu_K$ . Let  $\mathcal{F}_0$  be the collection of all framings of all points



$x \in X_0(\mathbb{C})$ . The set  $\mathcal{F}_0$  is naturally a complex manifold, by arguments similar to those used in [ACT02b]. Note that Corollary 2.4 implies that  $\mathcal{F}_0$  is connected, hence

$$\mathcal{F}_0 \rightarrow X_0(\mathbb{C}) \quad (15)$$

is a covering, with Galois group  $P\Gamma$ .

**Lemma 2.5.** *The spaces  $P\Gamma_\theta \backslash \mathcal{F}_0$  and  $\mathcal{N}_0$  are isomorphic as covering spaces of  $X_0(\mathbb{C})$ . In particular, there is a covering map  $\mathcal{F}_0 \rightarrow \mathcal{N}_0$  with Galois group  $P\Gamma_\theta$ .*

*Proof.* We have  $P\Gamma/P\Gamma_\theta \cong \mathfrak{S}_5$  as quotients of  $P\Gamma$ , see Proposition 2.3.  $\square$

**Lemma 2.6.**  $\Delta := X_s(\mathbb{C}) - X_0(\mathbb{C})$  is an irreducible normal crossings divisor of  $X_s(\mathbb{C})$ .

*Proof.* The proof is similar to the proof of Proposition 6.7 in [Bea09].  $\square$

**Lemma 2.7.** *The local monodromy transformations of  $\mathcal{F}_0 \rightarrow X_0(\mathbb{C})$  around every  $x \in \Delta$  are of finite order. More precisely, if  $x \in \Delta$  lies on the intersection of  $k$  local components of  $\Delta$ , then the local monodromy group around  $x$  is isomorphic to  $(\mathbb{Z}/10)^k$ .*

*Proof.* See [DM86, Proposition 9.2] or [CT99, Proposition 6.1] for the generic case, i.e. when a quintic  $Z = \{F = 0\} \subset \mathbb{P}_{\mathbb{C}}^1$  acquires only one node. In this case, the local equation of the singularity is  $x^2 = 0$ , hence the curve  $C_F$  acquires a singularity of the form  $y^5 + x^2 = 0$ . If the quintic acquires two nodes, then  $C_F$  acquires two such singularities; the vanishing cohomology splits as an orthogonal direct sum, hence the local monodromy transformations commute.  $\square$

In the following corollary, we let  $D = \{z \in \mathbb{C} \mid |z| < 1\}$  denote the open unit disc, and  $D^* = D - \{0\}$  the punctured open unit disc.

**Corollary 2.8.** *There is an essentially unique branched cover  $\pi : \mathcal{F}_s \rightarrow X_s(\mathbb{C})$ , with  $\mathcal{F}_s$  a complex manifold, such that for any  $x \in \Delta$ , any open  $x \in U \subset X_s(\mathbb{C})$  with  $U \cong D^k \times D^{6-k}$  and  $U \cap X_0(\mathbb{C}) \cong (D^*)^k \times D^{6-k}$ , and any connected component  $U'$  of  $\pi^{-1}(U) \subset \mathcal{F}_s$ , there is an isomorphism  $U' \cong D^k \times D^{6-k}$  such that the composition*

$$D^k \times D^{6-k} \cong U' \rightarrow U \cong D^6 \text{ is the map } (z_1, \dots, z_6) \mapsto (z_1^{10}, \dots, z_k^{10}, z_{k+1}, \dots, z_6).$$

*Proof.* See [Bea09, Lemma 7.2]. See also [Fox57] and [DM86, Section 8].  $\square$

The group  $\mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$  (see (9)) acts on  $\mathcal{F}_0$  over its action on  $X_0$ . Explicitly, if  $g \in \mathbb{G}(\mathbb{C})$  and if  $([\phi], \phi : \mathbb{V}_F \cong \Lambda)$  is a framing of  $F \in X_0(\mathbb{C})$ , then

$$([\phi \circ g^*], \phi \circ g^* : \mathbb{V}_{g \cdot F} \rightarrow \Lambda)$$

is a framing of  $g \cdot F \in X_0(\mathbb{C})$ . This is a left action. The group  $P\Gamma$  also acts on  $\mathcal{F}_0$  from the left, and the actions of  $P\Gamma$  and  $\mathbb{G}(\mathbb{C})$  on  $\mathcal{F}_0$  commute. By functoriality of the Fox completion, the action of  $\mathbb{G}(\mathbb{C})$  on  $\mathcal{F}_0$  extends to an action of  $\mathbb{G}(\mathbb{C})$  on  $\mathcal{F}_s$ .

**Lemma 2.9.** *The group  $\mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$  acts freely on  $\mathcal{F}_s$ .*

*Proof.* The functoriality of the Fox completion gives an action of  $\mathbb{G}(\mathbb{C})$  on  $\mathcal{N}_s$  such that, by Lemma 2.5, there is a  $\mathbb{G}(\mathbb{C})$ -equivariant commutative diagram

$$\begin{array}{ccc} P\Gamma_\theta \backslash \mathcal{F}_s & \xrightarrow{\sim} & \mathcal{N}_s \\ & \searrow & \swarrow \\ & X_s(\mathbb{C}) & \end{array}$$

In particular, it suffices to show that  $\mathbb{G}(\mathbb{C})$  acts freely on  $\mathcal{N}_s$ . Note that  $\mathcal{N}_0$  admits a natural  $\mathbb{G}_m$ -covering map  $\mathcal{N}_0 \rightarrow P_0$  where  $P_0 \subset \mathbb{P}^1(\mathbb{C})^5$  is the space of distinct ordered five-tuples in  $\mathbb{P}^1(\mathbb{C})$  introduced in Section 1. Consequently, there is a  $\mathbb{G}_m$ -quotient map  $\mathcal{N}_s \rightarrow P_s$ , where  $P_s$  is the space of stable ordered five-tuples, and this map is equivariant for the homomorphism  $\mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{PGL}_2(\mathbb{C})$ .

Let  $g \in \mathrm{GL}_2(\mathbb{C})$  and  $x \in \mathcal{N}_s$  such that  $gx = x$ . It is clear that  $\mathrm{PGL}_2(\mathbb{C})$  acts freely on  $P_s$ . Therefore,  $g = \lambda \in \mathbb{C}^*$ . Let  $F \in X_s(\mathbb{C})$  be the image of  $x \in \mathcal{N}_s$ ; then

$$gF(x, y) = F(g^{-1}(x, y)) = F(\lambda^{-1}x, \lambda^{-1}y) = \lambda^{-5}F(x, y).$$

The equality  $gF = F$  implies that  $\lambda^5 = 1 \in \mathbb{C}$ , from which we conclude that  $\lambda \in \langle \zeta \rangle$ . Therefore,  $[g] = [\mathrm{id}] \in \mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$ .  $\square$

**2.5 Complex uniformization.** Consider the hermitian space  $V_1 = \Lambda \otimes_{\mathcal{O}_K, \tau_1} \mathbb{C}$  and define  $\mathbb{C}H^2$  to be the space of negative lines in  $V_1$ . Using [GF23, Proposition 4.7] we see that the abelian scheme  $J \rightarrow X_0$  induces a  $\mathbb{G}(\mathbb{C})$ -equivariant morphism  $\mathcal{P} : \mathcal{F}_0 \rightarrow \mathbb{C}H^2$ . Explicitly, if  $(F, [f]) \in \mathcal{F}_0$  is the framing  $[f : H^1(C_F(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \Lambda]$  of the binary quintic  $F \in X_0(\mathbb{C})$ , and  $A_F$  is the Jacobian of the curve  $C_F$ , then

$$\mathcal{P} : \mathcal{F}_0 \rightarrow \mathbb{C}H^2, \quad \mathcal{P}(F, [f]) = f(H^{0,-1}(A_F)_{\tau_1}) = f(H^{1,0}(C_F)_{\zeta^3}) \in \mathbb{C}H^2. \quad (16)$$

The map  $\mathcal{P}$  is holomorphic, and descends to a morphism of complex analytic spaces

$$\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \backslash X_0(\mathbb{C}) \rightarrow P\Gamma \backslash \mathbb{C}H^2.$$

By Riemann extension, (16) extends to a  $\mathbb{G}(\mathbb{C})$ -equivariant holomorphic map

$$\overline{\mathcal{P}} : \mathcal{F}_s \rightarrow \mathbb{C}H^2. \quad (17)$$

**Theorem 2.10** (Deligne–Mostow). *The period map (17) induces an isomorphism of complex manifolds*

$$\mathcal{M}_s^f(\mathbb{C}) := \mathbb{G}(\mathbb{C}) \backslash \mathcal{F}_s \cong \mathbb{C}H^2. \quad (18)$$

Taking  $P\Gamma$ -quotients gives an isomorphism of complex analytic spaces

$$\mathcal{M}_s(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \backslash X_s(\mathbb{C}) \cong P\Gamma \backslash \mathbb{C}H^2. \quad (19)$$

*Proof.* In [DM86], Deligne and Mostow define  $\tilde{Q} \rightarrow Q$  to be the covering space corresponding to the monodromy representation  $\pi_1(Q, 0) \rightarrow \mathrm{PU}(2, 1)$ ; since the image of this homomorphism is  $P\Gamma_\theta$  (see the proof of Proposition 2.3), it follows that  $\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_0 \cong \tilde{Q}$  as covering spaces of  $Q$ . Consequently, if  $\tilde{Q}_{\mathrm{st}}$  is the Fox completion of the spread

$$\tilde{Q} \rightarrow Q \rightarrow Q_{\mathrm{st}} := \mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s = \mathrm{PGL}_2(\mathbb{C}) \setminus P_s,$$

then there is an isomorphism  $\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_s \cong \tilde{Q}_{\mathrm{st}}$  of branched covering spaces of  $Q_{\mathrm{st}}$ . We obtain commutative diagrams, where the lower right morphism uses (14):

$$\begin{array}{ccccc} \mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_s & \xrightarrow{\sim} & \tilde{Q}_{\mathrm{st}} & \longrightarrow & \mathbb{C}H^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s & \xrightarrow{\sim} & Q_{\mathrm{st}} & \longrightarrow & P\Gamma_\theta \setminus \mathbb{C}H^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}(\mathbb{C}) \setminus X_s(\mathbb{C}) & \xrightarrow{\sim} & Q_{\mathrm{st}}/\mathfrak{S}_5 & \longrightarrow & P\Gamma \setminus \mathbb{C}H^2. \end{array}$$

The map  $\tilde{Q}_{\mathrm{st}} \rightarrow \mathbb{C}H^2$  is an isomorphism by [DM86, (3.11)]. Therefore, we are done if the composition  $\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_0 \rightarrow \tilde{Q} \rightarrow \mathbb{C}H^2$  agrees with the period map  $\mathcal{P}$  of Equation (16). This follows from [DM86, (2.23) and (12.9)].  $\square$

**Proposition 2.11.** *The isomorphism (19) induces an isomorphism of complex analytic spaces*

$$\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_0(\mathbb{C}) \cong P\Gamma \setminus (\mathbb{C}H^2 - \mathcal{H}).$$

*Proof.* We have  $\overline{\mathcal{P}}(\mathcal{F}_0) \subset \mathbb{C}H^2 - \mathcal{H}$  by [GF23, Proposition 4.10], because the Jacobian of a smooth curve cannot contain an abelian subvariety whose induced polarization is principal. Therefore, we have  $\overline{\mathcal{P}}^{-1}(\mathcal{H}) \subset \mathcal{F}_s - \mathcal{F}_0$ . Since  $\mathcal{F}_s$  is irreducible (it is smooth by Corollary 2.8 and connected by Corollary 2.4), the analytic space  $\overline{\mathcal{P}}^{-1}(\mathcal{H})$  is a divisor. Since  $\mathcal{F}_s - \mathcal{F}_0$  is also a divisor by Corollary 2.8, we have

$$\overline{\mathcal{P}}^{-1}(\mathcal{H}) = \mathcal{F}_s - \mathcal{F}_0$$

and we are done.

Alternatively, let  $H_{0,5}$  be the moduli space of degree 5 covers of  $\mathbb{P}^1$  ramified along five distinct marked points [HM98, §2.G]. The period map

$$H_{0,5}(\mathbb{C}) \rightarrow P\Gamma \setminus \mathbb{C}H^2,$$

that sends the moduli point of a curve  $C \rightarrow \mathbb{P}^1$  to the moduli point of the  $\mathbb{Z}[\zeta]$ -linear Jacobian  $J(C)$ , extends to the stable compactification  $\overline{H}_{0,5}(\mathbb{C}) \supset H_{0,5}(\mathbb{C})$  because the curves in the limit are of compact type. Since the divisor  $\mathcal{H} \subset \mathbb{C}H^2$  parametrizes abelian varieties that are products of lower dimensional ones by [GF23, Proposition 4.10], the image of the boundary is exactly  $P\Gamma \setminus \mathcal{H}$ .  $\square$

### 3 Moduli of real binary quintics

Having understood the period map for complex binary quintics, we turn to the period map for real binary quintics in Section 3.

3.1 **The period map for stable real binary quintics.** Define  $\kappa$  as the anti-holomorphic involution

$$\kappa: X_0(\mathbb{C}) \rightarrow X_0(\mathbb{C}), \quad F(x, y) = \sum_{i+j=5} a_{ij} x^i y^j \mapsto \overline{F(x, y)} = \sum_{i+j=5} \overline{a_{ij}} x^i y^j.$$

Let  $\mathcal{A}$  be the set of anti-unitary involutions  $\alpha: \Lambda \rightarrow \Lambda$ . Define  $P\mathcal{A} = \mu_K \setminus \mathcal{A}$  and  $C\mathcal{A} = P\Gamma \setminus P\mathcal{A}$ , c.f. [GF23, Section 2.1]. For each  $\alpha \in P\mathcal{A}$ , there is a natural anti-holomorphic involution  $\alpha: \mathcal{F}_0 \rightarrow \mathcal{F}_0$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\alpha} & \mathcal{F}_0 \\ \downarrow & & \downarrow \\ X_0(\mathbb{C}) & \xrightarrow{\kappa} & X_0(\mathbb{C}). \end{array}$$

It is defined as follows. Consider a framed binary quintic  $(F, [f]) \in \mathcal{F}_0$ , where  $f: \mathbb{V}_F \rightarrow \Lambda$  is an  $\mathcal{O}_K$ -linear isometry. Let  $C_F \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the quintic cover defined by a smooth binary quintic  $F \in X_0(\mathbb{C})$ . Complex conjugation  $\text{conj}: \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$  induces an anti-holomorphic map

$$\sigma_F: C_F(\mathbb{C}) \rightarrow C_{\kappa(F)}(\mathbb{C}).$$

Consider the pull-back  $\sigma_F^*: \mathbb{V}_{\kappa(F)} \rightarrow \mathbb{V}_F$  of  $\sigma$ . The composition

$$\mathbb{V}_{\kappa(F)} \xrightarrow{\sigma_F^*} \mathbb{V}_F \xrightarrow{f} \Lambda \xrightarrow{\alpha} \Lambda$$

induces a framing of  $\kappa(F) \in X_0(\mathbb{C})$ , and we define

$$\alpha(F, [f]) := (\kappa(F), [\alpha \circ f \circ \sigma_F^*]) \in \mathcal{F}_0.$$

Although we have chosen a representative  $\alpha \in \mathcal{A}$  of the class  $\alpha \in P\mathcal{A}$ , the element  $\alpha(F, [f]) \in \mathcal{F}_0$  does not depend on this choice.

Consider the covering map  $\mathcal{F}_0 \rightarrow X_0(\mathbb{C})$  introduced in (3), and define

$$\mathcal{F}_0(\mathbb{R}) = \bigsqcup_{\alpha \in P\mathcal{A}} \mathcal{F}_0^\alpha \subset \mathcal{F}_0 \tag{20}$$

as the preimage of  $X_0(\mathbb{R})$  in the space  $\mathcal{F}_0$ . To see why the union on the left hand side of (20) is disjoint, observe that

$$\mathcal{F}_0^\alpha = \{(F, [f]) \in \mathcal{F}_0 : \kappa(F) = F \text{ and } [f \circ \sigma_F^* \circ f^{-1}] = \alpha\}.$$

Thus, for  $\alpha, \beta \in P\mathcal{A}$  and  $(F, [f]) \in \mathcal{F}_0^\alpha \cap \mathcal{F}_0^\beta$ , we have  $\alpha = [f \circ \sigma \circ f^{-1}] = \beta$ .

**Lemma 3.1.** *The anti-holomorphic involution  $\alpha: \mathcal{F}_0 \rightarrow \mathcal{F}_0$  defined by  $\alpha \in P\mathcal{A}$  makes the period map  $\mathcal{P}: \mathcal{F}_0 \rightarrow \mathbb{C}H^2$  equivariant for the  $\mathbb{G}(\mathbb{C})$ -actions on both sides.*

*Proof.* Indeed, if  $\text{conj}: \mathbb{C} \rightarrow \mathbb{C}$  is complex conjugation, then for any  $F \in X_0(\mathbb{C})$ , the induced map

$$\sigma_F^* \otimes \text{conj}: \mathbb{V}_{\kappa(F)} \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \mathbb{V}_F \otimes_{\mathbb{Z}} \mathbb{C}$$

is anti-linear, preserves the Hodge decompositions [Sil89, Chapter I, Lemma 2.4] as well as the eigenspace decompositions.  $\square$

We obtain a *real period map*

$$\mathcal{P}_{\mathbb{R}}: \mathcal{F}_0(\mathbb{R}) \longleftarrow \coprod_{\alpha \in P\mathcal{A}} \mathcal{F}_0^{\alpha} \longrightarrow \coprod_{\alpha \in P\mathcal{A}} \mathbb{R}H_{\alpha}^2 \longleftarrow \tilde{Y}. \quad (21)$$

Define  $\mathbb{G}(\mathbb{R}) = \text{GL}_2(\mathbb{R})$ . The map (21) is constant on  $\mathbb{G}(\mathbb{R})$ -orbits, since the same is true for the complex period map  $\mathcal{P}: \mathcal{F}_0 \rightarrow \mathbb{C}H^2$ .

By abuse of notation, we write  $\mathbb{R}H_{\alpha}^2 - \mathcal{H} = \mathbb{R}H_{\alpha}^2 - (\mathcal{H} \cap \mathbb{R}H_{\alpha}^2)$  for  $\alpha \in P\mathcal{A}$ .

**Proposition 3.2.** *The real period map (21) descends to a  $P\Gamma$ -equivariant diffeomorphism*

$$\mathcal{M}_0(\mathbb{R})^f := \mathbb{G}(\mathbb{R}) \backslash \mathcal{F}_0(\mathbb{R}) \cong \coprod_{\alpha \in P\mathcal{A}} \mathbb{R}H_{\alpha}^2 - \mathcal{H}. \quad (22)$$

By  $P\Gamma$ -equivariance, the map (22) induces an isomorphism of real-analytic orbifolds

$$\mathcal{P}_{\mathbb{R}}: \mathcal{M}_0(\mathbb{R}) = \mathbb{G}(\mathbb{R}) \backslash X_0(\mathbb{R}) \cong \coprod_{\alpha \in C\mathcal{A}} P\Gamma_{\alpha} \backslash (\mathbb{R}H_{\alpha}^2 - \mathcal{H}). \quad (23)$$

*Proof.* This follows from [ACT10, proof of Theorem 3.3]. It is crucial that the actions of  $G$  and  $P\Gamma$  on  $\mathcal{F}_0$  commute and are free, which is the case, see Corollary 2.9.  $\square$

**3.2 The period map for smooth real binary quintics.** Our next goal will be to prove the real analogue of the isomorphisms (18) and (19) in Theorem 2.10.

Consider the CM-type  $\Psi = \{\tau_1, \tau_2\}$  defined in (5), the hermitian  $\mathcal{O}_K$ -lattice  $(\Lambda, \mathfrak{h})$  defined in (7), and the following sets [GF23, Section 2.1]:

$$\mathcal{H} = \{H_r \subset \mathbb{C}H^2 \mid r \in \mathcal{R}\}, \quad \text{and} \quad \mathcal{H} = \bigcup_{H \in \mathcal{H}} H \subset \mathbb{C}H^2.$$

Here,  $\mathcal{R} \subset \Lambda$  is the set of short roots, i.e. the set of  $r \in \Lambda$  with  $\mathfrak{h}(r, r) = 1$  (see [GF23, Section 2.1]).

**Lemma 3.3.** *The hyperplane arrangement  $\mathcal{H} \subset \mathbb{C}H^2$  satisfies Condition 2.4 in [GF23], that is: any two distinct  $H_1, H_2 \in \mathcal{H}$  either meet orthogonally, or not at all.*

*Proof.* Consider Conditions 4.11 in [GF23]. Condition 4.11.1 in *loc. cit.* holds because  $K$  does not contain proper CM-subfields. By Proposition [GF23, Proposition 4.14], we have that Condition 4.11.2 is satisfied. By Equation (8), Condition 4.11.3 holds. Thus, by [GF23, Theorem 4.12], we obtain the desired result.  $\square$

**Definition 3.4.** 1. For  $k = 1, 2$ , define  $\Delta_k \subset \Delta = X_s(\mathbb{C}) - X_0(\mathbb{C})$  to be the locus of stable binary quintics with exactly  $k$  nodes. Define  $\tilde{\Delta} = \mathcal{F}_s - \mathcal{F}_0$ , and let  $\tilde{\Delta}_k \subset \tilde{\Delta}$  be the inverse image of  $\Delta_k$  in  $\tilde{\Delta}$  under the map  $\tilde{\Delta} \rightarrow \Delta$ .

2. For  $k = 1, 2$ , define  $\mathcal{H}_k \subset \mathcal{H}$  as the set  $\mathcal{H}_k = \{x \in \mathbb{C}H^2 : |\mathcal{H}(x)| = k\}$ . Thus, this is the locus of points in  $\mathcal{H}$  where exactly  $k$  hyperplanes meet.

**Lemma 3.5.** 1. The period map  $\overline{\mathcal{P}}$  of (17) satisfies  $\overline{\mathcal{P}}(\tilde{\Delta}_k) \subset \mathcal{H}_k$ .

2. Let  $f \in \tilde{\Delta}_k$  and  $x = \overline{\mathcal{P}}(f) \in \mathbb{C}H^2$ . Then  $\overline{\mathcal{P}}$  induces a group isomorphism  $P\Gamma_f \cong G(x)$ , where  $G(x) \cong (\mathbb{Z}/10)^k$  is as in [GF23, Definition 2.6].  $\square$

The naturality of the Fox completion implies that for  $\alpha \in P\mathcal{A}$ , the anti-holomorphic involution  $\alpha : \mathcal{F}_0 \rightarrow \mathcal{F}_0$  extends to an anti-holomorphic involution  $\alpha : \mathcal{F}_s \rightarrow \mathcal{F}_s$ .

**Lemma 3.6.** For every  $\alpha \in P\mathcal{A}$ , the restriction of  $\overline{\mathcal{P}} : \mathcal{F}_s \rightarrow \mathbb{C}H^2$  to  $\mathcal{F}_s^\alpha$  defines a diffeomorphism  $\mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s^\alpha \cong \mathbb{R}H_\alpha^2$ .

*Proof.* See [ACT10, Lemma 11.3]. It is essential that  $G$  acts freely on  $\mathcal{F}_s$ , which holds by Corollary 2.9.  $\square$

We arrive at the main theorem of Section 3. Consider the map  $\pi : \mathcal{F}_s \rightarrow X_s(\mathbb{C})$  (see Corollary 2.8) and define

$$\mathcal{F}_s(\mathbb{R}) = \bigcup_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha = \pi^{-1}(X_s(\mathbb{R})).$$

This is not a manifold because of the ramification of  $\pi$ , but a union of embedded submanifolds.

**Theorem 3.7.** There is a smooth map

$$\overline{\mathcal{P}}_{\mathbb{R}} : \prod_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha \rightarrow \prod_{\alpha \in P\mathcal{A}} \mathbb{R}H_\alpha^2 = \tilde{Y} \quad (24)$$

that extends the real period map (21). The map (24) induces the following commutative diagram of topological spaces, in which  $\mathcal{P}_{\mathbb{R}}$  and  $\mathfrak{P}_{\mathbb{R}}$  are homeomorphisms:

$$\begin{array}{ccccc} \prod_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha & \xrightarrow{\overline{\mathcal{P}}_{\mathbb{R}}} & \tilde{Y} = \prod_{\alpha \in P\mathcal{A}} \mathbb{R}H_\alpha^2 & & \\ \downarrow & & \downarrow & & \\ \mathcal{F}_s(\mathbb{R}) & \xrightarrow{\overline{\mathcal{P}}_{\mathbb{R}}} & \tilde{Y} & & \\ \downarrow & & \downarrow & & \\ \mathcal{M}_s(\mathbb{R})^f & \xrightarrow{\mathcal{P}_{\mathbb{R}}} & \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R}) & \xrightarrow{\sim} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_s(\mathbb{R}) & \xrightarrow{\mathfrak{P}_{\mathbb{R}}} & \mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R}) & \xrightarrow{\sim} & P\Gamma \setminus Y. \end{array}$$

*Proof.* The existence of  $\overline{\mathcal{P}}_{\mathbb{R}}$  follows from the compatibility between  $\overline{\mathcal{P}}$  and the involutions  $\alpha \in P\mathcal{A}$ . We first show that the composition

$$\coprod_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha \xrightarrow{\overline{\mathcal{P}}_{\mathbb{R}}} \widetilde{Y} \xrightarrow{p} Y$$

factors through  $\mathcal{F}_s(\mathbb{R})$ . Two elements  $f_\alpha$  and  $g_\beta \in \coprod_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha$  have the same image in  $\mathcal{F}_s(\mathbb{R})$  if and only if  $f = g \in \mathcal{F}_s^\alpha \cap \mathcal{F}_s^\beta$ , in which case  $x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{R}H_\alpha^2 \cap \mathbb{R}H_\beta^2$ . We need to show that  $x_\alpha \sim x_\beta \in \widetilde{Y}$ . Note that  $\alpha\beta \in P\Gamma_f \cong (\mathbb{Z}/10)^k$ , and  $\overline{\mathcal{P}}$  induces an isomorphism  $P\Gamma_f \cong G(x)$  by Lemma 3.5. Hence  $\alpha\beta \in G(x)$  so that  $x_\alpha \sim x_\beta$ .

Let us prove the  $\mathbb{G}(\mathbb{R})$ -equivariance of  $\overline{\mathcal{P}}_{\mathbb{R}}$ . Suppose that  $f \in \mathcal{F}_s^\alpha, g \in \mathcal{F}_s^\beta$  such that  $a \cdot f = g \in \mathcal{F}_s(\mathbb{R})$  for some  $a \in \mathbb{G}(\mathbb{R})$ . Then  $x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{C}H^2$ , so we need to show that  $\alpha\beta \in G(x)$ . The actions of  $\mathbb{G}(\mathbb{C})$  and  $P\Gamma$  on  $\mathbb{C}H^2$  commute, and the same holds for the actions of  $\mathbb{G}(\mathbb{R})$  and  $P\Gamma'$  on  $\mathcal{F}_s^{\mathbb{R}}$ , where  $P\Gamma'$  is as in [GF23, Section 2.1]. It follows that  $\alpha(g) = \alpha(a \cdot f) = a \cdot \alpha(f) = a \cdot f = g$ , hence  $g \in \mathcal{F}_s^\alpha \cap \mathcal{F}_s^\beta$ . This implies in turn that  $\alpha\beta(g) = g$ , hence  $\alpha\beta \in P\Gamma_g \cong G(x)$ , so that indeed,  $x_\alpha \sim x_\beta$ .

To prove that  $\mathcal{P}_{\mathbb{R}}$  is injective, let again  $f_\alpha, g_\beta \in \coprod_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha$  and suppose that these elements have the same image in  $Y$ . This implies that  $x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{R}H_\alpha^2 \cap \mathbb{R}H_\beta^2$ , and that  $\beta = \phi \circ \alpha$  for some  $\phi \in G(x)$ . We have  $\phi \in G(x) \cong P\Gamma_f$  (Lemma 3.5) hence

$$\beta(f) = \phi(\alpha(f)) = \phi(f) = f. \quad (25)$$

Therefore  $f, g \in \mathcal{F}_s^\beta$ ; since  $\overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g)$ , it follows from Lemma 3.6 that there exists  $a \in \mathbb{G}(\mathbb{R})$  such that  $a \cdot f = g$ . This proves injectivity of  $\mathcal{P}_{\mathbb{R}}$ , as desired.

The surjectivity of  $\mathcal{P}_{\mathbb{R}} : \mathbb{G}(\mathbb{R}) \backslash \mathcal{F}_s(\mathbb{R}) \rightarrow Y$  is straightforward, using the surjectivity of  $\overline{\mathcal{P}}_{\mathbb{R}}$ , which follows from Lemma 3.6.

Finally, we claim that  $\mathcal{P}_{\mathbb{R}}$  is open. Let  $U \subset \mathbb{G}(\mathbb{R}) \backslash \mathcal{F}_s^{\mathbb{R}}$  be open. Let  $V$  be the preimage of  $U$  in  $\coprod_{\alpha \in P\mathcal{A}} \mathcal{F}_s^\alpha$ . Then  $V = \overline{\mathcal{P}}_{\mathbb{R}}^{-1}(p^{-1}(\mathcal{P}_{\mathbb{R}}(U)))$ , and hence

$$\overline{\mathcal{P}}_{\mathbb{R}}(V) = p^{-1}(\mathcal{P}_{\mathbb{R}}(U)).$$

The map  $\overline{\mathcal{P}}_{\mathbb{R}}$  is open, being the coproduct of the maps  $\mathcal{F}_s^\alpha \rightarrow \mathbb{R}H_\alpha^2$ , which are open since they have surjective differential at each point. Thus  $\mathcal{P}_{\mathbb{R}}(U)$  is open in  $Y$ .  $\square$

**Corollary 3.8.** *There is a lattice  $P\Gamma_{\mathbb{R}} \subset \text{PO}(2, 1)$ , an inclusion of orbifolds*

$$\coprod_{\alpha \in C\mathcal{A}} P\Gamma_\alpha \backslash (\mathbb{R}H_\alpha^2 - \mathcal{H}) \hookrightarrow P\Gamma_{\mathbb{R}} \backslash \mathbb{R}H^2, \quad (26)$$

and a homeomorphism

$$\mathcal{M}_s(\mathbb{R}) = \mathbb{G}(\mathbb{R}) \backslash X_s(\mathbb{R}) \cong P\Gamma_{\mathbb{R}} \backslash \mathbb{R}H^2 \quad (27)$$

such that (27) restricts to (23) with respect to (26).

*Proof.* This follows directly from [GF23, Theorem 3.1] and Theorem 3.7 above.  $\square$

**Remark 3.9.** The proof of Theorem 3.7 also shows that  $\mathcal{M}_s(\mathbb{R})$  is homeomorphic to the glued space  $P\Gamma \backslash Y$  (see [GF23, Definition 2.17]) if  $\mathcal{M}_s$  is the stack of cubic surfaces or of binary sextics. This strategy to uniformize the real moduli space differs from the one used in [ACT10; ACT06; ACT07], since we first glue together the real ball quotients, and then prove that our real moduli space is homeomorphic to the result.



**3.3 Automorphism groups of stable real binary quintics.** Before we can finish the proof of Theorem 1.2, we need to understand the orbifold structure of  $\mathcal{M}_s(\mathbb{R})$ , and how this structure differs from the orbifold structure of the glued space  $P\Gamma \setminus Y$ . In this Section 3.3 we start by analyzing the orbifold structure of  $\mathcal{M}_s(\mathbb{R})$ , by listing its stabilizer groups. There is a canonical orbifold isomorphism  $\mathcal{M}_s(\mathbb{R}) = \mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R}) = (P_s/\mathfrak{S}_5)(\mathbb{R})$ . Therefore, to list those groups occurring as automorphism group of a binary quintic is to list those  $x = [\alpha_1, \dots, \alpha_5] \in (P_s/\mathfrak{S}_5)(\mathbb{R})$  whose stabilizer  $\mathrm{PGL}_2(\mathbb{R})_x$  is non-trivial, and calculate  $\mathrm{PGL}_2(\mathbb{R})_x$  in these cases. This will be our next goal.

**Proposition 3.10.** *All stabilizer groups  $\mathrm{PGL}_2(\mathbb{R})_x \subset \mathrm{PGL}_2(\mathbb{R})$  for  $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$  are among  $\mathbb{Z}/2, D_3, D_5$ . For  $n \in \{3, 5\}$ , there is a unique  $\mathrm{PGL}_2(\mathbb{R})$ -orbit in  $(P_s/\mathfrak{S}_5)(\mathbb{R})$  of points  $x$  with stabilizer  $D_n$ .*

*Proof.* We have an injection  $(P_s/\mathfrak{S}_5)(\mathbb{R}) \hookrightarrow P_s/\mathfrak{S}_5$  which is equivariant for the embedding  $\mathrm{PGL}_2(\mathbb{R}) \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ . In particular,  $\mathrm{PGL}_2(\mathbb{R})_x \subset \mathrm{PGL}_2(\mathbb{C})_x$  for every  $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ . The groups  $\mathrm{PGL}_2(\mathbb{C})_x$  for points  $x \in P_0/\mathfrak{S}_5$  are calculated in [WX17, Theorem 22], and each such a group is isomorphic to either  $\mathbb{Z}/2, D_3, \mathbb{Z}/4$  or  $D_5$ . None of these groups have subgroups isomorphic to  $D_2 = \mathbb{Z}/2 \rtimes \mathbb{Z}/2$  or  $D_4 = \mathbb{Z}/2 \rtimes \mathbb{Z}/4$ .

Define an involution

$$\nu := (z \mapsto 1/z) \in \mathrm{PGL}_2(\mathbb{R}).$$

The proof of Proposition 3.10 will follow from the following four lemmas.

**Lemma 3.11.** *Let  $\tau \in \mathrm{PGL}_2(\mathbb{R})$ . Consider a subset  $S = \{x, y, z\} \subset \mathbb{P}^1(\mathbb{C})$  stabilized by complex conjugation, such that  $\tau(x) = x, \tau(y) = z$  and  $\tau(z) = y$ . There is a transformation  $g \in \mathrm{PGL}_2(\mathbb{R})$  that maps  $S$  to either  $\{-1, 0, \infty\}$  or  $\{-1, i, -i\}$ , and that satisfies  $g\tau g^{-1} = \nu = (z \mapsto 1/z) \in \mathrm{PGL}_2(\mathbb{R})$ . In particular,  $\tau^2 = \mathrm{id}$ .*

*Proof.* Indeed, two transformations  $g, h \in \mathrm{PGL}_2(\mathbb{C})$  that satisfy  $g(x_i) = h(x_i)$  for three different points  $x_1, x_2, x_3 \in \mathbb{P}^1(\mathbb{C})$  are necessarily equal.  $\square$

**Lemma 3.12.** *There is no  $\phi \in \mathrm{PGL}_2(\mathbb{R})$  of order four that stabilizes a point  $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ .*

*Proof.* By [Bea10, Theorem 4.2], all subgroups  $G \subset \mathrm{PGL}_2(\mathbb{R})$  that are isomorphic to  $\mathbb{Z}/4$  are conjugate to each other. Since the transformation  $I : z \mapsto (z - 1)/(z + 1)$  is of order four, it gives a representative  $G_I = \langle I \rangle$  of this conjugacy class. Assume that there exists  $x$  and  $\phi$  as in the lemma. To get a contradiction, we may assume that  $\phi = I$ . It is easily shown that  $I$  cannot fix any point  $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ .  $\square$

Define

$$\rho \in \mathrm{PGL}_2(\mathbb{R}), \quad \rho(z) = \frac{-1}{z + 1}.$$

**Lemma 3.13.** *Let  $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ . Suppose that  $\phi(x) = x$  for some  $\phi \in \mathrm{PGL}_2(\mathbb{R})$  of order three. There is a transformation  $g \in \mathrm{PGL}_2(\mathbb{R})$  mapping  $x$  to  $(-1, \infty, 0, \omega, \omega^2)$  with  $\omega$  a primitive third root of unity. The stabilizer of  $x$  to the subgroup of  $\mathrm{PGL}_2(\mathbb{R})$  generated by  $\rho$  and  $\nu$ . In particular, we have  $\mathrm{PGL}_2(\mathbb{R})_x \cong D_3$ .*



*Proof.* By Lemma 3.11, there are elements  $x_1, x_2, x_3$  which form an orbit under  $\phi$ . Since complex conjugation preserves this orbit, one element in it is real; since  $g$  is defined over  $\mathbb{R}$ , they are all real. Let  $g \in \mathrm{PGL}_2(\mathbb{R})$  such that  $g(x_1) = -1$ ,  $g(x_2) = \infty$  and  $g(x_3) = 0$ . Define  $\kappa = g\phi g^{-1}$ . Then  $\kappa^3 = \mathrm{id}$ , and  $\kappa$  preserves  $\{-1, \infty, 0\}$  and sends  $-1$  to  $\infty$  and  $\infty$  to  $0$ . Consequently,  $\kappa(0) = -1$ , and it follows that  $\kappa = \rho$ . Hence  $x$  is equivalent to an element of the form  $(-1, \infty, 0, \alpha, \beta)$  with  $\beta = \bar{\alpha}$  and  $\alpha^2 + \alpha + 1 = 0$ .  $\square$

Recall that  $\zeta_5 = e^{2i\pi/5} \in \mathbb{C}$  and define

$$\lambda = \zeta_5 + \zeta_5^{-1} \in \mathbb{R}, \quad \gamma \in \mathrm{PGL}_2(\mathbb{R}) \quad \text{with} \quad \gamma(z) = \frac{(\lambda + 1)z - 1}{z + 1} \quad \text{for} \quad z \in \mathbb{P}^1(\mathbb{C}).$$

**Lemma 3.14.** *Let  $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ . Suppose  $x$  is stabilized by a subgroup of  $\mathrm{PGL}_2(\mathbb{R})$  of order five. There is a transformation  $g \in \mathrm{PGL}_2(\mathbb{R})$  mapping  $x$  to  $z = (0, -1, \infty, \lambda + 1, \lambda)$  and identifying the stabilizer of  $x$  with the subgroup of  $\mathrm{PGL}_2(\mathbb{R})$  generated by  $\gamma$  and  $\nu$ . In particular, the stabilizer  $\mathrm{PGL}_2(\mathbb{R})_x$  of  $x$  is isomorphic to  $D_5$ .*

*Proof.* Let  $\phi \in \mathrm{PGL}_2(\mathbb{R})_x$  be an element of order five. Using Lemma 3.11 one shows that the  $x_i$  are pairwise distinct, and we may assume that  $x_i = \phi^{i-1}(x_1)$  for  $i = 2, \dots, 5$ . Since there is one real  $x_i$  and  $\phi$  is defined over  $\mathbb{R}$ , all  $x_i$  are real. Note that  $z = \{0, -1, \infty, \lambda + 1, \lambda\}$  is the orbit of  $0$  under  $\gamma : z \mapsto ((\lambda + 1)z - 1)/(z + 1)$ . The reflection  $\nu : z \mapsto 1/z$  preserves  $z$  as well: we have  $\lambda + 1 = -(\zeta_5^2 + \zeta_5^{-2}) = -\lambda^2 + 2$ , so that  $\lambda(\lambda + 1) = 1$ . So we have  $\mathrm{PGL}_2(\mathbb{R})_z \cong D_5$ . By [WX17, Theorem 22], the point  $z$  with stabilizer  $\mathrm{PGL}_2(\mathbb{R})_z$  is equivalent under  $\mathrm{PGL}_2(\mathbb{C})$  to the point  $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$  with stabilizer  $\langle x \mapsto \zeta x, x \mapsto 1/x \rangle$ . Thus, there exists  $g \in \mathrm{PGL}_2(\mathbb{C})$  such that  $g(x_1) = 0$ ,  $g(x_2) = -1$ ,  $g(x_3) = \infty$ ,  $g(x_4) = \lambda + 1$  and  $g(x_5) = \lambda$ , and such that  $g\mathrm{PGL}_2(\mathbb{R})_x g^{-1} = \mathrm{PGL}_2(\mathbb{R})_z$ . Since all  $x_i$  and  $z_i \in z$  are real, we see that  $\bar{g}(x_i) = z_i$  for every  $i$ , hence  $g$  and  $\bar{g}$  coincide on more than two points, which implies that  $g = \bar{g} \in \mathrm{PGL}_2(\mathbb{R})$ .  $\square$

Proposition 3.10 follows.  $\square$

3.4 **Binary quintics with automorphism group of order two.** The goal of Section 3.4 is to prove that there are no cone points in the orbifold  $\mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R})$ , i.e. orbifold points whose stabilizer group is  $\mathbb{Z}/n$  for some  $n \geq 2$  acting on the orbifold chart by rotations. By Proposition 3.10, this fact will follow from the following:

**Proposition 3.15.** *Let  $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$  such that  $\mathrm{PGL}_2(\mathbb{R})_x = \langle \tau \rangle$  has order two. There is a  $\mathrm{PGL}_2(\mathbb{R})_x$ -stable open neighborhood  $U \subset (P_s/\mathfrak{S}_5)(\mathbb{R})$  of  $x$  such that  $\mathrm{PGL}_2(\mathbb{R})_x \setminus U \rightarrow \mathcal{M}_s(\mathbb{R})$  is injective, and a homeomorphism  $\phi : (U, x) \rightarrow (B, 0)$  for  $0 \in B \subset \mathbb{R}^2$  an open ball, such that  $\phi$  identifies  $\mathrm{PGL}_2(\mathbb{R})_x$  with  $\mathbb{Z}/2$  acting on  $B$  by reflections in a line through  $0$ .*

*Proof.* Using Lemma 3.11, one checks that the only possibilities for the element  $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$  are  $(-1, 0, \infty, \beta, \beta^{-1})$ ,  $(-1, i, -i, \beta, \beta^{-1})$ ,  $(-1, -1, \beta, 0, \infty)$ ,  $(-1, -1, \beta, i, -i)$ ,  $(0, 0, \infty, \infty, -1)$  and  $(-1, i, i, -i, -i)$ .  $\square$

3.5 **Comparing the orbifold structures.** Consider the moduli space  $\mathcal{M}_s(\mathbb{R})$  of real stable binary quintics.

**Definition 3.16.** Let  $\overline{\mathcal{M}}_{\mathbb{R}}$  be the hyperbolic orbifold with  $\mathcal{M}_s(\mathbb{R})$  as underlying space, whose orbifold structure is induced by the homeomorphism (27) in Corollary 3.8 and the natural orbifold structure of  $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^2$ .

There are two orbifold structures on the space  $\mathcal{M}_s(\mathbb{R})$ : the natural orbifold structure of  $\mathcal{M}_s(\mathbb{R})$ , see [GF22, Proposition 2.12] (i.e. the orbifold structure of the quotient  $\mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R})$ ), and the orbifold structure  $\overline{\mathcal{M}}_{\mathbb{R}}$  introduced in Definition 3.16.

**Proposition 3.17.** 1. *The orbifold structures of  $\mathcal{M}_s(\mathbb{R})$  and  $\overline{\mathcal{M}}_{\mathbb{R}}$  differ only at the moduli point  $x_0 \in \mathcal{M}_s(\mathbb{R})$  attached to the five-tuple  $(\infty, i, i, -i, -i)$ .*

2. *The stabilizer group of  $\mathcal{M}_s(\mathbb{R})$  at the point  $x_0$  is isomorphic to  $\mathbb{Z}/2$ , whereas the stabilizer group of  $\overline{\mathcal{M}}_{\mathbb{R}}$  at  $x_0$  is isomorphic to the dihedral group  $D_{10}$  of order twenty.*

3. *The orbifold  $\overline{\mathcal{M}}_{\mathbb{R}}$  has no cone points and three corner reflectors, whose orders are  $\pi/3, \pi/5$  and  $\pi/10$ .*

*Proof.* The statements can be deduced from [GF23, Proposition 3.14]. The notation of that proposition is as follows: for  $f \in Y \cong \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R})$  (see Theorem 3.7) the group  $A_f \subset P\Gamma$  is the stabilizer of  $f \in K$ . Moreover, if  $\tilde{f} \in \mathcal{F}_s(\mathbb{R})$  represents  $f$  and if  $F = [\tilde{f}] \in X_s(\mathbb{R})$  has  $k = 2a + b$  nodes, then the image  $x \in \mathbb{C}H^2$  has  $k = 2a + b$  nodes in the sense of [GF23, Definition 2.6].

If  $F$  has no nodes ( $k = 0$ ), then  $G(x)$  is trivial by [GF23, Proposition 3.14.1] and  $G_F = A_f = \Gamma_f$ . If  $F$  has only real nodes, then  $B_f = G(x)$  hence  $G_F = A_f/G(x) = A_f/B_f = \Gamma_f$ .

Suppose that  $a = 1$  and  $b = 0$ : the equation  $F$  defines a pair of complex conjugate nodes. In other words, the zero set of  $F$  defines a 5-tuple  $\underline{\alpha} = (\alpha_1, \dots, \alpha_5) \in \mathbb{P}^1(\mathbb{C})$ , well-defined up to the  $\mathrm{PGL}_2(\mathbb{R}) \times \mathfrak{S}_5$  action on  $\mathbb{P}^1(\mathbb{C})$ , where  $\alpha_1 \in \mathbb{P}^1(\mathbb{R})$  and  $\alpha_3 = \bar{\alpha}_2 = \alpha_5 = \bar{\alpha}_4 \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ . So we may write  $\underline{\alpha} = (\beta, \alpha, \bar{\alpha}, \alpha, \bar{\alpha})$  with  $\beta \in \mathbb{P}^1(\mathbb{R})$  and  $\alpha \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ . There is a unique  $T \in \mathrm{PGL}_2(\mathbb{R})$  such that  $T(\beta) = \infty$  and  $T(\alpha) = i$ . But this gives  $T(x) = (\infty, i, -i, i, -i)$  hence  $F$  is unique up to isomorphism.

As for the stabilizer  $G_F = A_f/G(x)$ , we have  $G(x) \cong (\mathbb{Z}/10)^2$ . Since there are no real nodes,  $B_f$  is trivial. By [GF23, Proposition 3.14.3], the set  $K_f$  is the union of ten copies of  $\mathbb{B}^2(\mathbb{R})$  meeting along a common point  $\mathbb{B}^0(\mathbb{R})$ . In fact, in the local coordinates  $(t_1, t_2)$  around  $f$ , the  $\alpha_j : \mathbb{B}^2(\mathbb{C}) \rightarrow \mathbb{B}^2(\mathbb{C})$  are defined as  $(t_1, t_2) \mapsto (\bar{t}_2 \zeta^j, \bar{t}_1 \zeta^j)$ , for  $j \in \mathbb{Z}/10$ , and so the fixed points sets are given by  $\mathbb{R}H_j^2 = \{t_2 = \bar{t}_1 \zeta^j\} \subset \mathbb{B}^2(\mathbb{C})$ .

The subgroup  $E \subset G(x)$  that stabilizes  $\mathbb{R}H_j^2$  is the cyclic group of order ten generated by the transformations  $(t_1, t_2) \mapsto (\zeta t_1, \zeta^{-1} t_2)$ . There is only one non-trivial transformation  $T \in \mathrm{PGL}_2(\mathbb{R})$  that fixes  $\infty$  and sends the subset  $\{i, -i\} \subset \mathbb{P}^1(\mathbb{C})$  to itself, and  $T$  is of order five. Hence  $G_F = \mathbb{Z}/2$  so that we have an exact sequence  $0 \rightarrow \mathbb{Z}/10 \rightarrow \Gamma_f \rightarrow \mathbb{Z}/2 \rightarrow 0$ . This splits since  $G_F$  is a subgroup of  $\Gamma_f$ . We are done by Propositions 3.10 and 3.15.  $\square$

**3.6 The real moduli space as a hyperbolic triangle.** The goal of Section 3.6 is to show that  $\overline{\mathcal{M}}_{\mathbb{R}}$  (see Definition 3.16) is isomorphic, as hyperbolic orbifolds, to the triangle  $\Delta_{3,5,10}$  in the real hyperbolic plane  $\mathbb{R}H^2$  with angles  $\pi/3, \pi/5$  and  $\pi/10$ . The results in the above Sections 3.3, 3.4 and 3.5 give the orbifold singularities of  $\overline{\mathcal{M}}_{\mathbb{R}}$  together with their stabilizer groups. In order to determine the hyperbolic orbifold structure of  $\overline{\mathcal{M}}_{\mathbb{R}}$ , we also need to know the underlying topological space  $\mathcal{M}_s(\mathbb{R})$  of  $\overline{\mathcal{M}}_{\mathbb{R}}$ . The first observation is that  $\mathcal{M}_s(\mathbb{R})$  is compact. Indeed, it is classical that the topological space  $\mathcal{M}_s(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_s(\mathbb{C})$ , parametrizing complex stable binary quintics, is compact. This follows from the fact that it is homeomorphic to  $\overline{M}_{0,5}(\mathbb{C})/\mathfrak{S}_5$ , and the stack of stable five-pointed curves  $\overline{M}_{0,5}$  is proper [Knu83], or from the fact that it is homeomorphic to a compact ball quotient [Shi64]. Moreover, the map  $\mathcal{M}_s(\mathbb{R}) \rightarrow \mathcal{M}_s(\mathbb{C})$  is proper, which proves the compactness of  $\mathcal{M}_s(\mathbb{R})$ .

The second observation is that  $\mathcal{M}_s(\mathbb{R})$  is connected, since  $X_s(\mathbb{R})$  is obtained from the euclidean space  $\{F \in \mathbb{R}[x, y] : F \text{ homogeneous and } \deg(F) = 5\}$  by removing a subspace of codimension at least two. We can prove more:

**Lemma 3.18.** *The moduli space  $\mathcal{M}_s(\mathbb{R})$  of real stable binary quintics is homeomorphic to a closed disc  $\overline{D} \subset \mathbb{R}^2$ .*

*Proof.* The idea is to show that the following holds:

1. For each  $i \in \{0, 1, 2\}$ , the embedding  $\mathcal{M}_i \hookrightarrow \overline{\mathcal{M}}_i \subset \mathcal{M}_s(\mathbb{R})$  of the connected component  $\mathcal{M}_i$  of  $\mathcal{M}_0(\mathbb{R})$  into its closure in  $\mathcal{M}_s(\mathbb{R})$  is homeomorphic to the embedding  $D \hookrightarrow \overline{D}$  of the open unit disc into the closed unit disc in  $\mathbb{R}^2$ .
2. We have  $\mathcal{M}_s(\mathbb{R}) = \overline{\mathcal{M}}_0 \cup \overline{\mathcal{M}}_1 \cup \overline{\mathcal{M}}_2$ , and this glueing corresponds up to homeomorphism to the glueing of three closed discs  $\overline{D}_i \subset \mathbb{R}^2$  as in Figure 1.

To prove this, one considers the moduli spaces of real smooth (resp. stable) genus zero curves with five real marked points [Knu83], as well as twists of this space. Define two anti-holomorphic involutions  $\sigma_i : \mathbb{P}^1(\mathbb{C})^5 \rightarrow \mathbb{P}^1(\mathbb{C})^5$  by  $\sigma_1(x_1, x_2, x_3, x_4, x_5) = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_5, \bar{x}_4)$ , and  $\sigma(x_1, x_2, x_3, x_4, x_5) = (\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_5, \bar{x}_4)$ . Then define

$$P_0^1(\mathbb{R}) = P_0^{\sigma_1}, \quad P_s^1(\mathbb{R}) = P_1(\mathbb{C})^{\sigma_1}, \quad P_0^2(\mathbb{R}) = P_0^{\sigma_2}, \quad P_s^2(\mathbb{R}) = P_1(\mathbb{C})^{\sigma_2}.$$

It is clear that  $\mathcal{M}_0 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R})/\mathfrak{S}_5$ . Similarly, we have:

$$\mathcal{M}_1 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0^1(\mathbb{R})/\mathfrak{S}_3 \times \mathfrak{S}_2 \quad \text{and} \quad \mathcal{M}_2 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0^2(\mathbb{R})/\mathfrak{S}_2 \times \mathfrak{S}_2.$$

Moreover, we have  $\overline{\mathcal{M}}_0 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s(\mathbb{R})/\mathfrak{S}_5$ . We define

$$\overline{\mathcal{M}}_1 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s^1(\mathbb{R})/\mathfrak{S}_3 \times \mathfrak{S}_2, \quad \text{and} \quad \overline{\mathcal{M}}_2 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s^2(\mathbb{R})/\mathfrak{S}_2 \times \mathfrak{S}_2.$$

Each  $\overline{\mathcal{M}}_i$  is homeomorphic to a closed disc in  $\mathbb{R}^2$ . Moreover, the natural maps  $\overline{\mathcal{M}}_i \rightarrow \mathcal{M}_s(\mathbb{R})$  are closed embeddings of topological spaces, and one can check that the images glue to form  $\mathcal{M}_s(\mathbb{R})$  in the prescribed way. We leave the details to the reader.  $\square$

*Proof of Theorem 1.2.* To any closed two-dimensional orbifold  $O$  one can associate a set of natural numbers  $S_O = \{n_1, \dots, n_k; m_1, \dots, m_l\}$  by letting  $k$  be the number of cone points of  $X_O$ ,  $l$  the number of corner reflectors,  $n_i$  the order of the  $i$ -th cone point and  $2m_j$  the order of the  $j$ -th corner reflector. A closed two-dimensional orbifold  $O$  is determined, up to orbifold-structure preserving homeomorphism, by its underlying space  $X_O$  and the set  $S_O$  [Thu80]. By Lemma 3.18,  $\overline{\mathcal{M}}_{\mathbb{R}}$  is homeomorphic to a closed disc in  $\mathbb{R}^2$ . By Proposition 3.17,  $\overline{\mathcal{M}}_{\mathbb{R}}$  has no cone points and three corner reflectors whose orders are  $\pi/3, \pi/5$  and  $\pi/10$ . This implies  $\overline{\mathcal{M}}_{\mathbb{R}}$  and  $\Delta_{3,5,10}$  are isomorphic as topological orbifolds. Consequently, the orbifold fundamental group of  $\overline{\mathcal{M}}_{\mathbb{R}}$  is abstractly isomorphic to the group  $\Gamma_{3,5,10}$  defined in (2).

Let  $\phi : \Gamma_{3,5,10} \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$  be any embedding such that  $X := \phi(\Gamma_{3,5,10}) \backslash \mathbb{R}H^2$  is a hyperbolic orbifold; we claim that there is a fundamental domain  $\Delta$  for  $X$  isometric to  $\Delta_{3,5,10}$ . Consider the generator  $a \in \Gamma_{3,5,10}$ . Since  $\phi(a)^2 = 1$ , there exists a geodesic  $L_1 \subset \mathbb{R}H^2$  such that  $\phi(a) \in \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}(\mathbb{R}H^2)$  is the reflection across  $L_1$ . Next, consider the generator  $b \in \Gamma_{3,5,10}$ . There exists a geodesic  $L_2 \subset \mathbb{R}H^2$  such that  $\phi(b)$  is the reflection across  $L_2$ . One easily shows that  $L_2 \cap L_1 \neq \emptyset$ . Let  $x \in L_1 \cap L_2$ . Then  $\phi(a)\phi(b)$  is an element of order three that fixes  $x$ , hence  $\phi(a)\phi(b)$  is a rotation around  $x$ . Therefore, one of the angles between  $L_1$  and  $L_2$  must be  $\pi/3$ . Finally, we know that  $\phi(c)$  is an element of order two in  $\mathrm{PSL}_2(\mathbb{R})$ , hence a reflection across a line  $L_3$ . By the previous arguments,  $L_3 \cap L_2 \neq \emptyset$  and  $L_3 \cap L_1 \neq \emptyset$ . It also follows that  $x \in L_3 \cap L_2 \cap L_1 = \emptyset$ . Consequently, the three geodesics  $L_i \subset \mathbb{R}H^2$  enclose a hyperbolic triangle; the orders of  $\phi(a)\phi(b)$ ,  $\phi(a)\phi(c)$  and  $\phi(b)\phi(c)$  imply that the three interior angles of the triangle are  $\pi/3, \pi/5$  and  $\pi/10$ .  $\square$

## 4 The monodromy groups

In this section, we describe the monodromy group  $P\Gamma$ , as well as the groups  $P\Gamma_{\alpha}$  appearing in Proposition 3.2. As for the lattice  $(\Lambda, \mathfrak{h})$  (see (7)), we have:

**Theorem 4.1** (Shimura). *There is an isomorphism of hermitian  $\mathcal{O}_K$ -lattices*

$$(\Lambda, \mathfrak{h}) \cong \left( \mathcal{O}_K^3, \mathrm{diag} \left( \frac{1 - \sqrt{5}}{2}, 1, 1 \right) \right).$$

*Proof.* See [Shi64, Section 6] as well as item (5) in the table on page 1 of *loc. cit.*  $\square$

Let us write  $\Lambda = \mathcal{O}_K^3$  and  $\mathfrak{h} = \mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2})$  in the remaining part of Section 4. Write  $\alpha = \zeta_5 + \zeta_5^{-1} = \frac{\sqrt{5}-1}{2}$ . For the element  $\theta = \zeta_5 - \zeta_5^{-1} \in \mathcal{O}_K$  we have that  $|\theta|^2 = \frac{\sqrt{5}+5}{2}$ . Define three quadratic forms  $q_0, q_1$  and  $q_2$  on  $\mathbb{Z}[\alpha]^3$  as follows:

$$\begin{aligned} q_0(x_0, x_1, x_2) &= x_0^2 + x_1^2 - \alpha x_2^2, \\ q_1(x_0, x_1, x_2) &= |\theta|^2 x_0^2 + x_1^2 - \alpha x_2^2, \\ q_2(x_0, x_1, x_2) &= |\theta|^2 x_0^2 + |\theta|^2 x_1^2 - \alpha x_2^2. \end{aligned} \tag{28}$$

Consider  $\mathbb{Z}[\alpha]$  as a subring of  $\mathbb{R}$  via the standard embedding.

**Theorem 4.2.** *Consider the quadratic forms  $q_j$  defined in (28). There is a union of geodesic subspaces  $\mathcal{H}_j \subset \mathbb{R}H^2$ ,  $j \in \{0, 1, 2\}$ , and an isomorphism of hyperbolic orbifolds*

$$\mathcal{M}_0(\mathbb{R}) \cong \prod_{j=0}^2 \text{PO}(q_j, \mathbb{Z}[\alpha]) \setminus (\mathbb{R}H^2 - \mathcal{H}_j). \quad (29)$$

*Proof.* Recall that  $\theta = \zeta_5 - \zeta_5^{-1}$ ; we consider the  $\mathbb{F}_5$ -vector space  $W$  equipped with the quadratic form  $q = \mathfrak{h} \pmod{\theta}$ . Define three anti-isometric involutions as follows:

$$\begin{aligned} \alpha_0 &: (x_0, x_1, x_2) \mapsto (\bar{x}_0, \bar{x}_1, \bar{x}_2) \\ \alpha_1 &: (x_0, x_1, x_2) \mapsto (-\bar{x}_0, \bar{x}_1, \bar{x}_2) \\ \alpha_2 &: (x_0, x_1, x_2) \mapsto (-\bar{x}_0, -\bar{x}_1, \bar{x}_2). \end{aligned} \quad (30)$$

For isometries  $\alpha : W \rightarrow W$ , the dimension and determinant of the fixed space  $(W^\alpha, q|_{W^\alpha})$  are conjugacy-invariant. Using this, one easily shows that an anti-unitary involution of  $\Lambda$  is  $\Gamma$ -conjugate to exactly one of the  $\pm\alpha_j$ , hence  $C\mathcal{A}$  has cardinality three and the elements  $\alpha_0, \alpha_1, \alpha_2$  of (30) form a set of representatives for  $C\mathcal{A}$ . By Proposition 3.2, we obtain  $\mathcal{M}_0(\mathbb{R}) \cong \prod_{j=0}^2 P\Gamma_{\alpha_j} \setminus (\mathbb{R}H_{\alpha_j}^2 - \mathcal{H})$  where each quotient space  $P\Gamma_{\alpha_j} \setminus (\mathbb{R}H_{\alpha_j}^2 - \mathcal{H})$  is connected. Next, consider the fixed lattices

$$\begin{aligned} \Lambda_0 &:= \Lambda^{\alpha_0} = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \\ \Lambda_1 &:= \Lambda^{\alpha_1} = \theta\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \\ \Lambda_2 &:= \Lambda^{\alpha_2} = \theta\mathbb{Z}[\alpha] \oplus \theta\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha]. \end{aligned} \quad (31)$$

One easily shows that  $P\Gamma_{\alpha_j} = N_{P\Gamma}(\alpha_j)$  for the normalizer  $N_{P\Gamma}(\alpha_j)$  of  $\alpha_j$  in  $P\Gamma$ . Moreover, if  $h_j$  denotes the restriction of  $\mathfrak{h}$  to  $\Lambda^{\alpha_j}$ , then there is a natural embedding

$$\iota : N_{P\Gamma}(\alpha_j) \hookrightarrow \text{PO}(\Lambda_j, h_j, \mathbb{Z}[\alpha]). \quad (32)$$

We claim that  $\iota$  is an isomorphism. This follows from the fact that the natural homomorphism  $\pi : N_\Gamma(\alpha_j) \rightarrow O(\Lambda_j, h_j)$  is surjective, where  $N_\Gamma(\alpha_j) = \{g \in \Gamma : g \circ \alpha_j = \alpha_j \circ g\}$  is the normalizer of  $\alpha_j$  in  $\Gamma$ . The surjectivity of  $\pi$  follows from the equality

$$\Lambda = \mathcal{O}_K \cdot \Lambda_j + \mathcal{O}_K \cdot \theta\Lambda_j^\vee \subset K^3,$$

which follows from (31). Since  $\text{PO}(\Lambda_j, h_j, \mathbb{Z}[\alpha]) = \text{PO}(q_j, \mathbb{Z}[\alpha])$ , we are done.  $\square$

## References

- [ACT02a] Daniel Allcock, James Carlson, and Domingo Toledo. ‘‘Orthogonal complex hyperbolic arrangements’’. In: *Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000)*. Vol. 312. Contemporary Mathematics. American Mathematical Society, 2002, pp. 1–8.
- [ACT02b] Daniel Allcock, James Carlson, and Domingo Toledo. ‘‘The complex hyperbolic geometry of the moduli space of cubic surfaces’’. In: *Journal of Algebraic Geometry* 11.4 (2002), pp. 659–724.

- [ACT06] Daniel Allcock, James Carlson, and Domingo Toledo. “Nonarithmetic uniformization of some real moduli spaces”. In: *Geometriae Dedicata* 122 (2006), pp. 159–169.
- [ACT07] Daniel Allcock, James Carlson, and Domingo Toledo. “Hyperbolic geometry and the moduli space of real binary sextics”. In: *Arithmetic and geometry around hypergeometric functions*. Vol. 260. Progress in Mathematics. Birkhäuser, 2007, pp. 1–22.
- [ACT10] Daniel Allcock, James Carlson, and Domingo Toledo. “Hyperbolic geometry and moduli of real cubic surfaces”. In: *Annales Scientifiques de l’École Normale Supérieure* 43.1 (2010), pp. 69–115.
- [Bea09] Arnaud Beauville. “Moduli of cubic surfaces and Hodge theory (after Allcock, Carlson, Toledo)”. In: *Géométries à courbure négative ou nulle, groupes discrets et rigidités*. Vol. 18. Séminaires et Congrès. Société Mathématique de France, 2009, pp. 445–466.
- [Bea10] Arnaud Beauville. “Finite subgroups of  $\mathrm{PGL}_2(K)$ ”. In: *Vector bundles and complex geometry*. Vol. 522. Contemporary Mathematics. American Mathematical Society, 2010, pp. 23–29.
- [Bea86] Arnaud Beauville. “Le groupe de monodromie des familles universelles d’hypersurfaces et d’intersections complètes”. In: *Complex analysis and algebraic geometry (Göttingen, 1985)*. Vol. 1194. Lecture Notes in Mathematics. Springer, 1986, pp. 8–18.
- [CT99] James Carlson and Domingo Toledo. “Discriminant complements and kernels of monodromy representations”. In: *Duke Mathematical Journal* 97.3 (1999), pp. 621–648.
- [DM86] Pierre Deligne and George Mostow. “Monodromy of hypergeometric functions and nonlattice integral monodromy”. In: *Institut des Hautes Études Scientifiques. Publications Mathématiques* 63 (1986), pp. 5–89.
- [Fox57] Ralph Fox. “Covering spaces with singularities”. In: *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*. Princeton University Press, 1957, pp. 243–257.
- [GF22] Olivier de Gaay Fortman. “Moduli spaces and algebraic cycles in real algebraic geometry”. PhD thesis. École normale supérieure de Paris, 2022.
- [GF23] Olivier de Gaay Fortman. “Non-arithmetic uniformization of metric spaces attached to unitary Shimura varieties”. In: *arXiv e-prints* (2023), arXiv: 2301.01598.
- [GH81] Benedict Gross and Joe Harris. “Real algebraic curves”. In: *Annales Scientifiques de l’École Normale Supérieure* 14.2 (1981), pp. 157–182.
- [HM98] Joe Harris and Ian Morrison. *Moduli of Curves*. Vol. 187. Graduate Texts in Mathematics. Springer, 1998, pp. xiv+366.

- [HR18] Gert Heckman and Sander Rieken. “Hyperbolic geometry and moduli of real curves of genus three”. In: *Mathematische Annalen* 370.3-4 (2018), pp. 1321–1360.
- [Knu83] Finn Knudsen. “The projectivity of the moduli space of stable curves. II. The stacks  $M_{g,n}$ ”. In: *Mathematica Scandinavica* 52.2 (1983), pp. 161–199.
- [Kon07] Shigeyuki Kondō. “The moduli space of 5 points on  $\mathbb{P}^1$  and  $K3$  surfaces”. In: *Arithmetic and Geometry around Hypergeometric Functions*. Vol. 260. Progress in Mathematics. Birkhäuser, 2007, pp. 189–206.
- [MFK94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*. Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1994, pp. xiv+292.
- [MO13] Ben Moonen and Frans Oort. “The Torelli locus and special subvarieties”. In: *Handbook of moduli. Vol. II*. Vol. 25. Advanced Lectures in Mathematics. International Press of Boston, 2013, pp. 549–594.
- [MS72] David Mumford and Kalevi Suominen. “Introduction to the theory of moduli”. In: *Algebraic Geometry, Oslo 1970*. Wolters-Noordhoff, 1972, pp. 171–222.
- [Neu99] Jürgen Neukirch. *Algebraic Number Theory*. Vol. 322. Grundlehren der mathematischen Wissenschaften. Springer, 1999, pp. xviii+571.
- [Nik79] Viacheslav Nikulin. “Integer symmetric bilinear forms and some of their geometric applications”. In: *Izvestiya Rossiiskoi Akademii Nauk, Seriya Matematicheskaya* 43.1 (1979), pp. 111–177, 238.
- [Shi64] Goro Shimura. “On purely transcendental fields automorphic functions of several variable”. In: *Osaka Mathematical Journal* 1.1 (1964), pp. 1–14.
- [Sil89] Robert Silhol. *Real Algebraic Surfaces*. Vol. 1392. Lecture Notes in Mathematics. Springer, 1989, pp. x+215.
- [SS89] Mika Seppälä and Robert Silhol. “Moduli spaces for real algebraic curves and real abelian varieties”. In: *Mathematische Zeitschrift* 201.2 (1989), pp. 151–165.
- [Tak77] Kisao Takeuchi. “Arithmetic triangle groups”. In: *Journal of the Mathematical Society of Japan* 29.1 (1977), pp. 91–106.
- [Thu80] William Thurston. *The Geometry and Topology of Three-Manifolds*. Princeton University Press, 1980.
- [WX17] Yue Wu and Bin Xu. “Moduli spaces of unordered  $n \geq 5$  points on the Riemann sphere and their singularities”. In: *arXiv e-prints* (2017), arXiv: 1704.07583.

OLIVIER DE GAAY FORTMAN, INSTITUTE OF ALGEBRAIC GEOMETRY, LEIBNIZ UNIVERSITY HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY  
*E-mail address:* degaayfortman@math.uni-hannover.de