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Non-arithmetic uniformization of metric spaces attached to unitary Shimura varieties

Olivier de Gaay Fortman · February 23, 2023

1 Introduction

In various cases, Hodge theory provides a bridge between hyperbolic geometry and algebraic geometry: there are moduli spaces of complex varieties whose period map identifies it with a quotient of complex hyperbolic space. The ball quotients that arise in this way are often (but not always) arithmetic [Pic83; Shi64; DM86; Kon00; ACT02b]. Similar constructions can be carried out to uniformize moduli of real varieties [AY98; ACT06; ACT10; Chu11; HR18]. A striking result in this direction was given by Allcock–Carlson–Toledo, who identified the space of stable real cubic surfaces with a four-dimensional non-arithmetic real ball quotient [ACT10]. Their space is assembled from various pieces, each of which an arithmetic quotient containing an open subset that parametrizes moduli of smooth real cubic surfaces of one topological type.

The starting point of this paper is the question whether one can glue the fixed loci of anti-holomorphic involutions on an arbitrary unitary Shimura variety, in a way that does not depend on real moduli theory, and such that the resulting space is naturally a real ball quotient. Our first goal is to prove that such glueing can be done, leading to a construction which generalizes [ACT07; ACT10] and seems an orbifold analogue of the construction of Gromov–Piatetski-Shapiro [GPS87]. For indeed, our second goal is to show that many of the lattices that arise in this way, are in fact non-arithmetic.

To explain this, let $n \in \mathbb{N}$ and consider a CM field K and a hermitian \mathcal{O}_K -lattice Λ_n of signature $(n, 1)$ for one infinite place of K and definite for others. To Λ_n one can associate a quotient $\Gamma \backslash \mathbb{B}^n(\mathbb{C})$, and each anti-unitary involution $\alpha: \Lambda_n \rightarrow \Lambda_n$ defines a real arithmetic ball quotient $\Gamma_\alpha \backslash \mathbb{B}_\alpha^n(\mathbb{R}) := \text{Stab}_\Gamma(\mathbb{B}^n(\mathbb{C})^\alpha) \backslash \mathbb{B}^n(\mathbb{C})^\alpha$. We glue these real quotients together along a certain orthogonal hyperplane arrangement $\mathcal{H} \subset \mathbb{B}^n(\mathbb{C})$. The resulting space $X(\Lambda_n)$ is a complete metric space (c.f. Proposition 3.7). It turns out that, moreover, $X(\Lambda_n)$ carries a complete hyperbolic orbifold structure:

Theorem 1.1 (c.f. Theorem 3.1). *For each connected component $X(\Lambda_n)^+ \subset X(\Lambda_n)$ there exists a lattice $\Gamma_n^+ \subset \text{PO}(n, 1)$ and a canonical isometry $X(\Lambda_n)^+ \cong \Gamma_n^+ \backslash \mathbb{B}^n(\mathbb{R})$.*

Applying Theorem 1.1 to the case where $K = \mathbb{Q}(\zeta_3)$ and

$$\Lambda_n = \mathbb{Z}[\zeta_3]^{n,1} = (\mathbb{Z}[\zeta_3]^{n+1}, \text{diag}(-1, 1, \dots, 1)),$$

one obtains a sequence of real hyperbolic orbifolds $X(\Lambda_n)$ with interesting properties. Indeed, $X(\Lambda_2)$ is connected, its orbifold fundamental group Γ_2^+ is non-arithmetic, and

$X(\Lambda_2)$ immerses totally geodesically into a connected component $X(\Lambda_n)^+$ of $X(\Lambda_n)$ for each $n \in \mathbb{Z}_{\geq 2}$ (Theorem 7.3). By [BC05, Proposition 15.2.2], this implies that $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ is non-arithmetic, which leads to the second main result of this paper.

Theorem 1.2 (c.f. Theorem 7.10). *Let $n \in \mathbb{Z}_{\geq 2}$ and let $\Lambda_n = \mathbb{Z}[\zeta_3]^{n,1}$. There exists a connected component $X(\Lambda_n)^+ \subset X(\Lambda_n)$ such that the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ underlying the complete hyperbolic orbifold $X(\Lambda_n)^+$ is non-arithmetic.*

Theorem 1.1 also has applications to the theory of moduli spaces of real algebraic varieties. Indeed, for some moduli stacks of GIT stable hypersurfaces \mathcal{M}_s , one can consider the hermitian lattice Λ that arises as the cohomology of the cover of projective space ramified along a member of the moduli space, and define an isomorphism $\mathcal{M}_s(\mathbb{R}) \cong X(\Lambda_n)$ for $n = \dim(\mathcal{M}_s)$. An application of Theorem 1.1 gives then a uniformization of $\mathcal{M}_s(\mathbb{R})$ by real hyperbolic space. For instance, for cubic surfaces and binary sextics, one retrieves the main results of [ACT10; ACT06; ACT07] in this way.

Applying Theorem 1.1 to $K = \mathbb{Q}(\zeta_5)$ and $\Lambda_n = (\mathbb{Z}[\zeta_5]^{n+1}, \mathrm{diag}(\frac{1-\sqrt{5}}{2}, 1, \dots, 1))$ yields further examples of non-arithmetic lattices. In a subsequent paper [GF21], we will use the theory developed in this paper to investigate the structure of the moduli space of stable real binary quintics. An analysis of the period map for binary quintics and an application of Theorem 1.1 shows that, also in this case, $X(\Lambda_2)$ is connected and underlies a non-arithmetic ball quotient. As before, this has the following consequence.

Theorem 1.3. *Let $K = \mathbb{Q}(\zeta_5)$, and define lattices Λ_n and a CM type Ψ as follows:*

$$\left(\Lambda_n = (\mathbb{Z}[\zeta_5]^{n+1}, \mathrm{diag}((1 - \sqrt{5})/2, 1, \dots, 1)), \quad \Psi = \{ \tau_1, \tau_2: K \rightarrow \mathbb{C} \mid \tau_i(\zeta_5) = \zeta_5^i \} \right).$$

Then for each $n \in \mathbb{Z}_{\geq 2}$, there is a connected component $X(\Lambda_n)^+ \subset X(\Lambda_n)$ such that the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ attached to $X(\Lambda_n)^+$ via Theorem 1.1 is non-arithmetic.

Theorem 1.3 will be proved in [GF21]. In the remaining part of the introduction, we will explain what the main ingredients are that go into the proof of Theorem 1.1.

- 1.1 **Outline of the glueing construction.** Let (K, Ψ) be a CM type, with K a CM field of degree $2g$ over \mathbb{Q} with ring of integers \mathcal{O}_K , and let Λ be a hermitian \mathcal{O}_K -lattice, i.e. a finite free \mathcal{O}_K -module equipped with a hermitian form $\Lambda \times \Lambda \rightarrow \mathcal{O}_K$. Assume that the hermitian form has signature $(n, 1)$ with respect to one embedding $(\tau : K \rightarrow \mathbb{C}) \in \Psi$ and is positive definite for the other elements in Ψ . Let $\mathbb{C}H^n$ be the space of negative lines in $\Lambda \otimes_{\mathcal{O}_K, \tau} \mathbb{C}$ and let $P\Gamma = \mathrm{Aut}(\Lambda)/\mu_K$, where $\mu_K \subset \mathcal{O}_K^*$ is the group of finite units in \mathcal{O}_K . Let $P\mathcal{A}$ be the quotient of the set of anti-unitary involutions $\alpha : \Lambda \rightarrow \Lambda$ by μ_K . Attached to the hermitian lattice Λ there is a Shimura variety $\mathrm{Sh}_K(G, X)$ (c.f. [Ach20, Section 5.3]) with complex uniformization $\mathrm{Sh}_K(G, X)(\mathbb{C}) = \Gamma \backslash \mathbb{C}H^n$, see Corollary 4.9 or [Shi63, Theorem 2]. Consider then the hyperplane arrangement

$$\mathcal{H} = \bigcup_{\substack{r \in \Lambda: \\ h(r,r)=1}} \langle r_{\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n$$

and assume that the following holds:

Condition 1.4 (c.f. [ACT02a]). *Any two different hyperplanes $\langle r_{\mathbb{C}} \rangle^{\perp}, \langle t_{\mathbb{C}} \rangle^{\perp} \subset \mathcal{H}$ intersect orthogonally or not at all, i.e. either $h(r, t) = 0$ or $\langle r_{\mathbb{C}} \rangle^{\perp} \cap \langle t_{\mathbb{C}} \rangle^{\perp} = \emptyset$.*

In many cases, Condition 1.4 is automatically satisfied:

Theorem 1.5 (c.f. Theorem 5.2). *If the different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a purely imaginary element $\eta \in \mathfrak{D}_K$ such that $\Im(\varphi(\eta)) > 0$ for every $\varphi \in \Psi$, then Condition 1.4 holds.*

The condition that $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a non-zero purely imaginary element $\eta \in \mathcal{O}_K$ holds whenever K is cyclotomic or imaginary quadratic, see Proposition 5.4.

The glueing construction is performed by assembling the different copies

$$\mathbb{R}H_{\alpha}^n := (\mathbb{C}H^n)^{\alpha} \subset \mathbb{C}H^n, \quad \alpha \in P\mathcal{A}$$

of real hyperbolic space $\mathbb{R}H^n$ along the hyperplane arrangement \mathcal{H} . See Definition 2.15 and Remark 2.16 for the formulation of the equivalence relation. This gives a topological space $Y(\Lambda)$, acted upon by $P\Gamma$. Define $P\Gamma_{\alpha} \subset P\Gamma$ to be the stabilizer of $\mathbb{R}H_{\alpha}^n$. Our first main result (Theorem 1.1) can then be reformulated as follows.


Theorem 1.6 (c.f. Theorem 3.1). *The space $X(\Lambda) = P\Gamma \backslash Y(\Lambda)$ admits a complete path metric such that the natural map $X(\Lambda) \rightarrow P\Gamma \backslash \mathbb{C}H^n$ is a local isometry. This metric on $X(\Lambda)$ extends to a real hyperbolic orbifold structure, such that the following conditions hold: the subspace*

$$\coprod_{\alpha \in P\Gamma \backslash P\mathcal{A}} P\Gamma_{\alpha} \backslash (\mathbb{R}H_{\alpha}^n - \mathcal{H}) \subset X(\Lambda)$$

is an open suborbifold, and for each connected component $X(\Lambda)^+ \subset X(\Lambda)$ there is a lattice $\Gamma^+ \subset \mathrm{PO}(n, 1)$ and an isomorphism of hyperbolic orbifolds $X(\Lambda)^+ \cong \Gamma^+ \backslash \mathbb{R}H^n$.

Remark 1.7. Our construction relies on Condition 1.4, saying that the hyperplane arrangement $\mathcal{H} \subset \mathbb{C}H^n$ is an *orthogonal arrangement* in the sense of [ACT02a]. In fact, there is always a canonical orthogonal arrangement $\mathcal{H} \subset \mathbb{C}H^n$ attached to h in such a way that $\mathcal{H} = \mathcal{H}$ when \mathcal{H} is orthogonal (c.f. Remark 5.5). Moreover, one can glue the different copies $\mathbb{R}H_{\alpha}^n$ of real hyperbolic space along the arrangement \mathcal{H} to obtain a complete hyperbolic orbifold as in Theorem 1.6, but we will not prove this.

1.2 **Overview of this paper.** This paper is structured as follows. In Section 2, we construct the glued space $X(\Lambda) = P\Gamma \backslash Y(\Lambda)$ starting from a hermitian \mathcal{O}_K -lattice Λ of hyperbolic signature. In Section 3, we provide the glued space $X(\Lambda)$ with a real hyperbolic orbifold structure. Theorem 1.6 follows readily from this. The goal of Section 4 will be to provide the ball quotient $P\Gamma \backslash \mathbb{C}H^n$ with a modular interpretation, which will be used to prove Theorem 1.5. Finally, in Section 6, we prove Theorem 1.2.

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2 Glueing real hyperbolic orbifolds

In this section, we construct a topological space by glueing together several arithmetic hyperbolic pieces. Afterwards, in Section 3, we provide this topological space first with a path metric and then with a complete hyperbolic orbifold structure.

2.1 **Arithmetic ball quotients and anti-unitary involutions.** The goal of Sections 2.1 - 2.4 is to introduce all the definitions needed for the glueing construction, and to prove some preliminary results.

Let g and n be positive integers. Let K be a CM field of degree $2g$ over \mathbb{Q} , with totally real subfield $F \subset K$ and non-trivial element $\sigma \in \text{Gal}(K/F)$. Let \mathcal{O}_K (resp. \mathcal{O}_F) be the ring of integers of K (resp. F). Moreover, assume that the following holds:

Condition 2.1. *The different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a non-zero element $\eta \in \mathfrak{D}_K$ that satisfies $\sigma(\eta) = -\eta$.*

For an element $x \in K$, we will sometimes use the notation $\bar{x} = \sigma(x)$. Fix an element $\eta \in \mathcal{O}_K$ as in Condition 2.1, and let Ψ be a set of embeddings

$$\Psi = \{\tau_i: K \rightarrow \mathbb{C}\}_{1 \leq i \leq g} \quad \text{such that} \quad \Psi \cup \Psi\sigma = \{\tau_i, \tau_i\sigma\}_{1 \leq i \leq g} = \text{Hom}(K, \mathbb{C}), \quad (1)$$

and such that

$$\Im(\tau_i(\eta)) > 0 \quad \text{for each} \quad i \in \{1, \dots, g\}.$$

Let Λ be a free \mathcal{O}_K -module of rank $n + 1$ equipped with a hermitian form

$$h: \Lambda \times \Lambda \rightarrow \mathcal{O}_K$$

of the following signature (r_i, s_i) with respect to τ_i : we have

$$(r_i, s_i) = \begin{cases} (n, 1) & \text{if } i = 1, \\ (n + 1, 0) & \text{if } 2 \leq i \leq g. \end{cases}$$

Thus, h is \mathcal{O}_K -linear in its first argument, satisfies $h(y, x) = \sigma(h(x, y))$, and the complex vector space $\Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C}$ admits a basis $\{e_i\}$ such that $(h^{\tau_i}(e_i, e_j))_{ij}$ is a diagonal matrix with r_i diagonal entries equal to 1 and s_i diagonal entries equal to -1 . Here

$$h^{\tau_i}: \Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C} \times \Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C} \rightarrow \mathbb{C}$$

is the hermitian form attached to h via the embedding τ_i . Define

$$\tau = \tau_1 : K \rightarrow \mathbb{C}, \quad \text{and} \quad V = \Lambda \otimes_{\mathcal{O}_K, \tau} \mathbb{C}.$$

Let m be the largest positive integer for which the m -th cyclotomic field $\mathbb{Q}(\zeta_m)$ can be embedded in K , where $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$. Let $\zeta \in K$ be a primitive m -th root of unity in K , and define

$$\mu_K = \langle \zeta \rangle \subset \mathcal{O}_K^* \subset \mathcal{O}_K.$$

Moreover, define Γ to be the unitary group of Λ , and $P\Gamma$ as its quotient by μ_K :

$$\Gamma = U(\Lambda)(\mathcal{O}_K) = \text{Aut}_{\mathcal{O}_K}(\Lambda, h) \quad \text{and} \quad P\Gamma = \Gamma/\mu_K.$$

A norm one vector $r \in \Lambda$ is called a *short root*. Let $\mathcal{R} \subset \Lambda$ be the set of short roots. For $r \in \mathcal{R}$, define isometries $\phi_r^i : V \rightarrow V$ as follows:

$$\phi_r(x) = x - (1 - \zeta)h(x, r) \cdot r, \quad \phi_r^i(x) = x - (1 - \zeta^i)h(x, r) \cdot r, \quad i \in (\mathbb{Z}/m)^*.$$

Note that $\phi_r^i \in \Gamma$ for $r \in \mathcal{R}$, and that $\phi_r^i = \phi_r \circ \cdots \circ \phi_r$ (i times). In particular, $\phi_r^m = \text{id}$. Let $\mathbb{P}(V)$ be the projective space of lines in V , and let

$$\mathbb{C}H^n = \{\ell = [v] \in \mathbb{P}(V) \mid h(v, v) < 0\} \subset \mathbb{P}(V);$$

this is the space of negative lines in V . Define

$$H_r = \{x \in \mathbb{C}H^n : h(x, r) = 0\} \text{ for } r \in \mathcal{R}, \quad \text{and} \quad \mathcal{H} = \bigcup_{r \in \mathcal{R}} H_r \subset \mathbb{C}H^n.$$

Lemma 2.2. *The family of hyperplanes $(H_r)_{r \in \mathcal{R}}$ is locally finite, so that the hyperplane arrangement $\mathcal{H} \subset \mathbb{C}H^n$ is a divisor of $\mathbb{C}H^n$.*

Proof. See [Bea09, Lemma 5.3]. □

2.2 Orthogonality of the hyperplane arrangement. It turns out that Condition 2.1 implies that the following condition is satisfied, which is why we imposed it:

Condition 2.3. *If $r, t \in \mathcal{R}$ are such that $H_r \neq H_t$ and $H_r \cap H_t \neq \emptyset$, then $h(r, t) = 0$.*

The fact that Condition 2.3 follows from Condition 2.1 (in combination with the rest of the assumptions made above) will be proved in Section 4, see Theorem 5.2.

Remarks 2.4. 1. Condition 2.1 is satisfied by quadratic and cyclotomic CM fields K , see Proposition 5.4 in Section 4.

2. Our glueing construction depends heavily on Condition 2.3. However, Condition 2.1 was only imposed to ensure Condition 2.3, so alternatively, one could omit Condition 2.1 and simply assume Condition 2.3 instead.

3. By Theorem 5.2, Condition 2.1 implies Condition 2.3. We do not know if the converse is true, i.e. whether Condition 2.1 is strictly necessary.

4. There is a way to avoid Condition 2.3 altogether, see Remark 5.5, but the definitions become more complicated and we will not pursue that direction in this paper.

2.3 **Anti-unitary involutions.** Define an \mathcal{O}_F -linear map $\alpha : \Lambda \rightarrow \Lambda$ to be *anti-unitary* if for all $x, y \in \Lambda$ and $\lambda \in \mathcal{O}_K$, one has $\alpha(\lambda x) = \sigma(\lambda) \cdot \alpha(x)$ and $h(\alpha(x), \alpha(y)) = \sigma(h(x, y)) \in \mathcal{O}_K$. Define Γ' to be the group of unitary and anti-unitary \mathcal{O}_F -linear bijections $\Lambda \xrightarrow{\sim} \Lambda$. Let $\mathcal{A} \subset \Gamma'$ be the set of anti-unitary involutions $\alpha : \Lambda \rightarrow \Lambda$. Then

$$\mu_K \subset \Gamma \subset \Gamma' \quad - \quad \text{define} \quad P\Gamma' = \Gamma' / \mu_K. \quad (2)$$

Let $\lambda \in K^*$. Observe that

$$(\lambda \in \mathcal{O}_K^* \text{ and } |\lambda|^2 = \lambda \cdot \sigma(\lambda) = 1) \quad \iff \quad (\lambda \in \mu_K). \quad (3)$$

Indeed, we have, for any embedding $\varphi : K \rightarrow \mathbb{C}$, that

$$|\varphi(\lambda)|^2 = \varphi(\lambda) \cdot \overline{\varphi(\lambda)} = \varphi(\lambda) \cdot \varphi(\sigma(\lambda)) = \varphi(\lambda \cdot \sigma(\lambda)),$$

where $\overline{\varphi(\lambda)} = \varphi(\sigma(\lambda))$ by [Mil20, Proposition 1.4]. Moreover, we have $|\varphi(\lambda)| = 1$ for each $\varphi : K \rightarrow \mathbb{C}$ if and only if λ is a root of 1, see [Mil08, Corollary 5.6].

Lemma 2.5. *Let $\text{Isom}(\mathbb{C}H^n)$ be the group of isometries $f : \mathbb{C}H^n \xrightarrow{\sim} \mathbb{C}H^n$. The natural homomorphism $P\Gamma' \rightarrow \text{Isom}(\mathbb{C}H^n)$ is injective.*

Proof. This follows readily from (3). □

The group μ_K acts on \mathcal{A} by multiplication; define

$$P\mathcal{A} = \mu_K \backslash \mathcal{A}, \quad \text{and} \quad C\mathcal{A} = P\Gamma \backslash P\mathcal{A},$$

where $P\Gamma$ acts on $P\mathcal{A}$ by conjugation. Any $\alpha \in P\mathcal{A}$ defines an anti-holomorphic involution

$$\alpha : \mathbb{C}H^n \rightarrow \mathbb{C}H^n; \quad \text{define} \quad \mathbb{R}H_\alpha^n = (\mathbb{C}H^n)^\alpha \subset \mathbb{C}H^n.$$

For any element $\alpha \in \mathcal{A}$, the quadratic form $h|_{V^\alpha}$ on the real vector space

$$V^\alpha = \Lambda^\alpha \otimes_{\mathcal{O}_F, \tau|_F} \mathbb{C}$$

has hyperbolic signature. The following lemma is readily proved:

Lemma 2.6. *For $\alpha \in \mathcal{A}$, let $\mathbb{P}(V^\alpha)$ be the real projective space of lines in V^α , and let $\mathbb{R}H(V^\alpha) \subset \mathbb{P}(V^\alpha)$ be the space of negative lines in V^α . The canonical isomorphism $\mathbb{P}(V^\alpha) \cong \mathbb{P}(V)^\alpha$ restricts to an isomorphism $\mathbb{R}H(V^\alpha) \cong \mathbb{R}H_\alpha^n$. □*

We conclude that $\mathbb{R}H_\alpha^n \subset \mathbb{C}H^n$ is isometric to the real hyperbolic space of dimension n . Finally, we define

$$P\Gamma_\alpha = \text{Stab}_{P\Gamma}(\mathbb{R}H_\alpha^n) \subset P\Gamma \quad (\text{the stabilizer of } \mathbb{R}H_\alpha^n \text{ in } P\Gamma).$$

2.4 **Orthogonal hyperplanes and complex reflections.** Note that Condition 2.3 implies that if H_{r_1}, \dots, H_{r_k} for $r_i \in \mathcal{R}$ are mutually distinct, and if their common intersection is non-empty, then $\cap_{i=1}^k H_{r_i} \subset \mathbb{C}H^n$ is a totally geodesic subspace of codimension k . Note also that for any $r \in \mathcal{R}$, the element $\phi_r \in \Gamma$ generates a finite subgroup $\langle \phi_r \rangle \subset \Gamma$ of order m , and that the restriction of the quotient map $\pi: \Gamma \rightarrow P\Gamma$ to this subgroup $\langle \phi_r \rangle \subset \Gamma$ is injective. We will abuse notation, by letting $\phi_r \in P\Gamma$ denote the image of $\phi_r \in \Gamma$ in $P\Gamma$.

Definition 2.7. Let $\mathcal{H} = \{H_r \mid r \in \mathcal{R}\}$. For $x \in \mathbb{C}H^n$, define

$$\mathcal{H}(x) = \{H \in \mathcal{H} \mid x \in H\}, \quad G(x) = \langle \phi_r^i \in P\Gamma \text{ with } r \in \mathcal{R}, i \in \mathbb{Z}/m \mid x \in H_r \rangle.$$

The hyperplanes $H \in \mathcal{H}(x)$ are called the *nodes* of x . We say that x has k nodes if the cardinality of $\mathcal{H}(x)$ is k .

Lemma 2.8. Let $x \in \mathbb{C}H^n$ and suppose that x has k nodes. Then $G(x) \cong (\mathbb{Z}/m)^k$.

Proof. Let $r, t \in \mathcal{R}$. Then, for $z \in \Lambda$, one has

$$\begin{aligned} \phi_r^i(\phi_t^j(z)) &= \phi_r^i(z - (1 - \zeta^j)h(z, t) \cdot t) \\ &= z - (1 - \zeta^j)h(z, t) \cdot t - (1 - \zeta^i)h((z - (1 - \zeta^j)h(z, t) \cdot t), r) \cdot r \\ &= z - (1 - \zeta^j)h(z, t) \cdot t - (1 - \zeta^i)h(z, r) \cdot r + (1 - \zeta^i)(1 - \zeta^j)h(z, t)h(t, r) \cdot r. \end{aligned} \tag{4}$$

Now suppose that $H_r, H_t \in \mathcal{H}(x)$, with $H_r \neq H_t$. By Condition 2.3, we have $h(r, t) = 0$; by (4), this implies that $\phi_r^i \circ \phi_t^j = \phi_t^j \circ \phi_r^i$ for each $i, j \in \mathbb{Z}/m$. We conclude that the group $G(x)$ is abelian.

Next, suppose that $H_r = H_t \in \mathcal{H}(x)$. To finish the proof, it suffices to show that $\phi_t = \lambda \cdot \phi_r^i$ for some $i \in \mathbb{Z}/m$ and $\lambda \in \mu_K$. This follows from Lemma 2.9 below. \square

Lemma 2.9. Let $r \in \mathcal{R}$. Let $\phi: \mathbb{C}H^n \rightarrow \mathbb{C}H^n$ be an isometry of order m that restricts to the identity on $H_r \subset \mathbb{C}H^n$. Then $\phi = \phi_r^i$ for some $i \in \mathbb{Z}/m$.

Proof. Let $\mathbb{H}_{\mathbb{C}}^n$ be the hyperbolic space attached to the standard hermitian space $\mathbb{C}^{n,1}$ of dimension $n+1$. It is classical that

$$\text{Stab}_{U(n,1)}(\mathbb{H}_{\mathbb{C}}^{n-1}) = U(n-1, 1) \times U(1).$$

Thus, any $\phi \in U(n, 1)$ that fixes $\mathbb{H}_{\mathbb{C}}^{n-1}$ pointwise lies in $\mathbb{C}^* \times U(1)$, where $U(1)$ denotes $\{z \in \mathbb{C}^* : |z|^2 = 1\}$. If $\phi^m \in \mathbb{C}^* \times \{1\}$, then $\phi \in \mathbb{C}^* \times \langle \zeta \rangle \subset U(n-1, 1) \times U(1)$. \square

Lemma 2.5 allows us to view $P\mathcal{A}$ as a subset of $\text{Isom}(\mathbb{C}H^n)$, and also to view the groups $G(x) \subset P\Gamma \subset P\Gamma'$ (for any $x \in \mathbb{C}H^n$) as subgroups of $\text{Isom}(\mathbb{C}H^n)$. Define

$$\tilde{Y} = \coprod_{\alpha \in P\mathcal{A}} \mathbb{R}H_{\alpha}^n.$$

We will glue the different hyperbolic spaces $\mathbb{R}H_{\alpha}^n$, by defining an equivalence relation \sim on \tilde{Y} . Before we define it, we state and prove a couple of trivial results.

Lemma 2.10. *Let $\alpha \in \mathcal{A}$ and $r \in \mathcal{R}$. Then $\alpha \circ \phi_r^i = \phi_{\alpha(r)}^{-i} \circ \alpha$.*

Proof. Indeed, for $x \in \Lambda$, we have

$$\begin{aligned} \alpha(\phi_r^i(x)) &= \alpha(x - (1 - \zeta^i)h(x, r) \cdot r) = \alpha(x) - (1 - \zeta^{-i})\overline{h(x, r)} \cdot \alpha(r) \\ &= \alpha(x) - (1 - \zeta^{-i})h(\alpha(x), \alpha(r)) \cdot \alpha(r) = \phi_{\alpha(r)}^{-i}(\alpha(x)). \end{aligned}$$

□

Lemma 2.11. *Let $x \in \mathbb{R}H_\alpha^n$ and write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$ for some $r_i \in \mathcal{R}$. Then for each $i \in \{1, \dots, k\}$ there is a unique $j \in \{1, \dots, k\}$ such that $\alpha(H_{r_i}) = H_{\alpha(r_i)} = H_{r_j}$.*

Proof. Indeed, we have, for any $\beta \in \mathcal{A}$ and $r \in \mathcal{R}$, that $\beta(H_r) = H_{\beta(r)}$. Since $x \in H_{r_i}$, we have $x = \alpha(x) \in \alpha(H_{r_i}) = H_{\alpha(r_i)}$ for every i . In particular, we have $H_{\alpha(r_i)} \in \mathcal{H}(x)$ (see Definition 2.7), so that $H_{\alpha(r_i)} = H_{r_j}$ for some j . □

Proposition 2.12. *Let $r, t \in \mathcal{R}$. The following are equivalent:*

1. *One has $\phi_r = \phi_t \in \Gamma$.*
2. *There exist $i, j \in \mathbb{Z}/m - \{0\}$ such that $\phi_r^i = \phi_t^j \in \Gamma$.*
3. *There exist $a, b \in \mathcal{O}_K - \{0\}$ with $|a|^2 = |b|^2$ such that $a \cdot r = b \cdot t \in \mathcal{O}_K$.*

Proof. This follows readily from the definitions. □

Definition 2.13. Let $\alpha \in P\mathcal{A}$ and $x \in \mathbb{R}H_\alpha^n$. Write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$, see Definition 2.7. By Lemma 2.11, the involution α induces an involution on the set $\mathcal{H}(x)$. Define $\alpha: I \rightarrow I$ as the resulting involution on the set $I = \{1, \dots, k\}$.

Proposition 2.14. *Let $\alpha \in P\mathcal{A}$ and $x \in \mathbb{R}H_\alpha^n$. Write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$, and let $g = \phi_{r_1}^{i_1} \circ \dots \circ \phi_{r_k}^{i_k} \in G(x)$ for some $i_\nu \in \mathbb{Z}/m$. The following are equivalent:*

1. *We have $g \circ \alpha \in P\mathcal{A}$. (In other words, $g \circ \alpha$ is an involution.)*
2. *For each $\nu \in I$, we have $i_\nu \equiv i_{\alpha(\nu)} \pmod{m}$.*

Proof. This follows from Lemma's 2.11, 2.9 and 2.10 and Proposition 2.12. □

2.5 **The definition of the glued space.** We are now in position to formulate for which $\alpha, \beta \in P\mathcal{A}$ we will glue $\mathbb{R}H_\alpha^n$ to $\mathbb{R}H_\beta^n$, and how we are going to glue these spaces.

Definition 2.15. Define a relation $R \subset \tilde{Y} \times \tilde{Y}$ as follows. An element

$$(x_\alpha, y_\beta) \in \mathbb{R}H_\alpha^n \times \mathbb{R}H_\beta^n \subset \tilde{Y} \times \tilde{Y}$$

is an element of R if the following conditions are satisfied:

1. The images of x_α and y_β in $\mathbb{C}H^n$ agree.

2. If $\alpha \neq \beta$, then

- (a) $x_\alpha = y_\beta$ lies in \mathcal{H} ; and
- (b) $\beta = g \circ \alpha \in P\mathcal{A}$ for some $g \in G(x_\alpha) = G(y_\beta)$ (c.f. Lemma 2.5).

Remark 2.16. Conditions 1 and 2 in Definition 2.15 say that we are identifying points of $\mathbb{R}H_\alpha^n \cap \mathcal{H}$ and $\mathbb{R}H_\beta^n \cap \mathcal{H}$ that have the same image in $\mathbb{C}H^n$. But we do not glue all such points: the real structures α and β are required to differ by complex reflections in the hyperplanes that pass through x . In fact, we will see below (see Lemma 3.2) that the glueing rules can be rephrased as follows: we glue $\mathbb{R}H_\alpha^n$ and $\mathbb{R}H_\beta^n$ along their intersection, provided that for some (equivalently, any) $x \in \mathbb{R}H_\alpha^n \cap \mathbb{R}H_\beta^n$, the real structures α and β differ by reflections in hyperplanes $H_r \subset \mathcal{H}$ that pass through x .

Lemma 2.17. *R is an equivalence relation.*

Proof. Consider three elements $x_\alpha, y_\beta, z_\gamma \in \tilde{Y}$. The fact that $x_\alpha \sim x_\alpha$ is clear.

Suppose that $x_\alpha \sim y_\beta$. If $\alpha = \beta$ then $x_\alpha = y_\beta \in \tilde{Y}$ hence $y_\beta \sim x_\alpha$. If $\alpha \neq \beta$ then $x_\alpha = y_\beta \in \mathcal{H} \subset \mathbb{C}H^n$, and $\beta = g \circ \alpha$ for $g \in G(x_\alpha) = G(y_\beta)$ as in Definition 2.15. Since $\alpha = g^{-1} \circ \beta$ with $g^{-1} \in G(x_\alpha)$, this shows that $y_\beta \sim x_\alpha$.

Suppose that $x_\alpha \sim y_\beta$ and $y_\beta \sim z_\gamma$; we claim that $x_\alpha \sim z_\gamma$. We may and do assume that α, β and γ are different, which implies that $x_\alpha = y_\beta = z_\gamma \in \mathcal{H}$, that $\gamma = h \circ \beta$ for some $h \in G(y_\beta)$, and that $\beta = g \circ \alpha$ for some $g \in G(x_\alpha)$. We obtain $\gamma = h \circ \beta = h \circ g \circ \alpha$ for $h \circ g \in G(x_\alpha) = G(y_\beta) = G(z_\gamma)$. \square

Lemma 2.18. *The action of $P\Gamma$ on $\mathbb{C}H^n$ induces an action of $P\Gamma$ on \tilde{Y} which is compatible with the equivalence relation R . Therefore, $P\Gamma$ acts naturally on Y . Moreover, $P\Gamma \backslash \tilde{Y} = \coprod_{\alpha \in \mathcal{C}\mathcal{A}} P\Gamma_\alpha \backslash \mathbb{R}H_\alpha^n$.*

Proof. If $\phi \in P\Gamma$, then $\phi(\mathbb{R}H_\alpha^n) = \mathbb{R}H_{\phi\alpha\phi^{-1}}^n$ hence $P\Gamma$ acts on $\tilde{Y} = \coprod_{\alpha \in P\mathcal{A}} \mathbb{R}H_\alpha^n$, and

$$P\Gamma \backslash \tilde{Y} = P\Gamma \backslash \coprod_{\alpha \in P\mathcal{A}} \mathbb{R}H_\alpha^n = \coprod_{\alpha \in \mathcal{C}\mathcal{A}} P\Gamma_\alpha \backslash \mathbb{R}H_\alpha^n.$$

Now suppose that $x_\alpha \sim y_\beta \in \tilde{Y}$ and $f \in P\Gamma$. Then $f(x_\alpha) \in \mathbb{R}H_{f\alpha f^{-1}}^n$ and $f(y_\beta) \in \mathbb{R}H_{f\beta f^{-1}}^n$. We claim that $f(x_\alpha)_{f\alpha f^{-1}} \sim f(y_\beta)_{f\beta f^{-1}}$. For this, we may and do assume that $x_\alpha \neq y_\beta$, hence $x_\alpha = y_\beta \in \mathcal{H}$ and $\beta = g \circ \alpha$ for some $g \in G(x_\alpha)$ as in Definition 2.15. In particular, $f(x_\alpha) = f(y_\beta)$. Since $f \circ \phi_r^i \circ f^{-1} = \phi_{f(r)}^i$ for each $r \in \mathcal{R}$ and $i \in \mathbb{Z}/m$, and $h(x, r) = 0$ if and only if $h(f(x), f(r)) = 0$, we have $fG(x)f^{-1} = G(f(x))$ for each $x \in \mathbb{C}H^n$. This proves that $f\beta f^{-1} = f(g \circ \alpha)f^{-1} = fgf^{-1} \circ (f\alpha f^{-1})$ with $fgf^{-1} \in G(f(x_\alpha))$, and therefore $f(x_\alpha) \sim f(y_\beta) \in \tilde{Y}$. \square

Definition 2.19. Define $Y(\Lambda)$ as the quotient of \tilde{Y} by the equivalence relation R introduced in Definition 2.15, and equip it with the quotient topology. By Lemma 2.18, the group $P\Gamma$ acts on $Y(\Lambda)$. We define

$$X(\Lambda) = P\Gamma \backslash Y(\Lambda),$$

and call $X(\Lambda)$ the *glued space* attached to the hermitian \mathcal{O}_K -lattice Λ .

3 The hyperbolic orbifold structure of the glued space

Section 3 is devoted to the proof of the following theorem.

Theorem 3.1. 1. *The glued space $X(\Lambda) = P\Gamma \backslash Y(\Lambda)$ admits a metric that makes it a complete path metric space. The natural map $X(\Lambda) \rightarrow P\Gamma \backslash \mathbb{C}H^n$ is a local isometry.*

2. *Each point $x \in X(\Lambda)$ admits an open neighborhood $U \subset X(\Lambda)$ which is isometric to the quotient of an open subset $V \subset \mathbb{R}H^n$ by a finite group of isometries. Therefore, the glued space $X(\Lambda)$ has a real hyperbolic orbifold structure.*

3. *One has $\coprod_{\alpha \in C\mathcal{A}} P\Gamma_\alpha \backslash (\mathbb{R}H_\alpha^n - \mathcal{H}) \subset X(\Lambda)$ as an open suborbifold.*

4. *The connected components of the real hyperbolic orbifold $X(\Lambda)$ are uniformized by $\mathbb{R}H^n$: for each connected component $C \subset X(\Lambda)$ there exists a lattice $P\Gamma_C \subset \text{PO}(n, 1)$ and an isomorphism of real hyperbolic orbifolds $C \cong P\Gamma_C \backslash \mathbb{R}H^n$. Consequently,*

$$X(\Lambda) \cong \coprod_{C \in \pi_0(P\Gamma \backslash K)} P\Gamma_C \backslash \mathbb{R}H^n.$$

It can happen that $X(\Lambda)$ is connected: such is the case when $K = \mathbb{Q}(\zeta_3)$ and $\Lambda = \mathbb{Z}[\zeta_3]^{2,1}$ or $\Lambda = \mathbb{Z}[\zeta_3]^{4,1}$, see [ACT06; ACT10]. If $K = \mathbb{Q}(\zeta_3)$ and $\Lambda = \mathbb{Z}[\zeta_3]^{3,1}$, then $X(\Lambda)$ has two components, see [ACT07, Remark 6].

3.1 **The path metric on the glued space.** We start with a lemma. We will need it in the proof of Lemma 3.4 below, which will be used to define a path metric on $X(\Lambda)$ making it locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries. It also serves as a sanity check: if there exists one element $x \in \mathbb{R}H_\alpha^n \cap \mathbb{R}H_\beta^n$ such that $x_\alpha \sim x_\beta$, then one glues the entire space $\mathbb{R}H_\alpha^n$ to the space $\mathbb{R}H_\beta^n$ along their intersection in $\mathbb{C}H^n$.

Lemma 3.2. 1. *Let $g = \prod_{\nu=1}^k \phi_{r_\nu}^{i_\nu} \in \Gamma$ for some set $\{r_\nu\} \subset \mathcal{R}$ of mutually orthogonal short roots r_ν , where $i_\nu \not\equiv 0 \pmod m$ for each ν . Then $(\mathbb{C}H^n)^g = \cap_{\nu=1}^k H_{r_\nu}$.*

2. *Let $\alpha, \beta \in P\mathcal{A}$ and $x \in \mathbb{R}H_\alpha^n \cap \mathbb{R}H_\beta^n$ such that $x_\alpha \sim x_\beta$. Then $y_\alpha \sim y_\beta$ for every $y \in \mathbb{R}H_\alpha^n \cap \mathbb{R}H_\beta^n$.*

3. *The natural map $\tilde{Y} \rightarrow \mathbb{C}H^n$ descends to a $P\Gamma$ -equivariant map $\mathcal{P} : Y(\Lambda) \rightarrow \mathbb{C}H^n$.*

Proof. 1. Let $y \in V$ be representing an element in $(\mathbb{C}H^n)^\phi$. Since the r_i are orthogonal, and $g(y) = \lambda$ for some $\lambda \in \mathbb{C}^*$, we have

$$g(y) = \prod_{\nu=1}^k \phi_{r_\nu}^{i_\nu}(y) = y - \sum_{\nu=1}^k (1 - \zeta^{i_\nu}) h(y, r_\nu) r_\nu = \lambda y, \quad (5)$$

hence $(1 - \lambda)y = \sum_{i=1}^k (1 - \zeta^{i_\nu}) h(y, r_\nu) r_\nu \in V$. But y spans a negative definite subspace of V while the r_ν span a positive definite subspace, so that we must have $1 - \lambda = 0 = \sum_{\nu=1}^k (1 - \zeta^{i_\nu}) h(y, r_\nu) r_\nu$. Since the r_ν are mutually orthogonal, they

are linearly independent; since $\zeta^{i\nu} \neq 1$ we find $h(y, r_\nu) = 0$ for each ν . Conversely, if $x \in \cap H_{r_\nu}$, then $\phi_{r_\nu}^{i\nu}(x) = x$ for each ν .

2. Since $x_\alpha \sim x_\beta$, there exists $g \in G(x)$ such that $\beta = g \circ \alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$. Let $y \in \mathbb{R}H_\alpha^n \cap \mathbb{R}H_\beta^n$. Then $\alpha(y) = \beta(y) = y$ implies that $g(y) = y$. In particular, $y \in \cap_\nu H_{r_\nu}$ by Part 1, which implies that $\mathcal{H}(x) \subset \mathcal{H}(y)$, which in turn implies that $G(x) \subset G(y)$. We conclude that $g \in G(y)$. Hence $y_\alpha \sim y_\beta$.

3. If $x_\alpha \sim y_\beta$, then $x = y \in \mathbb{C}H^n$. □

By Lemma 3.2, we obtain continuous maps

$$\mathcal{P}: Y(\Lambda) \rightarrow \mathbb{C}H^n \quad \text{and} \quad \overline{\mathcal{P}}: X(\Lambda) = P\Gamma \backslash Y(\Lambda) \rightarrow P\Gamma \backslash \mathbb{C}H^n.$$

Our next goal is to prove that each point $x \in Y(\Lambda)$ has a neighbourhood $V \subset Y(\Lambda)$ that maps homeomorphically onto a finite union $\cup_{i=1}^\ell \mathbb{R}H_{\alpha_i}^n \subset \mathbb{C}H^n$. Hence x has an open neighbourhood $x \in U \subset V$ that identifies with an open set in a union of copies of $\mathbb{R}H^n$ in $\mathbb{C}H^n$ under the map \mathcal{P} . This allows us to define a metric on $Y(\Lambda)$ by pulling back the metric on $\mathbb{C}H^n$.

Lemma 3.3. *Each compact set $Z \subset \mathbb{C}H^n$ meets only finitely many $\mathbb{R}H_\alpha^n$, $\alpha \in P\mathcal{A}$.*

Proof. Recall the subgroup $P\Gamma' \subset \text{Isom}(\mathbb{C}H^n)$ (see (2) and Lemma 2.5). We have that $P\Gamma'$ acts properly discontinuously on $\mathbb{C}H^n$. So if S is the set of $\alpha \in P\mathcal{A}$ such that $\alpha Z \cap Z \neq \emptyset$, then S is finite. In particular, Z meets only finitely many $\mathbb{R}H_\alpha^n$. □

Fix a point $f \in Y(\Lambda)$ and a point $x_\alpha \in \tilde{Y}$ lying above f . Let $\alpha_1, \dots, \alpha_\ell$ be the elements in $P\mathcal{A}$ such that $x_{\alpha_i} \sim x_\alpha$ for each $i \in I := \{1, \dots, \ell\}$ (since the group $G(x)$ is finite by Lemma 2.8, these are finite in number).

Let $p: \tilde{Y} \rightarrow Y(\Lambda)$ be the quotient map, and define

$$Y_f = p \left(\prod_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n \right) \subset Y(\Lambda). \quad (6)$$

We prove that $Y(\Lambda)$ is locally isometric to opens in unions of real hyperbolic subspaces of $\mathbb{C}H^n$. Indeed, we have the following:

Lemma 3.4. 1. *The set Y_f is closed in $Y(\Lambda)$.*

2. *We have $\mathcal{P}(Y_f) = \cup_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n \subset \mathbb{C}H^n$, and the map*

$$\mathcal{P}_f: Y_f \rightarrow \bigcup_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n$$

induced by \mathcal{P} is a homeomorphism.

3. *The set $Y_f \subset Y(\Lambda)$ contains an open neighborhood U_f of f in $Y(\Lambda)$.*

Proof. 1. One has

$$p^{-1}(Y_f) = p^{-1}\left(p\left(\prod_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n\right)\right) = \bigcup_{i=1}^{\ell} p^{-1}(p(\mathbb{R}H_{\alpha_i}^n)) \subset \tilde{Y}.$$

Therefore, it suffices to show that $p^{-1}(p(\mathbb{R}H_{\alpha_i}^n))$ is closed in \tilde{Y} . But notice that $x_{\beta} \in p^{-1}(p(\mathbb{R}H_{\alpha}^n))$ if and only if $x \in \mathbb{R}H_{\alpha}^n$ and $x_{\alpha} \sim x_{\beta}$, which implies (Lemma 3.2) that $\mathbb{R}H_{\alpha}^n \cap \mathbb{R}H_{\beta}^n \subset p^{-1}(p(\mathbb{R}H_{\alpha}^n))$. Hence for any $\alpha \in P\mathcal{A}$, one has

$$p^{-1}(p(\mathbb{R}H_{\alpha}^n)) = \prod_{\beta \sim \alpha} \mathbb{R}H_{\alpha}^n \cap \mathbb{R}H_{\beta}^n,$$

where $\beta \sim \alpha$ if and only if there exists $x \in \mathbb{R}H_{\alpha}^n \cap \mathbb{R}H_{\beta}^n$ such that $x_{\alpha} \sim x_{\beta}$. It follows that $p^{-1}(p(\mathbb{R}H_{\alpha}^n)) \cap \mathbb{R}H_{\beta}^n$ is closed in $\mathbb{R}H_{\beta}^n$ for every $\beta \in P\mathcal{A}$. But the $\mathbb{R}H_{\beta}^n$ are open in \tilde{Y} and cover \tilde{Y} , so that $p^{-1}(p(\mathbb{R}H_{\alpha}^n))$ is closed in \tilde{Y} .

2. We have

$$\mathcal{P}_f(Y_f) = \mathcal{P}\left(p\left(\prod_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n\right)\right) = \tilde{\mathcal{P}}\left(\prod_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n\right) = \bigcup_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n \subset \mathbb{C}H^n.$$

To prove injectivity, let $x_{\alpha_i}, y_{\alpha_j} \in \tilde{Y}$ and suppose that $x = y \in \mathbb{C}H^n$. Then indeed, $x_{\alpha_i} \sim y_{\alpha_j}$ because \sim is an equivalence relation by Lemma 2.17.

Let $Z \subset \mathbb{C}H^n$ be a compact set. Write

$$\tilde{\mathcal{P}} : \tilde{Y} \rightarrow \mathbb{C}H^n$$

for the canonical map. Remark that Z meets only finitely many of the $\mathbb{R}H_{\alpha}^n$ for $\alpha \in P\mathcal{A}$, see Lemma 3.3. Each $Z \cap \mathbb{R}H_{\alpha}^n$ is closed in Z since $\mathbb{R}H_{\alpha}^n$ is closed in $\mathbb{C}H^n$, so each $Z \cap \mathbb{R}H_{\alpha}^n$ is compact. We conclude that $\tilde{\mathcal{P}}^{-1}(Z) = \prod Z \cap \mathbb{R}H_{\alpha}^n$ is compact. In particular, $\tilde{\mathcal{P}}$ is closed [Lee13, Theorem A.57].

Finally, we prove that \mathcal{P}_f is closed. Let $Z \subset Y_f$ be a closed set. Then Z is closed in Y by Part 1, hence $p^{-1}(Z)$ is closed in \tilde{Y} , hence $\tilde{\mathcal{P}}(p^{-1}(Z))$ is closed in $\mathbb{C}H^n$, so that

$$\mathcal{P}_f(Z) = \mathcal{P}(Z) = \tilde{\mathcal{P}}(p^{-1}(Z)) = \left(\tilde{\mathcal{P}}(p^{-1}(Z))\right) \cap \left(\bigcup_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n\right)$$

is closed in $\bigcup_{i=1}^{\ell} \mathbb{R}H_{\alpha_i}^n$.

3. Let $x = \mathcal{P}(f) \in \mathbb{C}H^n$. Since $\mathbb{C}H^n$ is locally compact, there exists a compact set $Z \subset \mathbb{C}H^n$ and an open set $U \subset \mathbb{C}H^n$ with $x \in U \subset Z$. Since Z is compact, it meets only finitely many of the $\mathbb{R}H_{\beta}^n \subset \mathbb{C}H^n$ (Lemma 3.3). Consequently, the same holds for U ; define $V = \mathcal{P}^{-1}(U) \subset Y(\Lambda)$. Define

$$\mathcal{B} = \{\beta \in P\mathcal{A} : U \cap \mathbb{R}H_{\beta}^n \neq \emptyset\}.$$

Also define, for $\beta \in P\mathcal{A}$, $Z_{\beta} = p(\mathbb{R}H_{\beta}^n) \subset Y(\Lambda)$. Then

$$f \in V \subset \bigcup_{\beta \in \mathcal{B}} Z_{\beta} = \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta(x)=x}} Z_{\beta} \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta(x) \neq x}} Z_{\beta}.$$

Since each Z_β is closed in $Y(\Lambda)$ by the proof of part 1, there is an open $V' \subset V$ with

$$f \in V' \subset \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta(x)=x}} Z_\beta = \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta \cup \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta(x)=x \\ x_\beta \not\sim x_\alpha}} Z_\beta$$

Hence again there exists an open subset $V'' \subset V'$ with

$$f \in V'' \subset \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta \subset \bigcup_{\substack{\beta \in P\mathcal{A} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta = Y_f.$$

Therefore, $U_f := V'' \subset Y$ satisfies the requirements. \square

We need one further lemma:

Lemma 3.5. *The topological space $Y(\Lambda)$ is Hausdorff.*

Proof. Let $f, f' \in Y(\Lambda)$ be elements such that $f \neq f'$. First suppose that $f \notin Y_{f'}$. Since $Y_{f'}$ is closed in $Y(\Lambda)$ by Lemma 3.4, there is an open neighbourhood U of f such that $U \cap U_{f'} \subset U \cap Y_{f'} = \emptyset$.

Next, suppose that $f \in Y_{f'}$. Lift f and f' to elements $x_\alpha, y_\beta \in \tilde{Y}$. Assume first that $x = y$. This means that $\mathcal{P}(f) = \mathcal{P}(f')$. Since $\mathcal{P} : Y_{f'} \rightarrow \mathbb{C}H^n$ is injective, this implies that $f = f'$, contradiction. So we have $x \neq y \in \mathbb{C}H^n$. But $\mathbb{C}H^n$ is Hausdorff, so there are open subsets $(U \subset \mathbb{C}H^n, V \subset \mathbb{C}H^n)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Then $\mathcal{P}^{-1}(U) \cap \mathcal{P}^{-1}(V) = \emptyset$. \square

We then obtain:

Proposition 3.6. *$Y(\Lambda)$ is naturally a path metric space, piecewise isometric to $\mathbb{R}H^n$.*

Proof. For each $f \in Y(\Lambda)$ there exists an open neighborhood $f \in U_f \subset Y(\Lambda)$ such that \mathcal{P} induces a homeomorphism $Y \supset U_f \xrightarrow{\sim} \mathcal{P}(U_f) \subset \mathbb{C}H^n$. Indeed, this follows from Lemma 3.4. Pull back the metric on $\mathcal{P}(U_f)$ to obtain a metric on U_f . Then define a metric on $Y(\Lambda)$ as the largest metric which is compatible with the metric on each open set U_f and which preserves the lengths of paths. \square

Proposition 3.7. *The path metric on $Y(\Lambda)$ descends to a path metric on $X(\Lambda)$.*

Proof. The metric on $Y(\Lambda)$ descends in any case to a pseudo-metric on $X(\Lambda)$, and by [Gro07, Chapter 1], this is a metric if $P\Gamma$ acts by isometries on $Y(\Lambda)$ with closed orbits. This is true: the fact that $P\Gamma$ acts isometrically on $Y(\Lambda)$ comes from the $P\Gamma$ -equivariance of $\mathcal{P} : Y \rightarrow \mathbb{C}H^n$ (Lemma 3.2) together with the construction of the metric on $Y(\Lambda)$ (Proposition 3.6). To check that the $P\Gamma$ -orbits are closed in $Y(\Lambda)$, let $f \in Y(\Lambda)$ with representative $x_\alpha \in \tilde{Y}$. By equivariance of $p : \tilde{Y} \rightarrow Y(\Lambda)$, we have $p^{-1}(P\Gamma \cdot f) = P\Gamma \cdot (p^{-1}f)$, so since p is a quotient map, it suffices to show that

$$P\Gamma \cdot (p^{-1}f) = P\Gamma \cdot \bigcup_{x_\beta \sim x_\alpha} x_\beta = \bigcup_{x_\beta \sim x_\alpha} P\Gamma \cdot x_\beta$$

is closed in \tilde{Y} , thus that each orbit $P\Gamma \cdot x_\beta$ is closed in \tilde{Y} . Since $P\Gamma$ is discrete, it suffices to show that $P\Gamma$ acts properly on \tilde{Y} . So let $Z \subset \tilde{Y}$ be any compact set: we claim that $\{g \in P\Gamma : gZ \cap Z \neq \emptyset\}$ is a finite set. Indeed, for each $g \in P\Gamma$, one has $\tilde{\mathcal{P}}(gZ \cap Z) \subset g\tilde{\mathcal{P}}(Z) \cap \tilde{\mathcal{P}}(Z)$, and the latter is non-empty for only finitely many $g \in P\Gamma$, by properness of the action of $P\Gamma$ on $\mathbb{C}H^n$.

Since the metric on $Y(\Lambda)$ is a path metric, so is the metric on $P\Gamma \backslash Y(\Lambda)$ [Gro07]. \square

3.2 The orbifold structure on the glued space. The next step is to prove that the glued space $X(\Lambda) = P\Gamma \backslash Y(\Lambda)$ (see Definition 2.19) is locally isometric to quotients of open sets in $\mathbb{R}H^n$ by finite groups of isometries.

Definition 3.8. Let $f \in Y(\Lambda)$ with representative $x_\alpha \in \tilde{Y}$. Thus, x is an element in $\mathbb{C}H^n$, and $\alpha \in P\mathcal{A}$ is the class of an anti-unitary involution such that $\alpha(x) = x$.

1. The *nodes* of f are by definition the nodes of x_α (see Definition 2.7). Thus, these are the hyperplanes $H \in \mathcal{H}(x)$, i.e. the hyperplanes $H_r \in \mathcal{H}$ defined by short roots $r \in \mathcal{R}$ such that $x \in H_r$ (equivalently, such that $h(x, r) = 0$).

2. The number of nodes of f is the cardinality of $\mathcal{H}(x)$.

3. The anti-unitary involution $\alpha \in P\mathcal{A}$ induces an involution on the set $\mathcal{H}(x)$ by Lemma 2.11. Let $H \in \mathcal{H}(x)$ be a node. We call H a *real node* of f if $\alpha(H) = H$. We call $(H, \alpha(H))$ a *pair of complex conjugate nodes* of f if $\alpha(H) \neq H$.

4. If k is the number of nodes of f , we generally write $k = 2a + b$, with a the number of pairs of complex conjugate nodes of f , and b the number of real nodes of f .

Fix again a point $f \in Y(\Lambda)$ and a point $x_\alpha \in \tilde{Y}$ lying above f . Let $k = 2a + b$ be the number of nodes of f . Thus $x \in \mathbb{R}H_\alpha^n$, and there exist $r_1, \dots, r_k \in \mathcal{R}$ such that

$$\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}, \quad G(x) = \langle \phi_{r_1}, \dots, \phi_{r_k} \rangle \cong (\mathbb{Z}/m)^k.$$

For $\beta \in P\mathcal{A}$, observe that $x_\beta \sim x_\alpha$ if and only if $\alpha \circ \beta \in G(x)$. We relabel the r_i so that they satisfy the following condition:

$$\begin{aligned} \alpha(H_{r_i}) &= H_{r_{i+1}} \text{ for } i \text{ odd and } i \leq 2a, \\ \alpha(H_{r_i}) &= H_{r_{i-1}} \text{ for } i \text{ even and } i \leq 2a, \text{ and} \\ \alpha(H_{r_i}) &= H_{r_i} \text{ for } i \in \{2a+1, \dots, k\}. \end{aligned} \tag{7}$$

In other words, H_{r_i} is a real root if and only if $i > 2a$, and $(H_{r_i}, H_{r_{i+1}})$ is a pair of complex conjugate roots if and only if $i < 2a$ is odd.

Lemma 3.9. *Continue with the notation from above.*

1. Let $\beta \in P\mathcal{A}$ be such that $x_\beta \sim x_\alpha$. Then

$$\beta = \prod_{i=1}^a (\phi_{r_{2i-1}} \circ \phi_{r_{2i}})^{j_i} \circ \prod_{i=2a+1}^k \phi_{r_i}^{j_i} \circ \alpha$$

for some $j_1, \dots, j_a, j_{2a+1}, \dots, j_k \in \mathbb{Z}/m$. In particular, there are m^{a+b} such β .

2. There is an isometry $\mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$ identifying x with the origin, ϕ_{r_i} with the map

$$\mathbb{B}^n(\mathbb{C}) \rightarrow \mathbb{B}^n(\mathbb{C}), \quad (t_1, \dots, t_i, \dots, t_n) \mapsto (t_1, \dots, \zeta t_i, \dots, t_n),$$

and α with the map defined by

$$t_i \mapsto \begin{cases} \bar{t}_{i+1} & \text{for } i \text{ odd and } i \leq 2a \\ \bar{t}_{i-1} & \text{for } i \text{ even and } i \leq 2a \\ \bar{t}_i & \text{for } i > 2a. \end{cases} \quad (8)$$

Proof. 1. This follows readily from Proposition 2.14.

2. Since the H_{r_i} are orthogonal by Condition 2.3, and their intersection contains x , we can find coordinates t_1, \dots, t_{n+1} on V that induce an identification $(V, h) \cong \mathbb{C}^{n,1} := (\mathbb{C}^{n+1}, H)$ with $H(x, x) = |x_1|^2 + \dots + |x_n|^2 - |x_{n+1}|^2$, in such a way that $H_{r_i} \subset V$ is identified with the hyperplane $\{t_i = 0\} \subset \mathbb{C}^{n+1}$ and $x \in \cap_i H_{r_i}$ with the point $(0, 0, \dots, 0, 1)$. We will do this in the following way. Define

$$T = \langle x \rangle \oplus \langle r_1 \rangle \oplus \dots \oplus \langle r_k \rangle \subset V, \quad W = T^\perp = \{w \in V \mid h(w, t) = 0 \ \forall t \in T\}.$$

For each $i \in I = \{1, \dots, k\}$, we have $\alpha(r_i) = \lambda_i \cdot r_{\alpha(i)}$ for some $\lambda_i \in K$ (see Lemma 2.11, Definition 2.13, and Lemmas 2.9 and 2.8). Observe that $\alpha(W) = W$. Since $W \subset \langle x \rangle^\perp$, the hermitian space $(W, h|_W)$ is positive definite. Let $\{w_1, \dots, w_{n-k}\} \subset W$ be an orthonormal basis such that $\alpha(w_i) = w_i$, which exists by the elementary

Lemma 3.10. *Let (W, h) a non-degenerate hermitian vector space of dimension $n \geq 1$ and let $\alpha: W \rightarrow W$ be an anti-linear involution with $h(\alpha(x), \alpha(y)) = \overline{h(x, y)}$ for $x, y \in W$. For each positive integer $m \leq n$, there exists a linearly independent set $\{w_i\}_{i=1}^m \subset W$ such that $h(w_i, w_j) = \pm \delta_{ij}$ and $\alpha(w_i) = w_i$ for each $i = 1, \dots, m$. \square*

Let $\{e_i\}_{i=1}^{n+1}$ be the standard basis of \mathbb{C}^{n+1} , and define a \mathbb{C} -linear isomorphism

$$\Phi: V \xrightarrow{\sim} \mathbb{C}^{n+1}, \quad \left(\frac{x}{h(x, x)} \mapsto e_{n+1}, \quad r_i \mapsto e_i, \quad w_i \mapsto e_i \right). \quad (9)$$

By (7), we have that $\alpha(r_i) = \lambda_i \cdot r_{i+1}$ for i odd and $i \leq 2a$, that $\alpha(r_i) = \lambda_i \cdot r_{i-1}$ for i even and $i \leq 2a$, and that $\alpha(r_i) = \lambda_i \cdot r_i$ for $i > 2a$. We conclude that the anti-linear involution on \mathbb{C}^{n+1} induced by α and (9) corresponds to the matrix

$$\alpha = \begin{pmatrix} 0 & \alpha_1 & \dots & 0 & \dots & \dots & 0 \\ \alpha_2 & 0 & 0 & 0 & \dots & \dots & \vdots \\ 0 & 0 & 0 & \alpha_4 & \dots & \dots & \vdots \\ 0 & 0 & \alpha_3 & 0 & \dots & \dots & \vdots \\ \vdots & 0 & \ddots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & 0 & \vdots & \dots & \alpha_n & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & \alpha_{n+1} \end{pmatrix}$$

where each α_i is an anti-linear involution $\mathbb{C} \rightarrow \mathbb{C}$, and $\alpha_i = \alpha_{i+1}$ for $i < 2a$ odd. If $\alpha_i(1) = \mu_i \in \mathbb{C}^*$, then $\mu_i^{-1} \cdot \alpha_i = \text{conj}: \mathbb{C} \rightarrow \mathbb{C}$ (complex conjugation). Since $|\mu_i| = 1$, there exists $\rho_i \in \mathbb{C}$ such that $\mu_i = \bar{\rho}_i/\rho_i$ and $|\rho_i| = 1$. This gives $\mu_i^{-1} \cdot \alpha_i = \rho_i \cdot \alpha_i \cdot \rho_i^{-1} = \text{conj}: \mathbb{C} \rightarrow \mathbb{C}$. The composition of $\Phi: V \rightarrow \mathbb{C}^{n+1}$ with the diagonal linear transformation $\text{diag}(\rho_1, \dots, \rho_{n+1}): \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ induces an isomorphism $\mathbb{C}H^n \cong \mathbb{B}^n(\mathbb{C})$ with the required properties. \square

Definition 3.11. 1. Define $A_f = \text{Stab}_{P\Gamma}(f)$ to be the subgroup of $P\Gamma$ fixing $f \in Y(\Lambda)$. This contains the group $G(x) \cong (\mathbb{Z}/m)^k$.

2. Define B_f as the subgroup of $G(x)$ generated by the order m complex reflections associated to the real nodes of f . Hence $B_f = \langle \phi_{r_i} \rangle_{i>2a} \cong (\mathbb{Z}/m)^b$.

Recall the quotient map $p: \tilde{Y} \rightarrow Y(\Lambda)$, the definition (6) of Y_f , and Lemma 3.4.

Lemma 3.12. *The stabilizer A_f of $f \in Y(\Lambda)$ preserves the subset $Y_f \subset Y(\Lambda)$.*

Proof. Let $\psi \in A_f$, with $f = p(x_\alpha) \in Y(\Lambda)$, $x \in \cap_i H_{r_i}$. Then $\psi(x)_{\psi\alpha\psi^{-1}} \sim x_\alpha$. Now let $p(y_\beta) \in Y_f$. Then $\beta(x) = x$ and $x_\alpha \sim x_\beta$. Hence $x_\alpha \sim \psi \cdot x_\alpha \sim \psi \cdot x_\beta = \psi(x)_{\psi\beta\psi^{-1}}$. This implies that $\psi\beta\psi^{-1} \circ \alpha \in G(x)$, so that $p(\psi(y)_{\psi\beta\psi^{-1}}) \in Y_f$. \square

We also need the following lemma. Write $m = 2^a k$ with $k \not\equiv 0 \pmod{2}$.

Lemma 3.13. *Let $T = \{t \in \mathbb{C} : t^m \in \mathbb{R}\}$. Then $G = \langle \zeta_m \rangle$ acts on T by multiplication. Each element in T/G has a unique representative $\zeta_{2^{a+1}}^\epsilon \cdot r \in T$ for $r \geq 0$ and $\epsilon \in \{0, 1\}$.*

Proof. Therefore, we have $a \geq 1$. Next, observe that $t = r\zeta_{2^m}^j$ for some $j \in \mathbb{Z}$ and $r \in \mathbb{R}$ if and only if $t^m \in \mathbb{R}$. One easily shows that since $\gcd(2, k) = 1$, we have $\zeta_{2^{a+1}} \cdot \zeta_{2^a k} = (\zeta_{2^{a+1}k})^{k+2}$. Raising both sides to the power $b = (k+2)^{-1} \in (\mathbb{Z}/m)^*$ gives $\zeta_{2^m} = \zeta_{2^{a+1}}^b \cdot \zeta_m^b$. Consequently, $t^m \in \mathbb{R}$ if and only if $t = r \cdot \zeta_{2^{a+1}}^{bj} \cdot \zeta_m^{bj}$ for some $r \in \mathbb{R}$. Finally, $\zeta_{2^{a+1}}^u \cdot \zeta_{2^a}^v = \zeta_{2^{a+1}}^{u+2v}$ hence $\langle \zeta_{2^{a+1}} \rangle / \langle \zeta_{2^a} \rangle \cong \mathbb{Z}/2$. \square

We obtain the key to Theorem 3.1.

Proposition 3.14. *Keep the above notations, and consider $Y_f \subset Y(\Lambda)$ (see (6)).*

1. *If f has no nodes, then $G(x) = B_f$ is trivial, and $Y_f = \mathbb{R}H_\alpha^n \cong \mathbb{B}^n(\mathbb{R})$.*
2. *If f has only real nodes, then $B_f \setminus Y_f$ is isometric to $\mathbb{B}^n(\mathbb{R})$.*
3. *If f has a pairs of complex conjugate nodes ($k = 2a$), and no other nodes, then $B_f \setminus Y_f = Y_f$ is the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$, any two of which meet along a $\mathbb{B}^{2c}(\mathbb{R})$ for some integer c with $0 \leq c \leq a$.*
4. *If f has $2a$ complex conjugate nodes and b real nodes, then there is an isometry between $B_f \setminus Y_f$ and the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$ identified along common $\mathbb{B}^{2c}(\mathbb{R})$'s, that is, the set Y_f of case 3 above.*

5. In each case, A_f acts transitively on the indicated copies of $\mathbb{B}^n(\mathbb{R})$. If $\mathbb{B}^n(\mathbb{R})$ is any one of them, and $\Gamma_f = (A_f/B_f)_{\mathbb{B}^n(\mathbb{R})}$ its stabilizer, then the natural map

$$\Gamma_f \backslash \mathbb{B}^n(\mathbb{R}) \rightarrow (A_f/B_f) \backslash (B_f \backslash Y_f) = A_f \backslash Y_f$$

is an isometry of path metrics.

Proof. 1. This is clear.

2. Suppose then that f has k real nodes. Then in the local coordinates t_i of Lemma 3.9.2, we have that $\alpha : \mathbb{B}^n(\mathbb{C}) \rightarrow \mathbb{B}^n(\mathbb{C})$ is defined by $\alpha(t_i) = \bar{t}_i$. Part 1 of the same lemma shows that any $\beta \in P\mathcal{A}$ fixing x such that $x_\alpha \sim x_\beta$ is of the form

$$\mathbb{B}^n(\mathbb{C}) \rightarrow \mathbb{B}^n(\mathbb{C}), \quad (t_1, \dots, t_i, \dots, t_n) \mapsto (\bar{t}_1 \zeta^{j_1}, \dots, \bar{t}_k \zeta^{j_k}, \bar{t}_{k+1}, \dots, \bar{t}_n).$$

Since f has k real nodes and no complex conjugate nodes, we have (writing $j = (j_1, \dots, j_k)$ and $\alpha_j = \prod_{i=1}^k \phi_{r_i}^{j_i} \circ \alpha$):

$$Y_f \cong \bigcup_{j_1, \dots, j_k=1}^m \mathbb{R}H_{\alpha_j}^n \cong \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_1^m, \dots, t_k^m, t_{k+1}, \dots, t_n \in \mathbb{R}\}.$$

Each of the 2^k subsets

$$K_{f, \epsilon_1, \dots, \epsilon_k} := \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : \zeta_{2^{a+1}}^{-\epsilon_1} t_1, \dots, \zeta_{2^{a+1}}^{-\epsilon_k} t_k \in \mathbb{R}_{\geq 0} \text{ and } t_{k+1}, \dots, t_n \in \mathbb{R}\},$$

indexed by $\epsilon_1, \dots, \epsilon_k \in \{0, 1\}$, is isometric to the closed region in $\mathbb{B}^n(\mathbb{R})$ bounded by k mutually orthogonal hyperplanes. By Lemma 3.13, their union U is a fundamental domain for B_f , in the sense that it maps homeomorphically and piecewise-isometrically onto $B_f \backslash Y_f$. Under its path metric, $U = \cup K_{f, \epsilon_1, \dots, \epsilon_k}$ is isometric to $\mathbb{B}^n(\mathbb{R})$ by the following map:

$$U \rightarrow \mathbb{B}^n(\mathbb{R}), \quad (t_1, \dots, t_k) \mapsto ((-\zeta_{2^{a+1}})^{-\epsilon_1} t_1, \dots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \dots, t_n).$$

This identifies $B_f \backslash Y_f$ with the standard $\mathbb{B}^n(\mathbb{R}) \subset \mathbb{B}^n(\mathbb{C})$.

3. Now suppose f has $k = 2a$ nodes $H_{r_1}, \dots, H_{r_{2a}}$. There are now m^a anti-isometric involutions α_{j_i} fixing x and such that $x_{\alpha_{j_i}} \sim x_\alpha$: they are given in the coordinates t_i as follows, taking $j = (j_1, \dots, j_a) \in (\mathbb{Z}/m)^a$:

$$\alpha_j : (t_1, \dots, t_n) \mapsto (\bar{t}_2 \zeta^{j_1}, \bar{t}_1 \zeta^{j_1}, \dots, \bar{t}_{2a} \zeta^{j_a}, \bar{t}_{2a-1} \zeta^{j_a}, \bar{t}_{2a+1}, \dots, \bar{t}_n).$$

So any fixed-point set $\mathbb{R}H_{\alpha_j}^n$ is identified with

$$\mathbb{B}^n(\mathbb{R})_{\alpha_j} := \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i = \bar{t}_{i-1} \zeta^{j_i} \text{ for } 1 \leq i \leq 2a \text{ even, } t_i \in \mathbb{R} \text{ for } i > 2a\},$$

and we have

$$Y_f \cong \bigcup_j \mathbb{B}^n(\mathbb{R})_{\alpha_j} = \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i^m = \bar{t}_{i-1}^m \text{ for } i \leq 2a \text{ even, } t_i \in \mathbb{R} \text{ for } i > 2a\}.$$

These m^a copies of $\mathbb{B}^n(\mathbb{R})$ meet at the origin of $\mathbb{B}^n(\mathbb{C})$; in fact, for $j \neq j'$, the space $\mathbb{B}^n(\mathbb{R})_{\alpha_j}$ meets $\mathbb{B}^n(\mathbb{R})_{\alpha_{j'}}$ in a $\mathbb{B}^{2c}(\mathbb{R})$ if c is the number of pairs (j_i, j'_i) with $j_i = j'_i$.

4. Now we treat the general case. In the local coordinates t_i , any anti-unitary involutions fixing x and equivalent to α is of the form

$$\alpha_j: (t_1, \dots, t_n) \mapsto (\bar{t}_2 \zeta^{j_1}, \bar{t}_1 \zeta^{j_1}, \dots, \bar{t}_{2a} \zeta^{j_a}, \bar{t}_{2a-1} \zeta^{j_a}, \bar{t}_{2a+1} \zeta^{j_{2a+1}}, \dots, \bar{t}_k \zeta^{j_k}, \bar{t}_{k+1}, \dots, \bar{t}_n)$$

for some $j = (j_1, \dots, j_a, j_{2a+1}, \dots, j_k) \in (\mathbb{Z}/m)^{a+b}$. We now have $B_f \cong (\mathbb{Z}/m)^b$ acting by multiplying the t_i for $2a+1 \leq i \leq k$ by powers of ζ , and there are m^{a+b} anti-unitary involutions α_j . We have

$$Y_f \cong \bigcup_{j_1, \dots, j_k=1}^m \mathbb{R}H_{\alpha_j}^n \cong \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^m = \bar{t}_1^m, \dots, t_{2a}^m = \bar{t}_{2a-1}^m, t_{2a+1}^m, \dots, t_k^m, t_{k+1}, \dots, t_n \in \mathbb{R}\}.$$

We look at subsets $K_{f, \epsilon_1, \dots, \epsilon_k} \subset Y_f$ again, this time defined as

$$\begin{aligned} K_{f, \epsilon} &= K_{f, \epsilon_1, \dots, \epsilon_k} \\ &= \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid \\ &\quad t_i^m = \bar{t}_{i-1}^m \text{ } i \leq 2a \text{ even, } \zeta_{2a+1}^{-\epsilon_i} t_i \in \mathbb{R}_{\geq 0} \text{ } 2a < i \leq k, t_i \in \mathbb{R}, i > k\}. \end{aligned}$$

As before, we have that the natural map $U := \bigcup_{\epsilon} K_{f, \epsilon} \rightarrow B_f \setminus Y_f$ is an isometry. Define

$$\tilde{Y}_f = \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i^m = \bar{t}_{i-1}^m \text{ for } i \leq 2a \text{ even, } t_i \in \mathbb{R}, \text{ for } i > 2a\}.$$

Under its path metric, $U = \bigcup_{\epsilon} K_{f, \epsilon_1, \dots, \epsilon_k}$ is isometric to \tilde{Y}_f by the following map:

$$U \rightarrow \tilde{Y}_f, \quad (t_1, \dots, t_k) \mapsto (t_1, \dots, t_{2a}, (-\zeta_{2a+1})^{-\epsilon_1} t_{2a+1}, \dots, (-\zeta_{2a+1})^{-\epsilon_k} t_k, t_{k+1}, \dots, t_n).$$

Hence $B_f \setminus Y_f \cong \tilde{Y}_f$; but since \tilde{Y}_f is what Y_f was in case 3, we are done.

5. The transitivity of A_f on the copies of $\mathbb{B}^n(\mathbb{R})$ follows from the fact that $G(x) \subset A_f$ contains transformations multiplying t_1, \dots, t_{2a} by powers of ζ , hence the map $(t_i \mapsto \zeta^u t_i, t_{i-1} \mapsto t_{i-1})$ maps those (t_{i-1}, t_i) with $t_i = \bar{t}_{i-1} \zeta^{j_i}$ to those (t_{i-1}, t_i) with $t_i = \bar{t}_{i-1} \zeta^{j_i+u}$. So if B is any one of the copies of $\mathbb{B}^n(\mathbb{R})$, and $G = (A_f/B_f)_H$ is its stabilizer, then it remains to prove that $G \setminus B \rightarrow A_f \setminus Y_f$ is an isometry. Surjectivity follows from the transitivity of A_f on the $\mathbb{B}^n(\mathbb{R})$'s. It is a piecewise isometry so we only need to prove injectivity. This will follow from the following elementary lemma.

Lemma 3.15. *Let a group G act on a set X , let Y and I be sets, and let $\{\phi_i : Y \hookrightarrow X\}_{i \in I}$ be a set of embeddings. Write $Y_i = \phi_i(Y)$ and suppose that $X = \bigcup_i Y_i$. Fix $0 \in I$. Let $H \subset G$ be the stabilizer of Y_0 . Suppose that for all $y \in X$, the stabilizer of y in G acts transitively on the sets Y_i containing y . Then $H \setminus Y_0 \rightarrow G \setminus X$ is injective.*

Proof. Let $x, y \in Y_0$ and $g \in G$ such that $g \cdot x = y$. Then $y = gx \in gY_0$. Since also $y \in Y_0$, there is an element $h \in \text{Stab}_G(y)$ such that $hgY_0 = Y_0$ and $hg(x) = h(y) = y$. Let $f = hg$; then $f \in H$ and $f \cdot x = y$, which proves what we want. \square

Now let us use the lemma: suppose that $y \in B_f \setminus Y_f$. We need to prove that $\text{Stab}_{A_f/B_f}(y)$ acts transitively on the copies of $\mathbb{B}^n(\mathbb{R})$ containing y . There exists

$$j = (j_1, \dots, j_a, j_{2a+1}, \dots, j_k) \in (\mathbb{Z}/m)^{a+b}$$

such that $y = (t_1, \dots, t_n)$ with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $i \leq 2a$ even, $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $2a < i \leq k$, and $t_i \in \mathbb{R}$ for $i > k$. If all t_i are non-zero, then $y \in \cup_{j'} \mathbb{R}H_{\alpha_{j'}}$ is only contained in $\mathbb{R}H_{\alpha_j}$, so there is nothing to prove. Let us suppose that $t_1 = t_2 = 0$ and the other t_i are non-zero. Then y is contained in all the $\mathbb{R}H_{\alpha_{j'}}$ with $j'_i = j_i$ for $i \geq 2$; there are m of them. The stabilizer of y multiplies t_1 and t_2 by powers of ζ and leaves the other t_i invariant; it acts transitively on the $\mathbb{R}H_{\alpha_{j'}}$ containing y for if $t_2 = \bar{t}_1\zeta^{j'_1}$ then $\zeta^{(j'_1 - j_1)}t_2 = \bar{t}_1\zeta^{j_1}$. The general case is similar. \square

We need one more lemma before we can prove Theorem 3.1:

Lemma 3.16. *The maps $\mathcal{P}: Y(\Lambda) \rightarrow \mathbb{C}H^n$ and $\overline{\mathcal{P}}: X(\Lambda) \rightarrow P\Gamma \setminus \mathbb{C}H^n$ are proper.*

Proof. The map $\mathcal{P}: Y(\Lambda) \rightarrow \mathbb{C}H^n$ is proper because any compact set in $\mathbb{C}H^n$ meets only finitely many $(\mathbb{R}H_\alpha)'$ s, $\alpha \in P\mathcal{A}$ (Lemma 3.3), and \mathcal{P} carries each $H_\alpha = p(\mathbb{R}H_\alpha)$ homeomorphically onto $\mathbb{R}H_\alpha$. To prove that $\overline{\mathcal{P}}$ is proper, let π (resp. q) be the quotient map $\mathbb{C}H^n \rightarrow P\Gamma \setminus \mathbb{C}H^n$ (resp. $Y(\Lambda) \rightarrow X(\Lambda)$), and let $Z \subset X(\Lambda) = P\Gamma \setminus Y(\Lambda)$ be a closed subset. Its inverse image $W = q^{-1}(Z) \subset Y(\Lambda)$ in $Y(\Lambda)$ is a closed, $P\Gamma$ -invariant subset. Since \mathcal{P} is proper and $P\Gamma$ -equivariant, $\mathcal{P}(W) \subset \mathbb{C}H^n$ is a closed, $P\Gamma$ -invariant subset of $\mathbb{C}H^n$ and we have

$$\mathcal{P}(W) = \pi^{-1}\pi(\mathcal{P}(W)) = \pi^{-1}(\overline{\mathcal{P}}(Z)).$$

Hence $\overline{\mathcal{P}}(Z)$ is closed in $P\Gamma \setminus \mathbb{C}H^n$, which proves that $\overline{\mathcal{P}}$ is closed. Therefore, to prove that $\overline{\mathcal{P}}$ is proper, it suffices to show that it has finite fibers.

Let $y \in P\Gamma \setminus \mathbb{C}H^n$ and let $V = \overline{\mathcal{P}}^{-1}(y) \subset X(\Lambda)$. For each $v \in V$, choose an element $u_v \in Y(\Lambda)$ such that $q(u_v) = v$. This gives $q^{-1}(V) = \coprod_{v \in V} P\Gamma \cdot u_v \subset Y$. Moreover, we have that

$$\mathcal{P}(P\Gamma \cdot u_v) = P\Gamma \cdot \mathcal{P}(u_v) = \pi^{-1}\pi(\mathcal{P}(u_v)) = \pi^{-1}(q(v)) = \pi^{-1}(y)$$

for each $v \in V$. This gives a map $\mathcal{P}: \coprod_{v \in V} P\Gamma \cdot u_v \rightarrow \pi^{-1}(y)$ which is surjective on each $P\Gamma \cdot u_v$. By properness of \mathcal{P} , for any $z \in \pi^{-1}(y)$, the inverse image

$$\mathcal{P}^{-1}(z) \subset \coprod_{v \in V} P\Gamma \cdot u_v \subset Y(\Lambda)$$

is finite. Since $\mathcal{P}^{-1}(z)$ meets every orbit $P\Gamma \cdot u_v$, it follows that V is finite. \square

Proof of Theorem 3.1. 1. The path metric on $X(\Lambda)$ is given by Proposition 3.7. Note that the map $\mathcal{P}: Y \rightarrow \mathbb{C}H^n$ is a local embedding by Lemma 3.4, which was used to define the metric on $Y(\Lambda)$ (Proposition 3.6). Thus, almost by definition, \mathcal{P} is a local isometry. For each $f \in Y(\Lambda)$ we can find a $P\Gamma_f$ -invariant open neighborhood

$U_f \subset Y_f \subset Y(\Lambda)$ such that $P\Gamma_f \setminus U_f \subset P\Gamma \setminus Y(\Lambda)$, with U_f mapping bijectively onto an open subset V_f in the closed subset $\mathcal{P}(Y_f) = \cup_i \mathbb{R}H_{\alpha_i}^n \subset \mathbb{C}H^n$. By $P\Gamma$ -equivariance of \mathcal{P} , the set V_f is $P\Gamma_f$ -invariant, and we have $P\Gamma_f \setminus V_f \subset P\Gamma \setminus \mathbb{C}H^n$. Thus

$$\overline{\mathcal{P}}: X(\Lambda) = P\Gamma \setminus Y(\Lambda) \rightarrow P\Gamma \setminus \mathbb{C}H^n$$

is also a local isometry. The space $P\Gamma \setminus \mathbb{C}H^n$ is complete, and $\overline{\mathcal{P}}$ is proper by Lemma 3.16, so $P\Gamma \setminus Y(\Lambda)$ is complete as well.

2. $[f] \in P\Gamma \setminus Y(\Lambda)$ be the image of $f \in Y(\Lambda)$. Then $[f]$ has an open neighborhood isometric to the quotient of an open set W in $\mathbb{R}H^n$ by a finite group of isometries Γ_f . Indeed, take $Y_f \subset Y(\Lambda)$ as in Equation (6), and $f \in U_f \subset Y_f$ as in Lemma 3.4.2. We let $A_f = P\Gamma_f$ be the stabilizer of f in $P\Gamma$ as before, and take an A_f -equivariant open neighborhood $V_f \subset U_f$ such that $A_f \setminus V_f \subset P\Gamma \setminus Y(\Lambda)$. By Proposition 3.14.5, we know that $A_f \setminus V_f$ is isometric to $\Gamma_f \setminus \mathbb{R}H^n$ for some finite group of isometries of $\mathbb{R}H^n$. This implies that $A_f \setminus V_f$ is isometric to some open set W' in $\Gamma_f \setminus \mathbb{R}H^n$. Take $W \subset \mathbb{R}H^n$ to be the preimage of W' .

Claim: For any path metric space X locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries, there is a unique real-hyperbolic orbifold structure on X whose path metric is the given one.

Proof of the Claim: If U and U' are connected open subsets of $\mathbb{R}H^n$ and Γ and Γ' finite groups of isometries of $\mathbb{R}H^n$ preserving U and U' respectively, then any isometry $\bar{\phi}: \Gamma \setminus U \rightarrow \Gamma' \setminus U'$ extends to an isometry $\phi: \mathbb{R}H^n \rightarrow \mathbb{R}H^n$ such that $\phi(U) = U'$ and $\phi\Gamma\phi^{-1} = \Gamma' \subset \text{Isom}(\mathbb{R}H^n)$.

We conclude that $P\Gamma \setminus Y(\Lambda)$ is naturally a real hyperbolic orbifold.

3. Let us show that

$$O := \coprod_{\alpha \in \mathcal{C}\mathcal{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H_\alpha^n - \mathcal{H})] \subset X(\Lambda) = P\Gamma \setminus Y(\Lambda)$$

as hyperbolic orbifolds. It suffices to show the following:

Claim: For those $f = p(x_\alpha) \in Y(\Lambda)$ that have no nodes, the stabilizer $A_f = P\Gamma_f \subset P\Gamma$ of $f \in Y(\Lambda)$ and the stabilizer $P\Gamma_{\alpha,x} \subset P\Gamma_\alpha$ of $x \in \mathbb{R}H_\alpha^n$ agree as subgroups of $P\Gamma$.

Proof of the Claim: To prove that $A_f = P\Gamma_{\alpha,x}$, we first observe that $p: \tilde{Y} \rightarrow Y(\Lambda)$ induces an isomorphism between $P\Gamma_{x_\alpha}$, the stabilizer of $x_\alpha \in \tilde{Y}$ and $P\Gamma_f$, the stabilizer of $f = [x, \alpha] \in Y(\Lambda)$. So it suffices to show that $P\Gamma_{x_\alpha} = P\Gamma_{\alpha,x}$. For this we use that the normalizer $N_{P\Gamma}(\alpha)$ and the stabilizer $P\Gamma_\alpha \subset P\Gamma$ of α in $P\Gamma$ are equal, which implies that $P\Gamma_{\alpha,x} = P\Gamma_{x_\alpha}$ because

$$\begin{aligned} \{g \in P\Gamma_\alpha : gx = x\} &= \{g \in N_{P\Gamma}(\alpha) : gx = x\} \\ &= \{g \in P\Gamma : g \cdot x_\alpha = (g(x), g\alpha g^{-1}) = x_\alpha\}. \end{aligned}$$

The claim is proved. Part 3 of the theorem can be deduced from it as follows. Let $f = p(x_\alpha) \in Y(\Lambda)$ have no nodes. We have $Y_f = \mathbb{R}H_\alpha^n$, hence

$$A_f \setminus \mathbb{R}H_\alpha^n = A_f \setminus Y_f = \Gamma_f \setminus \mathbb{R}H^n \quad \text{with} \quad \Gamma_f = A_f \setminus B_f = A_f.$$

By construction, an orbifold chart of the glued space $P\Gamma \backslash Y(\Lambda)$ is given by

$$W \rightarrow A_f \backslash W \subset P\Gamma_\alpha \backslash \mathbb{R}H_\alpha^n \subset Y(\Lambda)$$

for an invariant open subset W of $\mathbb{R}H_\alpha^n$ containing x . Because $A_f = P\Gamma_{\alpha,x}$ by the claim, this is also an orbifold chart for O at the point x_α .

4. The real-hyperbolic orbifold $X(\Lambda) = P\Gamma \backslash Y(\Lambda)$ is complete by Part 1, so the uniformization of the connected components of $X(\Lambda)$ follows from the Ehresmann–Thurston uniformization theorem for (G, X) -orbifolds, see [Thu80, Proposition 13.3.2]. This concludes the proof of Theorem 3.1, and thereby also of Theorem 1.6. \square

4 Unitary Shimura varieties

The goal of this section is to prove Proposition 4.7, which describes the complex ball quotient $P\Gamma \backslash \mathbb{C}H^n$ defined in Section 2 in terms of moduli of abelian varieties with \mathcal{O}_M -action of hyperbolic signature, and Proposition 4.10, which interprets the divisor $P\Gamma \backslash \mathcal{H}$ as the locus of abelian varieties A that admit an \mathcal{O}_M -linear homomorphism $\mathbb{C}^g/\Phi(\mathcal{O}_M) \rightarrow A$ of polarized \mathcal{O}_M -linear abelian varieties. This has two applications:

1. Consider a relative uniform cyclic cover (see e.g. [AV04]) $\mathcal{X} \rightarrow P \rightarrow S$, where $P = \mathbb{P}_S^1$ (resp. \mathbb{P}_S^3), the fibers of $\mathcal{X} \rightarrow S$ are curves (resp. threefolds with $H^{0,3} = 0$) and the induced hermitian form on middle cohomology has hyperbolic signature. Since the image $\mathfrak{J} = P(S(\mathbb{C})) \subset P\Gamma \backslash \mathbb{C}H^n$ of the period map $P: S(\mathbb{C}) \rightarrow P\Gamma \backslash \mathbb{C}H^n$ is contained in the locus of abelian varieties whose theta divisor is irreducible, one has

$$\mathfrak{J} \subset P\Gamma \backslash (\mathbb{C}H^n - \mathcal{H}).$$

2. Using the theory of this section, we will be able to show that Condition 2.1 implies Condition 2.3, saying that the hyperplanes in the arrangement $\mathcal{H} \subset \mathbb{C}H^n$ are orthogonal along their intersection. See Theorem 5.2 in the next Section 5.

4.1 **Alternating and hermitian forms on the lattice.** The goal of this subsection is to prove two lemmas. They will later be used to show that the ball quotient $P\Gamma \backslash \mathbb{C}H^n$ of Section 2 is a moduli space of abelian varieties, and then to interpret the divisor $P\Gamma \backslash \mathcal{H}$ in terms of moduli of abelian varieties. This will be the key to Theorem 5.2.

Let M be a CM field, with ring of integers \mathcal{O}_M , and let $\sigma: M \rightarrow M$ be the involution induced by complex conjugation on \mathbb{C} . Let Λ be a free \mathcal{O}_M -module of rank $n+1$ for some $n \in \mathbb{N}$. Let $\mathfrak{D}_M \subset \mathcal{O}_M$ be the different ideal.

Lemma 4.1. *The assignment $T \mapsto \text{Tr}_{M/\mathbb{Q}} \circ T$ defines a bijection between:*

1. *The set of skew-hermitian forms $T: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow M$.*
2. *The set of alternating forms $E: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow \mathbb{Q}$ such that $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$.*

Under this correspondence, $T(\Lambda, \Lambda) \subset \mathfrak{D}_M^{-1}$ if and only if $E(\Lambda, \Lambda) \subset \mathbb{Z}$.

Proof. Let $T: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow M$ be as in 1. Define $E_T = \text{Tr}_{M/\mathbb{Q}} \circ T$. Since T is skew-hermitian, we have, for each $x, y \in \Lambda_{\mathbb{Q}}$, that

$$\text{Tr}_{M/\mathbb{Q}} T(x, y) = -\text{Tr}_{M/\mathbb{Q}} \overline{T(y, x)}.$$

Since M/\mathbb{Q} is separable, for any $x \in M$, we have $\text{Tr}_{M/\mathbb{Q}}(x) = \sum_{1 \leq i \leq g} (\tau_i(x) + \tau_i \sigma(x))$, see [Ste08, (7-1)]. Thus, we have $\text{Tr}_{M/\mathbb{Q}}(\sigma(x)) = \text{Tr}_{M/\mathbb{Q}}(x)$, so that $E_T(x, y) = -E_T(y, x)$ for any $x, y \in \Lambda_{\mathbb{Q}}$. The property in 2 is easily checked.

Conversely, let $E: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow \mathbb{Q}$ be as in 2. Choose a basis $\{b_1, \dots, b_{n+1}\} \subset \Lambda$ for Λ over \mathcal{O}_M . Define Q to be the induced map $M^{n+1} \times M^{n+1} \rightarrow \mathbb{Q}$ and consider the map $M \rightarrow \mathbb{Q}$, $a \mapsto Q(a \cdot e_i, e_j)$. Since the trace pairing

$$M \times M \rightarrow \mathbb{Q}, \quad (x, y) \mapsto \text{Tr}_{M/\mathbb{Q}}(xy)$$

is non-degenerate [Stacks, Tag 0BIE], there is a unique $t_{ij} \in M$ such that $Q(a \cdot e_i, e_j) = \text{Tr}_{M/\mathbb{Q}}(a \cdot t_{ij})$ for every $a \in M$. This gives a matrix $(t_{ij})_{ij} \in M_{n+1}(K)$ such that $\sigma(t_{ij}) = -t_{ji}$, and the basis $\{b_i\}$ induces a skew-hermitian form $T_E: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow M$.

The last claim from the definition of $\mathfrak{D}_M^{-1} \subset M$: it is the trace dual of \mathcal{O}_M , see [Ser79, Chapter III, §3]. \square

Examples 4.2. 1. Suppose that $M = \mathbb{Q}(\sqrt{\Delta})$ is imaginary quadratic over \mathbb{Q} , with discriminant Δ and non-trivial Galois automorphism $a \mapsto a^\sigma$. Let $E: \Lambda \times \Lambda \rightarrow \mathbb{Z}$ be an alternating form with $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$. The form $T: \Lambda \times \Lambda \rightarrow \mathfrak{D}_M^{-1} = (\sqrt{\Delta})^{-1}$ is defined as

$$T(x, y) = \frac{E(\sqrt{\Delta} \cdot x, y) + E(x, y)\sqrt{\Delta}}{2\sqrt{\Delta}}.$$

2. Suppose that $M = \mathbb{Q}(\zeta_p)$ where $\zeta_p = e^{2\pi i/p} \in \mathbb{C}$ for some prime number $p > 2$. Let $E: \Lambda \times \Lambda \rightarrow \mathbb{Z}$ be an alternating form with $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$. Then $\mathfrak{D}_M = (p/(\zeta_p - \zeta_p^{-1}))$ and

$$T: \Lambda \times \Lambda \rightarrow \mathfrak{D}_M^{-1} \quad \text{is defined as} \quad T(x, y) = \frac{1}{p} \sum_{j=0}^{p-1} \zeta_p^j \cdot E(x, \zeta_p^j \cdot y).$$

Consider a pair $(E: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow \mathbb{Q}, T: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow K)$ as in Lemma 4.1, and suppose that E is non-degenerate. To any embedding $\varphi: M \rightarrow \mathbb{C}$ be an embedding, one can associate a skew-hermitian form $T^\varphi: \Lambda \otimes_{\mathcal{O}_M, \varphi} \mathbb{C} \times \Lambda \otimes_{\mathcal{O}_M, \varphi} \mathbb{C} \rightarrow \mathbb{C}$ by putting

$$T^\varphi(x \otimes \lambda, y \otimes \mu) = \lambda \bar{\mu} \cdot \varphi(T(x, y))$$

and extending linearly. Remark that, on $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, one has the skew-hermitian form $A(x, y) = E_{\mathbb{C}}(x, \bar{y})$. The composition $(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi \rightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \Lambda \otimes_{\mathcal{O}_M, \varphi} \mathbb{C}$ is an isomorphism; we define A^φ as the restriction of A to the subspace

$$(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi = \Lambda \otimes_{\mathcal{O}_M, \varphi} \mathbb{C} \subset \Lambda_{\mathbb{C}}.$$

Note that $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{\phi: M \rightarrow \mathbb{C}} (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\phi$. For $x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, let $x_\phi \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\phi$ denote the image of x under the projection $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\phi$.

Lemma 4.3. *Let $\varphi: M \rightarrow \mathbb{C}$ be an embedding. We have an equality of skew-hermitian forms:*

$$T^\varphi = A^\varphi: (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi \times (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi \rightarrow \mathbb{C}.$$

More precisely, we have $A(x, y) = \sum_{\phi: M \rightarrow \mathbb{C}} T^\phi(x_\phi, y_\phi)$ for every $x, y \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$.

Proof. Write $V = \Lambda_{\mathbb{Q}}$. The lemma follows from the fact that the following diagram commutes:

$$\begin{array}{ccccc}
V \times V & \xrightarrow{T} & K & \xrightarrow{\text{Tr}_{M/\mathbb{Q}}} & \mathbb{Q} \\
\downarrow & & \downarrow & & \downarrow \\
V \otimes_{\mathbb{Q}} \mathbb{C} \times V \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{T_{\mathbb{C}}} & K \otimes_{\mathbb{Q}} \mathbb{C} & & \\
\parallel & \searrow^{A(x,y)} & \parallel & & \\
\oplus_{\phi} (V \otimes_{\mathbb{Q}} \mathbb{C})_{\phi} \times (V \otimes_{\mathbb{Q}} \mathbb{C})_{\phi} & \xrightarrow{\oplus T^{\phi}} & \oplus_{\phi} \mathbb{C}_{\phi} & \xrightarrow{\Sigma} & \mathbb{C}.
\end{array}$$

Here, ϕ ranges over the set of embeddings $M \rightarrow \mathbb{C}$, \mathbb{C}_{ϕ} is the M -module \mathbb{C} where M acts via ϕ , Σ is the sum homomorphism, and

$$T_{\mathbb{C}}: V \otimes_{\mathbb{Q}} \mathbb{C} \times V \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow K \otimes_{\mathbb{Q}} \mathbb{C}$$

is the map that sends $(v \otimes \lambda, x \otimes \mu)$ to $\lambda \bar{\mu} T(v, w)$. \square

4.2 Moduli of abelian varieties with an action by a CM field. In the rest of Section 4, we fix:

Notation 4.4. 1. a CM field M of degree $2g$ over \mathbb{Q} , and we define $\sigma: M \rightarrow M$ as the involution that corresponds to complex conjugation on \mathbb{C} ;

2. a non-zero element $\eta \in \mathcal{O}_M$ such that $\eta^{-1} \in \mathfrak{D}_M^{-1}$ and $\sigma(\eta) = -\eta$;

3. a free \mathcal{O}_M -module Λ of rank $n+1$ for some $n \in \mathbb{N}$ equipped with a non-degenerate hermitian form $\mathfrak{h}: \Lambda \times \Lambda \rightarrow \eta \cdot \mathfrak{D}_M^{-1} \subset K$; and

4. we let $\Phi = \{\tau_i: M \rightarrow \mathbb{C}\}_{i=1}^g$ be the CM type such that $\Im(\tau_i(\eta)) > 0$ for $1 \leq i \leq g$.

These data define a skew-hermitian form

$$T: \Lambda \times \Lambda \rightarrow \mathfrak{D}_M^{-1}, \quad \text{by setting} \quad T := \eta^{-1} \cdot \mathfrak{h}.$$

The form T is in turn attached to a symplectic form (see Lemma 4.1)

$$E: \Lambda \times \Lambda \rightarrow \mathbb{Z} \quad \text{such that} \quad E(ax, y) = E(x, a^\sigma y) \quad \text{for all } a \in \mathcal{O}_M, x, y \in \Lambda.$$

Write $V_i = \Lambda_{\mathbb{Q}} \otimes_{K, \tau_i} \mathbb{C}$ and define

$$\mathfrak{h}^{\tau_i}: V_i \times V_i \rightarrow \mathbb{C}$$

to be the hermitian form restricting to $\tau_i \circ \mathfrak{h}$ on Λ . Let (r_i, s_i) be the signature of the hermitian form \mathfrak{h}^{τ_i} .

Let A be a complex abelian variety, ι a homomorphism $\mathcal{O}_M \rightarrow \text{End}(A)$, and λ a polarization $A \rightarrow A^\vee$, satisfying the following (c.f. [KR14, Part I, §2.1]):

Conditions 4.5. 1. We have $\iota(a)^\dagger = i(a^\sigma)$ for the Rosati involution

$$\dagger: \text{End}(A)_\mathbb{Q} \rightarrow \text{End}(A)_\mathbb{Q}, \quad \text{and}$$

2. we have $\text{char}(t, \iota(a)|\text{Lie}(A)) = \prod_{\nu=1}^g (t - a^{\tau_i})^{r_i} \cdot (t - a^{\tau_i\sigma})^{s_i} \in \mathbb{C}[t]$.

Here, $\text{char}(t, \iota(a)|\text{Lie}(A)) \in \mathbb{C}[t]$ denotes the characteristic polynomial of $\iota(a)$. Remark that any abelian variety A as above satisfies $\dim A = g(n+1)$.

Define $E_A: \text{H}_1(A, \mathbb{Z}) \times \text{H}_1(A, \mathbb{Z}) \rightarrow \mathbb{Z}$ to be the alternating form corresponding to λ . The condition on the Rosati involution implies that $E_A(\iota(a)x, y) = E_A(x, \iota(a^\sigma)y)$ for $x, y \in \text{H}_1(A, \mathbb{Q})$. Define a hermitian form \mathfrak{h}_A on the \mathcal{O}_M -module $\text{H}_1(A, \mathbb{Z})$ as follows:

$$\mathfrak{h}_A = \eta \cdot T_A: \text{H}_1(A, \mathbb{Z}) \times \text{H}_1(A, \mathbb{Z}) \rightarrow \eta \cdot \mathfrak{D}_M^{-1} \subset K.$$

Here, $T_A: \text{H}_1(A, \mathbb{Z}) \times \text{H}_1(A, \mathbb{Z}) \rightarrow \mathfrak{D}_M^{-1}$ is the skew-hermitian form attached to E_A via Lemma 4.1.

Definition 4.6. 1. Let $\widetilde{\text{Sh}}_M(\Lambda)$ be the set of isomorphism classes of tuples (A, i, λ, j) , where (A, i, λ) is as above and satisfies Conditions 4.5, and where $j: \text{H}_1(A, \mathbb{Z}) \rightarrow \Lambda$ is an isomorphism of \mathcal{O}_M -modules compatible with the alternating forms E_A and E .

2. Let $\mathbb{D}(V_i)$ be the space of negative s_i -planes in the hermitian space $(V_i, \mathfrak{h}^{\tau_i})$.

We have the following proposition which is due to Shimura, see [Shi63, Theorem 2] or [Shi64, §1]. We give a different proof since it will imply Proposition 4.10 below, whereas we did not know how to deduce Proposition 4.10 from *loc. cit.* We remark that Shimura assumes Λ to be an R -module for any order $R \subset \mathcal{O}_M$; our proof carries over, but we do not need this generalization.

Proposition 4.7. *There is a canonical bijection*

$$\widetilde{\text{Sh}}_M(\Lambda) \cong \mathbb{D}(V_1) \times \cdots \times \mathbb{D}(V_g).$$

Proof. Let (A, i, λ, j) be a representative of an isomorphism class that defines a point in $\widetilde{\text{Sh}}_M(\Lambda)$. Consider the Hodge decomposition $\text{H}_1(A, \mathbb{C}) = \text{H}^{-1,0} \oplus \text{H}^{0,-1}$. For $1 \leq i \leq g$ there is a decomposition

$$\text{H}_1(A, \mathbb{C})_{\tau_i} = \text{H}_{\tau_i}^{-1,0} \oplus \text{H}_{\tau_i}^{0,-1}, \quad (10)$$

with $\dim \text{H}_{\tau_i}^{-1,0} = r_i$ and $\dim \text{H}_{\tau_i}^{0,-1} = s_i$. The latter holds because $\overline{\text{H}_{\tau_i\sigma}^{-1,0}} = \text{H}_{\tau_i}^{0,-1}$. By Lemma 4.3, the forms $\tau_i(\eta)E_{A,\mathbb{C}}(x, \bar{y})$ and $\mathfrak{h}_{A,\mathbb{C}}^{\tau_i}(x, y)$ agree as hermitian forms on the complex vector space $\text{H}_1(A, \mathbb{Z}) \otimes_{\mathcal{O}_M, \tau_i} \mathbb{C}$. Since $\Im(\tau_i(\eta)) > 0$ for every i , the decomposition of $\text{H}_1(A, \mathbb{C})_{\tau_i}$ in (10) is a decomposition into a positive definite r_i -dimensional subspace and a negative definite s_i -dimensional subspace. The isomorphism $j: \text{H}_1(A, \mathbb{Q}) \rightarrow \Lambda_\mathbb{Q}$ induces an isometry $j_i: \text{H}_1(A, \mathbb{C})_{\tau_i} \rightarrow V_i$ for every i , and so we obtain a negative s_i -plane $j_i(\text{H}_{\tau_i}^{0,-1})$ in the hermitian space V_i for all i .

Reversing the argument shows that given a negative s_i -plane $X_i \subset V_i$ for every i , there is a canonical polarized abelian variety $A = \text{H}^{-1,0}/\Lambda$, acted upon by \mathcal{O}_M and inducing the planes $X_i \subset V_i$. \square

Definition 4.8. 1. Let $\text{Sh}_M(\Lambda)$ be the set of isomorphism classes of polarized abelian varieties with \mathcal{O}_M -action (A, i, λ) , satisfying Conditions 4.5, such that $H_1(A, \mathbb{Z})$ is isomorphic to Λ as hermitian \mathcal{O}_M -modules.

2. Let $\Gamma(\Lambda) = \text{Aut}_{\mathcal{O}_M}(\Lambda)$; this is the group of \mathcal{O}_M -linear automorphisms of Λ preserving our form $\mathfrak{h} : \Lambda \times \Lambda \rightarrow \eta \cdot \mathfrak{D}_M^{-1}$.

The bijection in Proposition 4.7 being $\Gamma(\Lambda)$ -equivariant, we obtain the following:

Corollary 4.9. *There is a canonical bijection $\text{Sh}_M(\Lambda) \cong \Gamma(\Lambda) \backslash \prod_{i=1}^g \mathbb{D}(V_i)$. \square*

4.3 **Moduli of abelian varieties in the hyperplane arrangement.** The set of embeddings Φ introduced in Notation 4.4 defines a map $\Phi : \mathcal{O}_M \rightarrow \mathbb{C}^g$, which yields a complex torus $\mathbb{C}^g / \Phi(\mathcal{O}_M)$. The map

$$Q : M \times M \rightarrow \mathbb{Q}, \quad Q(x, y) = \text{Tr}_{M/\mathbb{Q}}(\eta^{-1}x\bar{y}) \quad (11)$$

is a non-degenerate \mathbb{Q} -bilinear form such that $Q(ax, y) = Q(x, a^\sigma y)$ for every $a, x, y \in M$. Moreover, $Q(\mathcal{O}_M, \mathcal{O}_M) \subset \mathbb{Z}$ because $\eta^{-1} \in \mathfrak{D}_M^{-1}$. By [Mil20, Example 2.9 & Footnote 16], Q induces defines a Riemann form on the complex torus $\mathbb{C}^g / \Phi(\mathcal{O}_M)$.

Suppose from now on that

$$(r_i, s_i) = \begin{cases} (n, 1) & \text{if } i = 1, \\ (n + 1, 0) & \text{if } 2 \leq i \leq g. \end{cases} \quad (12)$$

Setting $\tau = \tau_1$, the space

$$W = \Lambda \otimes_{\mathcal{O}_M, \tau} \mathbb{C}.$$

is equipped with the hermitian form $\mathfrak{h}^\tau : W \times W \rightarrow \mathbb{C}$. Define

$$\mathbb{C}H(W) = \{\text{lines } \ell \subset W \mid \mathfrak{h}^\tau(x, x) < 0 \text{ for } 0 \neq x \in \ell\}, \quad \mathcal{H} = \cup_{\mathfrak{h}(r, r)=1} \langle r_{\mathbb{C}} \rangle^\perp \subset \mathbb{C}H(W).$$

Proposition 4.10. *Under the bijection $\widetilde{\text{Sh}}_M(\Lambda) \cong \mathbb{C}H(W)$ of Proposition 4.7, points in $\mathcal{H} \subset \mathbb{C}H(W)$ correspond to polarized marked \mathcal{O}_M -linear abelian varieties A admitting an \mathcal{O}_M -linear homomorphism $\mathbb{C}^g / \Phi(\mathcal{O}_M) \rightarrow A$ of polarized abelian varieties.*

Proof. Consider an isomorphism class $[(A, i, \lambda, y)] \in \widetilde{\text{Sh}}_M(\Lambda)$ corresponding to a point $[x] \in \mathbb{C}H^n$. We may assume that $A = H^{-1,0} / \Lambda$ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = H^{-1,0} \oplus H^{0,-1}$, and that $T_A = T$. Let

$$\phi : \mathbb{C}^g / \Phi(\mathcal{O}_M) \rightarrow A$$

be a homomorphism as in the proposition. We obtain a homomorphism

$$\mathcal{O}_M \rightarrow \Phi(\mathcal{O}_M) \rightarrow H_1(A, \mathbb{Z}) = \Lambda$$

which, for simplicity, we also denote by $\phi : \mathcal{O}_M \rightarrow \Lambda$. Let $r \in \Lambda$ be the image of $1 \in \mathcal{O}_M$. The fact that $Q = \phi^* E_A$ implies that $T_Q = \phi^* T_A = \phi^* T$. Therefore, we have

$$\eta^{-1} = T_Q(1, 1) = T_A(\phi(1), \phi(1)) = T(\phi(1), \phi(1)) = T(r, r),$$

so that $\mathfrak{h}(r, r) = \eta \cdot T(r, r) = 1$. We claim that $\mathfrak{h}(x, r_\tau) = 0$, where the element $r_\tau \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau$ is the image of $r \in \Lambda$. To see this, write

$$\Phi(\mathcal{O}_M) = L, \quad L \otimes \mathbb{C} = W^{-1,0} \oplus W^{0,-1},$$

and let $\alpha \in L$ correspond to $1 \in \mathcal{O}_M$. Notice that $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau = W_\tau^{-1,0}$. Consequently, since the composition

$$W_\tau^{-1,0} = (L \otimes_{\mathbb{Z}} \mathbb{C})_\tau \rightarrow (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = H_\tau^{-1,0} \oplus H_\tau^{0,-1}$$

factors through the inclusion of $H_\tau^{-1,0}$ into $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau$, we see that $r_\tau = r_\tau^{-1,0} \in H_\tau^{-1,0} = (H_\tau^{0,-1})^\perp = \langle x \rangle^\perp$, and the claim follows.

Conversely, let $[x] \in \langle r_\mathbb{C} \rangle^\perp \subset \mathcal{H}$ with $r \in \Lambda$ such that $\mathfrak{h}(r, r) = 1$ and consider the marked abelian variety $A = H^{-1,0}/\Lambda$ corresponding to $[x]$. Define a homomorphism $\phi: \mathcal{O}_M \rightarrow \Lambda$ by $\phi(1) = r$. Then ϕ can be shown to be a morphism of Hodge structures using the fact that its \mathbb{C} -linear extension preserves the eigenspace decompositions. We obtain an \mathcal{O}_M -linear homomorphism $\phi: \mathbb{C}^g/\Phi(\mathcal{O}_M) \rightarrow A$. The fact that $\mathfrak{h}(r, r) = 1$ implies that ϕ preserves the polarizations on both sides. \square

5 Orthogonal hyperplane arrangements

Let M be a CM field of degree $2g$ over \mathbb{Q} , with involution $\sigma: M \rightarrow M$ that corresponds to complex conjugation on \mathbb{C} . We assume that M satisfies the following:

Condition 5.1. *There exists an element $\eta \in \mathcal{O}_M$ satisfying the condition that*

$$\mathfrak{D}_M = (\eta) \quad \text{and} \quad \sigma(\eta) = -\eta. \quad (13)$$

Fix an element $\eta \in \mathcal{O}_M$ that satisfies (13). Let $\Phi \subset \text{Hom}(M, \mathbb{C})$ be the unique CM type such that $\Im(\varphi(\eta)) > 0$ for all $\varphi \in \Phi$. Let Λ be a free \mathcal{O}_M -module of rank $n+1$ equipped with a hermitian form $\mathfrak{h}: \Lambda \times \Lambda \rightarrow \mathcal{O}_M$ of signature $(n, 1)$ with respect to one $\tau \in \Phi$ and of signature $(n+1, 0)$ with respect to all other $\varphi \neq \tau \in \Phi$. Let $W = \Lambda \otimes_{\mathcal{O}_M, \tau} \mathbb{C}$ and define $\mathbb{C}H(W)$ as the space of negative lines in the hermitian space W . For an element $r \in \Lambda$, let r_τ be its image in $(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = \Lambda \otimes_{\mathcal{O}_M, \tau} \mathbb{C}$. If $\mathfrak{h}(r, r) = 1$, we define $\langle r_\tau \rangle^\perp \subset \mathbb{C}H(W)$ as the set of negative lines $\ell \subset W$ with $\mathfrak{h}(r_\tau, \ell) = 0$.

Theorem 5.2. *Assume Condition 5.1. Let $r, t \in \Lambda$ such that $\mathfrak{h}(r, r) = \mathfrak{h}(t, t) = 1$. Suppose that $\langle r_\tau \rangle^\perp \cap \langle t_\tau \rangle^\perp \neq \emptyset$ and that $\langle r_\tau \rangle^\perp \neq \langle t_\tau \rangle^\perp \subset \mathbb{C}H(W)$. Then $\mathfrak{h}(r, t) = 0$.*

Proof. Consider the complex torus $B = \mathbb{C}^g/\Phi(\mathcal{O}_M)$. Since the different ideal \mathfrak{D}_M is the principal ideal $(\eta) \subset \mathcal{O}_M$, we have

$$\begin{aligned} \{x \in M : \text{Tr}_{M/\mathbb{Q}}(x\eta^{-1}\mathcal{O}_M) \subset \mathbb{Z}\} &= \{x \in M : x \cdot \eta^{-1}\mathcal{O}_M \subset \eta^{-1}\mathcal{O}_M\} \\ &= \{x \in M : x\mathcal{O}_M \subset \mathcal{O}_M\} = \mathcal{O}_M. \end{aligned} \quad (14)$$

Therefore, the alternating form

$$Q: \Phi(\mathcal{O}_M) \times \Phi(\mathcal{O}_M) \rightarrow \mathbb{Z}, \quad (\Phi(x), \Phi(y)) \mapsto \text{Tr}_{M/\mathbb{Q}}(\eta^{-1} \cdot x \cdot y)$$

defines a *principal* polarization on $B = \mathbb{C}^g/\Phi(\mathcal{O}_M)$. See also (11) above.

Consider the moduli space $\text{Sh}_M(\Lambda)$ attached to the hermitian \mathcal{O}_M -lattice Λ as in Definition (4.8). Let $[x] \in \langle r_\tau \rangle^\perp \cap \langle t_\tau \rangle^\perp \subset \mathbb{C}H(W)$, and let $[(A, i, \lambda)]$ be the moduli point in $\text{Sh}_M(\Lambda)$ corresponding to the image of $[x]$ in $P\Gamma \backslash \mathbb{C}H(W)$, see Corollary 4.9. Define $B = \mathbb{C}^g/\Phi(\mathcal{O}_M)$. By Proposition 4.10, the roots r and t induce \mathcal{O}_M -linear embeddings $\phi_1 : B \hookrightarrow A$ and $\phi_2 : B \hookrightarrow A$ of polarized abelian varieties. By [BL04, Corollary 5.3.13], there exist abelian subvarieties $C_i \subset A$, $i = 1, 2$ such that

$$A \cong B \times C_1 \quad \text{and} \quad A \cong B \times C_2$$

as polarized abelian varieties, where B and C_i are endowed with the polarizations λ_B and λ_{C_i} induced by their embedding into A . Note that B is a simple abelian variety because $\text{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{O}_M \otimes_{\mathbb{Z}} \mathbb{Q} = K$ is a field. In particular (B, λ_B) is non-decomposable as a principally polarized principally polarized abelian variety.

By [Deb96], the decomposition of (A, λ) into non-decomposable polarized abelian subvarieties is unique, in the strong sense that if (A_i, λ_i) , $i \in \{1, \dots, r\}$ and (B_j, μ_j) , $j \in \{1, \dots, m\}$ are polarized abelian subvarieties such that the natural homomorphisms $\prod_i (A_i, \lambda_i) \rightarrow (A, \lambda)$ and $\prod_j (B_j, \mu_j) \rightarrow (A, \lambda)$ are isomorphisms, then $r = m$ and there exists a permutation σ on $\{1, \dots, r\}$ such that B_j and $A_{\sigma(j)}$ are *equal* as polarized abelian subvarieties of (A, λ) , for every $j \in \{1, \dots, r\}$. Thus, for the abelian subvarieties $B_i = \phi_i(B) \subset A$, we have either that $B_1 = B_2 \subset A$, or that $B_1 \cap B_2 = \{0\} \subset A$.

Suppose that $B_1 = B_2$. Then $\mathcal{O}_M \cdot r = \phi_1(\mathcal{O}_M) = \phi_2(\mathcal{O}_M) = \mathcal{O}_M \cdot t \subset \Lambda$, hence $r = \lambda t$ for some $\lambda \in \mathcal{O}_M^*$; but then $\langle r_\tau \rangle^\perp = \langle t_\tau \rangle^\perp \subset \mathbb{C}H(W)$ which is absurd.

Therefore, we must have $A = B_1 \times B_2 \times C$ as polarized abelian varieties, for some polarized abelian subvariety C of A . This implies that

$$H^{-1,0} = \text{Lie}(A) = \text{Lie}(B_1) \times \text{Lie}(B_2) \times \text{Lie}(C), \quad (15)$$

which is orthogonal for the positive definite hermitian form $iE_{\mathbb{C}}(x, \bar{y})$ on $H^{-1,0}$.

Observe that $r_\tau = r_\tau^{-1,0} \in H_\tau^{-1,0}$ and $t_\tau = t_\tau^{-1,0} \in H_\tau^{-1,0}$: see the proof of Proposition 4.10. By Lemma 4.3, we have

$$\begin{aligned} \mathfrak{h}(r, t) &= \mathfrak{h}^\tau(r_\tau, t_\tau) = \tau(\eta) \cdot T_{\mathbb{C}}^\tau(r_\tau, t_\tau) \\ &= \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau, \bar{t}_\tau) = \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}). \end{aligned}$$

Since $r_\tau^{-1,0} \in \text{Lie}(B_1)$ and $\overline{t_\tau^{-1,0}} \in \text{Lie}(B_2)$, we have $iE_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}) = 0$ by (15). We conclude that $\mathfrak{h}(r, t) = 0$ and the proof is finished. \square

Example 5.3. Suppose that $K = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ for some $d \in \mathbb{Z}$ with $d < 0$. There is an alternative, more elementary proof of Theorem 5.2 in this case, see [ACT02b, Lemma 7.29]. Namely, let $r, t \in \Lambda \subset \Lambda \otimes_{\mathcal{O}_K} \mathbb{C} = W$ and suppose that $\mathfrak{h}(r, r) = \mathfrak{h}(t, t) = 1$, that $\langle r \rangle^\perp \neq \langle t \rangle^\perp$ and that $\langle r \rangle^\perp \cap \langle t \rangle^\perp \neq \emptyset \subset \mathbb{C}H(W)$. Let $[x] \in \langle r \rangle^\perp \cap \langle t \rangle^\perp$ and consider the decomposition $W = \langle x \rangle \oplus \langle x \rangle^\perp$. Since $r, t \in \langle x \rangle^\perp$, and the signature of W is $(n, 1)$, they span a positive definite subspace $\langle r, t \rangle \subset W$. Therefore, the matrix

$$\begin{pmatrix} h(r, r) & h(r, t) \\ h(t, r) & h(t, t) \end{pmatrix} = \begin{pmatrix} 1 & h(r, t) \\ h(t, r) & 1 \end{pmatrix}$$

is positive definite, hence $|h(r, t)|^2 < 0$. Since $|h(r, t)|^2 \in \mathbb{Z}$, we have $h(r, t) = 0$.

We conclude Section 5 by showing that the condition on the different ideal $\mathfrak{D}_M \subset \mathcal{O}_M$ in Theorem 5.2 (see Condition 5.1) is satisfied in several natural cases.

Proposition 5.4. *Suppose that M/\mathbb{Q} is an imaginary quadratic extension, or that $M = \mathbb{Q}(\zeta_n)$ is a cyclotomic field for some integer $n \geq 3$. Then $\mathfrak{D}_M = (\beta) \subset \mathcal{O}_M$ for some element $\beta \in \mathcal{O}_M$ such that $\sigma(\beta) = -\beta$.*

Proof. If M/\mathbb{Q} is imaginary quadratic with discriminant Δ , then $\mathfrak{D}_M = (\sqrt{\Delta})$ and the assertion is immediate. Thus, let $n \geq 3$ be an integer, and consider the fields

$$M = \mathbb{Q}(\zeta_n) \supset F = \mathbb{Q}(\alpha), \quad \text{where} \quad \alpha = \zeta_n + \zeta_n^{-1}.$$

Since $\mathcal{O}_M = \mathbb{Z}[\zeta_n]$ by [Neu99, Chapter I, Proposition 10.2], we have $\mathcal{O}_M = \mathcal{O}_F[\zeta_n]$. Notice that $f(x) = x^2 - \alpha x + 1 \in \mathcal{O}_F[x]$ is the minimal polynomial of ζ_n over F . We have $f'(\zeta_n) = 2\zeta_n - \alpha = \zeta_n - \zeta_n^{-1}$. Therefore,

$$\mathfrak{D}_{M/F} = (f'(\zeta_n)) = (\zeta_n - \zeta_n^{-1}), \quad \text{see [Neu99, Chapter III, Proposition 2.4].}$$

By [Lia76], we know that $\mathcal{O}_F = \mathbb{Z}[\alpha]$. Moreover, if $g(x) \in \mathbb{Z}[x]$ is the minimal polynomial of α over \mathbb{Q} , then $\mathfrak{D}_{F/\mathbb{Q}} = (g'(\alpha))$. By [Neu99, Chapter III, Proposition 2.2], we have that $\mathfrak{D}_{M/\mathbb{Q}} = \mathfrak{D}_{M/F}\mathfrak{D}_{F/\mathbb{Q}}$. Combining all this yields

$$\mathfrak{D}_{M/\mathbb{Q}} = \mathfrak{D}_{M/F}\mathfrak{D}_{F/\mathbb{Q}} = (\zeta_n - \zeta_n^{-1}) \cdot (g'(\alpha)) = ((\zeta_n - \zeta_n^{-1})g'(\alpha)).$$

□

Remark 5.5. Consider triples (M, β, Λ) of the following form: M is a CM field of degree $2g$ with totally real subfield F , β is an element in $\mathcal{O}_M - \mathcal{O}_F$ such that $\beta^2 \in \mathcal{O}_F$, and Λ is a finite free \mathcal{O}_M -module equipped with a hermitian form $\mathfrak{h}: \Lambda \times \Lambda \rightarrow \beta \cdot \mathfrak{D}_M^{-1}$ of signature $(n, 1)$ with respect to one $\tau \in \Phi = \{\varphi \in \text{Hom}(K, \mathbb{C}) \mid \Im(\varphi(\beta)) > 0\}$ and positive definite with respect to every $\varphi \neq \tau \in \Phi$. It turns out that can associate an orthogonal hyperplane arrangement

$$\mathcal{H}(M, \beta, \Lambda) = \bigcup_f H_f \subset \mathbb{C}H^n = \{\text{negative lines in } \Lambda \otimes_{\mathcal{O}_M, \tau} \mathbb{C}\}$$

to every such triple (M, β, Λ) , in such a way that $\mathcal{H}(M, \beta, \Lambda) = \mathcal{H} = \cup_{\mathfrak{h}(r, r)=1} \langle r_{\mathbb{C}} \rangle^{\perp}$ if $\mathfrak{D}_M = (\beta)$. Indeed, for such a triple, let \mathcal{S} be the set of fractional ideals $\mathfrak{a} \subset M$ for which there exist $b \in \mathcal{O}_F$ such that $\mathfrak{D}_M \mathfrak{a} \bar{\mathfrak{a}} = (b\beta)$. By [Wam99, Theorem 4], the set \mathcal{S} is not empty. For each $\mathfrak{a} \in \mathcal{S}$, pick an element b as above and define $\eta(\mathfrak{a}) = b\beta \in \mathcal{O}_M$. The complex torus $B = \mathbb{C}^g / \Phi(\mathfrak{a})$ is polarized by the Riemann form $Q: \Phi(\mathfrak{a}) \times \Phi(\mathfrak{a}) \rightarrow \mathbb{Z}$, $(\Phi(x), \Phi(y)) \mapsto \text{Tr}_{M/\mathbb{Q}}(\eta(\mathfrak{a})^{-1} \cdot x\bar{y})$, and this is a principal polarization [Wam99, Theorem 3]. Let \mathcal{R} be the set of embeddings $f: \mathfrak{a} \rightarrow \Lambda$, $\mathfrak{a} \in \mathcal{S}$, such that $\mathfrak{h}(f(x), f(y)) = x\bar{y}$ for $x, y \in \mathfrak{a}$. For $f \in \mathcal{R}$, one obtains a hyperplane

$$H_f = \{x \in \mathbb{C}H^n : \mathfrak{h}^{\tau}(x, f(\mathfrak{a})) = 0\} \subset \mathbb{C}H^n.$$

Then $\mathcal{H}(M, \beta, \Lambda) = \cup_{f \in \mathcal{R}} H_f$ gives the arrangement, because $\mathcal{H}(M, \beta, \Lambda)$ is orthogonal by arguments similar to those used to prove Proposition 4.10 and Theorem 5.2.

6 The standard hermitian lattice over the Eisenstein integers

The goal of this section is to study particular examples of the hyperbolic spaces $X(\Lambda)$ obtained via Theorem 3.1, namely those obtained by considering the field $\mathbb{Q}(\zeta_3)$ and the hermitian lattice $\mathbb{Z}[\zeta_3]^{n,1}$ over $\mathbb{Z}[\zeta_3]^n$. The ultimate goal is to prove Theorem 1.2.

6.1 **The case of Eisenstein integers.** In Section 6, we fix the following notation: we let $K = \mathbb{Q}(\zeta_3)$, and for each $n \in \mathbb{Z}_{\geq 2}$, we define

$$\Lambda_n = (\mathbb{Z}[\zeta_3]^{n+1}, h_n), \quad h_n(x, y) = -x_0\bar{y}_0 + \sum_{i=1}^n x_i\bar{y}_i, \quad \mathcal{B}_n = \{r \in \Lambda_n \mid h_n(r, r) = 1\}.$$

As in Sections 2.1 - 2.3, we make the following definitions, although this time we label each object by the integer $n \in \mathbb{Z}_{\geq 2}$ that is naturally attached to it:

$$\begin{aligned} \Gamma(n) &= \text{Aut}(\Lambda_n) = \text{Aut}_{\mathbb{Z}[\zeta_3]}(\mathbb{Z}[\zeta_3]^{n+1}, h_n), & P\Gamma(n) &= \Gamma(n)/\mu_K \\ \mathcal{A}_n &= \{\text{anti-unitary involutions } \alpha: \Lambda_n \rightarrow \Lambda_n\}, & P\mathcal{A}_n &= \mathcal{A}_n/\mu_K, \\ C\mathcal{A}_n &= P\mathcal{A}_n/P\Gamma(n). \end{aligned}$$

By Theorem 5.2 and Proposition 5.4, the hyperplane arrangement $\mathcal{H}_n = \cup_{r \in \mathcal{B}_n} H_r$ is an orthogonal arrangement, i.e. Condition 2.3 is satisfied. Thus, we can perform the glueing construction of Section 2 to obtain a sequence of metric spaces $X(\Lambda_n)$, with $n \in \mathbb{Z}_{\geq 2}$. Moreover, for each $n \in \mathbb{Z}_{\geq 2}$, the metric on $X(\Lambda_n)$ extends to a complete real hyperbolic orbifold structure, hence its connected components are quotients of $\mathbb{R}H^n$ by discrete groups of isometries (see Theorem 3.1.4). The goal of Section 6 is to prove Theorem 1.2, saying that for every $n \geq 2$, there exists a connected component $X(\Lambda_n)^+$ of $X(\Lambda_n)$ such that the lattice $\Gamma_n^+ \subset \text{PO}(n, 1)$ underlying $X(\Lambda_n)^+$ is non-arithmetic.

6.2 **The case $n = 2$.** Define three anti-unitary involutions $\alpha_i: \Lambda_2 \rightarrow \Lambda_2$ as follows:

$$\begin{aligned} \alpha_0: (x_0, x_1, x_2) &\mapsto (\bar{x}_0, \bar{x}_1, \bar{x}_2) \\ \alpha_1: (x_0, x_1, x_2) &\mapsto (\bar{x}_0, -\bar{x}_1, \bar{x}_2) \\ \alpha_2: (x_0, x_1, x_2) &\mapsto (\bar{x}_0, -\bar{x}_1, -\bar{x}_2). \end{aligned} \tag{16}$$

Lemma 6.1. *The involutions $\alpha_0, \alpha_1, \alpha_2$ are pairwise not $\Gamma(2)$ -conjugate, each anti-unitary involution of Λ_2 is $\Gamma(2)$ -conjugate to exactly one of the $\pm\alpha_i$, the composition*

$$\prod_{i=0}^2 \mathbb{R}H_{\alpha_i}^2 \rightarrow Y(\Lambda_2) \rightarrow P\Gamma(2) \setminus Y(\Lambda_2) = X(\Lambda_2)$$

is surjective, and $X(\Lambda_2)$ is connected.

Proof. Let $\theta = \zeta_3 - \zeta_3^{-1} = \sqrt{-3} \in \mathbb{Z}[\zeta_3]$ and consider the vector space $W_2 = \Lambda_2/\theta\Lambda_2$. For $\alpha \in \mathcal{A}$, let $\alpha: W_2 \rightarrow W_2$ be the induced involution, let $q_2: W_2 \rightarrow \mathbb{F}_3$ be the quadratic form $q_2(x) = h_2(x, x) \bmod \theta$, and define $D(\alpha) = \dim(V^\alpha)$ and $T(\alpha) =$

$\det(q_2|_{V^\alpha}) \in \mathbb{F}_3^*/(\mathbb{F}_3^*)^2 = \{\pm 1\}$. The restrictions of q_2 to the fixed spaces $V^{\alpha_i} \subset V$ for α_0, α_1 and α_2 have pairwise distinct conjugacy invariant $(D(\alpha_i), T(\alpha_i))$ (see Lemma 6.2 below), which proves the first statement. Next, one observes that the subspace

$$\coprod_{\alpha \in \mathcal{C}\mathcal{A}} P\Gamma(2)_\alpha \setminus (\mathbb{R}H_\alpha^2 - \mathcal{H}_2) \subset X(\Lambda_2)$$

(see Theorem 3.1.3) is naturally homeomorphic to the moduli space of stable real binary sextics with one double root at ∞ and no other double roots (c.f. [ACT06, Section 5]). As in the proof of Theorem 4 of [ACT10], one deduces that each anti-unitary involution of Λ_2 must be $\Gamma(2)$ -conjugate to exactly one of the $\pm\alpha_i$. The third and fourth statement follow from this. \square

6.3 Conjugacy classes of anti-unitary involutions. Define anti-unitary involutions as follows:

$$\begin{aligned} \beta_0: \Lambda_n &\rightarrow \Lambda_n, & \beta_0(x_0, \dots, x_n) &= (\bar{x}_0, \dots, \bar{x}_n), \\ \beta_i: \Lambda_n &\rightarrow \Lambda_n, & \beta_i(x_0, \dots, x_n) &= (\bar{x}_0, -\bar{x}_1, \dots, -\bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n). \end{aligned} \quad (17)$$

Lemma 6.2. *The classes $\pm\beta_i \in \mathcal{A}$ for $i = 0, \dots, n$ are all pairwise non $\Gamma(n)$ -conjugate. In particular, the classes $\beta_i \in P\mathcal{A}_n$ are all pairwise non $P\Gamma(n)$ -conjugate.*

Proof. Consider the element $\theta = \zeta_3 - \zeta_3^{-1}$ and define $W_n = \Lambda_n/\theta\Lambda_n$. Consider the quadratic form $q_n: W_n \rightarrow \mathbb{F}_3$ defined as $q_n(x) = h_n(x, x) \bmod \theta$. Note that, for $i \geq 1$ and $x = (x_0, x_{i+1}, \dots, x_n) \in W_n^{\beta_i}$, one has $q_n(x) = -x_0^2 + x_{i+1}^2 + \dots + x_n^2$. Similarly, for $x = (x_1, \dots, x_i) \in V^{-\beta_i}$, one has $q_n(x) = x_1^2 + \dots + x_i^2$. For an anti-unitary involution $\alpha: \Lambda_n \rightarrow \Lambda_n$, define the following $\Gamma(n)$ -conjugacy invariants:

$$D(\alpha) = \dim(V^\alpha), \quad T(\alpha) = \det(q_n|_{V^\alpha}).$$

The above shows that $(D(\beta_i), T(\beta_i)) = (n - i + 1, -1)$ and $(D(-\beta_i), T(-\beta_i)) = (i, 1)$ for $i = 0, 1, \dots, n$. The lemma follows. \square

6.4 The map between the glued spaces. Consider the canonical embedding of hermitian \mathcal{O}_K -lattices $\Psi: \Lambda_2 \rightarrow \Lambda_n$ defined as $\Psi(x_0, x_1, x_2) = (x_0, x_1, x_2, 0, 0, \dots, 0)$. We view Λ_2 as a sublattice of Λ_n via Ψ , and write

$$\Lambda_n = \Lambda_2 \oplus (\Lambda_2)^\perp. \quad (18)$$

Using the canonical basis of Λ_n , we may view $\Gamma(n) = \text{Aut}(\Lambda_n)$ as a subgroup of $\text{GL}_{n+1}(\mathcal{O}_K)$. This gives an embedding

$$j: \Gamma(2) \rightarrow \Gamma(n), \quad M \mapsto (M, \mathbb{I}_{n-2}). \quad (19)$$

The natural totally geodesic embedding

$$\nu: \mathbb{C}H^2 \hookrightarrow \mathbb{C}H^n \quad (20)$$

induces a totally geodesic embedding

$$\nu: \tilde{Y}_2^\# := \prod_{i=0}^2 \mathbb{R}H_{\alpha_i}^2 \hookrightarrow \prod_{i=0}^2 \mathbb{R}H_{\beta_i}^n \subset \prod_{\alpha \in P\mathcal{A}_n} \mathbb{R}H_\alpha^n =: \tilde{Y}_n. \quad (21)$$

Since the composition $\mathrm{SO}(2,1) \rightarrow \mathrm{O}(2,1) \rightarrow \mathrm{PO}(2,1) = \mathrm{Isom}(\mathbb{R}H^2)$ is an isomorphism, there is a natural embedding $\mathrm{PO}(2,1) \hookrightarrow \mathrm{PO}(n,1)$. Moreover, since we have

$$P\Gamma(2)_{\alpha_i} = \mathrm{Stab}_{P\Gamma(2)}(\mathbb{R}H_{\alpha_i}^2) = \mathrm{PO}(\Lambda_2^{\alpha_i})$$

by [ACT06, (5.1)], the map (19) induces embeddings $j: P\Gamma(2)_{\alpha_i} \hookrightarrow P\Gamma(n)_{\beta_i}$ for $i = 0, 1, 2$ that make the map ν in (21) equivariant.

By Lemma 6.2, the classes of β_0, β_1 and β_2 in $P\mathcal{A}_n$ are pairwise not $P\Gamma(n)$ -conjugate, so that the induced map of hyperbolic orbifolds

$$O_2 := \prod_{i=0}^2 P\Gamma(2)_{\alpha_i} \setminus (\mathbb{R}H_{\alpha_i}^2 - \mathcal{H}_2) \rightarrow \prod_{\alpha \in C\mathcal{A}_n} P\Gamma(n)_\alpha \setminus (\mathbb{R}H_\alpha^n - \mathcal{H}_n) =: O_n \quad (22)$$

induces an injective map on sets of connected components $\pi_0(O_2) \rightarrow \pi_0(O_n)$.

Lemma 6.3. *For each integer $n \geq 2$, there exists a natural map of metric spaces*

$$\iota: X(\Lambda_2) \rightarrow X(\Lambda_n).$$

Proof. We claim that (22) extends to a commutative diagram of metric spaces

$$\begin{array}{ccc} X(\Lambda_2) = \Gamma(2) \backslash Y(\Lambda_2) & \xrightarrow{\iota} & \Gamma(n) \backslash Y(\Lambda_n) = X(\Lambda_n) \\ \uparrow & & \uparrow \\ O_2 & \longrightarrow & O_n. \end{array}$$

For this, it suffices to prove the following assertions:

1. The map $\tilde{Y}_2^\# \rightarrow \tilde{Y}_n$ defined in (21) is compatible with the equivalence relations on both sides (see Definition 2.15).

2. Let $Y(\Lambda_2)^\# \subset Y(\Lambda_2)$ be the image of $\tilde{Y}_2^\#$ in $Y(\Lambda_2)$ under the canonical map $\tilde{Y}_2 \rightarrow Y(\Lambda_2)$. The resulting map of metric spaces

$$\phi: Y(\Lambda_2)^\# \rightarrow Y(\Lambda_n) \quad (23)$$

is equivariant with respect to the morphism $j: \mathrm{Stab}_{\Gamma(2)}(\tilde{Y}_2^\#) \rightarrow \Gamma(n)$ induced by (19). Thus, ϕ descends to a map of metric spaces

$$\iota: X(\Lambda_2) = \Gamma(2) \backslash Y(\Lambda_2) = \mathrm{Stab}_{\Gamma(2)}(\tilde{Y}_2^\#) \backslash Y(\Lambda_2)^\# \rightarrow \Gamma(n) \backslash Y(\Lambda_n) = X(\Lambda_n),$$

where the second equality follows from Lemma 6.1.

As for 1, let $r \in \mathcal{R}_2$ be a short root in Λ_2 , which we also view as a short root in Λ_n . Let $k \in \mathbb{Z}/m$ and let $x = y + z \in \Lambda_n$ be any element in Λ_n , where we decomposed x using (18). Consider the reflection $\phi_r^k: \Lambda_2 \rightarrow \Lambda_2$. We have

$$j(\phi_r^k)(x) = j(\phi_r^k)(y + z) = x - (1 - \zeta_3^k)h_n(x, r) \cdot r.$$

Thus $j(\phi_r^k) = \phi_r^k$. Next, let $(x, \alpha_i), (x, \alpha_j) \in \tilde{Y}_2^\#$ be such that $(x, \alpha_i) \sim (x, \alpha_j)$. We want to show that $\phi_2(x, \alpha_i) \sim \phi_2(x, \alpha_j)$. We may assume that $\alpha_i \neq \alpha_j$. Therefore, $x \in \mathcal{H}_2$ and $\alpha_j = g \circ \alpha_i \in P\mathcal{A}_2$ for some $g \in G_2(x)$, where $G_2(x)$ is as in Definition 2.7. By the above, we have $j(G_2(x)) = G_n(\nu(x))$, where $\nu: \mathbb{C}H^2 \rightarrow \mathbb{C}H^n$ is as in (20). Since $\alpha_i \circ \alpha_j = g \in G_2(x)$, we have $f(\alpha_i) \circ f(\alpha_j) = j(g) \in G_n(\nu(x))$, which proves 1.

To prove 2, let $p_n: \tilde{Y}_n \rightarrow Y(\Lambda_n)$ be the quotient map, and similarly define $p_2: \tilde{Y}_2 \rightarrow Y(\Lambda_2)$. Let $(x, \alpha_i), (y, \alpha_j) \in \tilde{Y}_2^\#$ with images $v = p_2(x, \alpha_i), w = p_2(y, \alpha_j) \in Y(\Lambda_2)^\#$. Suppose that there exists an element $g \in \text{Stab}_{P\Gamma(2)}(\tilde{Y}_2^\#) \subset P\Gamma(2)$ such that

$$g \cdot v = p_2(g \cdot x, g\alpha_i g^{-1}) = w = p_2(y, \alpha_j).$$

We want to show that $j_2(g) \cdot \phi(v) = \phi(w)$. Since $g \in \text{Stab}_{P\Gamma(2)}(\tilde{Y}_2^\#)$, we have $g\alpha_i g^{-1} = \alpha_i$ by Lemma 6.1. We thus get that $p_2(g \cdot x, \alpha_i) = p_2(y, \alpha_i)$. Since the map $p_2: \tilde{Y}_2 \rightarrow Y(\Lambda_2)$ is injective when restricted to $\mathbb{R}H_{\alpha_i}^2$, it follows that

$$g \cdot (x, \alpha_i) = (g \cdot x, g\alpha_i g^{-1}) = (g \cdot x, \alpha_i) = (y, \alpha_i),$$

which implies that $j(g) \cdot (\nu(x), f(\alpha_i)) = \nu(g \cdot (x, \alpha_i)) = \nu(y, \alpha_i) = (\nu(y), f(\alpha_i))$. Since $\phi(v) = p_n(\nu(x), f(\alpha_i))$ and $\phi(w) = p_n(\nu(y), f(\alpha_j))$, we obtain $j(g) \cdot \phi(v) = \phi(w)$. \square

6.5 **The orbifold map between the glued spaces.** The goal of this section is to prove:

Proposition 6.4. *The map ι defined in Lemma 6.3 is a map of hyperbolic orbifolds.*

Proof. Let $\bar{f} \in X(\Lambda_2)$ and lift \bar{f} to an element $f \in Y(\Lambda_2)^\# \subset Y(\Lambda_2)$ (this is possible by Lemma 6.1). In turn, we can lift f to an element $(x, \alpha_{j_0}) \in \tilde{Y}_2^\#$ for some $j_0 \in \{0, 1, 2\}$ with α_{j_0} as in (16)). To prove that $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)$ is an orbifold map at \bar{f} , there are three cases to consider:

Case I: *The element $f \in Y(\Lambda_2)$ has zero nodes* (see Definition 2.7). In this case, \bar{f} lies in $O_2 \subset X(\Lambda_2)$ and the map $O_2 \rightarrow O_n$ of (22) is a morphism of orbifolds.

Case II: *The element $f \in Y(\Lambda_2)$ has one real node.* Let the node be defined by a short root $r \in \mathcal{R}_2$ such that $x \in H_r$. Consider the image $g = \phi(f) \in Y(\Lambda_n)$ of f in $Y(\Lambda_n)$ by the map $\phi: Y(\Lambda_2)^\# \rightarrow Y(\Lambda_n)$ defined in (23). It admits the lift $(\nu(x), \beta_{j_0}) \in \tilde{Y}_n$. For $p \in \{2, \dots, n-1\}$, let $r_p = (0, 0, 0, \dots, 1, \dots, 0) \in \mathcal{R}_n$, where the 1 is on the $(p+1)$ -th coordinate. Note that $\nu(x)$ has $n-1$ nodes. Define $r_1 = \Psi(r) \in \mathcal{R}_n$, so that

$$\nu(x) \in \bigcap_{i=1}^{n-1} H_{r_i} \subset \mathbb{C}H^n, \quad \text{and} \quad \mathcal{S} = \{r_1, r_2, \dots, r_{n-1}\} \subset \mathcal{R}_n$$

is a set of short roots of maximal cardinality such that $\nu(x) \in H_t$ for all $t \in \mathcal{S}$.

Lemma 6.5. Consider $x \in \mathbb{C}H^2$, $\nu(x) \in \mathbb{C}H^n$, $r \in \mathcal{R}_2$ and $\Psi(r) \in \mathcal{R}_n$ as above. There are isometries $\kappa_2: \mathbb{C}H^2 \xrightarrow{\sim} \mathbb{B}^2(\mathbb{C})$ and $\kappa_n: \mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$, identifying x (resp. $\nu(x)$) with the origin, ϕ_r (resp. ϕ_{r_1} , resp. ϕ_{r_i} for $i \geq 2$) with the respective maps

$$\begin{aligned} \mathbb{B}^2(\mathbb{C}) &\rightarrow \mathbb{B}^2(\mathbb{C}), & (t_1, t_2) &\mapsto (\zeta_6 \cdot t_1, t_2), \\ \mathbb{B}^n(\mathbb{C}) &\rightarrow \mathbb{B}^n(\mathbb{C}), & (t_1, \dots, t_n) &\mapsto (\zeta_6 \cdot t_1, \dots, t_n), \\ \mathbb{B}^n(\mathbb{C}) &\rightarrow \mathbb{B}^n(\mathbb{C}), & (t_1, \dots, t_n) &\mapsto (t_1, \dots, \zeta_6 \cdot t_{i+1}, \dots, t_n), \end{aligned}$$

and α_{j_0} (resp. β_{j_0}) with the respective maps

$$\begin{aligned} \mathbb{B}^2(\mathbb{C}) &\rightarrow \mathbb{B}^2(\mathbb{C}), & (t_1, t_2) &\mapsto (\bar{t}_1, \bar{t}_2), \\ \mathbb{B}^n(\mathbb{C}) &\rightarrow \mathbb{B}^n(\mathbb{C}), & (t_1, \dots, t_n) &\mapsto (\bar{t}_1, \dots, \bar{t}_n), \end{aligned}$$

and such that, for the canonical embedding $\rho: \mathbb{B}^2(\mathbb{C}) \hookrightarrow \mathbb{B}^n(\mathbb{C})$, one has $\rho \circ \kappa_2 = \kappa_n \circ \nu$.

Proof. Define V_n as the complex $n+1$ -dimensional hermitian space $V_n = (\Lambda \otimes \mathbb{C}, (h_n)_{\mathbb{C}})$. Let $x \in V_2$ be a lift of $x \in \mathbb{C}H^2 = \{\text{negative lines in } V_2\}$. Defining $\nu: V_2 \rightarrow V_n$ as the natural embedding allows us to consider x and r as elements of V_n . Then define

$$T_2 = \langle x \rangle \oplus \langle r \rangle \subset V_2, \quad R_2 = T_2^\perp \subset V_2, \quad T_n = \langle x \rangle \oplus \langle r \rangle \subset V_n, \quad R_n = T_n^\perp \subset V_n.$$

Choose suitable bases for R_2 and R_n scale appropriately as in Lemma 3.9. \square

Consider the coordinates $\mathbb{C}H^2 \xrightarrow{\sim} \mathbb{B}^2(\mathbb{C})$ of Lemma 6.5: they make the involution α_{j_0} correspond to $(t_1, t_2) \mapsto (\bar{t}_1, \bar{t}_2)$, and any $\xi \in P\mathcal{A}_2$ with $(x, \alpha) \sim (x, \xi)$ to an involution of the form $(t_1, t_2) \mapsto (\zeta_6^i \cdot \bar{t}_1, \bar{t}_2)$. Thus if the ξ_i for $i = 1, \dots, 6$ are the six anti-unitary involutions such that $(x, \alpha) \sim (x, \xi_i)$, then for the set Y_f defined in (6), one has $Y_f \cong \bigcup_{i=1}^6 \mathbb{R}H_{\xi_i}^2 \cong \{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) \mid t_1^6, t_2 \in \mathbb{R}\}$. The union of the subsets $K_{f,\epsilon} = \{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) \mid i^{-\epsilon} \cdot t_1 \in \mathbb{R}_{\geq 0}, t_2 \in \mathbb{R}\}$, indexed by $\epsilon \in \{0, 1\}$ is a fundamental domain for the action of B_f on Y_f , and $U = K_{f,0} \cup K_{f,1}$ is isometric to $\mathbb{B}^2(\mathbb{R})$. Write $i = (i_1, \dots, i_{n-1})$ and $\chi_i = \prod_{p=1}^{n-1} \phi_{r_p}^{i_p} \circ \beta_{j_0}$. In the coordinates of Lemma 6.5, we have that $Y_g \cong \bigcup_{i_1, \dots, i_{n-1}=1}^6 \mathbb{R}H_{\chi_i}^n \cong \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_1^6, t_2, t_3^6, \dots, t_n^6 \in \mathbb{R}\}$. Moreover, if we define

$$K_{g,\epsilon_1, \dots, \epsilon_{n-1}} = \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid i^{\epsilon_1} t_1, i^{-\epsilon_2} t_3, \dots, i^{-\epsilon_{n-1}} t_n \in \mathbb{R}_{\geq 0}, t_2 \in \mathbb{R}\},$$

then $U_g = \bigcup_{\epsilon_1, \dots, \epsilon_{n-1}=0}^1 K_{g,\epsilon_1, \dots, \epsilon_{n-1}}$ is a fundamental domain for the action of B_g on Y_g , and there is an isometry $U_g \xrightarrow{\sim} \mathbb{B}^n(\mathbb{R})$. Since $\xi_i = \phi_r^i \circ \alpha_{j_0}$ and $\xi_{(i,6,\dots,6)} = \phi_{r_1}^i \circ \beta_{j_0}$ for $i \in \{1, \dots, 6\}$, one has

$$x \in \mathbb{R}H_{\xi_i}^2 \iff (\phi_r^i \circ \alpha_{j_0})(x) = x \implies (\phi_{r_1}^i \circ \beta_{j_0})(\nu(x)) = \nu(x) \iff \nu(x) \in \mathbb{R}H_{\chi_{i,6,\dots,6}}^n.$$

Therefore, with respect to the natural embedding $\nu: \mathbb{C}H^2(\mathbb{C}) \rightarrow \mathbb{C}H^n(\mathbb{C})$, one has for each $i = 1, \dots, 6$, that $\nu(\mathbb{R}H_{\xi_i}^2) \subset \mathbb{R}H_{\chi_{i,6,\dots,6}}^n$. Thus, the maps defined above fit

together in the following commutative diagram of metric spaces:

$$\begin{array}{ccc}
B_f \setminus Y_f & \hookrightarrow & B_g \setminus Y_g \\
\downarrow \wr & & \downarrow \wr \\
B_f \setminus \bigcup_{i=1}^6 \mathbb{R}H_{\xi_i}^n & \hookrightarrow & B_g \setminus \bigcup_{i_1, \dots, i_{n-1}=1}^6 \mathbb{R}H_{\chi_i}^2 \\
\downarrow \wr & & \downarrow \wr \\
\langle \zeta_6 \rangle \setminus \{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) \mid t_1^6, t_2 \in \mathbb{R}\} & \hookrightarrow & \langle \zeta_6 \rangle^{n-1} \setminus \{(t_1, \dots, t_n) \mid t_1^6, t_2, t_3^6, \dots, t_n^6 \in \mathbb{R}\} \\
\downarrow \wr & & \downarrow \wr \\
\bigcup_{\epsilon \in \{0,1\}} K_{f,\epsilon} & \hookrightarrow & \bigcup_{\epsilon_1, \dots, \epsilon_{n-1}=0}^1 K_{g,\epsilon_1, \dots, \epsilon_{n-1}} \\
\downarrow \wr & & \downarrow \wr \\
\mathbb{B}^2(\mathbb{R}) & \hookrightarrow & \mathbb{B}^n(\mathbb{R}).
\end{array} \tag{24}$$

Since the map $Y(\Lambda_2)^\# \rightarrow Y(\Lambda_n)$ defined in (23) is equivariant with respect to $\Gamma(2) \hookrightarrow \Gamma(n)$, we obtain a map $\text{Stab}_{\Gamma(2)}(f) \hookrightarrow \text{Stab}_{\Gamma(n)}(g)$. Via the commutative diagram

$$\begin{array}{ccc}
\text{PO}(2, 1) = \text{Isom}(\mathbb{B}^2(\mathbb{R})) & \hookrightarrow & \text{Isom}(\mathbb{B}^n(\mathbb{R})) = \text{PO}(n, 1) \\
\uparrow & & \uparrow \\
\text{Stab}_{\Gamma(2)}(f) & \hookrightarrow & \text{Stab}_{\Gamma(n)}(g) \\
\uparrow & & \uparrow \\
B_f & \hookrightarrow & B_g
\end{array}$$

we obtain a well-defined embedding

$$A_f/B_f = \text{Stab}_{P\Gamma(2)}(f)/B_f \hookrightarrow \text{Stab}_{P\Gamma(n)}(g)/B_g = A_g/B_g. \tag{25}$$

The embedding $B_f \setminus Y_f \hookrightarrow B_g \setminus Y_g$ is equivariant with respect to (25). As in the proof of Theorem 3.1.2, we choose an A_f (resp. A_g)-equivariant open neighbourhood $f \in V_f \subset Y_f$ (resp. $g \in V_g \subset Y_g$) such that $A_f \setminus V_f \subset P\Gamma_f \setminus Y(\Lambda_2)$ (resp. such that $A_g \setminus V_g \subset P\Gamma_g \setminus Y(\Lambda_n)$). Using the above diagram (24), we get open neighbourhoods $W_f \subset \mathbb{B}^2(\mathbb{R})$ and $W_g \subset \mathbb{B}^n(\mathbb{R})$, acted upon by A_f/B_f and A_g/B_g respectively, such that $A_f \setminus V_f = (A_f/B_f) \setminus W_f$ and $A_g \setminus V_g = (A_g/B_g) \setminus W_g$, and such that there exists a totally geodesic embedding

$$\mathbb{B}^2(\mathbb{R}) \supset W_f \xrightarrow{\rho} W_g \subset \mathbb{B}^n(\mathbb{R}) \tag{26}$$

which is equivariant for (25) and makes the following diagram commute:

$$\begin{array}{ccc}
W_f & \xrightarrow{\rho} & W_g \\
\downarrow & & \downarrow \\
(A_f/B_f) \setminus W_f & \xrightarrow{\rho} & (A_g/B_g) \setminus W_g \\
\downarrow & & \downarrow \\
P\Gamma(2) \setminus Y(\Lambda_2) & \xrightarrow{\iota} & P\Gamma(n) \setminus Y(\Lambda_n).
\end{array}$$

Case III: *The element $f \in Y(\Lambda_2)$ has two real nodes.* This case is similar to Case II and we leave the proof to the reader.

Case IV: *The element $f \in Y(\Lambda_2)$ has a pair of complex conjugate nodes and no real nodes.* Let the two nodes be defined by short roots $r, t \in \mathcal{R}_2$ such that $x \in H_r \cap H_t$. Consider the images $g = \phi(f) \in Y(\Lambda_n)$ of f in $Y(\Lambda_n)$ by the map $\phi: Y(\Lambda_2)^\# \rightarrow Y(\Lambda_n)$ defined in (23). It admits the lift $(\nu(x), \beta_{j_0}) \in \tilde{Y}_n$. For $p \in \{3, \dots, n-1\}$, let $r_p = (0, 0, 0, \dots, 1, \dots, 0) \in \mathcal{R}_n$, where the 1 is on the $(p+1)$ -th coordinate. Note that $\nu(x)$ has n nodes: two complex conjugate nodes and $n-2$ real nodes.

Define $r_1 = \Psi(r) \in \mathcal{R}_n$ and $r_2 = \Psi(t) \in \mathcal{R}_n$. This gives $\nu(x) \in \cap_{i=1}^n H_{r_i} \subset \mathbb{C}H^n$, and $\mathcal{S} = \{r_1, r_2, \dots, r_n\} \subset \mathcal{R}_n$ is a set of short roots of maximal cardinality such that $\nu(x) \in H_{r'}$ for each $r' \in \mathcal{S}$. As in the proof of Proposition 3.14.4 (see also Case II above), we have that

$$Y_f = B_f \setminus Y_f \cong \{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) : t_2^6 = \bar{t}_1^6\} = \bigcup_{i=1}^6 \mathbb{B}^2(\mathbb{R})_{\xi_i},$$

where $\xi_i: \mathbb{B}^2(\mathbb{C}) \rightarrow \mathbb{B}^2(\mathbb{C})$ is defined as $\xi_i(t_1, t_2) = (\bar{t}_2 \zeta^i, \bar{t}_1 \zeta^i)$ for $i \in \mathbb{Z}/6$. Similarly, we can describe $B_g \setminus Y_g$ as follows: we have

$$\begin{aligned}
Y_g &\cong \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^6 = \bar{t}_1^6, t_3^6, \dots, t_n^6 \in \mathbb{R}\}, \quad \text{and} \\
B_g \setminus Y_g &\cong \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^6 = \bar{t}_1^6, t_3^6, \dots, t_n^6 \in \mathbb{R}\} = \bigcup_{i=1}^6 \mathbb{B}^n(\mathbb{R})_{\xi_i},
\end{aligned}$$

where $\xi_i: \mathbb{B}^n(\mathbb{C}) \rightarrow \mathbb{B}^n(\mathbb{C})$ is defined as $\xi_i(t_1, \dots, t_n) = (\bar{t}_2 \zeta^i, \bar{t}_1 \zeta^i, \bar{t}_3, \dots, \bar{t}_n)$ for $i \in \mathbb{Z}/6$. By Proposition 3.14.5, the group A_f (resp. A_g) acts transitively on the copies of $\mathbb{B}^2(\mathbb{R})$ (resp. $\mathbb{B}^n(\mathbb{R})$). Therefore, if we define $\Gamma_f = \text{Stab}_{A_f/B_f}(\mathbb{B}_{\xi_1}^2)$ and $\Gamma_g = \text{Stab}_{A_g/B_g}(\mathbb{B}_{\xi_1}^n)$ as the stabilizer of $\mathbb{B}^2(\mathbb{R})_{\xi_1}$ (resp. $\mathbb{B}^n(\mathbb{R})_{\xi_1}$), then we obtain a commutative diagram

$$\begin{array}{ccc}
\mathbb{B}^2(\mathbb{R})_{\xi_1} & \longrightarrow & \mathbb{B}^n(\mathbb{R})_{\xi_1} \\
\downarrow & & \downarrow \\
\Gamma_f \setminus \mathbb{B}^2(\mathbb{R})_{\xi_1} & \longrightarrow & \Gamma_g \setminus \mathbb{B}^n(\mathbb{R})_{\xi_1}.
\end{array}$$

By definition of the orbifold structure of $X(\Lambda_2)$ and $X(\Lambda_n)$ (see the proof of Theorem 3.1.2), this shows that $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)$ is a morphism of orbifolds at $\bar{f} \in X(\Lambda_2)$. \square

7 Totally geodesic immersions and non-arithmetic lattices

We continue with the notation of Section 6. The goal of Section 7 is twofold:

1. We start by proving Theorem 7.3, using the results of Sections 6.1 - 6.5. This theorem says that, for each $n \in \mathbb{Z}_{\geq 2}$, the map ι defined in Lemma 6.3 defines a proper totally geodesic immersion of $X(\Lambda_2)$ into the connected component $X(\Lambda_n)^+$ of $X(\Lambda_n)$.
2. We then use Theorem 7.3 to prove that the lattice Γ_n^+ underlying the complete hyperbolic orbifold $X(\Lambda_n)^+$ is non-arithmetic, see Theorem 7.10.

Recall the following (see e.g. [Bel+21, §2.3.1]):

Definition 7.1. Let $n, m \in \mathbb{N}$. A map $i: \mathbb{R}H^m/\Lambda \rightarrow \mathbb{R}H^n/\Gamma$ of hyperbolic orbifolds is a *totally geodesic immersion* if there exists a totally geodesic subspace $U \subset \mathbb{R}H^n$ and a lift $\tilde{i}: \mathbb{R}H^m \rightarrow \mathbb{R}H^n$ of i that factors through an isometry $\tilde{i}: \mathbb{R}H^m \xrightarrow{\sim} U$. In this setting, the orbifold $\mathbb{R}H^m/\Lambda$ is called a *totally geodesic suborbifold* of $\mathbb{R}H^n/\Gamma$.

Definition 7.2. Consider the map $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)$, recall that $X(\Lambda_2)$ is connected (Lemma 6.1) and let $X(\Lambda_n)^+$ be the connected component containing $\iota(X(\Lambda_2))$. Let $\Gamma_n^+ \subset \text{PO}(n, 1)$ be the lattice underlying the complete hyperbolic orbifold $X(\Lambda_n)^+$.

The first goal of Section 7 is to prove the following theorem.

Theorem 7.3. *For each integer $n \geq 2$, there exists a canonical proper totally geodesic immersion of complete connected hyperbolic orbifolds $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)^+$.*

To prove this, we need several results.

Lemma 7.4. *The map $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)$ is proper with finite fibers.*

Proof. Observe that for each $\alpha \in P\mathcal{A}_n$, the canonical map $f_\alpha: P\Gamma(n)_\alpha \backslash \mathbb{R}H^n \rightarrow X(\Lambda_n)$ is a closed immersion. Since $C\mathcal{A}_n = P\mathcal{A}_n/P\Gamma(n)$ is finite, $\bigcup_{\alpha \in C\mathcal{A}_n} f_\alpha(P\Gamma(n)_\alpha \backslash \mathbb{R}H^n) = X(\Lambda_n)$ forms a finite closed covering of $X(\Lambda_n)$. Moreover, the restriction of ι to the closed subset $f_{\alpha_i}(P\Gamma(2)_{\alpha_i} \backslash \mathbb{R}H^2) \subset X(\Lambda_2)$ is induced by the canonical map

$$P\Gamma(2)_{\alpha_i} \backslash \mathbb{R}H_{\alpha_i}^2 \rightarrow P\Gamma(n)_{\beta_i} \backslash \mathbb{R}H_{\beta_i}^n. \quad (27)$$

Here, α_i and β_i are as in (16) and (17). To prove that ι is proper with finite fibers, it therefore suffices to prove that (27) is proper with finite fibers for each $i \in \{0, 1, 2\}$. Define quadratic forms

$$\begin{aligned} Q_0^n(x_0, \dots, x_n) &= -x_0^2 + x_1^2 + \dots + x_n^2, \\ Q_1^n(x_0, \dots, x_n) &= -x_0^2 + 3x_1^2 + \dots + x_n^2, \\ Q_2^n(x_0, \dots, x_n) &= -x_0^2 + 3x_1^2 + 3x_2^2 + x_3^2 + \dots + x_n^2. \end{aligned} \quad (28)$$

For each $i = 0, 1, 2$, the canonical homomorphism $\text{PO}(Q_i^n, \mathbb{Z}) \rightarrow P\Gamma(n)_{\beta_i}$ is an isomorphism by [ACT10, Theorem 5.1], thus (27) is proper with finite fibers. \square

Lemma 7.5. *For any base point of $X(\Lambda_2)$, the induced map $\Gamma_2^+ \rightarrow \Gamma_n^+$ is injective.*

Proof. The natural map of orbifolds $X(\Lambda_n)^+ \rightarrow P\Gamma(n) \backslash \mathbb{C}H^n$ induces a homomorphism $\Gamma_n^+ \rightarrow P\Gamma(n) = \text{PU}(n, 1)(\mathbb{Z}[\zeta_3])$. In case $n = 2$, this map is injective and factors as $\Gamma_2^+ \hookrightarrow \text{PO}(2, 1)(\mathbb{Z}[\sqrt{3}]) \subset \text{PU}(2, 1)(\mathbb{Z}[\zeta_3])$, see [ACT06, p. 167-168]. Since $\text{SO}(2, 1) = \text{PO}(2, 1)$, this map $\Gamma_2^+ \rightarrow \text{PU}(2, 1)(\mathbb{Z}[\zeta_3])$ factors through an embedding $\Gamma_2^+ \hookrightarrow \text{U}(2, 1)(\mathbb{Z}[\zeta_3])$. The commutative diagram of orbifolds

$$\begin{array}{ccc} \Gamma_2^+ \backslash \mathbb{R}H^2 & \longrightarrow & \Gamma_n^+ \backslash \mathbb{R}H^n \\ \downarrow & & \downarrow \\ \text{U}(2, 1)(\mathbb{Z}[\zeta_3]) \backslash \mathbb{C}H^2 & \longrightarrow & \text{PU}(n, 1)(\mathbb{Z}[\zeta_3]) \backslash \mathbb{C}H^n \end{array}$$

gives rise to a commutative diagram of orbifold fundamental groups

$$\begin{array}{ccc} \Gamma_2^+ & \longrightarrow & \Gamma_n^+ \\ \downarrow & & \downarrow \\ \text{U}(2, 1)(\mathbb{Z}[\zeta_3]) & \longrightarrow & \text{PU}(n, 1)(\mathbb{Z}[\zeta_3]). \end{array}$$

The composition $\Gamma_2^+ \rightarrow \text{U}(2, 1)(\mathbb{Z}[\zeta_3]) \rightarrow \text{PU}(n, 1)(\mathbb{Z}[\zeta_3])$ is injective, hence the map $\Gamma_2^+ \rightarrow \Gamma_n^+$ is injective as desired. \square

Lemma 7.6. *Let X/Λ and Y/L be quotient orbifolds, where X and Y are simply connected manifolds, and Λ and L discrete groups acting smoothly, properly discontinuously on X and Y . Let $f: X/\Lambda \rightarrow Y/L$ be a proper map with finite fibers and suppose that $f_*: \Lambda \rightarrow L$ is injective. Then any lift $\tilde{f}: X \rightarrow Y$ of f is proper with finite fibers.*

Proof. The map \tilde{f} is locally given by maps $U/G \rightarrow V/H$ where U (resp. V) is connected open in X (resp. Y) and G and H are finite groups. Thus \tilde{f} is closed. To see that \tilde{f} has finite fibers, observe that $\Lambda \cdot x \rightarrow L \cdot \tilde{f}(x)$ is injective for each $x \in X$. \square

Proposition 7.7. *The morphism of hyperbolic orbifolds*

$$\iota: X(\Lambda_2) = \Gamma_2^+ \backslash \mathbb{R}H^2 \rightarrow \Gamma_n^+ \backslash \mathbb{R}H^n = X(\Lambda_n)^+$$

is proper with finite fibers, and the same holds for any lift $\tilde{\iota}: \mathbb{R}H^2 \rightarrow \mathbb{R}H^n$ of ι .

Proof. This follows from Lemma's 7.4, 7.5 and 7.6. \square

Lemma 7.8. *Let $N = \mathbb{R}H^m/\Lambda$ and $M = \mathbb{R}H^n/\Gamma$ be real hyperbolic orbifolds. Let $i: N \rightarrow M$ be a proper map of hyperbolic orbifolds, induced by a homomorphism $\phi: \Lambda \rightarrow \Gamma$ and an equivariant proper map $\tilde{i}: \mathbb{R}H^m \rightarrow \mathbb{R}H^n$. Suppose that, locally around each point of N , the morphism i is given by a diagram of the form*

$$\begin{array}{ccc} U & \xrightarrow{\tilde{i}} & V \\ \downarrow & & \downarrow \\ U/G & \xrightarrow{i} & V/H, \end{array} \quad G \xrightarrow{\phi} H, \quad (U \subset \mathbb{R}H^m, V \subset \mathbb{R}H^n \text{ connected open}) \quad (29)$$

where the map $\tilde{i}: U \rightarrow V$ is a proper totally geodesic immersion. Then $i: N \rightarrow M$ is a totally geodesic immersion of hyperbolic orbifolds.

Proof. Since \tilde{i} is proper and locally a totally geodesic immersion, it is a totally geodesic embedding. Thus $i: N \rightarrow M$ is a totally geodesic immersion, see Definition 7.1. \square

Proof of Theorem 7.3. Consider the map of metric spaces $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)^+$ defined in Lemma 6.3, where $X(\Lambda_n)^+$ is as in Definition 7.2. By Proposition 6.4, the map ι is a morphism of hyperbolic orbifolds. Moreover, locally ι lifts to a proper totally geodesic embedding $\rho: W_f \hookrightarrow W_g$ as in (26) for some connected open subsets $W_f \subset \mathbb{B}^2(\mathbb{R})$ and $W_g \subset \mathbb{B}^n(\mathbb{R})$, equivariant for the inclusion of stabilizer groups $A_f/B_f \hookrightarrow A_g/B_g$. By Proposition 7.7, the map $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)^+$ is proper with finite fibers, and the same holds for any lift $\tilde{\iota}: \mathbb{R}H^2 \rightarrow \mathbb{R}H^n$ of ι . Consequently, by Lemma 7.8, the map ι is a totally geodesic immersion of hyperbolic orbifolds. \square

7.1 **Non-arithmetic real hyperbolic lattices.** Consider the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$, see Definition 7.2. We are in position to prove that Γ_n^+ is non-arithmetic. The key is:

Theorem 7.9 (Bergeron–Clozel). *Let $\mathbb{R}H^m/\Lambda \rightarrow \mathbb{R}H^n/\Gamma$ be a totally geodesic immersion of real hyperbolic orbifolds of finite volume. If the lattice $\Gamma \subset \mathrm{PO}(n, 1)$ is arithmetic, then the lattice $\Lambda \subset \mathrm{PO}(m, 1)$ is arithmetic as well.*

Proof. See [BC05, Proposition 15.2.2] (compare also [Bel+21, Theorem 1.4]). \square

Theorem 7.10. *For each $n \in \mathbb{Z}_{\geq 2}$, the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ is non-arithmetic.*

Proof. By Theorem 7.3, there exists a proper totally geodesic immersion of hyperbolic orbifolds $\iota: X(\Lambda_2) \rightarrow X(\Lambda_n)^+$. Therefore, by Theorem 7.9, it suffices to show that $\Gamma_2^+ \subset \mathrm{PO}(2, 1)$ is non-arithmetic. This is shown in [ACT06, Section 5]. \square

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