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# Existence of optimal shapes in parabolic bilinear optimal control problems

Idriss Mazari-Fouquier

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## Abstract

The aim of this paper is to prove the existence of optimal shapes in bilinear parabolic optimal control problems. We consider a parabolic equation that writes  $\partial_t u_m - \Delta u_m = f(t, x, u_m) + m u_m$ . The set of admissible controls is given by  $A = \{m \in L^\infty, m_- \leq m \leq m_+ \text{ a.e., } \int_\Omega m(t, \cdot) = V_1(t)\}$  where  $m_\pm = m_\pm(t, x)$  are two reference functions in  $L^\infty((0, T) \times \Omega)$ , and where  $V_1 = V_1(t)$  is a reference integral constraint. The functional to optimise is  $J : m \mapsto \iint j_1(u_m) + \int_\Omega j_2(u_m(T))$ . Roughly speaking we prove that, if  $j_1$  and  $j_2$  are non-decreasing and if one is increasing any solution of  $\max_A J$  is bang-bang: any optimal  $m^*$  writes  $m^* = \mathbb{1}_E m_- + \mathbb{1}_{E^c} m_+$  for some  $E \subset (0, T) \times \Omega$ . From the point of view of shape optimization, this is a parabolic analog of the Buttazzo-Dal Maso theorem in shape optimisation. The proof is based on second-order criteria and on an approximation-localisation procedure for admissible perturbations. This last part uses the theory of parabolic equations with measure datum.

**Keywords:** Bilinear optimal control, qualitative properties of optimisation problems, parabolic equations with measure datum.

**AMS classification (2020):** 35K55, 49J30, 49K20, 49N99, 49Q10.

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## 1 Introduction

### 1.1 Model and problem under consideration

The goal of this paper is to analyse a constrained parabolic bilinear (also dubbed “multiplicative”) optimal control problems. More precisely, we focus on the pointwise properties of optimal controls. In doing so, we develop a localisation method to tackle certain qualitative properties of PDE constrained optimisation problems.

We fix a smooth bounded domain  $\Omega \subset \mathbb{R}^d$  ( $d \geq 1$ ), as well as an initial datum  $u_0 \in \mathcal{C}^1(\overline{\Omega})$ . We consider a non-linearity  $f = f(t, x, u)$ , the regularity properties of which are to be made precise

later. Finally, we consider a time horizon  $T > 0$ . For any given  $m \in L^\infty((0, T) \times \Omega)$  (acting as a control) we let  $u_m$  be the solution of

$$\begin{cases} \frac{\partial u_m}{\partial t} - \Delta u_m = f(t, x, u_m) + m u_m, \\ u_m(0, \cdot) = u_0, \end{cases}$$

with either Neumann or Robin boundary conditions on  $\partial\Omega$ . The control  $m$  is assumed to satisfy  $L^\infty$  bounds of the form

$$m_-(t, x) \leq m(t, x) \leq m_+(t, x) \text{ a.e. in } (0, T) \times \Omega$$

where  $m_\pm \in L^\infty((0, T) \times \Omega)$ , as well as an  $L^1$ -bound, either of the form

$$\iint_{(0, T) \times \Omega} m = V_0$$

for some  $V_0 \in \left( \iint_{(0, T) \times \Omega} m_-; \iint_{(0, T) \times \Omega} m_+ \right)$ , or

$$\int_{\Omega} m(t, \cdot) = V_0(t) \text{ a.e. in } (0; T)$$

for some function  $V_0 : t \mapsto V_0(t) \in \left( \int_{\Omega} m_-(t, \cdot); \int_{\Omega} m_+(t, \cdot) \right)$ . For two functions  $j_1, j_2$ , the optimal control problem reads

$$\max_{m \text{ satisfying the constraints above}} J(m) := \iint_{(0, T) \times \Omega} j_1(t, x, u_m(t, x)) dt dx + \int_{\Omega} j_2(x, u_m(T, x)) dx.$$

Despite growing interest in these problems and their relative ubiquity (see Section 1.4), even basic properties of this optimal control problem remained elusive. Among them, the first of interest is the bang-bang property, which amounts to saying that optimal controls  $m^*$  saturate the point wise constraints; in other words: if  $m^*$  is optimal is it true that

$$m^*(t, x) \in \{m_\pm(t, x)\} \text{ a.e. in } (0, T) \times \Omega?$$

Our main theorem establishes that in all dimensions, if the function  $J$  is increasing with respect to the control  $m$ , the bang-bang property is satisfied.

Although it is not the only relevant property in these problems, this bang-bang property is particularly important as it opens up the possibility of tackling some other issues of a more qualitative flavour, as well as to put forth efficient numerical methods. In the conclusion, we list some possible related research questions to be investigated.

## 1.2 Setting and statement of the problem

**Parabolic model** We let  $f = f(t, x, u)$  satisfy

$$\left\{ \begin{array}{l} f \text{ is bounded, uniformly in } x \in \bar{\Omega}, \text{ locally uniformly in } (t, u) \in \mathbb{R}_+ \times \mathbb{R}, \\ \text{Uniformly in } x \in \bar{\Omega}, \text{ locally uniformly in } t \in \mathbb{R}_+, f \text{ is } \mathcal{C}^2 \text{ in } u \\ \text{with } \partial_{uu}^2 f \text{ bounded uniformly in } x \in \bar{\Omega}, \text{ locally uniformly in } (t, u), \\ \text{For a.e. } (t, x) \in \mathbb{R}_+ \times \bar{\Omega}, f(t, x, 0) \geq 0, \\ \text{there exist } A, B \in \mathbb{R} \text{ such that, for a.e. } (t, x, u) \in \mathbb{R}_+ \times \bar{\Omega} \times \mathbb{R}_+, f(t, x, u) \leq Au + B. \end{array} \right. \quad (\mathbf{H}_f)$$

We let  $u_0 \in \mathcal{C}^1(\overline{\Omega})$  be a fixed initial condition with

$$u_0 \geq 0, u_0 \neq 0.$$

The boundary condition operator denoted by  $B$  is defined with the help of a constant  $b \in \mathbb{R}_+$  as

$$Bu := \frac{\partial u}{\partial \nu} + bu.$$

The case  $b = 0$  corresponds to Neumann boundary conditions.

For any  $m \in L^\infty((0, T) \times \Omega)$  we define  $u_m$  as the unique solution of

$$\begin{cases} \frac{\partial u_m}{\partial t} - \Delta u_m = f(t, x, u_m) + mu_m & \text{in } (0, T) \times \Omega, \\ Bu_m = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_m(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (1.1)$$

**Cost functional** We let  $j_1 : \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \ni (t, x, u) \mapsto j_1(t, x, u)$  and  $j_2 : \overline{\Omega} \times \mathbb{R} \ni (x, u) \mapsto j_2(x, u)$  satisfy the following regularity assumptions (where to alleviate notations we see  $j_2$  as a function of  $(t, x, u)$  that does not depend on  $t$ )

$$\begin{cases} j_1, j_2 \text{ are bounded, uniformly in } x \in \overline{\Omega}, \text{ locally uniformly in } (t, u) \in \mathbb{R}_+ \times \mathbb{R}, \\ \text{Uniformly in } x \in \overline{\Omega}, \text{ locally uniformly in } t \in \mathbb{R}_+, j_1, j_2 \text{ are } \mathcal{C}^2 \text{ in } u. \end{cases} \quad (1.2)$$

Typical examples include  $j_1(t, x, u) = \mathbb{1}_\omega(t, x)\phi(u)$  for some measurable  $\omega \subset (0, T) \times \Omega$  and a  $\mathcal{C}^2$  function  $\phi$ .

The cost functional under consideration is

$$J : L^\infty((0, T) \times \Omega) \ni m \mapsto \iint_{(0, T) \times \Omega} j_1(t, x, u_m(t, x)) dt dx + \int_{\Omega} j_2(x, u_m(T, x)) dx. \quad (1.3)$$

**Admissible controls** We let  $m_+, m_- \in L^\infty((0, T) \times \Omega)$  satisfy

$$m_- \leq m_+ \text{ a.e. in } (0, T) \times \Omega,$$

$$\int_{\Omega} m_-(t, \cdot) \leq \int_{\Omega} m_+(t, \cdot) \text{ for a.e. } t \in (0, T), \quad \iint_{(0, T) \times \Omega} m_- < \iint_{(0, T) \times \Omega} m_+.$$

To distinguish between global in time  $L^1$  constraints and pointwise in time  $L^1$  constraints, we introduce one constraint

$$V_0 \in \left[ \iint_{(0, T) \times \Omega} m_-; \iint_{(0, T) \times \Omega} m_+ \right]$$

and a function  $V_1 \in L^\infty((0, T))$ ,

$$V_1 : (0, T) \ni t \mapsto V_1(t) \in \left[ \int_{\Omega} m_-(t, \cdot); \int_{\Omega} m_+(t, \cdot) \right].$$

The two associated class of admissible controls are

$$A_{\text{global}} := \left\{ m \in L^\infty((0, T) \times \Omega), m_- \leq m \leq m_+ \text{ a.e.}, \iint_{(0, T) \times \Omega} m = V_0 \right\}, \quad (1.4)$$

and

$$A_{\text{pointwise}} := \left\{ m \in L^\infty((0, T) \times \Omega), m_- \leq m \leq m_+ \text{ a.e.}, \int_{\Omega} m(t, \cdot) = V_1(t) \text{ for a.e. } t \in (0; T) \right\}, \quad (1.5)$$

Of notable importance in these two admissible classes are bang-bang functions:

**Definition 1.** A bang-bang function  $m \in A_{\text{global}}$  or  $m \in A_{\text{pointwise}}$  is a function  $m$  such that  $m(t, x) \in \{m_{\pm}(t, x)\}$  a.e. in  $(0, T) \times \Omega$ . Alternatively, a function  $m$  is a bang-bang function if and only if there exists a measurable  $E \subset (0, T) \times \Omega$  such that

$$m = \mathbb{1}_E m_+ + \mathbb{1}_{E^c} m_-.$$

Bang-bang functions can be geometrically interpreted as the extreme points of the admissible classes under consideration, which are obviously convex sets.

The two optimisation problems we study in this article are

$$\max_{A_{\text{global}}} J \quad (1.6)$$

and

$$\max_{A_{\text{pointwise}}} J. \quad (1.7)$$

The existence of solutions of these optimisation problems is a simple consequence of the direct method in the calculus of variations.

### 1.3 Main result

The main theorem of this paper deals with the pointwise behaviour of solutions of (1.6)-(1.7).

**Theorem I.** Assume that  $j_1, j_2$  are non-decreasing in  $u$ , and that one of the following assumptions is satisfied:

- For a.e.  $(t, x) \in (0; T) \times \Omega$ , for any  $u > 0$ ,  $\frac{\partial j_1}{\partial u} > 0$ , or
- For a.e.  $x \in \Omega$ , for any  $u > 0$ ,  $\frac{\partial j_2}{\partial u} > 0$ , or
- $j_1$  writes  $j_1 : (t, x) \mapsto \mathbb{1}_{\omega_1}(t, x)\phi_1(u)$  with  $\omega_1 \subset (0, T) \times \Omega$ ,  $\text{Vol}(\omega_1) > 0$  and  $\phi_1 \in \mathcal{C}^2(\mathbb{R}_+)$  with  $\phi_1' > 0$  in  $(0; \infty)$ , or
- $j_2$  writes  $j_2 : x \mapsto \mathbb{1}_{\omega_2}(x)\phi_2(u)$  with  $\omega_2 \subset \Omega$ ,  $\text{Vol}(\omega_2) > 0$  and  $\phi_2 \in \mathcal{C}^2(\mathbb{R}_+)$  with  $\phi_2' > 0$  in  $(0; \infty)$ .

Then any solution  $m^*$  of either (1.6) or of (1.7) is a bang-bang function in the sense of Definition 1.

**Remark 2.** We will only prove Theorem I in the case of pointwise constraints (1.7), as the bang-bang property for (1.7) implies the bang-bang property for (1.6), see section 2.1.

**Remark 3.** As we observed in some earlier works [Maz23, MNP21] this is linked to an existence property for a shape optimisation problem. Indeed, assume  $m_- \equiv 0, m_+ \equiv 1$ . Theorem I then show that the optimisation problem

$$\max_{E \subset (0, T) \times \Omega \text{ with some volume constraints}} \mathcal{J}(E) = J(\mathbb{1}_E),$$

where the volume constraints are defined pointwise with  $V_1$  or globally with the constraint  $V_0$  admits a solution. As will be clear in the proof, the core argument is that of the monotonicity of the functional  $J$ , so that our theorem could be rephrased as: if  $\mathcal{J}$  is increasing for the set inclusion (which is implied by the monotonicity of  $J$ ), then there exists an optimal set for the volume constrained shape optimisation problem. This is of course an analog of the seminal result of Buttazzo-Dal Maso [BDM93]. We also refer to [Maz23, Section 1.4]

## 1.4 Motivation & bibliographical references

**Motivation** The underlying motivations behind looking for pointwise properties of optimal controls is linked to both theoretical and numerical questions. At a theoretical level, not much is known regarding the behaviour of such bilinear optimal control problems and this property is merely a recasting of the question: do the optimisers of a constrained optimal control problem saturate the constraints? Furthermore, deriving this property is often a crucial step in fully characterising optimisers. If, for instance, the problem is amenable to rearrangement arguments, the bang-bang property, combined with geometric properties derived through symmetrisation often fully solves the problem at hand; a typical situation where this occurs is that of parabolic problems with convex non-linearities, or energetic bilinear criteria [Ban80]. At the numerical level, the validity of this so-called “bang-bang property” for instance allows the use of classical fixed-point algorithms [HKL11, KLY08, LLNP16], that are linked to thresholding schemes. The fact that, in linear control problems, this property is not valid creates several major difficulties at the level of numerical analysis, parts of which were addressed in [MNTM21, MFN22, NM20].

**Related works** A typical field where the investigation of the validity-or lack thereof-of the bang-bang property is important is spatial ecology. In this setting, the control  $m$  is interpreted as a resources distribution, or as a *per capita* growth rate, and solving (1.6) or (1.7) amounts to determining the optimal resources distribution for a certain criterion. A paradigmatic example is the study of the optimisation of the total population size in logistic models: how should resources be spread so as to maximise the total population size? This is formulated, mathematically, as a bilinear optimal control problem with an elliptic equation as a state equation. Multiple papers studied several qualitative properties of this problem [HK21, LL12, LZ17] or of variants [NLY21]; regarding the bang-bang property, after partial results [MNP20, NY18], it was proved to hold in [MNP21]. In [MNP21], an oscillatory technique was introduced by the author, Nadin and Privat, which was then adapted in [Maz23] to the case of parabolic equations, with controls independent of time (or smooth in time). In the present paper, we tackle general time-dependent controls, and, surprisingly, the type of two-scale methods put in place here is simpler than in [Maz23]; this is due to the flexibility allowed for by the theory of parabolic equations with measure data. In particular, our result applies to the optimisation of the total population size as studied for instance in [BL20], and for which the validity of the bang-bang property remained unproved.

## 2 Proof of Theorem I

### 2.1 A preliminary remark

We note that it suffices to prove the bang-bang property for (1.7), as it gives the conclusion for (1.6). Indeed, consider a solution  $m_{\text{global}}^*$  of (1.6). Define, for a.e.  $t \in (0; T)$ ,  $V_1(t) := \int_{\Omega} m_{\text{global}}^*(t, \cdot)$ . Then it is clear that  $m_{\text{global}}^*$  is also a solution of (1.7) for this constraint  $V_1$ . Thus, in the upcoming sections, we focus on (1.7).

## 2.2 Basic regularity estimates on $u_m$

We start the proof by listing some regularity bounds on  $u_m$  that will be useful in later parts of the proof.

**Lemma 4.** *Assume  $f$  satisfies  $(\mathbf{H}_f)$ . For any  $\mathfrak{M} \in \mathbb{R}_+$ , for any  $p \in [1; +\infty)$ , there exists  $C_{\mathfrak{M},p}$  such that*

$$\forall m \in L^\infty((0, T) \times \Omega) \text{ such that } \|m\|_{L^\infty((0, T) \times \Omega)} \leq \mathfrak{M}$$

there holds

$$\forall t > 0, \min_{[t; T] \times \bar{\Omega}} u_m > 0 \quad (2.1)$$

and

$$\sup_{t \in [0; T]} \|u_m(t, \cdot)\|_{W^{1,p}(\Omega)} \leq C_{\mathfrak{M},p}. \quad (2.2)$$

*Proof of Lemma 4.* We first show that there exists a constant  $C_0 > 0$  depending on  $f, u_0$  and  $\mathfrak{M}$  such that, for any  $m$  satisfying  $\|m\|_{L^\infty((0, T) \times \Omega)} \leq \mathfrak{M}$  there holds

$$0 \leq u_m \leq C_0 \text{ a.e. in } (0, T) \times \Omega. \quad (2.3)$$

To prove (2.3) we first observe that choosing  $\bar{z} \equiv 0$ , the assumptions on  $f$  ( $\mathbf{H}_f$ ) ensure that  $\bar{z}$  is a subsolution of (1.1), whence  $u_m \geq 0$  almost everywhere.

Estimate (2.1) is a consequence of the strong maximum principle: let  $(A, B)$  be as in ( $\mathbf{H}_f$ ) and let  $\bar{m} = \max\{\|m_+\|_{L^\infty((0, T) \times \Omega)}, \|m_-\|_{L^\infty((0, T) \times \Omega)}\}$ . Then, as  $u_m \geq 0$ , the solution  $z$  of

$$\frac{\partial z}{\partial t} - \Delta z = (A + \bar{m})z + B, z(0, \cdot) = u_0$$

is a supersolution of (1.1), whence  $u_m \leq z$ . By standard parabolic estimates,  $z \in L^\infty((0, T) \times \Omega)$ , thereby establishing (2.3).

Second, defining  $F := f(t, x, u_m) + mu_m$ , it follows from (2.3) that there exists a constant  $C_{0, \mathfrak{M}}$  that depends on  $f, \|u_0\|_{L^\infty(\Omega)}, \mathfrak{M}$  such that

$$\|F\|_{L^\infty((0, T) \times \Omega)} \leq C_{0, \mathfrak{M}}.$$

Observe that  $u_m$  solves

$$\begin{cases} \frac{\partial u_m}{\partial t} - \Delta u_m = F \in L^\infty((0, T) \times \Omega) & \text{in } (0, T) \times \Omega, \\ Bu_m = 0 & \text{on } (0; T) \times \partial\Omega, \\ u_m(0, \cdot) \in \mathcal{C}^1(\bar{\Omega}). \end{cases}$$

Standard parabolic regularity theory (see [LSU68, Lie96] or [Lun11, Chapter 5]) then implies (2.2).  $\square$

## 2.3 First-order derivative of the criterion, first order optimality conditions and strict monotonicity

The proof of Theorem I relies on first and second-order necessary optimality conditions; accordingly, we begin with a detailed study of the Gateaux derivatives of the functional  $J$ . As the Gateaux differentiability of  $J$  follows from standard considerations in parabolic regularity, we do not dwell on it.

**Computation of the first-order Gateaux derivative** We let  $m \in L^\infty((0, T) \times \Omega)$  be a control (at this point, we do not need to specify the class of admissible controls we work in; this only comes in later, when discussing optimality conditions, see Lemma 5).

For a fixed  $m \in L^\infty((0, T) \times \Omega)$  and any  $h \in L^\infty((0, T) \times \Omega)$ , we let  $\dot{u}_m$  be the first order Gateaux derivative of the map  $L^\infty \ni m \mapsto u_m \in L^2(0, T; W^{1,2}(\Omega))$  at  $m$  in the direction  $h$ . Similarly, we let  $\dot{J}(m)[h]$  be the first order Gateaux derivative of the functional  $L^\infty((0, T) \times \Omega) \ni m \mapsto J(m)$  in the direction  $h$ . For notational convenience, we introduce

$$V_m := m + \frac{\partial f}{\partial u}(t, x, u_m). \quad (2.4)$$

By Lemma 4 and  $(\mathbf{H}_f)$  we have

$$V_m \in L^\infty((0, T) \times \Omega). \quad (2.5)$$

By standard computations,  $\dot{u}_m$  solves the equation

$$\begin{cases} \frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m = h u_m & \text{in } (0, T) \times \Omega, \\ B \dot{u}_m = 0 & \text{on } (0, T) \times \partial \Omega, \\ \dot{u}_m(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (2.6)$$

The first order Gateaux derivative of the functional  $J$  at  $m$  in the direction  $h$  writes

$$\dot{J}(m)[h] = \iint_{(0, T) \times \Omega} \dot{u}_m(t, x) \frac{\partial j_1}{\partial u}(t, x, u_m(t, x)) dx dt + \int_{\Omega} \dot{u}_m(T, x) \frac{\partial j_2}{\partial u}(x, u_m(T, x)) dx. \quad (2.7)$$

We introduce the adjoint-state  $p_m$  as the unique solution of the backwards parabolic equation

$$\begin{cases} -\frac{\partial p_m}{\partial t} - \Delta p_m - V_m p_m = \frac{\partial j_1}{\partial u}(t, x, u_m) & \text{in } (0, T) \times \Omega, \\ B p_m = 0 & \text{on } (0, T) \times \partial \Omega, \\ p_m(T, \cdot) = \frac{\partial j_2}{\partial u}(x, u_m) & \text{in } \Omega. \end{cases} \quad (2.8)$$

Using  $p_m$  as a test function in the weak formulation of (2.6) gives

$$\dot{J}(m)[h] = \iint_{(0, T) \times \Omega} h (u_m p_m) = \iint_{(0, T) \times \Omega} h \varphi_m, \quad (2.9)$$

where the function  $\varphi_m$  is defined as

$$\varphi_m = u_m p_m. \quad (2.10)$$

The function  $\varphi_m$  is dubbed the switch function of the functional  $J$ .

**First order optimality conditions** The expression (2.9) leads to the following first order optimality conditions for (1.7).

**Lemma 5.** *Let  $m^*$  be a solution of (1.7). There exists a measurable function  $c_{\text{pointwise}} : (0, T) \rightarrow \mathbb{R}$  such that*

1.  $\{\varphi_{m^*} > c_{\text{pointwise}}\} \subset \{m^* = m_+\}$ ,
2.  $\{\varphi_{m^*} < c_{\text{pointwise}}\} \subset \{m^* = m_-\}$ ,
3.  $\{m_- < m^* < m_+\} \subset \{\varphi_{m^*} = c_{\text{pointwise}}\}$ .

This lemma is a straightforward adaptation of [HP06, Theorem 7.2.22].



**Strict monotonicity of the functional** We now comment on the (strict) monotonicity of the functional  $J$ , by which we mean: if  $m_1 \leq m_2$ ,  $m_1 \neq m_2$ , then

$$J(m_1) < J(m_2).$$

By the mean-value theorem, for any such couple  $(m_1, m_2)$  there exists  $s \in [0; 1]$  such that

$$J(m_2) = J(m_1) + \iint_{(0,T) \times \Omega} (m_2 - m_1) \varphi_{m_1 + s(m_2 - m_1)}.$$

The strict monotonicity is thus implied by the following lemma:

**Lemma 6.** *Under the assumptions of Theorem I on  $(j_1, j_2)$ , for any  $m \in L^\infty((0, T) \times \Omega)$ , for any  $0 < t_0 < t_1 < T$ ,*

$$\min_{[t_0; t_1] \times \bar{\Omega}} \varphi_m > 0.$$

*Proof of Lemma 6.* Given Lemma 4, it suffices to prove that for any  $0 < t_0 < t_1 < T$  there holds

$$\min_{[t_0; t_1] \times \bar{\Omega}} p_m > 0. \tag{2.11}$$

By assumption,  $\frac{\partial j_1}{\partial u}, \frac{\partial j_2}{\partial u} \geq 0$ , whence  $p_m \geq 0$  by the maximum principle. If  $\frac{\partial j_2}{\partial u}(\cdot, u_m(T, \cdot)) > 0$  on a subset of positive measure of  $\Omega$ , the strong maximum principle implies (2.11). If this is not the case, it follows that for a.e.  $t \in (0; T)$  there holds  $\frac{\partial j_1}{\partial u}(t, \cdot, u_m(t, \cdot)) > 0$  in a subset of positive measure of  $\Omega$ . The strong maximum principle then also implies (2.11).  $\square$

**Consequence of the first-order optimality conditions and strategy of proof** We prove Theorem I arguing by contradiction: if  $m^*$ , a solution of either (1.6) or (1.7) is not a bang-bang function, then the set  $\{m_- < m^* < m_+\}$  has positive measure and, for any perturbation  $h$  (that is admissible in the sense that it satisfies certain conditions related to the constraints of the problem) there holds

$$\dot{J}(m)[h] = 0.$$

If we can exhibit an admissible perturbation  $h$ , supported in  $\{m_- < m^* < m_+\}$  and such that the second-order Gateaux derivative  $\ddot{J}(m^*)[h, h]$  is positive, a contradiction is reached. The next section is devoted to a careful analysis of the second-order Gateaux derivative of  $J$ .

## 2.4 Analysis of the second-order derivative of $J$

We let  $m, h \in L^\infty((0, T) \times \Omega)$  and we consider  $\ddot{u}_m$ , the second-order Gateaux derivative of the map  $L^\infty((0, T) \times \Omega) \ni m \mapsto u_m$ . Retaining the notation  $V_m$  (see (2.4)) and introducing

$$W_m := \frac{\partial^2 f}{\partial u^2}(t, x, u_m), \tag{2.12}$$

the function  $\ddot{u}_m$  solves

$$\begin{cases} \frac{\partial \ddot{u}_m}{\partial t} - \Delta \ddot{u}_m - V_m \ddot{u}_m = 2h \dot{u}_m + W_m \dot{u}_m^2 & \text{in } (0, T) \times \Omega, \\ B \ddot{u}_m = 0 & \text{on } (0; T) \times \partial\Omega, \\ \ddot{u}_m(0, \cdot) \equiv 0 & \text{in } \Omega. \end{cases} \tag{2.13}$$

By Lemma 4, we have

$$W_m \in L^\infty((0, T) \times \Omega). \quad (2.14)$$

The second order Gateaux derivative of  $J$  at  $m$  in the direction  $h$  writes

$$\begin{aligned} \ddot{J}(m)[h, h] &= \int_{\Omega} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \int_{\Omega} \ddot{u}_m(T, \cdot) \frac{\partial j_2}{\partial u} \Big|_{u=u_m(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \Omega} \dot{u}_m^2 \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + \iint_{(0, T) \times \Omega} \ddot{u}_m \frac{\partial j_1}{\partial u} \Big|_{u=u_m(T, \cdot)}. \end{aligned} \quad (2.15)$$

Recall that  $p_m$  solves (2.8). Using  $p_m$  as a test function in (2.13) gives

$$\begin{aligned} 2 \iint_{(0, T) \times \Omega} h \dot{u}_m p_m + \iint_{(0, T) \times \Omega} W_m p_m \dot{u}_m^2 \\ = \int_{\Omega} \ddot{u}_m(T, \cdot) \frac{\partial j_2}{\partial u} \Big|_{u=u_m(T, \cdot)} + \iint_{(0, T) \times \Omega} \ddot{u}_m(T, \cdot) \frac{\partial j_1}{\partial u} \Big|_{u=u_m}. \end{aligned} \quad (2.16)$$

This gives

$$\begin{aligned} \ddot{J}(m)[h, h] &= \int_{\Omega} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \Omega} \dot{u}_m^2 \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + 2 \iint_{(0, T) \times \Omega} h \dot{u}_m p_m + \iint_{(0, T) \times \Omega} W_m p_m \dot{u}_m^2. \end{aligned} \quad (2.17)$$

Rearranging the terms, we get

$$\begin{aligned} \ddot{J}(m)[h, h] &= 2 \iint_{(0, T) \times \Omega} h \dot{u}_m p_m + \int_{\Omega} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \Omega} \dot{u}_m^2 \left( \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + W_m p_m \right). \end{aligned} \quad (2.18)$$

We single out the term

$$\iint_{(0, T) \times \Omega} h \dot{u}_m p_m. \quad (2.19)$$

Using (2.6) we know that

$$h = \frac{\frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m}{u_m}.$$

We define

$$\Psi_m := \frac{p_m}{u_m}.$$

Observe that

$$\frac{\partial \Psi_m}{\partial \nu} = \frac{1}{u_m} \frac{\partial p_m}{\partial \nu} - \frac{p_m}{u_m^2} \cdot \frac{\partial u_m}{\partial \nu} = 0 \text{ on } \partial \Omega. \quad (2.20)$$

We obtain:

$$\iint_{(0, T) \times \Omega} h \dot{u}_m p_m = \iint_{(0, T) \times \Omega} \Psi_m \dot{u}_m \left( \frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - \dot{u}_m V_m \right)$$

$$\begin{aligned}
&= \frac{1}{2} \iint_{(0,T) \times \Omega} \Psi_m \frac{\partial \dot{u}_m^2}{\partial t} - \frac{1}{2} \iint_{\Omega} \Psi_m \Delta(\dot{u}_m^2) + \iint_{(0,T) \times \Omega} \Psi_m |\nabla \dot{u}_m|^2 \\
&\quad - \iint_{(0,T) \times \Omega} \Psi_m V_m \dot{u}_m^2 \\
&= -\frac{1}{2} \iint_{(0,T) \times \Omega} \frac{\partial \Psi_m}{\partial t} \dot{u}_m^2 + \frac{1}{2} \int_{\Omega} \Psi_m(T, \cdot) \dot{u}_m^2 - \frac{1}{2} \iint_{(0,T) \times \Omega} \dot{u}_m^2 \Delta \Psi_m \\
&\quad - \iint_{(0,T) \times \partial \Omega} \Psi_m \dot{u}_m \frac{\partial \dot{u}_m}{\partial \nu} + \iint_{(0,T) \times \Omega} \Psi_m |\nabla \dot{u}_m|^2 - \iint_{(0,T) \times \Omega} \Psi_m V_m \dot{u}_m^2 \\
&= \frac{1}{2} \int_{\Omega} \Psi_m(T, \cdot) \dot{u}_m^2 + \iint_{(0,T) \times \Omega} \Psi_m |\nabla \dot{u}_m|^2 + b \iint_{(0,T) \times \partial \Omega} \Psi_m \dot{u}_m^2 \\
&\quad + \iint_{(0,T) \times \Omega} \left( -\frac{1}{2} \cdot \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \Delta \Psi_m - \Psi_m V_m \right) \dot{u}_m^2.
\end{aligned}$$

Let  $Z_m$  be defined as

$$Z_m := -\frac{1}{2} \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \Delta \Psi_m - \Psi_m V_m + \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + W_m p_m.$$

We obtain the expression

$$\begin{aligned}
\ddot{J}(m)[h, h] &= \int_{\Omega} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \frac{1}{2} \int_{\Omega} \Psi_m(T, \cdot) \dot{u}_m^2 + b \iint_{(0,T) \times \partial \Omega} \Psi_m \dot{u}_m^2 \\
&\quad + \iint_{(0,T) \times \Omega} \Psi_m |\nabla \dot{u}_m|^2 + \iint_{(0,T) \times \Omega} Z_m \dot{u}_m^2. \quad (2.21)
\end{aligned}$$

We now rewrite  $Z_m$  as

$$Z_m = -\frac{1}{2} \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \Delta \Psi_m + Y_m \quad \text{where} \quad Y_m := -\Psi_m V_m + \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + W_m p_m.$$

From Lemma 4 and the assumptions on  $(f, j_1, j_2)$  there holds

$$Y_m \in L^\infty((0, T) \times \Omega). \quad (2.22)$$

We now simplify and estimate the terme involving

$$-\frac{\partial \Psi_m}{\partial t} - \Delta \Psi_m.$$

Direct computations give

$$\frac{\partial \Psi_m}{\partial t} = \frac{1}{u_m} \frac{\partial p_m}{\partial t} - \frac{p_m}{u_m^2} \frac{\partial u_m}{\partial t}, \quad (2.23)$$

and

$$\Delta \Psi_m = -2 \frac{1}{u_m^2} \langle \nabla p_m, \nabla u_m \rangle + \frac{1}{u_m} \Delta p_m - \frac{p_m}{u_m^2} \Delta u_m + 2 \frac{p_m}{u_m^3} |\nabla u_m|^2. \quad (2.24)$$

Using Eqs. (1.1)-(2.8) we derive

$$-\frac{\partial \Psi_m}{\partial t} - \Delta \Psi_m = -\frac{1}{u_m} \left( \frac{\partial p_m}{\partial t} + \Delta p_m \right) + \frac{p_m}{u_m^2} \left( \frac{\partial u_m}{\partial t} + \Delta u_m \right) + \frac{2}{u_m^2} \langle \nabla p_m, \nabla u_m \rangle - 2 \frac{p_m}{u_m} \cdot \frac{|\nabla u_m|^2}{u_m^2}$$

$$\begin{aligned}
&= \frac{p_m}{u_m} V_m - \frac{1}{u_m} \frac{\partial j_1}{\partial u} + \frac{p_m}{u_m^2} (f(t, x, u_m) + m u_m) + 2 \frac{p_m}{u_m^2} \Delta u_m \\
&+ \frac{2}{u_m^2} \langle \nabla p_m, \nabla u_m \rangle - 2 \frac{p_m}{u_m} \cdot \frac{|\nabla u_m|^2}{u_m^2} \\
&= \Psi_m \tilde{V}_m - \frac{1}{u_m} \frac{\partial j_1}{\partial u} + 2 \left( \frac{p_m}{u_m^2} \Delta u_m + \frac{1}{u_m^2} \langle \nabla p_m, \nabla u_m \rangle - \frac{p_m}{u_m} \cdot \frac{|\nabla u_m|^2}{u_m^2} \right)
\end{aligned}$$

with  $\tilde{V}_m := V_m + \frac{f(t, x, u_m)}{u_m} + m$ . By Lemma 4 and the smoothness assumptions on  $(f, j_1)$ , we have, for any  $t_0 > 1$ ,

$$\tilde{V}_m \in L^\infty((t_0; T) \times \Omega). \quad (2.25)$$

Finally, observe that

$$\frac{p_m}{u_m^2} \Delta u_m + \frac{1}{u_m^2} \langle \nabla p_m, \nabla u_m \rangle - \frac{p_m}{u_m} \cdot \frac{|\nabla u_m|^2}{u_m^2} = \nabla \cdot \left( \Psi_m \frac{\nabla u_m}{u_m} \right) + \Psi_m \cdot \frac{|\nabla u_m|^2}{u_m^2}.$$

Thus there holds

$$\begin{aligned}
\int_{(0;T) \times \Omega} \left( -\frac{1}{2} \cdot \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \cdot \Delta \Psi_m \right) \dot{u}_m^2 &= \frac{1}{2} \iint_{(0;T) \times \Omega} \Psi_m \tilde{V}_m \dot{u}_m^2 - \frac{1}{2} \iint_{(0;T) \times \Omega} \frac{1}{u_m} \frac{\partial j_1}{\partial u} \dot{u}_m^2 \\
&+ \iint_{(0;T) \times \Omega} \nabla \cdot \left( \Psi_m \frac{\nabla u_m}{u_m} \right) \dot{u}_m^2 \\
&+ \iint_{(0;T) \times \Omega} \Psi_m \frac{|\nabla u_m|^2}{u_m^2} \dot{u}_m^2,
\end{aligned}$$

from which we deduce

$$\begin{aligned}
\iint_{(0;T) \times \Omega} Z_m \dot{u}_m^2 &= \iint_{(0;T) \times \Omega} \left( Y_m + \frac{1}{2} \cdot \Psi_m \tilde{V}_m + \frac{1}{2u_m} \frac{\partial j_1}{\partial u} + \Psi_m \cdot \frac{|\nabla u_m|^2}{u_m^2} \right) \dot{u}_m^2 \\
&- 2 \iint_{(0;T) \times \Omega} \Psi_m \dot{u}_m \left\langle \frac{\nabla u_m}{u_m}, \nabla \dot{u}_m \right\rangle - b \iint_{(0;T) \times \partial \Omega} \Psi_m \dot{u}_m^2.
\end{aligned}$$

In the end, this provides the following expression for the second-order Gateaux-derivative of the functional:

$$\begin{aligned}
\ddot{J}(m)[h, h] &= \iint_{(0;T) \times \Omega} \Psi_m |\nabla \dot{u}_m|^2 + \int_{\Omega} \dot{u}_m^2(T, \cdot) \left( \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \frac{1}{2} \Psi_m(T, \cdot) \right) \\
&+ \iint_{(0;T) \times \Omega} \left( Y_m + \frac{1}{2} \cdot \Psi_m \tilde{V}_m + \frac{1}{2u_m} \frac{\partial j_1}{\partial u} - \Psi_m \cdot \frac{|\nabla u_m|^2}{u_m^2} \right) \dot{u}_m^2 \\
&- 2 \iint_{(0;T) \times \Omega} \Psi_m \dot{u}_m \left\langle \frac{\nabla u_m}{u_m}, \nabla \dot{u}_m \right\rangle. \quad (2.26)
\end{aligned}$$

We analyse this expression further in the next section.

## 2.5 Estimate on the second order Gateaux derivative of $J$

We give a lower bound on  $\ddot{J}(m)$  in terms of  $\dot{u}_m$ :

**Proposition 7.** *Let  $0 < t_0 < t_1 < T$ . There exist four positive constants  $\alpha_1, \alpha_2, \beta, \gamma > 0$  ( $\alpha_1$  depends on  $(t_0; t_1)$ ) such that for any  $h \in L^\infty((0, T) \times \Omega)$  supported in  $[t_0; T]$  (i.e.  $h = \mathbb{1}_{[t_0; T]}h$ ), we have*

$$\begin{aligned} \ddot{J}(m)[h, h] \geq & \alpha_1 \iint_{(t_0; t_1) \times \Omega} |\nabla \dot{u}_m|^2 - \alpha_2 \iint_{(t_1; T) \times \Omega} |\nabla \dot{u}_m|^2 \\ & - \beta \iint_{(t_0; T) \times \Omega} \dot{u}_m^2 - \gamma \int_{\Omega} \dot{u}_m^2(T, \cdot). \end{aligned} \quad (2.27)$$

*Proof of proposition 7.* Let  $0 < t_0 < t_1 < T$ . Since  $h$  is supported in  $[t_0; T]$ , we have  $\dot{u}_m \equiv 0$  in  $[0; t_0)$ , whence

$$\begin{aligned} \ddot{J}(m)[h, h] = & \iint_{[t_0; T]} \Psi_m |\nabla \dot{u}_m|^2 + \int_{\Omega} \dot{u}_m^2(T, \cdot) \left( \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \frac{1}{2} \Psi_m(T, \cdot) \right) \\ & + \iint_{(t_0; T) \times \Omega} \left( Y_m + \frac{1}{2} \cdot \Psi_m \tilde{V}_m + \frac{1}{2u_m} \frac{\partial j_1}{\partial u} - \Psi_m \cdot \frac{|\nabla u_m|^2}{u_m^2} \right) \dot{u}_m^2 \\ & - 2 \iint_{(t_0; T) \times \Omega} \Psi_m \dot{u}_m \left\langle \frac{\nabla u_m}{u_m}, \nabla \dot{u}_m \right\rangle. \end{aligned}$$

From Lemmata 4-6

$$\alpha := \min_{[t_0; t_1]} \Psi_m > 0.$$

Thus,

$$\iint_{(0, T) \times \Omega} \Psi_m |\nabla \dot{u}_m|^2 \geq \alpha \iint_{(t_0; t_1) \times \Omega} |\nabla \dot{u}_m|^2. \quad (2.28)$$

As  $j_2$  is  $\mathcal{C}^2$  in  $u$  and as  $u_m \in L^\infty((0, T) \times \Omega)$ , there exists  $\gamma > 0$  such that

$$\frac{\partial j_2}{\partial u}(\cdot, u_m(T, \cdot)) + \frac{1}{2} \Psi_m(T, \cdot) \geq -\gamma$$

whence

$$\int_{\Omega} \dot{u}_m^2(T, \cdot) \left( \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \frac{1}{2} \Psi_m(T, \cdot) \right) \geq -\gamma \int_{\Omega} \dot{u}_m^2(T, \cdot). \quad (2.29)$$

For the sake of brevity, define

$$\tilde{Y}_m := Y_m + \frac{1}{2} \cdot \Psi_m \tilde{V}_m + \frac{1}{2u_m} \frac{\partial j_1}{\partial u} - \Psi_m \cdot \frac{|\nabla u_m|^2}{u_m^2}.$$

Now, since  $\min_{(t_0; T)} u_m > 0$ , it follows from Lemma 4 that for any  $p \in [1; +\infty)$

$$A_p := \sup_{t \in [t_0; T]} \left\| \tilde{Y}_m(t, \cdot) \right\|_{L^p(\Omega)} < \infty.$$

Let  $r_1 \in (2; +\infty)$  be such that

$$W^{1,2}(\Omega) \hookrightarrow L^{r_1}(\Omega), \quad (2.30)$$

and let  $r_2 \in (1; +\infty)$  be such that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{2} = 1.$$

By the Hölder inequality, for any  $t \in (t_0; T)$ ,

$$\left| \int_{\Omega} \tilde{Y}_m \dot{u}_m^2 \right| \leq \|\tilde{Y}_m(t, \cdot)\|_{L^{r_2}(\Omega)} \cdot \|\dot{u}_m(t, \cdot)\|_{L^{r_1}(\Omega)} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2(\Omega)} \leq A_{r_2} \|\dot{u}_m(t, \cdot)\|_{L^{r_1}(\Omega)} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2(\Omega)}.$$

Thus, with a new constant  $A'$  that depends on the constant of the embedding (2.30) we have

$$\left| \int_{\Omega} \tilde{Y}_m \dot{u}_m^2 \right| \leq A' \left( \|\dot{u}_m(t, \cdot)\|_{L^2(\Omega)}^2 + \|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\Omega)} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2(\Omega)} \right).$$

Applying the arithmetic-geometric inequality we obtain, for any  $\delta_0 > 0$ ,

$$\left| \int_{\Omega} \tilde{Y}_m \dot{u}_m^2 \right| \leq A' \left( 1 + \frac{1}{\delta_0} \right) \|\dot{u}_m(t, \cdot)\|_{L^2(\Omega)}^2 + A' \delta_0 \|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\Omega)}^2. \quad (2.31)$$

Then, define

$$\tilde{\tilde{Y}}_m := 2\Psi_m \frac{\nabla u_m}{u_m}.$$

The arithmetic-geometric inequality implies that for any  $\delta_1 > 0$  and for any  $t \in [t_0; T)$  there holds

$$\left| \int_{\Omega} \dot{u}_m \langle \tilde{\tilde{Y}}_m, \nabla \dot{u}_m \rangle \right| \leq \delta_1 \int_{\Omega} |\nabla \dot{u}_m|^2 + \frac{1}{\delta_1} \int_{\Omega} |\tilde{\tilde{Y}}_m|^2 \cdot \dot{u}_m^2.$$

Lemma 4 implies that for any  $p \in [1; +\infty)$

$$B_p := \sup_{t \in [t_0; T)} \left\| |\tilde{\tilde{Y}}_m(t, \cdot)|^2 \right\|_{L^p(\Omega)} < \infty.$$

We choose the same  $(r_1, r_2)$  as before, and, by the same proof as that of (2.31) we obtain, for any  $t \in [t_0; T)$  and for some  $B'$  that depends on  $B_{r_2}$  and on the constant of the embedding (2.30), that for any  $\delta_2 > 0$ ,

$$\left| \int_{\Omega} |\tilde{\tilde{Y}}_m|^2 \cdot \dot{u}_m^2 \right| \leq B' \left( 1 + \frac{1}{\delta_2} \right) \|\dot{u}_m(t, \cdot)\|_{L^2(\Omega)}^2 + B' \delta_2 \|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\Omega)}^2.$$

Thus, there exists a constant  $C'$  such that, for any  $\delta_1, \delta_2 > 0$  there holds

$$\left| \iint_{(t_0; T) \times \Omega} \tilde{Y}_m \dot{u}_m^2 + \dot{u}_m \langle \tilde{\tilde{Y}}_m, \nabla \dot{u}_m \rangle \right| \leq C' \left( \left( 1 + \frac{1}{\delta_0} + \frac{1}{\delta_1} + \frac{1}{\delta_1 \delta_2} \right) \iint_{(t_0; T) \times \Omega} \dot{u}_m^2 + \left( \delta_0 + \delta_1 + \frac{\delta_2}{\delta_1} \right) \iint_{(t_0; T) \times \Omega} |\nabla \dot{u}_m|^2 \right). \quad (2.32)$$

For any  $\delta_3 > 0$  we choose  $\delta_0 = \delta_1 = \delta_3$  and  $\delta_2 = \delta_3^2$ . Since  $\delta_3$  will eventually be chosen small, we can further assume  $\delta_3 < 1$  so that there exists a constant  $C''$  such that, for any  $\delta_3 > 0$ ,

$$\left| \iint_{(t_0; T) \times \Omega} \tilde{Y}_m \dot{u}_m^2 + \dot{u}_m \langle \tilde{\tilde{Y}}_m, \nabla \dot{u}_m \rangle \right| \leq C'' \left( \frac{1}{\delta_3^3} \iint_{(t_0; T) \times \Omega} \dot{u}_m^2 + \delta_3 \iint_{(t_0; T) \times \Omega} |\nabla \dot{u}_m|^2 \right). \quad (2.33)$$

Summing (2.28)-(2.29)-(2.31)-(2.33) we obtain the existence of a constant  $D'$  such that for any  $\delta_3 > 0$ ,

$$\check{J}(m)[h, h] \geq \alpha \iint_{(t_0; t_1) \times \Omega} |\nabla \dot{u}_m|^2 - D' \delta_3 \iint_{(t_0; T) \times \Omega} |\nabla \dot{u}_m|^2$$

$$-\gamma \int_{\Omega} \dot{u}_m^2(T, \cdot) - \frac{D'}{\delta_3^3} \iint_{(t_0; T) \times \Omega} \dot{u}_m^2.$$

Taking, for  $\delta \in (0; 1)$ ,  $\delta_3 = \frac{\alpha}{D'} \delta$  and defining  $\beta = \frac{(D')^4}{\alpha^3}$  we finally obtain the estimate

$$\ddot{J}(m)[h, h] \geq \alpha(1 - \delta) \iint_{(t_0; t_1) \times \Omega} |\nabla \dot{u}_m|^2 - \alpha \delta \iint_{(t_1; T)} |\nabla \dot{u}_m|^2 - \frac{\beta}{\delta^3} \iint_{(t_0; T) \times \Omega} \dot{u}_m^2 - \gamma \int_{\Omega} \dot{u}_m^2(T, \cdot),$$

and taking for instance  $\delta = \frac{1}{2}$  yields the desired bound.  $\square$

## 2.6 Reduction to the study of an initial value problem

**Definition of the singular sets** In order to prove Theorem I we let  $m_{\text{pointwise}}^*$  be a solution of (1.7). The singular set is defined as

$$\omega_{\text{pointwise}, 0}^* := \{m_- < m^* < m_+\}.$$

We want to prove that this set has volume zero. Arguing by contradiction, if it does not, then there exists  $\eta > 0$  such that the set

$$\omega^* := \{m_- + \eta < m^* < m_+ - \eta\} \quad (2.34)$$

has positive volume:

$$\text{Vol}(\omega^*) > 0. \quad (2.35)$$

These notations and the constant  $\eta$  are fixed for the rest of the paper, and so is the assumption (2.35).

**Second order optimality conditions for (1.7)** For any  $h \in L^\infty((0, T) \times \Omega)$  supported in  $\omega^*$  and such that, for a.e.  $t \in (0; T)$ ,  $\int_{\Omega} h(t, \cdot) = 0$ ,  $m^* + \tau h \in A_{\text{pointwise}}$  for any  $\tau \in (-1; 1)$  small enough, whence  $\dot{J}(m^*)[h] = 0$  The second-order necessary conditions read:

**Lemma 8.** *For any  $h \in L^\infty((0, T) \times \Omega)$  supported in  $\omega^*$  and such that*

$$\text{for a.e. } t \in [0; T], \int_{\Omega} h(t, \cdot) = 0$$

*we have*

$$\ddot{J}(m^*)[h, h] \leq 0.$$

To conclude the proof of Theorem I, arguing by contradiction and assuming (2.35), it suffices to exhibit  $h \in L^\infty((0, T) \times \Omega)$  such that, for a.e.  $t \in (0; T)$ ,  $\int_{\Omega} h(t, \cdot) = 0$ , and such that  $\ddot{J}(m^*)[h, h] > 0$ . The lower bound provided by Proposition 7 indicates that this perturbation should be constructed in such a way that

$$\iint_{(t_0; T) \times \Omega} |\nabla \dot{u}_m|^2 \gg \iint_{(t_1; T) \times \Omega} |\nabla \dot{u}_m|^2 + \iint_{(0, T) \times \Omega} \dot{u}_m^2 + \int_{\Omega} \dot{u}_m^2(T, \cdot).$$

The difficulty in this endeavour comes from the fact that  $\dot{u}_m$  solves a linear PDE of the form  $\partial_t \dot{u}_m - \Delta \dot{u}_m - V \dot{u}_m = u_m h$ . For reasons explained below, this is not very convenient, and we would much rather work with a function  $h$  that writes  $h = \delta_{t=t_0} \otimes h'$ , for some fixed function  $h'$ , thus treating (2.6) as an equation where  $h$  is featured in the initial condition rather than in the source term. In order to make this statement precise, we need to recall some basic facts about Lebesgue points for functions in  $L^1((0, T) \times \Omega)$ .

### Lebesgue points of $\mathbb{1}_{\omega^*}$

**Definition 9.** We say that  $t \in (0; T)$  is a Lebesgue point of  $\mathbb{1}_{\omega^*}$  if

$$\lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon}^{t+\varepsilon} \|\mathbb{1}_{\omega^*}(s, \cdot) - \mathbb{1}_{\omega^*}(t, \cdot)\|_{L^1(\Omega)} ds = 0. \quad (2.36)$$

By [Leo19, Theorem 8.19], almost every  $t \in (0; T)$  is a Lebesgue point of  $\mathbb{1}_{\omega^*}$ . For any  $t \in (0; T)$ , we define

$$\omega_t^* := \omega^* \cap (\{t\} \times \Omega).$$

We now fix  $t_0 \in (0; T)$  such that:

1.  $\text{Vol}(\omega_{t_0}^*) > 0$ ,
2.  $t_0$  is a Lebesgue point of  $\mathbb{1}_{\omega^*}$ .

**Second order necessary conditions revisited** A central result is the following:

**Proposition 10.** We let  $t_1 \in (t_0; T)$  and we let  $\alpha_1, \alpha_2, \beta, \gamma$  be given by Proposition 7. For any  $h \in L^\infty(\Omega)$  supported in  $\omega_{t_0}^*$ , let  $\bar{v}_h$  be the solution of

$$\begin{cases} \frac{\partial \bar{v}_h}{\partial t} - \Delta \bar{v}_h - V_m \bar{v}_h = 0 & \text{in } (t_0; T), \\ B \bar{v}_h = 0 & \text{on } (t_0; T) \times \Omega, \\ \bar{v}_h(t_0, \cdot) = u_m(t_0, \cdot) h & \text{in } \Omega. \end{cases} \quad (2.37)$$

Then there holds

$$\alpha_1 \iint_{(t_0; t_1) \times \Omega} |\nabla \bar{v}_h|^2 - \alpha_2 \iint_{(t_1; T) \times \Omega} |\nabla \bar{v}_h|^2 - \beta \iint_{(t_0; T) \times \Omega} \bar{v}_h^2 - \gamma \int_{\Omega} \bar{v}_h^2(T, \cdot) \leq 0. \quad (2.38)$$

*Proof of Proposition 10.* To alleviate notations in this proof we write  $\omega_0 := \omega_{t_0}^*$ , and we define, for any  $\varepsilon > 0$  and any  $h \in L^\infty(\Omega)$  supported in  $\omega_0$ , the function

$$h_\varepsilon(t, x) := \frac{1}{2\varepsilon} \mathbb{1}_{[t_0-\varepsilon; t_0+\varepsilon]} \left( h \mathbb{1}_{\omega^*} - \int_{\omega_t^*} h \right)$$

Similarly, to alleviate notations, rather than  $\dot{u}_m$ , we let  $v_\varepsilon$  ( $\varepsilon > 0$ ) be the unique solution of

$$\begin{cases} \frac{\partial v_\varepsilon}{\partial t} - \Delta v_\varepsilon - V_m v_\varepsilon = u_m h_\varepsilon & \text{in } (0; T) \times \Omega, \\ B v_\varepsilon = 0 & \text{on } (0; T) \times \partial\Omega, \\ v_\varepsilon(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (2.39)$$

and  $\bar{v}_h$  be the solution of

$$\begin{cases} \frac{\partial \bar{v}_h}{\partial t} - \Delta \bar{v}_h - V_m \bar{v}_h = 0 & \text{in } (t_0; T) \times \Omega, \\ \bar{v}_h(t_0, \cdot) = u_m(t_0, \cdot) h & \text{in } \Omega, \\ B \bar{v}_h = 0 & \text{on } (0; T) \times \partial\Omega. \end{cases} \quad (2.40)$$

The core proposition is the following approximation result:



**Proposition 11.** *There holds*

$$v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_h$$

*strongly in  $L^2(0, T; L^2(\Omega))$ , and weakly in  $L^2(0, T; W^{1,2}(\Omega))$ . Furthermore,*

$$v_\varepsilon(T, \cdot) \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_h(T, \cdot)$$

*strongly in  $L^2(\Omega)$  and, for any  $\tau > t_0$ ,*

$$v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_h$$

*strongly in  $L^2(\tau, T; W^{1,2}(\Omega))$ .*

Assuming Proposition 11 is valid, it suffices to pass to the limit in the inequality

$$\begin{aligned} \alpha_1 \iint_{(t_0; t_1) \times \Omega} |\nabla v_\varepsilon|^2 - \alpha_2 \iint_{(t_1; T) \times \Omega} |\nabla v_\varepsilon|^2 \\ - \beta \iint_{(0; T) \times \Omega} v_\varepsilon^2 - \gamma \int_\Omega v_\varepsilon^2(T, \cdot) = \ddot{J}(m^*)[h_\varepsilon, h_\varepsilon] \leq 0 \end{aligned} \quad (2.41)$$

to obtain the conclusion of Proposition 10.

*Proof of Proposition 11.* 1. Uniform  $L^\infty$  and  $L^2$  bounds We first prove the following:

$$\sup_{\varepsilon \in (0; 1)} \|v_\varepsilon\|_{L^\infty((0, T) \times \Omega)} < +\infty, \quad \sup_{\varepsilon \in (0; 1)} \sup_{t \in (0; T)} \|v_\varepsilon(t, \cdot)\|_{L^2(\Omega)} < \infty. \quad (2.42)$$

To do so observe that, as  $V_m, u_m \in L^\infty((0, T) \times \Omega)$ , the maximum principle ensures that

$$-w_\varepsilon \leq v_\varepsilon \leq w_\varepsilon$$

where  $w_\varepsilon$  solves

$$\begin{cases} \frac{\partial w_\varepsilon}{\partial t} - \Delta w_\varepsilon - \|V_m\|_{L^\infty((0, T) \times \Omega)} w_\varepsilon = \|u_m\|_{L^\infty((0, T) \times \Omega)} \cdot |h_\varepsilon| & \text{in } (0; T) \times \Omega, \\ w_\varepsilon(0, \cdot) = 0 & \text{in } \Omega, \\ Bw_\varepsilon = 0 & \text{on } (0; T) \times \partial\Omega, \end{cases} \quad (2.43)$$

Introducing

$$z_\varepsilon : (t, x) \mapsto e^{-\|V_m\|_{L^\infty((0, T) \times \Omega)} t} w_\varepsilon(t, x)$$

it is clear (since  $t \in (0; T)$ ) that it suffices to prove that

$$\sup_{\varepsilon \in (0; 1)} \|z_\varepsilon\|_{L^\infty((0, T) \times \Omega)} < \infty. \quad (2.44)$$

Observe that  $z_\varepsilon$  satisfies

$$\begin{cases} \frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon = \|u_m\|_{L^\infty((0, T) \times \Omega)} \cdot |h_\varepsilon| & \text{in } (0; T) \times \Omega, \\ z_\varepsilon(0, \cdot) = 0 & \text{in } \Omega, \\ Bz_\varepsilon = 0 & \text{on } (0; T) \times \partial\Omega, \end{cases} \quad (2.45)$$

Let  $S$  be the heat semi-group of  $\Omega$ . As  $S$  is a contraction on  $L^p(\Omega)$  for any  $p \in [1; +\infty]$  [Dav89, Theorem 1.3.3] we have, for any  $z_0 \in L^\infty(\Omega)$  and any  $t \geq 0$ :

$$\|S(t)z_0\|_{L^\infty(\Omega)} \leq \|z_0\|_{L^\infty(\Omega)}, \quad \|S(t)z_0\|_{L^2(\Omega)} \leq \|z_0\|_{L^2(\Omega)}.$$

By the Duhamel formula we have

$$z_\varepsilon(t, \cdot) = \|u_m\|_{L^\infty((0,T)\times\Omega)} \int_0^t S(t-\xi)|h_\varepsilon|(\xi, \cdot) d\xi.$$

However, for any  $\xi \in (t_0 - \varepsilon; t_0 + \varepsilon)$ , we also have

$$2\varepsilon S(t-\xi)|h_\varepsilon|(\xi, \cdot) \leq |S(t-\xi)(h\mathbf{1}_{\omega_t^*})| + \left| \left( \int_{\omega_t} h \right) S(t-\xi)(1) \right|,$$

whence

$$\|S(t-\xi)|h_\varepsilon|\|_{L^\infty(\Omega)} \leq \frac{\|h\|_{L^\infty(\Omega)}}{\varepsilon},$$

so that we finally obtain

$$\|z_\varepsilon\|_{L^\infty} \leq 2\|u_m\|_{L^\infty((0,T)\times\Omega)} \frac{\|h\|_{L^\infty(\Omega)}}{2\varepsilon} \int_{\min(t_0-\varepsilon; t)}^{\min(t, t_0+\varepsilon)} d\xi = \|u_m\|_{L^\infty((0,T)\times\Omega)} \cdot \|h\|_{L^\infty(\Omega)}.$$

We obtain, similarly,

$$\sup_{\varepsilon \in (0;1)} \sup_{t \in [0;T]} \|z_\varepsilon\|_{L^2(\Omega)} < \infty,$$

2. Uniform  $L^2(0, T; W^{1,2}(\Omega))$  bound and weak convergence in  $L^2(0, T; W^{1,2}(\Omega))$  Using  $v_\varepsilon$  as a test function in (2.39), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} v_\varepsilon^2 + \int_{\Omega} |\nabla v_\varepsilon|^2 + b \int_{\partial\Omega} v_\varepsilon^2 &\leq \\ \|u_m\|_{L^\infty((0,T)\times\Omega)} \cdot \|h_\varepsilon(t, \cdot)\|_{L^2(\Omega)} \cdot \|v_\varepsilon\|_{L^2(\Omega)} + \|V_m\|_{L^\infty((0,T)\times\Omega)} \cdot \|v_\varepsilon(t, \cdot)\|_{L^2(\Omega)}^2. \end{aligned} \quad (2.46)$$

We deduce that

$$e^{\|V_m\|_{L^\infty} t} \frac{d}{dt} \left( e^{-\|V_m\|_{L^\infty} t} \|v_\varepsilon(t, \cdot)\|_{L^2(\Omega)} \right) \leq \frac{\mathbf{1}_{[t_0-\varepsilon; t_0+\varepsilon]}(t)}{2\varepsilon} \|u_m\|_{L^\infty((0,T)\times\Omega)} \cdot \|h\|_{L^\infty(\Omega)}.$$

Integrating (2.46) in time and using (2.42) provides

$$\sup_{\varepsilon \in (0;1)} \|v_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} < \infty.$$

Thus, there exists  $\underline{v} \in L^2(0, T; W^{1,2}(\Omega))$  such that

$$v_\varepsilon \rightharpoonup \underline{v}$$

weakly in  $L^2(0, T; W^{1,2}(\Omega))$ .

3. Weak  $L^2(0, T; W^{1,2}(\Omega))$  convergence to  $\bar{v}_h$  The goal here is to prove that  $\underline{v} = \bar{v}_h$ . The following convergence holds in the sense of measures:

$$u_m h_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \delta_{t=t_0} \otimes (u_m(t_0, \cdot)h);$$

we refer for instance to [MFN22, Measure approximation, Proof of Proposition 13]. By [BG89, Section IV] we deduce that, up to a subsequence,  $\{v_\varepsilon\}_{\varepsilon>0}$  converges weakly, as  $\varepsilon \rightarrow 0$ , to  $\bar{v}_h$  in  $L^q(0, T; W^{1,q}(\Omega))$  for any  $q < \frac{d+2}{d+1}$ . However, as  $v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \underline{v}$  in  $L^2(0, T; W^{1,2}(\Omega))$ , we deduce that  $\underline{v} = \bar{v}_h$ . By uniqueness of the closure point we deduce that

$$v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_h \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)).$$

4. Strong  $L^2$ -convergence The family  $\{v_\varepsilon\}_{\varepsilon \in (0,1)}$  is relatively compact in  $L^1((0;T) \times \Omega)$ , see [BG89, Page 168]. In particular

$$v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_h \text{ a.e.}$$

We write

$$\iint_{(0;T) \times \Omega} v_\varepsilon^2 - \iint_{(0;T) \times \Omega} \bar{v}_h^2 = \iint_{(0;T) \times \Omega} (v_\varepsilon - \bar{v}_h)(v_\varepsilon + \bar{v}_h),$$

whence (2.42) and the dominated convergence theorem ensures that

$$\lim_{\varepsilon \rightarrow 0} \iint_{(0;T) \times \Omega} v_\varepsilon^2 = \iint_{(0;T) \times \Omega} \bar{v}_h^2.$$

Since  $L^2(0, T; L^2(\Omega))$  is a Hilbert space, we deduce that

$$v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_h \text{ strongly in } L^2((0, T) \times \Omega). \quad (2.47)$$

5. Strong convergence at  $t = T$  and strong  $L^2(\tau, T; W^{1,2}(\Omega))$  convergence

From (2.47) we deduce that

$$\int_0^T \left( \int_\Omega |v_\varepsilon(t, \cdot) - \bar{v}_h(t, \cdot)|^2 \right) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

In particular, along a subsequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  converging to 0, we have, for almost every  $t \in (0; T)$ ,

$$\int_\Omega |v_\varepsilon(t, \cdot) - \bar{v}_h(t, \cdot)|^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (2.48)$$

We now fix any  $\tau > t_0$ , and  $\tau' \in (t_0; \tau)$  such that (2.48) holds for  $t = \tau'$ . We let  $\varepsilon_0 \in (0; \tau - t_0)$ . Then, for any  $\varepsilon \leq \varepsilon_0$ , the function  $z_\varepsilon := v_\varepsilon - \bar{v}_h$  satisfies

$$\begin{cases} \frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon - V_m z_\varepsilon = 0 & \text{in } (\tau'; T) \times \Omega, \\ B z_\varepsilon = 0 & \text{on } (\tau'; T) \times \partial\Omega, \\ z_\varepsilon(\tau, \cdot) = v_\varepsilon(\tau', \cdot) - \bar{v}_h(\tau', \cdot) & \text{in } \Omega. \end{cases} \quad (2.49)$$

Since  $V_m \in L^\infty((0, T) \times \Omega)$ , standard parabolic estimate ensure the existence of a constant  $C > 0$  such that for any  $t \geq \tau$  we have

$$\int_\Omega |v_\varepsilon(t, \cdot) - \bar{v}_h(t, \cdot)|^2 = \int_\Omega z_\varepsilon(t, \cdot)^2 \leq C \int_\Omega z_\varepsilon(\tau', \cdot)^2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Applying this at  $t = T$  we finally obtain

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega v_\varepsilon(T, \cdot)^2 = \int_\Omega \bar{v}_h(T, \cdot)^2.$$

Similarly, we also have, for some constant  $C > 0$ ,

$$\iint_{(\tau', T) \times \Omega} |\nabla z_\varepsilon|^2 \leq - \int_\Omega z_\varepsilon^2(T, \cdot) + \int_\Omega z_\varepsilon^2(\tau', \cdot) + C \iint_{(\tau', T) \times \Omega} z_\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} 0,$$

whence the conclusion. □

□

## 2.7 Construction of a suitable perturbation

We now study the function  $\bar{v}_h$  solution of (2.40) and choose a particular perturbation  $h$ .

**Construction of a sequence  $\{h_K\}_{K \in \mathbb{N}}$**  We recall that  $t_0 > 0$  is a fixed time. Let  $\{\psi_k, \lambda_k\}_{k \in \mathbb{N}}$  be the family of eigenpairs of the Laplace operator with boundary conditions  $B$ . By the same arguments as in [MNP21, Proof of Theorem 1], for any  $K \in \mathbb{N}$ , there exists  $h_K \in L^\infty(\Omega)$  supported in  $\omega_{t_0}^*$  such that in the basis  $\{\psi_k\}_{k \in \mathbb{N}}$  we have the decomposition

$$u_m(t_0, \cdot)h_K = \sum_{k=K}^{\infty} a_{k,K} \psi_k, \quad \sum_{k=0}^{\infty} a_{k,K}^2 = 1,$$

and such that the following condition is satisfied:

$$\int_{\Omega} h_K = 0.$$

We fix, for any  $K \in \mathbb{N}$ , such a perturbation  $h_K$  and we use the notation  $\bar{v}_K = \bar{v}_{h_K}$ .

**Asymptotic expansion of  $\bar{v}_K$**  For any  $K \in \mathbb{N}$  we define

$$w_K := \sum_{k=K}^{\infty} a_{k,K} \psi_k e^{-(t-t_0)\lambda_k}. \quad (2.50)$$

We now prove:

**Lemma 12.** *There exists a constant  $C > 0$  such that, for any  $K \in \mathbb{N}$ ,*

$$\|\bar{v}_K - w_K\|_{L^\infty(t_0, T; W^{1,2}(\Omega))} \leq C \left( \sum_{k=K}^{\infty} \frac{a_{k,K}^2}{\lambda_k} \right)^{\frac{1}{2}}.$$

*Proof of Lemma 12.* We let  $z_K := \bar{v}_K - w_K$ . The function  $z_K$  solves

$$\begin{cases} \frac{\partial z_K}{\partial t} - \Delta z_K - V_m z_K = V_m w_K & \text{in } (t_0; T) \times \Omega, \\ Bz_K = 0 & \text{on } (t_0; T) \times \partial\Omega, \\ z_K(t_0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (2.51)$$

By standard  $L^2$  parabolic estimates [Eva10, Chapter 7] it follows that

$$\|z_K\|_{L^\infty(t_0, T; W^{1,2}(\Omega))} \leq C \|w_K\|_{L^2((t_0; T) \times \Omega)}$$

with a constant  $C$  that depends on  $\Omega$  and  $\|V_m\|_{L^\infty((0, T) \times \Omega)}$ . An explicit computation yields

$$\iint_{(t_0; T) \times \Omega} w_K^2 = \sum_{k=K}^{\infty} \frac{a_{k,K}^2}{2\lambda_k} \left(1 - e^{-\lambda_k(T-t_0)}\right). \quad (2.52)$$

We conclude with a brutal bound. □

**Another estimate on  $w_K$**  Observe that

$$\iint_{(t_0;T) \times \Omega} |\nabla w_K|^2 + b \iint_{(t_0;T) \times \partial\Omega} w_K^2 = \sum_{k=K}^{\infty} \frac{a_{k,K}^2}{2} (1 - e^{-(T-t_0)\lambda_k}).$$

However, the trace operator  $\text{Tr}$  is continuous from the Besov space  $B_{2,1}^{\frac{1}{2}}(\Omega)$  to  $L^2(\partial\Omega)$ , and, seeing  $B_{2,1}^{\frac{1}{2}}(\Omega)$  as an interpolation space [Leo19, Section 17.3] between  $L^2(\Omega)$  and  $W^{1,2}(\Omega)$ , there exists a constant  $M$  such that, for any  $\phi \in W^{1,2}(\Omega)$ ,

$$\int_{\partial\Omega} \phi^2 \leq M \|\phi\|_{L^2(\Omega)} \cdot \|\nabla\phi\|_{W^{1,2}(\Omega)} = M \int_{\Omega} \phi^2 + M \|\phi\|_{L^2(\Omega)} \cdot \|\nabla\phi\|_{W^{1,2}(\Omega)}.$$

Thus, for any  $\delta' > 0$ ,

$$b \iint_{(t_0;T) \times \partial\Omega} w_K^2 \leq M \left(1 + \frac{1}{\delta'}\right) \iint_{(t_0;T) \times \Omega} w_K^2 + M\delta' \iint_{(t_0;T) \times \Omega} |\nabla w_K|^2.$$

Choosing  $\delta' > 0$  small enough we deduce that there exist two constant  $\underline{C}_0, \overline{C}_0 > 0$  such that

$$\begin{aligned} \underline{C}_0 \sum_{k=K}^{\infty} \frac{a_{k,K}^2}{\lambda_k} &\leq \iint_{(t_0;T) \times \Omega} w_K^2 \leq \overline{C}_0 \sum_{k=K}^{\infty} \frac{a_{k,K}^2}{\lambda_k} \\ \text{and } \underline{C}_0 \sum_{k=K}^{\infty} a_{k,K}^2 &\leq \iint_{(t_0;T) \times \Omega} |\nabla w_K|^2 \leq \overline{C}_0 \sum_{k=K}^{\infty} a_{k,K}^2. \end{aligned} \quad (2.53)$$

Furthermore, the same type of reasoning leads to

$$\text{for a.e. } t_1 \in (t_0;T), \iint_{(t_1;T) \times \Omega} |\nabla w_K|^2 \leq M' e^{-\lambda_K(t_1-t_0)} \sum_{k=K}^{\infty} a_{k,K}^2 = \underset{K \rightarrow \infty}{o} \left( \sum_{k=K}^{\infty} a_{k,K}^2 \right). \quad (2.54)$$

**Consequence of the asymptotic expansion** Proposition 10 and estimates (2.53)-(2.54) imply the existence of  $\underline{C} > 0$  such that

$$\iint_{(t_0;T) \times \Omega} |\nabla \bar{v}_K|^2 \geq \underline{C} \sum_{k=K}^{\infty} a_{k,K}^2, \quad (2.55)$$

$$\iint_{(t_1;T) \times \Omega} |\nabla \bar{v}_K|^2 = \underset{K \rightarrow \infty}{o} \left( \sum_{k=K}^{\infty} a_{k,K}^2 \right), \quad (2.56)$$

$$\iint_{(t_0;T) \times \Omega} \bar{v}_K^2 = \underset{K \rightarrow \infty}{o} \left( \sum_{k=K}^{\infty} a_{k,K}^2 \right), \quad (2.57)$$

$$\int_{\Omega} \bar{v}_K^2(T, \cdot) = \underset{K \rightarrow \infty}{o} \left( \sum_{k=K}^{\infty} a_{k,K}^2 \right). \quad (2.58)$$

## 2.8 Conclusion of the proof

We are now in a position to prove Theorem I.

*Proof of Theorem I.* Let  $m^*$  be a solution of (1.7). Arguing by contradiction, the set  $\omega^*$  defined in (2.34) satisfies (2.35) for  $\eta > 0$  small enough. Fix such an  $\eta$ . Let  $t_0 \in (0; T)$  be an  $L^1$ -Lebesgue point of  $\mathbb{1}_{\omega^*}$ . For any  $K \in \mathbb{N}$ , we choose  $h_K \in L^\infty(\Omega)$  supported in  $\omega_{t_0}^*$  such that

$$u_m(t_0, \cdot)h_K = \sum_{k=K}^{\infty} a_{k,K}\psi_k, \int_{\Omega} h_K = 0, \sum_{k=K}^{\infty} a_{k,K}^2 = 1.$$

Finally, let  $\alpha, \beta, \gamma$  be given by Proposition 7. By Proposition 10, for any  $\delta > 0$ ,

$$I_K := \alpha_1 \iint_{(t_0; t_1) \times \Omega} |\nabla \bar{v}_K|^2 - \alpha_2 \iint_{(t_1; T) \times \Omega} |\nabla \bar{v}_K|^2 - \beta \iint_{(t_0; T) \times \Omega} \bar{v}_K^2 - \gamma \int_{\Omega} \bar{v}_K^2(T, \cdot) \leq 0 \quad (2.59)$$

where  $\bar{v}_K$  solves (2.40) with  $h = h_K$ . By (2.55)-(2.56)-(2.56)-(2.58) there holds

$$I_K \geq \left( \sum_{k=K}^{\infty} a_{k,K}^2 \right) \left( \alpha_1 \underline{C} + \underset{K \rightarrow \infty}{o}(1) \right).$$

Taking  $K$  large enough we deduce that  $I_K > 0$ , in contradiction with (2.59). Theorem I is proved.  $\square$

## 3 Conclusion and open questions

As we mentioned in the introduction, the bang-bang property is one of the basic qualitative features of optimal control problems, but most other qualitative questions remain fully open. Two of these questions that seem particularly relevant to us:

- The geometry of maximisers Apart from the case of convex equations (in other words, for linear or convex non-linearities  $f$ ), where rearrangement techniques allow, in simple geometries, to fully characterise optimisers [ATL90, ALT86, Ban80, MR86], no results allow to derive geometric properties about optimisers.
- The regularity of optimisers This would actually be the next natural step: given that no rearrangement techniques are in general available, is it possible to obtain *a priori* regularity bounds on optimal controls? This question should be reformulated as follows: as we know that any optimal control writes  $m^* = \mathbb{1}_E m_- + \mathbb{1}_{E^c} m_+$ , is it possible to show that  $E^*$  has finite perimeter? Does it have an analytic boundary? Thus far, this property remains elusive, and is only settled for elliptic spectral optimal control problems [CKT08].

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