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A CHARACTERISATION OF HIGHER TORSION CLASSES

JENNY AUGUST, JOHANNE HAUGLAND, KARIN M. JACOBSEN, SONDRE KVAMME, YANN PALU, AND HIPOLITO TREFFINGER

Abstract. Let $\mathcal{A}$ be an abelian length category containing a d-cluster tilting subcategory $\mathcal{M}$. We prove that a subcategory of $\mathcal{M}$ is a d-torsion class if and only if it is closed under d-extensions and d-quotients. This generalises an important result for classical torsion classes. As an application, we prove that the d-torsion classes in $\mathcal{M}$ form a complete lattice. Moreover, we use the characterisation to classify the d-torsion classes associated to higher Auslander algebras of type $\mathbb{A}$, and give an algorithm to compute them explicitly. The classification is furthermore extended to the setup of higher Nakayama algebras.

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1. Introduction

The notion of torsion pairs was introduced for arbitrary abelian categories in [D] to generalise the properties of the class of torsion groups in the category of abelian groups. Since then, torsion theories, and the related notion of t-structures [BBD] for triangulated categories, have become ubiquitous in representation theory, homological algebra, and algebraic geometry. Within these areas, torsion theories and t-structures play a key role in topics such as (perverse) sheaves [BBD], tilting theory and its generalisations [BB, AIR, AHMV, AHLSV], and stability conditions [B].

Meanwhile, higher homological algebra has become an increasingly active field of research since its introduction in [I1, I2, I3]. It has found applications in algebraic K-theory [DJW], wrapped Floer theory in symplectic geometry [DJL], and in algebraic geometry where it was an important ingredient in the proof of the Donovan-Wemyss conjecture [JKM]. Originally motivated by Auslander–Reiten theory, cluster tilting theory and the classical Auslander correspondence, one studies categories where the role of short exact sequences (or distinguished triangles) is taken by longer sequences. Examples include d-abelian and d-exact categories [J2], $(d+2)$-angulated categories [GKO], and d-exangulated categories [HLN]. Here, the positive integer d controls the length of the important sequences, with $d = 1$ coinciding with the classical cases.

By work of [J2, Kv, EN-I], studying d-abelian categories is equivalent to studying so called d-cluster tilting subcategories of abelian categories. Our setup will be the latter, where we

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assume the ambient abelian category to be of finite length (see Section 2.1 for details). Many important classical concepts in representation theory generalise to this setting. In this paper, we focus on the higher analogue of torsion classes, namely d-torsion classes, introduced in [Jø] and further studied in [AJST].

A fundamental result in the study of torsion classes states that a subcategory of an abelian length category is a torsion class if and only if it is closed under extensions and quotients [D]. This result is of crucial importance, as it both allows for the detection of torsion classes, and moreover gives properties which play a key role in many proofs related to these objects. A higher-dimensional version of this classical characterisation would hence be a substantial advancement. The main result of this paper gives such a characterisation of d-torsion classes in terms of closure under d-extensions and d-quotients (see Definitions 3.6 and 3.7).

Theorem 1.1 (Theorem 3.17)**.** *Let $\mathcal{M}$ be a d-cluster tilting subcategory of an abelian length category $\mathcal{A}$. A subcategory $\mathcal{U} \subseteq \mathcal{M}$ is a d-torsion class if and only if it is closed under d-extensions and d-quotients.*

Note that while a different characterisation of d-torsion classes using the bounded derived category of $\mathcal{A}$ has been given in the special case where $\mathcal{A}$ is the module category of a d-representation finite d-hereditary algebra [Jø], the characterisation in Theorem 1.1 works generally, does not require use of derived categories, and is closer to the classical result for torsion classes.

The proof of Theorem 1.1 makes significant use of the main result in [AJST], which relates d-torsion classes in $\mathcal{M}$ to torsion classes in $\mathcal{A}$. This allows us to apply results about torsion classes also in the higher setup. As another key ingredient in the proof of Theorem 1.1, we give a higher generalisation of the classical factorisation of a morphism in an abelian category as the composition of an epimorphism followed by a monomorphism, see Proposition 3.13. This result is of independent interest, and we expect it to play a role in providing answers to other questions in higher homological algebra.

With Theorem 1.1 in hand, we are able to generalise other well-known results about torsion classes to the higher setting. For example, an immediate consequence of combining Theorem 1.1 with a result in [Kl], is that every d-torsion class carries the structure of a d-exact category, see Corollary 3.19.

Other important results concerning classical torsion classes include the study of their poset structure. The set $\operatorname{tors}(\mathcal{A})$ of torsion classes in $\mathcal{A}$ has a natural partial order given by inclusion, and this poset is actually a complete lattice, with meet given by intersection, see e.g. [IRTT, Proposition 2.3]. Theorem 1.1 allows us to give the following generalisation of this result.

Theorem 1.2 (Theorem 4.3)**.** *Let $\mathcal{M}$ be a d-cluster tilting subcategory of an abelian length category $\mathcal{A}$. Then the set $d\text{-}\operatorname{tors}(\mathcal{M})$ of all d-torsion classes in $\mathcal{M}$ is a complete lattice with meet given by intersection.*

In the classical setting, lattice-theoretic properties of $\operatorname{tors}(\mathcal{A})$ form an area of active research [AP, GM, J1], which is intimately related to representation theory [BCZ, DIRRT]. We refer to [T] for an introductory survey on the topic. Theorem 1.2 opens up a new avenue of research in higher homological algebra through the study of the lattice of higher torsion classes.

The classical characterisation of torsion classes as those subcategories that are closed under extensions and quotients, is often used to determine if a given subcategory is a torsion class, or to compute the smallest torsion class containing that given subcategory. Crucial to the success of this approach, is that we often have a good understanding of the middle terms in extensions. To use Theorem 1.1 in an analogous way to determine d-torsion classes, we therefore need an understanding of middle terms in d-extensions, which is notably more complicated. The following result simplifies this problem significantly.

Theorem 1.3 (Theorem 3.20)**.** *Let $\mathcal{M}$ be a d-cluster tilting subcategory of an abelian length category $\mathcal{A}$. Suppose $\mathcal{U} \subseteq \mathcal{M}$ is closed under d-extensions with indecomposable end terms and all d-quotients. Then $\mathcal{U}$ is closed under all d-extensions.*

This theorem is of independent interest, as extension closure is a useful concept across many areas of representation theory. In this paper, we focus on using Theorem 1.3 to classify d-torsion classes in concrete examples.

We apply our results to higher Auslander algebras of type $\mathbb{A}$ [I3] and higher Nakayama algebras of type $\mathbb{A}$ and $\mathbb{A}_\infty^\infty$ [JKPK]. The module category of each such algebra contains a d-cluster tilting subcategory, and their combinatorial descriptions due to [OT] and [JKPK] make them an ideal testing ground for new results in higher homological algebra. Higher Auslander algebras are particularly important, as their derived categories are equivalent to certain partially wrapped Fukaya categories [DJL]. We use Theorems 1.1 and 1.3 to give a combinatorial description of all d-torsion classes associated to these algebras, where the classification results for the three families are given in Theorems 5.13, 6.1 and 6.4, respectively. Moreover, in the case of higher Auslander algebras, we implement our results in algorithms that compute and count all d-torsion classes, see Table 2.

We expect that the results we present in this article will provide tools for any further study of d-torsion classes and will be of importance in building bridges between d-torsion classes and other subjects in representation theory and beyond. This is demonstrated in a forthcoming paper [AHJ+], where Theorem 1.1 is applied to establish a connection between functorially finite d-torsion classes, τ_d-tilting theory and $(d+1)$-term silting objects.

Structure of the paper. In Section 2 we give an overview of the definitions and background used in the rest of the paper. In Section 3 we prove Theorems 1.1 and 1.3, while Theorem 1.2 is shown in Section 4. Section 5 is dedicated to the study of d-torsion classes associated to higher Auslander algebras. Finally, we extend our view to higher Nakayama algebras in Section 6.

Conventions and notation. Throughout this paper, let d denote a positive integer and $\mathcal{A}$ an essentially small abelian category. We always assume $\mathcal{A}$ to be a finite length category, which implies that the Krull–Remak–Schmidt property is satisfied, see [Kr, Lemma 5.1 and Theorem 5.5].

We let $\Bbbk$ be a field. Given a finite dimensional $\Bbbk$-algebra A, the notation $\operatorname{mod} A$ is used for the category of finitely presented right A-modules.

Arrows in a quiver are composed from left to right, meaning that we write ab for the path starting in the source of a and ending in the target of b.

All subcategories are assumed to be full and closed under isomorphisms and finite direct sums. For a collection of objects $\mathcal{X}$ in an additive category $\mathcal{C}$, we denote by $\operatorname{add}(\mathcal{X})$ the smallest subcategory of $\mathcal{C}$ which contains $\mathcal{X}$ and is closed under finite direct sums and direct summands.

2. Background and preliminaries

In this section we provide an overview of definitions and results which give the foundation for the rest of the paper. Before we start discussing notions from higher homological algebra, we recall some terminology related to subcategories and approximations.

Let $\mathcal{X}$ be a subcategory of the abelian category $\mathcal{A}$. We say that $\mathcal{X}$ is *generating* if any object in $\mathcal{A}$ is a quotient of an object in $\mathcal{X}$; that is, for every $Y \in \mathcal{A}$, there exists an exact sequence $X \to Y \to 0$ with $X \in \mathcal{X}$. Dually, we can define the notion of *cogenerating*, and we call a subcategory *generating-cogenerating* if it is both generating and cogenerating.

Given an object $Y \in \mathcal{A}$, a morphism $f\colon Y \to X$ with $X \in \mathcal{X}$ is a *left $\mathcal{X}$-approximation* of Y if any morphism $Y \to X'$ with $X' \in \mathcal{X}$ factors through f. The subcategory $\mathcal{X}$ is called *covariantly finite* if every object in $\mathcal{A}$ admits a left $\mathcal{X}$-approximation. The notions of *right $\mathcal{X}$-approximations* and *contravariantly finite* subcategories are defined dually, and a subcategory is *functorially finite* if it is both covariantly and contravariantly finite.

Recall that a morphism $f\colon X \to Y$ is called *left minimal* if any endomorphism g of Y satisfying $g \circ f = f$ is an isomorphism. A *minimal left $\mathcal{X}$-approximation* is a left $\mathcal{X}$-approximation which is also left minimal. Right minimal morphisms and minimal right $\mathcal{X}$-approximations are defined

dually. Since $\mathcal{A}$ is Krull–Schmidt, an object has a left (right) $\mathcal{X}$-approximation if and only if it has a minimal left (right) $\mathcal{X}$-approximation.

2.1. **d-cluster tilting subcategories and d-abelian categories.** The theory of higher homological algebra originated in [I1, I2] with the study of *d-cluster tilting subcategories*. The definition is given below.

Definition 2.1. A functorially finite generating-cogenerating subcategory $\mathcal{M}$ of the abelian category $\mathcal{A}$ is *d-cluster tilting* if

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X, M) = 0 \text{ for } M \in \mathcal{M} \text{ and } i = 1, \dots, d-1\}\\ &= \{Y \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(M, Y) = 0 \text{ for } M \in \mathcal{M} \text{ and } i = 1, \dots, d-1\}.\end{aligned}$$

To formalise the homological structure of d-cluster tilting subcategories, Jasso introduced *d-abelian categories*, where the case $d = 1$ recovers the classical notion of abelian categories [J2]. To give a precise definition, we first recall some terminology.

Let $\mathcal{M}$ be an additive category and recall that a *weak cokernel* of a morphism $f\colon X \to Y$ in $\mathcal{M}$ is a morphism $g\colon Y \to Z$ in $\mathcal{M}$ for which the induced sequence

$$\operatorname{Hom}_{\mathcal{M}}(Z, M) \to \operatorname{Hom}_{\mathcal{M}}(Y, M) \to \operatorname{Hom}_{\mathcal{M}}(X, M)$$

is exact for any $M \in \mathcal{M}$. This is equivalent to saying that $g \circ f = 0$ and that for any $g'\colon Y \to M$ with $g' \circ f = 0$, there exists a (not necessarily unique) morphism $h\colon Z \to M$ such that $h \circ g = g'$. We call a morphism g a weak cokernel if there exists a morphism f such that g is a weak cokernel of f.

A *d-cokernel* of a morphism $f_0\colon X_0 \to X_1$ in $\mathcal{M}$ is given by a sequence of morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} X_{d+1} \to 0$$

in $\mathcal{M}$ such that for every M in $\mathcal{M}$, the sequence

$$0 \to \operatorname{Hom}_{\mathcal{M}}(X_{d+1}, M) \to \operatorname{Hom}_{\mathcal{M}}(X_d, M) \to \cdots \to \operatorname{Hom}_{\mathcal{M}}(X_1, M) \to \operatorname{Hom}_{\mathcal{M}}(X_0, M)$$

of abelian groups is exact. Such a d-cokernel is sometimes simply denoted by $(f_1, \dots, f_d)$, and $(f_1, \dots, f_d)$ is a d-cokernel of f_0 if and only if f_i is a weak cokernel of f_{i-1} for $i = 1, \dots, d-1$ and f_d is the cokernel of f_{d-1}. The notion of a *d-kernel* in $\mathcal{M}$ is defined dually. A sequence

$$0 \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} X_{d+1} \to 0 \tag{1}$$

in $\mathcal{M}$ is called a *d-exact sequence* or a *d-extension* if $(f_1, \dots, f_d)$ is a d-cokernel of f_0 and $(f_0, \dots, f_{d-1})$ is a d-kernel of f_d.

Definition 2.2. [J2, Definition 3.1] An additive category $\mathcal{M}$ is *d-abelian* if it is idempotent complete, every morphism admits a d-kernel and a d-cokernel, and every monomorphism f_0 (resp. epimorphism f_d) fits into a d-exact sequence of the form (1).

A d-exact sequence of the form (1) is said to be *equivalent* to a d-exact sequence

$$0 \to X_0 \xrightarrow{f'_0} X'_1 \xrightarrow{f'_1} \dots \xrightarrow{f'_{d-1}} X'_d \xrightarrow{f'_d} X_{d+1} \to 0$$

if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d] & \cdots \arrow[r, "f_{d-1}"] & X_d \arrow[r, "f_d"] \arrow[d] & X_{d+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "f'_0"] & X'_1 \arrow[r, "f'_1"] & \cdots \arrow[r, "f'_{d-1}"] & X'_d \arrow[r, "f'_d"] & X_{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Note that this defines an equivalence relation on the class of d-exact sequences whenever the category $\mathcal{M}$ is d-abelian [J2, Proposition 4.10].

When $\mathcal{M} \subseteq \mathcal{A}$ is a d-cluster tilting subcategory, d-exact sequences coincide precisely with exact sequences of the form (1) where all terms are in $\mathcal{M}$. Moreover, any exact sequence in $\mathcal{A}$ of the form (1) with end terms in $\mathcal{M}$ is equivalent to one where all terms are in $\mathcal{M}$ [I1, A.1].

In a d-abelian category, we also find higher analogues of the classical notions of pushouts and pullbacks. For more details on the construction of d-pushouts and d-pullbacks, we refer the reader to [J2, Section 2.3].

Jasso proved the following theorem, which shows that d-abelian categories capture the homological structure of d-cluster tilting subcategories.

Theorem 2.3. [J2, Theorem 3.16] *Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{A}$. Then $\mathcal{M}$ is d-abelian.*

It has recently been shown that the converse of Theorem 2.3 also holds, i.e. that any d-abelian category is equivalent to a d-cluster tilting subcategory of an abelian category [EN-I, Kv]. Therefore, all d-abelian categories may be treated as d-cluster tilting subcategories.

We end this subsection by introducing a running example that will help us illustrate many of the results in this paper.

Example 2.4. Consider the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Let A denote the path algebra of this quiver modulo the ideal generated by the relation $\alpha\beta$. Figure 1 shows the Auslander–Reiten quiver of $\operatorname{mod} A$, where the dashed arrows indicate the Auslander–Reiten translation. The subcategory

$$\mathcal{M} = \operatorname{add}\left\{ {}_3 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus {}_1 \right\}$$

is 2-cluster tilting in $\operatorname{mod} A$, and $\mathcal{M}$ is hence an example of a 2-abelian category. The indecomposable objects of $\operatorname{mod} A$ that generate $\mathcal{M}$ are marked in Figure 1.

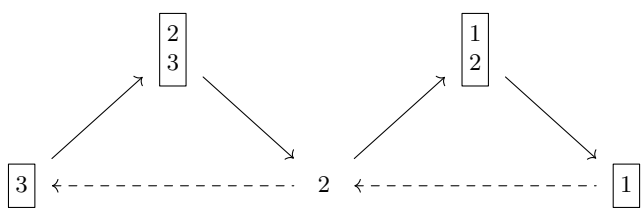

FIGURE 1. The Auslander–Reiten quiver of the module category considered in Example 2.4, with the generators of the 2-cluster tilting subcategory $\mathcal{M}$ marked.

2.2. **Minimality.** The d-kernels and d-cokernels in a d-abelian category are unique only up to homotopy. Many of our proofs require a stronger sense of uniqueness, which is why we recall the concept of minimality in this section.

The Jacobson radical of the abelian category $\mathcal{A}$ consists of the morphisms

$$\operatorname{Rad}_{\mathcal{A}}(X, Y) = \{f \in \operatorname{Hom}_{\mathcal{A}}(X, Y) \mid 1_X - g \circ f \text{ is invertible for all } g \in \operatorname{Hom}_{\mathcal{A}}(Y, X)\}.$$

For more details and basic properties, see [ASS, A.3].

Definition 2.5. [HJ, Definition 2.5] Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{A}$.

(1) A d-cokernel

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} X_{d+1} \to 0$$

of a morphism $f_0 \colon X_0 \to X_1$ in $\mathcal{M}$ is called *minimal* if $f_i \in \operatorname{Rad}_{\mathcal{A}}(X_i, X_{i+1})$ for $i = 2, \dots, d$.

(2) A d-kernel
$$0 \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{d-2}} X_{d-1} \xrightarrow{f_{d-1}} X_d$$
of a morphism $f_d \colon X_d \to X_{d+1}$ in $\mathcal{M}$ is called *minimal* if $f_i \in \operatorname{Rad}_{\mathcal{A}}(X_i, X_{i+1})$ for $i = 0, \dots, d-2$.

(3) A d-extension
$$0 \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} X_{d+1} \to 0$$
in $\mathcal{M}$ is called *minimal* if $f_i \in \operatorname{Rad}_{\mathcal{A}}(X_i, X_{i+1})$ for $i = 1, \dots, d-1$.

Proposition 2.6 justifies the terminology in the definition above, and shows that minimal d-cokernels, d-kernels and d-extensions exist and are unique up to isomorphism. When we say that a complex in a category $\mathcal{M}$ is a *direct summand* of another complex, this means that it is a direct summand in the category of complexes over $\mathcal{M}$.

Proposition 2.6. [HJ, Proposition 2.4] *Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{A}$.*

(1) *Given a morphism f in $\mathcal{M}$, there exists a minimal d-cokernel (resp. d-kernel) of f. This d-cokernel (resp. d-kernel) is a direct summand of any other d-cokernel (resp. d-kernel) of f.*

(2) *Given a d-extension in $\mathcal{M}$, there exists an equivalent minimal d-extension. This minimal d-extension is a direct summand of every d-extension in the associated equivalence class.*

Remark 2.7. Since [J2, Lemma 2.1] implies that any two d-cokernels of a morphism are isomorphic in the homotopy category, Proposition 2.6 implies that any d-cokernel is isomorphic to the direct sum of the minimal d-cokernel and a contractible exact sequence. Moreover, for an abelian category, any contractible complex is the direct sum of complexes of the form $N \xrightarrow{1_N} N$. In particular, given a d-cokernel
$$X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} X_{d+1} \to 0$$
of $f_0 \colon X_0 \to X_1$ where $f_i \notin \operatorname{Rad}_{\mathcal{A}}(X_i, X_{i+1})$ for some $2 \leq i \leq d$, there is an isomorphic d-cokernel where f_i is replaced by
$$\begin{pmatrix} f_i' & 0 \\ 0 & 1_N \end{pmatrix} : X_i' \oplus N \to X_{i+1}' \oplus N$$
with $f_i' \in \operatorname{Rad}_{\mathcal{A}}(X_i', X_{i+1}')$. Similar statements hold for d-kernels and d-extensions.

The terminology in Definition 2.5 is further justified by the following connection to minimal morphisms.

Lemma 2.8. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a complex in $\mathcal{A}$ for which the induced sequence*
$$\operatorname{Hom}_{\mathcal{A}}(Z, Y) \to \operatorname{Hom}_{\mathcal{A}}(Y, Y) \to \operatorname{Hom}_{\mathcal{A}}(X, Y)$$
is exact. The morphism f is left minimal if and only if $g \in \operatorname{Rad}_{\mathcal{A}}(Y, Z)$.

Proof. See [JK, Lemma 1.1] □

The construction of a minimal d-cokernel is frequently used throughout the paper. We discuss it in more detail in the following.

Construction 2.9. The minimal d-cokernel of a morphism $f \colon X \to Y$ in a d-cluster tilting subcategory $\mathcal{M} \subseteq \mathcal{A}$ can be constructed as follows:

(1) Set $C_1 = \operatorname{Coker} f$ and let $g_1 \colon C_1 \to M_1$ be the minimal left $\mathcal{M}$-approximation of C_1. Set $f_1 \colon Y \to M_1$ to be the composition $Y \to C_1 \xrightarrow{g_1} M_1$.

(2) Repeat on $f_1 \colon Y \to M_1$ to construct $f_2 \colon M_1 \to M_2$.

(3) Iterate the procedure, which must terminate and result in a d-cokernel by [J2, Proposition 3.17].

Since each morphism f_i is the composition of an epimorphism and a left minimal morphism, they are all left minimal, and hence Lemma 2.8 shows that this is the minimal d-cokernel of f. Note that each f_i can equivalently be described as the left minimal weak cokernel of the previous morphism.

Lemma 2.10. *Suppose we have a d-extension*

$$0 \to X \to E_1 \to E_2 \to \cdots \to E_d \to Y \to 0.$$

Then, for any morphism $h\colon X \to F_0$, there is a d-pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "h"] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_d \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & F_d \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

such that the bottom row is a minimal d-extension.

Proof. The fact that a d-pushout diagram exists is precisely [J2, Theorem 3.8]. It follows from [J2, Proposition 4.8] that the bottom row is a d-extension with last term equal to Y. If this d-extension is minimal, then we are done. Otherwise, it has a minimal d-extension as a direct summand by Proposition 2.6. Composing the morphism in the statement with the projection onto this minimal d-extension gives another commutative diagram, which is a d-pushout by [J2, Proposition 4.8]. The bottom row of this new diagram is a minimal d-extension as required. □

Note that there is a dual version of Lemma 2.10 which will also be used.

2.3. **Torsion and d-torsion classes.** Generalising the properties of the class of torsion groups in the category of abelian groups, the notion of a *torsion pair* was introduced in [D] as follows.

Definition 2.11. A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{A}$ is a *torsion pair* if the following conditions are satisfied:

(1) For every $X \in \mathcal{A}$, there exists a short exact sequence

$$0 \to tX \to X \to fX \to 0$$

where $tX \in \mathcal{T}$ and $fX \in \mathcal{F}$.

(2) $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$, we say that $\mathcal{T}$ is a *torsion class* and $\mathcal{F}$ a *torsion free class*.

Note that the short exact sequence from the definition above is unique up to isomorphism. The following classical result characterises torsion classes in $\mathcal{A}$ as those subcategories which are closed under extensions and quotients.

Theorem 2.12. [D, Theorem 2.3] *A subcategory $\mathcal{T}$ of $\mathcal{A}$ is a torsion class if and only if $\mathcal{T}$ is closed under extensions and quotients.*

We denote by $\mathrm{tors}(\mathcal{A})$ the poset of all torsion classes in $\mathcal{A}$ ordered by inclusion. It is well known that $\mathrm{tors}(\mathcal{A})$ is a complete lattice where the meet is given by intersection, see e.g. [IRTT, Proposition 2.3].

In light of the development of higher homological algebra, it is natural to consider higher analogues of torsion classes. The notion of d-torsion classes was introduced in [Jø].

Definition 2.13. [Jø, Definition 1.1] Let $\mathcal{M}$ be a d-abelian category. A subcategory $\mathcal{U}$ of $\mathcal{M}$ is a *d-torsion class* if for every M in $\mathcal{M}$, there exists a d-exact sequence

$$0 \to U_M \to M \to V_1 \to \cdots \to V_d \to 0$$

such that the following conditions are satisfied:

(1) The object U_M is in $\mathcal{U}$.

(2) The sequence $0 \to \mathrm{Hom}_{\mathcal{M}}(U, V_1) \to \cdots \to \mathrm{Hom}_{\mathcal{M}}(U, V_d) \to 0$ is exact for every U in $\mathcal{U}$.

The object U_M above is known as the *d-torsion subobject* of M with respect to $\mathcal{U}$ and a complex $V_1 \to \cdots \to V_d$ in $\mathcal{M}$ is called *$\mathcal{U}$-exact* when it satisfies the exactness property described in (2).

Note that when $d = 1$, the previous definition is equivalent to the definition of a torsion class, c.f. Definition 2.11.

Remark 2.14. It is shown in [Jø, Lemma 2.7(iii)] that d-torsion classes are closed under direct sums and summands, and thus any d-torsion class is completely determined by its indecomposable objects.

A recent paper [AJST] showed that there is a strong relationship between the torsion classes in an abelian category $\mathcal{A}$ and the d-torsion classes in a d-cluster tilting subcategory $\mathcal{M}$ of $\mathcal{A}$.

Theorem 2.15. [AJST, Theorem 1.1] *Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{A}$. Then a torsion class $\mathcal{T}$ in $\mathcal{A}$ is the minimal torsion class containing a given d-torsion class in $\mathcal{M}$ if and only if the following conditions are satisfied:*

1. *For every $M \in \mathcal{M}$, we have $tM \in \mathcal{M}$.*
2. *$\mathcal{T}$ is the smallest torsion class containing all tM for $M \in \mathcal{M}$.*
3. *For any $M, M' \in \mathcal{M}$, we have $\operatorname{Ext}_{\mathcal{A}}^{d-1}(tM, fM') = 0$.*

Moreover, in this case $\mathcal{U} := \mathcal{T} \cap \mathcal{M}$ is a d-torsion class and $\mathcal{T}$ is the minimal torsion class containing it. Furthermore, we have $U_M = tM$ for every object $M \in \mathcal{M}$.

We now illustrate Theorem 2.15 in our running example.

Example 2.16. Let A and $\mathcal{M}$ be as in Example 2.4. In Table 1 we give the complete list of 2-torsion classes $\mathcal{U}$ in $\mathcal{M}$ and the corresponding minimal torsion classes $\mathcal{T}$ in $\operatorname{mod} A$ such that $\mathcal{U} = \mathcal{T} \cap \mathcal{M}$.

2-torsion classes in $\mathcal{M}$	Corresponding minimal torsion classes in $\operatorname{mod} A$
$\mathcal{M}$	$\operatorname{mod} A$
$\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$	$\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$
$\operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$	$\operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$
$\operatorname{add}\{1\}$	$\operatorname{add}\{1\}$
$\operatorname{add}\{3\}$	$\operatorname{add}\{3\}$
$\{0\}$	$\{0\}$

TABLE 1. The 2-torsion classes considered in Example 2.16 and their corresponding torsion classes.

On one hand, we note that although Theorem 2.15 gives a complete characterisation of the *minimal* torsion classes $\mathcal{T}$ in $\operatorname{mod} A$ such that $\mathcal{T} \cap \mathcal{M}$ is a 2-torsion class in $\mathcal{M}$, there are other torsion classes in $\operatorname{mod} A$ with this property. For instance, we see that $\mathcal{T} = \operatorname{add}\left\{2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$ is a torsion class in $\operatorname{mod} A$ and $\mathcal{T} \cap \mathcal{M} = \operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$ is a 2-torsion class in $\mathcal{M}$.

On the other hand, it is not true that $\mathcal{T} \cap \mathcal{M}$ is a 2-torsion class in $\mathcal{M}$ for every torsion class $\mathcal{T}$ in $\operatorname{mod} A$. Two such examples are $\mathcal{T}_1 = \operatorname{add}\{3 \oplus 1\}$ and $\mathcal{T}_2 = \operatorname{add}\left\{3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2\right\}$. The intersections $\mathcal{T}_1 \cap \mathcal{M}$ and $\mathcal{T}_2 \cap \mathcal{M}$ are not 2-torsion classes in $\mathcal{M}$, for reasons that will be explained in Example 3.18.

3. Closure under d-extensions and d-quotients

Throughout this section, let $\mathcal{M}$ be a d-cluster tilting subcategory of the abelian category $\mathcal{A}$. We also introduce the following setup, which will not be assumed unless explicitly stated.

Setup 3.1. Let $\mathcal{U}$ be a d-torsion class in $\mathcal{M}$. Let $\mathcal{T}$ be a torsion class in $\mathcal{A}$ such that $\mathcal{U} = \mathcal{T} \cap \mathcal{M}$ and the torsion subobject tM of M belongs to $\mathcal{M}$ for all $M \in \mathcal{M}$.

We note that Theorem 2.15 implies that for every d-torsion class $\mathcal{U}$ in $\mathcal{M}$, we can find a torsion class $\mathcal{T}$ satisfying Setup 3.1.

Section 3 is divided into three subsections. We first verify some consequences of the setup above in Section 3.1. In Section 3.2 we state and prove the main result of this paper, namely the characterisation of higher torsion classes given in Theorem 1.1. The aim of Section 3.3 is to prove Theorem 1.3.

3.1. **Approximations by d-torsion classes.** Throughout this subsection we assume Setup 3.1.

Lemma 3.2. *For any $M \in \mathcal{M}$, there is an isomorphism $U_M \cong tM$ commuting with the inclusion to M.*

Proof. By the definition of a d-torsion class, the morphism $U_M \to M$ is a right $\mathcal{U}$-approximation. Since the morphism $tM \to M$ is a right $\mathcal{T}$-approximation and $tM \in \mathcal{U}$, it must also be a right $\mathcal{U}$-approximation. Hence, the inclusions $U_M \to M$ and $tM \to M$ must factor through each other, which implies that $U_M \cong tM$. □

By Lemma 3.2, we can assume $U_M = tM$ whenever we are in Setup 3.1, and we will do this from now on. Given Setup 3.1, we gain additional control of left $\mathcal{M}$-approximations of objects in $\mathcal{T}$.

Lemma 3.3. *Let $\phi_X \colon X \to M$ be the minimal left $\mathcal{M}$-approximation of an object X in $\mathcal{T}$. Then M is in $\mathcal{U}$.*

Proof. Consider the short exact sequence

$$0 \to tM \xrightarrow{\iota} M \to fM \to 0$$

associated to the torsion pair $(\mathcal{T}, \mathcal{F})$. The morphism ι is a right $\mathcal{T}$-approximation of M, and $tM \in \mathcal{M}$ by assumption. As $X \in \mathcal{T}$, there exists a morphism $\psi_X \colon X \to tM$ such that $\iota\psi_X = \phi_X$. Notice that since ϕ_X is a left $\mathcal{M}$-approximation of X, so is ψ_X. Since ϕ_X is minimal and ι is a monomorphism, this implies that M is isomorphic to tM. Consequently, one obtains that M is contained in $\mathcal{T}$, and hence also in $\mathcal{U}$. □

Corollary 3.4. *If $X \in \mathcal{T}$, then the minimal left $\mathcal{M}$-approximation of X is also the minimal left $\mathcal{U}$-approximation of X. Moreover, this approximation is a monomorphism.*

Proof. Let $\phi_X \colon X \to M$ be the minimal left $\mathcal{M}$-approximation of X. By Lemma 3.3, we know that $M \in \mathcal{U}$, and it follows that ϕ_X is also a left $\mathcal{U}$-approximation. Since ϕ_X is left minimal, it is the minimal left $\mathcal{U}$-approximation of X. Finally, since $\mathcal{M}$ is cogenerating, any left $\mathcal{M}$-approximation is a monomorphism. □

Remark 3.5. Note that the previous corollary implies that the d-torsion class $\mathcal{U}$ is always covariantly finite within the torsion class $\mathcal{T}$, even if $\mathcal{U}$ is not covariantly finite in $\mathcal{M}$ or $\mathcal{A}$.

3.2. **Characterising d-torsion classes.** In order to formulate our results, we need higher analogues of what it means for a subcategory to be closed under extensions and quotients. Recall the notions of d-cokernels and d-extensions from Section 2.1.

Definition 3.6. A subcategory $\mathcal{U}$ of $\mathcal{M}$ is called *closed under d-extensions* if for any d-extension

$$0 \to X \to E_1 \to \cdots \to E_d \to Y \to 0$$

in $\mathcal{M}$ with X and Y in $\mathcal{U}$, there exists an equivalent d-extension

$$0 \to X \to E_1' \to \cdots \to E_d' \to Y \to 0$$

where all the objects are in $\mathcal{U}$.

Definition 3.7. A subcategory $\mathcal{U}$ of $\mathcal{M}$ is called *closed under d-quotients* if for any morphism $f\colon M \to U$ in $\mathcal{M}$ with U in $\mathcal{U}$, there exists a d-cokernel

$$M \xrightarrow{f} U \to E_1 \to E_2 \to \cdots \to E_d \to 0$$

of f with E_i in $\mathcal{U}$ for all $i = 1, \dots, d$. If this condition is only assumed to hold when both M and U belong to $\mathcal{U}$, we say that $\mathcal{U}$ is *closed under d-cokernels*.

It will further be convenient to define a *(minimal) d-quotient of $U \in \mathcal{M}$* as a (minimal) d-cokernel of some morphism $f\colon M \to U$ in $\mathcal{M}$.

Remark 3.8. Since every d-extension is equivalent to a minimal one by Proposition 2.6, to prove that a subcategory $\mathcal{U} \subseteq \mathcal{M}$ is closed under d-extensions, it is enough to show that any minimal d-extension

$$0 \to X \to E_1 \to \cdots \to E_d \to Y \to 0$$

with $X, Y \in \mathcal{U}$ satisfies $E_1, \dots, E_d \in \mathcal{U}$. We call this latter property being *closed under minimal d-extensions*. Conversely, if $\mathcal{U}$ is closed under direct summands and d-extensions, then $\mathcal{U}$ is closed under minimal d-extensions. Analogous statements hold for minimal d-quotients.

Remark 3.9. Since $\mathcal{M}$ is closed under direct summands, it follows that any subcategory $\mathcal{U} \subseteq \mathcal{M}$ which is closed under d-quotients is also closed under direct summands.

We often implicitly use that the subcategories we work with are closed under direct summands. This is not a standing assumption, but will hold either due to Remark 2.14 or Remark 3.9.

The following lemma shows that when checking if a subcategory is closed under d-extensions, it is often sufficient to consider the first middle term. This simplifies the proof of Proposition 3.11 and is also an important step towards the main result in Section 3.3.

Lemma 3.10. *Let $\mathcal{U} \subseteq \mathcal{M}$ be closed under d-quotients. Then for any minimal d-extension*

$$0 \to X \xrightarrow{f} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{d-1}} E_d \xrightarrow{g} Y \to 0$$

in $\mathcal{M}$ with $X, E_1, Y \in \mathcal{U}$, it follows that $E_i \in \mathcal{U}$ for $i = 2, \dots, d$.

Proof. The minimality of the d-extension (see Definition 2.5) gives that $e_i \in \operatorname{Rad}_{\mathcal{A}}(E_i, E_{i+1})$ for all $i = 1, \dots, d-1$. Moreover, the sequence

$$E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{d-1}} E_d \xrightarrow{g} Y \to 0 \tag{2}$$

is a d-cokernel of f. If $g \in \operatorname{Rad}_{\mathcal{A}}(E_d, Y)$, then this d-cokernel is minimal. Since $E_1 \in \mathcal{U}$, the result then follows from $\mathcal{U}$ being closed under d-quotients.

Suppose hence that $g \notin \operatorname{Rad}_{\mathcal{A}}(E_d, Y)$. Recall from Remark 2.7 that the sequence (2) is isomorphic to the direct sum of the minimal d-cokernel of f and shifted complexes of the form $N \xrightarrow{1_N} N$. However, since $e_i \in \operatorname{Rad}_A(E_i, E_{i+1})$ for $i = 1, \dots d-1$, it follows that (2) is isomorphic to

$$E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{d-2}} E_{d-1} \xrightarrow{e'_{d-1} = \binom{h_1}{h_2}} E'_d \oplus Y'' \xrightarrow{\begin{pmatrix} g' & 0 \\ 0 & 1_{Y''} \end{pmatrix}} Y' \oplus Y'' \to 0,$$

where e'_{d-1} is e_{d-1} composed with an isomorphism and

$$E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{d-2}} E_{d-1} \xrightarrow{h_1} E'_d \xrightarrow{g'} Y' \to 0$$

is a minimal d-cokernel of f. In particular, the objects $E_2, \dots, E_{d-1}, E'_d$ are in $\mathcal{U}$ as $E_1 \in \mathcal{U}$ and $\mathcal{U}$ is closed under d-quotients. Since $Y' \oplus Y'' = Y \in \mathcal{U}$, we moreover see that $E_d = E'_d \oplus Y'' \in \mathcal{U}$, completing the proof. $\square$

We are now ready to prove the first part of our characterisation result. Recall that a complex $V_1 \to \cdots \to V_d$ in $\mathcal{M}$ is said to be $\mathcal{U}$*-exact* with respect to a subcategory $\mathcal{U} \subseteq \mathcal{M}$ if

$$0 \to \operatorname{Hom}_{\mathcal{A}}(U, V_1) \to \operatorname{Hom}_{\mathcal{A}}(U, V_2) \to \cdots \to \operatorname{Hom}_{\mathcal{A}}(U, V_d) \to 0$$

is exact for all objects U in $\mathcal{U}$.

Proposition 3.11. *Let $\mathcal{U} \subseteq \mathcal{M}$ be a d-torsion class. Then $\mathcal{U}$ is closed under d-extensions and d-quotients.*

Proof. As $\mathcal{U}$ is a d-torsion class in $\mathcal{M}$, we have that $\mathcal{U} = \mathcal{T} \cap \mathcal{M}$ for some torsion class $\mathcal{T}$ in $\mathcal{A}$ as in Setup 3.1. Recall from Remark 3.8 that it suffices to consider minimal d-extensions and minimal d-quotients.

We first show that $\mathcal{U}$ is closed under d-quotients. Let $f\colon M \to U$ be a morphism in $\mathcal{M}$ with $U \in \mathcal{U}$ and consider its minimal d-cokernel

$$M \xrightarrow{f} U \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} V_{d-1} \xrightarrow{f_d} V_d \to 0.$$

By construction of the minimal d-cokernel (see Construction 2.9), we have that V_i arises as the minimal left $\mathcal{M}$-approximation of $\operatorname{Coker} f_{i-1}$ for all $i = 1, \dots, d$, where we set $f_0 = f$. As $U \in \mathcal{U} = \mathcal{T} \cap \mathcal{M}$ and $\mathcal{T}$ is closed under quotients, Lemma 3.3 implies that $V_i \in \mathcal{U}$ for all $i = 1, \dots, d$. This shows that $\mathcal{U}$ is closed under d-quotients.

We next prove that $\mathcal{U}$ is closed under d-extensions. Consider a minimal d-exact sequence

$$0 \to X \xrightarrow{e_0} E_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{d-1}} E_d \xrightarrow{e_d} Y \to 0$$

in $\mathcal{M}$ with X and Y in $\mathcal{U}$. By Lemma 2.8, the morphism e_i is left minimal for $i = 0, \dots, d-2$. As $\mathcal{U}$ is a d-torsion class, we obtain the solid part of the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "e_0"] \arrow[d, dashed, "g_0"] & E_1 \arrow[r, "e_1"] \arrow[dl, dashed, "h_0"'] \arrow[d, equal] & E_2 \arrow[r, "e_2"] \arrow[dl, dashed, "h_1"'] \arrow[d, dashed, "g_2"] & \cdots \arrow[r, "e_{d-1}"] & E_d \arrow[r, "e_d"] \arrow[dl, dashed, "h_{d-1}"'] \arrow[d, dashed, "g_d"] & Y \arrow[r] \arrow[dl, dashed, "h_d"'] \arrow[d, dashed, "g_{d+1}"] & 0 \\
0 \arrow[r] & tE_1 \arrow[r, "\iota"] & E_1 \arrow[r, "w_0"] & W_1 \arrow[r, "w_1"] & \cdots \arrow[r, "w_{d-2}"] & W_{d-1} \arrow[r, "w_{d-1}"] & W_d \arrow[r] & 0,
\end{tikzcd}$$

where the bottom row is the d-exact sequence associated to E_1 by $\mathcal{U}$. In particular, the object $tE_1 \in \mathcal{U}$ is the d-torsion subobject of E_1 with respect to $\mathcal{U}$ by Lemma 3.2, and the sequence $W_1 \to \cdots \to W_d$ is $\mathcal{U}$-exact. As ι is a right $\mathcal{T}$-approximation and $X \in \mathcal{U} = \mathcal{T} \cap \mathcal{M}$, there exists a morphism $g_0\colon X \to tE_1$ making the left square commute. We can hence complete the diagram to a morphism g of d-exact sequences by using the factorisation property for weak cokernels, see Section 2.1.

Since the sequence $W_1 \to \cdots \to W_d$ is $\mathcal{U}$-exact, the morphism

$$w_{d-1} \circ -\colon \operatorname{Hom}_{\mathcal{A}}(Y, W_{d-1}) \to \operatorname{Hom}_{\mathcal{A}}(Y, W_d)$$

is surjective. Hence, there exists a morphism $h_d\colon Y \to W_{d-1}$ with $w_{d-1}h_d = g_{d+1}$. As the bottom row is d-exact, there is an exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(E_d, W_{d-2}) \to \operatorname{Hom}_{\mathcal{A}}(E_d, W_{d-1}) \to \operatorname{Hom}_{\mathcal{A}}(E_d, W_d).$$

Using the commutativity of the rightmost square, we get $w_{d-1}(g_d - h_d e_d) = 0$, so $g_d - h_d e_d$ is in the kernel of the second morphism. By exactness, there exists a morphism $h_{d-1}\colon E_d \to W_{d-2}$ such that $g_d - h_d e_d = w_{d-2}h_{d-1}$, or equivalently $g_d = h_d e_d + w_{d-2}h_{d-1}$.

We can repeat this process to obtain a homotopy of the map of complexes g. In particular, there are morphisms h_0 and h_1 such that $g_1 = 1_{E_1} = \iota h_0 + h_1 e_1$. This implies that $\iota h_0 e_0 = e_0$, so ιh_0 is an isomorphism by the left minimality of e_0. The morphism h_0 is hence a split monomorphism, so E_1 is contained in $\mathcal{U}$. By Lemma 3.10, this implies that $E_i \in \mathcal{U}$ for $i = 2, \dots, d$, so the subcategory $\mathcal{U}$ is closed under d-extensions. □

Remark 3.12. Proposition 3.11 implies that every d-torsion class is closed under d-cokernels.

The remainder of this subsection is devoted to proving the converse of Proposition 3.11. Since any d-torsion class in $\mathcal{M}$ is contravariantly finite in $\mathcal{M}$, we first establish that for a subcategory $\mathcal{U} \subseteq \mathcal{M}$ to be contravariantly finite, it is enough to assume closure under d-quotients.

We first need a result on coimage factorisation in a d-cluster tilting subcategory. Recall from Section 2.1 that a morphism g in $\mathcal{M}$ is a weak cokernel if there exists a morphism f in $\mathcal{M}$ such that g is a weak cokernel of f.

Proposition 3.13. *Let $f\colon M \to N$ be a morphism in $\mathcal{M}$. Then there exists a factorisation $f = f_2 \circ f_1$ in $\mathcal{M}$ where f_2 is a monomorphism and f_1 is a composite of left minimal weak cokernels.*

Proof. Let E_0 denote the image of f, and let $\iota_0\colon E_0 \to M_0$ be a minimal left $\mathcal{M}$-approximation. The inclusion $E_0 \to N$ lifts via ι_0 to a morphism $g_0\colon M_0 \to N$. Let E_1 be the image of g_0, and note that $E_0 \subseteq E_1$. Repeating this procedure, we get a subobject $E_i \subseteq N$, a minimal left approximation $\iota_i\colon E_i \to M_i$, and a lift $g_i\colon M_i \to N$ for each $i \geq 0$. In particular, the E_i's form an increasing sequence $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots E_i \subseteq \cdots$ of subobjects of N.

Since $\mathcal{A}$ is of finite length, this sequence has to stabilise, say at E_j. This implies that the image of $g_j\colon M_j \to N$ is E_j. But then the inclusion $\iota_j\colon E_j \to M_j$ is a split monomorphism, and hence an isomorphism since it is also left minimal. This shows that $E_j \in \mathcal{M}$. Now let $f_2\colon E_j \to N$ be the inclusion, and let $f_1\colon M \to E_j$ be the composite $M \to E_0 \to E_j$. Note that this is equal to the composite

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, two heads] & & M_0 \arrow[rr] \arrow[dr, two heads] & & M_1 \arrow[rr] \arrow[dr, two heads] & & \cdots \arrow[rr] \arrow[dr, two heads] & & M_{j-1} \arrow[rr] \arrow[dr, two heads] & & M_j. \\
& E_0 \arrow[ur, tail, "\iota_0"] \arrow[rr, tail] & & E_1 \arrow[ur, tail, "\iota_1"] \arrow[rr, tail] & & E_2 \arrow[ur, tail, "\iota_2"] & \cdots & E_{j-1} \arrow[ur, tail, "\iota_{j-1}"] \arrow[rr, tail] & & E_j \arrow[ur, tail, "\cong"] &
\end{tikzcd}$$

By construction, the morphisms $M \to M_0$ and $M_i \to M_{i+1}$ for $i = 0, \ldots, j-1$ are left minimal weak cokernels. Since f_1 is a composite of such morphisms and the isomorphism $M_j \cong E_j$, this proves the claim. □

Lemma 3.14. *Let $\mathcal{U} \subseteq \mathcal{M}$ be closed under d-quotients. If $g\colon M \to N$ is a left minimal weak cokernel with $M \in \mathcal{U}$, then $N \in \mathcal{U}$.*

Proof. Assume g is a weak cokernel of a morphism f. Then g is part of a d-cokernel of f, and if g is left minimal, then it is part of the minimal d-cokernel of f; see Construction 2.9. This proves the claim. □

We now apply Proposition 3.13 and Lemma 3.14 to show that being closed under d-quotients implies being contravariantly finite.

Proposition 3.15. *Let $\mathcal{U} \subseteq \mathcal{M}$ be closed under d-quotients and consider $M \in \mathcal{M}$. Then there exists a minimal right $\mathcal{U}$-approximation $U \to M$ which is a monomorphism. In particular, the subcategory $\mathcal{U}$ is contravariantly finite in $\mathcal{A}$.*

Proof. Consider the set of subobjects $U \subseteq M$ with $U \in \mathcal{U}$. Note that this set is non-empty since $0 \in \mathcal{U}$.

We first prove that this set has a unique maximal element. Indeed, since $\mathcal{A}$ has finite length, we can choose $U \subseteq M$ with $U \in \mathcal{U}$ and where U is of maximal length with this property. Now let $V \subseteq M$ with $V \in \mathcal{U}$, and consider the induced morphism $U \oplus V \to M$. By Proposition 3.13, there exists $W \subseteq M$ with $W \in \mathcal{M}$ such that $U \subseteq W \subseteq M$ and $V \subseteq W \subseteq M$, and such that the induced morphism $U \oplus V \to W$ is a composite of left minimal weak cokernels. Since $\mathcal{U}$ is closed under d-quotients, it follows from Lemma 3.14 that $W \in \mathcal{U}$. But since U is maximal with respect to the property that $U \in \mathcal{U}$ and $U \subseteq M$, it follows that $U = W$. This implies that V must be contained in U, and hence U is the unique maximal subobject $U \subseteq M$ satisfying $U \in \mathcal{U}$.

Now let $U' \to M$ be an arbitrary morphism with $U' \in \mathcal{U}$. By Proposition 3.13, there exists an object V' such that $U' \to M$ factors through V' and where the morphism $U' \to V'$ is a

composite of left minimal weak cokernels and the morphism $V' \to M$ is a monomorphism. It follows that $V' \in \mathcal{U}$ and V' is a subobject of M.

By the maximality of U, we get that $V' \subseteq U \subseteq M$, so the morphism $U' \to M$ also factors through U. This shows that the inclusion $U \subseteq M$ is a right $\mathcal{U}$-approximation, which is minimal since it is a monomorphism. This proves the first claim.

Finally, the fact that $\mathcal{U}$ is contravariantly finite in $\mathcal{A}$ follows from the fact that $\mathcal{M}$ is contravariantly finite in $\mathcal{A}$ and $\mathcal{U}$ is contravariantly finite in $\mathcal{M}$. □

Using the proposition above, we now show the converse of Proposition 3.11.

Proposition 3.16. *Let $\mathcal{U} \subseteq \mathcal{M}$ be closed under d-extensions and d-quotients. Then $\mathcal{U}$ is a d-torsion class in $\mathcal{M}$.*

Proof. Consider an object $M \in \mathcal{M}$. By the definition of a d-torsion class (see Definition 2.13), we need to show that there exists a d-exact sequence

$$0 \to U_M \to M \to V_1 \to \cdots \to V_d \to 0$$

where $U_M \in \mathcal{U}$ and the sequence $0 \to \mathrm{Hom}_{\mathcal{M}}(U, V_1) \to \cdots \to \mathrm{Hom}_{\mathcal{M}}(U, V_d) \to 0$ is exact for every U in $\mathcal{U}$.

Since $\mathcal{U}$ is closed under d-quotients, Proposition 3.15 shows that we may take a minimal right $\mathcal{U}$-approximation $f\colon U_M \to M$ which is a monomorphism. Taking the minimal d-cokernel of f gives a d-exact sequence

$$0 \to U_M \xrightarrow{f} M \to V_1 \to \cdots \to V_d \to 0. \tag{3}$$

Let $U \in \mathcal{U}$. As f is a right $\mathcal{U}$-approximation, we know that $\mathrm{Hom}_{\mathcal{M}}(U, f)$ is an epimorphism. Thus, it follows from d-exactness of the sequence (3) that

$$0 \to \mathrm{Hom}_{\mathcal{M}}(U, V_1) \to \cdots \to \mathrm{Hom}_{\mathcal{M}}(U, V_{d-1}) \to \mathrm{Hom}_{\mathcal{M}}(U, V_d)$$

is exact. To finish our proof, we hence need to show that the rightmost morphism in this sequence is an epimorphism.

Consider $h_d \in \mathrm{Hom}_{\mathcal{M}}(U, V_d)$ and take a d-pullback of (3) along h_d. This yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U_M \arrow[r, "f_0"] \arrow[d, equal] & W_0 \arrow[r] \arrow[d, "h_0"] & W_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & W_{d-1} \arrow[r, "f_d"] \arrow[d] & U \arrow[r] \arrow[d, "h_d"] & 0 \\
0 \arrow[r] & U_M \arrow[r, "f"] & M \arrow[r] & V_1 \arrow[r] & \cdots \arrow[r] & V_{d-1} \arrow[r] & V_d \arrow[r] & 0,
\end{tikzcd}$$

where the upper row is a d-exact sequence. By the dual of Lemma 2.10, this d-extension can be assumed to be minimal, and then closedness of $\mathcal{U}$ under d-extensions implies $W_i \in \mathcal{U}$ for all $i = 0, \ldots, d-1$ by Remark 3.8. As f is a right $\mathcal{U}$-approximation, the morphism h_0 factors through f, so f_0 is a split monomorphism. It follows that f_d is a split epimorphism, and hence h_d factors through V_{d-1}. In particular, the morphism $\mathrm{Hom}_{\mathcal{M}}(U, V_{d-1}) \to \mathrm{Hom}_{\mathcal{M}}(U, V_d)$ is an epimorphism as required. □

We can now generalise the classical characterisation of torsion classes, cf. Theorem 2.12.

Theorem 3.17. *Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{A}$. A subcategory $\mathcal{U} \subseteq \mathcal{M}$ is a d-torsion class if and only if it is closed under both d-extensions and d-quotients.*

Proof. The necessity follows from Proposition 3.11, while the sufficiency follows from Proposition 3.16. □

We now demonstrate the use of Theorem 3.17 in our running example.

Example 3.18. Let A and $\mathcal{M}$ be as described in Example 2.4. In Example 2.16 we claimed that $\mathcal{T}_1 = \operatorname{add}\{3 \oplus 1\}$ and $\mathcal{T}_2 = \operatorname{add}\left\{3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2\right\}$ are torsion classes for which $\mathcal{T}_1 \cap \mathcal{M} = \operatorname{add}\{3 \oplus 1\}$ and $\mathcal{T}_2 \cap \mathcal{M} = \operatorname{add}\left\{3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$ are not 2-torsion classes in $\mathcal{M}$. We now use Theorem 3.17 to explain why this is the case.

Consider the exact sequence

$$0 \longrightarrow 3 \longrightarrow \begin{smallmatrix}2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 1 \longrightarrow 0.$$

It is straightforward to check that this is a minimal 2-extension in $\mathcal{M}$. This implies that $\mathcal{T}_1 \cap \mathcal{M}$ is not closed under 2-extensions, so it is not a 2-torsion class in $\mathcal{M}$ by Theorem 3.17.

Similarly, using the same sequence, one can see that $\mathcal{T}_2 \cap \mathcal{M}$ is not closed under 2-quotients. Therefore, Theorem 3.17 implies that $\mathcal{T}_2 \cap \mathcal{M}$ is not a 2-torsion class.

A d-exact category is a pair $(\mathcal{C}, \mathcal{X})$ consisting of an additive category $\mathcal{C}$ and a class $\mathcal{X}$ of d-exact sequences in $\mathcal{C}$ satisfying certain axioms, see [J2, Definition 4.2]. One immediate consequence of our characterisation result is that any d-torsion class $\mathcal{U}$ in $\mathcal{M}$ carries the structure of a d-exact category.

Corollary 3.19. *Let $\mathcal{U} \subseteq \mathcal{M}$ be a d-torsion class. Consider the class $\mathcal{X}$ of d-exact sequences in $\mathcal{M}$ where all the terms are in $\mathcal{U}$. Then $(\mathcal{U}, \mathcal{X})$ is a d-exact category.*

Proof. By Theorem 3.17, the subcategory $\mathcal{U}$ is closed under d-extensions in $\mathcal{M}$. The result hence follows by applying [Kl, Corollary 4.12]. □

Note that when viewing $\mathcal{M}$ and $\mathcal{U}$ as d-exangulated categories, see [HLN], Corollary 3.19 moreover implies that $\mathcal{U}$ is a d-exangulated subcategory of $\mathcal{M}$ in the sense of [H, Definition 3.7].

3.3. **Closure under d-extensions.** To check if a subcategory $\mathcal{U} \subseteq \mathcal{M}$ is closed under d-extensions, it is necessary to determine the middle terms of any minimal d-extension between any two (not necessarily indecomposable) objects in $\mathcal{U}$. In this subsection we show that under certain conditions, it is enough to understand the d-extensions between indecomposable objects. The main result is the following.

Theorem 3.20. *Suppose $\mathcal{U} \subseteq \mathcal{M}$ is closed under d-extensions with indecomposable end terms and all d-quotients. Then $\mathcal{U}$ is closed under all d-extensions.*

We apply the theorem above in Sections 5 and 6, where we use it to give a combinatorial description of d-torsion classes of higher Auslander algebras of type $\mathbb{A}$ and higher Nakayama algebras of type $\mathbb{A}$ and $\mathbb{A}_\infty^\infty$.

In order to prove Theorem 3.20, recall first from Remark 3.8 that we may focus our attention purely on minimal d-extensions and minimal d-quotients. Our first step is to show that when closing a subcategory under d-extensions, it may suffice to consider d-extensions where the first term is indecomposable.

Lemma 3.21. *Suppose $\mathcal{U} \subseteq \mathcal{M}$ is closed under d-quotients. If $\mathcal{U}$ is closed under d-extensions with indecomposable first term, then $\mathcal{U}$ is closed under all d-extensions.*

Proof. Assume that $\mathcal{U}$ is closed under d-quotients and under d-extensions with indecomposable first term. Let

$$0 \to X \to E_1 \to \cdots \to E_d \to Y \to 0$$

be a minimal d-extension with $X, Y \in \mathcal{U}$. We want to show that $E_i \in \mathcal{U}$ for all $i = 1, \ldots, d$. By Lemma 3.10, it is sufficient to check that $E_1 \in \mathcal{U}$.

If X is indecomposable, we are done by assumption. Suppose hence that $X = X_1 \oplus X_2$, where X_1 is indecomposable and $X_2 \neq 0$. We take a d-pushout of the d-exact sequence along

the projection $\pi\colon X \to X_1$. This yields a commutative diagram

$$\begin{tikzcd}
 & X_2 \arrow[r, "\cong"] \arrow[d] & X_2 \arrow[d, "\iota"] & & & & \\
0 \arrow[r] & X \arrow[r] \arrow[d, "\pi"] & E_1 \arrow[r] \arrow[d, "g"] & \cdots \arrow[r] & E_d \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & F_d \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is d-exact and can be assumed to be minimal by Lemma 2.10. In the diagram, we have included the kernels of π and g, which are isomorphic. As X_1 is indecomposable, we have $F_1 \in \mathcal{U}$.

We claim that g is left minimal. Indeed, the morphism $X_1 \to F_1$ is left minimal since the lower sequence is a minimal d-extension. Since $\pi\colon X \to X_1$ is an epimorphism, the composite $X \xrightarrow{\pi} X_1 \to F_1$ is also left minimal. Hence the composite $X \to E_1 \xrightarrow{g} F_1$ is left minimal, so g must be left minimal. It follows that $g\colon E_1 \to F_1$ is the first morphism in the minimal d-cokernel of $\iota\colon X_2 \to E_1$. Using that ι is a monomorphism, this yields a d-exact sequence

$$0 \to X_2 \xrightarrow{\iota} E_1 \xrightarrow{g} F_1 \to G_2 \to \cdots \to G_d \to 0.$$

All morphisms but ι and g are in the radical by the definition of a minimal d-cokernel. We now show that we may also assume that g is in the radical, and thus that the d-extension above is minimal. Otherwise, we know by Remark 2.7 that the d-extension is isomorphic to

$$0 \to X_2 \xrightarrow{\iota' = \left(\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right)} E_1' \oplus E_1'' \xrightarrow{\left(\begin{smallmatrix} g' & 0 \\ 0 & 1 \end{smallmatrix}\right)} F_1' \oplus E_1'' \to G_2 \to \cdots \to G_d \to 0,$$

where ι' is ι composed with an isomorphism and the exact sequence

$$0 \to X_2 \xrightarrow{j_1} E_1' \xrightarrow{g'} F_1' \to G_2 \to \cdots \to G_d \to 0$$

is a minimal d-extension. As $F_1 \cong F_1' \oplus E_1'' \in \mathcal{U}$, we have $E_1'' \in \mathcal{U}$, and so it follows that $E_1 \in \mathcal{U}$ if and only if $E_1' \in \mathcal{U}$. Consequently, it suffices to assume that g was already in the radical. Now, the sequence $G_2 \to \cdots \to G_d \to 0$ is a minimal d-cokernel of g by construction, and so, since $F_1 \in \mathcal{U}$ and $\mathcal{U}$ is closed under d-quotients, we see that $G_2, \ldots, G_d \in \mathcal{U}$.

Obviously, the object X_2 has fewer indecomposable summands than X. If X_2 is indecomposable, we know that $E_1 \in \mathcal{U}$, as the end terms X_2 and G_d of the above d-extension are in $\mathcal{U}$. If not, we repeat the argument to eventually show that $E_1 \in \mathcal{U}$. □

We are now ready to give the proof of Theorem 3.20.

Proof of Theorem 3.20. Suppose that we have a minimal d-extension

$$0 \to X \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{d-1}} E_d \xrightarrow{e_d} Y \to 0$$

with $X, Y \in \mathcal{U}$. By Lemma 3.21, we may assume that X is indecomposable. Let $Y = \bigoplus_{j=1}^{t} Y_j$, where each Y_j is indecomposable. For each inclusion $\iota_j\colon Y_j \to Y$, consider a d-pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f_{j,0}"] \arrow[d, equal, "h_{j,0}"] & F_{j,1} \arrow[r, "f_{j,1}"] \arrow[d, "h_{j,1}"] & \cdots \arrow[r, "f_{j,d-1}"] & F_{j,d} \arrow[r, "f_{j,d}"] \arrow[d, "h_{j,d}"] & Y_j \arrow[r] \arrow[d, "\iota_j"] & 0 \\
0 \arrow[r] & X \arrow[r, "e_0"] & E_1 \arrow[r, "e_1"] & \cdots \arrow[r, "e_{d-1}"] & E_d \arrow[r, "e_d"] & Y \arrow[r] & 0.
\end{tikzcd} \tag{4}$$

By the dual of Lemma 2.10, the top d-extension can be chosen to be minimal. Consequently, each of the morphisms $f_{j,1}, \ldots, f_{j,d-1}$ is in the Jacobson radical. Moreover, the middle objects $F_{j,1}, \ldots, F_{j,d}$ are in $\mathcal{U}$ since $\mathcal{U}$ is closed under d-extensions between indecomposables.

Now look at the d-extension

$$0 \to \bigoplus_{j=1}^{t} X \xrightarrow{f_0} \bigoplus_{j=1}^{t} F_{j,1} \xrightarrow{f_1} \cdots \xrightarrow{f_{d-1}} \bigoplus_{j=1}^{t} F_{j,d} \xrightarrow{f_d} \bigoplus_{j=1}^{t} Y_j \to 0$$

given by the direct sum of all the upper d-extensions obtained as in (4). Consider the induced map

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j=1}^{t} X \arrow[r, "f_0"] \arrow[d, "h_0"] & \bigoplus_{j=1}^{t} F_{j,1} \arrow[r, "f_1"] \arrow[d, "h_1"] & \cdots \arrow[r, "f_{d-1}"] & \bigoplus_{j=1}^{t} F_{j,d} \arrow[r, "f_d"] \arrow[d, "h_d"] & \bigoplus_{j=1}^{t} Y_j \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "e_0"] & E_1 \arrow[r, "e_1"] & \cdots \arrow[r, "e_{d-1}"] & E_d \arrow[r, "e_d"] & Y \arrow[r] & 0
\end{tikzcd}$$

with $h_i = \left(h_{1,i} \ \cdots \ h_{t,i} \right)$ for $i = 0, \ldots, d$. This is a d-pushout diagram by [J2, Proposition 4.8], and thus the associated mapping cone

$$0 \to \bigoplus_{j=1}^{t} X \xrightarrow{\left(\begin{smallmatrix} h_0 \\ -f_0 \end{smallmatrix}\right)} X \oplus \bigoplus_{j=1}^{t} F_{j,1} \xrightarrow{\left(\begin{smallmatrix} e_0 & h_1 \\ 0 & -f_1 \end{smallmatrix}\right)} E_1 \oplus \bigoplus_{j=1}^{t} F_{j,2} \to \cdots \to E_d \to 0 \tag{5}$$

is a d-extension. Note that the term $X \oplus \bigoplus_{j=1}^{t} F_{j,1}$ lies in $\mathcal{U}$. If (5) is given by the minimal d-cokernel of the first morphism, we are hence done by closure under minimal d-quotients. So suppose this d-cokernel is not minimal. By Remark 2.7, it is then isomorphic to the direct sum of the minimal d-cokernel and shifted complexes of the form $N \xrightarrow{1} N$. In particular, if

$$X \oplus \bigoplus_{j=1}^{t} F_{j,1} \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_2} M_2 \to \cdots \to M_d \to 0$$

is the minimal d-cokernel of $\left(\begin{smallmatrix} h_0 \\ -f_0 \end{smallmatrix}\right)$, then there is a commutative diagram

$$\begin{tikzcd}
E_1 \oplus \bigoplus_{j=1}^{t} F_{j,2} \arrow[r, "\left(\begin{smallmatrix} e_1 & h_2 \\ 0 & -f_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"] & E_2 \oplus \bigoplus_{j=1}^{t} F_{j,3} \arrow[d, "\phi"] \\
M_1 \oplus N_1 \arrow[r, "\left(\begin{smallmatrix} \partial_2 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & M_2 \oplus N_1 \oplus N_2,
\end{tikzcd}$$

where the vertical maps are isomorphisms. Since $e_1 \in \operatorname{Rad}_{\mathcal{A}}(E_1, E_2)$, the morphism

$$\phi \circ \begin{pmatrix} e_1 \\ 0 \end{pmatrix} : E_1 \to M_2 \oplus N_1 \oplus N_2$$

also lies in the radical, and thus so does

$$\phi \circ \begin{pmatrix} e_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \partial_2 \circ a \\ c \\ 0 \end{pmatrix}.$$

This shows that $c \in \operatorname{Rad}_{\mathcal{A}}(E_1, N_1)$. Now let $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ denote the inverse of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have $\alpha a + \beta c = 1_{E_1}$, or equivalently $\alpha a = 1_{E_1} - \beta c$. It follows from the definition of the radical that this is an isomorphism, as $c \in \operatorname{Rad}_{\mathcal{A}}(E_1, N_1)$. This implies that E_1 is a direct summand of M_1. But $M_1 \in \mathcal{U}$ since $\mathcal{U}$ is closed under minimal d-quotients, and hence E_1 also lies in $\mathcal{U}$. It then follows from Lemma 3.10 that $E_2, \ldots, E_d \in \mathcal{U}$, so we can conclude that $\mathcal{U}$ is closed under d-extensions as required. $\square$

4. The lattice of d-torsion classes

The torsion classes in $\mathcal{A}$ form a complete lattice with meet given by intersection, see e.g. [IRTT, Proposition 2.3]. In this section, we use the characterisation of higher torsion classes given in Theorem 3.17 to show that an analogous statement holds for higher torsion classes.

Let us first recall some relevant definitions.

Definition 4.1. Let P be a poset. For an arbitrary subset $H \subseteq P$, the *join* of H, if it exists, is the least upper bound of H. Dually, the *meet* of H, if it exists, is the greatest lower bound of H. The poset P is a *complete lattice* if for any subset $H \subseteq P$, the join and the meet of H exist.

For the sake of clarity, note that a least upper bound is unique as it is smaller than any other upper bound, and similarly for greatest lower bounds. Note also that a complete lattice is bounded, i.e. it has a minimum and a maximum, obtained by letting H in the definition be empty. The following lemma is well-known, see e.g. [G, Lemma I.3.34].

Lemma 4.2. *Let P be a poset. If any subset of P admits a meet or if any subset of P admits a join, then P is a complete lattice.*

For a d-cluster tilting subcategory $\mathcal{M}$ of $\mathcal{A}$, we let d-tors$(\mathcal{M})$ denote the poset of d-torsion classes in $\mathcal{M}$ ordered by inclusion.

Theorem 4.3. *Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{A}$. Then d-tors$(\mathcal{M})$ is a complete lattice with meet given by intersection.*

Proof. To show that d-tors$(\mathcal{M})$ is a complete lattice, it suffices to show that it has arbitrary meets by Lemma 4.2. We note that if d-tors$(\mathcal{M})$ is closed under arbitrary intersections, then meets are given by intersections, and closure under arbitrary meets follows. Therefore, we only need to show that for any subset S of d-tors$(\mathcal{M})$, the intersection

$$\mathcal{V} := \bigcap_{\mathcal{U} \in S} \mathcal{U}$$

is a d-torsion class. By Theorem 3.17 and Remark 3.8, it is enough to show that $\mathcal{V}$ is closed under minimal d-extensions and minimal d-quotients. This follows from the fact that each $\mathcal{U} \in S$ is closed under d-extensions and d-quotients by Theorem 3.17. □

We illustrate the lattice structure on the set of higher torsion classes in our running example.

Example 4.4. In the setting of Example 2.4, the set of all 2-torsion classes in $\mathcal{M}$ is listed in Table 1. By Theorem 4.3, we know that the poset of all 2-torsion classes in $\mathcal{M}$ ordered by inclusion forms a complete lattice. We include the Hasse diagram in Figure 2.

Another example, namely the complete lattice of 3-torsion classes for the higher Auslander algebra A_3^3, can be found in Example 5.21. This example will demonstrate that, unlike in the classical setting, the lattice of d-torsion classes may not be Hasse-regular or semi-distributive (see e.g. [DIRRT] for the definitions).

It follows from Theorem 2.15 that there is an injective, order-preserving map

$$d\text{-tors}(\mathcal{M}) \to \text{tors}(\mathcal{A}) \tag{6}$$

which takes a d-torsion class $\mathcal{U} \subseteq \mathcal{M}$ to the smallest torsion class in $\mathcal{A}$ containing $\mathcal{U}$ (see [AJST, Corollary 3.3]). We now give an example that demonstrates that this map is not a morphism of lattices.

Example 4.5. Continuing with Example 4.4, we see that the 2-torsion classes add$\{3\}$ and add$\{1\}$ are both torsion classes in $\mathcal{A}$, and thus are sent to themselves under the map (6). However, Figure 2 shows that the join in 2-tors$(\mathcal{M})$ is $\mathcal{M}$, which is sent to $\mathcal{A}$ under (6), while the join in tors$(\mathcal{A})$ is simply add$\{3 \oplus 1\}$.

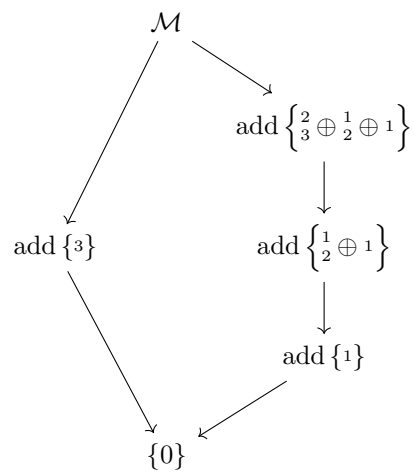

FIGURE 2. The Hasse diagram of the lattice of 2-torsion classes in Example 4.4.

We finish by investigating when the intersection with a d-cluster tilting subcategory gives a map of posets. The result will be applied in the context of higher Nakayama algebras in Section 6.

Proposition 4.6. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be abelian categories of finite length, and let $\mathcal{M}_1 \subseteq \mathcal{A}_1$ and $\mathcal{M}_2 \subseteq \mathcal{A}_2$ be d-cluster tilting subcategories. Assume we have an exact inclusion $\mathcal{A}_2 \subseteq \mathcal{A}_1$ such that $\mathcal{M}_2 \subseteq \mathcal{M}_1$. The following statements hold:*

(1) *If $\mathcal{U}$ is a d-torsion class in $\mathcal{M}_1$, then $\mathcal{U} \cap \mathcal{M}_2$ is a d-torsion class in $\mathcal{M}_2$.*
(2) *Intersecting with $\mathcal{M}_2$ gives a map of posets*
$$d\text{-tors}(\mathcal{M}_1) \to d\text{-tors}(\mathcal{M}_2)$$
which preserves meets.
(3) *If $\mathcal{A}_2$ is closed under quotients in $\mathcal{A}_1$, then $\mathcal{M}_2$ is closed under d-quotients in $\mathcal{M}_1$.*

Proof. Assume that $\mathcal{U}$ is a d-torsion class in $\mathcal{M}_1$. Since the inclusion $\mathcal{A}_2 \subseteq \mathcal{A}_1$ is exact, it must send d-cokernels and d-kernels in $\mathcal{M}_2$ to d-cokernels and d-kernels in $\mathcal{M}_1$, respectively. In particular, it preserves d-quotients and d-extensions. Therefore, the subcategory $\mathcal{U} \cap \mathcal{M}_2$ must be closed under d-extensions and d-quotients in $\mathcal{M}_2$, since $\mathcal{U}$ is closed under d-extensions and d-quotients in $\mathcal{M}_1$. This proves (1).

Part (2) follows from (1) and the fact that meets in d-tors$(\mathcal{M}_1)$ and d-tors$(\mathcal{M}_2)$ are given by intersection.

For part (3), note that giving a d-quotient in $\mathcal{M}_1$ of an object $X \in \mathcal{M}_2$ is the same as giving a d-cokernel in $\mathcal{M}_1$ of a morphism $Y \to X$ with $Y \in \mathcal{M}_1$. This is again equivalent to giving an exact sequence
$$0 \to C \to M_1 \to M_2 \to \cdots \to M_d \to 0$$
where C is the cokernel of $Y \to X$ and each M_i is in $\mathcal{M}_1$. Now since $\mathcal{A}_2$ is closed under quotients in $\mathcal{A}_1$, the cokernel C must be contained in $\mathcal{A}_2$. As $\mathcal{M}_2$ is d-cluster tilting in $\mathcal{A}_2$, we can construct an exact sequence
$$0 \to C \to N_1 \to N_2 \to \cdots \to N_d \to 0$$
where each N_i is in $\mathcal{M}_2$, see [J2, Proposition 3.17]. Since $\mathcal{M}_2 \subseteq \mathcal{M}_1$, this must give a d-cokernel of $Y \to X$ in $\mathcal{M}_1$ by the observation above. This proves the claim. □

Remark 4.7. Note that if $\mathcal{A}_2$ is closed under quotients in $\mathcal{A}_1$, it follows from Proposition 4.6(3) that any minimal d-cokernel in $\mathcal{M}_1$ of a morphism $X \to Y$ with $Y \in \mathcal{M}_2$ is a minimal d-cokernel

of a morphism in $\mathcal{M}_2$. In particular, a subcategory of $\mathcal{M}_2$ is closed under d-quotients in $\mathcal{M}_2$ if and only if it is closed under d-quotients in $\mathcal{M}_1$.

5. Higher Auslander algebras

In this section, we apply Theorem 3.17 to classify and count the d-torsion classes associated to higher Auslander algebras of type $\mathbb{A}$. Higher Auslander algebras were introduced in [I3] and constitute an important class of algebras in higher homological algebra. The module category of each such algebra contains a d-cluster tilting subcategory, which was described combinatorially in [I3] and [OT].

Recall from Theorem 3.17 that d-torsion classes in a d-cluster tilting subcategory are precisely the subcategories which are closed under d-extensions and d-quotients. In Sections 5.1 and 5.2 we present results on closure under d-extensions and d-quotients for higher Auslander algebras of type $\mathbb{A}$, culiminating in a combinatorial characterisation of their higher torsion classes in Theorem 5.13. In Section 5.3 we employ our results to write an algorithm which computes and counts all these d-torsion classes.

5.1. **Background on higher Auslander algebras.** We start by providing a brief introduction to higher Auslander algebras, highlighting combinatorial descriptions which will be important throughout Section 5. We mostly follow the notation and terminology from [JKPK].

For positive integers n and d, let $N_n = \{0, 1, \dots, n-1\}$ with the natural poset structure. Consider the set

$$N_n^d = \underbrace{N_n \times \cdots \times N_n}_{d \text{ times}}$$

of d-tuples $x = (x_0, \dots, x_{d-1})$ over N_n. We endow N_n^d with the product order, meaning that $x \leq y$ in N_n^d if and only if $x_i \leq y_i$ for all $i = 0, 1, \dots, d-1$. We can also consider N_n^d as a category whose objects are the elements of N_n^d, and whose morphisms are given by the poset relations of N_n^d. Taking the $\Bbbk$-linearisation of this category, we get a finite-dimensional $\Bbbk$-algebra, see [JKPK, Section 1.2] for more details. By abuse of notation, we also denote this algebra by N_n^d.

Let os_n^d be the subset of N_n^d of non-decreasing d-tuples over N_n. In particular, an element of os_n^d is a tuple $x = (x_0, \dots, x_{d-1})$ with $x_0 \leq x_1 \leq \cdots \leq x_{d-1}$. The *higher Auslander algebra* A_n^d is defined as the idempotent quotient

$$A_n^d := N_n^d / (N_n^d \setminus \mathrm{os}_n^d),$$

where we consider N_n^d as a finite-dimensional $\Bbbk$-algebra as above. Note that A_n^d is equivalently given by the opposite of a quiver $Q^{n,d}$ whose vertices are the elements of the set os_n^d, and where there is an arrow from vertex x to vertex y if we have $y_i = x_i + 1$ for exactly one $0 \leq i \leq d-1$ and $y_j = x_j$ for $j \neq i$. The relations of A_n^d are given by an admissible ideal $I_{n,d}$ making squares commutative and sending certain compositions of two arrows to zero, see [HJ].

Remark 5.1. The notation we use is similar to that in [JKPK]. It relates to the notation in [HJ] in the following way: What we call $Q^{n,d}$, A_n^d and $I_{n,d}$ corresponds to what is denoted by $Q^{n,d-1}$, A_n^{d-1} and $I_{n,d-1}$ in [HJ]. To see this, note that the poset os_n^d is isomorphic to the poset $\mathcal{V}_{n,d-1}$ of *increasing* d-tuples $x' = (x'_0, \dots, x'_{d-1})$ over $\{1, 2, \dots, n+d-1\}$ used in [HJ]. The isomorphism is given by

$$\mathrm{os}_n^d \to \mathcal{V}_{n,d-1} \quad (x_0, \dots, x_{d-1}) \mapsto (x_0 + 1, x_1 + 2, \dots, x_{d-1} + d).$$

The module category of A_n^d has a d-cluster tilting subcategory

$$\mathcal{M}_n^d := \mathrm{add}(M_n^d) \subseteq \mathrm{mod}\, A_n^d$$

where $M_n^d = \bigoplus_{x \in \mathrm{os}_n^{d+1}} M_x$. Here, the notation M_x is used for the indecomposable A_n^d-module with support in all vertices $y \in \mathrm{os}_n^d$ such that $x_0 \leq y_0 \leq x_1 \leq \cdots \leq x_{d-1} \leq y_{d-1} \leq x_d$. Note that the d-cluster tilting subcategory $\mathcal{M}_n^d$ contains finitely many indecomposable objects, indexed by os_n^{d+1}. It is known that $\mathrm{End}_{A_n^d}(M_n^d)$ and A_n^{d+1} are isomorphic as algebras by [I3,

Corollary 1.16], see also [JKPK, Theorem 2.3]. For examples of the quivers $Q^{n,d}$ and relevant modules, see [JKPK, Section 2.1] and [HJ, Example 2.13].

Next we define the relation

$$x \rightsquigarrow y \text{ if and only if } x_0 \leq y_0 \leq x_1 \leq y_1 \leq \cdots \leq x_d \leq y_d$$

on the set $\mathbb{Z}^{d+1}$ of all $(d+1)$-tuples over $\mathbb{Z}$. Note that $x \rightsquigarrow y$ implies $x \leq y$. Using the relation $\rightsquigarrow$, one can determine the Hom-spaces between indecomposable modules in $\mathcal{M}_n^d$.

Proposition 5.2. [OT, Theorem 3.6(3)], [JKPK, Proposition 2.8] *Let $x, y \in \mathrm{os}_n^{d+1}$. Then*

$$\dim \mathrm{Hom}_{A_n^d}(M_x, M_y) = \begin{cases} 1 & \text{if } x \rightsquigarrow y \\ 0 & \text{otherwise.} \end{cases}$$

The d-extensions in $\mathcal{M}_n^d$ have a similar combinatorial description, including a description of all the middle terms. To this end, we define $\tau_d \colon \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}$ by

$$\tau_d(x_0, \ldots, x_d) = (x_0 - 1, x_1 - 1, \ldots, x_d - 1).$$

The notation is motivated by the fact that if $x \in \mathrm{os}_n^{d+1}$ with $x_0 > 0$, then $\tau_d(M_x) \cong M_{\tau_d(x)}$ by [JKPK, Proposition 2.7 (iii)], where $\tau_d(M_x)$ is the higher Auslander–Reiten translate of M_x.

Proposition 5.3. [OT, Theorem 3.6(4) and 3.8], [JKPK, Proposition 2.8] *Let $x, y \in \mathrm{os}_n^{d+1}$. Then*

$$\dim \mathrm{Ext}^d_{A_n^d}(M_y, M_x) = \begin{cases} 1 & \text{if } x \rightsquigarrow \tau_d(y) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $x \rightsquigarrow \tau_d(y)$, there is a non-trivial d-extension

$$0 \to M_x \to E_1 \to \cdots \to E_d \to M_y \to 0 \tag{7}$$

where $E_k = \bigoplus_{z \in Z_k} M_z$ for

$$Z_k = \{z \in \mathrm{os}_n^{d+1} \mid z_i \in \{x_i, y_i\} \text{ for each } i \text{ and } |\{i \mid z_i = y_i\}| = k\}.$$

Remark 5.4. The d-extension (7) in Proposition 5.3 is minimal. This is seen by combining the fact that E_k and E_{k+1} have no isomorphic direct summands for $k = 1, \ldots, d-1$ with Proposition 2.6 and Remark 2.7.

We will use the following lemma, which can be seen as an immediate consequence of the description of d-extensions in Proposition 5.3.

Lemma 5.5. *Suppose that $x, y \in \mathrm{os}_n^{d+1}$ with $x_i = y_i$ for all $i = 0, \ldots, d-1$ and $x_d \leq y_d$. Then any non-zero morphism $M_x \to M_y$ is a monomorphism.*

Proof. If $x_d = y_d$, the only non-zero morphism (up to multiplication by a scalar) is the identity, which is a monomorphism. If $x_d < y_d$, the result follows from Proposition 5.3 when looking at the extension between M_x and M_z, where $z = (x_1 + 1, \ldots, x_d + 1, y_d)$. □

Although Proposition 5.3 is limited to describing the middle terms in d-extensions with indecomposable end terms, we know from Theorem 3.20 that this knowledge is sufficient for producing d-torsion classes if we have already established closure under d-quotients.

5.2. **A combinatorial characterisation of d-torsion classes in $\mathcal{M}_n^d$.** The main result of this subsection is a characterisation of higher torsion classes associated to higher Auslander algebras of type $\mathbb{A}$, see Theorem 5.13. The key ingredient in the proof of this result is a combinatorial description of how to close a subcategory of $\mathcal{M}_n^d$ under d-quotients.

Given $M \in \mathcal{M}_n^d$, we write $\mathrm{dq}(M)$ for the smallest subcategory of $\mathcal{M}_n^d$ which contains M and is closed under d-quotients (see Definition 3.7). We often refer to $\mathrm{dq}(M)$ as the *d-quotient closure* of M. It is clear that if $N \in \mathrm{dq}(M)$, then $\mathrm{dq}(N) \subseteq \mathrm{dq}(M)$.

We filter $\mathrm{dq}(M)$ as follows:

- Set $\mathrm{dq}(M)_0 = \mathrm{add}(M)$.
- For $i \geq 0$, set

$$\mathrm{dq}(M)_{i+1} = \mathrm{add}\left\{ N \in \mathcal{M}_n^d \;\middle|\; \begin{array}{l} \exists \text{ minimal } d\text{-quotient } X \to Y \to C_1 \to \cdots \to C_d \to 0 \text{ in} \\ \mathcal{M}_{n,d} \text{ with } Y \in \mathrm{dq}(M)_i \text{ and } N \cong C_j \text{ for some } 1 \leq j \leq d \end{array} \right\}.$$

We see that $\mathrm{dq}(M)_0 \subseteq \mathrm{dq}(M)_1 \subseteq \dots$ and that the chain must stabilise with $\mathrm{dq}(M)_t = \mathrm{dq}(M)_{t+1}$ for some $t \in \mathbb{N}$ because $\mathcal{M}_n^d$ has finitely many indecomposables. By definition, we have $\mathrm{dq}(M) = \mathrm{dq}(M)_t$.

To completely determine the subcategory $\mathrm{dq}(M)$, it is sufficient to describe the indecomposable modules it contains. With this in mind, we begin by identifying certain indecomposables which must be contained in d-quotient closures.

Lemma 5.6. *Let $x \in \mathrm{os}_n^{d+1}$ be such that $x_i + 1 \leq x_{i+1}$ for some $0 \leq i \leq d-1$ and set $y = (x_0, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_d)$. Then $M_y \in \mathrm{dq}(M_x)$.*

Proof. Define $z = (z_0, \dots, z_d)$ such that

$$z_j = \begin{cases} x_j & \text{if } j \neq i+1 \\ x_i + 1 & \text{if } j = i+1. \end{cases}$$

We then have $z \rightsquigarrow x$, so there is a non-zero morphism $M_z \to M_x$ by Proposition 5.2. The module M_y is hence in $\mathrm{dq}(M_x)$ by [HJ, Lemma 3.8(2)]. □

Lemma 5.6 yields the following corollary.

Corollary 5.7. *Given any $x \in \mathrm{os}_n^{d+1}$, the set*

$$\left\{ M_y \in \mathcal{M}_n^d \mid y \in \mathrm{os}_n^{d+1},\ x \leq y \text{ and } x_d = y_d \right\}$$

is contained in $\mathrm{dq}(M_x)$.

Proof. Suppose $y \in \mathrm{os}_n^{d+1}$ satisfies $x \leq y$ and $x_d = y_d$. Construct a sequence

$$x = z^0, z^1, z^2, \dots, z^m = y$$

in os_n^{d+1}, where the element z^{i+1} is constructed from z^i as follows. If $z^i = y$, then we are finished. Otherwise, there exists a maximal j such that $z_j^i < y_j$, and we must have $j < d$. Then $z_j^i + 1 \leq y_j \leq y_{j+1} = z_{j+1}^i$ and we define

$$z_k^{i+1} = \begin{cases} z_k^i + 1 & \text{if } k = j \\ z_k^i & \text{if } k \neq j. \end{cases}$$

Notice that $M_{z^{i+1}} \in \mathrm{dq}(M_{z^i})$ by Lemma 5.6 and that this process must terminate with $z^m = y$. Thus, we get

$$M_y \in \mathrm{dq}(M_{z^{m-1}}) \subseteq \cdots \subseteq \mathrm{dq}(M_x)$$

as required. □

We now present the key technical lemma needed to describe $\mathrm{dq}(M_x)$ completely.

Lemma 5.8. *Suppose that $C_0 \to C_1 \to \cdots \to C_d \to 0$ is a minimal d-cokernel of a morphism $C_{-1} \to C_0$ in $\mathcal{M}_n^d$. If $M_z \in \mathrm{add}(C_i)$ for some $1 \leq i \leq d$, then there exists $M_x \in \mathrm{add}(C_0)$ such that $x \leq z$ and $x_d = z_d$.*

Proof. Since each $C_i \in \mathcal{M}_n^d$, we may assume that every C_i is equal to a direct sum of modules of the form M_z for $z \in \mathrm{os}_n^{d+1}$. Choose $M_z \in \mathrm{add}(C_i)$. Write the morphism $C_{i-1} \to C_i$ as

$$\begin{pmatrix} f \\ g \end{pmatrix} : C_{i-1} \to M_z \oplus C_i'.$$

By the construction of minimal d-cokernels, this morphism factors through the cokernel K of the morphism $C_{i-2} \to C_{i-1}$ as indicated in the diagram

$$\begin{tikzcd} C_{i-1} \arrow[rr, "\binom{f}{g}"] \arrow[dr, two heads, "\pi"] & & M_z \oplus C_i', \\ & K \arrow[ur, "\binom{f'}{g'}"'] & \end{tikzcd}$$

where $\binom{f'}{g'}$ is a minimal left $\mathcal{M}_n^d$-approximation of K.

There exists some $M_w \in \operatorname{add}(C_{i-1})$ with a non-zero morphism $M_w \to M_z$. Indeed, if this is not the case, then $f = 0$. However, since π is an epimorphism, this would imply $f' = 0$, contradicting $\binom{f'}{g'}$ being left minimal. Consequently, we may write $C_{i-1} = D \oplus D'$, where $D \neq 0$ and D' is the largest summand of C_{i-1} that maps to zero under f. This means that the morphism $\binom{f}{g}$ may be written as

$$\begin{pmatrix} f_1 & 0 \\ g_1 & g_2 \end{pmatrix} : D \oplus D' \to M_z \oplus C_i'.$$

For every $M_v \in \operatorname{add}(D)$, there is a non-zero morphism to M_z. Hence, Proposition 5.2 shows that $v \rightsquigarrow z$. Since $D \neq 0$, we may choose $M_y \in \operatorname{add}(D)$ with y_d maximal, i.e. for all $M_v \in \operatorname{add}(D)$, we have $v_d \leq y_d$.

With z and y fixed and knowing that $y \rightsquigarrow z$, we may now consider $z' = (z_0, z_1, \ldots, z_{d-1}, y_d)$. Notice that $z' \rightsquigarrow z$. Proposition 5.2 and Lemma 5.5 thus yield that there exists a monomorphism $\iota \colon M_{z'} \to M_z$. Moreover, by the maximality of y_d, we have $v \rightsquigarrow z'$ for all $M_v \in \operatorname{add}(D)$. Hence, by Proposition 5.2, there exists a morphism $h \colon D \to M_{z'}$ such that $f_1 = \iota \circ h$. In particular, the solid part of the diagram

$$\begin{tikzcd} D \oplus D' \arrow[r, "\left(\begin{smallmatrix} h & 0 \\ g_1 & g_2 \end{smallmatrix}\right)"] \arrow[d, two heads, "\pi = \left(\begin{smallmatrix} \pi_1 \\ \pi_2 \end{smallmatrix}\right)"'] & M_{z'} \oplus C_i' \arrow[d, tail, "\left(\begin{smallmatrix} \iota & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\ K \arrow[r, tail, "\binom{f'}{g'}"'] \arrow[ur, dashed, "t"] & M_z \oplus C_i' \end{tikzcd}$$

commutes. Since $\left(\begin{smallmatrix} \iota & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a monomorphism, this implies that the composition of the morphism $C_{i-2} \to C_{i-1}$ and $\left(\begin{smallmatrix} h & 0 \\ g_1 & g_2 \end{smallmatrix}\right)$ is zero. Thus, there is a morphism $t \colon K \to M_{z'} \oplus C_i'$ making the upper triangle in the diagram above commute.

Note that the lower triangle also commutes as π is an epimorphism. Finally, since $\binom{f'}{g'}$ is an $\mathcal{M}_n^d$-approximation, there exists a morphism $s \colon M_z \oplus C_i' \to M_{z'} \oplus C_i'$ such that

$$t = s \circ \begin{pmatrix} f' \\ g' \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} \iota & 0 \\ 0 & 1 \end{pmatrix} \circ t = \begin{pmatrix} \iota & 0 \\ 0 & 1 \end{pmatrix} \circ s \circ \begin{pmatrix} f' \\ g' \end{pmatrix},$$

and thus $\left(\begin{smallmatrix} \iota & 0 \\ 0 & 1 \end{smallmatrix}\right) \circ s$ is an isomorphism, since $\binom{f'}{g'}$ is left minimal. Therefore, the monomorphism ι is also an epimorphism, so it must be an isomorphism and we have $M_z \cong M_{z'}$. This shows that $y_d = z_d$, and hence $M_y \in \operatorname{add}(C_{i-1})$ satisfies both $y \rightsquigarrow z$ (and thus $y \leq z$) and $y_d = z_d$.

We can now repeat the argument with M_y and keep going until we get $M_x \in \operatorname{add}(C_0)$ with $x \leq y \leq z$ and $x_d = y_d = z_d$. □

Lemma 5.8 enables us to fully describe $\operatorname{dq}(M_x)$ for an indecomposable module $M_x \in \mathcal{M}_n^d$.

Corollary 5.9. *Given any $x \in \mathrm{os}_n^{d+1}$, we have*

$$\mathrm{dq}(M_x) = \mathrm{add}\left\{M_y \in \mathcal{M}_n^d \mid y \in \mathrm{os}_n^{d+1},\ x \leq y \text{ and } x_d = y_d\right\}.$$

Proof. By Corollary 5.7, it suffices to show that for any $M_y \in \mathrm{dq}(M_x)$, we must have $x \leq y$ and $x_d = y_d$. Let $i \geq 0$ be such that $M_y \in \mathrm{dq}(M_x)_i$. If $i = 0$, we have $M_y \in \mathrm{dq}(M_x)_0 = \mathrm{add}(M_x)$, so $M_y = M_x$ and the statement holds. Assume hence $i > 0$. By Lemma 5.8, there exists $M_z \in \mathrm{dq}(M_x)_{i-1}$ such that $z \leq y$ and $z_d = y_d$.

Repeating this argument, we will eventually find some $M_w \in \mathrm{dq}(M_x)_0$ such that $w \leq y$ and $w_d = y_d$. As $\mathrm{dq}(M_x)_0 = \mathrm{add}(M_x)$ yields $M_w = M_x$, the result follows. □

Remark 5.10. It follows from Corollary 5.9 that any $M_y, M_z \in \mathrm{dq}(M_x)$ satisfy $y_d = x_d = z_d$, and thus there are no non-trivial d-extensions between them by Proposition 5.3. In particular, the subcategory $\mathrm{dq}(M_x)$ is closed under d-extensions, and it is hence the smallest d-torsion class containing M_x by Theorem 3.17.

We now consider the d-quotient closure of a set of indecomposables.

Proposition 5.11. *Given any subset $I \subseteq \mathrm{os}_n^{d+1}$, suppose that $M_y \in \mathrm{dq}\big(\bigoplus_{x\in I} M_x\big)$. Then $M_y \in \mathrm{dq}(M_x)$ for some $x \in I$.*

Proof. We prove this by induction on the filtration of $\mathrm{dq}(\bigoplus_{x\in I} M_x)$. If

$$M_y \in \mathrm{dq}\left(\bigoplus_{x\in I} M_x\right)_0 = \mathrm{add}\left(\bigoplus_{x\in I} M_x\right),$$

the statement clearly holds.

Now suppose $M_y \in \mathrm{dq}\big(\bigoplus_{x\in I} M_x\big)_i$ for some $i > 0$, and that the result is known for all $M_z \in \mathrm{dq}\big(\bigoplus_{x\in I} M_x\big)_{i-1}$. By construction, there must exist a minimal d-quotient

$$X \xrightarrow{f} Y \to C_1 \to \cdots \to C_d \to 0$$

in $\mathcal{M}_n^d$ of some morphism f such that $Y \in \mathrm{dq}\big(\bigoplus_{x\in I} M_x\big)_{i-1}$ and $M_y \in \mathrm{add}(C_j)$ for some $1 \leq j \leq d$. Hence, combining Lemma 5.8 with Corollary 5.7 shows that there exists

$$M_z \in \mathrm{add}(Y) \subseteq \mathrm{dq}\left(\bigoplus_{x\in I} M_x\right)_{i-1}$$

such that $M_y \in \mathrm{dq}(M_z)$. By the induction hypothesis, we have $M_z \in \mathrm{dq}(M_x)$ for some $x \in I$, and thus it follows that $M_y \in \mathrm{dq}(M_z) \subseteq \mathrm{dq}(M_x)$ as required. □

Using Proposition 5.11, we are able to give a complete description of the d-quotient closure of any module in $\mathcal{M}_n^d$.

Corollary 5.12. *Given a subset $I \subseteq \mathrm{os}_n^{d+1}$, the set of indecomposable modules in $\mathrm{dq}\big(\bigoplus_{x\in I} M_x\big)$ is precisely*

$$\bigcup_{x\in I} \left\{M_y \in \mathcal{M}_n^d \mid y \in \mathrm{os}_n^{d+1},\ x \leq y \text{ and } x_d = y_d\right\}.$$

Proof. Since it is clear that $\mathrm{dq}(M_x) \subseteq \mathrm{dq}\big(\bigoplus_{x\in I} M_x\big)$ for each $x \in I$, this is a direct consequence of Corollary 5.9 and Proposition 5.11. □

For a set of indecomposable modules in $\mathcal{M}_n^d$, Corollary 5.12 gives a purely combinatorial description of all the indecomposables in their d-quotient closure. Unlike what we saw in Remark 5.10, there may now exist non-trivial d-extensions, so the d-quotient closure is not necessarily a d-torsion class. However, using the results developed in this subsection, we give a full combinatorial description of the subsets corresponding to d-torsion classes. Note that

any subcategory of $\mathcal{M}_n^d$ which is closed under direct summands is uniquely determined by its indecomposable modules, i.e. by a subset of os_n^{d+1}. We hence use the notation

$$\mathcal{U}_I := \mathrm{add}\{M_y \in \mathcal{M}_n^d \mid y \in I\}$$

for the subcategory of $\mathcal{M}_n^d$ associated to a subset $I \subseteq \mathrm{os}_n^{d+1}$.

Theorem 5.13. *Consider a subset $I \subseteq \mathrm{os}_n^{d+1}$. The subcategory $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_n^d$ if and only if the following hold for any elements $x, z \in \mathrm{os}_n^{d+1}$:*

(1) *If $x \leq z$ and $x_d = z_d$, then $x \in I$ implies $z \in I$.*
(2) *If $x \rightsquigarrow \tau_d(z)$ and $x, z \in I$, then any $y \in \mathrm{os}_n^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i must be in I.*

Remark 5.14. The product order on N_n^{d+1} restricts to a partial order on the subset

$$\{y \in \mathrm{os}_n^{d+1} \mid y_d = m\} \subseteq N_n^{d+1}.$$

Theorem 5.13 (1) tells us that $\{y \in I \mid y_d = m\}$ is an *upper set* in $\{y \in \mathrm{os}_n^{d+1} \mid y_d = m\}$ for each $m = 0, \ldots, n-1$.

Proof of Theorem 5.13. By Corollary 5.12, condition (1) holds if and only if $\mathcal{U}_I$ is closed under d-quotients. It follows from Proposition 5.3 and Remark 5.4 that condition (2) is equivalent to $\mathcal{U}_I$ being closed under d-extensions by indecomposables. Moreover, if $\mathcal{U}_I$ is closed under d-quotients, it is closed under d-extensions by indecomposables if and only if it is closed under all d-extensions by Theorem 3.20. From Theorem 3.17, we know that $\mathcal{U}_I$ is a d-torsion class if and only if it is closed under d-extensions and d-quotients, which proves the claim. □

5.3. **Computational results.** We are now ready to present two algorithms for explicitly computing higher torsion classes associated to higher Auslander algebras of type $\mathbb{A}$. The first algorithm computes the minimal d-torsion class containing a given module. The second computes all d-torsion classes associated to a higher Auslander algebra. The code for the algorithms is available as a Google Colab notebook.[1]

We extend our notation for the sake of readability. In particular, for an element $x \in \mathrm{os}_n^{d+1}$, we set $\mathrm{dq}(x) = \{y \in \mathrm{os}_n^{d+1} \mid M_y \in \mathrm{dq}(M_x)\}$. By Corollary 5.9, we know that

$$\mathrm{dq}(x) = \left\{y \in \mathrm{os}_n^{d+1} \mid x \leq y \text{ and } x_d = y_d\right\}.$$

Given a set of indecomposable modules $M_{x^1}, \ldots, M_{x^r}$ in $\mathcal{M}_n^d$, we let $\mathcal{U}(M_{x^1}, \ldots, M_{x^r})$ denote the smallest d-torsion class in $\mathcal{M}_n^d$ containing $M_{x^1}, \ldots, M_{x^r}$. For a set $X = \{x^1, \ldots, x^r\} \subseteq \mathrm{os}_n^{d+1}$, we let $U(X)$ be the set of $(d+1)$-tuples corresponding to the indecomposables in $\mathcal{U}(M_{x^1}, \ldots, M_{x^r})$. If the set X is empty, then $U(X)$ is also empty and corresponds to the trivial d-torsion class $\{0\}$.

Algorithm 5.15. Given a set of initial indecomposable modules in $\mathcal{M}_n^d$, or equivalently a subset of os_n^{d+1}, this algorithm computes the minimal d-torsion class containing those modules.

Input: Integers $d \geq 1$, $n \geq 1$ and a set $X = \{x^1, \ldots, x^r\} \subseteq \mathrm{os}_n^{d+1}$.

(1) Let $I = X$.
(2) For each pair $x, y \in I$ such that $x \rightsquigarrow \tau_d(y)$, add the $(d+1)$-tuple $(x_0, \ldots, x_{d-1}, y_d)$ to I.
(3) For every $x \in I$, add the elements of $\mathrm{dq}(x)$ to I.
(4) If new elements were added to I in step (2) or (3), repeat from step (2). Otherwise, terminate the process.

Output: The set I.

Since os_n^{d+1} is a finite set, Algorithm 5.15 will always terminate and give a subset $I \subseteq \mathrm{os}_n^{d+1}$ as output. Recall that we use the notation

$$\mathcal{U}_I := \mathrm{add}\{M_y \in \mathcal{M}_n^d \mid y \in I\}$$

for the corresponding subcategory of $\mathcal{M}_n^d$. Proposition 5.16 shows that the set I produced in Algorithm 5.15 indeed corresponds to the minimal d-torsion class containing the indecomposable modules we started with.

[1] https://colab.research.google.com/drive/172Q-UZHvdPOhngGkl1T_xdYLzntg31dY

Proposition 5.16. *The set I constructed in Algorithm 5.15 satisfies $\mathcal{U}_I = \mathcal{U}(M_{x^1}, \dots, M_{x^r})$.*

Proof. We need to show that $\mathcal{U}_I$ is the minimal d-torsion class containing $M_{x^1}, \dots, M_{x^r}$. Step (1) of the algorithm ensures that $M_{x^1}, \dots, M_{x^r} \in \mathcal{U}_I$. By Proposition 5.11, step (3) implies that $\mathcal{U}_I$ is closed under d-quotients.

Consider two indecomposable modules $M_x, M_y \in \mathcal{U}_I$ with $\operatorname{Ext}^d_{A_n^d}(M_y, M_x) \neq 0$. This means that the pair $x, y \in I$ satisfies $x \rightsquigarrow \tau_d(y)$ by Proposition 5.3. Moreover, if M_z is a direct summand in one of the middle terms in the non-trivial d-extension from M_x to M_y described in Proposition 5.3, then either $z \in \operatorname{dq}(x)$ or $z \in \operatorname{dq}((x_1, \dots, x_{d-1}, y_d))$. Step (2) followed by step (3) thus ensures that $M_z \in \mathcal{U}_I$, so $\mathcal{U}_I$ is closed under d-extensions with indecomposable end terms. By Theorem 3.20, this implies that $\mathcal{U}_I$ is closed under all d-extensions. We can hence conclude that $\mathcal{U}_I$ is a d-torsion class by Theorem 3.17.

We now know that $\mathcal{U}_I$ is a d-torsion class containing the modules $M_{x^1}, \dots, M_{x^r}$. However, objects added to I in the algorithm corresponds to either $M_{x^1}, \dots, M_{x^r}$ or to indecomposable direct summands obtained from minimal d-quotients or minimal d-extensions, see Remark 5.4, and the result follows. □

Remark 5.17. An improvement of Algorithm 5.15, omitted for the sake of readability, is that on subsequent iterations it suffices to only consider $(d+1)$-tuples added in the previous iteration in step (3). Similarly, in step (2), one only needs to to consider pairs where at least one $(d+1)$-tuple was added in the previous iteration.

Building on Algorithm 5.15, we give an algorithm that determines all d-torsion classes in $\mathcal{M}_n^d$.

Algorithm 5.18. This algorithm computes all higher torsion classes associated to a higher Auslander algebra.

Input: Integers $d \geq 1$, $n \geq 1$.

(1) Let $\mathfrak{U}$ be the singleton set containing the empty set. Set $l = 1$.
(2) For all sets X consisting of l distinct $(d+1)$-tuples in $\operatorname{os}_n^{d+1}$, compute $U(X)$ using Algorithm 5.15 and add it to $\mathfrak{U}$.
(3) If new elements were added to $\mathfrak{U}$ in step (2), increase l by one and repeat from step (2). Otherwise, terminate the process.

Output: The set $\mathfrak{U}$.

Similarly as in the case of Algorithm 5.15, it should be noted that the algorithm above must terminate as $\operatorname{os}_n^{d+1}$ is a finite set.

Proposition 5.19. *The d-torsion classes in $\mathcal{M}_n^d$ are indexed by $\mathfrak{U}$. In other words,*

$$d\text{-tors}(\mathcal{M}_n^d) = \{\mathcal{U}_I \mid I \in \mathfrak{U}\}.$$

Proof. First observe that the trivial d-torsion class $\{0\} = \mathcal{U}_\emptyset$, and $\emptyset \in \mathfrak{U}$. As $\mathcal{M}_n^d$ has finitely many indecomposable objects, any non-trivial d-torsion class $\mathcal{V}$ in $\mathcal{M}_n^d$ can be written as $\mathcal{V} = \mathcal{U}(M_{x^1}, \dots, M_{x^m})$ for some positive integer m.

Let r be the lowest positive integer for which any d-torsion class of the form $\mathcal{U}(M_{x^1}, \dots, M_{x^r})$ can also be written as $\mathcal{U}(M_{y^1}, \dots, M_{y^s})$ for some $s < r$. By construction, the set $\mathfrak{U}$ indexes all d-torsion classes of the form $\mathcal{U}(M_{x^1}, \dots, M_{x^l})$ for $l < r$.

Suppose that $\mathcal{V}$ is a non-trivial d-torsion class, and let $m \geq 0$ be minimal such that

$$\mathcal{V} = \mathcal{U}(M_{x^1}, \dots, M_{x^m}).$$

We claim that $\mathcal{V} = \mathcal{U}_I$ for some $I \in \mathfrak{U}$, i.e. that $m < r$. Indeed, if $m \geq r$, consider the d-torsion class $\mathcal{V}' = \mathcal{U}(M_{x^1}, \dots, M_{x^r})$. By the assumption on r, we can write $\mathcal{V}'$ as $\mathcal{U}(M_{z^1}, \dots, M_{z^s})$ for some $s < r$. But then $\mathcal{V}$ is of the form $\mathcal{U}(M_{z^1}, \dots M_{z^s}, M_{x^{r+1}}, \dots M_{x^m})$, and can hence be generated by $m + s - r < m$ indecomposable modules. This contradicts the minimality of m, and the result follows. □

Remark 5.20. As before, Algorithm 5.18 is presented in its simplest form for the sake of readability. For efficient computations, we use the following improvements:

- We precalculate whether $x \rightsquigarrow y$ and $x \rightsquigarrow \tau_d(y)$ for all $x, y \in \mathrm{os}_n^{d+1}$.
- If a set X contains x, y with $y \in \mathrm{dq}(x)$, then $U(X)$ has already been added to $\mathfrak{U}$, so we skip the computation of $U(X)$ in step (2).

Using Algorithm 5.18, we can compute the number of higher torsion classes associated to a higher Auslander algebra. These computational results are summarised in Table 2.

		n					
		1	2	3	4	5	6
d	1	2	5	14	42	132	429
	2	2	6	25	140	1036	10040
	3	2	7	46	643	22224	
	4	2	8	87	4147		
	5	2	9	168	36543		
	6	2	10	329	427527		
	7	2	11	650			
	8	2	12	1291			

TABLE 2. The number of d-torsion classes in the d-cluster tilting subcategory $\mathcal{M}_n^d$ of the higher Auslander algebra A_n^d.

In addition to computing the full set of d-torsion classes in $\mathcal{M}_n^d$, our code also produces the associated Hasse diagram. Note that it gives a fully annotated version of the Hasse diagram, specifying the indecomposable modules contained in each d-torsion class.

Example 5.21. Consider the higher Auslander algebra A_3^3. The Hasse diagram of the 3-torsion classes in the 3-cluster tilting subcategory $\mathcal{M}_3^3$ is shown in Figure 3. We note that the vertices labelled w, x and v have valency 3, 4 and 5, respectively, so the lattice is not Hasse-regular. We moreover note that the lattice is not semi-distributive. Indeed, using the notation $\vee$ for join and $\wedge$ for meet, we see that $x \wedge y = v = x \wedge z$, but that $x \wedge (y \vee z) = x \wedge w = x \neq v$.

6. Higher Nakayama algebras

Higher Nakayama algebras were introduced in [JKPK] as a higher-dimensional generalisation of classical Nakayama algebras. In this section we extend the combinatorial description of higher torsion classes from Theorem 5.13 to the setup of higher Nakayama algebras. We first consider higher Nakayama algebras of type $\mathbb{A}$ in Section 6.1, before moving on to type $\mathbb{A}_\infty^\infty$ in Section 6.2.

6.1. Higher Nakayama algebras of type $\mathbb{A}$.

We start by giving a brief introduction to the construction of higher Nakayama algebras of type $\mathbb{A}$. Let n and d be positive integers, and recall the definitions of os_n^d, A_n^d, $\mathcal{M}_n^d$ and M_x from Section 5.1.

A *(connected) Kupisch series of type* $\mathbb{A}_n$ is a tuple $\underline{\ell} = (\ell_0, \ell_1, \ldots, \ell_{n-1})$ of positive integers satisfying

$$\ell_0 = 1 \quad \text{and} \quad 2 \leq \ell_i \leq \ell_{i-1} + 1 \quad \text{for} \quad i = 1, \ldots, n-1.$$

Given such a Kupisch series $\underline{\ell}$, consider the subset

$$\mathrm{os}_{\underline{\ell}}^{d+1} := \{y \in \mathrm{os}_n^{d+1} \mid \ell\ell(y) \leq \ell_{y_d}\}$$

where $\ell\ell(y) = y_d - y_0 + 1$. Note that $\ell\ell(y)$ is equal to the Loewy length of the module M_y in $\operatorname{mod} A_n^d$, see [JKPK, Lemma 2.9]. For $d \geq 2$, the *d-th Nakayama algebra with Kupisch series* $\underline{\ell}$ is the idempotent quotient

$$A_{\underline{\ell}}^d := A_n^d / (\mathrm{os}_n^d \setminus \mathrm{os}_{\underline{\ell}}^d).$$

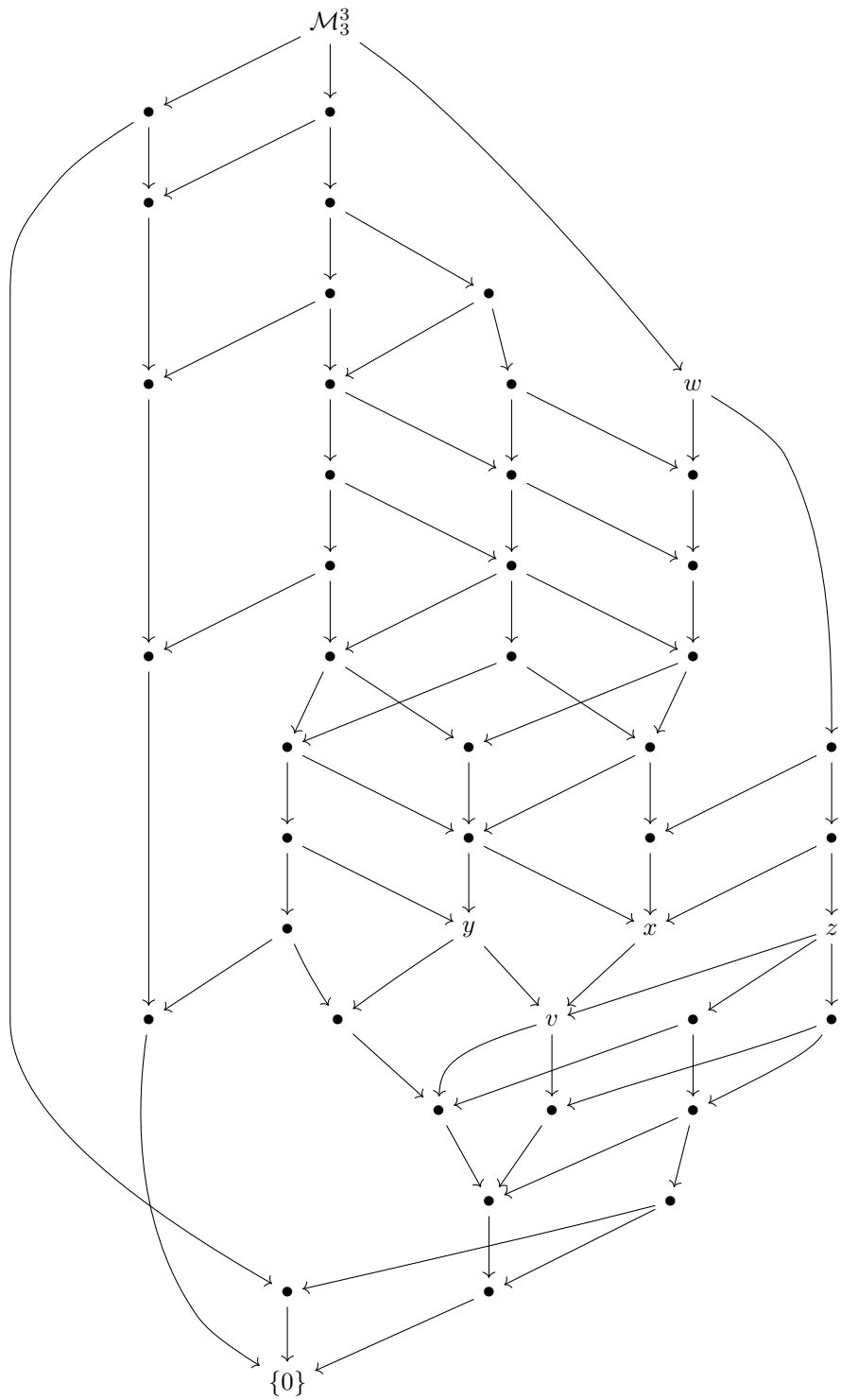

FIGURE 3. The Hasse diagram of the 3-torsion classes in the 3-cluster tilting subcategory $\mathcal{M}_3^3$ of the higher Auslander Algebra A_3^3.

In other words, if we let $e_{\underline{\ell}}$ be the idempotent consisting of the sum of the vertices in $\mathrm{os}_n^d \setminus \mathrm{os}_{\underline{\ell}}^d$, then $A_{\underline{\ell}}^d$ is isomorphic to $A_n^d / A_n^d e_{\underline{\ell}} A_n^d$. It follows from [JKPK, Proposition 2.24] that the subcategory $\mathcal{M}_{\underline{\ell}}^d := \mathcal{M}_n^d \cap \operatorname{mod} A_{\underline{\ell}}^d$ is d-cluster tilting in $\operatorname{mod} A_{\underline{\ell}}^d$, and that $\mathcal{M}_{\underline{\ell}}^d = \operatorname{add}(M_{\underline{\ell}}^d)$ for $M_{\underline{\ell}}^d = \bigoplus_{x \in \mathrm{os}_{\underline{\ell}}^{d+1}} M_x$.

Note that the isomorphism classes of indecomposable modules in $\mathcal{M}_{\underline{\ell}}^d$ are in bijection with elements of $\mathrm{os}_{\underline{\ell}}^{d+1}$. Using our results, we can characterise the subsets of $\mathrm{os}_{\underline{\ell}}^{d+1}$ which correspond to higher torsion classes in $\mathcal{M}_{\underline{\ell}}^d$. We use the notation

$$\mathcal{U}_I := \operatorname{add}\{M_y \in \mathcal{M}_{\underline{\ell}}^d \mid y \in I\}$$

for the subcategory of $\mathcal{M}_{\underline{\ell}}^d$ associated to a subset $I \subseteq \mathrm{os}_{\underline{\ell}}^{d+1}$.

Theorem 6.1. *Let $\underline{\ell}$ be a Kupisch series of type $\mathbb{A}_n$ and consider a subset $I \subseteq \mathrm{os}_{\underline{\ell}}^{d+1}$. The subcategory $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}}^d$ if and only if the following hold for any elements $x, z \in \mathrm{os}_{\underline{\ell}}^{d+1}$:*

1. *If $x \leq z$ and $x_d = z_d$, then $x \in I$ implies $z \in I$.*
2. *If $x \rightsquigarrow \tau_d(z)$ and $x, z \in I$, then any $y \in \mathrm{os}_{\underline{\ell}}^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i must be in I.*

Proof. By Corollary 5.12, condition (1) is equivalent to $\mathcal{U}_I$ being closed under d-quotients in $\mathcal{M}_n^d$. For objects in $\mathcal{M}_{\underline{\ell}}^d$, the minimal d-quotients in $\mathcal{M}_{\underline{\ell}}^d$ are the same as the minimal d-quotients in $\mathcal{M}_n^d$ by Proposition 4.6 (3) and Remark 4.7. Hence, condition (1) is also equivalent to $\mathcal{U}_I$ being closed under d-quotients in $\mathcal{M}_{\underline{\ell}}^d$.

Note next that d-extensions in $\mathcal{M}_{\underline{\ell}}^d$ coincide with d-extensions in $\mathcal{M}_n^d$ with all terms in $\mathcal{M}_{\underline{\ell}}^d$. By Proposition 5.3 and Remark 5.4, condition (2) hence implies that $\mathcal{U}_I$ is closed under d-extensions by indecomposables in $\mathcal{M}_{\underline{\ell}}^d$. Assuming both (1) and (2) thus yields that $\mathcal{U}_I$ is closed under all d-extensions in $\mathcal{M}_{\underline{\ell}}^d$ by Theorem 3.20, so $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}}^d$ by Theorem 3.17.

It remains to show that if $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}}^d$, then condition (2) is satisfied. We will use that we already know condition (1) holds. Consider $x, z \in \mathrm{os}_{\underline{\ell}}^{d+1}$ with $x \rightsquigarrow \tau_d(z)$, and suppose that $y \in \mathrm{os}_{\underline{\ell}}^{d+1}$ satisfies $y_i \in \{x_i, z_i\}$ for each i. We need to show that $y \in I$.

As $x \rightsquigarrow \tau_d(z)$, there is a minimal d-extension

$$0 \to M_x \to E_1 \to \cdots \to E_d \to M_z \to 0 \tag{8}$$

in $\mathcal{M}_n^d$ with M_y as a direct summand in one of the middle terms by Proposition 5.3 and Remark 5.4. Let us first assume $z_d - x_0 + 1 \leq \ell_{z_d}$. Using that $x, z \in \mathrm{os}_{\underline{\ell}}^{d+1}$ and $x_0 \leq z_0 - 1$, we see that all the terms in (8) are in $\mathcal{M}_{\underline{\ell}}^d$ in this case. Hence, we must have $M_y \in \mathcal{U}_I$ and $y \in I$ as $\mathcal{U}_I$ is closed under d-extensions in $\mathcal{M}_{\underline{\ell}}^d$.

Consider now the case $z_d - x_0 + 1 > \ell_{z_d}$. Note that we have $x \leq y$. If $y_d = x_d$, condition (1) hence yields that $y \in I$, so we can assume $y_d = z_d$. Now $y_0 = x_0$ would contradict the assumption $z_d - x_0 + 1 > \ell_{z_d}$ as $y \in \mathrm{os}_{\underline{\ell}}^{d+1}$, so we must have $y_0 = z_0$.

If $y = z$, we have $y \in I$, so assume that $y_i = x_i$ for some $1 \leq i \leq d-1$. This ensures that $k := \min\{i \mid z_{i-1} \leq x_i\}$ exists. Let $w = (z_0, z_1, \ldots, z_{k-1}, x_k, \ldots, x_{d-1}, z_d)$, and observe that $w \in \mathrm{os}_{\underline{\ell}}^{d+1}$ with $w \leq y$ and $w_d = y_d$. Consequently, it suffices to show that $w \in I$, as this implies $y \in I$ by condition (1).

To this end, define

$$\begin{aligned} \overline{x} &= (z_0, z_1, \ldots, z_{k-1}, x_k, \ldots, x_d) \\ \overline{z} &= (z_0 + 1, z_1 + 1, \ldots, z_{k-1} + 1, z_k, \ldots, z_d). \end{aligned}$$

We see that $\overline{x}, \overline{z} \in \mathrm{os}_{\underline{\ell}}^{d+1}$. Observe moreover that $x \leq \overline{x}$ with $\overline{x}_d = x_d$ and $z \leq \overline{z}$ with $\overline{z}_d = z_d$, so $\overline{x}, \overline{z} \in I$ by condition (1). Furthermore, we have $\overline{x} \rightsquigarrow \tau_d(\overline{z})$ and $w_i \in \{\overline{x}_i, \overline{z}_i\}$ for all i. Finally,

notice that $\overline{z}_d - \overline{x}_0 + 1 = z_d - z_0 + 1 \leq \ell_{\overline{z}_d}$. It follows that $w \in I$ by the same argument as earlier in this proof, and we can conclude that $y \in I$. $\square$

The following example illustrates the use of Theorem 6.1.

Example 6.2. Fix $n = 4$, $d = 2$ and $\underline{\ell} = (1, 2, 2, 3)$. The higher Nakayama algebra $A^2_{\underline{\ell}}$ has a 2-cluster tilting subcategory $\mathcal{M}^2_{\underline{\ell}}$, whose associated quiver can be found in Figure 4. Note that we identify the indecomposable objects of $\mathcal{M}^2_{\underline{\ell}}$ with $\mathrm{os}^3_{\underline{\ell}}$, and that we use a shortened notation for the sake of simplicity. The following subcategories are examples of 2-torsion classes in $\mathcal{M}^2_{\underline{\ell}}$:

- $\mathrm{add}\{0\}$
- $\mathcal{M}^2_{\underline{\ell}}$
- $\mathrm{add}\{000, 133, 222, 223, 233, 333\}$
- $\mathrm{add}\{112, 113, 122, 123, 222, 133, 223, 233, 333\}$

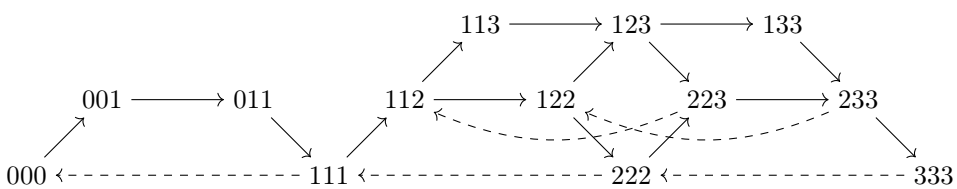

FIGURE 4. The quiver of the 2-cluster tilting subcategory $\mathcal{M}^2_{\underline{\ell}}$ in Example 6.2

.

6.2. **Higher Nakayama algebras of type $\mathbb{A}^\infty_\infty$.** The goal of this subsection is to further extend the combinatorial description of d-torsion classes to higher Nakayama algebras of type $\mathbb{A}^\infty_\infty$ as introduced in [JKPK]. We start by giving a brief introduction to this class of algebras.

In contrast to earlier in this paper, we now need to consider quivers with infinitely many vertices. Hence, associated to a quiver with relations is a category $\mathcal{C}$ rather than an algebra. The objects of $\mathcal{C}$ are the vertices of the quiver, and a basis of the morphisms spaces are given by the arrows modulo the given relations. A *right module over* $\mathcal{C}$ is a $\Bbbk$-linear functor $M \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\,\Bbbk$. The module M is called *finite-dimensional* if the sum $\bigoplus_{x \in \mathcal{C}} M(x)$ is finite-dimensional. We let $\mathrm{fd}\,\mathcal{C}$ denote the category of finite-dimensional right modules over $\mathcal{C}$. For more details, see [JKPK, Sections 1.1 and 1.2].

Let os^d denote the set of non-decreasing d-tuples $x = (x_0, \dots, x_{d-1})$ over $\mathbb{Z}$. Consider $\mathbb{Z}^d$ as a $\Bbbk$-linear category given by the $\Bbbk$-linearisation of the poset $\mathbb{Z}^d$ endowed with the product order. The *mesh category of type* $\mathbb{Z}\mathbb{A}^{d-1}_\infty$ is defined to be the additive quotient

$$A^d_\infty := \mathbb{Z}^d / (\mathbb{Z}^d \setminus \mathrm{os}^d).$$

Note that A^d_∞ can be represented by the opposite of an infinite quiver Q^d with relations. The vertices of Q^d are elements in os^d, and there is an arrow from vertex x to vertex y if $y_i = x_i + 1$ for exactly one $i = 0, \dots, d-1$ and $y_j = x_j$ for $j \neq i$. The relations of A^d_∞ are given by a certain admissible ideal making squares commutative and sending certain compositions of two arrows to zero, see [JKPK, Section 3.1] for more details.

Following [JKPK, Appendix B], we say that a *Kupisch series of type* $\mathbb{A}^\infty_\infty$ is an infinite tuple $\underline{\ell} = (\dots, \ell_{-1}, \ell_0, \ell_1, \dots)$ where ℓ_i is either a non-negative integer or equal to ∞, and where the inequality

$$\ell_i \leq \ell_{i-1} + 1$$

holds for all $i \in \mathbb{Z}$. We write $\mathrm{KS}(\mathbb{A}^\infty_\infty)$ for the set of Kupisch series of type $\mathbb{A}^\infty_\infty$. Given such a Kupisch series $\underline{\ell}$, define the subset

$$\mathrm{os}^{d+1}_{\underline{\ell}} := \{y \in \mathrm{os}^{d+1} \mid \ell\ell(y) \leq \ell_{y_d}\},$$

where $\ell\ell(y) = y_d - y_0 + 1$ as before. For $d \geq 2$, the category $A^d_{\underline{\ell}}$ is defined as the idempotent quotient

$$A^d_{\underline{\ell}} := A^d_\infty / (\mathrm{os}^d \setminus \mathrm{os}^d_{\underline{\ell}}).$$

Note that we have an inclusion $\mathrm{fd}\, A^d_{\underline{\ell}} \to \mathrm{fd}\, A^d_\infty$ between the categories of finite-dimensional modules. Associated to the Kupisch series $\underline{\ell}$, we also have the subcategory

$$\mathcal{M}^d_{\underline{\ell}} := \mathrm{add}\{M_x \in \mathrm{fd}\, A^d_\infty \mid x \in \mathrm{os}^{d+1}_{\underline{\ell}}\}$$

of $\mathrm{fd}\, A^d_\infty$. Here, we use the notation M_x for the indecomposable A^d_∞-module with support in all vertices $y \in \mathrm{os}^d$ satisfying $x_0 \leq y_0 \leq x_1 \leq \cdots \leq x_{d-1} \leq y_{d-1} \leq x_d$. This extends the analogous definition in Section 5.1. By [JKPK, Appendix B], the subcategory $\mathcal{M}^d_{\underline{\ell}}$ is d-cluster tilting in $\mathrm{fd}\, A^d_{\underline{\ell}}$. For certain choices of $\underline{\ell}$, this gives an example of a d-cluster tilting subcategory of an abelian category with no projective or injective objects, e.g. if $\ell_i = \infty$ for all $i \in \mathbb{Z}$, see [JKPK, Proposition 3.6].

Now consider the partial order on the set $\mathrm{KS}(\mathbb{A}^\infty_\infty)$ given by the product order. If $\underline{\ell} \leq \underline{\ell}'$, then we have a natural functor $A^d_{\underline{\ell}'} \to A^d_{\underline{\ell}}$ of categories. This gives an inclusion $\mathrm{fd}\, A^d_{\underline{\ell}} \to \mathrm{fd}\, A^d_{\underline{\ell}'}$ such that the equality

$$\mathcal{M}^d_{\underline{\ell}} = \mathcal{M}^d_{\underline{\ell}'} \cap \mathrm{fd}\, A^d_{\underline{\ell}}$$

holds. Similarly, if $\underline{\ell}^1 \leq \underline{\ell}^2 \leq \cdots$ is an increasing sequence of Kupisch series which converges to $\underline{\ell} \in \mathrm{KS}(\mathbb{A}^\infty_\infty)$ (in the natural way), then

$$\mathrm{fd}\, A^d_{\underline{\ell}} = \bigcup_{i \geq 1} \mathrm{fd}\, A^d_{\underline{\ell}^i} \quad \text{and} \quad \mathcal{M}^d_{\underline{\ell}} = \bigcup_{i \geq 1} \mathcal{M}^d_{\underline{\ell}^i}.$$

A Kupisch series $\underline{\ell} \in \mathrm{KS}(\mathbb{A}^\infty_\infty)$ is called *finite* if $\ell_j = 0$ for all but finitely many $j \in \mathbb{Z}$. Note that for any $\underline{\ell} \in \mathrm{KS}(\mathbb{A}^\infty_\infty)$, we can find a sequence $\underline{\ell}^1 \leq \underline{\ell}^2 \leq \cdots$ of finite Kupisch series in $\mathrm{KS}(\mathbb{A}^\infty_\infty)$ which converges to $\underline{\ell}$. We use this to give a characterisation of the d-torsion classes in $\mathcal{M}^d_{\underline{\ell}}$ in Theorem 6.4 below.

Remark 6.3. Let $\underline{\ell}$ be a Kupisch series of type $\mathbb{A}_n$. Then $\underline{\ell}$ can be identified with a finite Kupisch series $\underline{\ell}'$ of type $\mathbb{A}^\infty_\infty$ which is non-zero only in positions $0, \ldots, n-1$. In this case there is a bijection between the sets $\mathrm{os}^d_{\underline{\ell}}$ and $\mathrm{os}^d_{\underline{\ell}'}$, so the algebra $A^d_{\underline{\ell}}$ and the category $A^d_{\underline{\ell}'}$ can be naturally identified. Note that up to isomorphism, the set $\mathrm{os}^d_{\underline{\ell}'}$ and the category $A^d_{\underline{\ell}'}$ remain unchanged when shifting $\underline{\ell}'$ some number of steps to the left or right.

In general, if $\underline{\ell}$ is a finite Kupisch series of type $\mathbb{A}^\infty_\infty$, then $\underline{\ell}$ is obtained by gluing together shifts of Kupisch series of type $\mathbb{A}$. Hence, the set $\mathrm{os}^d_{\underline{\ell}}$ is in bijection with a disjoint union $\bigcup_{j=1}^m \mathrm{os}^d_{\underline{\ell}_j}$ where $\underline{\ell}_j$ is a Kupisch series of type $\mathbb{A}_{n_j}$ for some integer $n_j \geq 1$. The associated category $A^d_{\underline{\ell}}$ can thus be identified with a finite product

$$A^d_{\underline{\ell}_1} \times \cdots \times A^d_{\underline{\ell}_m}$$

where $A^d_{\underline{\ell}_j}$ is a higher Nakayama algebras of type $\mathbb{A}_{n_j}$. With this identification, the category $\mathrm{fd}\, A^d_{\underline{\ell}}$ is equivalent to the product $\mathrm{mod}\, A^d_{\underline{\ell}_1} \times \cdots \times \mathrm{mod}\, A^d_{\underline{\ell}_m}$, and the d-cluster tilting subcategory $\mathcal{M}^d_{\underline{\ell}}$ is equivalent to the product $\mathcal{M}^d_{\underline{\ell}_1} \times \cdots \times \mathcal{M}^d_{\underline{\ell}_m}$. Since any Kupisch series $\underline{\ell}$ of type $\mathbb{A}^\infty_\infty$ can be represented by a converging sequence of finite Kupisch series, it follows that the associated d-cluster tilting subcategory $\mathcal{M}^d_{\underline{\ell}}$ is the union of finite products of d-cluster tilting subcategories of higher Nakayama algebras of type $\mathbb{A}$.

As before, we use the notation

$$\mathcal{U}_I := \mathrm{add}\{M_y \in \mathcal{M}^d_{\underline{\ell}} \mid y \in I\}$$

for the subcategory of $\mathcal{M}^d_{\underline{\ell}}$ associated to a subset $I \subseteq \mathrm{os}^{d+1}_{\underline{\ell}}$.

Theorem 6.4. *Let $\underline{\ell}$ be a Kupisch series of type $\mathbb{A}_\infty^\infty$ and consider a subset $I \subseteq \mathrm{os}_{\underline{\ell}}^{d+1}$. The subcategory $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}}^d$ if and only if the following hold for any elements $x, z \in \mathrm{os}_{\underline{\ell}}^{d+1}$:*

(1) *If $x \leq z$ and $x_d = z_d$, then $x \in I$ implies $z \in I$.*
(2) *If $x \rightsquigarrow \tau_d(z)$ and $x, z \in I$, then any $y \in \mathrm{os}_{\underline{\ell}}^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i must be in I.*

Proof. Choose an increasing sequence $\underline{\ell}^1 \leq \underline{\ell}^2 \leq \cdots$ of finite Kupisch series which converges to $\underline{\ell}$. Then we have that

$$\mathcal{U}_I \cap \mathcal{M}_{\underline{\ell}^i}^d = \mathcal{U}_{I \cap \mathrm{os}_{\underline{\ell}^i}^{d+1}}$$

for each $i \geq 1$. Fix $i \geq 1$, and set $J^i = I \cap \mathrm{os}_{\underline{\ell}^i}^{d+1}$. We have an equivalence between $\mathcal{M}_{\underline{\ell}^i}^d$ and a finite product $\mathcal{M}_{\underline{\ell}_1^i}^d \times \cdots \times \mathcal{M}_{\underline{\ell}_m^i}^d$ as in Remark 6.3. The subcategory $\mathcal{U}_{J^i}$ of $\mathcal{M}_{\underline{\ell}^i}^d$ is hence equivalent to a product $\mathcal{U}_1 \times \cdots \times \mathcal{U}_m$, where $\mathcal{U}_k$ is a subcategory of $\mathcal{M}_{\underline{\ell}_k^i}^d$ for each k. Note that $\mathrm{os}_{\underline{\ell}^i}^{d+1}$ is in bijection with the disjoint union $\bigcup_{k=1}^m \mathrm{os}_{\underline{\ell}_k^i}^{d+1}$ as in Remark 6.3. We let J_k^i denote the intersection of $\mathrm{os}_{\underline{\ell}_k^i}^{d+1}$ with the image of J^i under this bijection. Then we get $\mathcal{U}_k = \mathcal{U}_{J_k^i}$.

Note that I satisfies the conditions (1) and (2) in the statement if and only if the set J^i satisfies the same conditions for each $i \geq 1$. Furthermore, this holds if and only if each J_k^i satisfies the conditions of Theorem 6.1. By Theorem 6.1, this is again equivalent to $\mathcal{U}_{J_k^i}$ being a d-torsion class in $\mathcal{M}_{\underline{\ell}_k^i}^d$ for all i and k, i.e. that $\mathcal{U}_{J^i} \simeq \mathcal{U}_{J_1^i} \times \cdots \times \mathcal{U}_{J_m^i}$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}^i}^d$ for all $i \geq 1$. Hence, it suffices to show that $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}}^d$ if and only if $\mathcal{U}_{J^i}$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}^i}^d$ for all $i \geq 1$.

If $\mathcal{U}_I$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}}^d$, then $\mathcal{U}_{J^i} = \mathcal{U}_I \cap \mathcal{M}_{\underline{\ell}^i}^d$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}^i}^d = \mathcal{M}_{\underline{\ell}}^d \cap \mathrm{fd}\, A_{\underline{\ell}^i}^d$ for all $i \geq 1$ by Proposition 4.6(1). Conversely, assume $\mathcal{U}_{J^i}$ is a d-torsion class in $\mathcal{M}_{\underline{\ell}^i}^d$ for all $i \geq 1$. Since we have

$$\mathcal{U}_{J^1} \subseteq \mathcal{U}_{J^2} \subseteq \cdots \quad \text{and} \quad \mathcal{U}_I = \bigcup_{i \geq 1} \mathcal{U}_{J^i},$$

the subcategory $\mathcal{U}_I$ must be closed both under d-extensions and d-quotients in $\mathcal{M}_{\underline{\ell}}^d = \bigcup_{i \geq 1} \mathcal{M}_{\underline{\ell}^i}^d$, since $\mathcal{U}_{J^i}$ is closed under d-extensions and d-quotients in $\mathcal{M}_{\underline{\ell}^i}^d$ for all $i \geq 1$. By Theorem 3.17, this shows that $\mathcal{U}_I$ is a d-torsion class. □

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