



**HAL**  
open science

# Semigroups generated in $L_p$ -spaces by some dispersal process including semi-permeability conditions at the interface

Laïd Djilali, Rabah Labbas, Ahmed Medeghri, Abdallah Menad, Alexandre Thorel

► **To cite this version:**

Laïd Djilali, Rabah Labbas, Ahmed Medeghri, Abdallah Menad, Alexandre Thorel. Semigroups generated in  $L_p$ -spaces by some dispersal process including semi-permeability conditions at the interface. 2022. hal-04001816v2

**HAL Id: hal-04001816**

**<https://hal.science/hal-04001816v2>**

Preprint submitted on 12 Jun 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Semigroups generated in $L^p$ -spaces by some dispersal process including semi-permeability conditions at the interface

Laïd Djilali, Rabah Labbas, Ahmed Medeghri, Abdallah Menad & Alexandre Thorel

R. L and A.T: Université Le Havre Normandie, Normandie Univ, LMAH UR 3821, 76600 Le Havre, France.

rabah.labbas@univ-lehavre.fr, alexandre.thorel@univ-lehavre.fr

L. D and A. M: Laboratoire de Mathématiques Pures et Appliquées,

Université Abdelhamid Ibn Badis, 27000 Mostaganem, Algérie,

A. M: National Higher School of Mathematics, Scientific and Technology Hub of Sidi Abdellah,

P.O. Box 75, Algiers 16093, Algeria.

laid.djilali@univ-mosta.dz, ahmed.medeghri@nhsm.edu.dz, abellah.menad@univ-mosta.dz

## Abstract

We study an elliptic differential equation set in two habitats under semi-permeability conditions at the interface. This equation describes some dispersal process in population dynamics. Using the well-known Dore-Venni theorem, some useful results in [6] and [21], we show that the associated space operator generates an analytic semigroup in  $L^p$ -spaces.

**Key Words and Phrases:** Abstract elliptic differential equations; transmission problem; functional calculus; semi-permeability conditions; population dynamics.

**2020 Mathematics Subject Classification:** 34G10, 35J25, 47A60, 47D06, 92D25.

## 1 Introduction

In this work, we are interested in the study of a reaction-diffusion model for population dynamics with a dispersal process in two habitats. This model takes into account the reaction effect of individuals on the interface of these habitats.

Our aim is to show that this dispersal process generates an analytic semigroup in the framework of  $L^p$ -spaces. The present work completes naturally the results obtained in [10].

Many authors have worked on different models of reaction-diffusion problems related to biology or various environmental phenomena. We were inspired, in particular, by the work in [5], where the model considered incorporates the response of individuals on the interface between the habitats. This study was detailed in one space dimension and focused mainly on the spectral aspect of the dispersal process, taking into account the dimension of the habitats.

Here, the concrete example which illustrates our theory, is modelled by partial differential equations of parabolic type set in two juxtaposed habitats  $\Omega_-$  and  $\Omega_+$ :

$$\Omega = \Omega_- \cup \Gamma_0 \cup \Omega_+,$$

where

$$\begin{cases} \Omega_- = (-\ell, 0) \times (0, 1), \\ \Gamma_0 = \{0\} \times (0, 1), \\ \Omega_+ = (0, L) \times (0, 1), \\ \Gamma_{\pm} = \partial\Omega_{\pm} \setminus \Gamma_0, \end{cases} \quad (1)$$

and  $\ell, L > 0$ . The reaction-diffusion equation is

$$\frac{\partial u}{\partial t}(t, x, y) = \begin{cases} d_- \Delta u_-(t, x, y) + F_-(u_-(t, x, y)) & \text{in } (0, T) \times \Omega_-, \\ d_+ \Delta u_+(t, x, y) + F_+(u_+(t, x, y)) & \text{in } (0, T) \times \Omega_+, \end{cases} \quad (2)$$

under the initial data

$$u(0, \cdot) = \begin{cases} \varphi_- & \text{in } \Omega_-, \\ \varphi_+ & \text{in } \Omega_+, \end{cases} \quad (3)$$

the boundary conditions

$$\begin{cases} u_- = 0 & \text{on } \Gamma_-, \\ u_+ = 0 & \text{on } \Gamma_+, \end{cases} \quad (4)$$

and the interface conditions

$$\begin{cases} d_- \frac{\partial u_-}{\partial x} = q(u_+ - u_-) & \text{on } \Gamma_0, \\ d_+ \frac{\partial u_+}{\partial x} = q(u_+ - u_-) & \text{on } \Gamma_0; \end{cases} \quad (5)$$

where  $d_{\pm} > 0$  is the diffusion coefficient,  $q > 0$  is given and  $u_{\pm}$  represents a population density in  $\Omega_{\pm}$ .

These two last conditions in (5) mean that the flux at the interface depends on the density of the population. They are of the Robin type and express the semi-permeability of the interface  $\Gamma_0$ .

In [10], the authors have considered different interface conditions in the framework of the space of continuous functions which are the following: the non-continuity of the flux and the continuity of the dispersal at the interface:

$$\begin{cases} (1 - \alpha)d_- \frac{\partial u_-}{\partial x} = \alpha d_+ \frac{\partial u_+}{\partial x} & \text{on } \Gamma_0, \\ d_- \Delta u_- + F_-(u_-) = d_+ \Delta u_+ + F_+(u_+) & \text{on } \Gamma_0, \end{cases}$$

where  $\alpha \in (0, 1)$  with  $\alpha \neq 1/2$ .

Here, we consider only the linearized part of logistic functions, that is

$$\begin{cases} F_-(u_-) = r_- u_- & \text{on } (-\ell, 0) \times (0, 1), \\ F_+(u_+) = r_+ u_+ & \text{on } (0, L) \times (0, 1), \end{cases}$$

where  $r_{\pm} > 0$ .

The study of the above reaction-diffusion problem depends essentially on the nature of operator  $\mathcal{P}$  defined, in the stationary case, by

$$\begin{cases} D(\mathcal{P}) = \left\{ u \in L^p(\Omega) : u_- \in W^{2,p}(\Omega_-), u_+ \in W^{2,p}(\Omega_+), u_{\pm} = 0 \text{ on } \Gamma_{\pm} \right\}, \\ \mathcal{P}u = \begin{cases} d_- \Delta u_- - r_- u_- & \text{in } \Omega_-, \\ d_+ \Delta u_+ - r_+ u_+ & \text{in } \Omega_+, \end{cases} \end{cases}$$

where  $p \in (1, +\infty)$ . Note that the transmission conditions (5) are well defined since, for example, if  $u_- \in W^{2,p}(\Omega_-)$ , then

$$\frac{\partial u_-}{\partial x} \Big|_{\{0\} \times (0,1)} \in W^{1-1/p,p}(0,1),$$

see [11], Corollary 1, p. 682.

Now, let us write the above parabolic equation in an abstract formulation. First, let us introduce, in the Banach space  $E_0 = L^p(0, 1)$ , operator  $A_0$  defined by

$$\begin{cases} D(A_0) = \{\varphi \in W^{2,p}(0, 1) : \varphi(0) = \varphi(1) = 0\}, \\ (A_0\varphi)(y) = \varphi''(y). \end{cases} \quad (6)$$

It is known that this operator is closed linear with a dense domain and verifies the two following properties :

$$\exists C > 0 : \forall z \in S_{\pi-\varepsilon} \cup \{0\}, \quad \|(zI - A_0)^{-1}\|_{\mathcal{L}(E_0)} \leq \frac{C}{1 + |z|}, \quad (7)$$

where

$$S_{\pi-\varepsilon} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi - \varepsilon\},$$

for any small  $\varepsilon > 0$  such that  $\pi - \varepsilon > 0$  and there exists a ball  $B(0, \delta)$ , such that  $\rho(A_0) \supset \overline{B(0, \delta)}$  and the above estimate is still true in  $S_{\pi-\varepsilon} \cup \overline{B(0, \delta)}$ ; and

$$\begin{cases} \forall s \in \mathbb{R}, (-A_0)^{is} \in \mathcal{L}(E_0), \forall \varepsilon > 0 : \\ \sup_{s \in \mathbb{R}} \|e^{-\varepsilon|s|} (-A_0)^{is}\|_{\mathcal{L}(E_0)} < +\infty, \end{cases} \quad (8)$$

see for instance the method used in [14], Proposition 3.1, p. 191.

**Remark 1.** *The above problem can be considered in dimension  $n$  by setting*

$$\Omega = (-\ell, L) \times \varpi,$$

where  $\varpi \subset \mathbb{R}^{n-1}$ ,  $n > 1$ , is a bounded regular open set,  $E_0 = L^p(\varpi)$  and

$$\begin{cases} D(A_0) = \{\varphi \in W^{2,p}(\varpi) : \varphi|_{\partial\varpi} = 0\}, \\ A_0\varphi = \Delta_{\varpi}\varphi; \end{cases}$$

here  $\Delta_{\varpi}$  denotes the Laplace operator related to variables on  $\varpi$ .

We will also use the following usual operational notation of vector-valued functions:

$$u_{\pm}(t, x)(y) := u_{\pm}(t, x, y), \quad t \in (0, T), \quad (x, y) \in \Omega_{\pm}.$$

So, we have to analyze the abstract Cauchy problem

$$\begin{cases} u'(t) = \mathcal{S}_0 u(t), \\ u(0) = u_0, \end{cases} \quad (9)$$

set in the Banach space  $L^p(-\ell, L; E_0)$ , where

$$\left\{ \begin{array}{l} D(\mathcal{S}_0) = \left\{ \begin{array}{l} w \in L^p(-\ell, L; E_0) : w_- \in W^{2,p}(-\ell, 0; E_0), w_+ \in W^{2,p}(0, L; E_0), \\ w_- \in W^{2,p}(-\ell, 0; D(A_0)), w_+ \in W^{2,p}(0, L; D(A_0)), \\ w_-(-\ell) = 0, w_+(L) = 0, d_- w'_-(0) = q(w_+(0) - w_-(0)) \\ \text{and } d_+ w'_+(0) = q(w_+(0) - w_-(0)) \end{array} \right\}, \\ (\mathcal{S}_0 w)(x) = \begin{cases} d_- w''_-(x) + d_- A_0 w_-(x) - r_- w_-(x) & \text{in } (-\ell, 0), \\ d_+ w''_+(x) + d_+ A_0 w_+(x) - r_+ w_+(x) & \text{in } (0, L). \end{cases} \end{array} \right.$$

We then consider a more general operator  $\mathcal{S}$  instead of  $\mathcal{S}_0$  where  $A_0$  is replaced by a closed linear operator  $A$  in a Banach space  $E$  satisfying some assumptions specified in Section 4.

Our method is essentially based on the use of abstract differential equations of elliptic type. The merit of this method lies above all in the fact of having the explicit formula of the resolvent operator of  $\mathcal{S}$  (and therefore of  $\mathcal{S}_0$ ), see Section 6.

This paper is organized as follows.

In Section 2, we only show that problem  $-\mathcal{P}u = g$  can be studied in the variational framework. In Section 3, we recall some useful notions on sectorial operators. Section 4 contains our assumptions and the main results. Section 5 is devoted to the establishment of some useful properties on complex numbers. Section 6 is composed of two subsections. In the first subsection, we explain the spectral equation of  $\mathcal{S}$  which leads to an abstract system to be solved. Thus, we invert the determinant operator of the above system by using among others the  $H^\infty$ -calculus. This leads us to obtain the explicit solution of the spectral equation. Then, we study the optimal regularity of this solution. In the second subsection, we give some sharp estimates which lead us to analyze the behaviour of the resolvent operator of  $\mathcal{S}$ . We then obtain our main results which state among others that  $\mathcal{S}$  generates an analytic semigroup in  $L^p(-\ell, L; E)$ , for  $p \in (1, +\infty)$ .

## 2 Variational formulation of $-\mathcal{P}u = g$

Let  $a, b \in \mathbb{R}$ , with  $a < b$ . For any

$$\begin{aligned} \varphi : (a, b) &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi(x), \end{aligned}$$

we set

$$\begin{aligned} \tilde{\varphi} : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{for } x \in (a, b), \\ 0 & \text{for } x \in \mathbb{R} \setminus (a, b), \end{cases} \end{aligned}$$

and we define a subspace  $\widetilde{H^{1/2}}(a, b)$  of  $H^{1/2}(a, b)$  by

$$\widetilde{H^{1/2}}(a, b) = \left\{ \varphi \in H^{1/2}(a, b) : \tilde{\varphi} \in H^{1/2}(\mathbb{R}) \right\}.$$

In [15], the authors denoted this space by  $H_{00}^{1/2}(a, b)$  which also coincides with the following particular interpolation space

$$\left( H_0^1(a, b), L^2(a, b) \right)_{1/2, 2}.$$

The interpolation spaces are described, for instance, in [11].

Set

$$H_{\Gamma_\pm}^1(\Omega_\pm) = \left\{ u_\pm \in H^1(\Omega_\pm) : u_\pm|_{\Gamma_\pm} = 0 \right\};$$

for  $v_\pm$  in  $H_{\Gamma_\pm}^1(\Omega_\pm)$ , it is clear that  $v_\pm|_{\Gamma_0}$  is in  $\widetilde{H^{1/2}}(\Gamma_0)$ .

Problem  $-\mathcal{P}u = g$  writes in the form

$$\left\{ \begin{array}{ll} -\operatorname{div}(d_+ \nabla u_+) + r_+ u_+ = g_+ & \text{in } \Omega_+, \\ -\operatorname{div}(d_- \nabla u_-) + r_- u_- = g_- & \text{in } \Omega_-, \\ d_+ \frac{\partial u_+}{\partial \nu} = q(u_+ - u_-) & \text{on } \Gamma_0, \\ d_- \frac{\partial u_-}{\partial \nu} = q(u_+ - u_-) & \text{on } \Gamma_0, \\ u_\pm = 0 & \text{on } \Gamma_\pm, \end{array} \right.$$

where  $\nu$  is the normal unit vector oriented towards the interior of  $\Omega_+$ . The variational formulation is set in the hilbertian space

$$\mathbb{V} = H_{\Gamma_+}^1(\Omega_+) \times H_{\Gamma_-}^1(\Omega_-),$$

with

$$\begin{aligned} a((u_+, u_-), (w_+, w_-)) &= \int_{\Omega_+} (d_+ \nabla u_+ \cdot \nabla w_+ + r_+ u_+ w_+) dx dy \\ &\quad + \int_{\Omega_-} (d_- \nabla u_- \cdot \nabla w_- + r_- u_- w_-) dx dy, \end{aligned}$$

and

$$\begin{cases} b((u_+, u_-), (w_+, w_-)) = \int_{\Gamma_0} q(u_+ - u_-)(w_+ - w_-) dy, \\ l(w_+, w_-) = \int_{\Omega_+} g_+ w_+ dx dy + \int_{\Omega_-} g_- w_- dx dy, \end{cases}$$

then

$$a((u_+, u_+), (w_+, w_-)) + b((u_+, u_-), (w_+, w_-)) = l(w_+, w_-).$$

Now, taking  $w_+$  in  $\mathcal{D}(\Omega_+)$  and  $w_- = 0$ , we have in the sense of distributions

$$-\operatorname{div}(d_+ \nabla u_+) + r_+ u_+ = g_+ \text{ in } \Omega_+.$$

For  $g_+$  in  $L^2(\Omega_+)$ , the trace of  $d_+ \frac{\partial u_+}{\partial \nu}$  on  $\Gamma_0$ , can be defined in the dual space  $\left(\widetilde{H^{1/2}}(\Gamma_0)\right)'$  of  $\widetilde{H^{1/2}}(\Gamma_0)$ . In fact, operator  $\partial/\partial \nu$  maps continuously from  $H^1(0, 1)$  into  $L^2(0, 1)$  and  $L^2(0, 1)$  into  $H^{-1}(0, 1)$ , then, by interpolation it maps continuously from  $(H^1(0, 1); L^2(0, 1))_{1/2, 2}$  into  $(L^2(0, 1), H^{-1}(0, 1))_{1/2, 2}$ ; but we know that

$$\left(H^1(0, 1), L^2(0, 1)\right)_{1/2, 2} = H^{1/2}(0, 1),$$

and

$$\left(L^2(0, 1), H^{-1}(0, 1)\right)_{1/2, 2} = \left[\left(H_0^1(0, 1), L^2(0, 1)\right)_{1/2, 2}\right]' = \left(\widetilde{H^{1/2}}(a, b)\right)',$$

see [19] p. 160.

The Green's formula for  $w_+$  in  $H_{\Gamma_+}^1(\Omega_+)$  gives

$$\begin{aligned} \int_{\Omega_+} (d_+ \nabla u_+ \cdot \nabla w_+ + r_+ u_+ w_+) dx dy &= \int_{\Omega_+} (-\nabla \cdot (d_+ \nabla u_+) w_+ + r_+ u_+ w_+) dx dy \\ &\quad + \int_{\Gamma_0} \left(-d_+ \frac{\partial u_+}{\partial \nu}\right) w_+ dy, \end{aligned}$$

where the last integral means that

$$\int_{\Gamma_0} \left(-d_+ \frac{\partial u_+}{\partial \nu}\right) w_+ dy := \left\langle d_+ \frac{\partial u_+}{\partial \nu}; w_+ \right\rangle_{\left(\widetilde{H^{1/2}}(\Gamma_0)\right)' \times \left(\widetilde{H^{1/2}}(\Gamma_0)\right)}.$$

Similarly, for

$$-\operatorname{div}(d_- \nabla u_-) + r_- u_- = g_- \text{ in } \Omega_-,$$

we obtain

$$\begin{aligned} \int_{\Omega_-} (d_- \nabla u_- \cdot \nabla w_- + r_- u_- w_-) dx dy &= \int_{\Omega_-} (-\nabla \cdot (d_- \nabla u_-) w_- + r_- u_- w_-) dx dy \\ &\quad + \int_{\Gamma_0} \left(d_- \frac{\partial u_-}{\partial \nu}\right) w_- dy, \end{aligned}$$

as above, the last integral means that

$$\int_{\Gamma_0} \left( -d_- \frac{\partial u_-}{\partial \nu} \right) w_- dy := \left\langle d_- \frac{\partial u_-}{\partial \nu}; w_- \right\rangle_{\left( \widetilde{H^{1/2}}(\Gamma_0) \right)' \times \left( \widetilde{H^{1/2}}(\Gamma_0) \right)}.$$

It follows that

$$\int_{\Gamma_0} \left( -d_+ \frac{\partial u_+}{\partial \nu} \right) w_+ dy + \int_{\Gamma_0} \left( d_- \frac{\partial u_-}{\partial \nu} \right) w_- dy + \int_{\Gamma_0} q(u_+ - u_-)(w_+ - w_-) dy = 0;$$

taking  $w_- = 0$ , we deduce that

$$-d_+ \frac{\partial u_+}{\partial \nu} + q(u_+ - u_-) = 0 \text{ in } \left( \widetilde{H^{1/2}}(\Gamma_0) \right)',$$

in the same way,  $w_+ = 0$  gives

$$d_- \frac{\partial u_-}{\partial \nu} - q(u_+ - u_-) = 0 \text{ in } \left( \widetilde{H^{1/2}}(\Gamma_0) \right)'.$$

### 3 Recall on sectorial operators

Let  $\omega \in [0, \pi]$ . We put

$$S_\omega := \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\} & \text{if } \omega \in (0, \pi], \\ (0, +\infty) & \text{if } \omega = 0. \end{cases} \quad (10)$$

Let us recall some known results from [12].

**Definition 2.** Let  $\omega \in [0, \pi)$ . A linear operator  $\Lambda$  on a complex Banach space  $E$  is called sectorial of angle  $\omega$  if

1.  $\sigma(\Lambda) \subset \overline{S_\omega}$  and
2.  $M(\Lambda, \omega') := \sup_{\lambda \in \mathbb{C} \setminus \overline{S_{\omega'}}} \|\lambda(\Lambda - \lambda I)^{-1}\| < \infty$  for all  $\omega' \in (\omega, \pi)$ .

We then write:  $\Lambda \in \text{Sect}(\omega)$ . The following angle

$$\omega_\Lambda := \min \{ \omega \in [0, \pi) : \Lambda \in \text{Sect}(\omega) \},$$

is called the spectral angle of  $\Lambda$ .

We recall the following properties of the set  $\text{Sect}(\omega)$ . It is clear that Statement 2. implies necessarily that  $\Lambda$  is closed.

**Proposition 3.** If  $(-\infty, 0) \subset \rho(\Lambda)$  and

$$M(\Lambda) := M(\Lambda, \pi) := \sup_{t > 0} \|t(\Lambda + tI)^{-1}\| < \infty,$$

then  $M(\Lambda) \geq 1$  and

$$\Lambda \in \text{Sect}(\pi - \arcsin(1/M(\Lambda))).$$

**Proposition 4.** Let  $\Lambda \in \text{Sect}(\omega_\Lambda)$  and  $\nu \in (0, 1/2]$ . Then  $\Lambda^\nu \in \text{Sect}(\nu\omega_\Lambda)$ , and therefore  $-\Lambda^\nu$  generates an analytic semigroup.

For more details, see [12], p. 80-81.

**Definition 5.** We denote by  $BIP(E, \theta)$  (see [18]), the class of sectorial injective operators  $T$ , on the Banach space  $E$ , such that

- i)  $\overline{D(T)} = \overline{R(T)} = X$ ,
- ii)  $\forall s \in \mathbb{R}, T^{is} \in \mathcal{L}(X)$ ,
- iii)  $\exists C \geq 0, \forall s \in \mathbb{R}, \|T^{is}\|_{\mathcal{L}(X)} \leq Ce^{|s|\theta}$ .

**Definition 6.** A Banach space  $E$  is a UMD space if and only if for all  $1 < p < +\infty$  the Hilbert transform is continuous from  $L^p(\mathbb{R}; E)$  into itself, see [3] and [4].

Now, let us recall some important result on the well-known functional calculus. To this end, we set

$$H^\infty(S_\omega) = \{f : f \text{ is an holomorphic and bounded function on } S_\omega\},$$

with  $\omega \in (0, \pi)$ .

**Proposition 7.** Let  $\Lambda$  be an injective sectorial operator with dense range. If  $f \in H^\infty(S_\omega)$  is such that  $1/f \in H^\infty(S_\omega)$  and

$$(1/f)(\Lambda) \in \mathcal{L}(E),$$

then  $f(\Lambda)$  is boundedly invertible and

$$[f(\Lambda)]^{-1} = (1/f)(\Lambda). \quad (11)$$

This result is proved in [6] Proposition 3.3, p. 1873.

## 4 Assumptions and main results

Let  $A$  be a linear operator in a Banach space  $E$  and assume in all this paper that

$$E \text{ is a UMD space,} \quad (12)$$

$$\sigma(A) \subset (-\infty, 0) \quad \text{and} \quad \forall \theta \in (0, \pi), \quad \sup_{\lambda \in S_\theta} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(E)} < +\infty, \quad (13)$$

$$-A \in BIP(E, \varepsilon) \text{ for all } \varepsilon > 0. \quad (14)$$

**Remark 8.** In concrete examples, operator  $-A$  represents, for instance, an elliptic operator set in some bounded regular domain in  $L^p$ -spaces.

We define operator  $\mathcal{S}$  by

$$\left\{ \begin{array}{l} D(\mathcal{S}) = \left\{ \begin{array}{l} w \in L^p(-\ell, L; E) : w_- \in W^{2,p}(-\ell, 0; E), w_+ \in W^{2,p}(0, L; E) \\ w_-(-\ell) = 0, w_+(L) = 0, d_-w'_-(0) = q(w_+(0) - w_-(0)) \\ \text{and } d_+w'_+(0) = q(w_+(0) - w_-(0)) \end{array} \right\}, \\ (\mathcal{S}w)(x) = \left\{ \begin{array}{l} d_-w''(x) + d_-Aw_-(x) - r_-w_-(x) \quad \text{in } (-\ell, 0), \\ d_+w''(x) + d_+Aw_+(x) - r_+w_+(x) \quad \text{in } (0, L). \end{array} \right. \end{array} \right.$$

Thanks to the fact that the domain is cylindrical, we will give an explicit expression of the resolvent operator of  $\mathcal{S}$  by using essentially the analytic semigroups theory and the functional calculus.

Therefore, our aim results are the following:

**Theorem 9.** Assume that (12), (13) and (14) hold. Then, operator  $\mathcal{S}$  generates an analytic semigroup in  $L^p(-\ell, L; E)$ .

As corollaries, we obtain.

**Theorem 10.** Operator  $\mathcal{S}_0$  generates an analytic semigroup in  $L^p(-\ell, L; E_0)$ .

**Theorem 11.** Operator  $\mathcal{P}$  generates an analytic semigroup in  $L^p(\Omega)$ .



## 5 Preliminary results

In this section, we recall some useful results and we state some technical results.

**Proposition 12.** *Let  $c \in \mathbb{R} \setminus \{0\}$  and  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then, we have*

$$\begin{cases} 0 < |\arg(z+c)| < |\arg(z)| < \pi & \text{if } c > 0, \\ 0 < |\arg(z)| < |\arg(z+c)| < \pi & \text{if } c < 0. \end{cases}$$

*Proof.* For  $z \in \mathbb{C} \setminus \mathbb{R}$ , we have

$$\arg(z) = 2 \arctan \left( \frac{\operatorname{Im}(z)}{\operatorname{Re}(z) + |z|} \right), \quad (15)$$

and

1. if  $c > 0$ , then

$$\begin{aligned} |\arg(z)| &= 2 \arctan \left( \frac{|\operatorname{Im}(z+c)|}{\operatorname{Re}(z) + |z+c-c|} \right) \\ &> 2 \arctan \left( \frac{|\operatorname{Im}(z+c)|}{\operatorname{Re}(z) + c + |z+c|} \right) \\ &= |\arg(z+c)|. \end{aligned}$$

2. if  $c < 0$ , then

$$\begin{aligned} |\arg(z)| &= 2 \arctan \left( \frac{|\operatorname{Im}(z+c)|}{\operatorname{Re}(z) + c - c + |z|} \right) \\ &= 2 \arctan \left( \frac{|\operatorname{Im}(z+c)|}{\operatorname{Re}(z) + c + |c| + |z|} \right) \\ &< 2 \arctan \left( \frac{|\operatorname{Im}(z+c)|}{\operatorname{Re}(z) + c + |z+c|} \right) \\ &= |\arg(z+c)|. \end{aligned}$$

□

**Proposition 13.** *Let  $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ . Assume that  $z_1 + z_2 \neq 0$  and*

$$\begin{cases} \arg(-z_2) \leq \arg(z_1) \leq \arg(z_2) & \text{if } \arg(z_2) \in [0, \pi], \\ -\pi \leq \arg(z_1) \leq \arg(z_2) & \text{if } \arg(z_2) \in [-\pi, 0]. \end{cases}$$

*Then, we have*

$$\arg(z_1) \leq \arg(z_1 + z_2) \leq \arg(z_2).$$

*Proof.*

1. Assume that  $\arg(z_2) = \pi$ .

If  $0 < \arg(z_1) < \pi$ , from Proposition 12, we obtain the expected inequalities.

If  $\arg(z_1) = \arg(z_2) = \pi$ , then

$$\arg(z_1) = \arg(z_1 + z_2) = \arg(z_2). \quad (16)$$

If  $\arg(z_1) = 0$ , then the expected inequalities hold since we have

$$\arg(z_1 + z_2) = \begin{cases} \arg(z_1) & \text{if } |z_1| > |z_2|, \\ \arg(z_2) & \text{if } |z_2| > |z_1|. \end{cases}$$

2. Assume that  $\arg(z_2) = -\pi$ , then  $\arg(z_1) = \arg(z_2) = -\pi$  and (16) holds.
3. Assume that  $\arg(z_2) \in (-\pi, 0]$ . Then,  $\arg(z_1) \in (-\pi, \arg(z_2)]$ ,  $\text{Im}(z_2) \leq 0$  and  $\text{Im}(z_1) \leq 0$ . From (15), we have

$$\frac{\text{Im}(z_1)}{\text{Re}(z_1) + |z_1|} \leq \frac{\text{Im}(z_2)}{\text{Re}(z_2) + |z_2|}.$$

Hence

$$\text{Im}(z_1) (\text{Re}(z_2) + |z_2|) + \text{Im}(z_2) (\text{Re}(z_2) + |z_2|) \leq \text{Im}(z_2) (\text{Re}(z_1) + |z_1|) + \text{Im}(z_2) (\text{Re}(z_2) + |z_2|),$$

wich gives

$$\text{Im}(z_1 + z_2) (\text{Re}(z_2) + |z_2|) \leq \text{Im}(z_2) (\text{Re}(z_1 + z_2) + |z_1| + |z_2|).$$

Since  $\text{Im}(z_2) \leq 0$ , then we have

$$\text{Im}(z_2) (|z_1| + |z_2|) \leq \text{Im}(z_2) |z_1 + z_2|.$$

Therefore

$$\text{Im}(z_1 + z_2) (\text{Re}(z_2) + |z_2|) \leq \text{Im}(z_2) (\text{Re}(z_1 + z_2) + |z_1 + z_2|),$$

and due to (15), we obtain

$$\arg(z_1 + z_2) \leq \arg(z_2).$$

We extend the above result, by rotation, for  $\arg(z_2) \in (-\pi, \pi)$ .

4. Similarly, for  $\arg(z_1), \arg(z_2) \in [0, \pi)$ , following the same steps, we deduce that

$$\arg(z_1) \leq \arg(z_1 + z_2),$$

and, by rotation, we obtain the expected result. □

**Proposition 14.** *Let  $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ . We have*

$$|z_1 + z_2| \geq (|z_1| + |z_2|) \left| \cos \left( \frac{\arg(z_1) - \arg(z_2)}{2} \right) \right|.$$

This result is given by Proposition 4.9, p. 1879 in [6].

**Proposition 15.** *Let  $0 < \alpha < \pi/2$  and  $z \in S_\alpha$ . We have*

1.  $|\arg(1 - e^{-z}) - \arg(1 + e^{-z})| < \alpha$ .
2.  $|1 + e^{-z}| \geq 1 - e^{-\pi/(2 \tan(\alpha))}$ .
3.  $\frac{|z| \cos(\alpha)}{1 + |z| \cos(\alpha)} \leq |1 - e^{-z}| \leq \frac{2|z|}{1 + |z| \cos(\alpha)}$ .

This result is given in Proposition 4.10, p. 1880 in [6].

**Corollary 16.** *Let  $\varepsilon_0, \varepsilon_1 \in (0, \pi/2)$  with  $\varepsilon_0 < \varepsilon_1$  and  $\bar{L} > 0$ . Then, there exists  $C > 0$  such that for all  $z \in S_{\varepsilon_0}$  and all  $\mu \in S_{\pi - \varepsilon_1} \cup \{0\}$ , we have*

$$\left| \frac{1 - e^{-\bar{L}\sqrt{z+\mu}}}{\sqrt{z+\mu} (1 + e^{-\bar{L}\sqrt{z+\mu}})} \right| \leq \frac{C}{|z|^{1/2}}.$$

*Proof.* If  $-\varepsilon_1 < \arg(\mu) < \pi - \varepsilon_1$ , since  $|\arg(z)| < \varepsilon_0$ , then from Proposition 13, we have

$$-\varepsilon_1 < \arg(z + \mu) < \pi - \varepsilon_1,$$

and

$$-\frac{\varepsilon_1}{2} < \arg(\bar{L}\sqrt{z + \mu}) = \frac{\arg(z + \mu)}{2} < \frac{\pi - \varepsilon_1}{2}.$$

If  $-\pi + \varepsilon_1 < \arg(\mu) < \varepsilon_1$ , since  $|\arg(z)| < \varepsilon_0$ , then from Proposition 13, we have

$$-\pi + \varepsilon_1 < \arg(z + \mu) < \varepsilon_1,$$

and

$$\frac{-\pi + \varepsilon_1}{2} < \arg(\bar{L}\sqrt{z + \mu}) = \frac{\arg(z + \mu)}{2} < \frac{\varepsilon_1}{2}.$$

Therefore, we always have

$$\bar{L}\sqrt{z + \mu} \in S_{\frac{\pi}{2} - \frac{\varepsilon_1}{2}}.$$

From Proposition 15, there exists  $C > 0$  such that

$$\begin{aligned} \left| \frac{1 - e^{-\bar{L}\sqrt{z + \mu}}}{\sqrt{z + \mu}(1 + e^{-\bar{L}\sqrt{z + \mu}})} \right| &\leq \frac{2\bar{L}|\sqrt{z + \mu}|}{\left(1 + \bar{L}|\sqrt{z + \mu}| \cos\left(\frac{\pi}{2} - \frac{\varepsilon_1}{2}\right)\right) |\sqrt{z + \mu}| \left(1 - e^{-\frac{\pi}{2 \tan\left(\frac{\pi}{2} - \frac{\varepsilon_1}{2}\right)}\right)} \\ &\leq \frac{2\bar{L}}{\left(1 + \bar{L}|\sqrt{z + \mu}| \sin\left(\frac{\varepsilon_1}{2}\right)\right) \left(1 - e^{-\frac{\pi}{2} \tan\left(\frac{\varepsilon_1}{2}\right)}\right)} \\ &\leq \frac{C}{|\sqrt{z + \mu}|} = \frac{C}{\sqrt{|z + \mu|}}. \end{aligned}$$

Moreover, from statement 1 of Proposition 15, we have

$$\begin{aligned} \left| \frac{1 - e^{-\bar{L}\sqrt{z + \mu}}}{\sqrt{z + \mu}(1 + e^{-\bar{L}\sqrt{z + \mu}})} \right| &\leq \frac{C}{\sqrt{(|z| + |\mu|) \left| \cos\left(\frac{\arg(z) - \arg(\mu)}{2}\right) \right|}} \\ &\leq \frac{C}{|z|^{1/2} \left| \cos\left(\frac{\arg(z) - \arg(\mu)}{2}\right) \right|^{1/2}}. \end{aligned}$$

Since  $|\arg(\mu)| < \pi - \varepsilon_1$  and  $|\arg(z)| < \varepsilon_0$ , with  $\varepsilon_1 > \varepsilon_0$ , it follows that

$$|\arg(z) - \arg(\mu)| < \pi - (\varepsilon_1 - \varepsilon_0) < \pi,$$

and

$$\cos\left(\frac{\arg(z) - \arg(\mu)}{2}\right) \geq \cos\left(\frac{\pi}{2} - \frac{\varepsilon_1 - \varepsilon_0}{2}\right) = \sin\left(\frac{\varepsilon_1 - \varepsilon_0}{2}\right) > 0,$$

which gives the result.  $\square$

**Corollary 17.** Let  $\alpha \in (0, \pi/2]$ ,  $\beta \in [0, \alpha/2]$  and  $z \in \mathbb{C} \setminus \{0\}$  such that  $|\operatorname{Im}(z)| \leq \pi$ . Then

1. if  $-\beta \leq \arg(z) < \alpha - \beta$ , then we have

$$-\beta \leq \arg(1 - e^{-z}) - \arg(1 + e^{-z}) < \alpha - \beta.$$

2. if  $-\alpha + \beta < \arg(z) \leq \beta$ , then we have

$$-\alpha + \beta < \arg(1 - e^{-z}) - \arg(1 + e^{-z}) \leq \beta.$$

*Proof.* Since  $S_\alpha$  is an open sector, then Proposition 15 remains true for  $\alpha = \pi/2$ .

1. First, let  $\beta = 0$ . Then, we have  $0 \leq \arg(z) < \alpha$ . As in the proof of statement 1 of Proposition 4.10, p. 1880 in [6], we have  $\operatorname{Re}(1 - e^{-z}), \operatorname{Re}(1 + e^{-z}) > 0$ . Let us prove that

$$\arg(1 - e^{-z}) - \arg(1 + e^{-z}) \geq 0.$$

To this end, we must show that

$$\arctan\left(\frac{\operatorname{Im}(1 - e^{-z})}{\operatorname{Re}(1 - e^{-z})}\right) \geq \arctan\left(\frac{\operatorname{Im}(1 + e^{-z})}{\operatorname{Re}(1 + e^{-z})}\right),$$

that is

$$\frac{\operatorname{Im}(1 - e^{-z})}{\operatorname{Re}(1 - e^{-z})} \geq \frac{\operatorname{Im}(1 + e^{-z})}{\operatorname{Re}(1 + e^{-z})},$$

or

$$\operatorname{Im}(1 - e^{-z})\operatorname{Re}(1 + e^{-z}) \geq \operatorname{Im}(1 + e^{-z})\operatorname{Re}(1 - e^{-z}). \quad (17)$$

Since we have

$$\begin{cases} \operatorname{Re}(1 - e^{-z}) = 1 - e^{-\operatorname{Re}(z)} \cos(\operatorname{Im}(z)), \\ \operatorname{Re}(1 + e^{-z}) = 1 + e^{-\operatorname{Re}(z)} \cos(\operatorname{Im}(z)), \\ \operatorname{Im}(1 - e^{-z}) = e^{-\operatorname{Re}(z)} \sin(\operatorname{Im}(z)), \\ \operatorname{Im}(1 + e^{-z}) = -e^{-\operatorname{Re}(z)} \sin(\operatorname{Im}(z)), \end{cases}$$

then, (17) is equivalent to

$$e^{-\operatorname{Re}(z)} \sin(\operatorname{Im}(z)) \left(1 + e^{-\operatorname{Re}(z)} \cos(\operatorname{Im}(z))\right) \geq -e^{-\operatorname{Re}(z)} \sin(\operatorname{Im}(z)) \left(1 - e^{-\operatorname{Re}(z)} \cos(\operatorname{Im}(z))\right);$$

hence

$$\sin(\operatorname{Im}(z)) \geq 0.$$

which is true since  $0 \leq \operatorname{Im}(z) \leq \pi$ . Now, taking into account that  $0 \leq \arg(z) < \alpha$  and Proposition 15, we obtain

$$0 \leq \arg(1 - e^{-z}) - \arg(1 + e^{-z}) < \alpha. \quad (18)$$

Now, let  $\beta \in \left(0, \frac{\alpha}{2}\right]$  and  $-\beta \leq \arg(z) < \alpha - \beta$ .

If  $0 \leq \arg(z) < \alpha - \beta$ , then from (18), we deduce that

$$-\beta \leq 0 \leq \arg(1 - e^{-z}) - \arg(1 + e^{-z}) < \alpha - \beta.$$

If  $-\beta < \arg(z) < 0$ , then  $z \in S_\beta$  and from Proposition 15, we have

$$-\beta \leq \arg(1 - e^{-z}) - \arg(1 + e^{-z}) < \beta.$$

Note that, when  $\arg(z) = -\beta$ , then the previous inequality holds true since  $\beta < \alpha$ . Finally, the result follows since  $\beta \leq \frac{\alpha}{2}$ .

2. In the same way, we obtain the expected result.

□

## 6 Proof of Theorem 9

### 6.1 Spectral study of $\mathcal{S}$

In this sections we will focus ourselves to study the spectral equation

$$\mathcal{S}w - \lambda w = f \in \mathcal{E} = L^p(-\ell, L; E), \quad (19)$$

where  $p \in (1, +\infty)$ .

Our aim is to estimate the resolvent operator

$$\left\| (\mathcal{S} - \lambda I)^{-1} \right\|_{\mathcal{L}(L^p(-\ell, L; E))},$$

where  $\lambda$  is a complex number in some sector to specify. This estimate will allow us to prove that  $\mathcal{S}$  generates an analytic semigroup in  $\mathcal{E}$ . So, after the resolution of the spectral equation, we have to estimate  $\|w\|_{L^p(-\ell, L; E)}$  that is

$$\|w_-\|_{L^p(-\ell, 0; E)} \quad \text{and} \quad \|w_+\|_{L^p(0, L; E)}.$$

We recall that all the constants  $r_-, r_+, d_-, d_+, q$  are strictly positive. In the sequel, we will use the following notations:

$$\lambda_{\pm} = \frac{\lambda}{d_{\pm}}, \quad \rho_{\pm} = \frac{r_{\pm}}{d_{\pm}}, \quad q_{\pm} = \frac{q}{d_{\pm}} \quad \text{and} \quad g_{\pm} = \frac{f_{\pm}}{d_{\pm}}. \quad (20)$$

#### 6.1.1 The system verified by the spectral equation

Equation (19) can be formulated as

$$(P_A) \left\{ \begin{array}{l} \left\{ \begin{array}{l} w_-''(x) + (A - \rho_- I - \lambda_- I) w_-(x) = g_-(x) \quad \text{in } (-\ell, 0), \\ w_+''(x) + (A - \rho_+ I - \lambda_+ I) w_+(x) = g_+(x) \quad \text{in } (0, L), \end{array} \right. \\ \left\{ \begin{array}{l} w_-(-\ell) = 0, \\ w_+(L) = 0, \end{array} \right. \\ \left\{ \begin{array}{l} w_-'(0) = q_-(w_+(0) - w_-(0)), \\ w_+'(0) = q_+(w_+(0) - w_-(0)). \end{array} \right. \end{array} \right.$$

Assume that the complex  $\lambda$  satisfies:

$$|\arg(\lambda)| < \pi - \varepsilon, \quad (21)$$

where  $\varepsilon \in (0, \pi)$  as in (14). Set

$$A_- = A - \rho_- I - \lambda_- I \quad \text{and} \quad A_+ = A - \rho_+ I - \lambda_+ I,$$

so we have

$$D(A_-) = D(A_+) = D(A).$$

**Proposition 18.** *Operators  $-A_-$  and  $-A_+$  are sectorial and satisfy*

$$\left\{ \begin{array}{l} -A_- \in \text{Sect}(\pi - \arcsin(1/M(-A_-))), \\ -A_+ \in \text{Sect}(\pi - \arcsin(1/M(-A_+))), \end{array} \right.$$

where  $M(-A_{\pm})$  is defined in Proposition 3.

*Proof.* If  $\lambda \in \mathbb{R}_+$ , then due to (13),  $-A_-$  and  $-A_+$  are sectorial operators.

Now, let  $\lambda \in S_{\pi-\varepsilon} \setminus \mathbb{R}_+$  and  $t > 0$ . We will verify that  $-A_-$  is a sectorial operator in  $E$  and  $(-\infty, 0] \subset \rho(-A_-)$ . Since  $\rho_- + t > 0$ , from Proposition 12, we obtain

$$|\arg(\rho_- + t + \lambda_-)| < |\arg(\lambda_-)| = |\arg(\lambda)| < \pi - \varepsilon.$$

Then

$$M(-A_-) = \sup_{t>0} \|t(-A_- + tI)^{-1}\| = \sup_{t>0} t \|(-A + (\rho_- + t + \lambda_-)I)^{-1}\|_{\mathcal{L}(E)}.$$

Then, from (13) and Proposition 14, we have

$$\begin{aligned} \|(-A + (\rho_- + t + \lambda_-)I)^{-1}\|_{\mathcal{L}(E)} &\leq \frac{C_A}{1 + |\rho_- + t + \lambda_-|} \leq \frac{C_A}{|\rho_- + t + \lambda_-|} \\ &\leq \frac{C_A}{(|\rho_- + \lambda_-| + t) \cos\left(\frac{|\arg(\rho_- + \lambda_-)|}{2}\right)} \\ &\leq \frac{C_A}{t \cos\left(\frac{|\arg(\rho_- + \lambda_-)|}{2}\right)}. \end{aligned}$$

Moreover, from Proposition 12, we obtain

$$|\arg(|\rho_- + \lambda_-|)| < |\arg(\lambda_-)| = |\arg(\lambda)| < \pi - \varepsilon,$$

and thus

$$\cos\left(\frac{|\arg(\rho_- + \lambda_-)|}{2}\right) > \cos\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right) = \sin\left(\frac{\varepsilon}{2}\right) > 0.$$

Finally, we have

$$M(-A_-) \leq \sup_{t>0} \left( \frac{t C_A}{t \sin\left(\frac{\varepsilon}{2}\right)} \right) = \frac{C_A}{\sin\left(\frac{\varepsilon}{2}\right)} < +\infty.$$

Hence, due to Proposition 3, we obtain the expected result for  $-A_-$ . For  $-A_+$ , the proof is similar.  $\square$

Therefore, the following operators

$$B = -\sqrt{-A}, \quad B_- = -\sqrt{-A_-} \quad \text{and} \quad B_+ = -\sqrt{-A_+},$$

are well defined for all  $\lambda \in S_{\pi-\varepsilon} \cup \{0\}$  and generate analytic semigroups in  $E$ , see [2].

By using estimates (28)-(29) in Lemma 4.2, see [9], there exists  $C > 0$ , independent of  $\lambda$  such that for all  $z \in S_{\pi-\varepsilon} \cup \{0\}$ , we have

$$\begin{cases} \|(B_- - zI)^{-1}\|_{\mathcal{L}(E)} \leq \frac{C}{\sqrt{1 + |\lambda_- + \rho_-| + |z|}} \leq \frac{C}{1 + |z|}, \\ \|(B_+ - zI)^{-1}\|_{\mathcal{L}(E)} \leq \frac{C}{\sqrt{1 + |\lambda_+ + \rho_+| + |z|}} \leq \frac{C}{1 + |z|}, \end{cases} \quad (22)$$

from which it follows that

$$\|B_-^{-1}\|_{\mathcal{L}(E)} \leq \frac{C\sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}} \quad \text{and} \quad \|B_+^{-1}\|_{\mathcal{L}(E)} \leq \frac{C\sqrt{d_+}}{\sqrt{d_+ + |\lambda + r_+|}}. \quad (23)$$

Our problem ( $P_A$ ) can be written in the following form

$$\begin{cases} \begin{cases} w''_-(x) - B_-^2 w_-(x) = g_-(x) & \text{in } (-\ell, 0), \\ w''_+(x) - B_+^2 w_+(x) = g_+(x) & \text{in } (0, L), \end{cases} \\ \begin{cases} w_-(-\ell) = 0, \\ w_+(L) = 0, \end{cases} \\ \begin{cases} w'_-(0) = q_-(w_+(0) - w_-(0)), \\ w'_+(0) = q_+(w_+(0) - w_-(0)). \end{cases} \end{cases} \quad (24)$$

Then, we have

$$\begin{cases} w_-(x) = e^{(x+\ell)B_-} j_- + e^{-xB_-} k_- + v_-(g_-)(x), & x \in (-\ell, 0), \\ w_+(x) = e^{xB_+} j_+ + e^{(L-x)B_+} k_+ + v_+(g_+)(x), & x \in (0, L), \end{cases}$$

with  $j_{\pm}, k_{\pm} \in E$  and

$$\begin{cases} v_-(g_-)(x) = \frac{1}{2} B_-^{-1} \int_{-\ell}^x e^{(x-t)B_-} g_-(t) dt + \frac{1}{2} B_-^{-1} \int_x^0 e^{(t-x)B_-} g_-(t) dt, \\ v_+(g_+)(x) = \frac{1}{2} B_+^{-1} \int_0^x e^{(x-t)B_+} g_+(t) dt + \frac{1}{2} B_+^{-1} \int_x^L e^{(t-x)B_+} g_+(t) dt; \end{cases} \quad (25)$$

see for instance [Proposition 4.4, p. 1878](#) in [6].

Then, we deduce that

$$\begin{cases} w'_-(0) = B_- e^{\ell B_-} j_- - B_- k_- + v'_-(g_-)(0), \\ w'_+(0) = B_+ j_+ - B_+ e^{LB_+} k_+ + v'_+(g_+)(0), \end{cases}$$

where

$$v'_-(g_-)(0) = \frac{1}{2} \int_{-\ell}^0 e^{-tB_-} g_-(t) dt \quad \text{and} \quad v'_+(g_+)(0) = -\frac{1}{2} \int_0^L e^{tB_+} g_+(t) dt.$$

The boundary and the interface conditions give

$$\begin{cases} \begin{cases} j_- = -e^{\ell B_-} k_- - v_-(g_-)(-\ell), \\ k_+ = -e^{LB_+} j_+ - v_+(g_+)(L), \end{cases} \\ \begin{cases} d_- \left( B_- e^{\ell B_-} j_- - B_- k_- + v'_-(g_-)(0) \right) = \\ q \left[ \left( j_+ + e^{LB_+} k_+ + v_+(g_+)(0) \right) - \left( e^{\ell B_-} j_- + k_- + v_-(g_-)(0) \right) \right], \\ d_+ \left( B_+ j_+ - B_+ e^{LB_+} k_+ + v'_+(g_+)(0) \right) = \\ q \left[ \left( j_+ + e^{LB_+} k_+ + v_+(g_+)(0) \right) - \left( e^{\ell B_-} j_- + k_- + v_-(g_-)(0) \right) \right]; \end{cases} \end{cases} \quad (26)$$

then, the two last equations lead us to the following system

$$\begin{cases} \begin{aligned} & d_- \left[ B_- e^{\ell B_-} \left( -e^{\ell B_-} k_- - v_-(g_-)(-\ell) \right) - B_- k_- + v'_-(g_-)(0) \right] \\ & = q \left[ \left( j_+ + e^{LB_+} \left( -e^{LB_+} j_+ - v_+(g_+)(L) \right) + v_+(g_+)(0) \right) \right. \\ & \quad \left. - e^{\ell B_-} \left( -e^{\ell B_-} k_- - v_-(g_-)(-\ell) \right) - k_- - v_-(g_-)(0) \right], \\ & d_+ \left[ B_+ j_+ - B_+ e^{LB_+} \left( -e^{LB_+} j_+ - v_+(g_+)(L) \right) + v'_+(g_+)(0) \right] \\ & = q \left[ \left( j_+ + e^{LB_+} \left( -e^{LB_+} j_+ - v_+(g_+)(L) \right) + v_+(g_+)(0) \right) \right. \\ & \quad \left. - e^{\ell B_-} \left( -e^{\ell B_-} k_- - v_-(g_-)(-\ell) \right) - k_- - v_-(g_-)(0) \right], \end{aligned} \end{cases}$$

which is equivalent to

$$\left\{ \begin{array}{l} -B_- e^{2\ell B^-} k_- - B_- e^{\ell B^-} v_-(g_-)(-\ell) - B_- k_- + v'_-(g_-)(0) \\ = q_- j_+ - q_- e^{2LB^+} j_+ - q_- e^{LB^+} v_+(g_+)(L) + q_- v_+(g_+)(0) \\ \quad + q_- e^{2\ell B^-} k_- + q_- e^{\ell B^-} v_-(g_-)(-\ell) - q_- k_- - q_- v_-(g_-)(0), \\ \\ B_+ j_+ + B_+ e^{2LB^+} j_+ + B_+ e^{LB^+} v_+(g_+)(L) + v'_+(g_+)(0) \\ = q_+ j_+ - q_+ e^{2LB^+} j_+ - q_+ e^{LB^+} v_+(g_+)(L) + q_+ v_+(g_+)(0) \\ \quad + q_+ e^{2\ell B^-} k_- + q_+ e^{\ell B^-} v_-(g_-)(-\ell) - q_+ k_- - q_+ v_-(g_-)(0). \end{array} \right.$$

Therefore, the above system becomes

$$\left\{ \begin{array}{l} [B_- (I + e^{2\ell B^-}) - q_- (I - e^{2\ell B^-})] k_- + q_- (I - e^{2LB^+}) j_+ = (\Pi'), \\ q_+ (I - e^{2\ell B^-}) k_- + [B_+ (I + e^{2LB^+}) - q_+ (I - e^{2LB^+})] j_+ = (\Pi''), \end{array} \right.$$

where

$$\left\{ \begin{array}{l} (\Pi') = v'_-(g_-)(0) - B_- e^{\ell B^-} v_-(g_-)(-\ell) + q_- e^{LB^+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ \quad - q_- e^{\ell B^-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \\ (\Pi'') = -v'_+(g_+)(0) - B_+ e^{LB^+} v_+(g_+)(L) - q_+ e^{LB^+} v_+(g_+)(L) + q_+ v_+(g_+)(0) \\ \quad + q_+ e^{\ell B^-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0). \end{array} \right.$$

It follows

$$\left\{ \begin{array}{l} [(I + e^{2\ell B^-}) - q_- B_-^{-1} (I - e^{2\ell B^-})] k_- + q_- B_-^{-1} (I - e^{2LB^+}) j_+ = B_-^{-1} (\Pi'), \\ q_+ B_+^{-1} (I - e^{2\ell B^-}) k_- + [(I + e^{2LB^+}) - q_+ B_+^{-1} (I - e^{2LB^+})] j_+ = B_+^{-1} (\Pi''). \end{array} \right.$$

The abstract determinant of this system is

$$\begin{aligned} D &= [(I + e^{2\ell B^-}) - q_- B_-^{-1} (I - e^{2\ell B^-})] [(I + e^{2LB^+}) - q_+ B_+^{-1} (I - e^{2LB^+})] \\ &\quad - q_+ B_+^{-1} (I - e^{2\ell B^-}) q_- B_-^{-1} (I - e^{2LB^+}) \\ &= (I + e^{2\ell B^-}) (I + e^{2LB^+}) - (I + e^{2\ell B^-}) q_+ B_+^{-1} (I - e^{2LB^+}) \\ &\quad - q_- B_-^{-1} (I - e^{2\ell B^-}) (I + e^{2LB^+}) \\ &= (I + e^{2\ell B^-}) (I + e^{2LB^+}) D_*, \end{aligned}$$

where

$$D_* = \left[ I - q_+ B_+^{-1} (I - e^{2LB^+}) (I + e^{2LB^+})^{-1} - q_- B_-^{-1} (I - e^{2\ell B^-}) (I + e^{2\ell B^-})^{-1} \right].$$

We recall that operators  $(I - e^{2LB^+})$ ,  $(I + e^{2LB^+})$ ,  $(I - e^{2\ell B^-})$  and  $(I + e^{2\ell B^-})$  are boundedly invertible by using the same method in [16], Proposition 2.3.6, p. 60.



### 6.1.2 Invertibility of the determinant

In order to invert  $D_*$ , we will use the  $H^\infty$ -calculus for sectorial operators. To this end, we consider the following equation

$$f(z) = \left[ 1 + \frac{q_+ \left(1 - e^{-2L\sqrt{z+\lambda_++\rho_+}}\right)}{\sqrt{z+\lambda_++\rho_+} \left(1 + e^{-2L\sqrt{z+\lambda_++\rho_+}}\right)} + \frac{q_- \left(1 - e^{-2\ell\sqrt{z+\lambda_--\rho_-}}\right)}{\sqrt{z+\lambda_--\rho_-} \left(1 + e^{-2\ell\sqrt{z+\lambda_--\rho_-}}\right)} \right],$$

for  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$  and for all  $z \in S_{\varepsilon_0}$ , with a suitable  $\varepsilon_0 > 0$ , specified in Proposition 19 below. Recall that  $\rho_\pm$ ,  $q_\pm$  and  $\lambda_\pm$  are defined in (20).

**Proposition 19.** *There exists a constant  $R > 0$  such that for any  $\varepsilon_0 \in \left(0, \arctan\left(\frac{\pi^2}{2R} \min\left(\frac{1}{L^2}, \frac{1}{\ell^2}\right)\right)\right]$  fixed,  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$  and for all  $z \in S_{\varepsilon_0}$ , we have*

$$|f(z)| > \sin\left(\frac{\varepsilon_0}{2}\right).$$

*Proof.* The proof is divided in two steps. In the first step, we consider that

$$|\operatorname{Im}(z + \lambda_\pm)| \leq \frac{\pi^2}{2} \min\left(\frac{1}{L^2}, \frac{1}{\ell^2}\right). \quad (27)$$

Thus

$$\left| \operatorname{Im}\left(2L\sqrt{z + \lambda_+ + \rho_+}\right) \right| = 2L\sqrt{\frac{|z + \lambda_+ + \rho_+| - \operatorname{Re}(z + \lambda_+ + \rho_+)}{2}} \leq \pi,$$

and

$$\left| \operatorname{Im}\left(2\ell\sqrt{z + \lambda_- + \rho_-}\right) \right| = 2\ell\sqrt{\frac{|z + \lambda_- + \rho_-| - \operatorname{Re}(z + \lambda_- + \rho_-)}{2}} \leq \pi,$$

Let  $\lambda \neq 0$ . We have to consider the two following cases:

1.  $-\varepsilon_0 \leq \arg(\lambda_\pm) < \pi - \varepsilon_0$ ,
2.  $-\pi + \varepsilon_0 < \arg(\lambda_\pm) \leq \varepsilon_0$ .

Let  $-\varepsilon_0 \leq \arg(\lambda_\pm) < \pi - \varepsilon_0$ . Since  $|\arg(z)| < \varepsilon_0$ , from Proposition 13, we have

$$-\varepsilon_0 \leq \min(\arg(\lambda_\pm), \arg(z)) \leq \arg(z + \lambda_\pm) \leq \max(\arg(\lambda_\pm), \arg(z)) < \pi - \varepsilon_0,$$

and thus, if  $\arg(z + \lambda_\pm) \neq 0$ , from Proposition 12, we deduce that

$$|\arg(z + \lambda_\pm + \rho_\pm)| < |\arg(z + \lambda_\pm)| < \pi - \varepsilon_0,$$

and if  $\arg(z + \lambda_\pm) = 0$ , then  $\arg(z + \lambda_\pm + \rho_\pm) = 0$ . Moreover, when  $\arg(z + \lambda_\pm + \rho_\pm) < 0$ , then due to Proposition 12, we have

$$-\varepsilon_0 \leq \arg(z + \lambda_\pm) < \arg(z + \lambda_\pm + \rho_\pm).$$

Therefore, we obtain that

$$-\varepsilon_0 \leq \arg(z + \lambda_\pm + \rho_\pm) < \pi - \varepsilon_0.$$

Hence, setting  $\bar{L} = 2L$  or  $2\ell$ , we deduce that

$$-\frac{\varepsilon_0}{2} \leq \arg\left(\sqrt{z + \lambda_\pm + \rho_\pm}\right) = \arg\left(\bar{L}\sqrt{z + \lambda_\pm + \rho_\pm}\right) < \frac{\pi}{2} - \frac{\varepsilon_0}{2}. \quad (28)$$

We set

$$z_1 = \frac{q_+ \left(1 - e^{-2L\sqrt{z+\lambda_+\rho_+}}\right)}{\sqrt{z+\lambda_+\rho_+} \left(1 + e^{-2L\sqrt{z+\lambda_+\rho_+}}\right)} \quad \text{and} \quad z_2 = \frac{q_- \left(1 - e^{-2\ell\sqrt{z+\lambda_-\rho_-}}\right)}{\sqrt{z+\lambda_-\rho_-} \left(1 + e^{-2\ell\sqrt{z+\lambda_-\rho_-}}\right)}.$$

Then, due to Proposition 14, it follows that

$$|f(z)| = |1 + z_1 + z_2| \geq (1 + |z_1 + z_2|) \left| \cos\left(\frac{\arg(z_1 + z_2)}{2}\right) \right| > \left| \cos\left(\frac{\arg(z_1 + z_2)}{2}\right) \right|.$$

Recall that

$$\arg(z_1) = \arg\left(1 - e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) - \arg\left(1 + e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) - \arg\left(\sqrt{z+\lambda_+\rho_+}\right),$$

and

$$\arg(z_2) = \arg\left(1 - e^{-2\ell\sqrt{z+\lambda_-\rho_-}}\right) - \arg\left(1 + e^{-2\ell\sqrt{z+\lambda_-\rho_-}}\right) - \arg\left(\sqrt{z+\lambda_-\rho_-}\right).$$

Then, due to (27) and (28), we can use Corollary 17, with  $\alpha = \frac{\pi}{2}$  and  $\beta = \frac{\varepsilon_0}{2}$ ; it follows that

$$-\frac{\varepsilon_0}{2} \leq \arg\left(1 - e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) - \arg\left(1 + e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) < \frac{\pi}{2} - \frac{\varepsilon_0}{2},$$

with

$$-\frac{\varepsilon_0}{2} \leq \arg\left(1 - e^{-2\ell\sqrt{z+\lambda_-\rho_-}}\right) - \arg\left(1 + e^{-2\ell\sqrt{z+\lambda_-\rho_-}}\right) < \frac{\pi}{2} - \frac{\varepsilon_0}{2},$$

hence

$$-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_+\rho_+}\right) \leq \arg(z_1) < \frac{\pi}{2} - \frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_+\rho_+}\right),$$

and

$$-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-\rho_-}\right) \leq \arg(z_2) < \frac{\pi}{2} - \frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-\rho_-}\right).$$

Then, due to (28), we obtain

$$-\frac{\pi}{2} < -\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_+\rho_+}\right) \leq \arg(z_1) < \frac{\pi}{2} - \frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_+\rho_+}\right) \leq \frac{\pi}{2},$$

with

$$-\frac{\pi}{2} < -\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-\rho_-}\right) \leq \arg(z_2) < \frac{\pi}{2} - \frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-\rho_-}\right) \leq \frac{\pi}{2}.$$

Thus,  $\arg(z_1), \arg(z_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Then, in virtue of Proposition 13, we deduce that

$$\min(\arg(z_1), \arg(z_2)) \leq \arg(z_1 + z_2) \leq \max(\arg(z_1), \arg(z_2)).$$

Moreover, due to Proposition 15, we have

$$\begin{aligned} |\arg(z_1)| &= \left| \arg\left(1 - e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) - \arg\left(1 + e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) - \arg\left(\sqrt{z+\lambda_+\rho_+}\right) \right| \\ &\leq \left| \arg\left(1 - e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) - \arg\left(1 + e^{-2L\sqrt{z+\lambda_+\rho_+}}\right) \right| + \left| \arg\left(\sqrt{z+\lambda_+\rho_+}\right) \right| \\ &< \frac{\pi}{2} - \frac{\varepsilon_0}{2} + \frac{\pi}{2} - \frac{\varepsilon_0}{2} = \pi - \varepsilon_0, \end{aligned}$$

and, in the same way, we have  $|\arg(z_2)| < \pi - \varepsilon_0$ . We then obtain

$$|\arg(z_1 + z_2)| \leq \max(\arg(z_1), \arg(z_2)) < \pi - \varepsilon_0,$$

and

$$\cos\left(\frac{\arg(z_1 + z_2)}{2}\right) > \cos\left(\frac{\pi}{2} - \frac{\varepsilon_0}{2}\right) = \sin\left(\frac{\varepsilon_0}{2}\right) > 0.$$

Therefore, for all  $z \in S_{\varepsilon_0}$  such that  $|\operatorname{Im}(z + \lambda_{\pm})| \leq \frac{\pi^2}{2} \min\left(\frac{1}{L^2}, \frac{1}{\ell^2}\right)$ , we have

$$|f(z)| > \sin\left(\frac{\varepsilon_0}{2}\right).$$

Now, when  $-\pi + \varepsilon_0 < \arg(\lambda_{\pm}) \leq \varepsilon_0$ , then

$$-\frac{\pi}{2} + \frac{\varepsilon_0}{2} < \arg(\sqrt{z + \lambda_{\pm} + \rho_{\pm}}) = \arg(\bar{L}\sqrt{z + \lambda_{\pm} + \rho_{\pm}}) \leq \frac{\varepsilon_0}{2},$$

and due to Corollary 17, we have  $\arg(z_1), \arg(z_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Thus, from Proposition 13, we deduce the expected result. Finally, when  $\lambda = 0$ , it is clear that, following the same steps, we obtain a similar result.

Now, in the second step, for  $\operatorname{Re}(z)$  enough large, with  $z \in S_{\varepsilon_0}$ ,  $\lambda_{\pm} \in S_{\pi - \varepsilon_0}$  and  $\rho_{\pm} > 0$ , we will prove that

$$|f(z)| > \sin\left(\frac{\varepsilon_0}{2}\right).$$

Since  $\lambda_{\pm} \in S_{\pi - \varepsilon_0}$  and  $\rho_{\pm} > 0$ , due to Proposition 12, there exists  $\varepsilon_1 > \varepsilon_0$  such that

$$\lambda_{\pm} + \rho_{\pm} \in S_{\pi - \varepsilon_1}.$$

Since  $z \in S_{\varepsilon_0}$ , from Corollary 16, with  $\mu = \lambda_{\pm} + \rho_{\pm}$  and  $\bar{L} = 2L$  or  $2\ell$ , there exist  $R_+, R_- > 0$  such that

$$\left| \frac{q_+ \left(1 - e^{-2L\sqrt{z + \lambda_+ + \rho_+}}\right)}{\sqrt{z + \lambda_+ + \rho_+} \left(1 + e^{-2L\sqrt{z + \lambda_+ + \rho_+}}\right)} \right| < \frac{1}{2} - \frac{\sin\left(\frac{\varepsilon_0}{2}\right)}{2} \quad \text{for } |z| > R_+,$$

and

$$\left| \frac{q_- \left(1 - e^{-2\ell\sqrt{z + \lambda_- + \rho_-}}\right)}{\sqrt{z + \lambda_- + \rho_-} \left(1 + e^{-2\ell\sqrt{z + \lambda_- + \rho_-}}\right)} \right| < \frac{1}{2} - \frac{\sin\left(\frac{\varepsilon_0}{2}\right)}{2} \quad \text{for } |z| > R_-.$$

Therefore, for  $|z| > R = \max(R_+, R_-)$ , we obtain that

$$|f(z)| \geq |1 - |z_1 + z_2|| \geq |1 - |z_1| - |z_2|| > 1 - \frac{1}{2} + \frac{\sin\left(\frac{\varepsilon_0}{2}\right)}{2} - \frac{1}{2} + \frac{\sin\left(\frac{\varepsilon_0}{2}\right)}{2} = \sin\left(\frac{\varepsilon_0}{2}\right).$$

It follows that, for  $\operatorname{Re}(z) > R$ , the previous inequality holds.  $\square$

**Proposition 20.** *Let  $\lambda \in S_{\pi - \varepsilon_0} \cup \{0\}$ , where  $\varepsilon_0$  is specified in Proposition 19. Then operator  $D$  is boundedly invertible and*

$$D^{-1} = D_*^{-1} \left( I + e^{2LB_+} \right)^{-1} \left( I + e^{2\ell B_-} \right)^{-1}.$$

*Proof.* From Proposition 19,  $f$  does not vanish on  $S_{\varepsilon_0}$  and  $1/f$  is bounded. Moreover

$$\left\| \frac{1}{f} \right\|_{\infty} \leq \frac{1}{\sin(\varepsilon_0/2)},$$

and taking into account Corollary 16, we have

$$|f(z) - 1| = O\left(\frac{1}{|z|^{1/2}}\right), \quad z \in S_{\varepsilon_0} \quad \text{and} \quad |z| \rightarrow +\infty.$$

We deduce that

$$\begin{aligned} \left(\frac{1}{f}\right)(-A) &= \left(\frac{1-f}{f} + 1\right)(-A) \\ &= \left(\frac{1-f}{f}\right)(-A) + I \in \mathcal{L}(E), \end{aligned}$$

then by using (11), since  $-A \in \text{Sect}(\varepsilon_0)$ , we deduce that

$$D_*^{-1} = \left(\frac{1}{f}\right)(-A) \in \mathcal{L}(E) \quad \text{and} \quad \left\| D_*^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{1}{\sin(\varepsilon_0/2)}.$$

Finally, taking into account the fact that  $(I + e^{2\ell B_-})$  and  $(I + e^{2\ell B_+})$  are boundedly invertible, by using the same method as in [6], Lemma 5.2, p. 1883, we obtain the expected result.  $\square$

**Remark 21.** Now, from equality  $D_* A^{-1} = A^{-1} D_*$ , it follows that

$$D_*^{-1} A = A D_*^{-1},$$

on  $D(A)$ , hence  $D_*^{-1}$  is a bounded operator from  $D(A)$  into itself. Therefore, by interpolation  $D_*^{-1}$  is bounded from any interpolation space  $(D(A), E)_{\alpha, \beta}$ , for all  $\alpha \in (0, 1)$  and  $\beta \in [1, +\infty]$  (see the definition in [11]) into itself and clearly we have also the same estimate

$$\left\| D_*^{-1} \right\|_{\mathcal{L}((D(A), E)_{\alpha, \beta})} \leq \frac{1}{\sin(\varepsilon_0/2)}.$$

### 6.1.3 Resolution of the system

Assume that  $\lambda \in S_{\pi - \varepsilon_0} \cup \{0\}$ . Recall that

$$\begin{cases} \left[ (I + e^{2\ell B_-}) - q_- B_-^{-1} (I - e^{2\ell B_-}) \right] k_- + q_- B_-^{-1} (I - e^{2\ell B_+}) j_+ = B_-^{-1}(\Pi'), \\ q_+ B_+^{-1} (I - e^{2\ell B_-}) k_- + \left[ (I + e^{2\ell B_+}) - q_+ B_+^{-1} (I - e^{2\ell B_+}) \right] j_+ = B_+^{-1}(\Pi''), \end{cases}$$

where

$$\begin{cases} (\Pi') &= v'_-(g_-)(0) - B_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- e^{LB_+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ &\quad - q_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \\ (\Pi'') &= -v'_+(g_+)(0) - B_+ e^{LB_+} v_+(g_+)(L) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ v_+(g_+)(0), \\ &\quad + q_+ e^{\ell B_-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0), \end{cases}$$

therefore

$$\begin{aligned} k_- &= D^{-1} \begin{vmatrix} B_-^{-1}(\Pi') & q_- B_-^{-1}(I - e^{2\ell B_+}) \\ B_+^{-1}(\Pi'') & \left[ (I + e^{2\ell B_+}) - q_+ B_+^{-1} (I - e^{2\ell B_+}) \right] \end{vmatrix} \\ &= D^{-1} \left[ B_-^{-1} \left[ (I + e^{2\ell B_+}) - q_+ B_+^{-1} (I - e^{2\ell B_+}) \right] (\Pi') - q_- B_+^{-1} B_-^{-1} (I - e^{2\ell B_+}) (\Pi'') \right], \end{aligned}$$

and

$$\begin{aligned} j_+ &= D^{-1} \begin{vmatrix} \left[ (I + e^{2\ell B_-}) - q_- B_-^{-1} (I - e^{2\ell B_-}) \right] & B_-^{-1}(\Pi') \\ q_+ B_+^{-1} (I - e^{2\ell B_-}) & B_+^{-1}(\Pi'') \end{vmatrix} \\ &= D^{-1} \left[ \left[ (I + e^{2\ell B_-}) - q_- B_-^{-1} (I - e^{2\ell B_-}) \right] B_+^{-1}(\Pi'') - q_+ B_+^{-1} (I - e^{2\ell B_-}) B_-^{-1}(\Pi') \right]. \end{aligned}$$

We then deduce

$$\begin{cases} j_- &= -e^{\ell B_-} D^{-1} B_-^{-1} \left[ (I + e^{2\ell B_+}) - q_+ B_+^{-1} (I - e^{2\ell B_+}) \right] (\Pi') \\ &\quad + q_- e^{\ell B_-} D^{-1} B_+^{-1} B_-^{-1} (I - e^{2\ell B_+}) (\Pi'') - v_-(g_-)(-\ell), \\ k_+ &= -e^{L B_+} D^{-1} \left[ (I + e^{2\ell B_-}) - q_- B_-^{-1} (I - e^{2\ell B_-}) \right] B_+^{-1}(\Pi'') \\ &\quad + q_+ e^{L B_+} D^{-1} B_+^{-1} (I - e^{2\ell B_-}) B_-^{-1}(\Pi') - v_+(g_+)(L). \end{cases}$$

Finally, we obtain for *a.e.*  $x \in (-\ell, 0)$

$$\begin{aligned} w_-(x) &= D^{-1} \left( e^{-x B_-} - e^{(x+2\ell) B_-} \right) \left[ (I + e^{2\ell B_+}) - q_+ B_+^{-1} (I - e^{2\ell B_+}) \right] B_-^{-1}(\Pi') \\ &\quad + q_- D^{-1} \left( e^{(x+2\ell) B_-} - e^{-x B_-} \right) B_-^{-1} (I - e^{2\ell B_+}) B_+^{-1}(\Pi'') \\ &\quad - e^{(x+\ell) B_-} v_-(g_-)(-\ell) + v_-(g_-)(x), \end{aligned}$$

and for *a.e.*  $x \in (0, L)$

$$\begin{aligned} w_+(x) &= D^{-1} \left( e^{x B_+} - e^{(2L-x) B_+} \right) \left[ (I + e^{2\ell B_-}) - q_- B_-^{-1} (I - e^{2\ell B_-}) \right] B_+^{-1}(\Pi'') \\ &\quad + q_+ D^{-1} \left( e^{(2L-x) B_+} - e^{x B_+} \right) B_+^{-1} (I - e^{2\ell B_-}) B_-^{-1}(\Pi') \\ &\quad - e^{(L-x) B_+} v_+(g_+)(L) + v_+(g_+)(x). \end{aligned}$$

#### 6.1.4 Optimal regularity of $w_-$ and $w_+$

Let  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ . Since  $B_+$  generates an analytic semigroup in  $E$ , we recall the following known results

$$\begin{cases} x \mapsto e^{x B_+} \psi \in L^p(0, L; E) \text{ for all } \psi \in E, \\ x \mapsto B_+^n e^{x B_+} \psi \in L^p(0, L; E) \iff \psi \in (D(B_+^n), E)_{\frac{1}{np}, p}, \end{cases} \quad (29)$$

where  $p \in (1, +\infty)$  and  $n \in \mathbb{N} \setminus \{0\}$ ; see the Theorem in [20], p. 96.

We have the same result for  $B_-$  on  $(-\ell, 0)$ . Note that, for these two results, we do not need assumption (12).

Let us recall the following well-known important result proved in [7].

**Theorem 22.** *Let  $X$  be a UMD Banach space,  $-Q \in BIP(X, \theta)$  with  $\theta \in (0, \pi/2)$  and  $g \in L^p(a, b; X)$ . Then, for almost every  $x \in (a, b)$ , we have*

$$\int_a^x e^{(x-s)Q} g(s) ds \in D(Q) \quad \text{and} \quad \int_x^b e^{(s-x)Q} g(s) ds \in D(Q).$$

Moreover,

$$x \mapsto Q \int_a^x e^{(x-s)Q} g(s) ds \in L^p(a, b; X) \quad \text{and} \quad x \mapsto Q \int_x^b e^{(s-x)Q} g(s) ds \in L^p(a, b; X).$$

We are applying these results to our operators  $B_{\pm}$ , with  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ , where  $\varepsilon_0 \in (0, \pi/2)$  is specified in Proposition 19. We know that  $-A \in \text{BIP}(E, \varepsilon_0)$  and  $|\arg(\lambda_{\pm} + \rho_{\pm})| < \pi - \varepsilon_0$ . On the other hand, we have

$$\varepsilon_0 + |\arg(\lambda_{\pm} + \rho_{\pm})| < \varepsilon_0 + \pi - \varepsilon_0 = \pi.$$

In virtue of Theorem 2.4, p. 408 in [17], we obtain that  $-A_{\pm} \in \text{BIP}(E, \pi - \varepsilon_0)$ . Thus, we obtain that  $-B_{\pm} \in \text{BIP}(E, (\pi - \varepsilon_0)/2)$ , from Proposition 3.2.1, e), p. 71 in [12].

We then deduce the following lemma by taking  $Q = B_{\pm}$ .

**Lemma 23.** *Let  $h_- \in L^p(-\ell, 0; E)$  and  $h_+ \in L^p(0, L; E)$  with  $1 < p < +\infty$ . Assume that (12), (13) and (14) hold. Then, we have*

$$\begin{cases} x \mapsto B_- \int_{-\ell}^x e^{(x-s)B_-} h_-(s) ds \in L^p(-\ell, 0; E), \\ x \mapsto B_- \int_x^0 e^{(s-x)B_-} h_-(s) ds \in L^p(-\ell, 0; E), \end{cases}$$

and

$$\begin{cases} x \mapsto B_+ \int_0^x e^{(x-s)B_+} h_+(s) ds \in L^p(0, L; E), \\ x \mapsto B_+ \int_x^L e^{(s-x)B_+} h_+(s) ds \in L^p(0, L; E). \end{cases}$$

**Lemma 24.** *Let  $h_- \in L^p(-\ell, 0; E)$  and  $h_+ \in L^p(0, L; E)$  with  $1 < p < +\infty$ . Assume that (12), (13) and (14) hold. Then, we have*

1.  $\int_0^L e^{sB_+} h_+(s) ds$  and  $\int_0^L e^{(L-s)B_+} h_+(s) ds$  belong to  $(D(B_+), E)_{\frac{1}{p}, p} = (D(B), E)_{\frac{1}{p}, p}$ ,
2.  $\int_{-\ell}^0 e^{(s+\ell)B_-} h_-(s) ds$  and  $\int_{-\ell}^0 e^{-sB_-} h_-(s) ds$  belong to  $(D(B_-), E)_{\frac{1}{p}, p} = (D(B), E)_{\frac{1}{p}, p}$ .

*Proof.* Let us indicate the proof of the first statement for instance. Consider the function

$$\psi_1(x) = \int_0^x e^{(x-s)B_+} h_+(s) ds;$$

then, from Theorem 22, we know that

$$\psi_1 \in W^{1,p}(0, L; E) \cap L^p(0, L; D(B_+));$$

by using the notation in [11], pp. 677-678 for the spaces of traces, we deduce that

$$\psi_1(L) \in T_0^1(p, 0, D(B_+), E) = (D(B_+), E)_{\frac{1}{p}, p} = (D(B), E)_{\frac{1}{p}, p},$$

here, the Poulsen condition is verified since  $0 < 1/p < 1$ . By considering the function

$$\psi_2(x) = \int_x^L e^{(s-x)B_+} h_+(s) ds,$$

we get

$$\psi_2(0) \in T_0^1(p, 0, D(B_+), E) = (D(B_+), E)_{\frac{1}{p}, p} = (D(B), E)_{\frac{1}{p}, p}.$$

Statement 2 is obtained analogously. □

**Proposition 25.** *Let  $f \in L^p(-\ell, L; E)$  with  $1 < p < +\infty$ . Assume that (12), (13) and (14) hold. Then, for all  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ , there exists a unique solution  $w \in D(\mathcal{S})$  of equation (19).*

*Proof.* Now we must show that

$$\begin{cases} w_- \in W^{2,p}(-\ell, 0; E) \cap L^p(-\ell, 0; D(A)), \\ w_+ \in W^{2,p}(0, L; E) \cap L^p(0, L; D(A)). \end{cases}$$

It is not difficult to see that all the boundary and transmission conditions in (24) are verified by  $w_-$  and  $w_+$ .

From Proposition 4.4 in [6], p. 1878, to prove that

$$x \mapsto w_-(x) = e^{(x+\ell)B_-} j_- + e^{-xb_-} k_- + v_-(g_-)(x) \in W^{2,p}(-\ell, 0; E) \cap L^p(-\ell, 0; D(A)),$$

it suffices to show that  $j_-$  and  $k_-$  belong to  $(D(B_-^2), E)_{\frac{1}{2p}, p}$ . Recall that due to (26), we have

$$j_- = -e^{\ell B_-} k_- - v_-(g_-)(-\ell).$$

It is clear that

$$e^{\ell B_-} k_- \in D(B_-^2) \subset (D(B_-^2), E)_{\frac{1}{2p}, p}.$$

Moreover, due to (25), we have

$$v_-(g_-)(-\ell) = \frac{1}{2} B_-^{-1} \int_{-\ell}^0 e^{(t+\ell)B_-} g_-(t) dt,$$

thus, from Lemma 24, it follows that

$$B_- v_-(g_-)(-\ell) \in (D(B_-), E)_{\frac{1}{p}, p},$$

hence

$$v_-(g_-)(-\ell) \in (D(B_-), E)_{1+\frac{1}{p}, p} = (D(B_-^2), E)_{\frac{1}{2p}, p}.$$

Furthermore, recall that

$$k_- = D^{-1} \left[ B_-^{-1} \left[ (I + e^{2LB_+}) - q_+ B_+^{-1} (I - e^{2LB_+}) \right] (\Pi') - q_- B_+^{-1} B_-^{-1} (I - e^{2LB_+}) (\Pi'') \right],$$

where

$$\begin{aligned} (\Pi') &= -B_- e^{\ell B_-} v_-(g_-)(-\ell) + v'_-(g_-)(0) + q_- e^{LB_+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ &\quad - q_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \end{aligned}$$

and

$$\begin{aligned} (\Pi'') &= -B_+ e^{LB_+} v_+(g_+)(L) - v'_+(g_+)(0) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ v_+(g_+)(0) \\ &\quad + q_+ e^{\ell B_-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0). \end{aligned}$$

From Remark 21, interpolation spaces are invariant for  $D^{-1}$ , therefore, in order to prove that  $k_- \in (D(B_-^2), E)_{\frac{1}{2p}, p}$ , it is sufficient to show that

$$(\Pi'), (\Pi'') \in (D(B_-), E)_{\frac{1}{p}, p} = (D(B), E)_{\frac{1}{p}, p} = (D(B_+), E)_{\frac{1}{p}, p}.$$

For  $(\Pi')$ , we have

$$-B_- e^{\ell B_-} v_-(g_-)(-\ell) - q_- e^{\ell B_-} v_-(g_-)(-\ell) \in D(B_-^2) \subset (D(B_-), E)_{\frac{1}{p}, p},$$

and

$$q_- e^{LB_+} v_+(g_+)(L) \in D(B_+^2) \subset (D(B_+), E)_{\frac{1}{p}, p}.$$

Similarly, for  $(\Pi'')$ , we have

$$-B_+ e^{LB_+} v_+(g_+)(L) - q_+ e^{LB_+} v_+(g_+)(L) \in D(B_+^2) \subset (D(B_+), E)_{\frac{1}{p}, p},$$

and

$$q_+ e^{\ell B_-} v_-(g_-)(-\ell) \in D(B_-^2) \subset (D(B_-), E)_{\frac{1}{p}, p}.$$

Then, it remains to prove that

$$v'_-(g_-)(0) - q_- v_+(g_+)(0) + q_- v_-(g_-)(0) \in (D(B), E)_{\frac{1}{p}, p},$$

and

$$-v'_+(g_+)(0) + q_+ v_+(g_+)(0) - q_+ v_-(g_-)(0) \in (D(B), E)_{\frac{1}{p}, p}.$$

From Lemma 24, we have

$$v'_-(g_-)(0) = B_- v_-(g_-)(0) = \frac{1}{2} \int_{-\ell}^0 e^{-tB_-} g_-(t) dt \in (D(B_-), E)_{\frac{1}{p}, p},$$

and

$$-v'_+(g_+)(0) = B_+ v_+(g_+)(0) = \frac{1}{2} \int_0^L e^{tB_+} g_+(t) dt \in (D(B_+), E)_{\frac{1}{p}, p},$$

hence

$$v_-(g_-)(0) = \frac{1}{2} B_-^{-1} \int_{-\ell}^0 e^{-tB_-} g_-(t) dt \in (D(B_-), E)_{1+\frac{1}{p}, p} \subset (D(B_-), E)_{\frac{1}{p}, p},$$

and

$$v_+(g_+)(0) = \frac{1}{2} B_+^{-1} \int_0^L e^{tB_+} g_+(t) dt \in (D(B_+), E)_{1+\frac{1}{p}, p} \subset (D(B_+), E)_{\frac{1}{p}, p}.$$

The proof for  $w_+$  is analogous. Therefore  $w \in D(\mathcal{S})$ . □

## 6.2 Estimate of the resolvent operator

In all the sequel,  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ , where  $\varepsilon_0 \in \left(0, \arctan\left(\frac{\pi^2}{2R} \min\left(\frac{1}{L^2}, \frac{1}{\ell^2}\right)\right)\right]$  is fixed.

### 6.2.1 Some sharp estimates

Recall that

$$B_- = -\sqrt{-A + \rho_- I + \lambda_- I} \quad \text{and} \quad B_+ = -\sqrt{-A + \rho_+ I + \lambda_+ I},$$

where

$$\lambda_{\pm} = \frac{\lambda}{d_{\pm}} \quad \text{and} \quad \rho_{\pm} = \frac{r_{\pm}}{d_{\pm}}.$$

Recall that Lemma 2.6, p. 103 in [8] gives the following result

$$\left\{ \begin{array}{l} \exists C > 0, \exists c > 0, \forall \kappa \in \mathbb{R}, \forall t \geq t_0 > 0, \forall \lambda \in S_{\pi-\varepsilon_0} \cup \{0\} : \\ \left\| B_{\pm}^{\kappa} e^{tB_{\pm}} \right\|_{\mathcal{L}(E)} \leq C e^{-ct|\lambda_{\pm} + \rho_{\pm}|^{1/2}}. \end{array} \right. \quad (30)$$

**Lemma 26.** *There exists a constant  $C > 0$  independent of  $\lambda$ , such that operators  $I \pm e^{2\ell B_-}$  and  $I \pm e^{2LB_+}$  are boundedly invertible in  $\mathcal{L}(E)$  and*

$$\left\| \left( I \pm e^{2\ell B_-} \right)^{-1} \right\|_{\mathcal{L}(E)} \leq C \quad \text{and} \quad \left\| \left( I \pm e^{2LB_+} \right)^{-1} \right\|_{\mathcal{L}(E)} \leq C.$$



The proof is similar to the one of Lemma 5.1 in [9].

As a consequence of Proposition 20 and Lemma 26, we deduce the following Corollary.

**Corollary 27.** *There exists a constant  $C > 0$  independent of  $\lambda$ , such that*

$$\|D^{-1}\|_{\mathcal{L}(E)} \leq C.$$

**Proposition 28.** *Let  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ ,  $g \in L^p(0, L; E)$  and  $h \in L^p(-\ell, 0; E)$  with  $1 < p < +\infty$ . Set*

$$U(g)(x) = \int_0^L e^{|x-t|B_+} g(t) dt \quad \text{and} \quad V(h)(x) = \int_{-\ell}^0 e^{|x-t|B_-} h(t) dt.$$

*Then, there exists a constant  $C > 0$  independent of  $\lambda$  such that*

$$\begin{cases} \|U(g)\|_{L^p(0, L; E)} & \leq \frac{C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}} \|g\|_{L^p(0, L; E)}, \\ \|V(h)\|_{L^p(-\ell, 0; E)} & \leq \frac{C\sqrt{d_-}}{\sqrt{|\lambda + r_-| + d_-}} \|h\|_{L^p(-\ell, 0; E)}. \end{cases}$$

*Proof.* We will prove the estimate, for instance, for  $B_+$ . Let

$$g \in \mathcal{D}(0, L; \mathcal{L}(E)) \subset \mathcal{S}(\mathbb{R}; \mathcal{L}(E)),$$

where  $\mathcal{D}(0, L; \mathcal{L}(E))$  is the space of all vector-valued test functions on  $\mathcal{L}(E)$  and  $\mathcal{S}(\mathbb{R}; \mathcal{L}(E))$  is the Schwartz space of rapidly decreasing vector-valued smooth functions on  $\mathbb{R}$ . Then, we can write

$$U(g)(x) = \left( e^{|\cdot|B_+} * g \right) (x) = \left( g * e^{|\cdot|B_+} \right) (x), \quad x \in \mathbb{R}.$$

This abstract convolution is well defined, see [1].

Recall the abstract Fourier transform  $F$  defined by

$$F(\psi)(x) = \int_{-\infty}^{+\infty} e^{-2i\pi\xi x} \psi(\xi) d\xi,$$

for all  $\psi \in L^1(\mathbb{R}; \mathcal{L}(E))$  and the well known property

$$F^{-1}(F(\phi)) = \phi,$$

for all  $\phi \in \mathcal{S}(\mathbb{R}; \mathcal{L}(E))$ . We have

$$\begin{aligned} F\left(e^{|\cdot|B_+}\right)(\xi) &= \int_{-\infty}^0 e^{-2i\pi\xi x} e^{-xB_+} dx + \int_0^{+\infty} e^{-2i\pi\xi x} e^{xB_+} dx \\ &= -(B_+ + 2i\pi\xi I)^{-1} - (B_+ - 2i\pi\xi I)^{-1} \\ &= -2B_+ (B_+ + 2i\pi\xi I)^{-1} (B_+ - 2i\pi\xi I)^{-1}; \end{aligned}$$

the integrals are absolutely convergent from (30); the last equality holds from the resolvent identity. In virtue of Theorem 3.6, p. 17 in [1], we obtain

$$U(g)(x) = F^{-1}\left(F\left(e^{|\cdot|B_+}\right)F(g)\right)(x) = F^{-1}(mF(g))(x),$$

with the Fourier multiplier

$$m(\xi) = -2B_+ (B_+ + 2i\pi\xi I)^{-1} (B_+ - 2i\pi\xi I)^{-1} \in \mathcal{L}(E).$$

Using estimate (29), p. 14, in [9], we obtain

$$\begin{aligned}\|m(\xi)\|_{\mathcal{L}(E)} &= \left\| -2B_+ (B_+ + 2i\pi\xi I)^{-1} (B_+ - 2i\pi\xi I)^{-1} \right\| \\ &\leq 2C \left\| (B_+ - 2i\pi\xi I)^{-1} \right\| \\ &\leq \frac{2C}{\sqrt{\frac{|\lambda + r_+|}{d_+} + 1 + |2i\pi\xi|}}.\end{aligned}$$

Then

$$\|m(\xi)\|_{\mathcal{L}(E)} \leq \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+ + 2\pi\sqrt{d_+}|\xi|}} \leq \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}},$$

hence

$$\sup_{\xi \in \mathbb{R}} \|m(\xi)\|_{\mathcal{L}(E)} \leq \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}}.$$

Now, we must estimate

$$\sup_{\xi \in \mathbb{R}} |\xi| \|m'(\xi)\|_{\mathcal{L}(E)}.$$

Due to the analyticity of the resolvent operator of  $B_+$  on the imaginary axis, it follows that

$$m \in C^\infty(\mathbb{R}, \mathcal{L}(E)),$$

and

$$m'(\xi) = 2i\pi (B_+ + 2i\pi\xi I)^{-2} - 2i\pi (B_+ - 2i\pi\xi I)^{-2}.$$

Therefore, as above, we have

$$\begin{aligned}|\xi| \|m'(\xi)\|_{\mathcal{L}(E)} &\leq \left\| 2i\pi\xi (B_+ + 2i\pi\xi I)^{-2} \right\| + \left\| 2i\pi\xi (B_+ - 2i\pi\xi I)^{-2} \right\| \\ &\leq \left\| 2i\pi\xi (B_+ + 2i\pi\xi I)^{-1} \right\| \left\| (B_+ + 2i\pi\xi I)^{-1} \right\| \\ &\quad + \left\| 2i\pi\xi (B_+ - 2i\pi\xi I)^{-1} \right\| \left\| (B_+ - 2i\pi\xi I)^{-1} \right\| \\ &\leq \frac{2C}{\sqrt{\frac{|\lambda + r_+|}{d_+} + 1 + |2i\pi\xi|}},\end{aligned}$$

so

$$\sup_{\xi \in \mathbb{R}} |\xi| \|m'(\xi)\|_{\mathcal{L}(E)} \leq \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}}.$$

We do similarly with  $\xi \mapsto (\xi m'(\xi))'$ . Thus, from [21], Proposition 2.5, p. 739, the sets

$$\{m(\xi), \xi \in \mathbb{R} \setminus \{0\}\} \quad \text{and} \quad \{\xi m'(\xi), \xi \in \mathbb{R} \setminus \{0\}\},$$

are  $R$ -bounded. Moreover, applying Theorem 3.4, p. 746 in [21], we obtain

$$\begin{aligned}\|U(g)\|_{L^p(0,L;E)} &= \left\| F^{-1} (mF(g)) \right\|_{L^p(\mathbb{R};E)} \\ &\leq C \left[ \sup_{\xi \in \mathbb{R}} \|m(\xi)\| + \sup_{\xi \in \mathbb{R}} |\xi| \|m'(\xi)\| \right] \|g\|_{L^p(0,L;E)} \\ &\leq \frac{C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}} \|g\|_{L^p(0,L;E)},\end{aligned}$$

for all  $g \in \mathcal{D}(0, L; E)$ . The same estimate is true for all  $g \in L^p(0, L; E)$  by density.  $\square$

We will need also the following result, given by Lemma 4.12 in [13].

**Lemma 29.** *Let  $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ ,  $g_+ \in L^p(0, L; E)$  and  $g_- \in L^p(-\ell, 0; E)$  with  $1 < p < +\infty$ . Then, there exists  $C > 0$  independent of  $\lambda$  such that*

1.  $\left\| e^{(\cdot+\ell)B_-} \int_{-\ell}^0 e^{(t+\ell)B_-} g_-(t) dt \right\|_{L^p(-\ell, 0; E)} \leq \frac{C\sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}} \|g_-\|_{L^p(-\ell, 0; E)},$
2.  $\left\| e^{(\cdot+\ell)B_-} \int_{-\ell}^0 e^{-tB_-} g_-(t) dt \right\|_{L^p(-\ell, 0; E)} \leq \frac{C\sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}} \|g_-\|_{L^p(-\ell, 0; E)},$
3.  $\left\| e^{-\cdot B_-} \int_{-\ell}^0 e^{-tB_-} g_-(t) ds \right\|_{L^p(-\ell, 0; E)} \leq \frac{C\sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}} \|g_-\|_{L^p(-\ell, 0; E)},$
4.  $\left\| e^{-\cdot B_-} \int_{-\ell}^0 e^{(t+\ell)B_-} g_-(t) dt \right\|_{L^p(-\ell, 0; E)} \leq \frac{C\sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}} \|g_-\|_{L^p(-\ell, 0; E)},$
5.  $\left\| e^{\cdot B_+} \int_0^L e^{tB_+} g_+(t) dt \right\|_{L^p(0, L; E)} \leq \frac{C\sqrt{d_+}}{\sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0, L; E)},$
6.  $\left\| e^{\cdot B_+} \int_0^L e^{(L-t)B_+} g_+(t) dt \right\|_{L^p(0, L; E)} \leq \frac{C\sqrt{d_+}}{\sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0, L; E)},$
7.  $\left\| e^{(L-\cdot)B_+} \int_0^L e^{(L-t)B_+} g_+(t) ds \right\|_{L^p(0, L; E)} \leq \frac{C\sqrt{d_+}}{\sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0, L; E)},$
8.  $\left\| e^{(L-\cdot)B_+} \int_0^L e^{tB_+} g_+(t) dt \right\|_{L^p(0, L; E)} \leq \frac{C\sqrt{d_+}}{\sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0, L; E)}.$

### 6.2.2 Estimate of $\|w\|_{L^p(-\ell, L; E)}$

We have to estimate

$$\|w_-\|_{L^p(-\ell, 0; E)} + \|w_+\|_{L^p(0, L; E)},$$

where, due to Proposition 25,  $w$  is the unique solution of (19). Thus, it suffices to estimate  $\|w_-\|_{L^p(-\ell, 0; E)}$ . The same techniques apply to  $\|w_+\|_{L^p(0, L; E)}$ .

We have

$$\begin{aligned} w_-(x) &= D^{-1} \left( e^{-xB_-} - e^{(x+2\ell)B_-} \right) B_-^{-1} \left( I + e^{2LB_+} \right) (\Pi') \\ &\quad - q_+ D^{-1} \left( e^{-xB_-} - e^{(x+2\ell)B_-} \right) B_-^{-1} B_+^{-1} \left( I - e^{2LB_+} \right) (\Pi') \\ &\quad + q_- D^{-1} \left( e^{(x+2\ell)B_-} - e^{-xB_-} \right) B_+^{-1} B_-^{-1} \left( I - e^{2LB_+} \right) (\Pi'') \\ &\quad - e^{(x+\ell)B_-} v_-(g_-)(-\ell) \\ &\quad + v_-(g_-)(x) \\ &= \sum_{i=1}^5 a_i(x), \end{aligned}$$

where

$$\left\{ \begin{array}{l} (\Pi') = v'_-(g_-)(0) - B_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- e^{LB_+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ \quad - q_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \\ (\Pi'') = -v'_+(g_+)(0) - B_+ e^{LB_+} v_+(g_+)(L) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ v_+(g_+)(0) \\ \quad + q_+ e^{\ell B_-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0), \end{array} \right.$$

and

$$\left\{ \begin{array}{l} v_-(g_-)(x) = \frac{1}{2}B_-^{-1} \int_{-\ell}^x e^{(x-t)B_-} g_-(t) dt + \frac{1}{2}B_-^{-1} \int_x^0 e^{(t-x)B_-} g_-(t) dt, \\ v_+(g_+)(x) = \frac{1}{2}B_+^{-1} \int_0^x e^{(x-t)B_+} g_+(t) dt + \frac{1}{2}B_+^{-1} \int_x^L e^{(t-x)B_+} g_+(t) dt, \\ v'_-(g_-)(x) = \frac{1}{2} \int_{-\ell}^x e^{(x-t)B_-} g_-(t) dt - \frac{1}{2} \int_x^0 e^{(t-x)B_-} g_-(t) dt, \\ v'_+(g_+)(x) = \frac{1}{2} \int_0^x e^{(x-t)B_+} g_+(t) dt - \frac{1}{2} \int_x^L e^{(t-x)B_+} g_+(t) dt. \end{array} \right.$$

We will focus ourselves, for instance, on the first term  $a_1$ , that is

$$a_1(x) = D^{-1} \left( e^{-xB_-} - e^{(x+2\ell)B_-} \right) B_-^{-1} \left( I + e^{2LB_+} \right) (\Pi').$$

After replacing  $(\Pi')$  by its expression, we obtain explicitly

$$\begin{aligned} a_1(x) &= D^{-1}B_-^{-1}e^{-xB_-}v'_-(g_-)(0) + D^{-1}B_-^{-1}e^{-xB_-}e^{2LB_+}v'_-(g_-)(0) \\ &\quad - D^{-1}e^{(x+2\ell)B_-}B_-^{-1}v'_-(g_-)(0) - D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{2LB_+}v'_-(g_-)(0) \\ &\quad - D^{-1}e^{-xB_-}e^{\ell B_-}v_-(g_-)(-\ell) - D^{-1}e^{-xB_-}e^{2LB_+}e^{\ell B_-}v_-(g_-)(-\ell) \\ &\quad + D^{-1}e^{(x+2\ell)B_-}e^{\ell B_-}v_-(g_-)(-\ell) + D^{-1}e^{(x+2\ell)B_-}e^{2LB_+}e^{\ell B_-}v_-(g_-)(-\ell) \\ &\quad + q_-D^{-1}B_-^{-1}e^{-xB_-}e^{LB_+}v_+(g_+)(L) + q_-D^{-1}B_-^{-1}e^{-xB_-}e^{3LB_+}v_+(g_+)(L) \\ &\quad - q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{LB_+}v_+(g_+)(L) - q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{3LB_+}v_+(g_+)(L) \\ &\quad - q_-D^{-1}e^{-xB_-}B_-^{-1}v_+(g_+)(0) - q_-D^{-1}e^{-xB_-}B_-^{-1}e^{2LB_+}v_+(g_+)(0) \\ &\quad + q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}v_+(g_+)(0) + q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{2LB_+}v_+(g_+)(0) \\ &\quad - q_-D^{-1}e^{-xB_-}B_-^{-1}e^{\ell B_-}v_-(g_-)(-\ell) - q_-D^{-1}e^{-xB_-}B_-^{-1}e^{2LB_+}e^{\ell B_-}v_-(g_-)(-\ell) \\ &\quad + q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{\ell B_-}v_-(g_-)(-\ell) + q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{2LB_+}e^{\ell B_-}v_-(g_-)(-\ell) \\ &\quad + q_-D^{-1}e^{-xB_-}B_-^{-1}v_-(g_-)(0) + q_-D^{-1}e^{-xB_-}B_-^{-1}e^{2LB_+}v_-(g_-)(0) \\ &\quad - q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}v_-(g_-)(0) - q_-D^{-1}e^{(x+2\ell)B_-}B_-^{-1}e^{2LB_+}v_-(g_-)(0) \\ &= \sum_{k=1}^{12} b_k(x). \end{aligned}$$

Let us estimate, for instance, the four first terms. The others can be treated analogously.

$$\begin{aligned} \|b_1(\cdot)\|_{L^p(-\ell,0;E)} &= \left\| D^{-1}B_-^{-1}e^{-\cdot B_-}v'_-(g_-)(0) \right\|_{L^p(-\ell,0;E)} \\ &= \frac{1}{2} \left\| D^{-1}B_-^{-1}e^{-\cdot B_-} \int_{-\ell}^0 e^{-tB_-} g_-(t) dt \right\|_{L^p(-\ell,0;E)} \\ &\leq \frac{1}{2} \|D^{-1}\|_{\mathcal{L}(E)} \|B_-^{-1}\|_{\mathcal{L}(E)} \left[ \int_{-\ell}^0 \left\| e^{-xB_-} \int_{-\ell}^0 e^{-tB_-} g_-(t) dt \right\|_E^p dx \right]^{\frac{1}{p}}; \end{aligned}$$

then, from (23), Corollary 27 and Lemma 29 statement 3. (with  $[a, b] = [-\ell, 0]$ ), there exists a constant  $C > 0$  independent of  $\lambda$  such that

$$\begin{aligned} \|b_1(\cdot)\|_{L^p(-\ell,0;E)} &\leq \frac{C\sqrt{d_-}\sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}\sqrt{d_- + |\lambda + r_-|}} \|g_-\|_{L^p(-\ell,0;E)} \\ &= \frac{Cd_-}{d_- + |\lambda + r_-|} \|g_-\|_{L^p(-\ell,0;E)}. \end{aligned}$$

For the second term  $b_2$ , we have

$$\begin{aligned} \|b_2(\cdot)\|_{L^p(-\ell,0;E)} &= \left\| D^{-1} e^{-\cdot B_-} e^{2LB_+} e^{\ell B_-} v_-(g_-)(-\ell) \right\|_{L^p(-\ell,0;E)} \\ &= \left\| \frac{1}{2} D^{-1} B_-^{-1} e^{2LB_+} e^{2\ell B_-} e^{-\cdot B_-} \int_{-\ell}^0 e^{tB_-} g_-(t) dt \right\|_{L^p(-\ell,0;E)} ; \end{aligned}$$

using (23) and the fact that

$$\left\| e^{2LB_+} \right\|_{\mathcal{L}(E)} \leq C e^{-2cL|\lambda_+ + \rho_+|^{1/2}} \quad \text{and} \quad \left\| e^{2\ell B_-} \right\|_{\mathcal{L}(E)} \leq C e^{-2c\ell|\lambda_- + \rho_-|^{1/2}},$$

see (30), we obtain the existence of a constant  $C > 0$  independent of  $\lambda$  such that

$$\|b_2(\cdot)\|_{L^p(-\ell,0;E)} \leq \frac{C d_-}{d_- + |\lambda + r_-|} \|g_-\|_{L^p(-\ell,0;E)}.$$

For the third term  $b_3$ , we have

$$\begin{aligned} \|b_3(\cdot)\|_{L^p(-\ell,0;E)} &= \left\| q_- D^{-1} B_-^{-1} e^{2\ell B_-} e^{2LB_+} e^{\cdot B_-} v_+(g_+)(0) \right\|_{L^p(-\ell,0;E)} \\ &= \left\| \frac{1}{2} q_- D^{-1} B_-^{-1} B_+^{-1} e^{2\ell B_-} e^{2LB_+} e^{\cdot B_-} \int_0^L e^{tB_+} g_+(t) dt \right\|_{L^p(-\ell,0;E)} ; \end{aligned}$$

thus, using the same arguments as above, there exists a constant  $C > 0$  independent of  $\lambda$  such that

$$\|b_3(\cdot)\|_{L^p(-\ell,0;E)} \leq \frac{q C \sqrt{d_+/d_-}}{\sqrt{d_- + |\lambda + r_-|} \sqrt{d_+ + |\lambda + r_+|}} \left\| e^{\cdot B_-} \int_0^L e^{tB_+} g_+(t) dt \right\|_{L^p(-\ell,0;E)}.$$

Now, the boundedness of the semigroup  $e^{\cdot B_-}$  and the Hölder inequality lead us to obtain

$$\|b_3(\cdot)\|_{L^p(-\ell,0;E)} \leq \frac{q C \sqrt{d_+/d_-}}{\sqrt{d_- + |\lambda + r_-|} \sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0,L;E)}.$$

In the same way, for the fourth term  $b_4$ , we have

$$\begin{aligned} \|b_4(\cdot)\|_{L^p(-\ell,0;E)} &= \left\| q_- D^{-1} B_-^{-1} e^{(\cdot + 2\ell)B_-} e^{3LB_+} v_+(g_+)(L) \right\|_{L^p(-\ell,0;E)} \\ &= \left\| \frac{1}{2} D^{-1} B_-^{-1} B_+^{-1} e^{2\ell B_-} e^{4LB_+} e^{\cdot B_-} \int_0^L e^{-tB_+} g_+(t) dt \right\|_{L^p(-\ell,0;E)} ; \end{aligned}$$

then

$$\|b_4(\cdot)\|_{L^p(-\ell,0;E)} \leq \frac{q C \sqrt{d_+/d_-}}{\sqrt{d_- + |\lambda + r_-|} \sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0,L;E)}.$$

Therefore, we can conclude that there exists  $C > 0$ , independent of  $\lambda$  such that

$$\begin{aligned} \|a_1(\cdot)\|_{L^p(-\ell,L;E)} &\leq \frac{C d_-}{d_- + |\lambda + r_-|} \|g_-\|_{L^p(-\ell,0;E)} \\ &\quad + \frac{q C \sqrt{d_+/d_-}}{\sqrt{d_- + |\lambda + r_-|} \sqrt{d_+ + |\lambda + r_+|}} \|g_+\|_{L^p(0,L;E)}. \end{aligned}$$

The same techniques as above lead us to obtain similar estimates for the terms  $a_i$ ,  $i = 2, \dots, 4$  in  $w_-$ .

For the convolution term  $v_-(g_-)(\cdot)$ , using (23) and Proposition 28, we have

$$\|v_-(g_-)(\cdot)\|_{L^p(-\ell,0;E)} \leq \frac{Cd_-}{d_- + |\lambda + r_-|} \|g_-\|_{L^p(-\ell,0;E)}.$$

Hence we can conclude that there exists of a constant  $C > 0$ , independent of  $\lambda \in S_{\pi-\varepsilon_0}$ , such that

$$\|w_-\|_{L^p(-\ell,0;E)} \leq \frac{C}{|\lambda|} \left( \|g_-\|_{L^p(-\ell,0;E)} + \|g_+\|_{L^p(0,L;E)} \right).$$

Using the same calculus as above, we obtain the existence of a constant  $C > 0$ , independent of  $\lambda \in S_{\pi-\varepsilon_0}$ , such that

$$\|w_+\|_{L^p(0,L;E)} \leq \frac{C}{|\lambda|} \left( \|g_-\|_{L^p(-\ell,0;E)} + \|g_+\|_{L^p(0,L;E)} \right).$$

Summing up, we obtain

$$\|w\|_{L^p(-\ell,L;E)} \leq \frac{C}{|\lambda|} \|g\|_{L^p(-\ell,L;E)}.$$

Then, we conclude that there exists  $C > 0$ , such that, for all  $\lambda \in S_{\pi-\varepsilon_0}$ , we have

$$\left\| (\mathcal{S} - \lambda I)^{-1} \right\|_{\mathcal{L}(L^p(-\ell,L;E))} \leq \frac{C}{|\lambda|},$$

which implies that  $\mathcal{S}$  generates an analytic semigroup in  $L^p(-\ell, L; E)$ .

## Declarations

**Ethical Approval:** Not applicable.

**Competing interests:** There is no competing interests.

**Authors' contributions:** All the authors have written and reviewed this manuscript.

**Funding:** Not applicable.

**Availability of data and materials:** Not applicable.

## Acknowledgments

The authors would like to thank very much the referee for the valuable comments and corrections which have helped us a lot to improve this paper.

## References

- [1] H. AMANN, *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr., 186, (1997), 5-56.
- [2] A. V. BALAKRISHNAN, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Maths., 10 (1960), 419-437.
- [3] J. BOURGAIN, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat., 21 (1983), 163-168.
- [4] D. L. BURKHOLDER, *A geometrical characterization of Banach spaces in in which martingale difference sequences are unconditional*, Ann. Probab., 9 (1981), 997-1011.
- [5] R. S. CANTRELL & C. COSNER, *Diffusion models for population dynamics incorporating individual behavior at boundaries: applications to refuge design*, Theoretical Population Biology, 55, (1999), 189-207.

- [6] G. DORE, A. FAVINI, R. LABBAS & K. LEMRABET, *An abstract transmission problem in a thin layer, I: Sharp estimates*, Journal of Functional Analysis, 261 (2011), 1865-1922.
- [7] G. DORE & A. VENNI, *On the closedness of the sum of two closed operators*, Mathematische Zeitschrift, 196 (1987), 124-136.
- [8] G. DORE & S. YAKUBOV, *Semigroup estimates and noncoercive boundary value problems*, Semigroup Forum, Vol. 60 (2000), 93-121.
- [9] A. FAVINI, R. LABBAS, S. MAINGOT & A. THOREL, *Elliptic differential operator with an abstract Robin boundary condition and two spectral parameters*, submitted (2021) hal-02975665.
- [10] A. FAVINI, R. LABBAS, A. MEDEGHRI & A. MENAD, *Analytic semigroups generated by the dispersal process in two habitats incorporating individual behavior at the interface*, J. Math. Anal. Appl, Vol. 471, 1-2 (2019), 448-480.
- [11] P. GRISVARD, *Spazi di tracce e applicazioni*, Rendiconti di Matematica, (4) Vol. 5, série VI, (1972), 657-729.
- [12] M. HAASE, "The functional calculus for sectorial operators", Birkhauser (2006).
- [13] R. LABBAS, S. MAINGOT & A. THOREL, *Generation of analytic semigroup for some generalized diffusion operators in  $L_p$ -spaces*, Math. Ann., 384, (2022), 1-49.
- [14] R. LABBAS & M. MOUSSAOUI, *On the resolution of the heat equation with discontinuous coefficients*, Semigroup Forum, 60 (2000), 187-201.
- [15] J. L. LIONS & E. MAGENES, "Non-homogeneous boundary value problems and applications", Die Grundlehren der mathematischen Wissenschaften, vol. 1, (1972).
- [16] A. LUNARDI, "Analytic semigroups and optimal regularity in parabolic problems", Birkhäuser, (1995).
- [17] S. MONNIAUX, *A perturbation result for bounded imaginary powers*, Archiv der Mathematik, 68, (1997), 407-417.
- [18] J. PRÜSS & H. SOHR, *On operators with bounded imaginary powers in Banach spaces*, Math. Z., 203, (1990), 429-452.
- [19] L. TARTAR, "Introduction to Sobolev spaces and interpolation spaces", Springer Heidelberg New York, (2007).
- [20] H. TRIEBEL, "Interpolation theory, function spaces, differential operators", North-Holland, Amsterdam, (1978).
- [21] L. WEIS, *Operator-valued Fourier multiplier theorems and maximal regularity*, Math. Ann., 319 (2001), 735-758.