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Abstract

We study an elliptic differential equation set in two habitats under semi-permeability conditions at the interface. This equation describes some dispersal process in population dynamics. Using functional calculus and results in Lutz Weis [22] among others, we show that the associated space operator generates an analytic semigroup in L^p-spaces.

Key Words and Phrases: Abstract elliptic differential equations; transmission problem; functional calculus; semi-permeability conditions; population dynamics.

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1 Introduction

In this work, we are interested in the study of a reaction-diffusion model for population dynamics with a dispersal process in two habitats. This model takes into account the reaction effect of individuals on the interface of these habitats.

Our aim is to show that this dispersal process generates an analytic semigroup in the framework of L^p-spaces. The present work completes naturally the results obtained in [10].

Many authors have worked on different models of reaction-diffusion problems related to biology or various environmental phenomena. We were inspired, in particular, by the work in [4], where the model considered incorporates the response of individuals on the interface between the habitats. This study was detailed in one space dimension and focused mainly on the spectral aspect of the dispersal process, taking into account the dimension of the habitats.

Here, the concrete example which illustrates our theory, is modeled by partial differential equations of parabolic type set in two juxtaposed habitats Ω_- and Ω_+:

$$\Omega = \Omega_- \cup \Gamma_0 \cup \Omega_+,$$

where

$$\begin{cases} \Omega_- = (-\ell, 0) \times (0,1), \\ \Gamma_0 = \{0\} \times (0,1), \\ \Omega_+ = (0, L) \times (0,1), \\ \Gamma_\pm = \partial\Omega_\pm \backslash \Gamma_0, \end{cases} \tag{1}$$

and ℓ, $L > 0$. The reaction-diffusion equation is

$$\frac{\partial u}{\partial t}(t,x,y) = \begin{cases} d_-\Delta u_-(t,x,y) + F_-(u_-(t,x,y)) & \text{in } (0,T)\times\Omega_-, \\ d_+\Delta u_+(t,x,y) + F_+(u_+(t,x,y)) & \text{in } (0,T)\times\Omega_+, \end{cases} \tag{2}$$

under the initial data

$$u(0,.) = \begin{cases} \varphi_- & \text{in } \Omega_-, \\ \varphi_+ & \text{in } \Omega_+, \end{cases} \tag{3}$$

the boundary conditions

$$\begin{cases} u_- = 0 & \text{on } \Gamma_-, \\ u_+ = 0 & \text{on } \Gamma_+, \end{cases} \tag{4}$$

and the interface conditions

$$\begin{cases} d_- \dfrac{\partial u_-}{\partial x} = q\,(u_+ - u_-) & \text{on } \Gamma_0, \\ d_+ \dfrac{\partial u_+}{\partial x} = q\,(u_+ - u_-) & \text{on } \Gamma_0; \end{cases} \tag{5}$$

where $d_\pm > 0$ is the diffusion coefficient, $q > 0$ is given and $u_\pm$ represents a population density in $\Omega_\pm$.

These two last conditions in (5) mean that the flux at the interface depends on the density of the population. They are of the Robin type and express the semi-permeability of the interface Γ_0.

In [10], the authors have considered different interface conditions in the framework of the space of continuous functions which are the following: the non-continuity of the flux and the continuity of the dispersal at the interface:

$$\begin{cases} (1-\alpha) d_- \dfrac{\partial u_-}{\partial x} = \alpha d_+ \dfrac{\partial u_+}{\partial x} & \text{on } \Gamma_0, \\ d_- \Delta u_- + F_-(u_-) = d_+ \Delta u_+ + F_+(u_+) & \text{on } \Gamma_0, \end{cases}$$

where $\alpha \in (0,1)$ with $\alpha \neq 1/2$.

Here, we consider only the linearized part of logistic functions, that is

$$\begin{cases} F_-(u_-) = r_- u_- & \text{on } (-\ell, 0) \times (0,1), \\ F_+(u_+) = r_+ u_+ & \text{on } (0, L) \times (0,1), \end{cases}$$

where $r_\pm > 0$.

The study of the above reaction-diffusion problem depends essentially on the nature of operator $\mathcal{P}$ defined, in the stationary case, by

$$\begin{cases} D(\mathcal{P}) = \left\{ \begin{array}{l} u \in L^p(\Omega) :\ u_- \in W^{2,p}(\Omega_-),\ u_+ \in W^{2,p}(\Omega_+),\ u_\pm = 0 \text{ on } \Gamma_\pm \\ \text{and } u_\pm \text{ satisfies (5)} \end{array} \right\}, \\ \mathcal{P}u = \begin{cases} d_- \Delta u_- - r_- u_- & \text{in } \Omega_-, \\ d_+ \Delta u_+ - r_+ u_+ & \text{in } \Omega_+, \end{cases} \end{cases}$$

where $p \in (1, +\infty)$. Note that the transmission conditions (5) are well defined since, for example, if $u_- \in W^{2,p}(\Omega_-)$, then

$$\frac{\partial u_-}{\partial x}|_{\{0\}\times(0,1)} \in W^{1-1/p,p}(0,1),$$

see [11], Corollary 1, p. 682.

Now, let us write the above parabolic equation in an abstract formulation. First, let us introduce, in the Banach space $E_0 = L^p(0,1)$, operator A_0 defined by

$$\begin{cases} D(A_0) = \{\varphi \in W^{2,p}(0,1) : \varphi(0) = \varphi(1) = 0\}, \\ (A_0\varphi)(y) = \varphi''(y). \end{cases} \tag{6}$$

It is known that this operator is closed linear with a dense domain and verifies the two following properties :

$$\exists\, C>0 : \forall z \in S_{\pi-\varepsilon_0} \cup \{0\}\,, \quad \left\|(zI-A_0)^{-1}\right\|_{\mathcal{L}(E_0)} \leqslant \frac{C}{1+|z|}, \tag{7}$$

where

$$S_{\pi-\varepsilon_0} = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \pi - \varepsilon_0\}\,,$$

for any small $\varepsilon_0 > 0$ such that $\pi - \varepsilon_0 > 0$ and there exists a ball $B(0,\delta)$, such that $\rho(A_0) \supset \overline{B(0,\delta)}$ and the above estimate is still true in $S_{\pi-\varepsilon_0} \cup \overline{B(0,\delta)}$; and

$$\left\{\begin{array}{l} \forall s \in \mathbb{R},\ \ (-A_0)^{is} \in \mathcal{L}\,(E_0)\,,\ \exists\, \theta_{A_0} \in (0,\pi/2) : \\ \sup\limits_{s\in\mathbb{R}} \left\|e^{-\theta_{A_0}|s|}(-A_0)^{is}\right\|_{\mathcal{L}(E_0)} < +\infty, \end{array}\right. \tag{8}$$

see for instance the method used in [14], Proposition 3.1, p. 191.

Remark 1. *The above problem can be considered in dimension n by setting*

$$\Omega = (-\ell, L) \times \varpi,$$

where $\varpi \subset \mathbb{R}^{n-1}$, $n > 1$, is a bounded regular open set, $E_0 = L^p(\varpi)$ and

$$\left\{\begin{array}{l} D(A_0) = \{\varphi \in W^{2,p}(\varpi) : \varphi|_{\partial\varpi} = 0\}\,, \\ A_0\varphi = \Delta_\varpi \varphi; \end{array}\right.$$

here Δ_ϖ denotes the Laplace operator related to variables on ϖ.

We will also use the following usual operational notation of vector-valued functions:

$$u_\pm(t,x)(y) := u_\pm(t,x,y), \quad t \in (0,T),\ (x,y) \in \Omega_\pm.$$

So, we have to analyze the abstract Cauchy problem

$$\left\{\begin{array}{l} u'(t) = \mathcal{S}_0 u(t), \\ u(0) = u_0, \end{array}\right. \tag{9}$$

set in the Banach space $L^p(-\ell, L; E_0)$, where

$$\left\{\begin{array}{rcl} D\left(\mathcal{S}_0\right) & = & \left\{\begin{array}{l} w \in L^p(-\ell, L; E_0) : w_- \in W^{2,p}(-\ell, 0; E_0), w_+ \in W^{2,p}(0, L; E_0) \\ w_-(-\ell) = 0, w_+(L) = 0, d_- w_-'(0) = q\,(w_+(0) - w_-(0)) \\ \text{and } d_+ w_+'(0) = q\,(w_+(0) - w_-(0)) \end{array}\right\}, \\ \left(\mathcal{S}_0 w\right)(x) & = & \left\{\begin{array}{ll} d_- w_-''(x) + d_- A_0 w_-(x) - r_- w_-(x) & \text{in } (-\ell, 0), \\ d_+ w_+''(x) + d_+ A_0 w_+(x) - r_+ w_+(x) & \text{in } (0, L). \end{array}\right. \end{array}\right.$$

We then consider a more general operator $\mathcal{S}$ instead of $\mathcal{S}_0$ where A_0 is replaced by a closed linear operator A in a Banach space E satisfying some assumptions specified in Section 4.

Our method is essentially based on the use of abstract differential equations of elliptic type. The merit of this method lies above all in the fact of having the explicit formula of the resolvent operator of $\mathcal{S}$ (and therefore of $\mathcal{S}_0$), see Section 6.

This paper is organized as follows.

In Section 2, we only show that problem $-\mathcal{P}u = g$ can be studied in the variational framework. In Section 3, we recall some useful notions on sectorial operators. Section 4 contains our assumptions and the main results. Section 5 is devoted to the establishment of some useful properties on complex numbers. Section 6 is composed of two subsections. In the first subsection, we explain the spectral equation of $\mathcal{S}$ which leads to an abstract system to be solved. Thus, we invert the determinant operator of the above system by using among others the H^∞-calculus. This leads us to obtain the explicit solution of the spectral equation. Then, we study the optimal regularity of this solution. In the second subsection, we give some sharp estimates which lead us to analyze the behavior of the resolvent operator of $\mathcal{S}$. We then obtain our main results which state among others that $\mathcal{S}$ generates an analytic semigroup in $L^p(-\ell, L; E)$, for $p \in (1, +\infty)$.

2 Variational formulation of $-\mathcal{P}u = g$

Let $a, b \in \mathbb{R}$, with $a < b$. For any

$$\begin{array}{rccc} \varphi: & (a,b) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \varphi(x), \end{array}$$

we set

$$\begin{array}{rccc} \widetilde{\varphi}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \widetilde{\varphi}(x) = \left\{ \begin{array}{ll} \varphi(x) & \text{for } x \in (a,b), \\ 0 & \text{for } x \in \mathbb{R}\backslash(a,b), \end{array} \right. \end{array}$$

and we define a subspace $\widetilde{H^{1/2}}(a,b)$ of $H^{1/2}(a,b)$ by

$$\widetilde{H^{1/2}}(a,b) = \left\{ \varphi \in H^{1/2}(a,b) : \widetilde{\varphi} \in H^{1/2}(\mathbb{R}) \right\}.$$

In [15], the authors denoted this space by $H_{00}^{1/2}(a,b)$ which also coincides with the following particular interpolation space

$$\left(H_0^1(a,b), L^2(a,b) \right)_{1/2,2}.$$

The interpolation spaces are described, for instance, in [11].

Set

$$H^1_{\Gamma_\pm}(\Omega_\pm) = \left\{ u_\pm \in H^1(\Omega_\pm) : u_{\pm|\Gamma_\pm} = 0 \right\};$$

for $v_\pm$ in $H^1_{\Gamma_\pm}(\Omega_\pm)$, it is clear that $v_{\pm|\Gamma_0}$ is in $\widetilde{H^{1/2}}(\Gamma_0)$.

Problem $-\mathcal{P}u = g$ writes in the form

$$\left\{ \begin{array}{rcll} -\mathrm{div}\left(d_+ \nabla u_+\right) + r_+ u_+ &=& g_+ & \text{in } \Omega_+, \\ -\mathrm{div}\left(d_- \nabla u_-\right) + r_- u_- &=& g_- & \text{in } \Omega_-, \\ d_+ \dfrac{\partial u_+}{\partial \nu} &=& q(u_+ - u_-) & \text{on } \Gamma_0, \\ d_- \dfrac{\partial u_-}{\partial \nu} &=& q(u_+ - u_-) & \text{on } \Gamma_0, \\ u_\pm &=& 0 & \text{on } \Gamma_\pm, \end{array} \right.$$

where ν is the normal unit vector oriented towards the interior of Ω_+. The variational formulation is set in the hilbertian space

$$\mathbb{V} = H^1_{\Gamma_+}(\Omega_+) \times H^1_{\Gamma_-}(\Omega_-),$$

with

$$\begin{array}{rcl} a\left((u_+,u_-),(w_+,w_-)\right) &=& \displaystyle\int_{\Omega_+} \left(d_+ \nabla u_+ . \nabla w_+ + r_+ u_+ w_+\right) dxdy \\ && \displaystyle + \int_{\Omega_-} \left(d_- \nabla u_- . \nabla w_- + r_- u_- w_-\right) dxdy, \end{array}$$

and

$$\left\{ \begin{array}{l} b\left((u_+,u_-),(w_+,w_-)\right) = \displaystyle\int_{\Gamma_0} q(u_+ - u_-)(w_+ - w_-)\, dy, \\ l(w_+,w_-) = \displaystyle\int_{\Omega_+} g_+ w_+ dxdy + \int_{\Omega_-} g_- w_- dxdy, \end{array} \right.$$

then

$$a\left(\left(u_+,u_+\right),\left(w_+,w_-\right)\right)+b\left(\left(u_+,u_-\right),\left(w_+,w_-\right)\right)=l\left(w_+,w_-\right).$$

Now, taking w_+ in $\mathcal{D}\left(\Omega_+\right)$ and $w_-=0$, we have in the sense of distributions

$$-\mathrm{div}\left(p_+\nabla u_+\right)+r_+u_+=f_+ \text{ in } \Omega_+.$$

For g_+ in $L^2\left(\Omega_+\right)$, the trace of $d_+\dfrac{\partial u_+}{\partial\nu}$ on Γ_0, can be defined in the dual space $\left(\widetilde{H^{1/2}}\left(\Gamma_0\right)\right)'$ of $\widetilde{H^{1/2}}\left(\Gamma_0\right)$. In fact, operator $\partial/\partial\nu$ maps continuously from $H^1(0,1)$ into $L^2(0,1)$ and $L^2(0,1)$ into $H^{-1}(0,1)$, then, by interpolation it maps continuously from $\left(H^1(0,1);L^2(0,1)\right)_{1/2,2}$ into $\left(L^2(0,1),H^{-1}(0,1)\right)_{1/2,2}$; but we know that

$$\left(H^1(0,1),L^2(0,1)\right)_{1/2,2}=H^{1/2}(0,1),$$

and

$$\left(L^2(0,1),H^{-1}(0,1)\right)\right)_{1/2,2}=\left[\left(H_0^1(0,1),L^2(0,1)\right)_{1/2,2}\right]'=\left(\widetilde{H^{1/2}}\left(a,b\right)\right)',$$

see [17] p. 160.

The Green's formula for w_+ in $H^1_{\Gamma_+}\left(\Omega_+\right)$ gives

$$\begin{aligned}\int_{\Omega_+}\left(d_+\nabla u_+.\nabla w_++r_+u_+w_+\right)dxdy &= \int_{\Omega_+}\left(-\nabla.\left(d_+\nabla u_+\right)w_++r_+u_+w_+\right)dxdy\\ &\quad+\int_{\Gamma_0}\left(-d_+\frac{\partial u_+}{\partial\nu}\right)w_+dy,\end{aligned}$$

where the last integral means that

$$\int_{\Gamma_0}\left(-d_+\frac{\partial u_+}{\partial\nu}\right)w_+dy:=\left\langle d_+\frac{\partial u_+}{\partial\nu};w_+\right\rangle_{\left(\widetilde{H^{1/2}}(\Gamma_0)\right)'\times\left(\widetilde{H^{1/2}}(\Gamma_0)\right)}.$$

Similarly, for

$$-\mathrm{div}\left(d_-\nabla u_-\right)+r_-u_-=g_- \text{ in } \Omega_-,$$

we obtain

$$\begin{aligned}\int_{\Omega_-}\left(d_-\nabla u_-.\nabla w_-+r_-u_-w_-\right)dxdy &= \int_{\Omega_-}\left(-\nabla.\left(d_-\nabla u_-\right)w_-+r_-u_-w_-\right)dxdy\\ &\quad+\int_{\Gamma_0}\left(d_-\frac{\partial u_-}{\partial\nu}\right)w_-dy,\end{aligned}$$

as above, the last integral means that

$$\int_{\Gamma_0}\left(-d_-\frac{\partial u_-}{\partial\nu}\right)w_-dy:=\left\langle d_-\frac{\partial u_-}{\partial\nu};w_-\right\rangle_{\left(\widetilde{H^{1/2}}(\Gamma_0)\right)'\times\left(\widetilde{H^{1/2}}(\Gamma_0)\right)}.$$

It follows that

$$\int_{\Gamma_0}\left(-d_+\frac{\partial u_+}{\partial\nu}\right)w_+dy+\int_{\Gamma_0}\left(d_-\frac{\partial u_-}{\partial\nu}\right)w_-dy+\int_{\Gamma_0}q\left(u_+-u_-\right)\left(w_+-w_-\right)dy=0;$$

taking $w_-=0$, we deduce that

$$-d_+\frac{\partial u_+}{\partial\nu}+q\left(u_+-u_-\right)=0 \text{ in } \left(\widetilde{H^{1/2}}\left(\Gamma_0\right)\right)',$$

in the same way, $w_+=0$ gives

$$d_-\frac{\partial u_-}{\partial\nu}-q\left(u_+-u_-\right)=0 \text{ in } \left(\widetilde{H^{1/2}}\left(\Gamma_0\right)\right)'.$$

3 Recall on sectorial operators

Let $\omega \in [0,\pi]$. We put

$$S_\omega := \begin{cases} \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \omega\} & \text{if } \omega \in (0,\pi], \\ (0,+\infty) & \text{if } \omega = 0. \end{cases} \tag{10}$$

Let us recall some known results from [12].

Definition 2. *Let $\omega \in [0,\pi)$. A linear operator Λ on a complex Banach space E is called sectorial of angle ω if*

1. *$\sigma(\Lambda) \subset \overline{S_\omega}$ and*
2. *$M(\Lambda,\omega') := \sup\limits_{\lambda\in\mathbb{C}\setminus\overline{S_{\omega'}}} \|\lambda(\Lambda-\lambda I)^{-1}\| < \infty$ for all $\omega' \in (\omega,\pi)$.*

We then write: $\Lambda \in Sect(\omega)$. The following angle

$$\omega_\Lambda := \min\{\omega \in [0,\pi) : \Lambda \in Sect(\omega)\},$$

is called the spectral angle of Λ.

We recall the following properties of the set $Sect(\omega)$. It is clear that Statement 2. implies necessarily that Λ is closed.

Proposition 3. *If $(-\infty,0) \subset \rho(\Lambda)$ and*

$$M(\Lambda) := M(\Lambda,\pi) := \sup_{t>0} \left\|t(\Lambda+tI)^{-1}\right\| < \infty,$$

then $M(\Lambda) \geqslant 1$ and

$$\Lambda \in Sect\left(\pi - \arcsin(1/M(\Lambda))\right).$$

Proposition 4. *Let $\Lambda \in Sect(\omega_\Lambda)$ and $\nu \in (0,1/2]$. Then $\Lambda^\nu \in Sect(\nu\omega_\Lambda)$, and therefore $-\Lambda^\nu$ generates an analytic semigroup.*

For more details, see [12], p. 80-81.
Set

$$H^\infty(S_\omega) = \{f : f \text{ is an holomorphic and bounded function on } S_\omega\},$$

with $\omega \in (0,\pi)$; we recall that if $f \in H^\infty(S_\omega)$ is such that $1/f \in H^\infty(S_\omega)$ and

$$(1/f)(\Lambda) \in \mathcal{L}(E),$$

then $f(\Lambda)$ is boundedly invertible and

$$[f(\Lambda)]^{-1} = (1/f)(\Lambda), \tag{11}$$

see, for instance [5].

4 Assumptions and main results

We suppose that E is a complex Banach space satisfying the following geometrical property

$$E \text{ is a } UMD \text{ space;} \tag{12}$$

we recall that a Banach space E is a UMD space if and only if for all $1 < p < +\infty$ the Hilbert transform is continuous from $L^p(\mathbb{R}; E)$ into itself, see [3]. Consider a closed linear operator A satisfying:

$$\begin{cases} \exists\, \varepsilon_0 \in (0, \pi/2) : \ S_{\pi-\varepsilon_0} \cup \{0\} \subset \rho(A) \ \text{ and } \exists\, C_A > 0 : \\ \forall \lambda \in S_{\pi-\varepsilon_0}, \quad \left\| (A - \lambda I)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \dfrac{C_A}{1 + |\lambda|}, \end{cases} \tag{13}$$

$$\begin{cases} \forall s \in \mathbb{R}, \ \ (-A)^{is} \in \mathcal{L}(E), \ \exists\, \theta_A \in (0, \varepsilon_0] : \\ \sup\limits_{s \in \mathbb{R}} \left\| e^{-\theta_A |s|} (-A)^{is} \right\|_{\mathcal{L}(E)} < +\infty. \end{cases} \tag{14}$$

We deduce in particular that $-A \in \mathrm{BIP}(E, \theta_A)$, see the precise definition in [20]. Note that $-A_0$, defined in (6), satifies $-A_0 \in \mathrm{BIP}(L^p(0,1), \varepsilon_0)$.

We define operator $\mathcal{S}$ by

$$\begin{cases} D(\mathcal{S}) &= \left\{ \begin{array}{l} w \in L^p(-\ell, L; E) : w_- \in W^{2,p}(-\ell, 0; E), w_+ \in W^{2,p}(0, L; E) \\ w_-(-\ell) = 0, w_+(L) = 0, d_- w_-'(0) = q\,(w_+(0) - w_-(0)) \\ \text{and } d_+ w_+'(0) = q\,(w_+(0) - w_-(0)) \end{array} \right\}, \\ (\mathcal{S} w)(x) &= \left\{ \begin{array}{ll} d_- w_-''(x) + d_- A w_-(x) - r_- w_-(x) & \text{in } (-\ell, 0), \\ d_+ w_+''(x) + d_+ A w_+(x) - r_+ w_+(x) & \text{in } (0, L). \end{array} \right. \end{cases}$$

Thanks to the fact that the domain is cylindrical, we will give an explicit expression of the resolvent operator of $\mathcal{S}$ by using essentially the analytic semigroups theory and the functional calculus.

Therefore, our aim results are the following:

Theorem 5. *Assume that (12), (13) and (14) hold. Then, operator $\mathcal{S}$ generates an analytic semigroup in $L^p(-\ell, L; E)$.*

As corollaries, we obtain.

Theorem 6. *Operator $\mathcal{S}_0$ generates an analytic semigroup in $L^p(-\ell, L; E_0)$.*

Theorem 7. *Operator $\mathcal{P}$ generates an analytic semigroup in $L^p(\Omega)$.*

5 Preliminary results

In this section, we recall some useful results and we state some technical results.

Proposition 8. *Let $c \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then, we have*

$$\begin{cases} 0 < |\arg(z + c)| < |\arg(z)| < \pi & \text{if } c > 0, \\ 0 < |\arg(z)| < |\arg(z + c)| < \pi & \text{if } c < 0. \end{cases}$$

Proof. For $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\arg(z) = 2 \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z) + |z|} \right), \tag{15}$$

and

1. if $c > 0$, then

$$\begin{aligned} |\arg(z)| &= 2\arctan\left(\frac{|\mathrm{Im}(z+c)|}{\mathrm{Re}(z)+|z+c-c|}\right) \\ &> 2\arctan\left(\frac{|\mathrm{Im}(z+c)|}{\mathrm{Re}(z)+c+|z+c|}\right) \\ &= |\arg(z+c)|\,. \end{aligned}$$

2. if $c < 0$, then

$$\begin{aligned} |\arg(z)| &= 2\arctan\left(\frac{|\mathrm{Im}(z+c)|}{\mathrm{Re}(z)+c-c+|z|}\right) \\ &= 2\arctan\left(\frac{|\mathrm{Im}(z+c)|}{\mathrm{Re}(z)+c+|c|+|z|}\right) \\ &< 2\arctan\left(\frac{|\mathrm{Im}(z+c)|}{\mathrm{Re}(z)+c+|z+c|}\right) \\ &= |\arg(z+c)|\,. \end{aligned}$$

□

Proposition 9. *Let $z_1, z_2 \in \mathbb{C}\setminus\{0\}$. We have*

$$|z_1+z_2| \geqslant (|z_1|+|z_2|)\left|\cos\left(\frac{\arg(z_1)-\arg(z_2)}{2}\right)\right|.$$

This result is given by Proposition 4.9, p. 1879 in [6].

Proposition 10. *Let $0 < \alpha < \pi/2$ and $z \in S_\alpha$. We have*

1. $|\arg(1-e^{-z}) - \arg(1+e^{-z})| < \alpha$.
2. $|1+e^{-z}| \geqslant 1 - e^{-\pi/(2\tan(\alpha))}$.

This result is given in Proposition 4.10, p. 1880 in [6].

Corollary 11. *Let $\alpha \in (0, \pi/2]$, $\beta \in [0, \alpha/2]$ and $z \in \mathbb{C}\setminus\{0\}$. Then*

1. *if $-\beta \leqslant \arg(z) < \alpha - \beta$, then we have*

$$-\beta \leqslant \arg(1-e^{-z}) - \arg(1+e^{-z}) < \alpha - \beta.$$

2. *if $-\alpha + \beta < \arg(z) \leqslant \beta$, then we have*

$$-\alpha + \beta < \arg(1-e^{-z}) - \arg(1+e^{-z}) \leqslant \beta.$$

Proof. Since S_α is an open sector, then Proposition 10 remains true for $\alpha = \pi/2$.

1. First, let $\beta = 0$. Then, we have $0 \leqslant \arg(z) < \alpha$. As in the proof of statement 1, Proposition 4.10, p. 1880 in [6], we have $\mathrm{Re}(1-e^{-z}), \mathrm{Re}(1+e^{-z}) > 0$. Let us prove that

$$\arg(1-e^{-z}) - \arg(1+e^{-z}) \geqslant 0.$$

To this end, we must show that

$$\arctan\left(\frac{\mathrm{Im}(1-e^{-z})}{\mathrm{Re}(1-e^{-z})}\right) \geqslant \arctan\left(\frac{\mathrm{Im}(1+e^{-z})}{\mathrm{Re}(1+e^{-z})}\right),$$

that is

$$\frac{\mathrm{Im}(1-e^{-z})}{\mathrm{Re}(1-e^{-z})} \geqslant \frac{\mathrm{Im}(1+e^{-z})}{\mathrm{Re}(1+e^{-z})},$$

or

$$\mathrm{Im}(1-e^{-z})\mathrm{Re}(1+e^{-z}) \geqslant \mathrm{Im}(1+e^{-z})\mathrm{Re}(1-e^{-z}). \tag{16}$$

Since we have

$$\begin{cases} \mathrm{Re}(1-e^{-z}) = 1-|z|\cos(\arg(-z)), \\ \mathrm{Re}(1+e^{-z}) = 1+|z|\cos(\arg(-z)), \\ \mathrm{Im}(1-e^{-z}) = -|z|\sin(\arg(-z)), \\ \mathrm{Im}(1+e^{-z}) = |z|\sin(\arg(-z)), \end{cases}$$

then, (16) is equivalent to

$$-|z|\sin(\arg(-z))\left(1+|z|\cos(\arg(-z))\right) \geqslant |z|\sin(\arg(-z))\left(1-|z|\cos(\arg(-z))\right);$$

which is true since $\sin(\arg(-z)) \leqslant 0$. Now, taking into account that $0 \leqslant \arg(z) < \alpha$ and Proposition 10, we obtain

$$0 \leqslant \arg\left(1-e^{-z}\right) - \arg\left(1+e^{-z}\right) < \alpha. \tag{17}$$

Now, let $\beta \in \left(0, \frac{\alpha}{2}\right]$ and $-\beta \leqslant \arg(z) < \alpha - \beta$.

If $0 \leqslant \arg(z) < \alpha - \beta$, then from (17), we deduce that

$$-\beta \leqslant 0 \leqslant \arg\left(1-e^{-z}\right) - \arg\left(1+e^{-z}\right) < \alpha - \beta.$$

If $-\beta < \arg(z) < 0$, then $z \in S_\beta$ and from Proposition 10, we have

$$-\beta \leqslant \arg\left(1-e^{-z}\right) - \arg\left(1+e^{-z}\right) < \beta.$$

Note that, when $\arg(z) = -\beta$, then the previous inequality holds true since $\beta < \alpha$. Finally, the result follows since $\beta \leqslant \frac{\alpha}{2}$.

2. In the same way, we obtain the expected result.

□

Proposition 12. *Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Assume that $z_1 + z_2 \neq 0$ and*

$$\begin{cases} \arg(-z_2) \leqslant \arg(z_1) \leqslant \arg(z_2) & \text{if } \arg(z_2) \in [0, \pi], \\ -\pi \leqslant \arg(z_1) \leqslant \arg(z_2) & \text{if } \arg(z_2) \in [-\pi, 0]. \end{cases}$$

Then, we have

$$\arg(z_1) \leqslant \arg(z_1 + z_2) \leqslant \arg(z_2).$$

Proof.

1. Assume that $\arg(z_2) = \pi$.

   If $0 < \arg(z_1) < \pi$, from Proposition 8, we obtain the expected inequalities.

   If $\arg(z_1) = \arg(z_2) = \pi$, then

   $$\arg(z_1) = \arg(z_1 + z_2) = \arg(z_2). \tag{18}$$

   If $\arg(z_1) = 0$, then the expected inequalities hold since we have

   $$\arg(z_1 + z_2) = \begin{cases} \arg(z_1) & \text{if } z_1 > |z_2|, \\ \arg(z_2) & \text{if } |z_2| > z_1. \end{cases}$$

2. Assume that $\arg(z_2) = -\pi$, then $\arg(z_1) = \arg(z_2) = -\pi$ and (18) holds.

3. Assume that $\arg(z_2) \in (-\pi, 0]$. Then, $\arg(z_1) \in (-\pi, \arg(z_2)]$, $\mathrm{Im}(z_2) \leqslant 0$ and $\mathrm{Im}(z_1) \leqslant 0$. From (15), we have
$$\frac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_1) + |z_1|} \leqslant \frac{\mathrm{Im}(z_2)}{\mathrm{Re}(z_2) + |z_2|}.$$
Hence
$$\mathrm{Im}(z_1)\left(\mathrm{Re}(z_2) + |z_2|\right) + \mathrm{Im}(z_2)\left(\mathrm{Re}(z_2) + |z_2|\right) \leqslant \mathrm{Im}(z_2)\left(\mathrm{Re}(z_1) + |z_1|\right) + \mathrm{Im}(z_2)\left(\mathrm{Re}(z_2) + |z_2|\right),$$
wich gives
$$\mathrm{Im}(z_1 + z_2)\left(\mathrm{Re}(z_2) + |z_2|\right) \leqslant \mathrm{Im}(z_2)\left(\mathrm{Re}(z_1 + z_2) + |z_1| + |z_2|\right).$$
Since $\mathrm{Im}(z_2) \leqslant 0$, then we have
$$\mathrm{Im}(z_2)\left(|z_1| + |z_2|\right) \leqslant \mathrm{Im}(z_2)|z_1 + z_2|.$$
Therefore
$$\mathrm{Im}(z_1 + z_2)\left(\mathrm{Re}(z_2) + |z_2|\right) \leqslant \mathrm{Im}(z_2)\left(\mathrm{Re}(z_1 + z_2) + |z_1 + z_2|\right),$$
and due to (15), we obtain
$$\arg(z_1 + z_2) \leqslant \arg(z_2).$$
We extend the above result, by rotation, for $\arg(z_2) \in (-\pi, \pi)$.

4. Similarly, for $\arg(z_1)$, $\arg(z_2) \in [0, \pi)$, following the same steps, we deduce that
$$\arg(z_1) \leqslant \arg(z_1 + z_2),$$
and, by rotation, we obtain the expected result.

□

6 Proof of Theorem 5

6.1 Spectral study of $\mathcal{S}$

In this sections we will focus ourselves to study the spectral equation
$$\mathcal{S}w - \lambda w = f \in \mathcal{E} = L^p(-\ell, L; E), \tag{19}$$
where $p \in (1, +\infty)$.

Our aim is to estimate the resolvent operator
$$\left\|(\mathcal{S} - \lambda I)^{-1}\right\|_{\mathcal{L}(L^p(-\ell,L;E))},$$
where λ is a complex number in some sector to specify. This estimate will allow us to prove that $\mathcal{S}$ generates an analytic semigroup in $\mathcal{E}$. So, after the resolution of the spectral equation, we have to estimate $\|w\|_{L^p(-\ell,L;E)}$ that is
$$\|w_-\|_{L^p(-\ell,0;E)} \quad \text{and} \quad \|w_+\|_{L^p(0,L;E)}.$$
We recall that all the constants r_-, r_+, d_-, d_+, q are strictly positive. In the sequel, we will use the following notations:
$$\lambda_\pm = \frac{\lambda}{d_\pm}, \quad \rho_\pm = \frac{r_\pm}{d_\pm}, \quad q_\pm = \frac{q}{d_\pm} \quad \text{and} \quad g_\pm = \frac{f_\pm}{d_\pm}. \tag{20}$$

6.1.1 The system verified by the spectral equation

Equation (19) can be formulated as

$$(P_A)\begin{cases} \begin{cases} w_-''(x)+(A-\rho_- I-\lambda_- I)\,w_-(x)=g_-(x) & \text{in } (-\ell,0),\\ w_+''(x)+(A-\rho_+ I-\lambda_+ I)\,w_+(x)=g_+(x) & \text{in } (0,L), \end{cases}\\ \begin{cases} w_-(-\ell)=0,\\ w_+(L)=0, \end{cases}\\ \begin{cases} w_-'(0)=q_-\left(w_+(0)-w_-(0)\right),\\ w_+'(0)=q_+\left(w_+(0)-w_-(0)\right). \end{cases} \end{cases}$$

Assume that the complex λ satisfies:

$$|\arg(\lambda)|<\pi-\varepsilon_0, \tag{21}$$

where ε_0 is a small fixed number. Set

$$A_-=A-\rho_- I-\lambda_- I \quad \text{and} \quad A_+=A-\rho_+ I-\lambda_+ I,$$

so we have

$$D(A_-)=D(A_+)=D(A).$$

Proposition 13. *Operators $-A_-$ and $-A_+$ are sectorial and satisfy*

$$\begin{cases} -A_-\in Sect\left(\pi-\arcsin\left(1/M(-A_-)\right)\right),\\ -A_+\in Sect\left(\pi-\arcsin\left(1/M(-A_+)\right)\right), \end{cases}$$

where $M(-A_\pm)$ is defined in Proposition 3.

Proof. If $\lambda\in\mathbb{R}_+$, then due to (13), $-A_-$ and $-A_+$ are sectorial operators.

Now, let $\lambda\in S_{\pi-\varepsilon_0}\setminus\mathbb{R}_+$ and $t>0$. We will verify that $-A_-$ is a sectorial operator in E and $(-\infty,0]\subset\rho(-A_-)$. Since $\rho_-+t>0$, from Proposition 8, we obtain

$$|\arg\left(\rho_-+t+\lambda_-\right)|<|\arg\left(\lambda_-\right)|=|\arg(\lambda)|<\pi-\varepsilon_0.$$

Then

$$M(-A_-)=\sup_{t>0}\left\|t(-A_-+tI)^{-1}\right\|=\sup_{t>0}t\left\|\left(-A+\left(\rho_-+t+\lambda_-\right)I\right)^{-1}\right\|_{\mathcal{L}(E)}.$$

Then, from (13) and Proposition 9, we have

$$\begin{aligned}\left\|\left(-A+\left(\rho_-+t+\lambda_-\right)I\right)^{-1}\right\|_{\mathcal{L}(E)} &\leqslant \frac{C_A}{1+|\rho_-+t+\lambda_-|}\leqslant\frac{C_A}{|\rho_-+t+\lambda_-|}\\ &\leqslant \frac{C_A}{(|\rho_-+\lambda_-|+t)\cos\left(\frac{\arg(\rho_-+\lambda_-)}{2}\right)}\\ &\leqslant \frac{C_A}{t\cos\left(\frac{|\arg(\rho_-+\lambda_-)|}{2}\right)}.\end{aligned}$$

Moreover, from Proposition 8, we obtain

$$|\arg\left(|\rho_-+\lambda_-|\right)|<|\arg\left(\lambda_-\right)|=|\arg(\lambda)|<\pi-\varepsilon_0,$$

and thus

$$\cos\left(\frac{|\arg\left(\rho_-+\lambda_-\right)|}{2}\right)>\cos\left(\frac{\pi}{2}-\frac{\varepsilon_0}{2}\right)=\sin\left(\frac{\varepsilon_0}{2}\right)>0.$$

Finally, we have

$$M(-A_-) \leqslant \sup_{t>0} \left(\frac{t\, C_A}{t \sin\left(\frac{\varepsilon_0}{2}\right)} \right) = \frac{C_A}{\sin\left(\frac{\varepsilon_0}{2}\right)} < +\infty.$$

Hence, due to Proposition 3, we obtain the expected result for $-A_-$. For $-A_+$, the proof is similar. □

Therefore, the following operators

$$B = -\sqrt{-A}, \quad B_- = -\sqrt{-A_-} \quad \text{and} \quad B_+ = -\sqrt{-A_+},$$

of domain

$$D(B) = D(B_-) = D(B_+),$$

are well defined for all $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ and generate analytic semigroups in E, see [2].

By using estimates (28)-(29) in Lemma 4.2, see [9], there exist C_-, $C_+ > 0$, independent of λ such that for all $z \in S_{\pi-\varepsilon_0} \cup \{0\}$, we have

$$\begin{cases} \left\| (B_- - zI)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \dfrac{C_-}{\sqrt{1+|\lambda_- + \rho_-|} + |z|} \leqslant \dfrac{C_-}{1+|z|}, \\ \left\| (B_+ - zI)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \dfrac{C_+}{\sqrt{1+|\lambda_+ + \rho_+|} + |z|} \leqslant \dfrac{C_+}{1+|z|}, \end{cases} \tag{22}$$

from which it follows that

$$\left\| B_-^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{C_- \sqrt{d_-}}{\sqrt{d_- + |\lambda + r_-|}} \quad \text{and} \quad \left\| B_+^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{C_+ \sqrt{d_+}}{\sqrt{d_+ + |\lambda + r_+|}}. \tag{23}$$

Our problem (P_A) can be written in the following form

$$\begin{cases} \begin{cases} w_-''(x) - B_-^2 w_-(x) = g_-(x) & \text{in } (-\ell, 0), \\ w_+''(x) - B_+^2 w_+(x) = g_+(x) & \text{in } (0, L), \end{cases} \\ \begin{cases} w_-(-\ell) = 0, \\ w_+(L) = 0, \end{cases} \\ \begin{cases} w_-'(0) = q_- \left(w_+(0) - w_-(0) \right), \\ w_+'(0) = q_+ \left(w_+(0) - w_-(0) \right). \end{cases} \end{cases} \tag{24}$$

Then, we have

$$\begin{cases} w_-(x) &= e^{(x+\ell)B_-} j_- + e^{-xB_-} k_- + v_-(g_-)(x), \quad x \in (-\ell, 0), \\ w_+(x) &= e^{xB_+} j_+ + e^{(L-x)B_+} k_+ + v_+(g_+)(x), \quad x \in (0, L), \end{cases}$$

with $j_\pm$, $k_\pm \in E$ and

$$\begin{cases} v_-(g_-)(x) &= \dfrac{1}{2} B_-^{-1} \displaystyle\int_{-\ell}^{x} e^{(x-t)B_-} g_-(t)dt + \frac{1}{2} B_-^{-1} \int_x^0 e^{(t-x)B_-} g_-(t)dt, \\ v_+(g_+)(x) &= \dfrac{1}{2} B_+^{-1} \displaystyle\int_{0}^{x} e^{(x-t)B_+} g_+(t)dt + \frac{1}{2} B_+^{-1} \int_x^L e^{(t-x)B_+} g_+(t)dt; \end{cases}$$

see for instance Proposition 5.1, p. 1882 in [6].

Then, we deduce that

$$\begin{cases} w_-'(0) &= B_- e^{\ell B_-} j_- - B_- k_- + v_-'(g_-)(0), \\ w_+'(0) &= B_+ j_+ - B_+ e^{LB_+} k_+ + v_+'(g_+)(0), \end{cases}$$

where

$$v'_-(g_-)(0)=\frac{1}{2}\int_{-\ell}^{0}e^{-tB_-}g_-(t)dt \quad \text{and} \quad v'_+(g_+)(0)=-\frac{1}{2}\int_{0}^{L}e^{tB_+}g_+(t)dt.$$

The boundary and the interface conditions give

$$\begin{cases} j_-=-e^{\ell B_-}k_- - v_-(g_-)(-\ell),\\ k_+=-e^{LB_+}j_+ - v_+(g_+)(L),\\ \\ d_-\left(B_-e^{\ell B_-}j_- - B_-k_- + v'_-(g_-)(0)\right)=\\ q\left[\left(j_+ + e^{LB_+}k_+ + v_+(g_+)(0)\right)-\left(e^{\ell B_-}j_- + k_- + v_-(g_-)(0)\right)\right],\\ \\ d_+\left(B_+j_+ - B_+e^{LB_+}k_+ + v'_+(g_+)(0)\right)=\\ q\left[\left(j_+ + e^{LB_+}k_+ + v_+(g_+)(0)\right)-\left(e^{\ell B_-}j_- + k_- + v_-(g_-)(0)\right)\right]; \end{cases}$$

then, the two last equations lead us to the following system

$$\begin{cases} d_-\left[B_-e^{\ell B_-}\left(-e^{\ell B_-}k_- - v_-(g_-)(-\ell)\right) - B_-k_- + v'_-(g_-)(0)\right]\\ =q[\left(j_+ + e^{LB_+}\left(-e^{LB_+}j_+ - v_+(g_+)(L)\right) + v_+(g_+)(0)\right)\\ \quad - e^{\ell B_-}\left(-e^{\ell B_-}k_- - v_-(g_-)(-\ell)\right) - k_- - v_-(g_-)(0)],\\ \\ d_+\left[B_+j_+ - B_+e^{LB_+}\left(-e^{LB_+}j_+ - v_+(g_+)(L)\right) + v'_+(g_+)(0)\right]\\ =q[\left(j_+ + e^{LB_+}\left(-e^{LB_+}j_+ - v_+(g_+)(L)\right) + v_+(g_+)(0)\right)\\ \quad - e^{\ell B_-}\left(-e^{\ell B_-}k_- - v_-(g_-)(-\ell)\right) - k_- - v_-(g_-)(0), \end{cases}$$

which is equivalent to

$$\begin{cases} -B_-e^{2\ell B_-}k_- - B_-e^{\ell B_-}v_-(g_-)(-\ell) - B_-k_- + v'_-(g_-)(0)\\ =q_-j_+ - q_-e^{2LB_+}j_+ - q_-e^{LB_+}v_+(g_+)(L) + q_-v_+(g_+)(0)\\ \quad + q_-e^{2\ell B_-}k_- + q_-e^{\ell B_-}v_-(g_-)(-\ell) - q_-k_- - q_-v_-(g_-)(0),\\ \\ B_+j_+ + B_+e^{2LB_+}j_+ + B_+e^{LB_+}v_+(g_+)(L) + v'_+(g_+)(0)\\ =q_+j_+ - q_+e^{2LB_+}j_+ - q_+e^{LB_+}v_+(g_+)(L) + q_+v_+(g_+)(0)\\ \quad + q_+e^{2\ell B_-}k_- + q_+e^{\ell B_-}v_-(g_-)(-\ell) - q_+k_- - q_+v_-(g_-)(0). \end{cases}$$

Therefore, the above system becomes

$$\begin{cases} \left[B_-\left(I+e^{2\ell B_-}\right) - q_-\left(I-e^{2\ell B_-}\right)\right]k_- + q_-\left(I-e^{2LB_+}\right)j_+ = (\Pi'),\\ \\ q_+\left(I-e^{2\ell B_-}\right)k_- + \left[B_+\left(I+e^{2LB_+}\right) - q_+\left(I-e^{2LB_+}\right)\right]j_+ = (\Pi''), \end{cases}$$

where

$$\begin{cases} (\Pi') &= v'_-(g_-)(0) - B_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-e^{LB_+}v_+(g_+)(L) - q_-v_+(g_+)(0)\\ &\quad -q_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-v_-(g_-)(0),\\ \\ (\Pi'') &= -v'_+(g_+)(0) - B_+e^{LB_+}v_+(g_+)(L) - q_+e^{LB_+}v_+(g_+)(L) + q_+v_+(g_+)(0)\\ &\quad +q_+e^{\ell B_-}v_-(g_-)(-\ell) - q_+v_-(g_-)(0). \end{cases}$$

It follows

$$\begin{cases} \left[\left(I+e^{2\ell B_-}\right)-q_-B_-^{-1}\left(I-e^{2\ell B_-}\right)\right]k_- + q_-B_-^{-1}\left(I-e^{2LB_+}\right)j_+ = B_-^{-1}(\Pi'), \\ q_+B_+^{-1}\left(I-e^{2\ell B_-}\right)k_- + \left[\left(I+e^{2LB_+}\right)-q_+B_+^{-1}\left(I-e^{2LB_+}\right)\right]j_+ = B_+^{-1}(\Pi''). \end{cases}$$

The abstract determinant of this system is

$$\begin{aligned} D &= \left[\left(I+e^{2\ell B_-}\right)-q_-B_-^{-1}\left(I-e^{2\ell B_-}\right)\right]\left[\left(I+e^{2LB_+}\right)-q_+B_+^{-1}\left(I-e^{2LB_+}\right)\right] \\ &\quad -q_+B_+^{-1}\left(I-e^{2\ell B_-}\right)q_-B_-^{-1}\left(I-e^{2LB_+}\right) \\ &= \left(I+e^{2\ell B_-}\right)\left(I+e^{2LB_+}\right)-\left(I+e^{2\ell B_-}\right)q_+B_+^{-1}\left(I-e^{2LB_+}\right) \\ &\quad -q_-B_-^{-1}\left(I-e^{2\ell B_-}\right)\left(I+e^{2LB_+}\right) \\ &= \left(I+e^{2\ell B_-}\right)\left(I+e^{2LB_+}\right)D_*, \end{aligned}$$

where

$$D_* = \left[I-q_+B_+^{-1}\left(I-e^{2LB_+}\right)\left(I+e^{2LB_+}\right)^{-1}-q_-B_-^{-1}\left(I-e^{2\ell B_-}\right)\left(I+e^{2\ell B_-}\right)^{-1}\right].$$

We recall that operators $\left(I-e^{2LB_+}\right)$, $\left(I+e^{2LB_+}\right)$, $\left(I-e^{2\ell B_-}\right)$ and $\left(I+e^{2\ell B_-}\right)$ are boundedly invertible by using the same method in [16], Proposition 2.3.6, p. 60.

6.1.2 Invertibility of the determinant

In order to invert D_*, we will use the H^∞-calculus for sectorial operators. To this end, we fix $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$ and for all $z \in S_{\varepsilon_0}$, we consider the following function

$$f(z) = \left[1+\frac{q_+\left(1-e^{-2L\sqrt{z+\lambda_++\rho_+}}\right)}{\sqrt{z+\lambda_++\rho_+}\left(1+e^{-2L\sqrt{z+\lambda_++\rho_+}}\right)}+\frac{q_-\left(1-e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right)}{\sqrt{z+\lambda_-+\rho_-}\left(1+e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right)}\right],$$

where the constants $\rho_\pm$, $q_\pm$ and the complex parameter $\lambda_\pm$ are defined in (20).

Proposition 14. *Let $\varepsilon_0 \in (0,\pi/2)$ and $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$. Then, for all $z \in S_{\varepsilon_0}$, we have*

$$|f(z)| > \sin\left(\frac{\varepsilon_0}{2}\right).$$

Proof. Let $\lambda \neq 0$. We have to consider the two following cases:

1. $-\varepsilon_0 \leqslant \arg(\lambda_\pm) < \pi-\varepsilon_0$,
2. $-\pi+\varepsilon_0 < \arg(\lambda_\pm) \leqslant \varepsilon_0$.

Let $-\varepsilon_0 \leqslant \arg(\lambda_\pm) < \pi-\varepsilon_0$. Since $|\arg(z)| < \varepsilon_0$, it is clear that $\lambda_\pm$ and z are in the same half-plane and that $\lambda_\pm + z \neq 0$. Then, from Proposition 12, we have

$$-\varepsilon_0 \leqslant \min\left(\arg(\lambda_\pm),\arg(z)\right) \leqslant \arg(z+\lambda_\pm) \leqslant \max\left(\arg(\lambda_\pm),\arg(z)\right) < \pi-\varepsilon_0,$$

and thus, if $\arg(z+\lambda_\pm) \neq 0$, from Proposition 8, we deduce that

$$|\arg(z+\lambda_\pm+\rho_\pm)| < |\arg(z+\lambda_\pm)| < \pi-\varepsilon_0,$$

and if $\arg(z+\lambda_\pm)=0$, then $\arg(z+\lambda_\pm+\rho_\pm)=0$. Moreover, when $\arg(z+\lambda_\pm+\rho_\pm)<0$, then due to Proposition 8, we have

$$-\varepsilon_0 \leqslant \arg(z+\lambda_\pm) < \arg(z+\lambda_\pm+\rho_\pm).$$

Therefore, we obtain that

$$-\varepsilon_0 \leqslant \arg(z+\lambda_\pm+\rho_\pm) < \pi-\varepsilon_0.$$

Hence, setting $\overline{L}=2L$ or 2ℓ, we deduce that

$$-\frac{\varepsilon_0}{2} \leqslant \arg\left(\sqrt{z+\lambda_\pm+\rho_\pm}\right) = \arg\left(\overline{L}\sqrt{z+\lambda_\pm+\rho_\pm}\right) < \frac{\pi}{2}-\frac{\varepsilon_0}{2}. \tag{25}$$

We set

$$z_1 = \frac{q_+\left(1-e^{-2L\sqrt{z+\lambda_++\rho_+}}\right)}{\sqrt{z+\lambda_++\rho_+}\left(1+e^{-2L\sqrt{z+\lambda_++\rho_+}}\right)} \quad \text{and} \quad z_2 = \frac{q_-\left(1-e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right)}{\sqrt{z+\lambda_-+\rho_-}\left(1+e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right)}.$$

Then, due to Proposition 9, it follows that

$$|f(z)| = |1+z_1+z_2| \geqslant (1+|z_1+z_2|)\left|\cos\left(\frac{\arg(z_1+z_2)}{2}\right)\right| > \cos\left(\frac{\arg(z_1+z_2)}{2}\right).$$

Now, in order to use Proposition 12, we have to prove that z_1 and z_2 are in the same half-plane. Since we have

$$\arg(z_1) = \arg\left(1-e^{-2L\sqrt{z+\lambda_++\rho_+}}\right) - \arg\left(1+e^{-2L\sqrt{z+\lambda_++\rho_+}}\right) - \arg\left(\sqrt{z+\lambda_++\rho_+}\right),$$

and

$$\arg(z_2) = \arg\left(1-e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right) - \arg\left(1+e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right) - \arg\left(\sqrt{z+\lambda_-+\rho_-}\right),$$

from (25) and Corollary 11, with $\alpha=\dfrac{\pi}{2}$ and $\beta=\dfrac{\varepsilon_0}{2}$, it follows that

$$-\frac{\varepsilon_0}{2} \leqslant \arg\left(1-e^{-2L\sqrt{z+\lambda_++\rho_+}}\right) - \arg\left(1+e^{-2L\sqrt{z+\lambda_++\rho_+}}\right) < \frac{\pi}{2}-\frac{\varepsilon_0}{2},$$

with

$$-\frac{\varepsilon_0}{2} \leqslant \arg\left(1-e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right) - \arg\left(1+e^{-2\ell\sqrt{z+\lambda_-+\rho_-}}\right) < \frac{\pi}{2}-\frac{\varepsilon_0}{2},$$

hence

$$-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_++\rho_+}\right) \leqslant \arg(z_1) < \frac{\pi}{2}-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_++\rho_+}\right),$$

and

$$-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-+\rho_-}\right) \leqslant \arg(z_2) < \frac{\pi}{2}-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-+\rho_-}\right).$$

Then, due to (25), we obtain

$$-\frac{\pi}{2} < -\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_++\rho_+}\right) \leqslant \arg(z_1) < \frac{\pi}{2}-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_++\rho_+}\right) \leqslant \frac{\pi}{2},$$

with

$$-\frac{\pi}{2} < -\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-+\rho_-}\right) \leqslant \arg(z_2) < \frac{\pi}{2}-\frac{\varepsilon_0}{2} - \arg\left(\sqrt{z+\lambda_-+\rho_-}\right) \leqslant \frac{\pi}{2}.$$

Thus, $\arg(z_1), \arg(z_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, in virtue of Proposition 12, we deduce that

$$\min\left(\arg(z_1), \arg(z_2)\right) \leqslant \arg(z_1 + z_2) \leqslant \max\left(\arg(z_1), \arg(z_2)\right).$$

Moreover, due to Proposition 10, we have

$$\begin{aligned} |\arg(z_1)| &= \left|\arg\left(1 - e^{-2L\sqrt{z+\lambda_+ +\rho_+}}\right) - \arg\left(1 + e^{-2L\sqrt{z+\lambda_+ +\rho_+}}\right) - \arg\left(\sqrt{z+\lambda_+ +\rho_+}\right)\right| \\ &\leqslant \left|\arg\left(1 - e^{-2L\sqrt{z+\lambda_+ +\rho_+}}\right) - \arg\left(1 + e^{-2L\sqrt{z+\lambda_+ +\rho_+}}\right)\right| + \left|\arg\left(\sqrt{z+\lambda_+ +\rho_+}\right)\right| \\ &< \frac{\pi}{2} - \frac{\varepsilon_0}{2} + \frac{\pi}{2} - \frac{\varepsilon_0}{2} = \pi - \varepsilon_0, \end{aligned}$$

and, in the same way, we have $|\arg(z_2)| < \pi - \varepsilon_0$. We then obtain

$$|\arg(z_1 + z_2)| \leqslant \max\left(\arg(z_1), \arg(z_2)\right) < \pi - \varepsilon_0,$$

and

$$\cos\left(\frac{\arg(z_1 + z_2)}{2}\right) > \cos\left(\frac{\pi}{2} - \frac{\varepsilon_0}{2}\right) = \sin\left(\frac{\varepsilon_0}{2}\right) > 0.$$

Now, when $-\pi + \varepsilon_0 < \arg(\lambda_\pm) \leqslant \varepsilon_0$, then

$$-\frac{\pi}{2} + \frac{\varepsilon_0}{2} < \arg(\sqrt{z+\lambda_\pm +\rho_\pm}) = \arg(\overline{L}\sqrt{z+\lambda_\pm +\rho_\pm}) \leqslant \frac{\varepsilon_0}{2},$$

and due to Corollary 11, we have $\arg(z_1), \arg(z_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, from Proposition 12, we deduce the expected result. Finally, when $\lambda = 0$, it is clear that, following the same steps, we obtain a similar result. □

Proposition 15. *Operator D is boundedly invertible and*

$$D^{-1} = D_*^{-1}\left(I + e^{2LB_+}\right)^{-1}\left(I + e^{2\ell B_-}\right)^{-1}.$$

Proof. From Proposition 14, f does not vanish on S_{ε_0} and $1/f$ is bounded. Moreover

$$\left\|\frac{1}{f}\right\|_\infty \leqslant \frac{1}{\sin(\varepsilon_0/2)},$$

and taking into account Proposition 10, statement 2, we have

$$|f(z) - 1| = O\left(1/|z|^{1/2}\right), \quad z \in S_{\varepsilon_0}.$$

We deduce that

$$\begin{aligned} \left(\frac{1}{f}\right)(-A) &= \left(\frac{1-f}{f} + 1\right)(-A) \\ &= \left(\frac{1-f}{f}\right)(-A) + I \in \mathcal{L}(E), \end{aligned}$$

then by using (11), since $-A \in \mathrm{Sect}(\varepsilon_0)$, we deduce that

$$D_*^{-1} = \left(\frac{1}{f}\right)(-A) \in \mathcal{L}(E) \quad \text{and} \quad \left\|D_*^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{1}{\sin(\varepsilon_0/2)}.$$

Finally, since $(I + e^{2\ell B_-})$ and $(I + e^{2LB_+})$ are boundedly invertible, see [16], p. 60, Proposition 2.3.6, we obtain the expected result. □

Now, from equality $D_*A^{-1} = A^{-1}D_*$, it follows that

$$D_*^{-1}A = AD_*^{-1},$$

on $D(A)$, hence D_*^{-1} is a bounded operator from $D(A)$ into itself. Therefore, by interpolation D_*^{-1} is bounded from any interpolation space $(D(A), E)_{\alpha,\beta}$, for all $\alpha \in (0,1)$ and $\beta \in [1, +\infty]$ (see the definition in [11]) into itself and clearly we have also the same estimate

$$\left\| D_*^{-1} \right\|_{\mathcal{L}((D(A),E)_{\alpha,\beta})} \leqslant \frac{1}{\sin(\varepsilon_0/2)}.$$

6.1.3 Resolution of the system

Recall that

$$\begin{cases} \left[\left(I + e^{2\ell B_-}\right) - q_- B_-^{-1}\left(I - e^{2\ell B_-}\right)\right] k_- + q_- B_-^{-1}\left(I - e^{2LB_+}\right) j_+ = B_-^{-1}(\Pi'), \\ q_+ B_+^{-1}\left(I - e^{2\ell B_-}\right) k_- + \left[\left(I + e^{2LB_+}\right) - q_+ B_+^{-1}\left(I - e^{2LB_+}\right)\right] j_+ = B_+^{-1}(\Pi''), \end{cases}$$

where

$$\begin{cases} (\Pi') &= v_-'(g_-)(0) - B_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- e^{LB_+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ & \quad -q_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \\ (\Pi'') &= -v_+'(g_+)(0) - B_+ e^{LB_+} v_+(g_+)(L) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ v_+(g_+)(0), \\ & \quad +q_+ e^{\ell B_-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0), \end{cases}$$

therefore

$$\begin{aligned} k_- &= D^{-1} \begin{vmatrix} B_-^{-1}(\Pi') & q_- B_-^{-1}(I - e^{2LB_+}) \\ B_+^{-1}(\Pi'') & \left[\left(I + e^{2LB_+}\right) - q_+ B_+^{-1}\left(I - e^{2LB_+}\right)\right] \end{vmatrix} \\ &= D^{-1}\left[B_-^{-1}\left[\left(I + e^{2LB_+}\right) - q_+ B_+^{-1}\left(I - e^{2LB_+}\right)\right](\Pi') - q_- B_+^{-1} B_-^{-1}(I - e^{2LB_+})(\Pi'')\right], \end{aligned}$$

and

$$\begin{aligned} j_+ &= D^{-1} \begin{vmatrix} \left[\left(I + e^{2\ell B_-}\right) - q_- B_-^{-1}\left(I - e^{2\ell B_-}\right)\right] & B_-^{-1}(\Pi') \\ q_+ B_+^{-1}\left(I - e^{2\ell B_-}\right) & B_+^{-1}(\Pi'') \end{vmatrix} \\ &= D^{-1}\left[\left[\left(I + e^{2\ell B_-}\right) - q_- B_-^{-1}\left(I - e^{2\ell B_-}\right)\right] B_+^{-1}(\Pi'') - q_+ B_+^{-1}\left(I - e^{2\ell B_-}\right) B_-^{-1}(\Pi')\right]. \end{aligned}$$

We then deduce

$$\begin{cases} j_- &= -e^{\ell B_-} D^{-1} B_-^{-1}\left[\left(I + e^{2LB_+}\right) - q_+ B_+^{-1}\left(I - e^{2LB_+}\right)\right](\Pi') \\ & \quad +q_- e^{\ell B_-} D^{-1} B_+^{-1} B_-^{-1}(I - e^{2LB_+})(\Pi'') - v_-(g_-)(-\ell), \\ k_+ &= -e^{LB_+} D^{-1}\left[\left(I + e^{2\ell B_-}\right) - q_- B_-^{-1}\left(I - e^{2\ell B_-}\right)\right] B_+^{-1}(\Pi'') \\ & \quad +q_+ e^{LB_+} D^{-1} B_+^{-1}\left(I - e^{2\ell B_-}\right) B_-^{-1}(\Pi') - v_+(g_+)(L). \end{cases}$$

Finally, we obtain for a.e $x \in (-\ell, 0)$

$$\begin{aligned} w_-(x) &= D^{-1}\left(e^{-xB_-} - e^{(x+2\ell)B_-}\right)\left[\left(I + e^{2LB_+}\right) - q_+ B_+^{-1}\left(I - e^{2LB_+}\right)\right] B_-^{-1}(\Pi') \\ & \quad +q_- D^{-1}\left(e^{(x+2\ell)B_-} - e^{-xB_-}\right) B_-^{-1}(I - e^{2LB_+}) B_+^{-1}(\Pi'') \\ & \quad -e^{(x+\ell)B_-} v_-(g_-)(-\ell) + v_-(g_-)(x), \end{aligned}$$

and for all a.e $x \in (0, L)$

$$\begin{aligned} w_+(x) &= D^{-1}\left(e^{xB_+} - e^{(2L-x)B_+}\right)\left[\left(I + e^{2\ell B_-}\right) - q_- B_-^{-1}\left(I - e^{2\ell B_-}\right)\right] B_+^{-1}(\Pi'') \\ &\quad + q_+ D^{-1}\left(e^{(2L-x)B_+} - e^{xB_+}\right) B_+^{-1}\left(I - e^{2\ell B_-}\right) B_-^{-1}(\Pi') \\ &\quad - e^{(L-x)B_+} v_+(g_+)(L) + v_+(g_+)(x). \end{aligned}$$

6.1.4 Optimal regularity of w_- and w_+

Since B_+ generates an analytic semigroup in E, we recall the following known results

$$\begin{cases} x \longmapsto e^{xB_+}\psi \in L^p(0, L; E) \text{ for all } \psi \in E, \\ x \longmapsto B_+^n e^{xB_+}\psi \in L^p(0, L; E) \Longleftrightarrow \psi \in \left(D\left(B_+^n\right), E\right)_{\frac{1}{np}, p}, \end{cases} \tag{26}$$

where $p \in (1, +\infty)$ and $n \in \mathbb{N}\backslash\{0\}$; see the Theorem in [21], p. 96.

We have the same result for B_- on $(-\ell, 0)$. Note that, for these two results we do not need assumption (12).

Let us recall the following well-known important result proved in [7].

Theorem 16. *Let X be a UMD Banach space, $-Q \in BIP(X, \theta)$ with $\theta \in (0, \pi/2)$ and $g \in L^p(a, b; X)$. Then, for almost every $x \in (a, b)$, we have*

$$\int_a^x e^{(x-s)Q} g(s)\, ds \in D(Q) \quad \text{and} \quad \int_x^b e^{(s-x)Q} g(s)\, ds \in D(Q).$$

Moreover,

$$x \longmapsto Q\int_a^x e^{(x-s)Q} g(s)\, ds \in L^p(a, b; X) \quad \text{and} \quad x \longmapsto Q\int_x^b e^{(s-x)Q} g(s)\, ds \in L^p(a, b; X).$$

We are applying these results to our operators B_- and B_+. We know that $-A \in \mathrm{BIP}(E, \theta_A)$, with $\theta_A \in (0, \varepsilon_0]$, $0 < \varepsilon_0 < \pi/2$ and $|\arg(\lambda_\pm + \rho_\pm)| < \pi - \varepsilon_0$. On the other hand, we have

$$\theta_A + |\arg(\lambda_\pm + \rho_\pm)| < \varepsilon_0 + \pi - \varepsilon_0 = \pi,$$

since $\lambda_\pm$ is such that $|\arg(\lambda_\pm)| < \pi - \varepsilon_0$ and $\rho_\pm > 0$. In virtue of Theorem 2.4, p. 408 in [18], we obtain that $-A_\pm \in \mathrm{BIP}(E, \pi - \varepsilon_0)$. Thus $-B_\pm \in \mathrm{BIP}(E, (\pi - \varepsilon_0)/2)$, from Proposition 3.2.1, e), p. 71 in [12].

We then deduce the following lemma by taking $Q = B_\pm$.

Lemma 17. *Let $h_- \in L^p(-\ell, 0; E)$ and $h_+ \in L^p(0, L; E)$ with $1 < p < +\infty$. Assume that (12), (13) and (14) hold. Then, we have*

$$\begin{cases} x \longmapsto B_- \displaystyle\int_{-\ell}^x e^{(x-s)B_-} h_-(s) ds \in L^p(-\ell, 0; E), \\ x \longmapsto B_- \displaystyle\int_x^0 e^{(s-x)B_-} h_-(s)\, ds \in L^p(-\ell, 0; E), \\ x \longmapsto B_- \displaystyle\int_{-\ell}^0 e^{(x+s)B_-} h_-(s)\, ds \in L^p(-\ell, 0; E). \end{cases}$$

and

$$\begin{cases} x \longmapsto B_+ \displaystyle\int_0^x e^{(x-s)B_+} h_+(s)\, ds \in L^p(0, L; E), \\ x \longmapsto B_+ \displaystyle\int_x^L e^{(s-x)B_+} h_+(s)\, ds \in L^p(0, L; E), \\ x \longmapsto B_+ \displaystyle\int_0^L e^{(x+s)B_+} h_+(s)\, ds \in L^p(0, L; E). \end{cases}$$

Lemma 18. *Let $h_- \in L^p(-\ell, 0; E)$ and $h_+ \in L^p(0, L; E)$ with $1 < p < +\infty$. Assume that (12), (13) and (14) hold. Then, we have*

1. $\int_0^L e^{sB_+} h_+(s)\, ds$ *and* $\int_0^L e^{(L-s)B_+} h_+(s)\, ds$ *belong to* $(D(B_+), E)_{\frac{1}{p},p} = (D(B), E)_{\frac{1}{p},p}$,

2. $\int_{-\ell}^0 e^{sB_-} h_-(s)\, ds$ *and* $\int_{-\ell}^0 e^{-sB_-} h_-(s)\, ds$ *belong to* $(D(B_-), E)_{\frac{1}{p},p} = (D(B), E)_{\frac{1}{p},p}$.

Proof. Let us indicate the proof of the first statement for instance. Consider the function

$$\psi_1(x) = \int_0^x e^{(x-s)B_+} h_+(s)\, ds;$$

then, from Theorem 16, we know that

$$\psi_1 \in W^{1,p}(0, L; E) \cap L^p(0, L; D(B_+));$$

by using the notation in [11], pp. 677-678 for the spaces of traces, we deduce that

$$\psi_1(L) \in T_0^1(p, 0, D(B_+), E) = (D(B_+), E)_{\frac{1}{p},p} = (D(B), E)_{\frac{1}{p},p},$$

here, the Poulsen condition is verified since $0 < 1/p < 1$. By considering the function

$$\psi_2(x) = \int_x^L e^{(s-x)B_+} h_+(s)\, ds,$$

we get

$$\psi_2(0) \in T_0^1(p, 0, D(B_+), E) = (D(B_+), E)_{\frac{1}{p},p} = (D(B), E)_{\frac{1}{p},p}.$$

Statement 2 is obtained by the same way. □

Proposition 19. *Let $f \in L^p(-\ell, L; E)$ with $1 < p < +\infty$. Assume that (12), (13) and (14) hold. Then, for all $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$, there exists a unique solution $w \in D(\mathcal{S})$ of equation (19).*

Proof. Now we must show that

$$\begin{cases} w_- \in W^{2,p}(-\ell, 0; E) \cap L^p(-\ell, 0; D(A)), \\ w_+ \in W^{2,p}(0, L; E) \cap L^p(0, L; D(A)). \end{cases}$$

It is not difficult to see that all the boundary and transmission conditions in (24) are verified by w_- and w_+.

We recall that

$$\begin{aligned} (\Pi') \quad = \quad & -B_- e^{\ell B_-} v_-(g_-)(-\ell) + v_-'(g_-)(0) + q_- e^{LB_+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ & -q_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \end{aligned}$$

and

$$\begin{aligned} (\Pi'') \quad = \quad & -B_+ e^{LB_+} v_+(g_+)(L) - v_+'(g_+)(0) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ v_+(g_+)(0) \\ & +q_+ e^{\ell B_-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0). \end{aligned}$$

The terms

$$-B_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- e^{LB_+} v_+(g_+)(L) - q_- e^{\ell B_-} v_-(g_-)(-\ell),$$

in (Π') and

$$-B_+ e^{LB_+} v_+(g_+)(L) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ e^{\ell B_-} v_-(g_-)(-\ell),$$

in (Π'') are clearly regular.

From Lemma 18, the terms

$$v_-(g_-)(0) = \frac{1}{2}B_-^{-1}\int_{-\ell}^{0} e^{-tB_-}g_-(t)dt,$$

and

$$v_+(g_+)(0) = \frac{1}{2}B_+^{-1}\int_{0}^{L} e^{tB_+}g_+(t)\,dt,$$

satisfy

$$B_-v_-(g_-)(0) = \frac{1}{2}\int_{-\ell}^{0} e^{-tB_-}g_-(t)dt \in (D(B_-),E)_{\frac{1}{p},p},$$

and

$$B_+v_+(g_+)(0) = \frac{1}{2}\int_{0}^{L} e^{tB_+}g_+(t)dt \in (D(B_+),E)_{\frac{1}{p},p}.$$

We also obtain

$$v'_-(g_-)(0) = \frac{1}{2}\int_{-\ell}^{0} e^{-tB_-}g_-(t)dt \in (D(B_-),E)_{\frac{1}{p},p},$$

and

$$v'_+(g_+)(0) = -\frac{1}{2}\int_{0}^{L} e^{tB_+}g_+(t)dt \in (D(B_+),E)_{\frac{1}{p},p}.$$

Now, due to Proposition 15, we have for a.e $x \in (-\ell,0)$

$$\begin{aligned}
w_-(x) &= -D^{-1}e^{(x+2\ell)B_-}B_-^{-1}\left[\left(I+e^{2LB_+}\right)-q_+B_+^{-1}\left(I-e^{2LB_+}\right)\right](\Pi')\\
&\quad +q_-D^{-1}e^{(x+2\ell)B_-}B_+^{-1}B_-^{-1}(I-e^{2LB_+})(\Pi'')\\
&\quad -e^{(x+\ell)B_-}v_-(g_-)(-\ell)\\
&\quad +D^{-1}e^{-xB_-}B_-^{-1}\left[\left(I+e^{2LB_+}\right)-q_+B_+^{-1}\left(I-e^{2LB_+}\right)\right](\Pi')\\
&\quad -q_-D^{-1}e^{-xB_-}B_+^{-1}B_-^{-1}(I-e^{2LB_+})(\Pi'')\\
&\quad +\frac{1}{2}B_-^{-1}\int_{-\ell}^{x}e^{(x-t)B_-}g_-(t)dt+\frac{1}{2}B_-^{-1}\int_{x}^{0}e^{(t-x)B_-}g_-(t)dt,
\end{aligned}$$

so, we can write w_- as the following superposition of less and less regular functions:

$$\begin{aligned}
w_-(x) &= R(x)\\
&\quad -q_+D^{-1}B_+^{-1}e^{-xB_-}B_-^{-1}(\Pi')\\
&\quad -q_-D^{-1}B_-^{-1}e^{-xB_-}B_+^{-1}(\Pi'')\\
&\quad -e^{(x+\ell)B_-}v_-(g_-)(-\ell)\\
&\quad +D^{-1}e^{-xB_-}B_-^{-1}(\Pi')\\
&\quad +\frac{1}{2}B_-^{-1}\int_{-l}^{x}e^{(x-t)B_-}g_-(t)dt+\frac{1}{2}B_-^{-1}\int_{x}^{0}e^{(t-x)B_-}g_-(t)dt,
\end{aligned}$$

where, for all $x \in [-\ell,0]$

$$\begin{aligned}
R(x) &= -e^{(x+2\ell)B_-}D^{-1}B_-^{-1}\left[\left(I+e^{2LB_+}\right)-q_+B_+^{-1}\left(I-e^{2LB_+}\right)\right](\Pi')\\
&\quad +q_-e^{(x+2\ell)B_-}D^{-1}B_+^{-1}B_-^{-1}(I-e^{2LB_+})(\Pi'')\\
&\quad +e^{2LB_+}D^{-1}e^{-xB_-}B_-^{-1}\left[I+q_+B_+^{-1}\right](\Pi')\\
&\quad +q_-e^{2LB_+}D^{-1}e^{-xB_-}B_+^{-1}B_-^{-1}(\Pi''),
\end{aligned}$$

is clearly a regular term.

Replacing (Π') and (Π''), we obtain

$$w_-(x) = R_1(x) + S_1(x) + S_2(x) + I(x),$$

where

$$\begin{aligned} R_1(x) &= R(x) + D^{-1}B_-^{-1}e^{-xB_-}\left[(-B_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-e^{LB_+}v_+(g_+)(L)\right] \\ &+D^{-1}B_-^{-1}e^{-xB_-}\left[-q_-v_+(g_+)(0) - q_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-v_-(g_-)(0)\right] \\ &-q_+D^{-1}B_+^{-1}B_-^{-1}e^{-xB_-}\left[(-B_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-e^{LB_+}v_+(g_+)(L)\right] \\ &-q_+D^{-1}B_+^{-1}B_-^{-1}e^{-xB_-}\left[-q_-v_+(g_+)(0) - q_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-v_-(g_-)(0)\right] \\ &-q_-D^{-1}B_+^{-1}B_-^{-1}e^{-xB_-}\left[-B_+e^{LB_+}v_+(g_+)(L) - q_+e^{LB_+}v_+(g_+)(L)\right] \\ &-q_-D^{-1}B_+^{-1}B_-^{-1}e^{-xB_-}\left[q_+v_+(g_+)(0) - q_-e^{\ell B_-}v_-(g_-)(-\ell) + q_-v_-(g_-)(0)\right] \\ &+D^{-1}B_+^{-1}B_-^{-1}e^{-xB_-}\left(q_-v'_+(g_+)(0) - q_+v'_-(g_-)(0)\right), \end{aligned}$$

$$\begin{cases} S_1(x) &= D^{-1}B_-^{-1}e^{-xB_-}v'_-(g_-)(0), \\ S_2(x) &= -e^{(x+\ell)B_-}v_-(g_-)(-\ell), \end{cases}$$

and

$$I(x) = \frac{1}{2}B_-^{-1}\int_{-\ell}^{x} e^{(x-t)B_-}g_-(t)dt + \frac{1}{2}B_-^{-1}\int_{x}^{0} e^{(t-x)B_-}g_-(t)dt.$$

It is clear that for all $x \in [-\ell, 0]$, we have

$$R_1(x) \in D(B_-^2) \quad \text{and} \quad x \longmapsto B_-^2R_1(x) \in L^p\left(-\ell, 0; E\right).$$

Due to Lemma 18 and (26), for $a.e\ x \in (-\ell, 0)$, it follows that

$$\begin{cases} S_1(x) \in D(B_-^2) \quad \text{and} \quad x \longmapsto B_-^2S_1(x) \in L^p\left(-\ell, 0; E\right), \\ S_2(x) \in D(B_-^2) \quad \text{and} \quad x \longmapsto B_-^2S_2(x) \in L^p\left(-\ell, 0; E\right). \end{cases}$$

Finally, due to Lemma 17, for $a.e\ x \in (-\ell, 0)$, we have

$$I(x) \in D(B_-^2) \quad \text{and} \quad x \longmapsto I(x) \in L^p(-\ell, 0; E).$$

In the same way, for $a.e\ x \in (0, L)$, we obtain

$$w_+(x) \in D(B_+^2) \quad \text{and} \quad x \longmapsto B_+^2w_+\left(x\right) \in L^p\left(0, L; E\right).$$

Finally, we have proved that $w \in D(\mathcal{S})$. □

6.2 Estimate of the resolvent operator

6.2.1 Some sharp estimates

Recall that

$$B_- = -\sqrt{-A + \rho_-I + \lambda_-I} \quad \text{and} \quad B_+ = -\sqrt{-A + \rho_+I + \lambda_+I},$$

where

$$\lambda_\pm = \frac{\lambda}{d_\pm} \quad \text{and} \quad \rho_\pm = \frac{r_\pm}{d_\pm},$$

with $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$, $r_\pm, d_\pm > 0$.

In virtue of (22) and [19], p. 70, we have

$$\begin{cases} \exists\, M>0, \exists\, \delta>0, \forall\, t \geqslant 0, \forall \lambda \in S_{\pi-\varepsilon_0} \cup \{0\} : \\ \left\| e^{tB_\pm} \right\|_{\mathcal{L}(E)} \leqslant M e^{-\delta t} \end{cases} \tag{27}$$

On the other hand, using Lemma 2.6, p. 103 in [8], we have more precisely

$$\begin{cases} \exists\, K>0, \exists\, c>0, \forall\, t \geqslant t_0>0, \forall\, \lambda \in S_{\pi-\varepsilon_0} \cup \{0\} : \\ \left\| B_\pm e^{tB_\pm} \right\|_{\mathcal{L}(E)} \leqslant K e^{-ct|\lambda_\pm+\rho_\pm|^{1/2}}. \end{cases} \tag{28}$$

Lemma 20. *There exists a constant $M>0$ independent of $\lambda \in S_{\pi-\varepsilon_0}$, such that for any $\lambda \in S_{\pi-\varepsilon_0}$, operators $I \pm e^{2\ell B_-}$ and $I \pm e^{2LB_+}$ are boundedly invertible in $\mathcal{L}(E)$ and*

$$\left\| \left(I \pm e^{2\ell B_-} \right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant M \quad \text{and} \quad \left\| \left(I \pm e^{2LB_+} \right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant M.$$

The proof is similar to the one of Lemma 5.1 in [9].

As a consequence of Proposition 15 and Lemma 20, we deduce the following Corollary.

Corollary 21. *There exists a constant $M>0$ independent of $\lambda \in S_{\pi-\varepsilon_0}$, such that for any $\lambda \in S_{\pi-\varepsilon_0}$, we have*

$$\left\| D^{-1} \right\|_{\mathcal{L}(E)} \leqslant M.$$

Proposition 22. *Let $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$, $g \in L^p(0,L;E)$ and $h \in L^p(-\ell,0;E)$ with $1<p<+\infty$. Set*

$$U(g)(x) = \int_0^L e^{|x-t|B_+} g(t)dt \quad \text{and} \quad V(h)(x) = \int_{-\ell}^0 e^{|x-t|B_-} h(t)dt.$$

Then, there exists a constant $C>0$ independent of λ such that

$$\begin{cases} \|U(g)\|_{L^p(0,L;E)} & \leqslant & \dfrac{C\sqrt{d_+}}{\sqrt{|\lambda+r_+|+d_+}} \|g\|_{L^p(0,L;E)}, \\ \|V(h)\|_{L^p(-\ell,0;E)} & \leqslant & \dfrac{C\sqrt{d_-}}{\sqrt{|\lambda+r_-|+d_-}} \|h\|_{L^p(-\ell,0;E)}. \end{cases}$$

Proof. We will prove the estimate, for instance, for B_+. Let

$$g \in \mathcal{D}(0,L;\mathcal{L}(E)) \subset \mathscr{S}(\mathbb{R};\mathcal{L}(E)),$$

where $\mathcal{D}(0,L;\mathcal{L}(E))$ is the space of all vector-valued test functions on $\mathcal{L}(E)$ and $\mathscr{S}(\mathbb{R};\mathcal{L}(E))$ is the Schwartz space of rapidly decreasing vector-valued smooth functions on $\mathbb{R}$. Then, we can write

$$U(g)(x) = \left(e^{|\cdot|B_+} * g \right)(x) = \left(g * e^{|\cdot|B_+} \right)(x), \quad x \in \mathbb{R}.$$

This abstract convolution is well defined, see [1].

Recall the abstract Fourier transform F defined by

$$F(\psi)(x) = \int_{-\infty}^{+\infty} e^{-2i\pi\xi x} \psi(\xi) d\xi,$$

for all $\psi \in L^1(\mathbb{R};\mathcal{L}(E))$ and the well known property

$$F^{-1}(F(\phi)) = \phi,$$

for all $\phi \in \mathscr{S}(\mathbb{R};\mathcal{L}(E))$. We have

$$\begin{aligned} F\left(e^{|.|B_+}\right)(\xi) &= \int_{-\infty}^{0} e^{-2i\pi\xi x} e^{-xB_+} dx + \int_{0}^{+\infty} e^{-2i\pi\xi x} e^{xB_+} dx \\ &= -\left(B_+ + 2i\pi\xi I\right)^{-1} - \left(B_+ - 2i\pi\xi I\right)^{-1} \\ &= -2B_+ \left(B_+ + 2i\pi\xi I\right)^{-1} \left(B_+ - 2i\pi\xi I\right)^{-1}; \end{aligned}$$

the integrals are absolutely convergent from (27); the last equality holds from the resolvent identity. In virtue of Theorem 3.6, p. 17 in [1], we obtain

$$U(g)(x) = F^{-1}\left(F\left(e^{-|.|B_+}\right)F(g)\right)(x) = F^{-1}\left(mF(g)\right)(x),$$

with the Fourier multiplier

$$m(\xi) = -2B_+ \left(B_+ + 2i\pi\xi I\right)^{-1} \left(B_+ - 2i\pi\xi I\right)^{-1} \in \mathcal{L}(E).$$

Using estimate (29), p. 14, in [9], we obtain

$$\begin{aligned} \|m(\xi)\|_{\mathcal{L}(E)} &= \left\| -2B_+ \left(B_+ + 2i\pi\xi I\right)^{-1} \left(B_+ - 2i\pi\xi I\right)^{-1} \right\| \\ &\leqslant 2C \left\| \left(B_+ - 2i\pi\xi I\right)^{-1} \right\| \\ &\leqslant \frac{2C}{\sqrt{\dfrac{|\lambda + r_+|}{d_+} + 1} + |2i\pi\xi|}. \end{aligned}$$

Then

$$\|m(\xi)\|_{\mathcal{L}(E)} \leqslant \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+} + 2\pi\sqrt{d_+}\,|\xi|} \leqslant \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}},$$

hence

$$\sup_{\xi\in\mathbb{R}} \|m(\xi)\|_{\mathcal{L}(E)} \leqslant \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}}.$$

Now, we must estimate

$$\sup_{\xi\in\mathbb{R}} |\xi| \left\|m'(\xi)\right\|_{\mathcal{L}(E)}.$$

Due to the analyticity of the resolvent operator of B_+ on the imaginary axis, it follows that

$$m \in C^\infty(\mathbb{R}, \mathcal{L}(E)),$$

and

$$m'(\xi) = 2i\pi \left(B_+ + 2i\pi\xi I\right)^{-2} - 2i\pi \left(B_+ - 2i\pi\xi I\right)^{-2}.$$

Therefore, as above, we have

$$\begin{aligned} |\xi| \left\|m'(\xi)\right\|_{\mathcal{L}(E)} &\leqslant \left\| 2i\pi\xi \left(B_+ + 2i\pi\xi I\right)^{-2} \right\| + \left\| 2i\pi\xi \left(B_+ - 2i\pi\xi I\right)^{-2} \right\| \\ &\leqslant \left\| 2i\pi\xi \left(B_+ + 2i\pi\xi I\right)^{-1} \right\| \left\| \left(B_+ + 2i\pi\xi I\right)^{-1} \right\| \\ &\quad + \left\| 2i\pi\xi \left(B_+ - 2i\pi\xi I\right)^{-1} \right\| \left\| \left(B_+ - 2i\pi\xi I\right)^{-1} \right\| \\ &\leqslant \frac{2C}{\sqrt{\dfrac{|\lambda + r_+|}{d_+} + 1} + |2i\pi\xi|}, \end{aligned}$$

so

$$\sup_{\xi\in\mathbb{R}} |\xi| \left\|m'(\xi)\right\|_{\mathcal{L}(E)} \leqslant \frac{2C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}}.$$

We do similarly with $\xi \longmapsto (\xi m'(\xi))'$. Thus, from [22], Proposition 2.5, p. 739, the sets

$$\{m(\xi),\ \xi \in \mathbb{R} \setminus \{0\}\} \quad \text{and} \quad \{\xi m'(\xi),\ \xi \in \mathbb{R} \setminus \{0\}\},$$

are R-bounded. Moreover, applying Theorem 3.4, p. 746 in [22], we obtain

$$\begin{aligned} \|U(g)\|_{L^p(0,L;E)} &= \left\| F^{-1}\left(mF(g)\right)\right\|_{L^p(\mathbb{R};E)} \\ &\leqslant C\left[\sup_{\xi\in\mathbb{R}} \|m(\xi)\| + \sup_{\xi\in\mathbb{R}} |\xi|\,\|m'(\xi)\|\right] \|g\|_{L^p(0,L;E)} \\ &\leqslant \frac{C\sqrt{d_+}}{\sqrt{|\lambda + r_+| + d_+}} \|g\|_{L^p(0,L;E)}, \end{aligned}$$

for all $g \in \mathcal{D}(0,L;E)$. The same estimate is true for all $g \in L^p(0,L;E)$ by density. □

We will need also the following result, given by Lemma 4.12 in [13].

Lemma 23. *Let $\lambda \in S_{\pi-\varepsilon_0} \cup \{0\}$, $g_+ \in L^p(0,L;E)$ and $g_- \in L^p(-\ell,0;E)$ with $1 < p < +\infty$. Then, there exists $C > 0$, such that*

1. $$\left\| e^{(\,\bullet - a)B_\pm} \int_a^b e^{(t-a)B_\pm} g_\pm(t)\ dt \right\|_{L^p(a,b;E)} \leqslant \frac{C\sqrt{d_\pm}}{\sqrt{d_\pm + |\lambda + r_\pm|}} \|g_\pm\|_{L^p(a,b;E)},$$

2. $$\left\| e^{(\,\bullet - a)B_\pm} \int_a^b e^{(b-t)B_\pm} g_\pm(t)\ dt \right\|_{L^p(a,b;E)} \leqslant \frac{C\sqrt{d_\pm}}{\sqrt{d_\pm + |\lambda + r_\pm|}} \|g_\pm\|_{L^p(a,b;E)},$$

3. $$\left\| e^{(b-\,\bullet)B_\pm} \int_a^b e^{(b-t)B_\pm} g_\pm(t)\ ds \right\|_{L^p(a,b;E)} \leqslant \frac{C\sqrt{d_\pm}}{\sqrt{d_\pm + |\lambda + r_\pm|}} \|g_\pm\|_{L^p(a,b;E)},$$

4. $$\left\| e^{(b-\,\bullet)B_\pm} \int_a^b e^{(t-a)B_\pm} g_\pm(t)\ dt \right\|_{L^p(a,b;E)} \leqslant \frac{C\sqrt{d_\pm}}{\sqrt{d_\pm + |\lambda + r_\pm|}} \|g_\pm\|_{L^p(a,b;E)}.$$

6.2.2 Estimate of $\|w\|_{L^p(-\ell,L;E)}$

We have to estimate

$$\|w_-\|_{L^p(-\ell,0;E)} + \|w_+\|_{L^p(0,L;E)},$$

where, due to Proposition 19, w is the unique solution of (19). Thus, it suffices to estimate $\|w_-\|_{L^p(-\ell,0;E)}$. The same techniques apply to $\|w_+\|_{L^p(0,L;E)}$.

We have

$$\begin{aligned} w_-(x) &= D^{-1}\left(e^{-xB_-} - e^{(x+2\ell)B_-}\right) B_-^{-1}\left(I + e^{2LB_+}\right)(\Pi') \\ &\quad -q_+ D^{-1}\left(e^{-xB_-} - e^{(x+2\ell)B_-}\right) B_-^{-1} B_+^{-1}\left(I - e^{2LB_+}\right)(\Pi') \\ &\quad +q_- D^{-1}\left(e^{(x+2\ell)B_-} - e^{-xB_-}\right) B_+^{-1} B_-^{-1}(I - e^{2LB_+})(\Pi'') \\ &\quad -e^{(x+\ell)B_-} v_-(g_-)(-\ell) \\ &\quad +v_-(g_-)(x) \\ &= \sum_{i=1}^{5} a_i(x), \end{aligned}$$

where

$$\begin{cases} (\Pi') &= v'_-(g_-)(0) - B_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- e^{LB_+} v_+(g_+)(L) - q_- v_+(g_+)(0) \\ & \quad -q_- e^{\ell B_-} v_-(g_-)(-\ell) + q_- v_-(g_-)(0), \\ (\Pi'') &= -v'_+(g_+)(0) - B_+ e^{LB_+} v_+(g_+)(L) - q_+ e^{LB_+} v_+(g_+)(L) + q_+ v_+(g_+)(0) \\ & \quad +q_+ e^{\ell B_-} v_-(g_-)(-\ell) - q_+ v_-(g_-)(0), \end{cases}$$

and

$$\begin{cases} v_-(g_-)(x) &= \frac{1}{2} B_-^{-1} \int_{-\ell}^{x} e^{(x-t)B_-} g_-(t)\ dt + \frac{1}{2} B_-^{-1} \int_{x}^{0} e^{(t-x)B_-} g_-(t)\ dt, \\ v_+(g_+)(x) &= \frac{1}{2} B_+^{-1} \int_{0}^{x} e^{(x-t)B_+} g_+(t)\ dt + \frac{1}{2} B_+^{-1} \int_{x}^{L} e^{(t-x)B_+} g_+(t)\ dt, \\ v'_-(g_-)(x) &= \frac{1}{2} \int_{-\ell}^{x} e^{(x-t)B_-} g_-(t)\ dt - \frac{1}{2} \int_{x}^{0} e^{(t-x)B_-} g_-(t)\ dt, \\ v'_+(g_+)(x) &= \frac{1}{2} \int_{0}^{x} e^{(x-t)B_+} g_+(t)\ dt - \frac{1}{2} \int_{x}^{L} e^{(t-x)B_+} g_+(t)\ dt. \end{cases}$$

We will focus ourselves, for instance, on the first term a_1, that is

$$a_1(x) = D^{-1} \left(e^{-xB_-} - e^{(x+2\ell)B_-} \right) B_-^{-1} \left(I + e^{2LB_+} \right) (\Pi').$$

After replacing (Π') by it expression, we obtain explicitly

$$\begin{aligned} a_1(x) &= D^{-1} B_-^{-1} e^{-xB_-} v'_-(g_-)(0) + D^{-1} B_-^{-1} e^{-xB_-} e^{2LB_+} v'_-(g_-)(0) \\ &\quad -D^{-1} e^{(x+2\ell)B_-} B_-^{-1} v'_-(g_-)(0) - D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{2LB_+} v'_-(g_-)(0) \\ &\quad -D^{-1} e^{-xB_-} e^{\ell B_-} v_-(g_-)(-\ell) - D^{-1} e^{-xB_-} e^{2LB_+} e^{\ell B_-} v_-(g_-)(-\ell) \\ &\quad +D^{-1} e^{(x+2\ell)B_-} e^{\ell B_-} v_-(g_-)(-\ell) + D^{-1} e^{(x+2\ell)B_-} e^{2LB_+} e^{\ell B_-} v_-(g_-)(-\ell) \\ &\quad +q_- D^{-1} B_-^{-1} e^{-xB_-} e^{LB_+} v_+(g_+)(L) + q_- D^{-1} B_-^{-1} e^{-xB_-} e^{3LB_+} v_+(g_+)(L) \\ &\quad -q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{LB_+} v_+(g_+)(L) - q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{3LB_+} v_+(g_+)(L) \\ &\quad -q_- D^{-1} e^{-xB_-} B_-^{-1} v_+(g_+)(0) - q_- D^{-1} e^{-xB_-} B_-^{-1} e^{2LB_+} v_+(g_+)(0) \\ &\quad +q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} v_+(g_+)(0) + q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{2LB_+} v_+(g_+)(0) \\ &\quad -q_- D^{-1} e^{-xB_-} B_-^{-1} e^{\ell B_-} v_-(g_-)(-\ell) - q_- D^{-1} e^{-xB_-} B_-^{-1} e^{2LB_+} e^{\ell B_-} v_-(g_-)(-\ell) \\ &\quad +q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{\ell B_-} v_-(g_-)(-\ell) + q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{2LB_+} e^{\ell B_-} v_-(g_-)(-\ell) \\ &\quad +q_- D^{-1} e^{-xB_-} B_-^{-1} v_-(g_-)(0) + q_- D^{-1} e^{-xB_-} B_-^{-1} e^{2LB_+} v_-(g_-)(0) \\ &\quad -q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} v_-(g_-)(0) - q_- D^{-1} e^{(x+2\ell)B_-} B_-^{-1} e^{2LB_+} v_-(g_-)(0) \\ &= \sum_{k=1}^{12} b_k(x). \end{aligned}$$

Let us estimate, for instance, the four first terms. The others can be treated analogously.

$$\begin{aligned} \|b_1(.)\|_{L^p(-\ell,0;E)} &= \left\| D^{-1} B_-^{-1} e^{-\bullet B_-} v'_-(g_-)(0) \right\|_{L^p(-\ell,0;E)} \\ &= \frac{1}{2} \left\| D^{-1} B_-^{-1} e^{-\bullet B_-} \int_{-\ell}^{0} e^{-tB_-} g_-(t) dt \right\|_{L^p(-\ell,0;E)} \\ &\leqslant \frac{1}{2} \left\| D^{-1} \right\|_{\mathcal{L}(E)} \left\| B_-^{-1} \right\|_{\mathcal{L}(E)} \left[\int_{-\ell}^{0} \left\| e^{-xB_-} \int_{-\ell}^{0} e^{-tB_-} g_-(t) dt \right\|_E^p dx \right]^{\frac{1}{p}}; \end{aligned}$$

then, from (23), Corollary 21 and Lemma 23 statement 3. (with $[a,b]=[-\ell,0]$), there exists a constant $K>0$ independent of λ such that

$$\begin{aligned}\|b_1(.)\|_{L^p(-\ell,0;E)} &\leqslant \frac{K\sqrt{d_-}\sqrt{d_-}}{\sqrt{d_-+|\lambda+r_-|}\sqrt{d_-+|\lambda+r_-|}}\|g_-\|_{L^p(-\ell,0;E)}\\ &= \frac{Kd_-}{d_-+|\lambda+r_-|}\|g_-\|_{L^p(-\ell,0;E)}.\end{aligned}$$

For the second term b_2, we have

$$\begin{aligned}\|b_2(.)\|_{L^p(-\ell,0;E)} &= \left\|D^{-1}e^{-\bullet B_-}e^{2LB_+}e^{\ell B_-}v_-(g_-)(-\ell)\right\|_{L^p(-\ell,0;E)}\\ &= \left\|\frac{1}{2}D^{-1}B_-^{-1}e^{2LB_+}e^{2\ell B_-}e^{-\bullet B_-}\int_{-\ell}^{0}e^{tB_-}g_-(t)dt\right\|_{L^p(-\ell,0;E)};\end{aligned}$$

thus, using the same arguments as above, there exists a constant $K>0$ independent of λ such that

$$\|b_2(.)\|_{L^p(-\ell,0;E)} \leqslant \frac{Kd_-}{d_-+|\lambda+r_-|}\|g_-\|_{L^p(-\ell,0;E)}.$$

For the third term b_3, we have

$$\begin{aligned}\|b_3(.)\|_{L^p(-\ell,0;E)} &= \left\|q_-D^{-1}B_-^{-1}e^{2\ell B_-}e^{2LB_+}e^{.B_-}v_+(g_+)(0)\right\|_{L^p(-\ell,0;E)}\\ &= \left\|\frac{1}{2}q_-D^{-1}B_-^{-1}B_+^{-1}e^{2\ell B_-}e^{2LB_+}e^{.B_-}\int_0^L e^{tB_+}g_+(t)dt\right\|_{L^p(-\ell,0;E)};\end{aligned}$$

thus, using the same arguments as above, there exists a constant $K>0$ independent of λ such that

$$\|b_3(.)\|_{L^p(-\ell,0;E)} \leqslant \frac{qK\sqrt{d_+/d_-}}{\sqrt{d_-+|\lambda+r_-|}\sqrt{d_++|\lambda+r_+|}}\left\|e^{\bullet B_-}\int_0^L e^{tB_+}g_+(t)dt\right\|_{L^p(-\ell,0;E)}.$$

Now, the boundedness of the semigroup $e^{\bullet B_-}$ and the Hölder inequality lead us to obtain

$$\|b_3(.)\|_{L^p(-\ell,0;E)} \leqslant \frac{qK'\sqrt{d_+/d_-}}{\sqrt{d_-+|\lambda+r_-|}\sqrt{d_++|\lambda+r_+|}}\|g_+\|_{L^p(0,L;E)}.$$

In the same way, for the fourth term b_4, we have

$$\begin{aligned}\|b_4(.)\|_{L^p(-\ell,0;E)} &= \left\|q_-D^{-1}B_-^{-1}e^{(\bullet+2\ell)B_-}e^{3LB_+}v_+(g_+)(L)\right\|_{L^p(-\ell,0;E)}\\ &= \left\|\frac{1}{2}D^{-1}B_-^{-1}B_+^{-1}e^{2\ell B_-}e^{4LB_+}e^{\bullet B_-}\int_0^L e^{-tB_+}g_+(t)dt\right\|_{L^p(-\ell,0;E)};\end{aligned}$$

then

$$\|b_4(.)\|_{L^p(-\ell,0;E)} \leqslant \frac{qK'\sqrt{d_+/d_-}}{\sqrt{d_-+|\lambda+r_-|}\sqrt{d_++|\lambda+r_+|}}\|g_+\|_{L^p(0,L;E)}.$$

Therefore, we can conclude that there exists $M>0$, independent of λ such that

$$\begin{aligned}\|a_1\|_{L^p(-\ell,L;E)} &\leqslant \frac{Md_-}{d_-+|\lambda+r_-|}\|g_-\|_{L^p(-\ell,0;E)}\\ &\quad+\frac{qM\sqrt{d_+/d_-}}{\sqrt{d_-+|\lambda+r_-|}\sqrt{d_++|\lambda+r_+|}}\|g_+\|_{L^p(0,L;E)}.\end{aligned}$$

The same techniques as above lead us to obtain similar estimates for the terms a_i, $i = 2, ..., 4$ in w_-.

For the convolution term $v_-(g_-)(.)$, using Proposition 22, we have

$$\|v_-(g_-)(.)\|_{L^p(-\ell,0;E)} \leqslant \frac{Cd_-}{d_- + |\lambda + r_-|} \|g_-\|_{L^p(-\ell,0;E)}.$$

Hence we can conclude that there exists of a constant $C > 0$, independent of $\lambda \in S_{\pi-\varepsilon_0}$, such that

$$\|w_-\|_{L^p(-\ell,0,;E)} \leqslant \frac{C}{|\lambda|} \left(\|g_-\|_{L^p(-\ell,0;E)} + \|g_+\|_{L^p(0,L;E)} \right).$$

Using the same calculus as above, we obtain the existence of a constant $C > 0$, independent of $\lambda \in S_{\pi-\varepsilon_0}$, such that

$$\|w_+\|_{L^p(0,L;E)} \leqslant \frac{C}{|\lambda|} \left(\|g_-\|_{L^p(-\ell,0;E)} + \|g_+\|_{L^p(0,L;E)} \right).$$

Summing up, we obtain

$$\|w\|_{L^p(-\ell,L;E)} \leqslant \frac{C}{|\lambda|} \|g\|_{L^p(-\ell,L;E)}.$$

Then, we conclude that there exists $C > 0$, such that, for all $\lambda \in S_{\pi-\varepsilon_0}$, we have

$$\left\| (\mathcal{S} - \lambda I)^{-1} \right\|_{\mathcal{L}(L^p(-\ell,L;E))} \leqslant \frac{C}{|\lambda|},$$

which implies that $\mathcal{S}$ generates an analytic semigroup in $L^p(-\ell, L; E)$.

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References

[1] H. Amann, *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr., 186, (1997), 5-56.

[2] A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Maths., 10 (1960), 419-437.

[3] D. L. Burkholder, *A geometrical characterization of Banach spaces in in which martingale difference sequences are unconditional*, Ann. Probab., 9 (1981), 997-1011.

[4] R. S. Cantrell & C. Cosner, *Diffusion models for population dynamics incorporating individual behavior at boundaries: applications to refuge design*, Theoretical Population Biology, 55, (1999), 189-207.

[5] M. Cowling, I. Doust, A. McIntosh & A. Yagi, *Banach space operator with a bounded H^∞ functional calculus*, J. Austral. Math. Soc., Ser. A, 60 (1996), 51-89.

[6] G. Dore, A. Favini, R. Labbas & K. Lemrabet, *An abstract transmission problem in a thin layer, I: Sharp estimates*, Journal of Functional Analysis, 261 (2011), 1865-1922.

[7] G. Dore & A. Venni, *On the closedness of the sum of two closed operators*, Mathematicsche Zeitschrift, 196 (1987), 124-136.

[8] G. Dore & S. Yakubov, *Semigroup estimates and noncoercive boundary value problems*, Semigroup Forum, Vol. 60 (2000), 93-121.

[9] A. Favini, R. Labbas, S. Maingot & A. Thorel, *Elliptic differential operator with an abstract Robin boundary condition and two spectral parameters*, submitted (2021) `hal-02975665`.

[10] A. Favini, R. Labbas, A. Medeghri & A. Menad, *Analytic semigroups generated by the dispersal process in two habitats incorporating individual behavior at the interface*, J. Math. Anal. Appl, Vol. 471, 1-2 (2019), 448-480.

[11] P. Grisvard, *Spazi di tracce e applicazioni*, Rendiconti di Matematica, (4) Vol. 5, série VI, (1972), 657-729.

[12] M. Haase, "The functional calculus for sectorial operators", Birkhauser (2006).

[13] R. Labbas, S. Maingot & A. Thorel, *Generation of analytic semigroup for some generalized diffusion operators in Lp-spaces*, Math. Ann., (2021), published online.

[14] R. Labbas & M. Moussaoui, *On the resolution of the heat equation with discontinuous coefficients*, Semigroup Forum, 60 (2000), 187-201.

[15] J. L. Lions & E. Magenes, "Non-homogeneous boundary value problems and applications", Die Grundlehren der mathematischen Wissenschaften, vol. 1, (1972).

[16] A. Lunardi, "Analytic semigroups and optimal regularity in parabolic problems", Birkhäuser, (1995).

[17] L. Tartar, "Introduction to Sobolev spaces and interpolation spaces", Springer Heidelberg New York, (2007).

[18] S. Monniaux, *A perturbation result for bounded imaginary powers*, Archiv der Mathematik, 68, (1997), 407-417.

[19] A. Pazy, "Semigroups of linear operators and applications to partial differential equations", Springer-Verlag, New York, 119, (1983).

[20] J. Prüss & H. Sohr, *On operators with bounded imaginary powers in Banach spaces*, Math. Z., 203, (1990), 429-452.

[21] H. Triebel, "Interpolation theory, function spaces, differential operators", North-Holland, Amsterdam, (1978).

[22] L. Weis, *Operator-valued Fourier multiplier theorems and maximal regularity*, Math. Ann., 319 (2001), 735-758.