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# On Valuation Fans and the Real Holomorphy Ring
## *Dedicated to Professor Francisco Miraglia*

Danielle Gondard-Cozette

**Abstract.** In this survey we recall basic facts in Real Algebra, and study valuation fans, a refined notion of fans. We explain how valuation fans are involved in the theory of residually real-closed henselian fields. Then we deal with other mathematical notions, such as $\mathbb{R}-$*places* and the real holomorphy ring which are also linked to valuation fans. Finally we consider abstract spaces of orderings and Marshall's problem, recall some results, and present the complete real spectrum of a ring.

## 1. Background in Real Algebra.

### 1.1. Preorderings, orderings.

**Definition 1.1.** A preordering $T$ of $K$ is a subset $T \subseteq K$, satisfying:

$$\begin{array}{rcl} T+T & \subseteq & T,\ T\cdot T \subseteq T,\ 0,1 \in T,\ \ -1 \notin T \\ \text{and } T^* & = & T\backslash\{0\} \text{ is a subgroup of } K^* = K\backslash\{0\}. \end{array}$$

**Definition 1.2.** A preordering $T$ is called a quadratic preordering if $K^2 \subseteq T$. If $K^{2n} \subseteq T$, $T$ is said to be of level n. Preorderings with no level do exist.

Zorn's lemma shows the existence of maximal quadratic preorderings; these are just the usual orderings, and are characterized by:

**Definition 1.3.** A subset $P$ of $K$ is an ordering if:

$$P+P \subseteq P,\ P\cdot P \subseteq P,\ P \cup -P = K,\ \ -1 \notin P.$$

From these properties one can deduce that $0,1 \in P$, $P \cap -P = \{0\}$ and $\sum K^2 \subseteq P$. Here, and throughout the paper, $\sum K^{2n}$ denotes the set of all finite sums of $2n$-*th* powers.

We can also call $P$ a positive cone: to any such ordering $P$ one can associate a binary relation $\leq_P$. This is a total order relation compatible with the field structure, defined as follows:

$$b-a \in P \Leftrightarrow a \leq_P b.$$

Then $P$ is the set of elements positive for the order relation $\leq_P$ .

The set of orderings of a field $K$ will be denoted by $\chi(K)$; it is also denoted in the literature by $SperK$ (so as to coincide with the usual notation for rings).

The field $\mathbb{R}$ admits only one ordering, and its set of positive elements is $\mathbb{R}^2$.

The field $\mathbb{Q}(\sqrt[2]{2}) := \left\{a + b\sqrt[2]{2} \mid a, b \in \mathbb{Q}\right\}$ admits two orderings, one making $\sqrt[2]{2}$ positive and the other making $\sqrt[2]{2}$ negative.

$\mathbb{R}((X))$, the power series field, admits also two orderings making X infinitesimal positive or negative.

$\mathbb{R}(X)$ admits infinitely many orderings. For any $a \in \mathbb{R}$ one can define orderings $P_{a,+}$ and $P_{a,-}$ making $X - a$ infinitesimal positive or negative respectively. $\mathbb{R}(X)$ admits also the orderings $P_+$ and $P_-$ making $\frac{1}{X}$ infinitesimal positive or negative respectively.

### 1.2. Real Valuations.

**Definition 1.4.** A Krull valuation $v$ on a field $K$ is a surjective map

$$v : K^* \twoheadrightarrow \Gamma$$

where $\Gamma$ is a totally ordered abelian group (called the value group), such that
(1) $v(xy) = v(x) + v(y)$ for any $x, y$ in $K^*$;
(2) $v(x + y) \geq \min\{v(x), v(y)\}$, for any $x, y$ in $K^*$, with $x + y$ in $K^*$.

The *valuation ring* of $v$ is

$$A_v := \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

and its *maximal ideal* is

$$I_v := \{x \in K \mid x = 0 \text{ or } v(x) > 0\}.$$

$k_v := A_v/I_v$ is called the *residue field* of the valuation.
$U_v := A_v \setminus I_v$ denotes the *group of units.*

**Definition 1.5.** A valuation $v$ on a field $K$ is said to be real if and only if the residue field $k_v$ is real (meaning $-1 \notin \sum k_v^2$).

A field admits real valuations if and only if it is real. Of course a real field admits real valuations, at least the trivial one.

The converse implication follows from the Baer-Krull theorem which ensures that if $k_v$ admits an ordering, then $K$ admits also at least one ordering.

Given an ordering $P$ in a field $K$, the convex hull of $\mathbb{Q}$ in $K$ is:

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \quad r \pm x \in P\}.$$

$A(P)$ is a valuation ring in $K$ with unique maximal ideal:

$$I(P) := \left\{x \in K \mid \forall r \in \mathbb{Q}^{+*} \quad r \pm x \in P\right\}.$$

where $\mathbb{Q}^{+*} = \{r \in \mathbb{Q} \mid r > 0\}$.

$A(P)$ is clearly a subring of $K$; it is a valuation ring because $b \notin A(P)$ implies $b^{-1} \in A(P)$: let $b \notin A(P)$, assume $b > 0$, since $b \notin A(P)$ we have in particular $1 < b$, therefore $0 < b^{-1} < 1$ which implies that $b^{-1} \in A(P)$ because $A(P)$ is convex in $K$ with respect to $P$.

We will see below that the valuation associated to $A(P)$ is compatible with the ordering $P$ and pushes down on the residue field an (archimedean) ordering, hence this valuation is real.

### 1.3. Compatibility of an ordering with a valuation.

**Definition 1.6.** An ordering $P$ is said to be compatible with a valuation $v$ if and only if $1 + I_v \subset P$.

Then $\overline{P}$, induced by $P$ on the residue field $k_v$, is an ordering of $k_v$. Clearly $\overline{P}$ is closed under addition and multiplication and $\overline{P} \cup -\overline{P} = k_v$. If $-1$ was in $\overline{P}$ we would have $-1 = \overline{a}$ for some $a \in P \cap A(P)$. Then $1 + a \in I(P)$, hence $-a \in 1 + I(P) \subset P$, so we would get $a = 0$ which is impossible.

The trivial valuation, sending every non-zero element of $K$ to 0, is compatible with any ordering of $K$.

Note that the valuation $v$ associated to an ordering $P$ of $K$ with valuation ring

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \;\; r \pm x \in P\}$$

is compatible with $P$. In fact $I(P) := \{x \in K \mid \forall r \in \mathbb{Q}^{+*} \;\; r \pm x \in P\}$ being the maximal ideal of $A(P)$ we have $1 + I(P) \subset P$. Hence the valuation is compatible with $P$. Then $\overline{P}$ induced by $P$ on the residue field $k_v$ is an archimedean ordering; we already know that $\overline{P}$ is an ordering, this ordering $\overline{P}$ is archimedean: for any $x \in A(P)$ there exists some $r \in \mathbb{Z}$ such that $-r <_P x <_P r$, hence in the residue field we have $-r <_{\overline{P}} \overline{x} <_{\overline{P}} r$, and therefore $\overline{P}$ is an archimedean ordering of $k_v$.

**Theorem 1.7.** *Let $P$ be an ordering of $K$, and $v$ be a valuation on $K$; the following are equivalent:*

*(1) $0 <_P a \leq_P b \Rightarrow v(a) \geq v(b)$ in $\Gamma$ (the value group of $v$).*

*(2) The valuation ring $A_v$ is convex in $K$ with respect to $P$.*

*(3) The maximal ideal $I_v$ of $A_v$ is convex in $K$ with respect to $P$ .*

*(4) $v$ is compatible with $P$ (i.e. $1 + I_v \subset P$).*

**Proof.** (1) $\Rightarrow$ (2) $A_v$ convex in $K$ means that if $x <_P y <_P z$, with $x, z \in A_v$ then $y \in A_v$, or equivalently $0 <_P a <_P b$ with $b \in A_v$ implies $a \in A_v$.

From (1) we deduce that $v(a) \geq v(b) \geq 0$ in $\Gamma$ hence $a \in A_v$.

(2) $\Rightarrow$ (3) Assume $0 <_P a <_P b$ with $b \in I_v$ then $0 <_P b^{-1} <_P a^{-1}$. Since $b^{-1} \notin A_v$ using (2) we deduce $a^{-1} \notin A_v$, hence $a \in I_v$, $I_v$ being the ideal of non invertible elements of $A_v$.

(3) $\Rightarrow$ (4) Let $m \in I_v$, if $1 + m \notin P$ then $1 + m \in -P$, so $1 + m <_P 0$ hence $0 <_P 1 <_P -m$. Using the convexity of $I_v$ in $K$ for $P$, since $-m \in I_v$ too, this yields $1 \in I_v$ which is impossible.

(4) $\Rightarrow$ (1) Assume $0 <_P a \leq_P b$ and $v(a) < v(b)$ in $\Gamma$ ; then we deduce $0 < v(b) - v(a) = v(\frac{b}{a})$, hence $\frac{b}{a} \in I_v$, and also $-\frac{b}{a} \in I_v$ and $a \neq b$. From (4) we get $1 + (-\frac{b}{a}) \in P$, so $\frac{a-b}{a} >_P 0$, hence $a >_P b$ which is impossible.

**Theorem 1.8.** *Let $\mathcal{F}$ be the family of all valuation rings of $K$ compatible with a given ordering $P$, then:*
*(1) the valuation rings in $\mathcal{F}$ form a chain under inclusion;*
*(2) the smallest element of $\mathcal{F}$ is $A(P)$.*

**Proof.** (1) Suppose $A, B \in \mathcal{F}$ and $A \subsetneqq B$, let $a \in A \backslash B$ and $a > 0$. We prove that $B \subset A$. Consider $0 < b \in B$, by the convexity of $B$ in $K$ we cannot have $0 < a \leq b$, so we must have $0 < b \leq a$. From the convexity of $A$ in $K$, we deduce $b \in A$.

(2) Let $A \in \mathcal{F}$, $A$ is convex in $K$ and contains $\mathbb{Z}$, hence $A$ contains $A(P)$ the convex hull of $\mathbb{Q}$ in $K$.

Note that any subring of $K$ containing a valuation ring must itself be a valuation ring, hence $\mathcal{F}$ consists of all subrings of $K$ containing $A(P)$. Remark also that $A \subset A'$ implies $I' \subset I$.

**Theorem 1.9.** *(Baer-Krull) Let $A$ be a real valuation ring of $K$, and let $v$ be the associated valuation. Let $\overline{P}$ be an ordering in the residue field $k_v$. Denote $\chi_{v,\overline{P}}$ the set of all orderings $P_i$ in $K$ inducing the given $\overline{P}$ in $k_v$. Then there is a bijection between $\chi_{v,\overline{P}}$ and* $\mathrm{Hom}(\Gamma, \mathbb{Z}/2)$ *where $\Gamma$ denotes the value group of $v$.*

For the proof we refer the reader to [5].

## 2. Fans (level 1 case).

In this section we mainly follow the notations and proofs of [48].

**2.1. Quadratic preorderings.** The compatibility of a quadratic preordering with a valuation can be of two types. Given $T$ a quadratic preordering in a real field $K$, $v$ a valuation on $K$ is *compatible* with $T$ if it is compatible with *some* ordering $P$ containing $T$.

$v$ is called *fully compatible* with $T$ if it is compatible with *every* ordering $P$ containing $T$. In this case $T$ induces on the residue field $k_v$ a quadratic preordering $\overline{T}$. This pushdown preordering $\overline{T}$ is defined to be the image of $T \cap A_v$ under the natural map from the valuation ring $A_v$ to the residue field $k_v$.

Below we give alternative characterizations.

**Definition 2.1.** Given $T$ a quadratic preordering in a real field $K$, and $v$ a valuation on $K$ with unique maximal ideal $I_v$ in the associated valuation ring $A_v$:

(1) $v$ is fully compatible with $T$ if and only if $1 + I_v \subset T$.

(2) $v$ is compatible with $T$ if and only if $(1 + I_v) \cap -T = \emptyset$.

(3) $v$ is compatible with $T$ if and only if $\overline{T}$ is a preordering in the residue field $k_v$.

We set $\chi_{/T} := \{P \textit{ ordering} \mid P \supset T\}$.

A way of building fully compatible preorderings is to use the "wedge product" introduced in 1978 by Becker in [6], and by Becker and Brocker in [10].

**Definition 2.2.** Let $K$ be a real field, let $A$ be a valuation ring in $K$, and $\pi : A \longrightarrow k_v$ be the projection map. Let $T$ be a preordering of $K$ and let $S$ be a preordering of $k_v$ such that $S \supset \overline{T}$. The wedge product is defined by $T \wedge S := T \cdot \pi^{-1}(S \backslash \{0\})$.

We refer the reader to Lam's book ([48], p.21) to verify that $T \wedge S$ is a preordering in $K$, fully compatible with $v$, and such that residually $\overline{T \wedge S} = S$.

Again refering to [48] (3.3 p.22), remark that the wedge product $T \wedge S$ can also be defined for $S$ a preordering of $k_v$ and $T = T^* \cup \{0\}$ where $T^*$is a subgroup of $K^*$. Then $T \wedge S$ is a preordering in $K$, and if $K^2 \subseteq T$ then $T \wedge S$ is a quadratic preordering.

There is also an alternative definition for the wedge product:

$T \wedge S = \cap \left\{\text{orderings } P \ \mid \ P \supset T \text{ and } \overline{P} \in \chi_{/S}\right\}$

**2.2. Fans of level 1.** In the context of preorderings fans were first presented by Becker and Köpping in [15].

**Definition 2.3.** Let $K$ be a real field and let $T$ be a quadratic preordering in $K$. $T$ is a fan if and only if for any $S \supset T$, such that $-1 \notin S$ and such that $S^* = S \backslash \{0\}$ is a subgroup of $K^*$ satisfying $[K^* : S^*] = 2$, $S$ is an ordering in $K$.

Note that if $T$ is a fan any preordering containing $T$ is again a fan. There is an alternative useful characterization of a fan given in [48] (p.40), with proof of equivalence:

**Proposition 2.4.** *A preordering $T$ is a fan if and only if for any $a \in K^* \backslash -T$ we have $T + aT \subset T \cup aT$. Such an element $a$ is said to be $T$-rigid.*

First examples of fans are the trivial fans : these are orderings $P$ and intersection of two orderings $P_1 \cap P_2$.

Another example is the pullback $\widehat{S}$ of a trivial fan $S$ in $k_v$ . Namely $\widehat{S} = K^2 \wedge S = K^2 \cdot \pi^{-1}(S \backslash \{0\})$ is a fan in $K$. In fact Bröcker's trivialization theorem given later in 2.6 says that all fans arise in this way.

Fans are well behaved for compatibility with real valuations.

**Theorem 2.5.** *Let $K$ be a real field, $v$ a valuation on $K$, and $T$ a preordering in $K$ . Then the followings hold:*

*(a) If $v$ is compatible with $T$, $T$ is a fan implies that $\overline{T}$ is a fan in $k_v$;*

*(b) If $v$ is fully compatible with $T$, $T$ is a fan if and only if $\overline{T}$ is a fan in $k_v$.*

**Proof.**

**(a)** We use proposition 2.4 characterizing a fan. Let $b \in A \backslash I$ such that $\overline{b} \notin -\overline{T}$ we shall show that $\overline{b}$ is $\overline{T}$-*rigid*. $T$ being a fan let $t_1 + t_2 b \in T + bT \subset T \cup bT$ hence there exist $t_3$ or $t_4$ such that $t_1 + t_2 b = t_3$ or $t_1 + t_2 b = t_4 b$. Going down to $k_v$ we get $\overline{t_1} + \overline{t_2}\overline{b} = \overline{t_3}$ or $\overline{t_1} + \overline{t_2}\overline{b} = \overline{t_4}\overline{b}$ hence $\overline{t_1} + \overline{t_2}\overline{b} \in \overline{T} \cup \overline{b}\overline{T}$, and $\overline{T}$ is a fan.

**(b)** We use the definition of a fan. Assume $v$ is fully compatible with $T$ and $\overline{T}$ is a fan we have to prove that $T$ is a fan. Let $W \supset T$ be such that $-1 \notin W$, $W^* = W \backslash \{0\}$ is a subgroup of $K^*$ and $[K^* : W^*] = 2$, we have to prove that $W$ is an ordering. We first show that $\overline{W}$ is an ordering. If $-1 = \overline{w}$ for some $w \in W \cap A$, then $-1 = w + m$ for some $m \in I$, so $-w = 1 + m \in 1 + I \subset T \subset W$ hence $-1 \in W$ which is impossible. Since $\overline{T}$ is a fan and $\overline{W^*}$ a subgroup of $k_v^*$ such that $\left[k_v^* : \overline{W^*}\right] = 2$, $\overline{W}$ is an ordering. Form the wedge product $W \wedge \overline{W} = W \cdot \pi^{-1}(\overline{W} \backslash \{0\}) = W \cdot (1 + I) \subset W \cdot T \subset W$, since from [48] (p.22) $W \cdot \pi^{-1}(\overline{W} \backslash \{0\}) = W \cdot (1 + I)$ ); then $W \wedge \overline{W} \subset W$ holds, hence $W = W \wedge \overline{W}$ is an ordering.

**2.3. Trivialization of fans.** A remarkable result is Bröcker's theorem on trivialization of fans ([20]).

**Theorem 2.6.** *Let $K$ be a real field and $T \subset K$ be a fan. Then there exists a valuation $v$, fully compatible with $T$, such that the pushdown $\overline{T}$ in the residue field $k_v$ is a trivial fan.*

The theorem follows from propositions 2.7 and 2.8 below. We use the proof given by Lam ([48], p. 94).

**Proposition 2.7.** *Let $T$ be a non-trivial fan in the field $K$. Then there exists a non-trivial valuation $v$ on $K$, fully compatible with $T$.*

The proof of proposition 2.7 requires three lemmas.

**Lemma 1.** *Let $G$ be an ordered group (written additively), and $H$ be a subgroup of $G$. If $H$ does not contain a non-trivial convex subgroup of $G$, then for any positive element $h \in H$ there exists $g \in G \backslash H$ such that $0 < g < h$.*

**Proof of lemma 1.** Let $C := \{g \in G \mid \exists n \in \mathbb{N} \quad -nh \leq g \leq nh\}$. $C$ is the convex hull of the subgroup of $G$ generated by $h$, hence a convex subgroup. Assume there does not exist an element $g$ as in the statement, then for any $g \in G$, $0 \leq g \leq h$ implies $g \in H$. By easy induction on $n$ it follows that for any $n \in \mathbb{N}$, $-nh \leq g \leq nh$ implies $g \in H$. Hence $\{0\} \neq C \subseteq H$, contradicting the assumption that $H$ does not contain a non-trivial convex subgroup of $G$.

**Lemma 2.** *Let $T$ be a fan in the field $K$. Let $v_1$ be a valuation on $K$ with value group $\Gamma_1$; if $v_1(T^*)$ does not contain a non-trivial convex subgroup of $\Gamma_1$, then $v_1$ is fully compatible with $T$.*

**Proof of lemma 2.** We claim that the condition: "for every $m$ in the unique maximal ideal $\mathcal{M}_1$, and for every $t \in U_1 \cap T$, a unit belonging to $T$, $t + m \in T$ implies that $1 + \mathcal{M}_1 \subset T$" entails that $v_1$ is fully compatible with $T$.

We distinguish two cases:

*Case 1.* Assume $v_1(m) \notin v_1(T^*)$.

In this case $(T \cdot m) \cap U_1 = \emptyset$; so in particular $m \notin -T$, since $v_1(m) > 0$. Since $T$ is a fan, $t + m \in T + T \cdot m = T \cup T \cdot m$. We have to show that $t + m \in T$.

Clearly $t+m \in U_1$ because $v_1(t+m)=0$ since $v_1(t)=0$ and $v_1(m)>0$. Since $(T\cdot m)\cap U_1=\emptyset$ we get $t+m\notin T\cdot m$ hence $t+m\in T$.

*Case 2.* Assume $v_1(m)\in v_1(T^*)$.

Apply lemma 1 to $H:=v_1(T^*)$. Since $v_1(m)$ is a positive element of $H$ there exists $x$ such that $v_1(x)\notin H$ and $0<v_1(x)<v_1(m)$. Now let $t+m=t'+m'$ where $t':=t+x$ and $m'=m-x$. From $x\in\mathcal{M}_1$ we get $t'\in U_1$, and since $v_1(m')\notin v_1(T^*)$, case 1 gives $t'\in T$. Finally from $v_1(x)<v_1(m)$ we get $v_1(m')=v_1(m-x)=\min\{v_1(m),v_1(x)\}=v_1(x)\notin v_1(T^*)$. Thus using again case 1, we get $t'+m'\in T$, and hence $t+m\in T$.

**Lemma 3.** *Let $T\subset K$ be a non trivial fan and $P\in\chi_{/T}$. Let $v_P:K^*\longrightarrow\Gamma$ be the canonical valuation associated with $P$; then $v_P(T^*)\neq\Gamma$. In particular $v_P$ is not the trivial valuation so every ordering in $\chi_{/T}$ is non archimedean.*

For the proof of this last lemma we refer to Lam [48], corollary 12-11 of lemma 12-10 p. 95.

**Proof of proposition 2.7.** Given a non trivial fan $T\subset K$, fix $v_0:K^*\twoheadrightarrow\Gamma_0$ such that $v_0(T^*)\neq\Gamma_0$ (for instance, take $P\in\chi_{/T}$ and let $v_0$ be the valuation $v_P$ associated with $A(P)$). Now consider the convex subgroups of $\Gamma_0$ contained in $v_0(T^*)$; they form a chain under inclusion. The union of them $\Delta$ is the largest convex subgroup contained in $v_0(T^*)$. By quotienting we can coarsen the valuation $v_0$ into a valuation $v_1:K^*\twoheadrightarrow\Gamma_1:=\Gamma_0/\Delta$. Then $v_1(T^*)$ cannot contain a non-trivial convex subgroup of $\Gamma_1$. Hence, by lemma 2, $v_1$ is fully compatible with $T$. Since $[\Gamma_1:v_1(T^*)]=[\Gamma_0:v_0(T^*)]>1$, $v_1$ is a non trivial valuation.

**Proposition 2.8.** *For any preordering $T$ in a field $K$, the followings are equivalent:*

*(1) $T$ is a fan in $K$.*

*(2) There exists a valuation $v_1$ on $K$, fully compatible with $T$, such that, with respect to $v_1$, $T$ pushes down to a trivial fan, hence $\left[\overline{K}^*:\overline{T}^*\right]\leq 4$.*

**Proof of proposition 2.8.**

(2)$\Rightarrow$(1) Trivially if $v_1$ exists, is fully compatible with $T$, and pushes down to a trivial fan $\overline{T}$, then $T$ is a fan.

(1)$\Rightarrow$ (2) From the previous proposition we know that there exists a valuation $v$ fully compatible with $T$, hence $\overline{T}$ is a fan in the residue field $k_v$.

If $\left[k_v^*:\overline{T}^*\right]\geq 8$, then $\overline{T}$ would be a non-trivial fan, and applying lemma 3 to $\overline{T}$ in $k_v$ we would get a non-trivial valuation on $k_v$ fully compatible with $\overline{T}$. But from proposition 12-3 in [48], $k_v$ has no non-trivial valuation fully compatible with $\overline{T}$. Then just take $v_1=v$.

For the geometric point of view on fans we refer to [2] and [1].

## 3. Valuation fans and examples.

From now on preorderings are no more supposed to be quadratic.

Let us recall the definition of a general preordering. A preordering $T$ in a field $K$ is a subset $T \subseteq K$, satifying:

$T + T \subseteq T$, $T \cdot T \subseteq T$, $0, 1 \in T$, $-1 \notin T$, $T^* = T \backslash \{0\}$ is a subgroup of $K^*$.

### 3.1. Valuation fans (of any level).

**Definition 3.1.** (Jacob, [41]). Let $K$ be a field; a valuation fan in $K$ is a preordering $T$ such that there exists $v$ a real valuation on $K$, $v$ fully compatible with $T$ (meaning $1 + I_v \subset T$), and $T$ induces an archimedean ordering on the residue field $k_v$.

More precisely, a preordering $T$ in $K$ is a valuation fan if and only if $A(T) = \{x \in K \mid \exists r \in \mathbb{Q}\ r \pm x \in T\}$ is a valuation ring with associated valuation $v$ fully compatible with $T$, and $\overline{T}$ in $k_v$ is an (archimedean) ordering.

There is an alternative characterization for valuation fans given in [42], which is sometimes useful in model theory:

**Proposition 3.2.** *A preordering $T$ in a field $K$ is a valuation fan if and only if for any $x \notin \pm T$ we have either $1 \pm x \in T$ or $1 \pm x^{-1} \in T$.*

Usual orderings $P$ are valuation fans (of level 1, i.e. $\sum K^2 \subset P$).

It is I think important for real algebraic geometry to understand minimal valuation fans of level 1. They are defined as valuation fans not properly containing any valuation fan which is a quadratic preordering. Of course such a minimal valuation fan $T_0$ pushes down an archimedean ordering in the residue field of $K$ for the valuation associated to the valuation ring given by: $A(T_0) = \{x \in K \mid \exists r \in \mathbb{Q}\ r \pm x \in T_0\}$.

**3.2. Orderings of higher level.** Further examples of valuation fans are provided by Becker's orderings of higher level.

**Definition 3.3.** (Becker, [6]). Let $K$ be a commutative real field, $P \subset K$ is an ordering of level $n$ if: $\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P$, $-1 \notin P$, $P^*$ is a subgroup of $K^*$ and $K^*/P^*$is cyclic.

When $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$, then the ordering is said to be of exact level $n$.

A very interesting paper on sums of d-th powers in rings with some relation to orderings of higher level is [43].

The orderings of level 1 are the usual total orderings.

If $K = \mathbb{R}((X))$, there exist two usual orderings:

$$P_+ := K^2 \cup XK^2, P_- := K^2 \cup -XK^2$$

And for every integer $n \geq 1$ there exist two orderings of exact level $n$:

$$P_{n,+} := K^{2n} \cup X^n K^{2n}, P_{n,-} := K^{2n} \cup -X^n K^{2n}.$$

These higher level orderings have important links with sums of powers; we refer the reader to [9] and just mention the following important theorems from [6] :

**Theorem 3.4.** *(Becker, [6]) Let $K$ be a real field, then:*
$\sum K^{2n} = \cap \{P_i \mid P_i$ *ordering of level dividing* $n\}$ .

**Theorem 3.5.** *(Becker, [6]). Let $K$ be a real field, and let $p$ be a prime. The followings are equivalent:*
*(1)* $\sum K^2 \neq \sum K^{2p}$.
*(2) $K$ admits an ordering of exact level $p$.*

**3.3. Another approach with signatures.** Usual orderings can also be studied in terms of signatures. A signature is a group morphism, $\sigma : K^* \longrightarrow \{\pm 1\}$, with additivily closed kernel; then $P = ker\sigma \cup \{0\}$ is an ordering of $K$.

This notion of a signature has a higher level analog:

**Definition 3.6.** (Becker, [8]). A signature of level $n$ on a field $K$ is a morphism of abelian groups:

$$\sigma : K^* \to \mu_{2n}$$

such that the kernel is additively closed, where $\mu_{2n}$ denotes the group of $2n$-th roots of 1.

Clearly if $\sigma$ is a signature of level $n$, then $P = ker\sigma \cup \{0\}$ is an ordering of higher level with exact level dividing $n$.

But there exists also a much more general notion of signature involving valuation fans:

**Definition 3.7.** ( Schwartz, [55]). A generalized signature in a field $K$ is a morphism of abelian groups, $\sigma : K^* \to G$, such that the kernel is a valuation fan.

## 4. Algebraic closure of a field equipped with a valuation fan.

Several notions of a closure, under algebraic extensions, of a field equipped with either higher level orderings or higher level signatures, either valuation fans or generalized signatures, have been introduced and studied in the literature.

All these notions of closure can be unified in one theory, the theory of Henselian Residually Real-Closed fields (HRRC fields).

In this section we present, without any proof, the main features of this theory, from an algebraic point of view.

**Definition 4.1.** (Becker, Berr, G. , [11]). A field $K$ is henselian residually real-closed (HRRC) if and only if it admits an henselian valuation $v$ with real-closed residue field $k_v$.

Recall that a valuation $v$ on a field $K$, with valuation ring $A_v$, is *henselian* if it satisfies Hensel's lemma : *"For any monic polynomial $f \in A_v[X]$, if $\overline{f}$ has a simple root $\beta \in k_v$, then $f$ has a root $b \in A_v$ such that $\overline{b} = \beta$ "*. The henselian residually real-closed fields have been variously named in the literature: they are called real henselian fields in Brown [16], [17], fields real-closed with respect to a signature in Schwartz [55] and almost real-closed fields in Delon-Farre [27].

**4.1. Examples of HRRC fields.** The basic examples of henselian residually real-closed fields arise in a classical way as follows (see [33]): given $R$ a real-closed field, and $\Gamma$ a totally ordered abelian group, let $R((\Gamma)) = \{\sum_\gamma a_\gamma t^\gamma \mid \gamma \in \Gamma, a_\gamma \in R\}$ be the set of generalized power series with support well ordered, where $support \sum_\gamma a_\gamma t^\gamma = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$. In $K = R((\Gamma))$ one can define:

- <u>Product</u> by: $t^\gamma t^\delta = t^{\gamma+\delta}$;
- <u>Addition</u> by: $\sum_\gamma a_\gamma t^\gamma + \sum_\delta b_\delta t^\delta = \sum_\alpha (a_\alpha + b_\alpha) t^\alpha$;
- <u>Order</u> by: $\sum_\gamma a_\gamma t^\gamma >_K 0 \Leftrightarrow a_m >_R 0$, where $m = \min(support \sum_\gamma a_\gamma t^\gamma)$;
- <u>Valuation</u> by: $v : R((\Gamma)) \to \Gamma$ and $v(\sum_\gamma a_\gamma t^\gamma) = m = \min(support \sum_\gamma a_\gamma t^\gamma)$.

It is well-known that $R((\Gamma))$ is a field, admitting $v$ as a henselian valuation with real-closed residue field $R$ and value group $\Gamma$; hence $R((\Gamma))$ is an HRRC field.

**4.2. Subtheories of the theory of HRRC fields.** Let $v$ be a real valuation on a field $K$, $k_v$ its residue field, $\Gamma_v$ its value group, and let $S$ be a set of primes. Relations between various subtheories of the theory of HRRC fields are described by the following diagram where arrows indicate subtheories.

*<u>Henselian Residually Real-Closed Fields (HRRC)</u>*
$v$ henselian valuation , $k_v$ real-closed field
closed for **generalized signature**, or for **valuation fan**

$\downarrow$

*<u>HRRC fields of type $S$</u>* ($p \notin S \Rightarrow \Gamma_v$ $p$-divisible)

$\swarrow \quad \searrow$

*<u>$S-$generalized real-closed fields ($S$ finite)</u>*
if $p \notin S$ then $\Gamma_v$ is $p$-divisible
if $p \in S$ and $\Gamma_v$ is not $p$-divisible then
$\Gamma_v/p\Gamma_v \simeq \mathbb{Z}/p\mathbb{Z}$ ; closed for
**higher level ordering** or **chain signature**

*<u>Rolle fields</u>*
$\Gamma_v$ odd divisible
HRRC field of type $\{2\}$

$\downarrow \quad \searrow$

*<u>Real-closed fields</u>*
$\emptyset$-generalized real-closed
$\Gamma_v$ divisible
closed for a **usual order**

$\swarrow \quad \downarrow$

*<u>Chain-closed fields</u>*
$\{2\}$-generalized real-closed
$\Gamma_v/2\Gamma_v \simeq \mathbb{Z}/2\mathbb{Z}$
closed for an **ordering of level** $2^k$

In the diagram above, most of the theories correspond to some notion of closure, under algebraic extensions, of a field equipped with some object. With an *ordering* (real-closed field), with an *ordering of exact level a power of* 2 (chain-closed field), with an *ordering of exact level a power of p where p is prime* ($\{p\}$-real-closed fields), with an *ordering of exact level n* ($S$-generalized real-closed fields of exact type $S$ ($p \in S \Leftrightarrow \Gamma_v$ not $p$-divisible, and for all $p \in S$, $p \mid n$ ), or with a *valuation fan* (henselian residually real-closed field).

**4.3. On the question of the uniqueness of closure.** For a field equipped with a usual ordering it is well known that the real closure is unique up to $K$-isomorphism.

Even for chain-closed fields this is not true anymore. In order to recover the uniqueness of the closure, up to $K$-isomorphism, one needs to consider a closure for a whole chain of orderings with levels powers of 2 in the sense of Harman:

**Definition 4.2.** (Harman, [40]). A 2-primary chain of orderings in a field $K$ is:

$$(P_n)_{n\in\mathbb{N}} = (P_0, P_1, ..., P_n, ...)$$

$P_0$ being a usual ordering and $P_n$ an ordering of level $2^{n-1}$, such that

$$P_n \cup -P_n = (P_0 \cap P_{n-1}) \cup -(P_0 \cap P_{n-1}).$$

**Theorem 4.3.** *A field $K$ equipped with a 2-primary chain of orderings admits a closure under algebraic extensions unique up to $K$ -isomorphism. The closure is called a chain-closed field and it is equal to the intersection of two real-closures of $K$ for $P_0$ and $P_1$.*

For generalized real-closed fields, in order to recover the uniqueness up to $K$-isomorphism, Niels Schwartz has introduced the notion of chain signature.

**Definition 4.4.** (Schwartz, [54]). A chain signature on a field $K$ is a homomorphism:

$$\varphi : K^* \to \{1, -1\} \times \widehat{\mathbb{Z}}$$

such that $\ker\varphi$ is a valuation fan, where $\widehat{\mathbb{Z}} = \prod \widehat{\mathbb{Z}}_p$ and $\widehat{\mathbb{Z}}_p$ denotes the additive group of p-adic integers.

One can recover orderings of higher level by taking:
$P_n(\varphi) = \varphi^{-1}(1 \times n\widehat{\mathbb{Z}}) \cup \{0\}$ .

**Theorem 4.5.** *A field $K$ equipped with a chain signature $\varphi$ admits a closure under algebraic extensions unique up to $K$ -isomorphism. This closure is a HHRC field.*

In the more general situation of a field equipped with a valuation fan we can also ensure the uniqueness of the closure by considering a field equipped, not only with a single valuation fan, but with a whole chain of valuation fans.

From Brown's work we can derive the following:

**Theorem 4.6.** *Let $R$ and $R'$ be two HRRC fields, algebraic extensions of a field $K$, then the followings are equivalent:*
*(1) $R$ and $R'$ are $K$-isomophic.*
*(2) $R^{2n} \cap K = R'^{2n} \cap K$ for all $n \in \mathbb{N}$ .*

In fact these $T_n = R^{2n} \cap K$ are valuation fans, which form a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$ as defined below; this chain is said to be induced on $K$ by $R$.

**Definition 4.7.** (Becker, Berr, G. [11]). A chain of valuation fans in a field $K$ is defined as $(T_n)_{n\in\mathbb{N}}$ such that:
(1) $K^{2n} \subset T_n$;
(2) $T_{n.m} \subset T_n$;
(3) $(T_n)^m \subset T_{n.m}$;
(4) $T_n^*/T_{n.m}^* \subset T_1^*/T_{n.m}^*$ is the subgroup of elements of exponent $m$.

With this notion we have been able in [11] to obtain the following theorem:

**Theorem 4.8.** *Any field $K$, equipped with a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$, admits a closure under algebraic extensions $R$, unique up to $K$-isomorphism. Then $R$ is a HRRC field, and $R$ induces on $K$ a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$ (i.e. $T_n = R^{2n} \cap K$ for all $n$).*

**4.4. Properties of HRRC fields.** Henselian residually real-closed fields have a lot of nice properties ; we list, again without any proof, some of them below. Main reference is [11].

Let $K$ be an HRRC field then:

(1) $K$ is a real field;

(2) Every algebraic extension of $K$ is a radical extension;

(3) $K$ a HRRC field of type $S$ has no real extension of degree $p \in \mathbb{P}\backslash S$.
Note that whenever $2 \in S$, one can replace (3) by (3') "$K$ has no extension of degree $p \in \mathbb{P}\backslash S$ ";

(4) $\forall n \in \mathbb{N}$ , $K$ is *n-pythagorean* : $K^{2n} + K^{2n} = K^{2n}$;

(5) $K$ is *hereditarily pythagorean*, i.e., every algebraic extension is again a pythagorean field;

(6) $\forall n \in \mathbb{N}$ , $K^{2n}$ is a *fan* (refer to definion 2.3, or to characteriztion 2.4 for such preorderings);

(7) $\forall n \in \mathbb{N}$ , $K^{2n}$ is a *valuation fan*, i.e. it is a preordering such that:
$\forall x \notin \pm K^{2n}$ either $1 \pm x \in K^{2n}$ or $1 \pm x^{-1} \in K^{2n}$;

(8) All real valuations on $K$ are henselian;

(9) The set of real valuation rings in $K$ is totally ordered by inclusion;

(10) The *smallest* real valuation ring in $K$ is:

$$A(K^2) = A(K^{2n}) = H(K)$$

where $A(T) = \{x \in K \mid \exists n \in \mathbb{N} \;\; n \pm x \in T\}$, $T$ being a valuation fan, and where $H(K)$ is the real holomorphy ring (i.e. the intersection of all real valuation rings);

(11) $K$ admits a unique $\mathbb{R}$-place which can be defined using the valuation ring $A(K^2)$ and the associated valuation;

(12) Jacob's ring $J(\bigcap_{n\in\mathbb{N}} K^{2n})$ is the *biggest* valuation ring with real-closed residue field. This ring is defined as follows. If $T$ is a valuation fan, the ring $J(T)$ is equal to $J_1(T) \cup J_2(T)$ where:

$$\begin{array}{rcl} J_1(T) & = & \{x \in K \mid x \notin \pm T \text{ and } 1 + x \in T\} \\ & & \text{and} \\ J_2(T) & = & \{x \in K \mid x \in \pm T \text{ and } xJ_1(T) \subset J_1(T)\}). \end{array}$$

**4.5. On the model theory of HRRC fields.** These fields have been studied from a model theoretic point of view; the previous theories are all elementary theories, with nice first order axiomatizations (see [11], [24], [25], [28], [34] and [36]).

A Rolle field is an ordered field where Rolle theorem holds for polynomials. These fields have been introduced by Brown, Craven and Pelling [21]. Below is an axiomatization for the theory of Rolle fields; these axioms are first order in the language of fields, hence the theory is elementary.

**Theorem 4.9.** *(G. [34]) :*

*(1) axioms for commutative fields ;*

*(2) "K formally real " :*

*for each* $n \geqslant 1$

$$\forall x_1 ... \forall x_n \; \rceil(-1 = x_1^2 + ... + x_n^2)$$

*(3) "K does not have any algebraic extension of odd degree" :*

*for each* $p \geqslant 0$

$$\begin{array}{rcl} & & \forall x_0 ... \forall x_{2p+1} \exists y \\ (x_{2p+1} & = & 0 \vee x_0 + x_1 y + ... + x_{2p+1} y^{2p+1} = 0) \end{array}$$

*(4) "$K^2$ is a fan " :*

$$\forall x \forall y \forall z \exists t (x = -t^2 \vee y^2 + xz^2 = t^2 \vee y^2 + xz^2 = xt^2)$$

*(5)"K is pythagorean at level* 2 *" :*

$$\forall x \forall y \exists z (x^4 + y^4 = z^4)$$

Remark that the three first sets of axioms are the same as in the theory of real-closed fields; to get a real-closed field axiomatization, just replace (4) and (5) by

$$\forall x \exists y (x = y^2 \vee x = -y^2)$$

In [G1] it is also shown that :

**Theorem 4.10.** *For any Rolle field $K$ having a finite number of orders $2^n$, there exists $n+1$ orders $P_i$, such that $K$ is the intersection of $n+1$ real closures $R_i$ of $K$ ordered by $P_i$.*

The theory of HRRC fields is also elementary and the next theorem gives a first order axiomatization.

**Theorem 4.11.** *(Becker, Berr, G. [11]) : The class of HRRC fields admits the following axiomatization :*
*(1) $R$ is a real commutative field ;*
*(2) $R$ is a hereditarily pythagorean field ;*
*(3) for all $n \in \mathbb{N}$, $R^{2n}$ is a valuation fan.*

**Corollary 4.12.** *The class of HRRC fields is an elementary class.*

**Remark 4.13.** The class of HRRC fields of type S is also an elementary class, just add to the axiomatization in theorem 4.11 :
(4) for all $p \in \mathbb{P} \backslash S$ , $K^2 = K^{2p}$.

Corollary 4.11 follows from B. Jacob ([41]), who first proved that the class of hereditarily pythagorean fields is elementary.

An alternative proof from [11] for "the class of hereditarily-pythagorean fields is elementary" is given below. It uses the characterization by Becker ([6], thm. 4, p. 94) of hereditarily pythagorean fields :

$$\sum K(X)^2 = K(X)^2 + K(X)^2$$

which is equivalent to :

$$\sum K[X]^2 \subset K(X)^2 + K(X)^2$$

By Cassel's theorem this is also equivalent to :

$$\sum K[X]^2 = K[X]^2 + K[X]^2 \quad (*)$$

Remark that if $f, g, h \in K[X]$ satisfy $f^2 = g^2 + h^2$, the degrees of $g$ and $h$ are less or equal to the degree of $f$ because $K$ is formally real.

Hence $(*)$ is expressible by an infinite sequence of first order sentences in the language of fields.

The significance of Jacob's ring for the model theory of these fields appears in [42], and also later with the transfer theorem obtained by Delon and Farre [27] and given below.

We first recall that in the following theorem, $\equiv$ denotes elementary equivalence and $\prec$ elementary inclusion. The second symbol $\prec$ means that every closed first order formula with parameters in the smaller model holds in one model if and only if it holds in the other.

**Theorem 4.14.** *(Delon, Farré, [27]) : Let $K$ and $L$ be HRRC fields, then :*

*(i) $K \equiv L \Leftrightarrow \Gamma_{J(K)} \equiv \Gamma_{J(L)}$ ;*

*(ii) if $K \subset L$ then $K \prec L \Leftrightarrow \Gamma_{J(L)}$ extends $\Gamma_{J(K)}$, and $\Gamma_{J(K)} \prec \Gamma_{J(L)}$ , where the $\Gamma's$ are the value groups of the Jacob rings of $K$ and $L$.*

In [27] the authors established a bijection between theories of HHRC fields and certain theories of ordered abelian groups. This bijection preserves completeness and sometimes decidability. Finally they proved that the only model-complete theory among these is the theory of real-closed fields.

They also characterized definable real valuation rings in such fields and have shown that these valuation rings were in bijection with the definable convex subgroups of the value group of the Becker ring.

In case there is only one real (henselian) valuation ring with real-closed residue field, i.e. the Becker ring equals the Jacob ring, then the model theory works well, and we are able to get real algebraic results such as a Nullstellensatz or Hilbert's 17th problem at level n; we refer the reader to [12], [13], [26].

## 5. $\mathbb{R}$-places, and the real holomorphy ring.

**5.1. $\mathbb{R}$-place associated to an ordering.** For a complete presentation of these notions one can refer to [48], or in a more geometrical setting to [57], [58] and [59].

Let $K$ be a real field and $P$ be an ordering on $K$. Let $v$ denote the valuation associated to the valuation ring $A(P)$. From previous results we know that $(k_v, \overline{P})$ can be uniquely embedded in $(\mathbb{R}, \mathbb{R}^2)$ since $\overline{P}$ is archimedean. Denote this embedding by $i$ and let $\pi$ be the canonical mapping from $K$ into $k_v \cup \{\infty\}$ (where if $a \notin A(P)$, then $\pi(a) = \infty$).

**Definition 5.1.** The $\mathbb{R}$-place associated to $P$ is $\lambda_P : K \to \mathbb{R} \cup \{\infty\}$ defined by the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\lambda_P} & \mathbb{R} \cup \{\infty\} \\ \pi \searrow & & \nearrow i \\ & k_v \cup \{\infty\} & \end{array}$$

Explicitly $\lambda_P(a) = \infty$ when $a \notin A(P)$, and $\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\} = \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$ if $a \in A(P)$. In fact it is known that any $\mathbb{R}$-place arises in this way from some ordering $P$ (see [48], 9.1).

**5.2. The space of $\mathbb{R}$-places.** The space of $\mathbb{R}$-places of a field $K$ is the set $M(K) = \{\lambda_P \mid P \in \chi(K)\}$, where $\chi(K)$ denotes the space of orderings of $K$. $M(K)$ is equipped with the coarsest topology making continuous the evaluation mappings defined for every $a \in K$ by:

$$e_a \ : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda_P \mapsto \lambda_P(a)$$

Recall that the usual topology on $\chi(K)$ is the Harrison topology generated by the open-closed Harrison sets:

$$\mathcal{H}(a) = \{P \in \chi(K) \mid a \in P\}.$$

With this topology $\chi(K)$ is a compact totally disconnected space. Craven has shown in [23] that every compact totally disconnected space is homeomorphic to the space of orderings $\chi(K)$ of some field $K$.

Now consider the mapping $\Lambda$ defined by:

$$\Lambda : \chi(K) \longrightarrow M(K)$$

$$P \mapsto \lambda_P$$

With the previous topologies on $\chi(K)$ and $M(K)$ the mapping $\Lambda$ is a continuous, surjective and closed mapping.

$M(K)$ equipped with the above topology is a compact Hausdorff space. Remark that this topology on $M(K)$ is also the quotient topology inherited from the above topology on $\chi(K)$.

**5.3. The Real Holomorphy Ring.** We now provide some facts on the real holomorphy ring which has heavy links with orderings and $\mathbb{R}$-places.

**Definition 5.2.** The real holomorphy ring, denoted $H(K)$, is the intersection of all real valuation rings of $K$.

From the results in part 1 we obtain $H(K) = \bigcap_{P \in \chi(K)} A(P)$.

We also have:

$$H(K) = A(\sum K^2) = \{a \in K \mid \exists n \in \mathbb{N},\ n \geq 1,\ n \pm a \in \sum K^2\}.$$

$H(K)$ is a Prüfer ring with quotient field $K$ (see [48], p.85). Recall that a Prüfer ring is a ring $R \subset K$ such that, for any prime ideal $\mathfrak{p}$ in $R$, the localization $R_{\mathfrak{p}}$ is a valuation ring in $K$ .

In the sequel we denote the real spectrum of the real holomorphy ring of $K$ by:

$$Sper(H(K)) = \{\alpha = (\mathfrak{p}, \overline{\alpha}),\ \mathfrak{p} \in Spec(H(K)),\ \overline{\alpha}\ ordering\ of\ quot(H(K)/\mathfrak{p})\}.$$

Relations between $\chi(K)$, $M(K)$ and $H(K)$ are given in [14] by the next theorem.

**Theorem 5.3.** *(Becker, G. [14]) The following diagram is commutative:*

$$\begin{array}{ccc} \chi(K) & \overset{sper\ i}{\longrightarrow} & MinSperH(K) \\ \downarrow \Lambda & & \downarrow sp \\ M(K) \overset{res}{\to} \mathrm{Hom}(H(K),\mathbb{R}) & \overset{j}{\to} & MaxSperH(K) \end{array}$$

*where the horizontal mappings are homeomorphisms, and the vertical ones continuous surjective mappings (see definitions below).*

Hence $\chi(K)$ the space of orderings of $K$ is homeomorphic to $MinSperH(K)$, and the space $M(K)$ of $\mathbb{R}$-places on $K$ is homeomorphic to $MaxSperH(K)$.

The mappings in the above diagram are defined as follows:

$\Lambda : \chi(K) \longrightarrow M(K)$ is given by $P \mapsto \lambda_P$.

$sper\ i : \chi(K) \longrightarrow MinSperH(K)$ is given by $P \mapsto P \cap H(K)$.

$sp : MinSperH(K) \longrightarrow MaxSperH(K)$ is given by $\alpha \longmapsto \alpha^{\max}$ ,
where $\alpha^{\max}$ is the unique maximal specialization of $\alpha$.

$res : M(K) \longrightarrow \mathrm{Hom}(H(K),\mathbb{R})$ is given by $\lambda \mapsto \lambda_{|H(K)}$.

$j : \mathrm{Hom}(H(K),\mathbb{R}) \longrightarrow MaxSperH(K)$ is given by $\varphi \mapsto \alpha_\varphi$,
where $\alpha_\varphi = \varphi^{-1}(\ \mathbb{R}^2)$ or, using the notation for the real spectrum,
$\alpha_\varphi = (\ker\varphi, \overline{\alpha})$ with $\overline{\alpha} = \mathbb{R}^2 \cap quot(\varphi(H(K))$.

All the spaces in the diagram are compact and the topologies of $M(K)$ and $MaxSperH(K)$ are the quotient topologies inherited through $\Lambda$ and $sp$.

**Remark 5.4.** $\chi(K)$ may be seen as a space of maximal valuation fans of level 1, and $M(K)$ might be associated to the space of minimal valuation fans of level 1 using the preorderings $T_\lambda = \cap\left\{P_i \mid P_i \in \Lambda^{-1}(\lambda)\right\}$, where $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$.

## 6. On the Abstract Side.

The space of orderings of a field, studied in relation with quadratic forms and real valuations, have been the origin of the theory of abstract spaces of orderings (1979-80) and of Marshall's problem:

*"Is every abstract space of orderings the space of orderings of some field ?"*

**6.1. Abstract spaces of orderings (level 1 case).** Abstract space of orderings have been introduced using signatures by Marshall in [50]:

**Definition 6.1.** An abstract space of orderings is $(X, G)$, where $G$ is a group of exponent 2 (hence abelian), -1 a distinguished element of $G$, and $X$ a subset of $Hom(G, \{1, -1\})$ such that:

(1) $X$ is a closed subset of $Hom(G, \{1, -1\})$;

(2) $\forall \sigma \in X \quad \sigma(-1) = -1$;

(3) $\bigcap_{\sigma \in X} \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);

(4) For any $f$ and $g$ quadratic forms over $G$:

$$D_X(f \oplus g) = \cup\{D_X \langle x, y\rangle \mid x \in D_X(f),\ y \in D_X(g)\}.$$

In the above definition $D_X(f)$ denotes the set $\{a \in G$ represented by $f\}$, i.e. there exists $g$ such that $f \equiv_X \langle a\rangle \oplus g$ where $f \equiv_X h$ if and only if $f$ and $h$ have same dimension, and have for any $\sigma \in X$ same signature.

On the side of fans, seen as sets of signatures on a field, a four elements fan of level 1 is characterized by: $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ and it corresponds to the fan seen as a preordering: $T = \bigcap_{i=0}^{3} \ker \sigma_i \cup \{0\}$.

In the abstract situation, abstract fans have been defined by Marshall.

**Definition 6.2.** An abstract fan is an abstract space of orderings $(X, G)$ such that $X = \{\sigma \in Hom(G, \{1, -1\}) \mid \sigma(-1) = -1\}$.

It is also characterized by: if $\sigma_0, \sigma_1, \sigma_2 \in X$ then the product $\sigma_0\sigma_1\sigma_2 \in X$.

What was expected to correspond to the space of $\mathbb{R}$-places of the field case in the context of abstract spaces of orderings is called a $P$-structure and has been defined as follows by Marshall in [51].

**Definition 6.3.** A $P$-structure is an equivalence relation on a space of orderings $(X, G)$ such that the canonical mapping $\Lambda : X \to M$, where $M$ is the set of equivalence classes, satisfies:

(1) Each fiber is a fan;

(2) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ then $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ has a non empty intersection with at most two fibers.

Marshall has proved in [51] that every abstract space of orderings has a $P$ -structure, generally not unique. But unlike the case of the space of $\mathbb{R}$-places in a field, this $P$-structure $M$, equipped with the quotient topology, is not always Hausdorff. Hence we have to improve this notion to fit with the space of $\mathbb{R}$-places in the field case.

**6.2. Abstract spaces of signatures (higher level).** In the higher level case, one can also define abstract spaces of signatures (similar to 3.3 in the field case).

**Definition 6.4.** An abstract space of signatures of level $2^n$ is $(X, G)$, $G$ abelian group of exponent $2^n$, $X \subset \mathrm{Hom}(G, \mu_{2^n})$ such that:

(0) $\forall \sigma \in X$, $\forall k \in \mathbb{N}$ with $k$ odd, $\sigma^k \in X$;
(1) $X$ is a closed subset of $\mathrm{Hom}(G, \mu_{2^n})$ ;
(2) $\forall \sigma \in X$ $\sigma(-1) = -1$ ($-1$ distinguished element of $\mu_{2^n}$);
(3) $\bigcap_{\sigma \in X} \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);
(4) For any $f$ and $g$ forms over $G$

$$D_X(f \oplus g) = \cup\{D_X \langle x, y\rangle \mid x \in D_X(f),\ y \in D_X(g)\}.$$

In fields, the space of $\mathbb{R}$-places is known as soon as one knows the usual orderings and the orderings of level 2. Using this idea in the abstract situation we have been able to obtain in [38] a theorem which can be seen as the first case of a $P$-structure which looks like an abstract space of $\mathbb{R}$-places.

**Theorem 6.5.** *Let $(X, G)$ be a subspace of a space of signatures $(X', G')$ with 2-power exponent.*

*For $\sigma_0, \sigma_1 \in X$, define $\sigma_0 \sim \sigma_1$ if $\sigma_0\sigma_1 = \tau^2 \in X'^2$.*

*Then the followings are equivalent:*

*(1) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$, then either $\sigma_0$ is in relation by $\sim$ with exactly one of the $\sigma_1, \sigma_2, \sigma_3$, or $\sigma_0$ is in relation by $\sim$ with everyone of the $\sigma_1, \sigma_2, \sigma_3$.*

*(2) $\sim$ defines a $P$-structure on $X$.*

Moreover in this case the induced $P$-structure defined on $X$ by $\sim$ has a Hausdorff topology.

The key idea for proving the theorem is that, in the field case studied by Harman in [40], for any $P_2$, ordering of level 2, holds for some orderings $P_0, P_1$:

$a^2 \in P_2 \Longleftrightarrow a \in P_2 \cup -P_2 = (P_0 \cap P_1) \cup (-(P_0 \cap P_1))$.

Hence on the side of abstract signatures we get $\tau(a^2) = \tau(a)^2 = \sigma_0(a)\sigma_1(a)$.

## 7. Some open problems and the complete real spectrum

**7.1. The space of valuation fans.** Study in the field case the space of level 1 valuation fans $\mathcal{VF}(K)$, and its relation with $SperH(K)$. The motivation comes from the fact that $\chi(K)$, isomorphic to $MinSperH(K)$, consists of valuation fans $P_i$, and that to a $\mathbb{R}$-place $\lambda$ in $M(K)$, which is isomorphic to $MaxSperH(K)$, can be associated a valuation fan of level 1: $T_\lambda = \cap\left\{P_i \mid P_i \in \Lambda^{-1}(\lambda)\right\}$ where $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$.

Then we could define a notion of abstract space of valuation fans of level 1, or of minimal valuation fans of level 1. Then use abstract space of valuation fans of level 1 to solve Marshall's problem of realizability of abstract spaces of orderings, or state Marshall's problem in other terms.

**7.2. The space of $\mathbb{R}$-places.** Construct a finer theory for abstract spaces of orderings taking into account the $\mathbb{R}$-places. For example, $\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$ have isomorphic spaces of orderings, but the first one has two $\mathbb{R}$-places and no ordering of level 2, and the second one has only one $\mathbb{R}$-place but has a 2-primary chain of higher level orderings. A preliminary step is to characterize the topological spaces which are realizable as spaces of $\mathbb{R}$-places. Partial results in that direction have been recently obtained in [30], [45], [46] and [49].

It will be useful to study for a field $K$ the space of connected components of the space of $\mathbb{R}$-places of $K$, $\pi_0(M(K))$. This might be some kind of space of orderings. Another question in this area is: in which cases are the connected components of $M(K)$ homeomorphic?

**7.3. The complete real spectrum.** With Murray Marshall, in [39], we considered rings instead of fields.

Let $A$ be any commutative ring with 1. We define a big object $\mathrm{Sper}^c A$ which we call the *complete real spectrum* of $A$. There are various connections between this and the valuation spectra considered by R. Huber and M. Knebush in Contemporary Math. 155 (1994). Roughly speaking, the complete real spectrum is related to the valuation spectrum in the same way that the real spectrum is related to the prime spectrum. We define a topology on $\mathrm{Sper}^c A$ and prove that $\mathrm{Sper}^c A$, with this topology, is a spectral space.

In [51, Sect. 8.6] another sort of attempt is made to overcome shortcomings of the real spectrum of $A$ by introducing the space of real places of $A$, which we denote here by $M_A$. By definition, $M_A$ consists of pairs $(\mathfrak{p}, \lambda)$ where $\mathfrak{p}$ is a real prime of $A$ and $\lambda$ is a place from the residue field $k(\mathfrak{p})$ into the field of real numbers. This takes care of the real places in a satisfactory way but does not keep track of all real valuations on the $k(\mathfrak{p})$ and all the orderings on the corresponding residue fields of $k(\mathfrak{p})$. Still, the $M_A$ construction in [51] is closely related to the complete real spectrum construction described below.

**Definition 7.1.** (Marshall, G. [39]) The elements of $\mathrm{Sper}^c A$ are triples $(\mathfrak{p}, v, P)$ where $\mathfrak{p}$ is a real prime of $A$, $v$ is a real valuation (more precisely, an equivalence class of real valuations) on the residue field $k(\mathfrak{p})$, and $P$ is an ordering on the residue field $B_v/M_v$ of $v$. Here, $B_v \subseteq k(\mathfrak{p})$ denotes the valuation ring of $v$ and $M_v$ its maximal ideal.

It is possible to give another definition of the complete real spectrum.

**Definition 7.2.** The elements of $\mathrm{Sper}^c A$ are pairs $(\mathfrak{p}, Q)$ where $\mathfrak{p}$ is a real prime of $A$ and $Q$ is an element of $\mathrm{Sper}\, H_{k(\mathfrak{p})}$.

There are natural maps

$$(\mathfrak{p}, v, P) \mapsto \mathfrak{p}, \quad (\mathfrak{p}, v, P) \mapsto (\mathfrak{p}, v)$$

from $\mathrm{Sper}^c A$ into $\mathrm{Spec}\, A$ (the prime spectrum of $A$) and from $\mathrm{Sper}^c A$ into $\mathrm{Spv}\, A$ (the valuation spectrum of $A$), and a natural map

$$(\mathfrak{p}, P) \mapsto (\mathfrak{p}, 0, P)$$

(where 0 denotes the trivial valuation on $k(\mathfrak{p})$) from $\mathrm{Sper}\, A$ into $\mathrm{Sper}^c A$.

There is also the specialization map

$$(\mathfrak{p}, Q) \mapsto (\mathfrak{p}, Q')$$

from $\mathrm{Sper}^c A$ onto the space of real places $M = M_A$ defined in [51]. Here, $Q'$ denotes the unique maximal specialization of $Q$ in $\mathrm{Sper}\, H_{k(\mathfrak{p})}$; also see [14]. The composite map $\mathrm{Sper}\, A \to M_A$ is just the P-structure map $\Lambda$ considered in [51].

$\mathrm{Sper}^c A$ is given a topology and turned to be a spectral space. Subbasic open sets in $\mathrm{Sper}^c A$ are defined using pairs of elements of $A$. For $(a, b) \in A \times A$, we define:

$$U(a,b) = \{(\mathfrak{p}, v, P) \in \ \mathrm{Sper}^c A : \ v(a) = v(b) \neq \infty \text{ and } \frac{a+\mathfrak{p}}{b+\mathfrak{p}} + M_v > 0 \text{ at } P\}$$

Here, $v(a)$ is standard shorthand notation for $v(a + \mathfrak{p})$.

**Remark 7.3.** It is interesting to note that the complete real spectrum of a formally real field $K$ is naturally identified with the real spectrum of its real holomorphy ring $H_K$ involved in the diagram 5.3.

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