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Algorithm for connectivity queries on real algebraic curves

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Abstract

We consider the problem of answering connectivity queries on a real algebraic curve. The curve is given as the real trace of an algebraic curve, assumed to be in generic position, and being defined by some rational parametrizations. The query points are given by a zero-dimensional parametrization

We design an algorithm which counts the number of connected components of the real curve under study, and decides which query point lie in which connected component, in time log-linear in N^6 , where N is the maximum of the degrees and coefficient bit-sizes of the polynomials given as input. This matches the currently best-known bound for computing the topology of real plane curves.

The main novelty of this algorithm is the avoidance of the computation of the complete topology of the curve.

1 Introduction

Motivation and problem statement In this paper, we consider a real field \mathbf{Q} , its real closure \mathbf{R} and its algebraic closure \mathbf{C} . Let $n \geq 1$, and $\mathbf{X} = (x_1, \dots, x_n)$ be a sequence of indeterminates. Further, $\mathbf{Q}[\mathbf{X}]$ and $\mathbf{C}[\mathbf{X}]$ denote the rings of multivariate polynomials in the x_i 's, with coefficients in respectively \mathbf{Q} and \mathbf{C} . An *algebraic set* $\mathcal{C} \subset \mathbf{C}^n$ is defined as the set of common zeros of a finite sequence polynomials $(f_1, \dots, f_p) \subset \mathbf{C}[\mathbf{X}]$ which is denoted by $\mathbf{V}(f_1, \dots, f_p)$. We denote by $\mathbf{I}(\mathcal{C}) \subset \mathbf{C}[\mathbf{X}]$ the radical of the ideal $\langle f_1, \dots, f_p \rangle$ generated by the f_j 's, that is the *ideal of definition* of \mathcal{C} . The *function ring* $\mathbf{C}[\mathcal{C}]$ of polynomial functions defined on \mathcal{C} is exactly the quotient ring $\mathbf{C}[\mathbf{X}]/\mathbf{I}(\mathcal{C})$. When, in addition, $\mathbf{I}(\mathcal{C}) \subset \mathbf{Q}[\mathbf{X}]$, then $\mathbf{Q}[\mathcal{C}]$ will denote $\mathbf{Q}[\mathbf{X}]/\mathbf{I}(\mathcal{C})$. Finally, we will say that \mathcal{C} is an *algebraic curve* if the ideal $\mathbf{I}(\mathcal{C})$ is equidimensional of dimension 1, and that \mathcal{C} is *plane*, if it is contained in some plane of \mathbf{C}^n .

In this whole work, \mathcal{C} is an algebraic curve such that $I(\mathcal{C}) \subset \mathbf{Q}[\mathbf{X}]$. Given a generating system \mathbf{f} of $I(\mathcal{C})$, let $\text{Jac}(\mathbf{f})$ be the Jacobian matrix of \mathbf{f} . We denote by $\text{sing}(\mathcal{C})$ the set of singular points of \mathcal{C} , that is the points where $\text{Jac}(\mathbf{f})$ has rank less than $n - 2$; it is finite subset of \mathcal{C} . Besides, we note $\text{reg}(\mathcal{C}) = \mathcal{C} - \text{sing}(\mathcal{C})$ and for all $\mathbf{x} \in \text{reg}(\mathcal{C})$, let $T_{\mathbf{x}}\mathcal{C}$ be the right-kernel of $\text{Jac}(\mathbf{f})$: it is the tangent line of \mathcal{C} at \mathbf{x} . For $1 \leq i \leq n$ we denote by $\pi_i : \mathbf{C}^n \rightarrow \mathbf{C}^i$ the canonical projection on the first i variables. Let $\mathcal{C}_2 \subset \mathbf{C}^2$ be the Zariski closure of $\pi_2(\mathcal{C})$. Then the set of apparent singularities of \mathcal{C}_2 is

$$\text{app}(\mathcal{C}) = \text{sing}(\mathcal{C}_2) - \pi_2(\text{sing}(\mathcal{C})).$$

These are the singularities introduced by π_2 . A singular point of \mathcal{C}_2 is called a node if it is an ordinary double point (see e.g. [31, §3.1]). We refer to [48] for definitions and propositions about algebraic sets.

For any $\varphi \in \mathbf{C}[\mathcal{C}]$, we denote by $\mathcal{W}^\circ(\varphi, \mathcal{C})$ the set of *critical points of φ on \mathcal{C}* , that is the set of points $\mathbf{x} \in \text{reg}(\mathcal{C})$ such that $d_{\mathbf{x}}\varphi : T_{\mathbf{x}}\mathcal{C} \rightarrow \mathbf{C}$ is not surjective. Then we note $\mathcal{K}(\varphi, \mathcal{C})$ the *singular points of φ on \mathcal{C}* , that is the union of $\mathcal{W}^\circ(\varphi, \mathcal{C})$ and $\text{sing}(\mathcal{C})$.

To satisfy some genericity assumptions, we will need to perform some linear changes of variables. Given $A \in \text{GL}_n(\mathbf{C})$, for $f \in \mathbf{C}[\mathbf{X}]$, f^A will denote the polynomial $f(A\mathbf{X})$. For $V \subset \mathbf{C}^n$, we denote by V^A the image of V by the map $\Phi_A : \mathbf{x} \mapsto A^{-1}\mathbf{x}$. Thus, for $\mathbf{f} = (f_1, \dots, f_p) \in \mathbf{C}[\mathbf{X}]$ we have

$$\mathbf{V}(\mathbf{f}^A) = \Phi_A(\mathbf{V}(\mathbf{f})) = \mathbf{V}(\mathbf{f})^A.$$

A *semi-algebraic (s.a.) set* $S \subset \mathbf{R}^n$ is the set of solutions of a finite system of polynomial equations and inequalities with coefficients in \mathbf{R} . We say that S is *s.a. connected* if for any $\mathbf{y}, \mathbf{y}' \in S$, \mathbf{y} and \mathbf{y}' can be connected by a *s.a. path* in S , that is an injective continuous s.a. function $\gamma : [0, 1] \rightarrow S$ such that $\gamma(0) = \mathbf{y}$ and $\gamma(1) = \mathbf{y}'$. A s.a. set S can be decomposed into finitely many *s.a. connected components* which are s.a. connected s.a. sets that are both closed and open in S . We refer to [4] and [8] for definitions and propositions about s.a. sets and functions. In this work, the s.a. sets in consideration will mainly be real traces of algebraic sets of \mathbf{C}^n (defined by polynomials with coefficients in \mathbf{R}). In particular, we will note e.g. $\mathcal{C}_{\mathbf{R}}$ and $\mathcal{C}_{2,\mathbf{R}}$, respectively the real traces of \mathcal{C} , \mathcal{C}_2 . Then, e.g. $\mathcal{K}(\pi_1, \mathcal{C}) \cap \mathbf{R}^n$ and $\mathcal{K}(\pi_1, \mathcal{C}_2) \cap \mathbf{R}^2$ will be denoted by $\mathcal{K}(\pi_1, \mathcal{C}_{\mathbf{R}})$ and $\mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbf{R}})$.

In this paper, we address the problem of designing an algorithm for answering connectivity queries on real algebraic curves in \mathbf{R}^n , defined as real traces of algebraic curves of \mathbf{C}^n . More precisely, given representations of an algebraic curve \mathcal{C} and a finite set \mathcal{P} of points of \mathcal{C} , we want to compute a partition of \mathcal{P} , grouping the points lying in the same s.a. connected component of $\mathcal{C}_{\mathbf{R}}$.

It is a problem of importance in symbolic computation, and more specifically, in effective real algebraic geometry. Indeed, using the notions of roadmaps, introduced by Canny in [12, 13], one can reduce connectivity queries in real algebraic sets of arbitrary dimension to the such queries on real algebraic curves. Moreover algorithms computing such roadmaps, on input a real algebraic set,

has been continuously improved in a series of recent works [3, 46, 5, 6], making now tractable challenging problems in applications such as robotics [12, 14, 16, 15].

Prior works A first approach to our problem consists in reducing this problem to piecewise linear approximations sharing the same topology as the curve under study.

The problem of computing the topology of plane algebraic curves in \mathbb{R}^2 , defined by a polynomial with integer coefficients has been extensively studied. Some rely on subdivision algorithm [11, 43], while most of them are based on some variants of Cylindrical Algebraic Decomposition methods [7, 17, 23, 22, 25, 29, 35, 42, 41, 44, 47, 26, 27]. There exists also hybrid approach such as [1]. [42, 23] obtain the best-known complexity bound in $O(\delta^5(\delta + \tau))$, where δ and τ is a bound on the degree of the polynomials defining the curve and the bit-size of its coefficients, respectively. This is done by computing quantitative bounds on (bivariate) real root isolation of the polynomials in consideration.

Computing the topology of space curves in \mathbb{R}^3 has been considered through many different angles by computing the topology of the projection on various planes [2, 33, 18] or lifting the plane projection by algebraic considerations [31, 21, 24]. Only few of these papers gives a complexity bound for the computation of such topology [18, 24], and [39] obtains the best-known complexity in $O(\delta^{19}(\delta + \tau))$. We further show, in this paper, that the case of curves in \mathbb{R}^n can be reduced to $n = 3$ by the mean of a generic projection.

Main result In this paper, we aim to *avoid the computation of the complete topology* of the curve by refining the approach developed in [38] and which is based on [31]. This consists in using the algorithm of [31] to compute the topology of the image $\mathcal{C}_{3,\mathbf{R}}$ of $\mathcal{C}_{\mathbf{R}}$ by the projection π_3 , i.e. an isotopic graph, assuming \mathcal{C} satisfies genericity assumptions that we make explicit below. These imply, amongst other things, that they share the same connectivity properties. Next, the topology of $\mathcal{C}_{3,\mathbf{R}}$ is deduced from that of $\mathcal{C}_{2,\mathbf{R}}$, again assuming some genericity properties.

It turns out that one does not need to compute the topology of $\mathcal{C}_{3,\mathbf{R}}$ in order to answer connectivity queries. A geometric outcome of this work is that the topological analysis needed to be done at some special points of $\mathcal{C}_{2,\mathbf{R}}$, which are called nodes, can be much simplified when one only needs to answer connectivity queries and this has a significant impact on the complexity. To state this complexity result, we need first to introduce how our geometric objects are encoded. For a univariate function φ , we denote by φ' its derivative. For a bivariate function ψ in the variables x_1 and x_2 , we denote by $\partial_{x_1}\psi$, $\partial_{x_2}\psi$, $\partial_{x_1}^2\psi$ and $\partial_{x_1x_2}^2\psi$ respectively the simple and double derivative with respect to the index variable(s).

To encode finite sets of points with algebraic coordinates over a field \mathbf{Q} , we use so called *zero-dimensional parametrizations*. A zero-dimensional parametrization $\mathcal{P} = (\Omega, \lambda)$ is a couple as follows:

- $\Omega = (\omega, \rho_1, \dots, \rho_n)$ of polynomials in $\mathbf{Q}[u]$ where u is a new variable and ω is a monic square-free polynomial and $\deg(\rho_i) < \deg(\omega)$;
- λ is a linear form $\lambda_1 x_1 + \dots + \lambda_n x_n$ in $\mathbf{Q}[x_1, \dots, x_n]$ such that

$$\lambda_1 \rho_1 + \dots + \lambda_n \rho_n = u \partial_u \omega \pmod{\omega}.$$

Such a data-structure encodes the finite set of points, denoted by $Z(\mathcal{P})$, defined as follows

$$Z(\mathcal{P}) = \left\{ \left(\frac{\rho_1(\vartheta)}{\partial_u \omega(\vartheta)}, \dots, \frac{\rho_n(\vartheta)}{\partial_u \omega(\vartheta)} \right) \in \mathbf{C}^n \mid \omega(\vartheta) = 0 \right\}.$$

We define the *degree* of such a parametrization \mathcal{P} as the degree of the polynomial ω . Similarly, we encode an algebraic curve with a one-dimensional parametrization over \mathbf{Q} that is $\mathcal{R} = (\Omega, (\lambda, \mu))$ which is a couple as follows:

- $\Omega = (\omega, \rho_1, \dots, \rho_n)$ of polynomials in $\mathbf{Q}[u, v]$ where u and v are new variables and ω is a monic in u and v , square-free polynomial and $\deg(\rho_i) < \deg(\omega)$;
- (λ, μ) is a couple of linear forms $\lambda_1 x_1 + \dots + \lambda_n x_n$ and $\mu_1 x_1 + \dots + \mu_n x_n$ in $\mathbf{Q}[x_1, \dots, x_n]$,

such that $\lambda_1 \rho_1 + \dots + \lambda_n \rho_n = u \partial_v \omega \pmod{\omega}$, and

$$\mu_1 \rho_1 + \dots + \mu_n \rho_n = v \partial_v \omega \pmod{\omega}.$$

Such a data-structure encodes the algebraic curve $Z(\mathcal{R})$, defined as the Zariski closure of the locally closed set of \mathbf{C}^n

$$\left\{ \left(\frac{\rho_1(\vartheta, \eta)}{\partial_v \omega(\vartheta, \eta)}, \dots, \frac{\rho_n(\vartheta, \eta)}{\partial_v \omega(\vartheta, \eta)} \right) \in \mathbf{C}^n \mid \omega(\vartheta, \eta) = 0, \quad \partial_v \omega(\vartheta, \eta) \neq 0 \right\}.$$

We define the *degree* of such a parametrization \mathcal{R} as the degree of ω which coincides with the degree of $Z(\mathcal{R})$. Note that such a parametrization \mathcal{R} of degree δ involves $O(n\delta^2)$ coefficients.

We say that a polynomials $f \in \mathbb{Z}[x_1, \dots, x_n]$ has magnitude (δ, τ) , if the total degree of f is bounded by δ and all coefficients have absolute values at most 2^τ . We naturally extend this definition to a sequence of polynomials by bounding all entries as above.

We can now state the aforementioned genericity properties, which can be seen as a generalization of the ones in [31] as follows.

Let $\mathcal{C} \subset \mathbf{C}^n$ be an algebraic curve and $\mathcal{P} \subset \text{reg}(\mathcal{C})$ be finite. We say that $(\mathcal{C}, \mathcal{P})$ satisfies assumption (H), that is:

- (H₁) for $1 \leq i \leq n$, $\mathbf{Q}[\mathcal{C}]$ is integral over $\mathbf{Q}[\mathcal{C}_i]$, where $\mathcal{C}_i = \pi_i(\mathcal{C})$ is an algebraic curve;
- (H₂) for all $\mathbf{x} \in \text{reg}(\mathcal{C})$, $\pi_2(T_{\mathbf{x}}\mathcal{C})$ is a tangent line to \mathcal{C} at $\pi_2(\mathbf{x})$;

- (H₃) the restriction of π_3 to \mathcal{C} is injective;
- (H₄) if $\mathbf{y} \in \text{app}(\mathcal{C}_2)$ then
 - (H_{4'}) $\pi_2^{-1}(\mathbf{y}) \cap \mathcal{C}$ has cardinality 2;
 - (H_{4''}) \mathbf{y} is a node of \mathcal{C}_2 ;
- (H₅) $\mathcal{K}(\pi_1, \mathcal{C}_2) \cup \pi_2(\mathcal{P})$ is finite and π_1 is injective on it;
- (H₆) for all $\mathbf{x} \in \mathcal{K}(\pi_1, \mathcal{C}) \cup \mathcal{P}$, $\pi_2^{-1}(\pi_2(\mathbf{x})) \cap \mathcal{C} = \{\mathbf{x}\}$;
- (H₇) there exists a one-dimensional parametrization \mathcal{R} encoding \mathcal{C} , of the form $(\mathcal{R}, (x_1, x_2))$, where $\Omega = (\omega, x_1, x_2, \rho_3, \dots, \rho_n)$ are polynomials in $\mathbb{Q}[x_1, x_2]$.

We omit \mathcal{P} when it is clear from the context. We prove further that, up to a generic linear change of coordinates, performed over \mathcal{C} , this property is satisfied.

We can now state the main result of this paper, assuming that (H) holds and that the parametrizations encoding \mathcal{C} and control points \mathcal{P} have been normalized to yield integer coefficients.

Theorem 1.1. *Let $\mathcal{R} \subset \mathbb{Z}[x_1, x_2]$ be a one-dimensional parametrization encoding an algebraic curve $\mathcal{C} \subset \mathbb{C}^n$ satisfying (H) and $\mathcal{P} \subset \mathbb{Z}[x_1]$ a zero-dimensional parametrization encoding a finite subset of \mathcal{C} . Let (δ, τ) and (μ, κ) the magnitude of \mathcal{R} and \mathcal{P} , respectively.*

Then, there exists an algorithm which, on input \mathcal{R} and \mathcal{P} , computes a partition of the points of $Z(\mathcal{P}) \cap \mathbb{R}^n$ lying in the same s.a. connected component of $\mathcal{C} \cap \mathbb{R}^n$, using

$$\tilde{O}(\delta^6 + \mu^6 + \delta^5\tau + \mu^5\kappa)$$

bit operations.

This is to be compared with the best complexity $O(\delta^{19}(\delta + \tau))$ known to analyse the topology of space curve.

Structure of the paper After some preliminary results we prove that up to a generic change coordinate, assumption (H) holds. Then, under these assumptions, we describe two steps of our algorithm that is identifying the finitely many points of the curve where there is connectivity ambiguity and resolving these ambiguities. Finally, we describe the main algorithm together with complexity bounds.

2 Curves in generic position

In this section, \mathbb{Q} can be any arbitrary field of characteristic 0. This section is devoted to prove that (H) generically holds for an algebraic curve \mathcal{C} that is, there is an open dense subset \mathfrak{A} of $\text{GL}_n(\mathbb{C})$ such that for any A the sheared curve \mathcal{C}^A satisfies (H).

2.1 Generic projections of affine curves

The results below are well known in the case of smooth projective curves (see e.g. [36, IV. Theorem 3.10] or [45, §7B.] for $\mathbf{C} = \mathbb{C}$), and have been generalized subsequently in e.g. [37, 40]. A version for singular affine curves is proved in [32, Proposition 5.2] in the case of three variables and with \mathbf{C} contained in \mathbb{C} , adapting suitably the arguments of [45, §7B.]. Moreover it does not consider singularities with high rank deficiency of the Jacobian matrix. We present here, a generalization of [32, Proposition 5.2] in our context, following the proof, but adapting the arguments using more general results from the literature referred above.

In the following, we consider, for $n \geq 3$, two sets in \mathbf{C}^n : an affine algebraic curve \mathcal{C} and a finite subset $\mathcal{P} \subset \mathcal{C}$. Let \mathbb{P}^n be the projective space $\mathbb{P}^n(\mathbf{C})$ of dimension n over \mathbf{C} , and let $[\mathbf{x}_0 : \cdots : \mathbf{x}_n]$, where $\mathbf{x}_i \in \mathbf{C}$, be its elements. We consider $\mathcal{H}^\infty = \{[\mathbf{x}_0 : \cdots : \mathbf{x}_n] \in \mathbb{P}^n \mid \mathbf{x}_0 = 0\}$, the hyperplane at infinity with respect to the affine open chart given by $\mathbb{P}^n \setminus \mathcal{H}^\infty$. Indeed \mathbf{C}^n can be identified with $\mathbb{P}^n - \mathcal{H}^\infty$ through the map $[\mathbf{x}_0 : \cdots : \mathbf{x}_n] \mapsto (\mathbf{x}_1/\mathbf{x}_0, \dots, \mathbf{x}_n/\mathbf{x}_0)$, and then embed the sets \mathcal{C} and \mathcal{P} in \mathbb{P}^n . We finally denote by $\overline{\mathcal{C}}$ the projective closure of \mathcal{C} in \mathbb{P}^n .

We denote by $\mathbb{G}(1, n) = G(2, n+1)$ the Grassmanian of lines in \mathbb{P}^n and for any \mathbf{x}, \mathbf{y} distinct points in \mathbb{P}^n , by $\mathcal{L}(\mathbf{x}, \mathbf{y}) \in \mathbb{G}(1, n)$ the line containing \mathbf{x} and \mathbf{y} . For \mathbf{x}, \mathbf{y} distinct points of $\overline{\mathcal{C}}$, the line $s = \mathcal{L}(\mathbf{x}, \mathbf{y})$ will be called the *secant line of $\overline{\mathcal{C}}$ determined by \mathbf{x} and \mathbf{y}* . If s intersects $\overline{\mathcal{C}}$ in a third point distinct from \mathbf{x} and \mathbf{y} , then it will be called a *trisecant line of $\overline{\mathcal{C}}$* . If there exist distinct $\mathbf{x}', \mathbf{y}' \in s \cap \text{reg}(\overline{\mathcal{C}})$ such that $T_{\mathbf{x}'}\overline{\mathcal{C}}$ and $T_{\mathbf{y}'}\overline{\mathcal{C}}$ are coplanar, then it will be called a *secant line with coplanar tangents of $\overline{\mathcal{C}}$* . Then we define $\text{Sec}(\overline{\mathcal{C}})$, $\text{Tri}(\overline{\mathcal{C}})$ and $\text{CoTg}(\overline{\mathcal{C}})$ as the set of points of \mathbb{P}^n that lie on respectively a secant, trisecant and secant with coplanar tangent of $\overline{\mathcal{C}}$. Finally, we denote by $\text{Tg}(\overline{\mathcal{C}})$ the set of points of \mathbb{P}^n that lie on the tangent line $T_{\mathbf{x}}\overline{\mathcal{C}}$ for some $\mathbf{x} \in \text{reg}(\overline{\mathcal{C}})$.

Lemma 2.1. *The sets $\text{Sec}(\overline{\mathcal{C}})$ and $\text{Tg}(\overline{\mathcal{C}})$ are algebraic sets of dimension ≤ 3 and ≤ 2 respectively. If, in addition, $\overline{\mathcal{C}}$ is not a plane curve, then $\text{Tri}(\overline{\mathcal{C}})$ and $\text{CoTg}(\overline{\mathcal{C}})$ are algebraic sets of dimension ≤ 2 . Finally, none of these sets contains \mathcal{H}^∞ .*

Proof. Let $\overline{\mathcal{C}}_1, \dots, \overline{\mathcal{C}}_m$ be the irreducible components of $\overline{\mathcal{C}}$. Let $i, j \in \{1, \dots, m\}$, possibly equal, and $\Sigma_{i,j} \subset \mathbb{G}(1, n)$ be the Zariski closure of the image of $\overline{\mathcal{C}}_i \times \overline{\mathcal{C}}_j - \{(\mathbf{y}, \mathbf{y}) \mid \mathbf{y} \in \overline{\mathcal{C}}_i \cap \overline{\mathcal{C}}_j\}$ through the map $(\mathbf{y}, \mathbf{z}) \mapsto \mathcal{L}(\mathbf{y}, \mathbf{z})$. Remark that, as the image of a Cartesian product of two irreducible curves, $\Sigma_{i,j}$ is an irreducible algebraic set. Besides, since such a secant is uniquely determined by fixing two points in the one-dimensional components $\overline{\mathcal{C}}_i$ and $\overline{\mathcal{C}}_j$, we deduce by [48, Theorem 1.25] that $\Sigma_{i,j}$ has dimension at most 2. Then if $\Sigma = \bigcup_{i,j} \Sigma_{i,j}$ is the secant variety of $\overline{\mathcal{C}}$, it has dimension at most 2 and contains the subset of $\mathbb{G}(1, n)$ of secant lines. Since elements of $\mathbb{G}(1, n)$ can be seen as algebraic sets of dimension 1, this proves that $\text{Sec}(\overline{\mathcal{C}})$ has Zariski closure of dimension at most 3.

Consider now, the subset $\Gamma_i \subset \mathbb{P}^n \times \overline{\mathcal{C}}_i$, consisting of points (\mathbf{u}, \mathbf{y}) such that $\mathbf{y} \in \text{reg}(\overline{\mathcal{C}})$ and $\mathbf{u} \in T_{\mathbf{y}}\overline{\mathcal{C}}$. Then, considering projections $\varphi_i: \Gamma_i \rightarrow \mathbb{P}^n$ and $\psi_i: \Gamma_i \rightarrow \overline{\mathcal{C}}_i$, then for all \mathbf{y} in the Zariski open subset $\text{reg}(\overline{\mathcal{C}}) \cap \overline{\mathcal{C}}_i$ of $\overline{\mathcal{C}}_i$, $\psi_i^{-1}(\mathbf{y})$ is exactly $T_{\mathbf{y}}\overline{\mathcal{C}}$ and then, has dimension 1. Hence, by [48, Theorem 1.25], Γ_i has Zariski closure of dimension 2 and the one of $\varphi_i(\Gamma_i)$ has dimension at most 2. Since $\text{Tg}(\overline{\mathcal{C}}) = \cup_i \varphi_i(\Gamma_i)$, we are done.

Assume now, that $\overline{\mathcal{C}}$ is not a plane curve then, by [40, Theorem 2], the set of trisecant lines of $\overline{\mathcal{C}}$ is a subset of $\mathbb{G}(1, n)$ whose Zariski closure has dimension at most 1. Then, as seen above, $\text{Tri}(\overline{\mathcal{C}})$ has Zariski closure of dimension at most 2. Let now $M_{i,j}$ be the subset of $\Sigma_{i,j}$ consisting of secant lines intersecting $\overline{\mathcal{C}}$ at points whose tangents are all contained in the same plane. We are going to prove that the Zariski closure of $M_{i,j}$ has dimension at most 1. Together with the dimension bound on $\text{Tri}(\overline{\mathcal{C}})$, this will allow us to deduce the statement on $\text{CoTg}(\overline{\mathcal{C}})$.

Suppose first that $\overline{\mathcal{C}}_i$ and $\overline{\mathcal{C}}_j$ are not coplanar components. Then, there exists $\mathbf{y} \in \overline{\mathcal{C}}_i - \text{sing}(\overline{\mathcal{C}})$ such that $l = T_{\mathbf{y}}\overline{\mathcal{C}}$ and $\overline{\mathcal{C}}_j$ are not coplanar. Then, let $\mathbf{p}_l: \mathbb{P}^n \rightarrow \mathbb{P}^{n-2}$ be the projection of center l . The Zariski closure of $\mathbf{p}_l(\overline{\mathcal{C}}_j)$ is not a point. Since $\overline{\mathcal{C}}_j$ is irreducible, and by [48, Theorem 1.25] the Zariski closure \mathcal{R} of $\mathbf{p}_l(\overline{\mathcal{C}}_j)$ has dimension at most 1, then \mathcal{R} is an irreducible algebraic subset of \mathbb{P}^{n-2} of dimension 1. Hence, by [48, Theorem 1.25] again, there exists a finite set $K_1 \subset \mathbb{P}^{n-2}$ such that for all $\mathbf{w} \in \mathcal{R} \setminus K_1$, $\mathbf{p}_l^{-1}(\mathbf{w}) \cap \overline{\mathcal{C}}_j$ is finite. Besides, by an algebraic geometric version of Sard's Theorem [48, Theorem 2.27], there exists a finite set $K_2 \subset \mathbb{P}^{n-2}$ such that $\mathcal{R} \setminus K_2$ does not contain any critical value of the restriction of \mathbf{p}_l to $\overline{\mathcal{C}}_j$. Now, let w in the non-empty Zariski open set $\mathcal{R} \setminus [K_1 \cup K_2 \cup \mathbf{p}_l(\text{sing}(\overline{\mathcal{C}}))]$. One has

$$\mathbf{p}_l^{-1}(w) \cap \overline{\mathcal{C}}_j = \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$$

where $k \geq 1$, and for all $i \in \{1, \dots, k\}$, $\mathbf{z}_i \in \text{reg}(\overline{\mathcal{C}})$ and $\mathbf{p}_l(T_{\mathbf{z}_i}\overline{\mathcal{C}})$ has dimension 1. Hence, \mathbf{y} and \mathbf{z}_i have no coplanar tangents for all $i \in \{1, \dots, k\}$. In particular, the secant line $\mathcal{L}(\mathbf{y}, \mathbf{z}_1) \in \mathbb{G}(1, n)$ of $\overline{\mathcal{C}}$ contains two points having no coplanar tangents so that $\mathcal{L}(\mathbf{y}, \mathbf{z}_1) \in \Sigma_{i,j} - M_{i,j}$ and $M_{i,j} \subsetneq \Sigma_{i,j}$. Therefore, the Zariski closure of $M_{i,j}$ is a proper algebraic set by the previous paragraph, and since $\Sigma_{i,j}$ is irreducible, $M_{i,j}$ has closure of dimension at most 1.

Now, if $\overline{\mathcal{C}}_i$ and $\overline{\mathcal{C}}_j$ are coplanar then $\Sigma_{i,j}$ is the Zariski closure of $M_{i,j}$ and one of the following holds. If $i = j$ and $\overline{\mathcal{C}}$ is a line, then $\Sigma_{i,j}$ is reduced to the line corresponding to $\overline{\mathcal{C}}_i$, and then of dimension 0. Else, there exists a unique plane $S_{i,j}$ containing the coplanar components $\overline{\mathcal{C}}_i$ and $\overline{\mathcal{C}}_j$ and any line of $\Sigma_{i,j}$ must be contained in $S_{i,j}$. Hence, in both cases, $\Sigma_{i,j}$, and then the closure of $M_{i,j}$ have dimension at most 1. Then, the Zariski closure of the union M of all $M_{i,j}$ for $i, j \in \{1, \dots, m\}$, is an algebraic subset of $\mathbb{G}(1, n)$ of dimension at most 1 as requested.

Remark now that a secant with coplanar tangents is either a trisecant, or a secant intersecting $\overline{\mathcal{C}}$ in exactly two regular points with coplanar tangents. Hence the set of secants with coplanar tangents of $\overline{\mathcal{C}}$ is contained in the union of M and the set of trisecant lines of $\overline{\mathcal{C}}$. By the above discussion it has dimension

1. As seen above, this proves that the Zariski closure of $\text{CoTg}(\overline{\mathcal{C}})$ has dimension ≤ 2 .

Since $\overline{\mathcal{C}} - \mathcal{H}^\infty$ can be identified with \mathcal{C} , it is a non-empty Zariski open subset of $\overline{\mathcal{C}}$, so that $\overline{\mathcal{C}} \cap \mathcal{H}^\infty$ is finite. In particular, \mathcal{H}^∞ contains finitely many secant and tangent lines of $\overline{\mathcal{C}}$ and then, cannot be contained in $\text{Sec}(\overline{\mathcal{C}})$ or $\text{Tg}(\overline{\mathcal{C}})$. Since $\text{Tri}(\overline{\mathcal{C}})$ and $\text{CoTg}(\overline{\mathcal{C}})$ are contained in $\text{Sec}(\overline{\mathcal{C}})$, they cannot contain \mathcal{H}^∞ as well. \square

In the following, for $0 \leq r \leq n-1$, we denote by $\mathbb{G}(r, n-1) = G(r+1, n)$ the set of r -dimensional projective linear subspaces of \mathcal{H}^∞ . Recall that, by e.g. [48, Example 1.24], using Plücker embedding, $\mathbb{G}(r, n-1)$ can be embedded in $\mathbb{P}^{\binom{n}{r+1}-1}$ as an irreducible algebraic set of dimension $(r+1)(n-r)$. The following lemma is then an immediate consequence of [48, Theorem 1.25].

Lemma 2.2. *Let $X \subset \mathcal{H}^\infty$ be an algebraic set of dimension $m \leq n-1$. Then, for any $i \geq m$ there exists a non-empty Zariski open subset \mathfrak{E}_i of $\mathbb{G}(n-1-i, n-1)$ such that for every $E \in \mathfrak{E}_i$, the set $E \cap X$ is finite and, if $i > m$, it is empty.*

Recall that \mathcal{P} is a finite set of control points in $\overline{\mathcal{C}} - \text{sing}(\overline{\mathcal{C}})$.

Proposition 2.3. *If $\overline{\mathcal{C}}$ is not a plane curve, then for all $1 \leq i \leq n-1$, there exists a non-empty Zariski open subset \mathfrak{E}_i of $\mathbb{G}(n-1-i, n-1)$ such that for all $E \in \mathfrak{E}_i$, the following holds. Let $\mathfrak{p}_E : \overline{\mathcal{C}} \rightarrow \mathbb{P}^i$ be the projection with center E , then \mathfrak{p}_E is a finite regular map and*

(i) *for all $\mathbf{x} \in \mathcal{P}$, $\mathfrak{p}_E(T_{\mathbf{x}}\overline{\mathcal{C}})$ is a projective line of \mathbb{P}^i .*

If, in addition, $i \geq 2$ then,

(ii) *item (i) holds for any $\mathbf{x} \in \text{reg}\overline{\mathcal{C}}$;*

(iii) *for any $\mathbf{x} \in \overline{\mathcal{C}}$, there exists at most one point $\mathbf{x}' \in \overline{\mathcal{C}}$, distinct from \mathbf{x} such that $\mathfrak{p}_E(\mathbf{x}) = \mathfrak{p}_E(\mathbf{x}')$;*

(iv) *there exists finitely many such couples $(\mathbf{x}, \mathbf{x}')$, all satisfying $\mathbf{x}, \mathbf{x}' \in \text{reg}(\overline{\mathcal{C}}) - \mathcal{P}$ and $\mathfrak{p}_E(T_{\mathbf{x}}\overline{\mathcal{C}}) \neq \mathfrak{p}_E(T_{\mathbf{x}'}\overline{\mathcal{C}})$;*

(v) *if $i \geq 3$, there is no such couple.*

Proof. Fix $1 \leq i \leq n-1$ and suppose that $\overline{\mathcal{C}}$ is not a plane curve. Let $X_1 = \mathcal{H}^\infty \cap \overline{\mathcal{C}}$. As a proper Zariski closed set of $\overline{\mathcal{C}}$, it is finite. Hence, since $i > 0$, then by Lemma 2.2, there exists a non-empty Zariski open subset \mathfrak{E}_1 of $\mathbb{G}(n-1-i, n-1)$ such that for all $E \in \mathfrak{E}_1$, $E \cap X_1$ is empty. Moreover, any $(n-i)$ -dimensional space containing E cannot contain an irreducible component of $\overline{\mathcal{C}}$. Indeed, such a component would be a line, intersecting E at some point of $E \cap \overline{\mathcal{C}} = E \cap X_1$, which is empty. Therefore, the projection with center $E \in \mathfrak{E}_1$ induces a finite map on $\overline{\mathcal{C}}$, regular by definition.

According to Lemma 2.1, the set of points lying on a tangent or a trisecant line of $\overline{\mathcal{C}}$ is an algebraic set of dimension at most 2. Since \mathcal{H}^∞ contains finitely many such tangents or trisecants, then

$$X_2 = (\text{Tg}(\overline{\mathcal{C}}) \cup \text{Tri}(\overline{\mathcal{C}})) \cap \mathcal{H}^\infty$$

has dimension at most 1. Hence, since $i \geq 1$, by Lemma 2.2, there exists a non-empty Zariski open subset \mathfrak{E}_2 of $\mathbb{G}(n-1-i, n-1)$ such that any $E \in \mathfrak{E}_2$ intersects finitely many points of $\text{Tg}(\overline{\mathcal{C}}) \cup \text{Tri}(\overline{\mathcal{C}})$. Besides, there is finitely many tangents intersecting the finite set \mathcal{P} , so that by Lemma 2.2, up to intersecting \mathfrak{E}_2 with a non-empty Zariski open subset of $\mathbb{G}(n-1-i, n-1)$, one can assume that none of these tangents intersect \mathcal{P} . This proves (i).

Assume now that $i \geq 2$ then, by Lemma 2.2, no $E \in \mathfrak{E}_2$ intersects points in $\text{Tg}(\overline{\mathcal{C}}) \cup \text{Tri}(\overline{\mathcal{C}})$. In particular, any $(n-i)$ -dimensional space containing E cannot contain a tangent nor a trisecant, and, as seen above, this means that no tangent, or three distinct points, are mapped to one point. This proves respectively (ii) and (iii).

Now, by Lemma 2.1, the set $X_3 = \text{Sec}(\overline{\mathcal{C}}) \cap \mathcal{H}^\infty$ of points of \mathcal{H}^∞ , lying on a secant line of $\overline{\mathcal{C}}$, is an algebraic set of dimension at most 2. Hence, since $i \geq 2$, by Lemma 2.2, there exists a non-empty Zariski open subset \mathfrak{E}_3 of $\mathbb{G}(n-1-i, n-1)$ such that any $E \in \mathfrak{E}_3$ contains finitely many points lying on a secant line of $\overline{\mathcal{C}}$, that is, as seen above, there are finitely many couples of points which are mapped to the same point in \mathbb{P}^i . Also, the set of secants intersecting the finite set $\text{sing}(\overline{\mathcal{C}}) \cup \mathcal{P}$ is a proper algebraic subset of the secant variety of $\overline{\mathcal{C}}$. Hence, by Lemma 2.2, up to intersecting \mathfrak{E}_3 with a non-empty Zariski open subset of $\mathbb{G}(n-1-i, n-1)$, one can assume that none of these secants intersect $\text{sing}(\overline{\mathcal{C}}) \cup \mathcal{P}$. Finally, since $\text{CoTg}(\overline{\mathcal{C}}) \cap \mathcal{H}^\infty$ has dimension ≤ 1 , then by Lemma 2.2, up to intersecting \mathfrak{E}_3 with a non-empty Zariski open subset of $\mathbb{G}(n-1-i, n-1)$, one can assume that these secants intersect $\overline{\mathcal{C}}$ at points with no coplanar tangents. Hence, these tangents cannot be mapped to the same line. All in all, any $E \in \mathfrak{E}_3$ satisfies assertion (iv).

If moreover, $i \geq 3$ then, by Lemma 2.2, no $E \in \mathfrak{E}_3$ intersects points in $\text{Sec}(\overline{\mathcal{C}})$ that is, as seen above, no two distinct points are mapped to the same image. This proves assertion (v). Taking \mathfrak{E}_i as the intersection of $\mathfrak{E}_1, \mathfrak{E}_2$ and \mathfrak{E}_3 ends the proof. \square

We can now state the affine counterpart of Proposition 2.3.

Corollary 2.4. *There exists a non-empty Zariski open set \mathfrak{A} of $\text{GL}_n(\mathbf{C})$ such that for all $A \in \mathfrak{A}$ and $1 \leq i \leq n$ the following holds. The restriction of π_i to \mathcal{C}^A is a finite morphism, and*

- (i) for all $\mathbf{x} \in \mathcal{P}^A$, $\pi_i(T_{\mathbf{x}}\mathcal{C}^A)$ is a line of \mathbf{C}^i .

If, in addition, $i \geq 2$ then,

- (ii) item (i) holds for any $\mathbf{x} \in \text{reg}(\mathcal{C}^A)$;

- (iii) the restriction of π_i to \mathcal{C}^A is not injective at \mathbf{x} if, and only if, $i = 2$ and $\pi_2(\mathbf{x}) \in \text{app}(\mathcal{C}_2^A)$;
- (iv) $\text{app}(\mathcal{C}_2^A)$ contains only nodes, with exactly two preimages through π_2 , none of them being in \mathcal{P}^A ;

Proof. If \mathcal{C} is a plane curve, then it can be isomorphically embedded into some plane and then, up to a change of variable into \mathbf{C}^i . In the following we suppose that $n \geq 3$ and that \mathcal{C} is not a plane curve. If $i = n$, there is nothing to prove, so let $1 \leq i \leq n-1$. Let $\overline{\mathcal{C}}$ be the projective closure of \mathcal{C} , which is not a plane curve too. Let \mathfrak{E}_i be the non-empty Zariski open subset of $\mathbb{G}(n-1-i, n-1)$ given by Proposition 2.3. According to Plücker embedding, there exists a surjective regular map from the set of i linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_i$ of \mathbf{C}^n to the set of $(n-1-i)$ -dimensional (projective) linear subspaces of \mathcal{H}^∞ , defined by the linear equations $x_0 = 0$ and $\mathbf{a}_{j,1}x_1 + \dots + \mathbf{a}_{j,n}x_n = 0$ for $1 \leq j \leq i$. Hence, there exists a non-empty Zariski open set \mathfrak{A}_i of $\text{GL}_n(\mathbf{C})$ of matrices A such that the first i rows of A^{-1} are mapped to some $E \in \mathfrak{E}_i$, through the above map. Moreover, for any $A \in \mathfrak{A}_i$ the following holds. Consider,

$$\tilde{A} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix},$$

and for $1 \leq j \leq n$, let $\mathbf{a}_j = (\mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,n})$ be the rows of A . If $L_0 = x_0$ and for $1 \leq j \leq i$, $L_j = \mathbf{a}_{j,1}x_1 + \dots + \mathbf{a}_{j,n}x_n$, then the equations L_0, \dots, L_i define a projective linear subspace E of \mathcal{H}^∞ , such that $E \in \mathfrak{E}_i$ and, by definition [48, Example 1.27],

$$\mathfrak{p}_E : \begin{array}{ccc} \overline{\mathcal{C}}^{\tilde{A}} & \rightarrow & \mathbb{P}^i \\ \mathbf{x} & \mapsto & [\mathbf{x}_0 : \dots : \mathbf{x}_i] \end{array}.$$

Therefore, the restriction of \mathfrak{p}_E to the affine chart $\mathbb{P}^n - \mathcal{H}^\infty$ can be identified with the restriction of π_i to \mathcal{C}^A . According to Proposition 2.3, the restriction of π_i to \mathcal{C}^A is a finite morphism satisfying item (i). Assume now that $i \geq 2$ then, assertion (ii) is a direct consequences of respectively item (ii) of Proposition 2.3.

Besides, let $\mathbf{x} \in \mathcal{C}^A$ such that there exists $\mathbf{x}' \in \mathcal{C}^A$, distinct from \mathbf{x} , such that $\pi_i(\mathbf{x}) = \pi_i(\mathbf{x}')$. Then, by item (iv) and (v) of Proposition 2.3, \mathbf{x}' is unique, both $\mathbf{x}, \mathbf{x}' \notin \text{sing}(\mathcal{C}^A) \cup \mathcal{P}^A$, and necessarily $i = 2$. Moreover, $T_{\mathbf{x}}\mathcal{C}^A$ and $T_{\mathbf{x}'}\mathcal{C}^A$ map to distinct lines of \mathbf{C}^2 crossing at $\pi_2(\mathbf{x})$: it is then a node. Hence, $\mathbf{x} \in \text{app}(\mathcal{C}_2^A)$ and $\pi_2(\mathbf{x})$ is a node, with exactly two preimages, none of them being in \mathcal{P}^A . Conversely, according to item (ii) of Proposition 2.3, all points of $\text{app}(\mathcal{C}_2^A)$ have at least two preimages in \mathcal{C}^A . This proves the last two points. Finally, taking $\mathfrak{A} = \bigcap_{i=1}^{n-1} \mathfrak{A}_i$ ends the proof. \square

2.2 Recovering (H)

Proposition 2.5. *Let $\mathcal{C} \subset \mathbf{C}^n$ be an algebraic curve and a finite subset $\mathcal{P} \subset \text{reg}(\mathcal{C})$. There exists a non-empty Zariski open set $\mathfrak{A} \subset \text{GL}_n(\mathbf{C})$ such that, for any $A \in \mathfrak{A}$, $(\mathcal{C}^A, \mathcal{P}^A)$ satisfies (H).*

Proof. Let $\mathfrak{A}_1 \subset \mathrm{GL}_n(\mathbf{C})$ be the non-empty Zariski open subset defined in Corollary 2.4 and $A \in \mathfrak{A}_1$. For all $1 \leq i \leq n$, the restriction of π_i to \mathcal{C}^A is a finite morphism and, in particular, $\mathcal{C}_i^A = \pi_i(\mathcal{C}^A)$ is an algebraic curve. Since \mathbf{C} is integral over \mathbf{Q} , the extension $\mathbf{Q}[\mathcal{C}_i^A] \hookrightarrow \mathbf{Q}[\mathcal{C}^A]$ is integral as well: (H₁) is satisfied. Applying Corollary 2.4, for $i = 3$ and $i = 2$ shows that the curve \mathcal{C}^A satisfies respectively (H₃) on the one hand and (H₂) and (H₄) on the other.

Let $\mathbf{A} = (a_{i,j})_{1 \leq i,j \leq n}$ and t be new indeterminates, the former ones standing for the entries of a square matrix of size $n \times n$. Since \mathfrak{A}_1 is non-empty and Zariski open, there exists a non-zero polynomial $F \in \mathbf{C}[\mathbf{A}]$, such that $A \in \mathfrak{A}_1$ if $F(A) \neq 0$. Besides, according to [9, §4.2] (or [35, §3.2]), there exists a non-zero polynomial $G \in \mathbf{C}[\mathbf{A}, t]$ such that, if $F(A) \neq 0$ and $G(A, b) \neq 0$ then, for

$$B = \begin{bmatrix} 1 & b & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-2} \end{bmatrix},$$

the curve \mathcal{C}_2^{BA} is a plane curve in generic position in the sense of [9, §4.2] and [31, Definition 3.3]. In particular, π_1 maps no tangent line of any singular point of \mathcal{C}_2 to a point and its restriction of π_1 to the finite set $\mathcal{W}^\circ(\pi_1, \mathcal{C}_2^{BA})$ is injective. Let $\mathcal{P}_2 = \pi_2(\mathcal{P})$, since $\mathcal{P}_2 \cup \mathrm{sing}(\mathcal{C}_2)$ is finite, we can assume that π_1 is injective on $\mathcal{P}_2^{BA} \cup \mathrm{sing}(\mathcal{C}_2^{BA})$ as well. But, for any $\mathbf{x} \in \mathcal{W}^\circ(\pi_1, \mathcal{C}_2^{BA})$, $\pi_1(\mathbf{x})$ is a point, so that \mathbf{x} is neither in $\mathrm{sing}(\mathcal{C}_2^{BA})$ nor \mathcal{P}_2^{BA} , by genericity of \mathcal{C}_2^{BA} and item (i) of Corollary 2.4 respectively. Then, let $b \in \mathbf{C}$ such that $G(\mathbf{A}, b)$ is not zero and let B be as above. The subset $\mathfrak{A}_2 \subset \mathrm{GL}_n(\mathbf{C})$ of elements of the form BA' where $F(A')G(A', b) \neq 0$ is a non-empty Zariski open subset. Moreover, for any $A \in \mathfrak{A}_2$, \mathcal{C}^A satisfies (H₅).

Take $A \in \mathfrak{A}_1 \cap \mathfrak{A}_2$ and let $\mathbf{x} \in \mathcal{K}(\pi_1, \mathcal{C}^A) \cup \mathcal{P}^A$ and $\mathbf{y} = \pi_2(\mathbf{x})$. Suppose there exists $\mathbf{x}' \in \mathcal{C}^A$, distinct from \mathbf{x} such that $\pi_2(\mathbf{x}') = \mathbf{y}$. By item (iii), $\mathbf{x} \in \mathcal{W}^\circ(\pi_1, \mathcal{C}^A)$ and \mathbf{y} is a node in $\mathrm{app}(\mathcal{C}_2^A)$, with vertical tangent line $\pi_2(T_{\mathbf{x}}\mathcal{C}^A)$: this is impossible by the above paragraph. Indeed $A \in \mathfrak{A}_2$ and then \mathcal{C}_2^A is in generic position. Therefore, for all $A \in \mathfrak{A}_1 \cap \mathfrak{A}_2$, \mathcal{C}^A satisfies (H₆).

We proceed the same way for (H₇). Let $A \in \mathfrak{A}_1$. By (H₁) the curve \mathcal{C}^A is in Noether position (for π_1). Let $\mathbf{D} = (\mathfrak{d}_3, \dots, \mathfrak{d}_n)$ be new variables, by [28, Corollary 3.4 and 3.5] there exists a non-zero polynomial $H \in \mathbf{C}[\mathbf{A}, \mathbf{D}]$ such that, if $F(A) \neq 0$ and $H(A, \mathbf{d}) \neq 0$ then the following holds. If $\mu_{\mathbf{d}} = x_2 + \mathfrak{d}_3 x_3 + \dots + \mathfrak{d}_n x_n$ is a linear form, then there exists $\mathcal{R} = (\omega, \rho_1, \dots, \rho_n) \subset \mathbf{Q}[x_1, v]$ such that $(\mathcal{R}, x_1, \mu_{\mathbf{d}})$ is a one-dimensional parametrization encoding \mathcal{C}^A . Letting $\mathbf{d} \in \mathbf{C}^{n-1}$ such that $H(\mathbf{A}, \mathbf{d})$ is not zero and

$$C = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 0 & 1 & \mathbf{d} \\ \mathbf{0} & \mathbf{0} & I_{n-2} \end{bmatrix},$$

the subset $\mathfrak{A}_3 \subset \mathrm{GL}_n(\mathbf{C})$ of elements, of the form CA' , where $F(A')$ and $H(A', \mathbf{c})$ are both not zero, is a non-empty Zariski open subset where \mathcal{C}^A satisfies (H₇).

Finally, let \mathfrak{A} be the non-empty Zariski open subset $\mathfrak{A}_1 \cap \mathfrak{A}_2 \cap \mathfrak{A}_3$ of $\mathrm{GL}_n(\mathbf{C})$. For $A \in \mathfrak{A}$, \mathcal{C}^A satisfies (H). \square

3 Witness apparent singularities

In this section, \mathbf{Q} can be any arbitrary field of characteristic 0. We modify the criterion of [31] used to identify apparent singularities in the projected plane curve.

For the rest of the document, we consider an algebraic curve $\mathcal{C} \subset \mathbf{C}^n$, with associated ideal $\mathbf{I}(\mathcal{C}) \subset \mathbf{Q}[X]$, and a finite subset $\mathcal{P} \subset \text{reg}(\mathcal{C})$, such that $(\mathcal{C}, \mathcal{P})$ satisfies (H). According to (H7), let $\mathcal{R} = (\omega, \rho_3, \dots, \rho_n) \subset \mathbf{Q}[x_1, x_2]$ be a one-dimensional parametrization encoding \mathcal{C} and let $\mathcal{C}_2 = \pi_2(\mathcal{C})$. It is a plane algebraic curve such that $\mathcal{C}_2 = \mathbf{V}(\omega) \subset \mathbf{C}^2$.

We start by an adapted version of [31, Lemma 4.1]. The equivalence relation modulo $\mathbf{I}(\mathcal{C})$ is denoted by \equiv .

Lemma 3.1. *Let (α, β) be a node of \mathcal{C}_2 , then the following holds. There exist exactly two power-series $y_1, y_2 \in \mathbf{C}[[x_1 - \alpha]]$ such that for $i = 1, 2$, if $z_i = \frac{\rho_3(x_1, y_i)}{\partial_{x_2}\omega(x_1, y_i)}$ then, the following holds:*

1. $\omega(x_1, y_i) \equiv 0$ and $y_i(\alpha) = \beta$ but $y'_1(\alpha) \neq y'_2(\alpha)$;
2. $h(x_1, y_i, z_i) \equiv 0$ for any $h \in \mathbf{I}(\mathcal{C}) \cap \mathbf{Q}[x_1, x_2, x_3]$ and we have $z_i \in \mathbf{C}[[x_1 - \alpha]]$.

Proof. Since \mathcal{C} satisfies (H), by (H5) and (H7), \mathcal{C}_2 is in generic position in the sense of [35, Definition 3.1]. Hence, by [35, Proposition 2.1 and Theorem 3.1], β is a double root of $\omega(\alpha, x_2)$, as (α, β) is a node of $\mathcal{C}_2 = \mathbf{V}(\omega)$. The Puiseux theorem (see e.g. [30, Corollary 13.16]) states that there exist exactly two Puiseux series, say y_1, y_2 , of \mathcal{C}_2 at (α, β) . Moreover, according to [31, §3.2], y_1, y_2 are formal power-series in $\mathbf{C}[[x_1 - \alpha]]$. Hence, by definition of such series $\omega(x_1, y_i) \equiv 0$ and $y_i(\alpha) = \beta$ where $i = 1, 2$. Besides, since (α, β) is a node, there are exactly two tangent lines at this point, that is $y'_1(\alpha) \neq y'_2(\alpha)$. This proves assertion (1).

Let now $h \in \mathbf{I}(\mathcal{C}) \cap \mathbf{Q}[x_1, x_2, x_3]$, by Euclidean division with respect to x_3 , there exist $u, r \in \mathbf{Q}[x_1, x_2]$ and $m \geq 0$ such that

$$(\partial_{x_2}\omega)^m \cdot h = u(\partial_{x_2}\omega \cdot x_3 - \rho_3) + r.$$

Since $\mathbf{I}(\mathcal{C}) \cap \mathbf{Q}[x_1, x_2] = \langle \omega \rangle$, then ω divides r in $\mathbf{Q}[x_1, x_2]$. Hence,

$$(\partial_{x_2}\omega(x_1, y_i))^m \cdot h(x_1, y_i, z_i) \equiv 0,$$

for any $i = 1, 2$. But $\partial_{x_2}\omega(x_1, y_i)$ cannot be identically zero, since $\mathcal{K}(\pi_1, \mathcal{C}_2)$ is finite by (H5) and then, $h(x_1, y_i, z_i) \equiv 0$.

Finally, by (H1), $\mathbf{Q}[\mathcal{C}_3]$ is integral over $\mathbf{Q}[\mathcal{C}_2]$. Hence, there exists a polynomial

$$h_0 \in \mathbf{I}(\mathcal{C}_3) = \mathbf{I}(\mathcal{C}) \cap \mathbf{Q}[x_1, x_2, x_3],$$

which is monic in x_3 . Since, by the above paragraph, for $i = 1, 2$, $h_0(x_1, y_i, z_i) \equiv 0$, then z_i is integral over $\mathbf{C}[[x_1 - \alpha]]$. But, since \mathbf{C} is an algebraically closed

field of characteristic 0, then by [30, Corollary 13.15], $\mathbf{C}[[x_1 - \alpha]]$ is integrally closed. Hence, as z_i is in its fraction field by definition, it belongs to $\mathbf{C}[[x_1 - \alpha]]$ by closure. \square

Proposition 3.2. *The following assertions are equivalent:*

1. $\mathbf{y} \in \text{app}(\mathcal{C}_2)$;
2. \mathbf{y} is a node of \mathcal{C}_2 and

$$(\partial_{x_2}^2 \omega \cdot \partial_{x_1} \rho_3 - \partial_{x_1 x_2}^2 \omega \cdot \partial_{x_2} \rho_3)(\mathbf{y}) \neq 0. \quad (1)$$

Proof. Assume first that $\mathbf{y} = (\alpha, \beta)$ is a node. We first prove that if (1) holds then, there are two distinct points of \mathcal{C} that project on \mathbf{y} . By Lemma 3.1, there exist two power-series $y_1, y_2 \in \mathbf{C}[[x_1 - \alpha]]$ such that $y_1(\alpha) = \beta$, $y_2(\alpha) = \beta$ and $y_1'(\alpha) \neq y_2'(\alpha)$ and $\omega(x_1, y_i) \equiv 0$ for $i = 1, 2$. Let $i = 1, 2$ and $z_i = \frac{\rho_3(x_1, y_i)}{\partial_{x_2} \omega(x_1, y_i)}$. By Lemma 3.1,

$$\partial_{x_2} \omega(x_1, y_i) \cdot z_i \equiv \rho_3(x_1, y_i)$$

since $\partial_{x_2} \omega \cdot x_3 - \rho_3 \in \mathbf{I}(\mathcal{C}) \cap \mathbf{Q}[x_1, x_2, x_3]$. Moreover, $z_i \in \mathbf{C}[[x_1 - \alpha]]$ so that, by derivation and evaluation in $x_1 = \alpha$,

$$(\partial_{x_1 x_2}^2 \omega(\mathbf{y}) + y_i'(\alpha) \partial_{x_2}^2 \omega(\mathbf{y})) z_i(\alpha) = \partial_{x_1} \rho_3(\mathbf{y}) + y_i'(\alpha) \partial_{x_2} \rho_3(\mathbf{y}). \quad (2)$$

By Lemma 3.1, $\omega(x_1, y_i) \equiv 0$. Hence, by differentiating twice and evaluating in $x_1 = \alpha$:

$$\partial_{x_1}^2 \omega(\mathbf{y}) + 2y_i'(\alpha) \partial_{x_1 x_2}^2 \omega(\mathbf{y}) + y_i'(\alpha)^2 \partial_{x_2}^2 \omega(\mathbf{y}) = 0.$$

Since $y_1'(\alpha) \neq y_2'(\alpha)$ (by Lemma 3.1), they are simple roots of the polynomial $\partial_{x_1}^2 \omega(\mathbf{y}) + 2U \partial_{x_1 x_2}^2 \omega(\mathbf{y}) + U^2 \partial_{x_2}^2 \omega(\mathbf{y}) \in \mathbf{C}[U]$. Therefore,

$$\partial_{x_1 x_2}^2 \omega(\mathbf{y}) + y_i'(\alpha) \partial_{x_2}^2 \omega(\mathbf{y}) \neq 0. \quad (3)$$

Now let $H: \mathbf{C} \rightarrow \mathbf{C}$ such that for all $t \in \mathbf{C}$

$$H(t) = \frac{\partial_{x_1} \rho_3(\mathbf{y}) + t \cdot \partial_{x_2} \rho_3(\mathbf{y})}{\partial_{x_1 x_2}^2 \omega(\mathbf{y}) + t \cdot \partial_{x_2}^2 \omega(\mathbf{y})}.$$

Using (2) and according to (3), $H(y_i'(\alpha)) = z_i(\alpha)$ for $i = 1, 2$. But one checks that H is either bijective or constant, whether (1) respectively holds or not. Since $y_1'(\alpha) \neq y_2'(\alpha)$, then (1) holds if, and only if, $z_1(\alpha) \neq z_2(\alpha)$. By the second statement of Lemma 3.1, $\mathbf{z}_1 = (\alpha, \beta, z_1(\alpha))$ and $\mathbf{z}_2 = (\alpha, \beta, z_2(\alpha))$ are points of \mathcal{C}_3 projecting on \mathbf{y} . Hence, by (H₃), there exist \mathbf{x} and \mathbf{x}' in \mathcal{C} that project on respectively \mathbf{z}_1 and \mathbf{z}_2 , and they are distinct if, and only if, (1) holds.

We can now prove the equivalence statement. We just proved that if \mathbf{y} is a node and (1) holds then, \mathbf{y} is the projection of two distinct points, that cannot be singular by (H₅). Conversely, either \mathbf{y} is not a node, and we conclude by (H₄) or, by the above discussion, it is the projection of a point of \mathcal{C} , with two distinct tangents lines (that project on the ones of \mathbf{y}). Hence, \mathbf{y} is the projection of a singular point, and then, not in $\text{app}(\mathcal{C}_2)$, by definition. \square

4 Connectivity recovery

We now investigate the connectivity relation between $\mathcal{C}_{\mathbf{R}}$ and $\mathcal{C}_{2,\mathbf{R}}$. The following lemma is partly adapted from [31, Lemma 6.2].

Lemma 4.1. *Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{K}(\pi_1, \mathcal{C})$, then $\mathbf{x} \in \mathbf{R}^n$ if and only if $\mathbf{x}_1 \in \mathbf{R}$ and*

$$\mathcal{K}(\pi_1, \mathcal{C}_2) - \text{app}(\mathcal{C}_2) = \pi_2(\mathcal{K}(\pi_1, \mathcal{C})).$$

Proof. The second point is a direct consequence of (H). Indeed, by (H₂), the non-singular critical points of \mathcal{C}_2 are exactly the projections of the ones of \mathcal{C} . Moreover, by (H₂) and (H₆) the singular points of \mathcal{C}_2 that are not in $\text{app}(\mathcal{C}_2)$ are exactly the projections of the singular points of \mathcal{C} .

Let $\mathbf{x} \in \mathcal{K}(\pi_1, \mathcal{C})$ and suppose that $\mathbf{x}_1 \in \mathbf{R}$. Since \mathcal{C} is in generic position, by [35, Proposition 3.1], computing sub-resultant sequences gives a rise to $\sigma_2 \in \mathbf{Q}[x_1]$ such that $\mathbf{x}_2 = \sigma_2(\mathbf{x}_1) \in \mathbf{R}$.

By (H₆), the line $\mathbf{V}(x_1 - \mathbf{x}_1, x_2 - \mathbf{x}_2)$ intersects \mathcal{C} at exactly one point. Hence, by [20, Theorem 3.2], computing a Gröbner basis of the ideal

$$\mathbf{I}(\mathcal{C}) + \langle x_1 - \mathbf{x}_1, x_2 - \mathbf{x}_2 \rangle \subset \mathbf{R}[\mathbf{X}]$$

with respect to the lexicographic order $x_1 \prec \dots \prec x_n$ gives a rise to $n - 2$ polynomials $\sigma_3, \dots, \sigma_n$ such that $\sigma_i \in \mathbf{R}[x_1, \dots, x_{i-1}]$ and $\sigma_i(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}) = \mathbf{x}_i$, for $3 \leq i \leq n$. Hence, the triangular system formed by the σ_i 's raises polynomials $\tau_2, \dots, \tau_n \in \mathbf{R}[x_1]$ such that $\mathbf{x}_i = \tau_i(\mathbf{x}_1)$ for $i \geq 2$ and then, $\mathbf{x} \in \mathbf{R}^n$. The converse is straightforward. \square

The following lemma shows that, except at apparent singularities, the real traces of \mathcal{C} and \mathcal{C}_2 share the same connectivity properties.

Lemma 4.2. *The restriction of π_2 to $\mathcal{C}_{\mathbf{R}} - \pi_2^{-1}(\text{app}(\mathcal{C}_2))$ is a s.a. homeomorphism of inverse φ_2 , defined on $\mathcal{C}_{2,\mathbf{R}} - \text{app}(\mathcal{C}_2)$.*

Moreover, for all $\mathbf{y} \notin \mathcal{K}(\pi_1, \mathcal{C}_2)$,

$$\varphi_2(\mathbf{y}) = \left(\mathbf{y}, \frac{\rho_3(\mathbf{y})}{\partial_{x_2}\omega(\mathbf{y})}, \dots, \frac{\rho_n(\mathbf{y})}{\partial_{x_2}\omega(\mathbf{y})} \right).$$

Proof. Let $\mathbf{y} \in \mathcal{C}_{2,\mathbf{R}} - \text{app}(\mathcal{C}_2)$ then, as $\mathcal{C}_2 = \mathbf{V}(\omega)$, either $\partial_{x_2}\omega(\mathbf{y}) \neq 0$ or

$$\mathbf{y} \in \mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbf{R}}) - \text{app}(\mathcal{C}_2).$$

Since $\mathbf{y} \notin \text{app}(\mathcal{C}_2)$ then by Lemma 4.1, in the latter case $\pi_2^{-1}(\mathbf{y}) \cap \mathcal{C} \subset \mathcal{K}(\pi_1, \mathcal{C}_{\mathbf{R}})$, and by (H₆) there exists a unique $\mathbf{x} \in \mathcal{K}(\pi_1, \mathcal{C}_{\mathbf{R}}) - \pi_2^{-1}(\text{app}(\mathcal{C}_2))$ such that $\pi_2(\mathbf{x}) = \mathbf{y}$. Then, let $\varphi_2 : \mathcal{C}_{2,\mathbf{R}} - \text{app}(\mathcal{C}_2) \rightarrow \mathbf{R}^n$ be the map defined by setting $\varphi_2(\mathbf{y})$ to be the unique \mathbf{x} satisfying $\pi_2(\mathbf{x}) = \mathbf{y}$ if $\mathbf{y} \in \mathcal{K}(\pi_1, \mathcal{C}_2) - \text{app}(\mathcal{C}_2)$ and

$$\varphi_2(\mathbf{y}) = (\mathbf{y}, (\rho_3/\partial_{x_2}\omega)(\mathbf{y}), \dots, (\rho_n/\partial_{x_2}\omega)(\mathbf{y})),$$

otherwise. Remark that φ_2 is a s.a. map since its graph is a s.a. set and, by construction and (H7), if $\mathbf{y} \in \mathcal{C}_{2,\mathbf{R}} - \text{app}(\mathcal{C}_2)$ then $\varphi_2(\mathbf{y})$ is the unique element of $\mathcal{C}_{\mathbf{R}} - \pi_2^{-1}(\text{app}(\mathcal{C}_2))$ such that $\pi_2(\varphi_2(\mathbf{y})) = \mathbf{y}$.

Remark that φ_2 is continuous on $\mathcal{C}_{2,\mathbf{R}} - \mathcal{K}(\pi_1, \mathcal{C}_2)$, since $\partial_{x_2}\omega(\mathbf{y})$ does not vanish anywhere, let us show that it is continuous everywhere. Let $\mathbf{y} \in \mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbf{R}}) - \text{app}(\mathcal{C}_2)$, suppose there exists a s.a. path $\gamma : [0, 1] \rightarrow \mathcal{C}_{2,\mathbf{R}}$, such that $\gamma(0) = \mathbf{y}$ and for all $t > 0$, $\gamma(t) \in \mathcal{C}_{2,\mathbf{R}} - \mathcal{K}(\pi_1, \mathcal{C}_2)$. Such a path exists since $\mathcal{K}(\pi_1, \mathcal{C}_2)$ is finite. Consider the s.a. path $\tau : t \in (0, 1] \mapsto \varphi_2(\gamma(t)) \in \mathcal{C}_{\mathbf{R}}$. Since π_2 is a proper map by (H1), τ is bounded. Therefore, by [4, Proposition 3.21], τ can be continuously extended in $t = 0$ and by continuity, $\tau(0) \in \mathcal{C}_{\mathbf{R}}$ and $\pi_2(\tau(0)) = \pi_2(\varphi_2(\mathbf{y})) = \mathbf{y}$. Hence, by uniqueness $\tau(0) = \varphi_2(\mathbf{y})$ and, by [4, Proposition 3.6 & 3.20], φ_2 is continuous in \mathbf{y} . If no such path γ exists, then both \mathbf{y} and \mathbf{x} are isolated points in $\mathcal{C}_{2,\mathbf{R}}$ and $\mathcal{C}_{\mathbf{R}}$, respectively, so that φ_2 is trivially continuous at \mathbf{y} .

In conclusion, φ_2 is a s.a. map, defined and continuous on $\mathcal{C}_{2,\mathbf{R}} - \text{app}(\mathcal{C}_2)$, of inverse the restriction of π_2 to $\mathcal{C}_{\mathbf{R}} - \pi_2^{-1}(\text{app}(\mathcal{C}_2))$ by Lemma 4.1. Therefore, this latter restriction is a s.a. homeomorphism as stated. \square

It remains to investigate how the connectivity the real traces of \mathcal{C} and \mathcal{C}_2 are related in some neighbourhood of apparent singularities.

Recall that an (ambient) isotopy of \mathbf{R}^n is a continuous application $\mathcal{H} : \mathbf{R}^n \times [0, 1] \rightarrow \mathbf{R}^n$ such that $\mathbf{y} \in \mathbf{R}^n \mapsto \mathcal{H}(\mathbf{y}, 0)$ is the identity map of \mathbf{R}^n and for all $t \in [0, 1]$, the map $\mathbf{y} \in \mathbf{R}^n \mapsto \mathcal{H}(\mathbf{y}, t)$ is a homeomorphism. Then two subsets Y and Z of \mathbf{R}^n are isotopy equivalent if there exists an isotopy \mathcal{H} of \mathbf{R}^n such that $\mathcal{H}(Y, 1) = Z$.

Recall also that a graph \mathcal{G} is the data of a set \mathcal{V} of vertices, together with a set \mathcal{E} of edges $\{\mathbf{v}, \mathbf{v}'\}$, where $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$. For any $\mathbf{y}, \mathbf{y}' \in \mathbf{R}^2$, we will denote by $[\mathbf{y}, \mathbf{y}']$, the closed line segment $\{(1-t)\mathbf{y} + t\mathbf{y}', t \in [0, 1]\}$. Then, if $\mathcal{V} \subset \mathbf{R}^2$, we call the piecewise linear curve $\mathcal{C}_{\mathcal{G}}$ associated to \mathcal{G} the union of $[\mathbf{v}, \mathbf{v}']$ for all $\{\mathbf{v}, \mathbf{v}'\} \in \mathcal{E}$. In the following, we let $\mathcal{P}_2 = \pi_2(\mathcal{P})$.

Definition 4.3. Let $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be a graph, with $\mathcal{V}_2 \subset \mathbf{R}^2$. Then we say that \mathcal{G}_2 is a *real topology graph* of $(\mathcal{C}_2, \mathcal{P}_2)$ if

1. the planar piecewise linear curve $\mathcal{C}_{\mathcal{G}_2}$ associated to \mathcal{G}_2 , is isotopy equivalent to $\mathcal{C}_{2,\mathbf{R}}$;
2. the points of $\mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbf{R}}) \cup \mathcal{P}_{2,\mathbf{R}}$ are embedded in \mathcal{V}_2 ;
3. no two points of $\mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbf{R}})$ have adjacent vertices in \mathcal{G} .

For the rest of this section, let \mathcal{G}_2 be such a *real topology graph* of $(\mathcal{C}_2, \mathcal{P}_2)$ and $\mathcal{C}_{\mathcal{G}_2}$ the associated planar piecewise linear curve. Let \mathcal{H} be the induced isotopy and, for all $t \in [0, 1]$, let $\mathcal{H}_t : \mathbf{y} \in \mathbf{R}^2 \rightarrow \mathcal{H}(\mathbf{y}, t)$, so that $\mathcal{H}_1(\mathcal{C}_{\mathcal{G}_2}) = \mathcal{C}_{2,\mathbf{R}}$.

Consider, s.a. paths $\gamma_1, \dots, \gamma_4$ in \mathbf{R}^2 intersecting each other at a point $\mathbf{p} \in \mathbf{R}^2$, and only there (see Figure 1). Hence, the γ_i 's can be pairwise associated with respect to their unique *opposite branch* at \mathbf{p} as follows. Given an orientation of \mathbf{R}^2 , and a sufficiently small circle centered at \mathbf{p} , we can arrange the γ_i 's around

\mathbf{p} , with respect to their unique intersection with this circle (see [8, Theorem 9.3.6]). Then, the γ_i 's can be pairwise associated to the one after next in the above arrangement. One remarks that it does not depend on the chosen orientation. Up to reindexing, we say that (γ_1, γ_3) and (γ_2, γ_4) are the *unique couples of opposite branches at \mathbf{p}* .

The following lemma is a direct consequence of classical results from knots and braids theory, we refer to [10, Proposition 1.9 and 1.10] for the key arguments.

Lemma 4.4. *Let the γ_i 's as above, and an isotopy $\tilde{\mathcal{H}}$ of \mathbf{R}^2 . The curves $(\tilde{\mathcal{H}}_1(\gamma_1), \tilde{\mathcal{H}}_1(\gamma_3))$ and $(\tilde{\mathcal{H}}_1(\gamma_2), \tilde{\mathcal{H}}_1(\gamma_4))$ do not intersect each other, but at $\tilde{\mathcal{H}}_1(\mathbf{p})$, and are the unique couples of opposite branches at this point.*

This property allows us to deduce relations between edges of \mathcal{G}_2 , from relations between the associated branches of $\mathcal{C}_{2,\mathbf{R}}$.

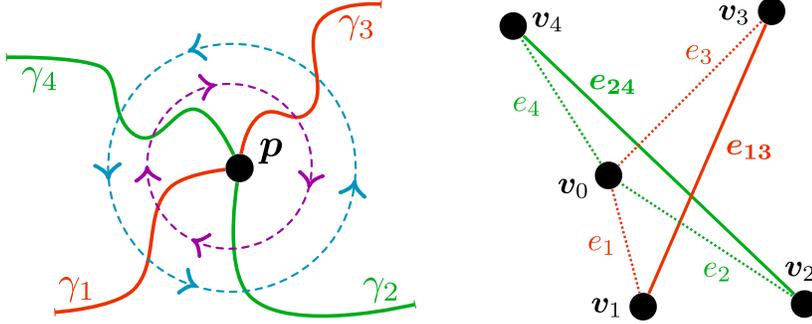


Figure 1: The left figure illustrates the construction considered in Lemma 4.4 showing two possible arrangements of the branches; the braid structure appears clearly. On the right, we illustrate the modification of \mathcal{G}_2 that NodeResolution operates at every vertex of \mathcal{V}_{app} ; in dotted and solid lines are the edges of \mathcal{G}_2 and \mathcal{G} .

Lemma 4.5. *Let $\mathbf{y} = (\alpha, \beta) \in \text{app}(\mathcal{C}_{2,\mathbf{R}})$, then there is exactly five vertices $\mathbf{v}_0, \dots, \mathbf{v}_4 \in \mathcal{V}_2$ such that $\mathcal{H}_1(\mathbf{v}_0) = \mathbf{y}$ and for $i \in \{1, \dots, 4\}$ the following holds:*

1. $\{\mathbf{v}_0, \mathbf{v}_i\} \in \mathcal{E}_2$ and $\mathcal{H}_1(\mathbf{v}_i) \notin \text{app}(\mathcal{C}_2)$;
2. if $e_i = [\mathbf{v}_0, \mathbf{v}_i]$, the e_i 's do not cross each other except at \mathbf{v}_0 ;
3. there exists unique s.a. paths τ_1, \dots, τ_4 such that for

$$\tau_i : [0, 1] \rightarrow \mathcal{C}_{\mathbf{R}}, \quad \text{and} \quad \begin{cases} \pi_2(\tau_i([0, 1])) = \mathcal{H}_1(e_i) \\ \pi_2(\tau_i(0)) = \mathbf{y} \end{cases}$$

4. assume, without loss of generality, that (e_1, e_3) and (e_2, e_4) are the two unique couples of opposite edges of \mathcal{G}_2 at \mathbf{v}_0 . Then, there exist $\mathbf{x}_1 \neq \mathbf{x}_2$ in $\pi_2^{-1}(\mathbf{y}) \cap \mathcal{C}_{\mathbf{R}}$, such that $\mathbf{x}_1 = \tau_1(0) = \tau_3(0)$ and $\mathbf{x}_2 = \tau_2(0) = \tau_4(0)$.

Proof. Let $\mathbf{v}_0 = \mathcal{H}_1^{-1}(\mathbf{y})$, since \mathbf{y} is a node, there exist exactly four distinct vertices $\mathbf{v}_1, \dots, \mathbf{v}_4 \in \mathcal{V}_2$ such that $\{\mathbf{v}_0, \mathbf{v}_i\} \in \mathcal{E}_2$, for all $i \in \{1, \dots, 4\}$. Indeed, for $i \in \{1, \dots, 4\}$, let

$$e_i : t \in [0, 1] \mapsto \mathbf{v}_0 + t(\mathbf{v}_i - \mathbf{v}_0) \in \mathbf{R}^2$$

and $\gamma_i = \mathcal{H}_1 \circ e_i$. Then the γ_i 's are the four branches of $\mathcal{C}_{2, \mathbf{R}}$ incident in \mathbf{y} . Remark that, by the third item of Definition 4.3, none of the $\mathcal{H}_1(\mathbf{v}_i)$'s lie in $\mathcal{K}(\pi_1, \mathcal{C}_{2, \mathbf{R}})$, since $\mathcal{H}_1(\mathbf{v}_0) = \mathbf{y}$ does. Besides, by the second item, the γ_i 's do not intersect $\mathcal{K}(\pi_1, \mathcal{C}_2)$, except in \mathbf{y} .

In particular, the γ_i 's do not contain points of $\text{app}(\mathcal{C}_2)$ and intersect each other only at \mathbf{y} . Hence, by Lemma 4.4, through \mathcal{H}_1 , the e_i 's intersect each other only at \mathbf{v}_0 .

Besides, for $0 < t \leq 1$ and $i \in \{1, \dots, 4\}$, let $\tau_i(t) = \varphi_2(\gamma_i(t))$, where φ_2 is defined in Lemma 4.2. It is a well defined s.a. path by the above discussion. Moreover, by Lemma 4.2, for all $0 < t \leq 1$ and $i \in \{1, \dots, 4\}$, $\tau_i(t) \in \mathcal{C}_{\mathbf{R}}$ and

$$\pi_2(\tau(t)) = \gamma(t) = \mathcal{H}_1(e_i(t)).$$

Since π_2 is a proper map by (H1), [4, Proposition 3.21] implies that, for all $i \in \{1, \dots, 4\}$, τ_i can be continuously extended in $t = 0$. Moreover, by continuity, $\pi_2(\tau_i(0)) = \mathbf{y}$.

Finally, since \mathbf{y} is a node, then there exist distinct $\theta_1 \neq \theta_2$ in \mathbf{R}^2 and $1 \leq i_1, i_2, i_3, i_4 \leq 4$ such that,

$$\theta_1 = \gamma'_{i_1}(0) = \gamma'_{i_3}(0) \quad \text{and} \quad \theta_2 = \gamma'_{i_2}(0) = \gamma'_{i_4}(0).$$

This means exactly that the branches $(\gamma_{i_1}, \gamma_{i_3})$ and $(\gamma_{i_2}, \gamma_{i_4})$ are the two couples of opposite branches of \mathcal{C}_2 at \mathbf{y} . Then, by Lemma 4.4, (e_{i_1}, e_{i_3}) and (e_{i_2}, e_{i_4}) are the two couples of opposite edges of \mathcal{G}_2 at \mathbf{y} . For the sake of clarity assume, without loss of generality that $i_k = k$ for all $k \in \{1, \dots, 4\}$. By continuity, there exist $\vartheta_1 \neq \vartheta_2$ in \mathbf{R}^n such that

$$\vartheta_1 = \tau'_1(0) = \tau'_3(0) \quad \text{and} \quad \vartheta_2 = \tau'_2(0) = \tau'_4(0),$$

and $\tau_i(0) \in \pi_2^{-1}(\mathbf{y}) \cap \mathcal{C}_{\mathbf{R}}$, for $i \in \{1, \dots, 4\}$. But, since $\mathbf{y} \in \text{app}(\mathcal{C}_2)$, $\pi_2^{-1}(\mathbf{y}) \cap \mathcal{C}$ contains two distinct non-singular points, of distinct tangent lines, by (H4). Since the $\tau'_i(0)$'s are tangent lines of \mathcal{C} , necessarily, $\tau_1(0)$ and $\tau_3(0)$ are equal to one of these points, while $\tau_2(0)$ and $\tau_4(0)$ are equal to the other one (indeed, if either more than two branches arrive at the same point or the tangent lines are different, then it is singular). \square

If $\mathcal{V}_{\text{app}} = \mathcal{H}_1^{-1}(\text{app}(\mathcal{C}_2)) \subset \mathcal{V}_2$, is the subset of apparent nodes, then Lemma 4.5 provides a procedure to compute a new graph \mathcal{G} , from which we can deduce connectivity queries on \mathcal{C} .

Definition 4.6. Let NodeResolution be the procedure that takes as input \mathcal{G}_2 and \mathcal{V}_{app} as above and outputs the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows, where we keep

the notations of Lemma 4.5.

For all $\mathbf{v} \in \mathcal{V}_{\text{app}}$, compute the adjacent vertices $\mathbf{v}_1, \dots, \mathbf{v}_4$ of \mathbf{v} , indexed such that (e_1, e_3) and (e_2, e_4) are opposite edges.

Then, remove \mathbf{v} from \mathcal{V}_2 and replace the four edges $(\{\mathbf{v}, \mathbf{v}_k\})_{1 \leq k \leq 4}$ by the two edges $(\{\mathbf{v}_j, \mathbf{v}_{j+2}\})_{k=1,2}$, as depicted in Figure 1.

We say that $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$ are connected in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if, there exists an ordered sequence $(\mathbf{v}_0, \dots, \mathbf{v}_{N+1})$ of vertices in \mathcal{V} such that $\mathbf{v}_0 = \mathbf{v}$, $\mathbf{v}_{N+1} = \mathbf{v}'$ and $\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \in \mathcal{E}$, for all $0 \leq i \leq N$.

Proposition 4.7. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph output by NodeResolution, on input \mathcal{G}_2 and \mathcal{V}_{app} . Then,*

1. $\pi_2(\mathcal{P}_{\mathbf{R}}) \subset \mathcal{H}_1(\mathcal{V})$;
2. $\mathbf{y}, \mathbf{y}' \in \mathcal{P}_{\mathbf{R}}$ are s.a. connected in $\mathcal{C}_{\mathbf{R}}$ if, and only if, $\mathcal{H}_1^{-1}(\pi_2(\mathbf{y}))$ and $\mathcal{H}_1^{-1}(\pi_2(\mathbf{y}'))$ are connected in \mathcal{G} .

Proof. Since \mathcal{C} satisfies (H), (H₅) and (H₆) implies that $\pi_2(\mathcal{P}) \cap \mathcal{H}_1(\mathcal{V}_{\text{app}}) = \emptyset$. Then $\mathcal{P}_{2,\mathbf{R}} = \pi_2(\mathcal{P}_{\mathbf{R}})$ as π_2 is injective on \mathcal{P} , and, by definition, $\mathcal{P}_{2,\mathbf{R}} \subset \mathcal{V}$.

We now deal with the second statement. Let $\mathbf{y}, \mathbf{y}' \in \mathcal{P}_{\mathbf{R}}$ and

$$\mathbf{v} = \mathcal{H}_1^{-1}(\pi_2(\mathbf{x})) \quad \text{and} \quad \mathbf{v}' = \mathcal{H}_1^{-1}(\pi_2(\mathbf{x}'))$$

in \mathcal{V} . Assume first that \mathbf{v} and \mathbf{v}' are connected in \mathcal{G} , then there exist $\mathbf{v}_1, \dots, \mathbf{v}_N \in \mathcal{V}$ such that, if $\mathbf{v}_0 = \mathbf{v}$ and $\mathbf{v}_{N+1} = \mathbf{v}'$, then $\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \in \mathcal{E}$ and $\mathcal{H}_1(\mathbf{v}_i) \notin \text{app}(\mathcal{C}_2)$ for every $i \in \{0, \dots, N+1\}$. Fix $i \in \{0, \dots, N\}$. By Lemma 4.2, $\mathbf{x}_i = \varphi_2(\mathcal{H}_1(\mathbf{v}_i))$ and $\mathbf{x}_{i+1} = \varphi_2(\mathcal{H}_1(\mathbf{v}_{i+1}))$ are well defined in $\mathcal{C}_{\mathbf{R}}$

If $\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \in \mathcal{E}_2$ then,

$$\mathcal{H}_1([\mathbf{v}_i, \mathbf{v}_{i+1}]) \cap \text{app}(\mathcal{C}_2) = \emptyset$$

and, by Lemma 4.2, \mathbf{x}_i and \mathbf{x}_{i+1} are s.a. connected in $\mathcal{C}_{\mathbf{R}}$ through φ_2 . Otherwise, $\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \notin \mathcal{E}_2$, and, by construction of \mathcal{G} , there exists $\mathbf{w} \in \mathcal{V}_{\text{app}}$ such that $\{\mathbf{v}_i, \mathbf{w}\}$ and $\{\mathbf{w}, \mathbf{v}_{i+1}\}$ are in \mathcal{E}_2 . However, since $\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \in \mathcal{E}$, then, according to the construction of \mathcal{G} , $e_i = [\mathbf{w}, \mathbf{v}_i]$ and $e_{i+1} = [\mathbf{w}, \mathbf{v}_{i+1}]$ are opposite edges of \mathcal{G}_2 at \mathbf{w} . Hence, by items (2) and (3) of Lemma 4.5, there exists a s.a. path $\tau : [-1, 1] \rightarrow \mathcal{C}_{\mathbf{R}}$ connecting \mathbf{x}_i to \mathbf{x}_{i+1} . All in all, by transitivity, $\mathbf{x}_0 = \mathbf{x}$ and $\mathbf{x}_{N+1} = \mathbf{x}'$ are s.a. connected in $\mathcal{C}_{\mathbf{R}}$, and we are done.

Conversely, suppose that \mathbf{x} and \mathbf{x}' are s.a. connected in $\mathcal{C}_{\mathbf{R}}$ and let $\tau : [0, 1] \rightarrow \mathcal{C}_{\mathbf{R}}$ be a s.a. path such that $\tau(0) = \mathbf{x}$ and $\tau(1) = \mathbf{x}'$. Let $\gamma = \pi_2 \circ \tau$, and

$$\{t_1, \dots, t_N\} = \gamma^{-1}(\mathcal{H}_1(\mathcal{V}_2)) \subset (0, 1)$$

such that $t_1 < \dots < t_N$. Let $t_0 = 0$ and $t_{N+1} = 1$ and for all $i \in \{0, \dots, N+1\}$, let $\mathbf{v}_i = \mathcal{H}_1^{-1}(\gamma(t_i)) \in \mathcal{V}_2$. By assumption, $\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \in \mathcal{E}_2$ for all $i \in \{0, \dots, N\}$. Let us prove by induction that for all $i \in \{0, \dots, N+1\}$, either $\mathbf{v}_i \in \mathcal{V}_{\text{app}}$ or \mathbf{v}_i is connected to \mathbf{v}_0 in \mathcal{G} . If $i = 0$, there is nothing to prove, so let $1 \leq i \leq N$ and suppose that the statement holds for all $0 \leq j \leq i$.

Assume that $\mathbf{v}_{i+1} \notin \mathcal{V}_{\text{app}}$. Then, either $\mathbf{v}_i \notin \mathcal{V}_{\text{app}}$, and, by induction hypothesis, \mathbf{v}_{i+1} and \mathbf{v}_0 are connected, through \mathbf{v}_i , in \mathcal{G} . Otherwise, $\mathbf{v}_i \in \mathcal{V}_{\text{app}}$ and, by Lemma 4.5, there exist exactly four distinct $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 \in \mathcal{V} - \mathcal{V}_{\text{app}}$ such that $\{\mathbf{v}_i, \mathbf{w}_j\} \in \mathcal{E}_2$, where $1 \leq j \leq 4$. Assume, without loss of generality, that $\mathbf{v}_{i+1} = \mathbf{w}_1$. Then, there exists $j_1 \in \{2, 3, 4\}$ such that $\mathbf{v}_{i-1} = \mathbf{w}_{j_1}$. Using the notation of Lemma 4.5, assume, without loss of generality, that $e_3 = [\mathbf{v}_i, \mathbf{w}_3]$ is the opposite branch of $e_1 = [\mathbf{v}_i, \mathbf{w}_1]$ in \mathcal{G}_2 at \mathbf{v}_i . Then, by items (2) and (3) of Lemma 4.5, we have necessarily $j_0 = 3$, since $\tau([t_{i-1}, t_i])$ is connected to $\tau([t_i, t_{i+1}])$. By construction of \mathcal{G} , $\mathbf{w}_1 = \mathbf{v}_{i+1}$ is connected to $\mathbf{w}_3 = \mathbf{v}_{i-1}$ in \mathcal{G} , so that, applying the induction hypothesis, \mathbf{v}_{i+1} is connected to \mathbf{v}_0 , through \mathbf{v}_{i-1} . Hence, by induction, $\mathbf{v} = \mathbf{v}_{N+1}$ and $\mathbf{v}' = \mathbf{v}_0$ are connected in \mathcal{G} , which proves the converse. \square

5 Algorithm

Now, we design an algorithm for solving connectivity queries over real algebraic curves, whose different steps correspond sequentially, except for one, to the different sections of this document.

On input a sequence of polynomials defining an algebraic curve, the first step is to perform a linear change of variable, generic enough to ensure assumption (H), and to compute a one-dimensional parametrization encoding it. Answering connectivity queries on the sheared curve is equivalent to do so on the original curve. By [34, Theorem 6.18] (or [46, Proposition 6.3]) computing such a parametrization has complexity cubic in the degree of the curve, that is bounded by the overall complexity of our algorithm. Besides, according to [46, Section J.], changing variables in zero and one-dimensional parametrizations has similar complexity. Hence, for the sake of clarity, we omit these two steps.

Following the state of the art of curve topology computation, we consider polynomials with integer coefficients, so that $\mathbf{Q} = \mathbb{Q}$, $\mathbf{R} = \mathbb{R}$ and $\mathbf{C} = \mathbb{C}$. Moreover, we denote by \preceq_1 the preorder on points of \mathbb{R}^n w.r.t. the first coordinate, when they are distinct.

5.1 Subroutines

We assume that $\mathcal{R} = (\omega, \rho_3, \dots, \rho_n)$, has coefficients in \mathbb{Z} , and magnitude (δ, τ) . We consider a zero-dimensional parametrization $\mathcal{P} = (\lambda, \vartheta_2, \dots, \vartheta_n)$, with coefficients in \mathbb{Z} , of magnitude (μ, κ) encoding \mathcal{P} . Remark that $\mathcal{R}_2 = (\omega, \rho_2)$ and $\mathcal{P}_2 = (\lambda, \vartheta_2)$ are parametrizations encoding \mathcal{C}_2 and \mathcal{P}_2 , respectively. We denote further $R = \text{Res}_{x_2}(\omega, \partial_{x_2}\omega)$. Since, by (H7), ω is monic in x_2 , its roots are exactly the abscissas of $\mathcal{K}(\pi_1, \mathcal{C}_2)$.

According (H5), points of $\text{app}(\mathcal{C}_2)$ can be identified by their abscissa. Following Proposition 3.2, it can be reduced to gcd computations.

Proposition 5.1. *There exists an algorithm `ApparentSingularities` taking as input \mathcal{R} , as above, and computing a square-free polynomial $q_{\text{app}} \in \mathbb{Z}[x_1]$, of mag-*

itude $(\delta^2, \tilde{O}(\delta^2 + \delta\tau))$ such that,

$$\text{app}(\mathcal{C}_2) = \{(\alpha, \beta) \in \mathcal{K}(\pi_1, \mathcal{C}_2) \mid q_{\text{app}}(\alpha) = 0\},$$

using $\tilde{O}(\delta^6 + \delta^5\tau)$ bit operations.

Proof. Let $(\alpha, \beta) \in \mathcal{K}(\pi_1, \mathcal{C}_2)$. According to [31, Theorem 3.2.(ii)], since \mathcal{C} satisfies (H), (α, β) is a node if, and only if, α is a double root of R , that is if, and only if, α is a root of

$$q = \gcd(R^*, R') / \gcd(R^*, R', R''),$$

where R^* is the square-free part of R . Moreover, let $(\text{sr}_1, \text{sr}_{1,0})$ be the first subresultant sequence of $(\omega, \partial_{x_2}\omega)$. By [35, Theorem 3.1], if $q(\alpha) = 0$ then, $\text{sr}_1(\alpha) \neq 0$, and

$$\text{sr}_1(\alpha) \cdot \beta = -\text{sr}_{1,0}(\alpha).$$

Let $A(x_1, x_2)$ be the polynomial on the left hand side of (1) in Proposition 3.2, and let u be a new indeterminate. Let $\tilde{A}(x_1, x_2, u)$ be the homogenization of A in x_2 , and $B = \tilde{A}(x_1, -\text{sr}_{1,0}, \text{sr}_1)$. Then, according to Proposition 3.2, the square-free polynomial

$$q_{\text{app}} = q / \gcd(q, B)$$

vanishes at α if, and only if, $(\alpha, \beta) \in \text{app}(\mathcal{C}_2)$, as required.

We now deal with the quantitative bounds. By [44, Lemma 14], R, R^*, sr_1 and $\text{sr}_{1,0}$ have magnitude $(\delta^2, \tilde{O}(\delta^2 + \delta\tau))$ and can be computed using $\tilde{O}(\delta^6 + \delta^5\tau)$ bit operations. Hence, by [49, Corollary 11.14] and [44, Lemma 12], computing $\gcd(R^*, R')$, $\gcd(R^*, R', R'')$ and then q can be done using $\tilde{O}(\delta^4 + \delta^3\tau)$ bit operations. Moreover, by [44, Lemma 11], q has magnitude $(\delta^2, \tilde{O}(\delta^2 + \delta\tau))$.

Besides, \tilde{A} has magnitude $(O(\delta), \tilde{O}(\tau))$, so that B has magnitude

$$\left(\tilde{O}(\delta^3), \tilde{O}(\delta^3 + \delta^2\tau) \right).$$

Hence, by [49, Corollary 11.14] computing, $\gcd(q, B)$ requires $\tilde{O}(\delta^6 + \delta^5\tau)$ bit operations. From this, computing q_{app} costs $\tilde{O}(\delta^4 + \delta^3\tau)$ bit operations, by [23, Proposition 2.15]. Finally, q_{app} has magnitude $(\delta^2, \tilde{O}(\delta^2 + \delta\tau))$, by [44, Lemma 11]. \square

Suppose now that the polynomial q_{app} , from Proposition 5.1, has been computed. We can compute a real topology graph of $(\mathcal{C}_2, \mathcal{P}_2)$, while identifying the vertices corresponding to $\text{app}(\mathcal{C}_2)$ and \mathcal{P}_2 .

Proposition 5.2. *There exists an algorithm `Topo2D` taking as input $\mathcal{R}, \mathcal{P}_2$ and q_{app} as above and computing $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a real topology graph of $(\mathcal{C}_2, \mathcal{P}_2)$, of size at most $O(\delta^3 + \delta\mu)$, using*

$$\tilde{O}(\delta^6 + \delta^5\tau + \mu^6 + \mu^5\kappa)$$

bit operations. It also outputs ordered sequences \mathcal{V}_{app} and $\mathcal{V}_{\mathcal{P}}$, of elements of \mathcal{V} , that are in one-to-one correspondence with respectively the points of $\text{app}(\mathcal{C}_{2,\mathbb{R}})$ and $\mathcal{P}_{2,\mathbb{R}}$, ordered with respect to \preceq_1 .

Proof. According to [42, Theorem 14], and more recently [23, Theorem 1.1], there exists an algorithm that computes a planar graph \mathcal{G} , whose associated piecewise-linear curve $\mathcal{C}_{\mathcal{G}}$, is isotopy equivalent to $\mathcal{C}_{2,\mathbb{R}}$, using $\tilde{O}(\delta^6 + \delta^5\tau)$ bit operations. We claim that, under slight modifications, these algorithms can compute the claimed output of Topo2D, within the same complexity bounds, w.r.t. the size of inputs. For the sake of clarity, we only consider the algorithm of [23], that we roughly describe hereafter.

Let $\alpha_1 < \dots < \alpha_N$ be the abscissas of the points of $\mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbb{R}})$, they are all distinct by (H₅). It first computes disjoint isolating intervals for each α_i by [23, Proposition 2.24]. Then, it isolates the ordinates of the points above each α_i , using [23, Proposition 3.13]. This process gives rise to isolating boxes, which stand for vertices in the final graph. The algorithm eventually connects these boxes to separating vertices above regular values in the intervals (α_j, α_{j+1}) . The latter is done by counting the number of incoming left and right branches in each box. For points of $\mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbb{R}})$, it is tackled by [23, Section 4.2-4], while for others it is straightforward: there is exactly one incoming branch from each side.

The above process computes a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, such that $\mathcal{C}_{\mathcal{G}}$ is isotopy equivalent to $\mathcal{C}_{2,\mathbb{R}}$. Remark that \mathcal{V} contains a subset $\mathcal{V}_{\mathcal{K}}$ of vertices associated to the unique point of $\mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbb{R}})$ above the α_i 's, all separated by vertices associated to regular points. Moreover, by Proposition 5.1, \mathcal{V}_{app} is exactly the subset of $\mathcal{V}_{\mathcal{K}}$, associated to the α_i 's where q_{app} vanishes. Then, according to [23, Proposition 2.24] and Proposition 5.1, one can compute disjoint isolating intervals of the roots of R and q_{app} and identify all common roots, using

$$\tilde{O}(\delta^6 + \delta^5\tau)$$

bit operations. This gives \mathcal{V}_{app} .

Hence, it remains to show that introducing vertices for control points $\mathcal{P}_{2,\mathbb{R}}$ (together with those above and below) can be done in the claimed bound. First, recall that $\mathcal{D} = (\lambda, \vartheta_2)$ encode \mathcal{P}_2 . According to [23, Proposition 2.24] again, we can compute disjoint isolating intervals for all distinct (by (H₅)) real roots of λ and R , using at most

$$\tilde{O}(\delta^6 + \delta^5\tau + \mu^6 + \mu^5\kappa)$$

bit operations. Next, let $g(x_1, x_2) = \lambda' \cdot x_2 - \vartheta_2$, it is a bivariate polynomial with magnitude (μ, κ) . Then, according to [23, Proposition 3.14], for each root β of λ , we can compute isolating intervals for all roots \mathbf{x}_2 of $(\omega \cdot g)(\beta, \mathbf{x}_2)$, and identify the unique common roots, within the same complexity bound. This gives $\mathcal{V}_{\mathcal{P}}$. Moreover, since $\mathcal{P} \cap \mathcal{K}(\pi_1, \mathcal{C}_{2,\mathbb{R}}) = \emptyset$, as seen above, the connection step for the introduced vertices is straightforward, and does not affect the complexity bound.

Finally, since we consider at most $\delta^2 + \mu$ fibers, each of them containing at most δ points then, taking in account the regular separating fibers, we get at most $O(\delta^3 + \delta\mu)$ vertices and edges. \square

5.2 The algorithm

We now state our main algorithm, namely Algorithm 1. Let `IndexConnectedComponents` be an algorithm taking as input a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and an ordered sequence $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_N)$ of vertices of \mathcal{G} . It outputs a partition I_1, \dots, I_s of $\{1, \dots, N\}$, grouping the indices of the \mathbf{v}_i 's lying in the same connected components of \mathcal{G} . By [19, Section 22.2], this can be done with bit complexity linear in the size of \mathcal{G} .

Algorithm 1 `ConnectCurve`

Input: $\mathcal{R} = (\omega, \rho_3, \dots, \rho_n) \subset \mathbb{Z}[x_1, x_2]$ encoding an algebraic curve $\mathcal{C} \subset \mathbb{C}^n$ and $\mathcal{P} = (\lambda, \vartheta_2, \dots, \vartheta_n) \subset \mathbb{Z}[x_1]$ encoding points $\mathbf{p}_1 \preceq_1 \dots \preceq_1 \mathbf{p}_\mu$ of $\mathcal{C}_{\mathbb{R}}$, such that $(\mathcal{C}, \mathcal{P})$ satisfies (H).

Output: a partition of $\{1, \dots, \mu\}$ grouping the indices of the \mathbf{p}_i 's lying in the same s.a. connected component of $\mathcal{C}_{\mathbb{R}}$.

- 1: $\mathcal{P}_2 \leftarrow (\lambda, \vartheta_2)$;
 - 2: $q_{\text{app}} \leftarrow \text{ApparentSingularities}(\mathcal{R})$
 - 3: $[\mathcal{G}_2, \mathcal{V}_{\text{app}}, \mathcal{V}_{\mathcal{P}}] \leftarrow \text{Topo2D}(\mathcal{R}, \mathcal{P}_2, q_{\text{app}})$;
 - 4: $\mathcal{G} \leftarrow \text{NodeResolution}(\mathcal{G}_2, \mathcal{V}_{\text{app}})$;
 - 5: **return** `IndexConnectedComponents` $(\mathcal{V}_{\mathcal{P}}, \mathcal{G})$;
-

Then, correction and complexity estimate of Algorithm 1 are direct consequences of Proposition 5.1, Proposition 5.2 and Proposition 4.7. In particular, this proves Theorem 1.1.

We conclude by mentioning two noticeable usages of Algorithm 1. Remark that, the number of connected components of the graph \mathcal{G} computed equals the number of s.a. connected components of $\mathcal{C}_{\mathbb{R}}$. Besides, for the case of the union of several curves, one can apply Algorithm 1 to each curve, adding, to the query points, the pairwise common intersection points. Then, one concludes by merging the output subsets having one point in common.

References

- [1] L. Alberti, B. Mourrain, and J. Wintz. Topology and arrangement computation of semi-algebraic planar curves. *Computer Aided Geometric Design*, 25(8):631–651, 2008.
- [2] J. G. Alcázar and J. R. Sendra. Computation of the topology of real algebraic space curves. *Journal of Symbolic Computation*, 39(6):719–744, 2005.
- [3] S. Basu, R. Pollack, and M.-F. Roy. Computing roadmaps of semi-algebraic sets on a variety. *Journal of the American Mathematical Society*, 13(1):55–82, 1999.

- [4] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry*. 10. Springer, Berlin, Heidelberg, 3 edition, 2016.
- [5] S. Basu and M.-F. Roy. Divide and conquer roadmap for algebraic sets. *Discrete & Computational Geometry*, 52(2):278–343, 2014.
- [6] S. Basu, M.-F. Roy, M. Safey El Din, and É. Schost. A baby step–giant step roadmap algorithm for general algebraic sets. *Foundations of Computational Mathematics*, 14(6):1117–1172, 2014.
- [7] E. Berberich, P. Emeliyanenko, A. Kobel, and M. Sagraloff. Exact symbolic–numeric computation of planar algebraic curves. *Theoretical Computer Science*, 491:1–32, 2013.
- [8] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, volume 36. Springer-Verlag Berlin Heidelberg, 1998.
- [9] D. Bouziane and M. El Kahoui. Computation of the dual of a plane projective curve. *Journal of Symbolic Computation*, 34(2):105–117, 2002.
- [10] G. Burde and H. Zieschang. *Knots*. De Gruyter, Berlin, New York, 2nd revised and extended 2003 edition, 2002.
- [11] M. Burr, S. W. Choi, B. Galehouse, and C. K. Yap. Complete subdivision algorithms, ii: Isotopic meshing of singular algebraic curves. In *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*, pages 87–94, 2008.
- [12] J. Canny. *The complexity of robot motion planning*. MIT press, 1988.
- [13] J. Canny. Computing roadmaps of general semi-algebraic sets. *The Computer Journal*, 36(5):504–514, 1993.
- [14] J. Capco, M. Din, Safey El Din, and J. Schicho. Robots, computer algebra and eight connected components. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, pages 62–69, 2020.
- [15] J. Capco, M. Safey El Din, and J. Schicho. Positive dimensional parametric polynomial systems, connectivity queries and applications in robotics. *Journal of Symbolic Computation*, 115:320–345, 2023.
- [16] D. Chablat, R. Prébet, M. Safey El Din, D. H. Salunkhe, and P. Wenger. Deciding cuspidality of manipulators through computer algebra and algorithms in real algebraic geometry. In *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*, pages 439–448, 2022.
- [17] J. Cheng, S. Lazard, L. Peñaranda, M. Pouget, F. Rouillier, and E. Tsigaridas. On the topology of real algebraic plane curves. *Mathematics in Computer Science*, 4:113–137, 2010.

- [18] J.-S. Cheng, K. Jin, and D. Lazard. Certified rational parametric approximation of real algebraic space curves with local generic position method. *Journal of Symbolic Computation*, 58:18–40, 2013.
- [19] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to algorithms*. MIT press, third edition, 2009.
- [20] D. A. Cox, J. Little, and D. O’Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Cham, 4 edition, 2015.
- [21] D. N. Daouda, B. Mourrain, and O. Ruatta. On the computation of the topology of a non-reduced implicit space curve. In *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*, pages 47–54, 2008.
- [22] D. N. Diatta. *Calcul effectif de la topologie de courbes et surfaces algébriques réelles*. PhD thesis, Université de Limoges, 2009.
- [23] D. N. Diatta, S. Diatta, F. Rouillier, M.-F. Roy, and M. Sagraloff. Bounds for polynomials on algebraic numbers and application to curve topology. *Discrete & Computational Geometry*, 67(3):631–697, 2022.
- [24] D. N. Diatta, B. Mourrain, and O. Ruatta. On the isotopic meshing of an algebraic implicit surface. *Journal of Symbolic Computation*, 47(8):903–925, 2012.
- [25] D. N. Diatta, F. Rouillier, and M.-F. Roy. On the computation of the topology of plane curves. In *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, pages 130–137, 2014.
- [26] D. I. Diochnos, I. Z. Emiris, and E. P. Tsigaridas. On the complexity of real solving bivariate systems. In *Proceedings of the 2007 International Symposium on Symbolic and Algebraic Computation*, ISSAC ’07, page 127134, New York, NY, USA, 2007. Association for Computing Machinery.
- [27] D. I. Diochnos, I. Z. Emiris, and E. P. Tsigaridas. On the asymptotic and practical complexity of solving bivariate systems over the reals. *Journal of Symbolic Computation*, 44(7):818–835, 2009. International Symposium on Symbolic and Algebraic Computation.
- [28] C. Durvy and G. Lecerf. A concise proof of the kronecker polynomial system solver from scratch. *Expositiones Mathematicae*, 26(2):101–139, 2008.
- [29] A. Eigenwillig, M. Kerber, and N. Wolpert. Fast and exact geometric analysis of real algebraic plane curves. In *Proceedings of the 2007 international symposium on Symbolic and algebraic computation*, pages 151–158, 2007.

- [30] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150. Springer-Verlag, New York, 1995.
- [31] M. El Kahoui. Topology of real algebraic space curves. *Journal of Symbolic Computation*, 43(4):235–258, 2008.
- [32] E. Fortuna, P. Gianni, and B. Trager. Generators of the ideal of an algebraic space curve. *Journal of Symbolic Computation*, 44(9):1234–1254, 2009. Effective Methods in Algebraic Geometry.
- [33] G. Gattellier, A. Labrouzy, B. Mourrain, and J.-P. T  court. Computing the topology of three-dimensional algebraic curves. *Computational methods for algebraic spline surfaces*, pages 27–43, 2005.
- [34] N. Gim  nez and G. Matera. On the bit complexity of polynomial system solving. *Journal of Complexity*, 51:20–67, 2019.
- [35] L. Gonz  lez-Vega and M. El Kahoui. An improved upper complexity bound for the topology computation of a real algebraic plane curve. *Journal of Complexity*, 12(4):527–544, 1996.
- [36] R. Hartshorne. Graduate texts in mathematics. *Algebraic Geometry*, 52, 1977.
- [37] A. Holme and J. Roberts. Pinch-points and multiple locus of generic projections of singular varieties. *Advances in Mathematics*, 33(3):212–256, 1979.
- [38] M. N. Islam and A. Poteaux. Connectivity queries on curves in \mathbb{R}^n . *ACM Communications in Computer Algebra*, 45(1/2):117–118, 2011.
- [39] K. Jin and J. Cheng. On the complexity of computing the topology of real algebraic space curves. *Journal of Systems Science and Complexity*, 34:809–826, 2021.
- [40] J. Y. Kaminski, A. Kanel-Belov, and M. Teicher. Trisecant lemma for non equidimensional varieties. *Journal of Mathematical Sciences*, 149(2):1087–1097, 2008.
- [41] M. Kerber and M. Sagraloff. A worst-case bound for topology computation of algebraic curves. *Journal of Symbolic Computation*, 47(3):239–258, 2012.
- [42] A. Kobel and M. Sagraloff. On the complexity of computing with planar algebraic curves. *Journal of Complexity*, 31(2):206–236, 2015.
- [43] C. Liang, B. Mourrain, and J.-P. Pavone. Subdivision methods for the topology of 2d and 3d implicit curves. *Geometric modeling and algebraic geometry*, pages 199–214, 2008.
- [44] K. Mehlhorn, M. Sagraloff, and P. Wang. From approximate factorization to root isolation with application to cylindrical algebraic decomposition. *Journal of Symbolic Computation*, 66:34–69, 2015.

- [45] D. Mumford. *Algebraic geometry I: complex projective varieties*. Springer Science & Business Media, 1995.
- [46] M. Safey El Din and É. Schost. A nearly optimal algorithm for deciding connectivity queries in smooth and bounded real algebraic sets. *Journal of the ACM (JACM)*, 63(6):1–37, 2017.
- [47] R. Seidel and N. Wolpert. On the exact computation of the topology of real algebraic curves. In *Proceedings of the twenty-first annual symposium on Computational geometry*, pages 107–115, 2005.
- [48] I. R. Shafarevich. *Basic algebraic geometry*, volume 1. Springer, 2013.
- [49] J. von zur Gathen and J. Gerhard. *Modern Computer Algebra*. Cambridge University Press, 3 edition, 2013.