HAL open science

A VARIATIONAL METHOD FOR FUNCTIONALS DEPENDING ON EIGENVALUES

Romain Petrides

To cite this version:

Romain Petrides. A VARIATIONAL METHOD FOR FUNCTIONALS DEPENDING ON EIGENVALUES. 2022. hal-04000377

HAL Id: hal-04000377

https://hal.science/hal-04000377v1

Preprint submitted on 22 Feb 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:2211.15632v1 [math.AP] 28 Nov 2022

A VARIATIONAL METHOD FOR FUNCTIONALS DEPENDING ON EIGENVALUES

ROMAIN PETRIDES

Abstract. We perform a systematic variational method for functionals depending on eigenvalues of Riemannian manifolds. It is based on a new concept of Palais Smale sequences that can be constructed thanks to a generalization of classical min-max methods on $\mathcal{C}^1$ functionals to locally-Lipschitz functionals. We prove convergence results on these Palais-Smale sequences emerging from combinations of Laplace eigenvalues or combinations of Steklov eigenvalues in dimension 2.

Optimization of eigenvalues of operators (Laplacian with Dirichlet or Neumann boundary conditions, Dirichlet-to-Neumann operator, bi-laplacian, magnetic Laplacian etc) is a common field of spectral geometry. We consider the eigenvalues as functionals depending on the shape and topology of the domain, on the operator, and/or on the geometric structure (Riemannian metrics, CR structure, sub-Riemannian metrics, etc). One old and celebrated problem was independently solved by Faber [Fab23] and Krahn [Kra25] in 1923: the domains minimizing the first Laplace eigenvalue with Dirichlet boundary conditions among domains of same volume in $\mathbb{R}^n$ are Euclidean balls. This problem is very similar to the classical problem of isoperimetry, and the proof of this result uses the isoperimetric inequality, so that even when the perimeter is not involved in the renormalization (by a prescribed area/perimeter/diameter or Cheeger constant etc) of an eigenvalue functional, shape optimization on it is often called an isoperimetric problem on the eigenvalue.

We can distinguish two main families of optimization of eigenvalues. In the first one, the ambiant geometry is prescribed (for instance, the Euclidean space $\mathbb{R}^n$, sphere, hyperbolic space, etc) and there is an optimization with respect to the shape and topology of a domain in this ambiant space. Emblematic results are the Faber-Krahn inequality [Fab23][Kra25] and the Szegö-Weinberger [Sze54][Wei56] inequality. In the second one, the ambiant topology is prescribed (on a fixed manifold) but the optimization holds with respect to the metric on the manifold, or potentials involved in the eigenvalue operator. An emblematic result is Hersch inequality [Her70]: the round sphere is the maximizer of the first Laplace eigenvalue among metrics of same area on the 2-sphere. In both problems, we look for bounds on eigenvalues, optimal inequalities and critical domains/metrics/potentials realizing these bounds.

The current paper is devoted to the second family of problems. In principle, the bigger the space of variations is, the richer the critical points of the functional are. For instance, *critical metrics* for combinations of Laplace eigenvalues over Riemannian metrics with prescribed volume are associated to minimal surfaces into ellipsoids (see [Pet21]), while *critical metrics* for Steklov eigenvalues with prescribed perimeter are associated to free boundary minimal surfaces into ellipsoids (see [Pet22]). If only one eigenvalue appears in the functional, the target ellipsoids are spheres/balls as was primarily noticed by Nadirashvili [Nad96] for Laplace eigenvalues and Fraser and Schoen for Steklov eigenvalues

[FS13][FS16]. This gives an elegant connexion with the theory of minimal surfaces. If we look for critical metrics with respect to variations in a conformal class, we only obtain harmonic maps instead of minimal immersions [ESI03][ESI08][FS13][Pet21][Pet22]. Other examples of critical metrics will be given in [PT22] thanks to a unified approach based on computations of subdifferentials (see e.g. [Cla13] and discussions below). Noticing that the harmonic maps enjoy a regularity theory (see e.g [Hel96][Riv08]), we can start a long story of investigations for variational aspects of eigenvalue functionals.

In the past decades, many variational methods have been proposed since the seminal works by Nadirashvili [Nad96] for the maximization of the first Laplace eigenvalues on tori and Fraser and Schoen [FS16] for the maximization of the first Steklov eigenvalues on surfaces with boundary of genus 0. We briefly explain the idea with the example of maximization of one eigenvalue in a conformal class $[g] = \{e^{2u}g; u \in \mathcal{C}^\infty(M)\}$

$$\Lambda_k(M, [g]) = \sup_{\tilde{g}\in[g]} \bar{\lambda}_k(\tilde{g})$$

where $\bar{\lambda}_k$ is a renormalized eigenvalue. Notice that conformal classes are convenient not only because the space of variation is a space of functions, but also because there are upper bounds on eigenvalues in this space [Kor93][Has11]. The main idea was to build a specific maximizing sequence of conformal factors that emerge from a *regularized* variational problem.

- In [Nad96], (Laplacian, dimension 2) the author maximizes the first eigenvalue $\bar{\lambda}_1$ on the smaller admissible space E_N of conformal factors $f \in \mathcal{C}^\infty(M)$ such that $0 \leq f \leq N$ for $N \in \mathbb{N}$, giving a maximizing sequence as $N \to +\infty$ of L^∞ factors $f_N \in E_N$ for $\Lambda_1(\Sigma, [g]) = \sup_{\tilde{g}\in[g]} \lambda_1(\tilde{g})$.
- In [FS16], [Pet14a], (Laplacian, dimension 2) the authors maximize a relaxed functional $f \mapsto \bar{\lambda}_1(K_\varepsilon(f)g)$, where $K_\varepsilon(f)$ is the solution at time $\varepsilon > 0$ of the heat equation with respect to g at time $\varepsilon > 0$ with initial data f, obtaining a maximizing sequence $K_\varepsilon(\nu_\varepsilon)$ of smooth positive conformal factors as $\varepsilon \to 0$, for some maximal probability measure ν_ε of the relaxed functional $\nu \mapsto \bar{\lambda}_1(K_\varepsilon(\nu)g)$.
- In [GP22], (Conformal Laplacian, dimension $n \geq 3$) the authors proposed to modify both the functional and the space of admissible variations.

Whatever the choice, the main difficulty is to obtain convergence of this maximizing sequence of conformal factors to a regular conformal factor. Since these maximizing sequences come from the maximization of a regularized variational problem, we obtain Euler-Lagrange equations expected to bring regularity estimates on the sequence, in order to pass to the limit. Of course, these expectations are only possible if sequences of critical metrics already *a priori* satisfy regularity estimates and compactness properties. This is the case for conformal factors associated to harmonic maps [Hel96][Riv08].

The second method (see [Pet14a]), improved in [Pet18] and [Pet19] (Laplace and Steklov eigenvalues with higher index) is now performed for combinations of eigenvalues [Pet21] [Pet22]. The first method (see [Nad96] [NS15]) was improved in [KNPP20] for Laplace eigenvalues of higher index. It is also worth mentioning that there is an indirect method to maximize first and second conformal Laplace eigenvalues [KS20] [KS22] based on min-max methods to build harmonic maps. While it is difficult to generalize it to higher eigenvalues or combinations, this gives a nice characterization of the maximizers, also leading to quantified inequalities on first and second eigenvalues [KNPS21].

In the current paper, we simplify, unify and generalize the previous variational methods by defining a notion of *Palais-Smale* sequences of conformal factors. It is a significative improvement, e.g for the following reasons:

- We can observe that maximizing sequences extracted by the maximization of relaxed functionals by the Heat kernel $e^{2u_\varepsilon} = K_\varepsilon[\nu_\varepsilon]$ (in [Pet14a][Pet18][Pet21][Pet22]) satisfy the properties of *Palais-Smale* sequences as $\varepsilon \to 0$. Notice that these sequences $(e^{2u_\varepsilon})_{\varepsilon>0}$ are canonical in the sense that they satisfy even more regularity properties (for instance, there are $\mathcal{C}^0$ a priori estimates on eigenfunctions) than a random *Palais Smale* sequence. However, working on these sequences requires an overly high technicality.
- All the previous methods are *ad hoc* methods while the concept of *Palais-Smale* sequences gives a systematic approach.
- *Palais-Smale* sequences can be extracted from min-max problems on combinations of eigenvalues, while the previous methods seem specific to maximizations, and for some of them specific to the maximization of only one eigenvalue.
- This new method is also used in [Pet22a] for variational problems on the Laplacian in higher dimensions and is promising to solve many other variational problems on eigenvalues.

Classically, *Palais-Smale* sequences on a $\mathcal{C}^1$ functional $E : X \to \mathbb{R}$ are sequences (x_n) such that $E(x_n) \to c$ and $|DE(x_n)| \to 0$. The main problem is that a functional involving eigenvalues (depending on a space X of metrics, conformal factors, potentials, etc) is not a $\mathcal{C}^1$ functional. Of course, it is a $\mathcal{C}^1$ functional at any point in which the involved eigenvalues are simple, but we often have multiplicity of eigenvalues at the critical points, corresponding to intersection of smooth branches of eigenvalues. However, thanks to F. Clarke (see e.g [Cla13]), the subdifferential $\partial E(x)$ plays the role of the differential for locally Lipschitz functionals. Roughly speaking, it is a space of subgradients containing all the informations on the first variation of the functional, and in particular on the derivatives corresponding to the smooth branches of eigenvalues at points of multiplicity (see [PT22] for more details). Then, criticality of E at x can be defined by $0 \in \partial E(x)$. The current paper is devoted to quantify this property by the definition of a pseudo-norm $|\partial E(x)|$ on subdifferentials such that $x \mapsto |\partial E(x)|$ is lower semi-continuous and criticality is characterized by $|\partial E(x)| = 0$. A *Palais-Smale* sequence (x_n) is then nothing but defined by $E(x_n) \to c$ and $|\partial E(x_n)| \to 0$.

With this point of view, extraction of Palais-Smale sequences from variational problems on eigenvalue functionals follows the classical constructions for $\mathcal{C}^1$ functionals (see for instance the nice book [Str08]). This method is explained in Section 1. In particular, we explain how this new point of view is available for min-max variational problems on combinations of eigenvalues. As in the previous methods, the main difficulty is to prove convergence of *Palais-Smale* sequences. A wide part of the current paper is devoted to prove this convergence for functionals depending on combinations of Laplace eigenvalues (proof or Proposition 2.1 in Section 2) or combinations of Steklov eigenvalues (proof of Proposition 3.1 in Section 3) in dimension 2. We rewrite here a proof of the main theorems in [Pet21] and [Pet22] to simplify and enlighten the techniques used there and we give new convergence results on Palais-Smale sequences (H^{-1} convergence of maximizing sequences, convergence of spectral indices to the spectral index of the limit). A short summary of the systematic technique we use for convergence of Palais-Smale sequences is given after

Proposition 2.1. This technique is also developped in [Pet22a] for eigenvalues of the Laplacian in higher dimensions, with all the specificities due to higher dimensions. We emphasize that this systematic approach is promising to solve many other variational problems on eigenvalues.

1. The variational approach and consequences

1.1. **Examples of functionals and spaces of variation.** We denote by $E : X \subset V \to \mathbb{R}$ a functional depending on combination of eigenvalues, where X is a subset of a Banach space V. E is a locally Lipschitz functional. We can define its subdifferential as

$$\partial E(f) = \{\zeta \in V^\star; \forall v \in V, E^\circ(f;v) \geq \langle \zeta, v\rangle\}$$

where

$$E^\circ(f;v) := \limsup_{\tilde{f}\to f; t\to 0} \frac{E(\tilde{f}+tv)-E(\tilde{f})}{t}.$$

is a generalized directional derivative of E.

Concerning Laplace eigenvalues on a compact surface without boundary Σ,

$$\lambda_0 \leq \lambda_1(g) \leq \cdots \leq \lambda_k(g) \leq \cdots,$$

we denote by $\bar{\lambda}_k(g) = \lambda_k(g)A_g(\Sigma)$ the renormalized eigenvalue where $A_g(\Sigma)$ is the area of Σ with respect to g, and $E_k(g)$ the set of (maybe multiple) eigenfunctions associated to $\lambda_k(g)$. We set

$$V = \mathcal{C}^0(\Sigma) \text{ and } X = \{f \in \mathcal{C}^0(\Sigma), f > 0\}$$

and for $f \in X$ we set

$$E(f) := F\left(\bar{\lambda}_1(fg), \cdots, \bar{\lambda}_m(fg)\right)$$

where $F : \mathbb{R}^m \to \mathbb{R}_+$ is a $\mathcal{C}^1$ function such that for all $i \in \{1, \cdots, m\}$, $\partial_i F \leq 0$ everywhere. We denote $\partial E(f)$ the subdifferential of E at f. We proved in [PT22]

$$\partial E(f) \subset \overline{co}_{(\phi_1,\cdots,\phi_m)\in\mathcal{O}_{E(f)}} \left\{\sum_{i=1}^m d_i\bar{\lambda}_i(fg)\left(1-(\phi_i)^2\right)\right\} \tag{1.1}$$

where $(\phi_1, \cdots, \phi_m)$ lies in the set $\mathcal{O}_{E(f)}$ of $L^2(\Sigma, fg)$-orthonormal families where $\phi_i \in E_i(fg)$ and $d_i = \partial_i F\left(\bar{\lambda}_1(fg), \cdots, \bar{\lambda}_m(fg)\right) \leq 0$. Notice that in this case, the subdifferential is a space of functions, while it is defined as a subspace of $V^\star$ the set of Radon measures: here, we identify a function ψ to the measure ψdA_g.

Concerning Steklov eigenvalues on a compact surface with boundary Σ,

$$\sigma_0 \leq \sigma_1(g) \leq \cdots \leq \sigma_k(g) \leq \cdots,$$

we denote by $\bar{\sigma}_k(g) = \sigma_k(g)L_g(\partial\Sigma)$ the renormalized eigenvalue where $L_g(\partial\Sigma)$ is the length of the boundary $\partial\Sigma$ of Σ with respect to g, and again $E_k(g)$ the set of (maybe multiple) eigenfunctions associated to $\sigma_k(g)$.

$$V = \mathcal{C}^0(\partial\Sigma) \text{ and } X = \{f \in \mathcal{C}^0(\partial\Sigma), f > 0\}$$

and for $f \in X$ we set

$$E(f) := F\left(\bar{\sigma}_1(\tilde{f}^2 g), \cdots, \bar{\sigma}_m(\tilde{f}^2 g)\right)$$

where $F : \mathbb{R}^m \to \mathbb{R}_+$ is a $\mathcal{C}^1$ function such that for all $i \in \{1, \cdots, m\}$, $\partial_i F \leq 0$ everywhere, and $\tilde{f} : \Sigma \to \mathbb{R}$ denotes one smooth extension of f to Σ, noticing that $\sigma_k(\tilde{f}^2 g)$ does not

depend on the extension of f by conformal invariance of the metric. We denote $\partial E(f)$ the subdifferential of E at f. We proved in [PT22]

$$\partial E(f) \subset \overline{co}_{(\phi_1,\cdots,\phi_m)\in\mathcal{O}_{E(f)}} \left\{ \sum_{i=1}^{m} d_i \bar{\sigma}_i(\tilde{f}^2 g) \left(1-(\phi_i)^2\right) \right\} \tag{1.2}$$

where $(\phi_1,\cdots,\phi_m)$ lies in the set $\mathcal{O}_{E(f)}$ of $L^2(\partial\Sigma, f dL_g)$-orthonormal families where $\phi_i \in E_i(\tilde{f}^2 g)$ and $d_i = \partial_i F\left(\bar{\sigma}_1(\tilde{f}^2 g),\cdots,\bar{\sigma}_m(\tilde{f}^2 g)\right) \leq 0$. Notice that in this case the subdifferential is a space of functions on $\partial\Sigma$, while it is defined as a subspace of $V^\star$, the set of Radon measures on $\partial\Sigma$: here, we identify a function ψ to the measure ψdL_g on $\partial\Sigma$.

1.2. **A pseudo-norm on subdifferentials.** We set

$$|\partial E(f)| = -\min_{\tau\in Y} \max_{\psi\in\partial E(f)} \langle \tau, \psi\rangle$$

the pseudo-norm of the subdifferential $\partial E(f)$, where $Y \subset V^\star$ is the unit sphere of $V^\star$. It is the set of probability measures on Σ for Laplace eigenvalues, or the set of probability measures on $\partial\Sigma$ for Steklov eigenvalues and $\langle .,. \rangle$ is the natural pairing between $V^\star$ and V. Notice that

- $|\partial E(f)| \geq 0$ because $\partial E(f)$ is a subset of a finite dimensional vector space and we can always find $\tau \in Y$ such that
$$\forall \psi \in \partial E(f), \langle \tau, \psi\rangle = 0$$
- If E is differentiable at f, $|\partial E(f)|$ is nothing but the norm of the linear form $f \to DE(f)$ in $V^\star$.
- The infimum in the definition of $|\partial E(f)|$ is a minimum because of the compactness of the sphere with respect to the weak $\star$ topology on $V^\star$ (the set of Radon measures) and because $\partial E(f)$ can be seen as a set of continuous functions.
- By a standard density theorem,
$$|\partial E(f)| = -\inf_{\tau\in\tilde{Y}} \max_{\psi\in\partial E(f)} \langle \tau, \psi\rangle$$
where $\tilde{Y} = \{hdA_g \in Y; h \in \mathcal{C}^0(\Sigma), h \geq 0, \int_\Sigma hdA_g = 1\}$, seeing naturally a function h as the measure hdA_g for Laplace eigenvalues. $\tilde{Y} = \{hdL_g \in Y; h \in \mathcal{C}^0(\Sigma), h \geq 0, \int_\Sigma hdL_g = 1\}$ for Steklov eigenvalues. $|\partial E(f)|$ has to be seen as the absolute value of the least right derivative of E among all the variations $v_t = th$ for $hdA_g \in \tilde{Y}$.

Claim 1.1. *The map $f \mapsto |\partial E(f)|$ is lower semi-continous.*

Proof. Let $f_n \to f$ in X. By definition of the subdifferential, we have the following semi-continuity result on subdifferentials

$$\partial E(f) \supset \left\{ \lim_{n\to+\infty} \varphi_n; (\varphi_n)_{n\in\mathbb{N}} \text{ is a convergent sequence with } \varphi_n \in \partial E(f_n) \right\}$$

Let $\tau \in Y$. $\partial E(f_n)$ is compact as a bounded subset of a finite dimensional set. Then, let $\psi_n \in \partial E(f_n)$ be such that

$$\langle \tau, \psi_n\rangle = \max_{\psi\in\partial E(f_n)} \langle \tau, \psi\rangle$$

By convergence of eigenvalues and eigenfunctions, up to a subsequence, ψ_n converges to some function ψ_∞ in $\mathcal{C}^0$ as $n \to +\infty$. More precisely, we have by ellipticity of the eigenfunction equations

$$|\langle \tau, \psi_\infty - \psi_n \rangle| \leq \|\psi - \psi_n\|_{\mathcal{C}^0} \leq C \left(\|f_n - f\|_{\mathcal{C}^0} + \sum_{j=1}^{N} \|\phi^j - \phi_n^j\|_{L^2} + \sum_{i=1}^{m} \left|\lambda_n^i - \lambda^i\right| \right),$$

where $\lambda_n^i = \lambda_i(f_n g)$, $\lambda^i = \lambda_i(fg)$, N is the dimension of $E_1(f_n g) + \cdots + E_m(f_n g)$ and $(\phi_n^1, \cdots, \phi_n^M)$ is an $L^2(f_n g)$-orthonormal basis of eigenfunctions for $f_n g$ in this space converging in L^2 to $(\phi^1, \cdots, \phi^M)$, an $L^2(fg)$-orthonormal family of eigenfunctions with respect for fg, and C is a function depending on $\|f\|_{\mathcal{C}^0}$, λ_i and $\sup \partial_i F$ at the neighborhood of $(\lambda^1, \cdots, \lambda^n)$. Knowing in addition that $\psi_\infty \in \partial E(f)$, we obtain that

$$\begin{aligned} -|\partial E(f_n)| &= \min_{\mu \in Y} \max_{\psi \in \partial E(f_n)} \langle \mu, \psi \rangle \leq \langle \tau, \psi_n \rangle \leq \langle \tau, \psi_\infty \rangle + |\langle \tau, \psi_\infty - \psi_n \rangle| \\ &\leq \max_{\psi \in \partial E(f)} \langle \tau, \psi \rangle + C \left(\|f_n - f\|_{\mathcal{C}^0} + \sum_{j=1}^{N} \|\phi^j - \phi_n^j\|_{L^2} + \sum_{i=1}^{m} \left|\lambda_n^i - \lambda^i\right| \right) \end{aligned}$$

Passing to the infimum on $\tau \in X$, we get

$$-|\partial E(f_n)| \leq -|\partial E(f)| + o(1)$$

as $n \to +\infty$, and finally we have

$$|\partial E(f)| \leq \liminf_{n \to +\infty} |\partial E(f_n)|$$

$\diamondsuit$

Claim 1.2. *Let $f \in X$. The following propositions are equivalent*

(i) $0 \in \partial E(f)$
(ii) $\partial E(f) \cap \{a \in \mathcal{C}^0(\Sigma); a \geq 0\} \neq \emptyset$
(iii) $|\partial E(f)| = 0$

Proof. Of course, $(i) \Rightarrow (ii)$. We also have that $(ii) \Rightarrow (i)$, indeed, if there is $\psi \in \partial E(f) \cap \{a \in \mathcal{C}^0(\Sigma); a \geq 0\}$, then knowing that any function in $\partial E(f)$ satisfies $\int_\Sigma \psi dA_g = 0$, we deduce that $\psi = 0$.

We prove now that $(iii) \Rightarrow (ii)$. If $\partial E(f) \cap \{a \in \mathcal{C}^0(\Sigma); a \geq 0\} = \emptyset$, then by Hahn-Banach's theorem there is a Radon measure τ such that

$$\forall \psi \in \partial E(f), \langle \tau, \psi \rangle < 0$$

and

$$\forall a \in \mathcal{C}^0(\Sigma), a \geq 0 \Rightarrow \langle \tau, a \rangle \geq 0$$

which implies that τ is a non-negative Radon measure. Up to renormalization, we can assume that τ is a probablility measure. We obtain that $|\partial E(f)| > 0$.

Finally, we prove that $(ii) \Rightarrow (iii)$. If $|\partial E(f)| > 0$, then, there is $\tau \in Y$ such that

$$\forall \psi \in \partial E(f), \langle \tau, \psi \rangle < 0$$

This positive measure τ also satisfies

$$\forall a \in \mathcal{C}^0(\Sigma), a \geq 0 \Rightarrow \langle \tau, a \rangle \geq 0$$

so that $\partial E(f) \cap \{a \in \mathcal{C}^0(\Sigma); a \geq 0\} = \emptyset$ $\diamondsuit$

The set of regular points of f is denoted

$$X_r = \{f \in X; |\partial E(f)| > 0\} = \{f \in X; 0 \notin \partial E(f)\}$$

and the set of critical points we are looking for all along the current paper:

$$X_c = \{f \in X; |\partial E(f)| = 0\} = \{f \in X; 0 \in \partial E(f)\}.$$

1.3. **A deformation Lemma.** Thanks to the definition of $|\partial E(f)|$ we have an adaptation of the classical deformation lemma (thanks to downhill directions) for $\mathcal{C}^1$ functionals to the functionals depending on eigenvalues:

Proposition 1.1. *If there is $\varepsilon_0 > 0$ and $\delta > 0$ such that*

$$\forall f \in X; \forall \varepsilon \in (0, \varepsilon_0), |E(f) - c| \leq \varepsilon \Rightarrow |\partial E(f)| \geq \delta \ ,$$

Then $\forall \varepsilon \in (0, \varepsilon_0)$, there is $\eta : X \to X$ such that

- *$\eta(f) = f$ for any $f \in \{E \geq c + \varepsilon_0\} \cup \{E \leq c - \varepsilon_0\}$*
- *$\forall f \in X, E(\eta(f)) \leq E(f)$*
- *$\eta(\{E \leq c + \varepsilon\}) \subset \{E \leq c - \varepsilon\}$*

We first build an adapted pseudo-vector field for our problem

Claim 1.3. *Let $\varepsilon > 0$. There is a locally Lipschitz vector field $v : X_r \to X$ such that for all $f \in X$*

- *$\|v(f)\|_1 < 2$*
- *$\forall \psi \in \partial E(f), \langle v(f), \psi \rangle < -\frac{|\partial E(f)|}{2}$*
- *$v(f) \geq 0$*

Proof. We first fix $f_0 \in X$ and we build an adapted image $v_0(f_0)$ satisfying the conclusion of the Claim.

Let $\tau_0 \in \tilde{Y}$ such that

$$-|\partial E(f_0)| \geq \max_{\psi \in \partial E(f_0)} \langle \tau_0, \psi \rangle - \frac{|\partial E(f_0)|}{4} \ .$$

We choose $v_0(f_0) = \tau_0$ so that

- $\|v_0(f_0)\|_1 \leq 1 < 2$
- $\forall \psi \in \partial E(f_0), \langle v_0(f_0), \psi \rangle \leq -\frac{3}{4} |\partial E(f_0)| < -\frac{|\partial E(f_0)|}{2}$
- $v_0(f_0) \geq 0$

Now, we aim at defining v by some transformation of v_0 in order to obtain a locally Lipschitz vector field $v : X_r \to X$.

Let $f_0 \in X_r$. Let Ω_{f_0} be an open neighborhood of f_0 in X_r such that for all $f \in \Omega_{f_0}$

$$\forall \psi \in \partial E(f), \langle \psi, v_0(f_0) \rangle < -\frac{|\partial E(f)|}{2}$$

We notice that

$$X_r = \bigcup_{f_0 \in X_r} \Omega_{f_0}.$$

Since X_r is paracompact, one has a family of open sets $(\omega_i)_{i \in I}$ such that

- $X_r = \bigcup_{i \in I} \omega_i$
- $\forall i \in I, \exists f_i \in X_r, \omega_i \subset \Omega_{f_i}$

- for all $u \in X_r$ there is an open set Ω such that $u \in \Omega$ and $\Omega \cap \omega_i = \emptyset$ except for a finite number of indices i.

We set $\psi_i(u) = d\left(u, X_r \setminus \omega_i\right)$ and $\eta_i(u) = \frac{\psi_i}{\sum_{j\in I}\psi_j}$ and the vectorfield

$$v(f) = \sum_{i\in I} \eta_i(f) v_0(f_i)$$

satisfies the conclusion of the claim. ♢

Proof. (of proposition 1.1) We define a vector-field $\Phi : X \to X$ as for $f \in X$,

$$\Phi(f) = \frac{d(f,A)}{d(f,A)+d(f,B)} v(f)$$

where $v : X_r \to X$ is given by Claim 1.3 and we define the sets

$$A = \{E \le c - \varepsilon_0\} \cup \{E \ge c + \varepsilon_0\}$$

and

$$B = \{c - \varepsilon \le E \le c + \varepsilon\} .$$

Let η be a solution of

$$\begin{cases} \frac{d}{dt}\eta_t(f) = \Phi\left(\eta_t(f)\right) \\ \eta_0(f) = f . \end{cases} \tag{1.3}$$

Such a solution η exists since Φ is locally Lipschitz. Moreover, η is well defined on $\mathbb{R}_+$ since Φ is bounded. Let $t \ge 0$, and $f \in X$. We have by elementary properties on the subdifferential that

$$\begin{aligned} \frac{d}{dt}E(\eta_t(f)) &\le \max\{\langle \Phi(\eta_t(f)), \psi\rangle ; \psi \in \partial E\left(\eta_t(f) g\right)\} \\ &\le -\frac{d(\eta_t(f),A)}{d(\eta_t(f),A)+d(\eta_t(f),B)} \frac{|\partial E(\eta_t(f))|}{2} \le 0 \end{aligned} \tag{1.4}$$

for any $t \ge 0$. It is clear that for any $t \ge 0$, $\eta_t(f) = f$ for $f \in A$ and that

$$\eta_t\left(\{E \le c - \varepsilon\}\right) \subset \{E \le c - \varepsilon\} .$$

It remains to prove that for $t_0 > 0$ small enough we also have that

$$\eta_{t_0}\left(B\right) \subset \{E \le c - \varepsilon\} .$$

Let $t_\star$ be the smallest time such that $\eta_{t_\star}(f) = c - \varepsilon$. Then, for $0 \le t \le t_\star$, we have

$$\eta_t(f) \in B \Rightarrow \frac{d}{dt}E(\eta_t(f)) \le -\frac{\delta}{2} .$$

since by assumption, $c - \varepsilon \le E(\eta_t(f)) \le c$ implies that $|\partial E(\eta_t(f))| \ge \delta$ and that $d(\eta_t(f), B) = 0$. We deduce by integration that

$$E(\eta_{t_\star}(f)) - E(f) \le -\frac{\delta}{2} t_\star$$

so that $t_\star \le \frac{4\varepsilon}{\delta}$ Then, letting $t_0 = \frac{2\varepsilon}{\delta}$ we obtain $\eta_{t_0}(B) \subset \{E \le c - \varepsilon\}$. Therefore,

$$\eta_{t_0}(\{E \le c + \varepsilon\}) \subset \eta_{t_0}(\{E \le c - \varepsilon\})$$

and we obtain the proposition. ♢

1.4. **Construction of Palais-Smale sequences.** Let $\mathcal{A} \subset \mathcal{P}(X)$. We assume that

$$c = \inf_{A \in \mathcal{A}} \sup_{f \in A} E(f).$$

is finite. We give a sufficient condition on $\mathcal{A}$ to define an associated Palais-Smale sequence:

Claim 1.4. *We assume that there is $\alpha > 0$ such that for any homeomorphism $\eta : X \to X$ such that $E(\eta(f)) \leq E(f)$, and such that $E(f) = f$ for any $f \in \{|E - c| > \alpha\}$,*

$$A \in \mathcal{A} \Rightarrow \eta(A) \in \mathcal{A}.$$

Then, there is a sequence $f_\varepsilon \in X$ such that $|E(f_\varepsilon) - c| \leq \varepsilon$ and $\delta_\varepsilon = |\partial E(f_\varepsilon)| \to 0$ as $\varepsilon \to 0$.

Proof. If it is not the case, then there is $\delta_0 > 0$ and $\varepsilon_0 > 0$ satisfying the assumptions of the deformation lemma. We can assume that $\varepsilon_0 \leq \alpha$. Then, for $0 < \varepsilon \leq \varepsilon_0$, we let $\eta : X \to X$ be the homeomorphism given by the deformation lemma. By definition of c, let $A \in \mathcal{A}$ be such that

$$\sup_{f \in \mathcal{A}} E(f) \leq c + \varepsilon$$

Then, we have that

$$\sup_{f \in \eta(\mathcal{A})} E(f) \leq c - \varepsilon$$

where $\eta(A) \in \mathcal{A}$ by assumption. This is a contradiction with the definition of c $\diamondsuit$

For instance:

- If $\mathcal{A} = \{\{f\}; f \in X\}$, we obtain a minimizing sequence satisfying the Palais-Smale condition.
- If we have two strict local minimizers f_1 and f_2 of E, then if

  $$\mathcal{A} = \{\gamma([0,1]); \gamma : [0,1] \to X \text{ continuous and } \gamma(0) = f_1, \gamma(1) = f_2\}$$

  we obtain a new Palais-Smale sequence.

Up to standard regularizations, we can assume that a Palais Smale sequence is a sequence of smooth positive functions e^{2u_ε} satisfying

$$\left|E(e^{2u_\varepsilon}) - c\right| \leq \varepsilon \text{ and } \delta_\varepsilon = |\partial E(f_\varepsilon)| \to 0 \text{ as } \varepsilon \to 0.$$

The equality $\delta_\varepsilon = |\partial E(f_\varepsilon)|$ can be rewritten as

$$\forall \tau \in Y, \exists \psi \in \partial E(e^{2u_\varepsilon}); -\langle \tau, \psi \rangle \leq \delta_\varepsilon$$

which can be rewritten as

$$\forall \tau \in Y, \exists \psi \in \partial E(e^{2u_\varepsilon}); -\langle \tau, \psi + \delta_\varepsilon \rangle \leq 0$$

where δ_ε is a constant function in V, meaning that

$$-\left(\partial E(e^{2u_\varepsilon}) + \{\delta_\varepsilon\}\right) \cap \{a \in V; a \leq 0\} \neq \emptyset. \tag{1.5}$$

Indeed, if not, we use the classical Hahn-Banach theorem to separate these two spaces (the first one is compact, the second one is closed in V) by $\tau \in V^\star$ satisfying

$$\forall \psi \in \partial E(e^{2u_\varepsilon}), \langle \tau, -(\psi + \delta_\varepsilon) \rangle > 0$$

and

$$\forall a \in V; a \leq 0, \langle \tau, a \rangle \leq 0.$$

The second condition implies that τ is a non-negative Radon measure. Up to a renormalization, we assume that $\tau \in Y$ is a probability measure and we obtain a contradiction. Therefore we obtain the Palais-Smale condition on the sequence e^{2u_ε}, given in the assumption of Proposition 2.1 (with (1.1) and (1.5)) and Proposition 3.1 (with (1.2) and (1.5)), up to a renormalization on δ_ε and to denote δ_ε another sequence converging to 0 as $\varepsilon \to 0$.

2. Convergence of Palais-Smale sequences for finite combination of Laplace eigenvalues in dimension 2

Proposition 2.1. *Let e^{2u_ε}, Λ_ε, $\Phi_\varepsilon : \Sigma \to \mathbb{R}^m$ be a smooth sequence of maps satisfying the Palais-Smale assumption (PS) as $\varepsilon \to 0$, that is*

- *$\Lambda_\varepsilon = diag(\lambda^\varepsilon_{k^\varepsilon_1}, \cdots, \lambda^\varepsilon_{k^\varepsilon_m})$ where the diagonal terms are eigenvalues associated to e^{2u_ε} with uniformly bounded spectral indices k^ε_i, $\lambda^\varepsilon_{k^\varepsilon_1} \leq \cdots \leq \lambda^\varepsilon_{k^\varepsilon_m}$ and $\Lambda_\varepsilon \to \Lambda = (\lambda_{k_1}, \cdots, \lambda_{k_m})$ as $\varepsilon \to 0$.*
- *$\Delta_g \Phi_\varepsilon = e^{2u_\varepsilon} \Lambda_\varepsilon \Phi_\varepsilon$*
- *$\int_\Sigma e^{2u_\varepsilon} dA_g = \int_\Sigma |\Phi_\varepsilon|^2_{\Lambda_\varepsilon} e^{2u_\varepsilon} dA_g = \int_\Sigma |\nabla \Phi_\varepsilon|^2_g dA_g = 1$*
- *For $i \in \{1, \cdots, m\}$, $t^\varepsilon_i = \int_\Sigma (\phi^\varepsilon_i)^2 e^{2u_\varepsilon} dA_g$ and $\sum \lambda^\varepsilon_{k_i} t^\varepsilon_i = 1$.*
- *$|\Phi_\varepsilon|^2_{\Lambda_\varepsilon} \geq 1 - \delta_\varepsilon$ uniformly, where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Then, up to a subsequence Φ_ε bubble tree converges in $W^{1,2}$ to $\Phi_0 : \Sigma \to \mathcal{E}_\Lambda$, $\Phi_j : \mathbb{S}^2 \to \mathcal{E}_\Lambda$ for $j = 1, \cdots, l$ $(l \geq 0)$ with an energy identity:

$$1 = \int_\Sigma |\nabla \Phi_0|^2_g dA_g + \sum_{j=1}^{l} \int_{\mathbb{S}^2} |\nabla \Phi_j|^2_h dA_h$$

Moreover, Φ_j are smooth harmonic maps for $j = 0, \cdots, l$ and their i-th coordinates are eigenfunctions associated to λ_{k_i} on the surface $\Sigma \cup \bigcup_{1 \leq i \leq l} \mathbb{S}^2$ with respect to the metrics $\frac{|\nabla \Phi_0|^2_{\Lambda,g}}{|\Lambda \Phi_0|^2} g$ on Σ and $\frac{|\nabla \Phi_j|^2_{\Lambda,h}}{|\Lambda \Phi_j|^2} h$ on $\mathbb{S}^2$.

We can summarize the proof in the following way: we have

$$\Delta_g \Phi_\varepsilon = e^{2u_\varepsilon} \Lambda_\varepsilon \Phi_\varepsilon \text{ and } |\Phi|^2_{\Lambda_\varepsilon} \geq 1 - \delta_\varepsilon \text{ and } \int_\Sigma e^{2u_\varepsilon} dA_g = \int_\Sigma |\Phi_\varepsilon|^2_{\Lambda_\varepsilon} e^{2u_\varepsilon} dA_g$$

where Φ_ε is harmonic if and only if $|\Phi_\varepsilon|^2_{\Lambda_\varepsilon} = 1$. We aim at proving that Φ_ε converges to Φ as $\varepsilon \to 0$ in a suitable space so that if $e^{2u_\varepsilon} dA_g \to_\star \nu$ with respect to weak$\star$ convergence of measures and $\Lambda_\varepsilon \to \Lambda$, the equation on Φ_ε passes to the limit and we get

$$\Delta_g \Phi = \nu \Lambda \Phi \text{ and } |\Phi|^2_\Lambda = 1,$$

which is exactly the equation for weak harmonic maps. By regularity of Φ, we obtain by computation of $0 = \Delta_g \left(|\Phi|^2_\Lambda \right)$ that $\nu = \frac{|\nabla \Phi|^2_\Lambda}{|\Phi|^2_\Lambda} dA_g$ is a regular measure. This idea has to be written up to a bubble tree (see Claim 2.1).

All the proof is based on local energy-convexity results for the harmonic replacement Ψ_ε of $\frac{\Phi_\varepsilon}{\omega_\varepsilon}$ where $\omega_\varepsilon := |\Phi_\varepsilon|^2_{\Lambda_\varepsilon}$ on disks $\mathbb{D}_r(p)$ of small energy of Φ_ε:

$$\frac{1}{2} \int_{\mathbb{D}_r(p)} \left| \nabla \left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon \right) \right|^2 \leq \int_{\mathbb{D}_r(p)} \left| \nabla \frac{\Phi_\varepsilon}{\omega_\varepsilon} \right|^2 - \int_{\mathbb{D}_r(p)} |\nabla \Psi_\varepsilon|^2 \tag{2.1}$$

where the harmonic replacement Ψ_ε is defined as the unique harmonic map into the ellipsoid $\mathcal{E}_{\Lambda_\varepsilon}$ such that $\Psi_\varepsilon = \frac{\Phi_\varepsilon}{\omega_\varepsilon}$ on $\partial\mathbb{D}_r(p)$ (see [CM08] [LP19]). Taking the harmonic replacement of $\frac{\Phi_\varepsilon}{\omega_\varepsilon}$ is justified because of the crucial assumption $\omega_\varepsilon \geq 1-\delta_\varepsilon$, and its consequences: ω_ε converges to 1 in some sense and $\nabla\left(\Phi_\varepsilon - \frac{\Phi_\varepsilon}{\omega_\varepsilon}\right)$ converges to 0 in L^2 (see (2.3) and (2.4)).

Therefore, we just have to prove that the right-hand term in (2.1) converges to 0 as $\varepsilon \to 0$. This comes from the equation on Φ_ε and an extra assumption on the disk $\mathbb{D}_r(p)$ of replacement: we assume that the first Laplace eigenvalue with Dirichlet boundary condition satisfies $\lambda_\star(\mathbb{D}_r(p)) \geq \lambda^\varepsilon_{k^\varepsilon_m}$. Then

$$\int_{\mathbb{D}_r(p)} \left|\nabla \frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2 - \int_{\mathbb{D}_r(p)} |\nabla\Psi_\varepsilon|^2 \to 0 \text{ as } \varepsilon \to 0. \tag{2.2}$$

Using (2.1), (2.2) and standard convergence theorems for harmonic maps, we obtain the strong H^1 convergence of $\frac{\Phi_\varepsilon}{\omega_\varepsilon}$ and Φ_ε to a function Φ satisfying the desired weak-harmonic map equation and the regularity of the mimiting measure ν. We globalize these local convergences noticing that they happen at a fixed neighborhood of any point except at most k_m+1 points that we call bad points (see Claim 2.3 and (2.5)). They do not interfere in the final regularity result and energy identities. The proof of local strong H^1 convergence is given in Claim 2.5. However, in this Claim, we need the assumption that all the eigenvalues involved in the Palais-Smale sequence are uniformly lower bounded because in this case, the local smallness of the energy of Φ_ε is implied by (2.5). Therefore, we prove an adaptation of this simple technique in the general case (see subsection 2.3.4).

All along the proof, every local computation is made in the exponential chart centered at points $p \in M$, defined on balls whose radius is controlled by the injectivity radius with respect to g: $inj_g(M)$. Without loss of generality, we can assume that $inj_g(M) \geq 2$ and make arguments on the unit disk $\mathbb{D}$ endowed with the metric $\exp_p^\star g$ still denoted g. We do not change the notations of the metrics and functions in the charts. Moreover sometimes, when there is not any embiguity, we do not precise the measures of integration associated to g, dA_g inside the integrals nor the index g in $|\nabla\varphi|^2_g$ in order to lighten the computations.

2.1. **A bubble tree structure.** For a Radon measure, μ on Σ (or $\mathbb{R}^2$), we denote $\tilde{\mu}$ the continuous part of μ, defined as the measure without atom such that

$$\mu = \tilde{\mu} + \sum_{x\in\mathcal{A}(\mu)} \alpha_x \delta_x$$

where $\mathcal{A}(\mu)$ is the set of atoms of μ and $\alpha_x \in \mathbb{R}$

Claim 2.1. *Let e^{2u_ε} be a sequence of smooth positive functins such that $\int_\Sigma e^{2u_\varepsilon} dA_g = 1$ and*

$$\liminf_{\varepsilon\to 0} \lambda_k(e^{2u_\varepsilon}) > 0$$

Then up to a subsequence on $\varepsilon \to 0$, there are an integer $l \geq 0$ and if $l > 0$ sequences of points q^i_ε for $i = 1, \cdots, l$ and scales $\alpha^l_\varepsilon \leq \cdots \leq \alpha^1_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $e^{2u_\varepsilon} dA_g$ weak-$\star$ converges in M to a non negative Radon measure μ_0 in Σ and $e^{2u_\varepsilon(\alpha^i_\varepsilon z + q^i_\varepsilon)}$ we have the weak-$\star$ convergence in (any compact set of) $\mathbb{R}^2$ to a non negative and non zero Radon

measure μ_i on $\mathbb{R}^2$ for any i. Moreover, their associated continuous parts preserve the mass before the limit:

$$\lim_{\varepsilon\to 0}\int_\Sigma e^{2u_\varepsilon}dA_g=\int_\Sigma d\tilde{\mu}_0+\sum_{i=1}^{l}\int_{\mathbb{R}^2}d\tilde{\mu}_i=1 \text{ and } \forall i\geq 1,\int_{\mathbb{R}^2}d\tilde{\mu}_i>0$$

and we have that $1\leq \mathbf{1}_{\tilde{\mu}_0\neq 0}+t\leq k$. In addition, setting $F_i=\{j>i;\left(\frac{d_g(q_i^\varepsilon,q_j^\varepsilon)}{\alpha_i^\varepsilon}\right)$ is bounded$\}$, we have for $j>i$,

$$j\in F_i\Rightarrow\frac{\alpha_j^\varepsilon}{\alpha_i^\varepsilon}\to 0 \text{ and } j\notin F_i\Rightarrow\frac{d_g(q_i^\varepsilon,q_j^\varepsilon)}{\alpha_i^\varepsilon}\to+\infty$$

The last condition also reads as

$$\frac{\alpha_i^\varepsilon}{\alpha_j^\varepsilon}+\frac{\alpha_j^\varepsilon}{\alpha_i^\varepsilon}+\frac{d_g(q_i^\varepsilon,q_j^\varepsilon)}{\alpha_i^\varepsilon+\alpha_j^\varepsilon}\to+\infty$$

which is the standard condition for a bubble tree. A wide part of the section is devoted to prove that the continuous measures $\tilde{\mu}_i$ for $i\in\{0,\cdots,t\}$ are absolutely continuous with respect to dA_g if $i=0$ and to the Euclidean metric if $i\geq 1$, with densities equal to densities of energy of harmonic maps. The proof of Claim 2.1 is already written in [Kok14] [Pet18] or [KNPP20]. The selection of scales of concentration is based on Hersch's trick because it uses the conformal group of the sphere to balance continuous measures on a sphere. Then the selection stops because of the min-max characterization of λ_k: If there are more than $k+1$ scales of concentration, we can build $k+1$ independant test functions with arbitrarily small rayleigh quotient, contradicting the first assumption of Claim 2.1.

2.2. **Some convergence of ω_ε to 1 and first replacement of Φ_ε.** We set $\omega_\varepsilon=|\Phi_\varepsilon|_{\Lambda_\varepsilon}$. We first prove that in some sense, ω_ε converges to 1 and that Φ_ε has a similar H^1 behaviour as $\frac{\Phi_\varepsilon}{\omega_\varepsilon}$

Claim 2.2. *We have that*

$$\int_\Sigma\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon}dA_g=O(\delta_\varepsilon) \tag{2.3}$$

$$\int_\Sigma\left|\nabla\left(\Phi_\varepsilon-\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right)\right|^2_{\Lambda_\varepsilon}=O\left(\delta_\varepsilon^{\frac{1}{2}}\right) \tag{2.4}$$

as $\varepsilon\to 0$.

Proof. We integrate $\Delta_g\Phi_\varepsilon=\Lambda_\varepsilon e^{2u_\varepsilon}\Phi_\varepsilon$ against $\Lambda_\varepsilon\Phi_\varepsilon$ and $\frac{\Lambda_\varepsilon\Phi_\varepsilon}{\omega_\varepsilon}$. We obtain

$$\int_\Sigma|\Lambda_\varepsilon\Phi_\varepsilon|^2e^{2u_\varepsilon}=\int_\Sigma|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}$$

and

$$\int_\Sigma\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon}e^{2u_\varepsilon}=\int_\Sigma\frac{\Lambda\Phi_\varepsilon}{\omega_\varepsilon}\Delta_g\Phi_\varepsilon=\int_\Sigma\left(\frac{|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}}{\omega_\varepsilon}-\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon}\right)=\int_\Sigma\omega_\varepsilon\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_{\Lambda_\varepsilon}$$

Therefore

$$\begin{aligned}\int_\Sigma \frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon}dA_g &= \int_\Sigma \left(\frac{|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}}{\omega_\varepsilon} - \omega_\varepsilon\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_{\Lambda_\varepsilon}\right)dA_g\\ &= \int_\Sigma \left(\frac{|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}}{\omega_\varepsilon} - e^{2u_\varepsilon}\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon}\right)dA_g\\ &= \int_\Sigma \left(\frac{|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}}{\omega_\varepsilon} - |\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}\right)dA_g - \int_\Sigma e^{2u_\varepsilon}\left(\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon} - |\Lambda_\varepsilon\Phi_\varepsilon|^2\right)\\ &= \int_\Sigma \frac{|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}}{\omega_\varepsilon}(1-\omega_\varepsilon)\,dA_g + \int_\Sigma \frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon^2}e^{2u_\varepsilon}(\omega_\varepsilon^2-\omega_\varepsilon)\end{aligned}$$

We know that $\omega_\varepsilon^2-\omega_\varepsilon=\omega_\varepsilon(\omega_\varepsilon-1)\geq 1-\delta_\varepsilon-\sqrt{1-\delta_\varepsilon}$ so that

$$\int_\Sigma \frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon^2}e^{2u_\varepsilon}(\omega_\varepsilon^2-\omega_\varepsilon) \leq \sup\left|\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon^2}\right|\left(\int_\Sigma e^{2u_\varepsilon}(\omega_\varepsilon^2-\omega_\varepsilon) + \left(\sqrt{1-\delta_\varepsilon}+\delta_\varepsilon-1\right)\right)$$

Noticing the crucial equality $\int_\Sigma e^{2u_\varepsilon}\omega_\varepsilon^2 = \int_\Sigma e^{2u_\varepsilon}=1$, we obtain that

$$\int_\Sigma e^{2u_\varepsilon}(\omega_\varepsilon^2-\omega_\varepsilon) = \int_\Sigma e^{2u_\varepsilon}(1-\omega_\varepsilon) \leq 1-\sqrt{1-\delta_\varepsilon}$$

so that

$$\begin{aligned}\int_\Sigma \frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon}dA_g &\leq \int_\Sigma \frac{|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}}{\omega_\varepsilon}(1-\omega_\varepsilon)\,dA_g + \delta_\varepsilon\sup\left|\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon^2}\right|\\ &\leq \int_\Sigma |\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}\frac{1-\sqrt{1-\delta_\varepsilon}}{\sqrt{1-\delta_\varepsilon}} + \delta_\varepsilon\sup\left|\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon^2}\right|\\ &\leq \max_j \lambda^\varepsilon_{k^\varepsilon_j}\left(\frac{1-\sqrt{1-\delta_\varepsilon}}{\sqrt{1-\delta_\varepsilon}}+\delta_\varepsilon\right) = O(\delta_\varepsilon)\end{aligned}$$

We obtain (2.3).

Let's prove (2.4). We have that

$$\nabla\left(\Phi_\varepsilon - \frac{\Phi_\varepsilon}{\omega_\varepsilon}\right) = \left(1-\frac{1}{\omega_\varepsilon}\right)\nabla\Phi_\varepsilon + \frac{\nabla\omega_\varepsilon}{\omega_\varepsilon^2}\Phi_\varepsilon$$

and then

$$\left|\nabla\left(\Phi_\varepsilon - \frac{\Phi_\varepsilon}{\omega_\varepsilon}\right)\right|^2_{\Lambda_\varepsilon} = |\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}\left(1-\frac{1}{\omega_\varepsilon}\right)^2 + \frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon^2} + 2\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon}\left(1-\frac{1}{\omega_\varepsilon}\right)$$

so that

$$\int_\Sigma\left|\nabla\left(\Phi_\varepsilon - \frac{\Phi_\varepsilon}{\omega_\varepsilon}\right)\right|^2_{\Lambda_\varepsilon} \leq \int_\Sigma |\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}\left(1-\frac{1}{\omega_\varepsilon}\right)^2 + O(\delta_\varepsilon)$$

as $\varepsilon \to 0$ and

$$\begin{aligned}\int_\Sigma |\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}\left(1-\frac{1}{\omega_\varepsilon}\right)^2 &= -\int_\Sigma div\left(\nabla\Phi_\varepsilon\left(1-\frac{1}{\omega_\varepsilon}\right)^2\right)\Lambda_\varepsilon\Phi_\varepsilon\\ &= \int_\Sigma\left(1-\frac{1}{\omega_\varepsilon}\right)^2|\Lambda_\varepsilon\Phi_\varepsilon|^2e^{2u}+2\int_\Sigma\nabla\left(\frac{1}{\omega_\varepsilon}\right)\omega_\varepsilon\nabla\omega_\varepsilon\left(1-\frac{1}{\omega_\varepsilon}\right)\\ &\le\left(\int_\Sigma\frac{|\Lambda_\varepsilon\Phi_\varepsilon|^4}{\omega_\varepsilon^2}e^{2u_\varepsilon}\right)^{\frac12}\left(\int_\Sigma e^{2u_\varepsilon}\left(\omega_\varepsilon-\frac{1}{\omega_\varepsilon}\right)^2\right)^{\frac12}-2\int_\Sigma\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon^2}(\omega_\varepsilon-1)\\ &\le\max_j\lambda^\varepsilon_{k_j}\left(\int_\Sigma e^{2u_\varepsilon}\left(\frac{\omega_\varepsilon^2-1}{\omega_\varepsilon^2}\right)dA_g\right)^{\frac12}+O\left(\delta_\varepsilon^2\right)\\ &\le O\left(\delta_\varepsilon^{\frac12}\right)\end{aligned}$$

as $\varepsilon \to 0$, where we crucially used again $\int_\Sigma e^{2u_\varepsilon}\omega_\varepsilon^2 = \int_\Sigma e^{2u_\varepsilon} = 1$. We then obtain (2.4). This completes the proof of the claim.

$\diamondsuit$

2.3. **Regularity of the limiting measures.** We apply Claim 2.1 to the Palais-Smale sequence given by Proposition 2.1. We choose to prove only the regularity of $\tilde\mu_0$. The regularity of $\tilde\mu_i$ will follow the same arguments because of the scale invariance of all the equations satisfied by the Palais-Smale sequence.

In order to apply locally the a priori estimates on the map Φ_ε, we have to detect points where the smallness asumptions on ε-regularity type results fail, that is

- Disks $\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon)$ satisfying $\lambda_\star\left(\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon), e^{2u_\varepsilon}dA_g\right)\le\lambda_{k^\varepsilon_m}$
- Points and scales of concentration of $\left(|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon}+e^{2u_\varepsilon}\right)dA_g$

We prove that in the first point, the equality occurs for finitely many disks thanks to the assumption that the indices k_i^ε of $\lambda_i^\varepsilon=\lambda_{k_i^\varepsilon}\left(e^{2u_\varepsilon}g\right)$ is uniformly bounded. We prove in the second point that thanks to global properties on the system of equation, the concentration function of this energy is controlled by $e^{2u_\varepsilon}dA_g$

2.3.1. *Disjoint small disks with small critical fundamental state and bad points.*

Claim 2.3. *Up to the extraction of a subsequence of $\{e^{2u_\varepsilon}g\}_{\varepsilon>0}$ we can find a maximal collection of points $p_1^\varepsilon,\cdots,p_s^\varepsilon\in\Sigma$ with $0\le s\le k_m$ such that $p_i^\varepsilon\to p_i$ as $\varepsilon\to0$ and positive scales $r_1^\varepsilon\le\cdots\le r_s^\varepsilon$ such that for any $1\le i\le s$, setting A_i^ε as*

$$A_i^\varepsilon=\left\{r>0;\overline{\mathbb{D}_r(p)}\subset\Sigma\setminus\left(\bigcup_{j=1}^i\overline{\mathbb{D}_{r_j^\varepsilon}\left(p_j^\varepsilon\right)}\right)\ \text{and}\ \lambda_\star\left(\mathbb{D}_r(p),e^{2\tilde u_\varepsilon}dx\right)\le\lambda_{k_m}(e^{2u_\varepsilon}g)\right\}$$

$$\lim_{\varepsilon\to0}\left(\inf A_s^\varepsilon\right)>0\ ,$$

$$r_i^\varepsilon:=\min A_i^\varepsilon\to0\ \text{as}\ \varepsilon\to0\ ,$$

$$\overline{\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon)}\subset\Sigma\setminus\left(\bigcup_{j=1}^{i-1}\overline{\mathbb{D}_{r_j^\varepsilon}\left(p_j^\varepsilon\right)}\right)\ ,$$

$$\lambda_\star\left(\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon),e^{2\tilde u_\varepsilon}dx\right)=\lambda_{k_m}(e^{2u_\varepsilon}g)\ .$$

Proof. We first define

$$A_0^\varepsilon = \left\{ r > 0; \overline{\mathbb{D}_r(p)} \subset \Sigma \text{ and } \lambda_\star\left(\mathbb{D}_r(p), e^{2\tilde{u}_\varepsilon} dx\right) \leq \lambda_{k_m}(e^{2u_\varepsilon} g) \right\}$$

Notice first that if $\lim_{\varepsilon\to 0} \inf A_0^\varepsilon > 0$ then $s = 0$ and there is not such a sequence and the claim is proved. Otherwise $\lim_{\varepsilon\to 0} \inf A_0^\varepsilon = 0$. We set $r_1^\varepsilon = \min A_0^\varepsilon$ (notice that the infimum is a minimum) and we chose p_1^ε such that $\overline{\mathbb{D}_{r_1^\varepsilon}(p_1^\varepsilon)} \subset \Sigma$ and $\lambda_\star\left(\mathbb{D}_{r_1^\varepsilon}(p_1^\varepsilon), e^{2\tilde{u}_\varepsilon} dx\right) \leq \lambda_{k_m}(e^{2u_\varepsilon} g)$. Since r_1^ε is a minimum, the previous inequality has to be an equality.

If for a given $i \geq 1$ the sequences $r_1^\varepsilon, \cdots, r_i^\varepsilon, p_1^\varepsilon, \cdots, p_i^\varepsilon$ are built, then if $\lim_{\varepsilon\to 0} \inf A_i^\varepsilon > 0$, the construction terminates and $s = i$. Otherwise $\lim_{\varepsilon\to 0} \inf A_i^\varepsilon = 0$ and we set $r_{i+1}^\varepsilon = \min A_i^\varepsilon$, and we chose p_1^ε such that $\overline{\mathbb{D}_{r_{i+1}^\varepsilon}(p_{i+1}^\varepsilon)} \subset \Sigma$ and $\lambda_\star\left(\mathbb{D}_{r_{i+1}^\varepsilon}(p_{i+1}^\varepsilon), e^{2\tilde{u}_\varepsilon} dx\right) \leq \lambda_{k_m}(e^{2u_\varepsilon} g)$. Since r_{i+1}^ε is a minimum, the previous inequality has to be an equality.

Finally we prove that this sequence terminates after k_m steps. Indeed, if not, we set $D_i^\varepsilon = \mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon)$ for $i \in \{1, \cdots, k_m + 1\}$. The domains D_i^ε are disjoint in Σ and we have $\lambda_\star\left(D_i^\varepsilon, e^{2u_\varepsilon}\right) = \lambda_{k_m}(e^{2u_\varepsilon} g)$. Let φ_i^ε be first Dirichlet eigenfunctions on D_i^ε extended by 0 on Σ. We use these functions as test functions for the variational characterization of $\lambda_{k_m}(e^{2u_\varepsilon}, g)$:

$$\begin{aligned}
\lambda_{k_m}(e^{2u_\varepsilon}, g) &= \inf_{E_{k_m+1}} \max_{\varphi \in E_{k_m+1}} \frac{\int_\Sigma |\nabla\varphi|_g^2 \, dA_g}{\int_\Sigma (\varphi)^2 e^{2u_\varepsilon} dA_g} \\
&\leq \max_{0\leq i\leq k_m} \left\{ \frac{\int_\Sigma |\nabla\varphi_i^\varepsilon|_g^2 \, dA_g}{\int_\Sigma (\varphi_i^\varepsilon)^2 e^{2u_\varepsilon} dA_g} \right\} = \lambda_{k_m}(e^{2u_\varepsilon}, g)
\end{aligned}$$

where the inequality holds because the test functions have support in disjoint sets.

Therefore, we obtain the case of equality in the variational characterization of $\lambda_{k_m}(e^{2u_\varepsilon}, g)$ and deduce that there is a linear combination of the φ_i which is an eigenfunction associated to $\lambda_m(e^{2u_\varepsilon}, g)$. Such a function vanishes on an open set : this is absurd. $\diamondsuit$

From now to the end of the proof, we set $r_\star(p) = \min_{i=1,\cdots,s}\{\inf A_s^\varepsilon; \frac{|p-p_i|}{2}\} > 0$, and we say that $p_1, \cdots, p_s$ are bad points.

2.3.2. *Non concentration of the energy far from bad points.*

Claim 2.4. *Let $p \in \Sigma \setminus \{p_1, \cdots, p_s\}$, then*
(2.5)

$$\lim_{r\to 0} \limsup_{\varepsilon\to 0} \int_{\mathbb{D}_r(p)} e^{2u_\varepsilon} = \lim_{r\to 0} \limsup_{\varepsilon\to 0} \int_{\mathbb{D}_r(p)} \left| \nabla \frac{\Phi_\varepsilon}{\omega_\varepsilon} \right|_{\Lambda_\varepsilon}^2 = \lim_{r\to 0} \limsup_{\varepsilon\to 0} \int_{\mathbb{D}_r(p)} |\nabla\Phi_\varepsilon|_{\Lambda_\varepsilon}^2 = 0$$

Proof. Let $\eta \in \mathcal{C}_c^\infty(\mathbb{D}_{\sqrt{r}}(p))$ with $0 \leq \eta \leq 1$, $\eta = 1$ in $\mathbb{D}_r(p)$ and $\int_\Sigma |\nabla\eta|_g^2 \leq \frac{C}{\ln\left(\frac{1}{r}\right)}$, and we first have

$$\int_\Sigma \eta e^{2u_\varepsilon} \leq \left(\int_\Sigma \eta^2 e^{2u_\varepsilon} \right)^{\frac{1}{2}} \leq \left(\frac{1}{\lambda_\star\left(\mathbb{D}_{\sqrt{r}}(p), e^{2u_\varepsilon}\right)} \int_{\mathbb{D}_{\sqrt{r}}(p)} |\nabla\eta|^2 \right)^{\frac{1}{2}} \leq \left(\frac{C}{\lambda_{k_m^\varepsilon} \ln\frac{1}{r}} \right)^{\frac{1}{2}}$$

Now, we integrate the equation $\Delta\Phi_\varepsilon = \Lambda_\varepsilon e^{2u_\varepsilon}\Phi_\varepsilon$ against $\eta\Lambda_\varepsilon\frac{\Phi_\varepsilon}{\omega_\varepsilon^2}$ and we obtain

$$\int_\Sigma \eta\left\langle \nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon^2}, \nabla\Phi_\varepsilon\right\rangle_{\Lambda_\varepsilon} + \int_\Sigma \nabla\eta\nabla\Lambda_\varepsilon\Phi_\varepsilon\frac{\Phi_\varepsilon}{\omega_\varepsilon^2} = \int_\Sigma \eta e^{2u_\varepsilon}$$

so that

$$\begin{aligned}\int_\Sigma \eta\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_{\Lambda_\varepsilon} &= \int_\Sigma \eta e^{2u_\varepsilon} - \int_\Sigma \eta\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon^2}\nabla\omega_\varepsilon\Lambda_\varepsilon\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \int_\Sigma \nabla\eta\nabla\Phi_\varepsilon\Lambda_\varepsilon\frac{\Phi_\varepsilon}{\omega_\varepsilon^2}\\ &\leq \left(\frac{C}{\lambda_{k_m^\varepsilon}\ln\frac{1}{r}}\right)^{\frac12} + \int_\Sigma \eta\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon} + \left(\int_{\mathbb{D}_{\sqrt{r}}(p)}\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon^2}\right)^{\frac12}\left(\int_\Sigma|\nabla\eta|^2\right)^{\frac12}\end{aligned}$$

where we used that $|\Phi_\varepsilon|_{\Lambda_\varepsilon} = \omega_\varepsilon$, so that

$$\int_{\mathbb{D}_{\sqrt{r}}(p)}\eta\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_{\Lambda_\varepsilon} \leq \left(\frac{C}{\ln\frac{1}{r}}\right)^{\frac12}\left(\lambda_{k_m^\varepsilon}^{-\frac12} + O\left(\delta_\varepsilon^{\frac12}\right)\right) + O\left(\delta_\varepsilon\right)$$

Now, to conclude, we have by (2.4) that for any $p\in\Sigma$,

$$\int_{\mathbb{D}_{\sqrt{r}}(p)}|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon} - \int_{\mathbb{D}_r(p)}\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_{\Lambda_\varepsilon} \leq O\left(\delta_\varepsilon^{\frac14}\right)$$

and since we know that $\int_\Sigma|\nabla\phi_i^\varepsilon|^2 = t_i^\varepsilon\lambda_{k_i^\varepsilon}$ we obtain that

$$\int_{\mathbb{D}_{\sqrt{r}}(p)}|\nabla\Phi_\varepsilon|^2_{\Lambda_\varepsilon} \leq \max_j\lambda_{k_j}^\varepsilon\int_{\mathbb{D}_{\sqrt{r}}(p)}\left|\nabla\frac{\Phi}{\omega}\right|^2_{\Lambda_\varepsilon} + \sum_{i;\lambda_{k_i^\varepsilon}\to 0}t_i^\varepsilon\lambda_{k_i^\varepsilon} + O\left(\delta_\varepsilon^{\frac14}\right)$$

which completes the proof of the Claim. ♢

2.3.3. *Local H^1-convergence of eigenfunctions by harmonic replacement, in the case $\lambda_{k_1}\neq 0$.* Since the case $\lambda_{k_i}\neq 0$ is simple, we assume it in the following claim in order to obtain the strong H^1 convergence of Φ_ε far from $p_1,\cdots,p_s$

Claim 2.5. *We assume that the eigenvalues $\lambda_{k_i^\varepsilon}$ are lower bounded by a positive constant. Then Φ_ε converges to a harmonic map $\Phi_0:\Sigma\to\mathcal{E}_\Lambda$ in $H^1_{loc}(\Sigma\setminus\{p_1,\cdots,p_s\})$.*

Proof. Let $p\in\Sigma\setminus\{p_1,\cdots,p_s\}$. Since we have (2.5), and that $\lambda_{k_i^\varepsilon}$ are lower bounded by a positive constant, let $r>0$ be such that for any $\varepsilon>0$,

$$\int_{\mathbb{D}_r(p)}\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_g < \varepsilon_0.$$

Let $\Psi_\varepsilon:\mathbb{D}_r(p)\to\mathcal{E}_{\Lambda_\varepsilon}$ be the harmonic replacement of $\frac{\Phi_\varepsilon}{\omega_\varepsilon}$ and we obtain by [CM08] Theorem 3.1 (see also [LP19] Theorem 1.2)

$$\int_{\mathbb{D}_r(p)}\left|\nabla\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon}-\Psi_\varepsilon\right)\right|^2_g \leq 2\left(\int_{\mathbb{D}_r(p)}\left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_g - \int_{\mathbb{D}_r(p)}|\nabla\Psi_\varepsilon|^2_g\right). \tag{2.6}$$

Let's prove that the right-hand term converges to 0 as $\varepsilon \to 0$. We test the function $\frac{\Phi_\varepsilon^i}{\omega_\varepsilon} - \Psi_\varepsilon^i$ in the variational characterization of $\lambda_\star := \lambda_\star\left(\mathbb{D}_r(p), e^{2u_\varepsilon}\right)$:

$$\lambda_{k_i^\varepsilon} \int_{\mathbb{D}_r(p)} \left(\frac{\Phi_\varepsilon^i}{\omega_\varepsilon} - \Psi_\varepsilon^i\right)^2 e^{2u_\varepsilon} \leq \lambda_\star \int_{\mathbb{D}_r(p)} \left(\frac{\Phi_\varepsilon^i}{\omega_\varepsilon} - \Psi_\varepsilon^i\right)^2 e^{2u_\varepsilon} \leq \int_{\mathbb{D}_r(p)} \left|\nabla\left(\frac{\Phi_\varepsilon^i}{\omega_\varepsilon} - \Psi_\varepsilon^i\right)\right|^2$$

and we sum on i to get

$$\int_{\mathbb{D}_r(p)} \left|\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right|_{\Lambda_\varepsilon}^2 e^{2u_\varepsilon} \leq \int_{\mathbb{D}_r(p)} \left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2 + \int_{\mathbb{D}_r(p)} |\nabla\Psi_\varepsilon|^2 - 2\int_{\mathbb{D}_r(p)} \nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\nabla\Psi_\varepsilon \tag{2.7}$$

Now we test the equation $\Delta\Phi_\varepsilon = \Lambda_\varepsilon e^{2u_\varepsilon}\Phi_\varepsilon$ against $\frac{\Phi_\varepsilon}{\omega_\varepsilon^2} - \frac{\Psi_\varepsilon}{\omega_\varepsilon}$ and we get after an integration by part knowing that $\frac{\Phi_\varepsilon}{\omega_\varepsilon^2} - \frac{\Psi_\varepsilon}{\omega_\varepsilon} = 0$ on $\partial\mathbb{D}_r(p)$

$$\int_{\mathbb{D}_r(p)} \left(\frac{1}{\omega_\varepsilon}\nabla\Phi_\varepsilon\nabla\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right) + \nabla\frac{1}{\omega_\varepsilon}\nabla\Phi_\varepsilon\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right)\right) = \int_{\mathbb{D}_r(p)} \left\langle\frac{\Phi_\varepsilon}{\omega_\varepsilon}, \frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right\rangle_{\Lambda_\varepsilon} e^{2u_\varepsilon}$$

so that,

$$\begin{aligned}\int_{\mathbb{D}_r(p)} \nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\nabla\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right) &= \int_{\mathbb{D}_r(p)} \left\langle\frac{\Phi_\varepsilon}{\omega_\varepsilon}, \frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right\rangle_{\Lambda_\varepsilon} e^{2u_\varepsilon} - \int_{\mathbb{D}_r(p)} \nabla\frac{1}{\omega_\varepsilon}\nabla\Phi_\varepsilon\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right) \\ &\quad + \int_{\mathbb{D}_r(p)} \Phi_\varepsilon\nabla\frac{1}{\omega_\varepsilon}\nabla\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right)\end{aligned} \tag{2.8}$$

Knowing that $\left|\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|_{\Lambda_\varepsilon} = |\Psi_\varepsilon|_{\Lambda_\varepsilon}$, it is clear that $2\left\langle\frac{\Phi_\varepsilon}{\omega_\varepsilon}, \frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right\rangle_{\Lambda_\varepsilon} = \left|\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right|_{\Lambda_\varepsilon}^2$ and multiplying (2.8) by 2, we obtain

$$\begin{aligned}&2\int_{\mathbb{D}_r(p)} \left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2 - 2\int_{\mathbb{D}_r(p)} \nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\nabla\Psi_\varepsilon \\ &= \int_{\mathbb{D}_r(p)} \left|\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right|_{\Lambda_\varepsilon}^2 e^{2u_\varepsilon} + 2\int_{\mathbb{D}_r(p)} \frac{\nabla\omega_\varepsilon}{\omega_\varepsilon}\left(\frac{\nabla\Phi_\varepsilon}{\omega_\varepsilon}\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right) - \frac{\Phi_\varepsilon}{\omega_\varepsilon}\nabla\left(\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right)\right) \\ &\leq \int_{\mathbb{D}_r(p)} \left|\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon\right|_{\Lambda_\varepsilon}^2 e^{2u_\varepsilon} + O\left(\frac{\delta_\varepsilon^{\frac{1}{2}}}{\lambda_{k_1^\varepsilon}^{\frac{1}{2}}}\right)\end{aligned} \tag{2.9}$$

Summing (2.7) and (2.9), we get

$$\int_{\mathbb{D}_r(p)} \left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|_g^2 - \int_{\mathbb{D}_r(p)} |\nabla\Psi_\varepsilon|_g^2 \leq O\left(\frac{\delta_\varepsilon^{\frac{1}{2}}}{\lambda_{k_1^\varepsilon}^{\frac{1}{2}}}\right)$$

By (2.6), we obtain that

$$\frac{\Phi_\varepsilon}{\omega_\varepsilon} - \Psi_\varepsilon \to 0 \text{ in } H^1\left(\mathbb{D}_r(p)\right)$$

Now since Ψ_ε converges in $\mathcal{C}^k\left(\mathbb{D}_{\frac{r}{2}}(p)\right)$ for any $k \in \mathbb{N}$ to some harmonic map Φ, and because the property is true for all $p \in \Sigma \setminus \{p_1, \cdots, p_s\}$, we have that $\frac{\Phi_\varepsilon}{\omega_\varepsilon}$ converges to some harmonic map Φ_0 in $H^1_{loc}(\Sigma \setminus \{p_1, \cdots, p_s\})$. Then by (2.4) Φ_ε converges to Φ_0 in $H^1_{loc}(\Sigma \setminus \{p_1, \cdots, p_s\})$.

$\diamondsuit$

2.3.4. *Local H^1-convergence of eigenfunctions by harmonic replacement, general case.* In the general case, the obstruction to perform the proof of Claim 2.5 is the lack of L^∞ control of $\frac{\phi^i_\varepsilon}{\omega_\varepsilon}$ in the case $\lambda_{k^\varepsilon_i} \to 0$. However, thanks to the Courant-Lebesgue lemma and the equation satisfied by ϕ^i_ε we can even assume the boundedness of ϕ^i_ε, but only on some circle. This proves the needed boundedness results on the harmonic replacement (see claim (2.7)). Before that, we need a bound on the eigenfunctions ϕ^i_ε in $L^2_{loc}\left(\Sigma \setminus \{p_1, \cdots, p_s\}, g\right)$, remembering that the $L^2\left(\Sigma, e^{2u_\varepsilon} g\right)$-norm of ϕ^i_ε is $\sqrt{t^\varepsilon_i}$. The Poincaré inequality used there is also a first step used in [Pet14a], [Pet18].

Claim 2.6. *We assume that*

$$\lim_{r\to 0} \liminf_{\varepsilon \to 0} \int_{\Sigma \setminus \bigcup_{j=1}^s \mathbb{D}_r(p_j)} e^{2u_\varepsilon} dA_g > 0$$

Then, for any $i \in \{1, \cdots, m\}$, for any $\rho > 0$ small enough, there is a constant $C(\rho) > 0$ such that for any ε,

$$\int_{\Sigma \setminus \bigcup_{j=1}^s \mathbb{D}_\rho(p_j)} \left(\phi^i_\varepsilon - m_{\varepsilon,i,\rho}\right)^2 dA_g \leq C(\rho) \int_\Sigma \left|\nabla \phi^i_\varepsilon\right|^2_g dA_g$$

where

$$m_{\varepsilon,i,\rho} = \int_{\Sigma \setminus \bigcup_{j=1}^s \mathbb{D}_\rho(p_j)} \phi^i_\varepsilon \frac{e^{2u_\varepsilon} dA_g}{\int_{\Sigma \setminus \bigcup_{j=1}^s \mathbb{D}_\rho(p_j)} e^{2u_\varepsilon} dA_g} \text{ and } \lim_{\rho \to 0} \lim_{\varepsilon \to 0} \frac{|m_{\varepsilon,i,\rho}|}{\sqrt{t^i_\varepsilon}} \leq 1$$

Proof. We have the following Poincaré inequality for any domain Ω of Σ

$$\int_\Omega \left(\zeta - \int_\Omega \zeta \frac{e^{2u_\varepsilon}}{\int_\Omega e^{2u_\varepsilon} dA_g} dA_g\right)^2 dA_g \leq C \left\| \frac{e^{2u_\varepsilon}}{\int_\Omega e^{2u_\varepsilon} dA_g} \right\|^2_{H^{-1}(\Omega)} \int_\Omega |\nabla \zeta|^2_g dA_g.$$

We choose $\Omega = \Sigma \setminus \bigcup_{j=1}^s \mathbb{D}_\rho(p_j)$ for $\rho > 0$, and we aim at proving that there is a constant $C(\rho) > 0$ such that

$$\left\| \frac{e^{2u_\varepsilon}}{\int_\Omega e^{2u_\varepsilon} dA_g} \right\|^2_{H^{-1}(\Omega)} \leq C(\rho).$$

With the first assumption of Claim 2.6, it suffices to prove that $\left\|e^{2u_\varepsilon}\right\|^2_{H^{-1}(\Omega)} \leq C(\rho)$. This property is true because of the globalization, up to partition of unity of the following estimate coming form Claim 2.3: for any $p \in \Sigma \setminus \{p_1, \cdots, p_s\}$ and $\zeta \in H^1_0\left(\mathbb{D}_{r_\star(p)}(p)\right)$,

$$\begin{aligned}\int_{\mathbb{D}_{r_\star(p)}(p)} \zeta e^{2u_\varepsilon} \leq \left(\int_{\mathbb{D}_{r_\star(p)}(p)} \zeta^2 e^{2u_\varepsilon}\right)^{\frac{1}{2}} &\leq \left(\frac{1}{\lambda_\star\left(\mathbb{D}_{r_\star(p)}(p), e^{2u_\varepsilon}\right)} \int_{\mathbb{D}_{r_\star(p)}(p)} |\nabla \zeta|^2\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\lambda_{k_m}\left(e^{2u_\varepsilon} g\right)}\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}_{r_\star(p)}(p)} |\nabla \zeta|^2\right)^{\frac{1}{2}}\end{aligned}$$

where $\lambda_{k_m}\left(e^{2u_\varepsilon} g\right)$ is uniformly lower bounded. The number of disks in the partition of unity give the dependence with respect to ρ of the upper bound of $\left\|e^{2u_\varepsilon}\right\|^2_{H^{-1}(\Omega)}$. Then, we apply the Poincaré inequality to all the eigenfunctions in order to obtain the Claim. $\diamondsuit$

Up to a subsequence, let $I \in \{1, \cdots, m\}$ be such that for $i \leq I$, $\lambda_\varepsilon^i \to 0$ and for $i \geq I+1$, λ_ε^i is uniformly lower bounded. Thanks to Claim 2.6, we know that for $1 \leq i \leq I$, there is a sequence of constant functions m_ε^i (which does not depend on ρ) such that

$$\phi_\varepsilon^i - m_\varepsilon^i \to 0 \text{ in } H^1_{loc}(\Sigma \setminus \{p_1, \cdots, p_s\}) \tag{2.10}$$

and

$$\forall \rho_0 > 0, \exists \varepsilon_0 > 0, \forall 0 < \varepsilon \leq \varepsilon_0, \left|m_\varepsilon^i\right| \leq \sqrt{t_\varepsilon^i}\,(1+\rho_0) \tag{2.11}$$

We denote $\Phi_\varepsilon = (\phi_\varepsilon^1, \cdots, \phi_\varepsilon^I, \widetilde{\Phi_\varepsilon})$. It remains to prove the H^1 convergence of $\widetilde{\Phi_\varepsilon}$. We assume that there is a constant c_0 such that for any $\varepsilon > 0$,

$$1 - \sum_{i=1}^{I} t_i^\varepsilon \lambda_i^\varepsilon \geq c_0 \tag{2.12}$$

because if it is not the case, we would directly have the H^1 convergence of $\widetilde{\Phi_\varepsilon}$ to 0. Let's fist prove that in this case, the harmonic replacement of $\widetilde{\Phi_\varepsilon}$ is well defined.

Claim 2.7. *For any $p \in \Sigma \setminus \{p_1, \cdots, p_s\}$, there is $r_0 > 0$ such that for any ε there is $r_0 \leq r_\varepsilon \leq r_\star(p)$, such that the harmonic replacement $\widetilde{\Psi_\varepsilon} : \Sigma \to \mathcal{E}_{\widetilde{\Lambda_\varepsilon}}$ of $\frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}}$ on $\mathbb{D}_{r_\varepsilon}(p)$ satisfies*

$$\int_{\mathbb{D}_{r_\varepsilon}(p)} \left|\nabla \widetilde{\Psi_\varepsilon}\right|_g^2 dA_g \leq \varepsilon_0 \tag{2.13}$$

where $\widetilde{\omega_\varepsilon}$ is defined on $\partial \mathbb{D}_{r_\varepsilon}(p)$ by

$$\widetilde{\omega_\varepsilon} = \sqrt{\sum_{i=I+1}^{m} \lambda_{k_i^\varepsilon} (\phi_i^\varepsilon)^2} = \sqrt{\omega_\varepsilon^2 - \sum_{i=1}^{I} \lambda_{k_i^\varepsilon} (\phi_i^\varepsilon)^2} \text{ and } \widetilde{\Lambda_\varepsilon} = diag\left(\lambda_\varepsilon^{I+1}, \cdots, \Lambda_\varepsilon^m\right)$$

and

$$\frac{\omega_\varepsilon}{\widetilde{\omega_\varepsilon}} \leq \sqrt{\frac{2}{c_0}} \tag{2.14}$$

Proof. Let $\eta > 0$ a small constant we explicit later. Up to a subsequence, let $I \in \{1, \cdots, m\}$ be such that for $i \leq I$, $\lambda_\varepsilon^i \to 0$ and for $i \geq I+1$, λ_ε^i is uniformly lower bounded. Thanks to (2.5), let $r > 0$ a positive radius such that $r \leq r_\star(p)$ and for any $\varepsilon > 0$,

$$\int_{\mathbb{D}_r(p)} |\nabla \Phi_\varepsilon|_g^2 \leq \eta \varepsilon_0.$$

By the Courant-Lebesgue lemma, let $\frac{r}{2} \leq r_\varepsilon \leq r$ be a radius such that

$$\int_{\partial \mathbb{D}_{r_\varepsilon}(p)} |\partial_\theta \Phi_\varepsilon|^2 \, d\theta \leq \frac{1}{\ln 2} \int_{\mathbb{D}_r(p)} |\nabla \Phi_\varepsilon|_g^2 \leq \frac{1}{\ln 2} \eta \varepsilon_0. \tag{2.15}$$

and as a consequence

$$\forall q, q' \in \partial \mathbb{D}_{r_\varepsilon}(p); \left|\Phi_\varepsilon(q) - \Phi_\varepsilon(q')\right|^2 \leq \frac{\pi}{\ln 2} \eta \varepsilon_0. \tag{2.16}$$

By the classical trace L^2 embedding into H^1 and (2.10), we have that for $1 \leq i \leq I$,

$$\int_{\partial \mathbb{R}_{r_\varepsilon}(p)} \left(\phi_\varepsilon^i - m_\varepsilon^i\right)^2 \to 0$$

as $\varepsilon \to 0$. Using this and (2.16), we obtain for $1 \leq i \leq I$ and ε small enough:

$$\forall q, q' \in \partial \mathbb{D}_{r_\varepsilon}(p); \left|\phi^i_\varepsilon(q) - m^i_\varepsilon\right|^2 \leq \frac{\pi}{\ln 2}\eta\varepsilon_0.$$

On $\partial \mathbb{D}_{r_\varepsilon}(p)$, we have from (2.13) with $\rho_0 = \frac{c_0}{4}$, $\varepsilon_0 > 0$ that for ε small enough,

$$\begin{aligned}\frac{\widetilde{\omega_\varepsilon}^2}{\omega_\varepsilon^2} = 1 - \frac{1}{\omega_\varepsilon^2}\sum_{i=1}^I \lambda_{k_i^\varepsilon}\left(\phi_i^\varepsilon\right)^2 &\geq 1 - \frac{1}{1+\delta_\varepsilon}\sum_{i=1}^I \lambda_{k_i^\varepsilon}\left(m_i^\varepsilon + \sqrt{\frac{\pi}{\ln 2}\eta\varepsilon_0}\right)^2 \\ &\geq 1 - \sum_{i=1}^I \lambda_{k_i^\varepsilon} t_i^\varepsilon(1+\rho_0) - O(\lambda_{k_I^\varepsilon} + \delta_\varepsilon) \geq \frac{3c_0}{4} - O(\lambda_{k_I^\varepsilon} + \delta_\varepsilon)\end{aligned}$$

and we obtain (2.14). Then $\widetilde{\Psi_\varepsilon}$ is well defined. Let's prove the energy bound (2.13). Let $H_\varepsilon : \mathbb{D}_{r_\varepsilon}(p) \to \mathbb{R}^{m-I}$ be the (Euclidean) harmonic extension of $\widetilde{\Phi_\varepsilon} : \partial\mathbb{D}_{r_\varepsilon}(p) \to \mathbb{R}^{m-I}$ on $\mathbb{D}_{r_\varepsilon}(p)$. We have that

$$\int_{\mathbb{D}_{r_\varepsilon}(p)} \left|\nabla \widetilde{\Psi_\varepsilon}\right|_g^2 dA_g \leq \int_{\mathbb{D}_{r_\varepsilon}(p)} \left|\nabla \frac{H_\varepsilon}{|H_\varepsilon|_{\widetilde{\Lambda_\varepsilon}}}\right|_g^2 dA_g$$

where we choose η such that $|H_\varepsilon|_{\widetilde{\Lambda_\varepsilon}}$ is uniformly lower bounded by a positive constant:

$$\min_{q\in\mathbb{D}_{r_\varepsilon}(p)} |H_\varepsilon(q)|^2_{\widetilde{\Lambda_\varepsilon}} \geq \sum_{i=I+1}^m \lambda_{k_i^\varepsilon} \min_{q\in\mathbb{D}_{r_\varepsilon}(p)} \left(H_i^\varepsilon(q)\right)^2 \geq \sum_{i=I+1}^m \lambda_{k_i^\varepsilon} \min_{q\in\partial\mathbb{D}_{r_\varepsilon}(p)} \left(\phi_i^\varepsilon(q)\right)^2$$

where the second inequality comes from the maximum principle for harmonic functions. Let $q^i_\varepsilon \in \partial\mathbb{D}_{r_\varepsilon}(p)$ be such that $\phi^i_\varepsilon(q^i_\varepsilon)^2 = \min_{q\in\partial\mathbb{D}_{r_\varepsilon}(p)} \left(\phi_i^\varepsilon(q)\right)^2$. We choose $q \in \partial\mathbb{D}_{r_\varepsilon}(p)$

$$|H_\varepsilon|_{\widetilde{\Lambda_\varepsilon}}(q) \geq \widetilde{\omega_\varepsilon}(q) - \sqrt{\sum_{i=I+1}^m \lambda_{k_i^\varepsilon}\left(\phi^i_\varepsilon(q) - \phi^i_\varepsilon(q^i_\varepsilon)\right)^2} \geq \sqrt{\frac{c_0}{2}} - \sqrt{\sum_{i=I+1}^m \lambda_{k_\varepsilon^i}\frac{\pi}{\ln 2}\eta\varepsilon_0}$$

so that choosing $\eta \leq \left(\sum_{i=I+1}^m \lambda_{k_\varepsilon^i}\frac{\pi}{\ln 2}\varepsilon_0\right)^{-1}\frac{c_0}{8}$, we obtain that $|H_\varepsilon|_{\widetilde{\Lambda_\varepsilon}}$ is uniformly lower bounded by $\sqrt{\frac{c_0}{8}}$. By a straightforward computation, we obtain that

$$\int_{\mathbb{D}_{r_\varepsilon}(p)} \left|\nabla \frac{H_\varepsilon}{|H_\varepsilon|_{\widetilde{\Lambda_\varepsilon}}}\right|_g^2 dA_g \leq K\int_{\mathbb{D}_{r_\varepsilon}(p)} |\nabla H_\varepsilon|_g^2 dA_g \leq K\int_{\mathbb{D}_{r_\varepsilon}(p)} \left|\nabla\widetilde{\Phi_\varepsilon}\right|_g^2 dA_g \leq K\eta\varepsilon_0$$

where K is independant of ε and η. Choosing $\eta \leq K^{-1}$ completes the proof of the Claim. ♢

In order to define a convenient replacement, we define for $i \in \{1, \cdots, I\}$ the harmonic function ψ^i_ε such that $\psi^i_\varepsilon = \phi^i_\varepsilon$ on $\partial\mathbb{D}_{r_\varepsilon}(p)$. We complete the definition of a replacement of Φ_ε in the whole target ellipsoid

$$\Psi_\varepsilon = \left(\psi^1_\varepsilon, \cdots, \psi^I_\varepsilon, \widetilde{\omega_\varepsilon}\widetilde{\Psi_\varepsilon}\right) : \mathbb{D}_{r_\varepsilon}(p) \to \mathcal{E}_{\Lambda_\varepsilon} \text{ where } \widetilde{\omega_\varepsilon} = \sqrt{\omega_\varepsilon^2 - \sum_{i=1}^I \lambda_{k_i^\varepsilon}\left(\psi^i_\varepsilon\right)^2} \tag{2.17}$$

We are now in position to prove the expected general result

Claim 2.8. *Φ_ε converges to a harmonic map $\Phi_0 : \Sigma \to \mathcal{E}_\Lambda$ in $H^1_{loc}(\Sigma \setminus \{p_1, \cdots, p_s\})$.*

Proof. Let $p \in \Sigma \setminus \{p_1, \cdots, p_s\}$ and let r_ε and Ψ_ε the replacement of Φ_ε on $\mathbb{D}_{r_\varepsilon}(p)$ be given by (2.17). We first test $\frac{\widetilde{\Phi_\varepsilon} - \widetilde{\omega_\varepsilon}\widetilde{\Psi_\varepsilon}}{\widetilde{\omega_\varepsilon}^2}$ against the equation satisfied by $\widetilde{\Phi_\varepsilon}$. We obtain

$$\begin{aligned}
\int_{\mathbb{D}_{r_\varepsilon}(p)} \left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \frac{\widetilde{\Phi_\varepsilon} - \widetilde{\omega_\varepsilon}\widetilde{\Psi_\varepsilon}}{\widetilde{\omega_\varepsilon}^2} \right\rangle &= \int_{\mathbb{D}_{r_\varepsilon}(p)} e^{2u_\varepsilon} \left\langle \widetilde{\Lambda_\varepsilon} \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}}, \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right\rangle \\
&= \frac{1}{2} \int_{\mathbb{D}_{r_\varepsilon}(p)} e^{2u_\varepsilon} \left| \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right|^2_{\widetilde{\Lambda_\varepsilon}} - \frac{1}{2} \sum_{i=1}^{I} \lambda_{k_i^\varepsilon} \int_{\mathbb{D}_{r_\varepsilon}(p)} \frac{e^{2u_\varepsilon}}{\widetilde{\omega_\varepsilon}^2} \left(\left(\phi_\varepsilon^i\right)^2 - \left(\psi_\varepsilon^i\right)^2 \right) \\
&\leq \frac{1}{2} \sum_{i=I+1}^{m} \frac{\lambda_{k_i^\varepsilon}}{\lambda_{k_m^\varepsilon}} \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\frac{\widetilde{\phi_\varepsilon^i}}{\widetilde{\omega_\varepsilon}} - \widetilde{\psi_\varepsilon^i} \right) \right|^2 \\
&\quad + \sum_{i=1}^{I} \frac{8}{c_0} \lambda_{k_i^\varepsilon} \left(\int_{\mathbb{D}_{r_\varepsilon}(p)} \left(\phi_\varepsilon^i + \psi_\varepsilon^i\right)^2 e^{2u_\varepsilon} \right)^{\frac{1}{2}} \left(\frac{1}{\lambda_{k_m^\varepsilon}} \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\phi_\varepsilon^i - \psi_\varepsilon^i\right) \right|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right) \right|^2 + C \sum_{i=1}^{I} \frac{8}{c_0} \frac{\lambda_{k_i^\varepsilon}}{\lambda_{k_m^\varepsilon}} \left(t_\varepsilon^i\right)^{\frac{1}{2}} \left(\lambda_\varepsilon^i t_\varepsilon^i\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right) \right|^2 + O\left(\left(\lambda_{k_\varepsilon^I}\right)^{\frac{1}{2}} \right)
\end{aligned}$$

which gives

$$\begin{aligned}
\int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right|^2 - \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \widetilde{\Psi_\varepsilon} \right|^2 &\leq O\left(\left(\lambda_{k_\varepsilon^I}\right)^{\frac{1}{2}} \right) \\
&\quad + \int_{\mathbb{D}_{r_\varepsilon}(p)} \frac{\nabla \widetilde{\omega_\varepsilon}}{\widetilde{\omega_\varepsilon}} \left(\frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \nabla \left(\frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right) - \left(\frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right) \frac{\nabla \widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \\
&\leq O\left(\left(\lambda_{k_\varepsilon^I}\right)^{\frac{1}{2}} + \delta_\varepsilon^{\frac{1}{2}} \right)
\end{aligned} \tag{2.18}$$

Now, we test $\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}}$ against the harmonic map equation satisfied by $\widetilde{\Psi_\varepsilon}$. We obtain in a similar way

$$\begin{aligned}
I := 2 \int_{\mathbb{D}_{r_\varepsilon}(p)} \left\langle \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right), \nabla \widetilde{\Psi_\varepsilon} \right\rangle &= \int_{\mathbb{D}_{r_\varepsilon}(p)} \frac{\left| \nabla \widetilde{\Psi_\varepsilon} \right|^2_{\widetilde{\Lambda_\varepsilon}}}{\left| \widetilde{\Lambda_\varepsilon} \widetilde{\Psi_\varepsilon} \right|^2} \left\langle \widetilde{\Lambda_\varepsilon} \widetilde{\Psi_\varepsilon}, \widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right\rangle \\
&= \int_{\mathbb{D}_{r_\varepsilon}(p)} \frac{\left| \nabla \widetilde{\Psi_\varepsilon} \right|^2_{\widetilde{\Lambda_\varepsilon}}}{\left| \widetilde{\Lambda_\varepsilon} \widetilde{\Psi_\varepsilon} \right|^2} \left(\left| \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} - \widetilde{\Psi_\varepsilon} \right|^2_{\widetilde{\Lambda_\varepsilon}} - \sum_{i=1}^{I} \frac{\lambda_{k_i^\varepsilon}}{\widetilde{\omega_\varepsilon}^2} \left(\left(\psi_\varepsilon^i\right)^2 - \left(\phi_\varepsilon^i\right)^2 \right) \right)
\end{aligned}$$

and using the consequence of the ε-regularity result on harmonic maps $|\nabla \Psi_\varepsilon|^2 \leq \frac{C}{(1-|x|)^2}$ and a Hardy inequality (this is the way we prove energy convexity results in [LP19]), we

obtain:

$$\begin{aligned} I \leq & C\epsilon_0 \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \right|^2 + C\epsilon_0 \sum_{i=1}^{I} \frac{8}{c_0} \lambda_{k_i^\varepsilon} \left(t_\varepsilon^i \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\psi_\varepsilon^i - \phi_\varepsilon^i \right) \right|^2 \right)^{\frac{1}{2}} \\ \leq & C\epsilon_0 \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \right|^2 + O\left(\left(\lambda_{k_I^\varepsilon} \right)^{\frac{1}{2}} \right). \end{aligned}$$

Letting ε_0 be small enough such that $1 - C\varepsilon_0 \geq \frac{1}{2}$ we obtain that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \right|^2 & \leq (1 - C\varepsilon_0) \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \right|^2 \\ & \leq \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \right|^2 - 2 \int_{\mathbb{D}_{r_\varepsilon}(p)} \left\langle \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right), \nabla \widetilde{\Psi_\varepsilon} \right\rangle - O\left(\left(\lambda_{k_I^\varepsilon} \right)^{\frac{1}{2}} \right) \\ & \leq \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right|^2 - \int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \widetilde{\Psi_\varepsilon} \right|^2 - O\left(\left(\lambda_{k_I^\varepsilon} \right)^{\frac{1}{2}} \right). \end{aligned}$$

It implies

$$\int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\widetilde{\omega_\varepsilon}} \right) \right|^2 = O\left(\left(\lambda_{k_I^\varepsilon} \right)^{\frac{1}{2}} + \delta_\varepsilon^{\frac{1}{2}} \right)$$

and we also obtain

$$\int_{\mathbb{D}_{r_\varepsilon}(p)} \left| \nabla \left(\frac{\widetilde{\omega_\varepsilon}}{\omega_\varepsilon} \widetilde{\Psi_\varepsilon} - \frac{\widetilde{\Phi_\varepsilon}}{\omega_\varepsilon} \right) \right|^2 = O\left(\left(\lambda_{k_I^\varepsilon} \right)^{\frac{1}{2}} + \delta_\varepsilon^{\frac{1}{2}} \right)$$

as $\varepsilon \to 0$. We obtain the expected local H^1 comparison of $\frac{\widetilde{\Phi_\varepsilon}}{\omega_\varepsilon}$ to a harmonic map into $\mathcal{E}_{\Lambda_\varepsilon}$ times a function $\frac{\widetilde{\omega_\varepsilon}}{\omega_\varepsilon}$ uniformly bounded by 1 and uniformly lower bounded by $\sqrt{\frac{c_0}{2}}$ which converges to a constant function in H^1 and we conclude like in the end of the proof of Claim 2.5. $\diamondsuit$

2.3.5. *The limiting measure has a smooth density.* Let $\zeta \in \mathcal{C}_c^\infty(\mathbb{D}_r(p))$, we have that

$$\begin{aligned} \int_\Sigma \zeta \left(\Lambda_\varepsilon \Phi_\varepsilon e^{2u_\varepsilon} dA_g - \Lambda \Phi_0 d\tilde{\mu}_0 \right) &= \int_\Sigma \zeta \left(\Lambda_\varepsilon \Phi_\varepsilon - \Lambda \Phi_0 \right) e^{2u_\varepsilon} dA_g \\ & \quad + \int_\Sigma \zeta \Lambda \Phi_0 \left(e^{2u_\varepsilon} dA_g - d\tilde{\mu}_0 \right). \end{aligned}$$

Then on the first right-hand term, we have that

$$\begin{aligned}\int_\Sigma \zeta\left(\Lambda_\varepsilon\Phi_\varepsilon-\Lambda\Phi_0\right)e^{2u_\varepsilon}dA_g &\leq \left(\int_{\mathbb{D}_r(p)}\zeta^2\left|\Lambda_\varepsilon\Phi_\varepsilon-\Lambda\Phi_0\right|^2e^{2u_\varepsilon}dA_g\right)^{\frac{1}{2}}\\ &\leq \left(\frac{1}{\lambda_\star(\mathbb{D}_r(p),e^{2u_\varepsilon})}\int_{\mathbb{D}_r(p)}\left|\nabla\left(\zeta\left|\Lambda_\varepsilon\Phi_\varepsilon-\Lambda\Phi_0\right|\right)\right|_g^2dA_g\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\mathbb{D}_r(p)}\left|\nabla\left(\Phi_\varepsilon-\Phi_0\right)\right|_g^2dA_g\right)^{\frac{1}{2}}\end{aligned}$$

for some constant C independent of ε. Letting $\varepsilon\to 0$ in a weak sense to the eigenvalue equation $\Delta_g\Phi_\varepsilon=\Lambda_\varepsilon\Phi_\varepsilon e^{2u_\varepsilon}$, we get

$$\Delta\Phi_0=\Lambda\Phi_0\tilde{\mu}_0$$

and since Φ_0 is harmonic, we obtain that $\tilde{\mu}_0=\frac{|\nabla\Phi_0|^2_\Lambda}{|\Lambda\Phi_0|^2_\Lambda}dA_g$.

With a same analysis in the scales α_ε^j centered at q_ε^j, we obtain limits Φ_j which are harmonic maps on $\mathbb{R}^2$ into $\mathcal{E}_\Lambda$ such that $\tilde{\mu}_j=\frac{|\nabla\Phi_j|^2_\Lambda}{|\Lambda\Phi_j|^2_\Lambda}dx$. Up to a stereographic projection and a standard point removability theorem on finite energy harmonic maps at the point ∞, these quantities for $1\leq j\leq l$ can be seen on $\mathbb{S}^2$.

2.4. **Extra convergence results.**

2.4.1. *Global H^1 convergence of eigenfunctions.* Notice that at that stage Φ_j for $0\leq j\leq l$ are weak limits of (the rescalings of) Φ_ε in $H^1(\Sigma)$ (or $H^1_{loc}\left(\mathbb{R}^2\right)$). We prove here that this weak limit of this bubble tree is strong, meaning that we have the energy identity given by Proposition 2.1. The main point is to prove that H^1 convergence also holds at the neighborhood of bad points. We set for $\rho>0$, and $1\leq j\leq l$ (l is given by Propositon 2.1)

$$U_0(\rho)=\Sigma\setminus\bigcup_{i=1}^{s_0}\mathbb{D}_\rho(p_i^0)\text{ and }U_j(\rho)=\mathbb{D}_{\frac{1}{\rho}}\setminus\bigcup_{i=1}^{s_j}\mathbb{D}_\rho(p_i^j)$$

where $\{p_1^0,\cdots,p_{s_0}^0\}=\{p_1,\cdots,p_s\}$ are the bad points given by Claim 2.3 and for $1\leq j\leq l$, $\{p_1^j,\cdots,p_{s_j}^j\}$ are the rescaled bad points: it is the set of the finite limits (up to a subsequence as $\varepsilon\to 0$) of $\frac{p_i^\varepsilon-q_i^\varepsilon}{\alpha_i^\varepsilon}$ where the indices i satisfy $\frac{r_i^\varepsilon}{\alpha_i^\varepsilon}\to 0$ as $\varepsilon\to 0$. We set for $1\leq j\leq l$, $e^{2u_j^\varepsilon(z)}=\left(\alpha_\varepsilon^j\right)^2e^{2u_\varepsilon(q_\varepsilon+\alpha_\varepsilon^jz)}$ and $\Phi_j^\varepsilon(z)=\Phi_\varepsilon(q_\varepsilon+\alpha_\varepsilon^jz)$, the rescaled quantities at j (for $j=0$, there is no rescaling). We denote $Q(\varepsilon,\rho)$ any quantity which satisfies $\lim_{\rho\to 0}\lim_{\varepsilon\to 0}Q(\varepsilon,\rho)=0$. By Proposition 2.1, we have:

$$\int_\Sigma|\nabla\Phi_\varepsilon|^2=\int_\Sigma e^{2u_\varepsilon}=\sum_{j=0}^l\int_{U_j(\sqrt{\rho})}e^{2u_j^\varepsilon}+Q(\varepsilon,\rho)=\int_\Sigma d\tilde{\mu}_0+\sum_{j=1}^l\int_{\mathbb{R}^2}d\tilde{\mu}_j+Q(\varepsilon,\rho)$$

and integrating the limiting equations on Φ_j (limit of Φ_j^ε as $\varepsilon\to 0$) against Φ_j, we obtain

$$\int_\Sigma|\nabla\Phi_\varepsilon|^2=\int_\Sigma|\nabla\Phi_0|^2dA_g+\sum_{j=1}^l\int_{\mathbb{R}^2}|\nabla\Phi_j|^2+Q(\varepsilon,\rho)$$

and letting $\varepsilon \to 0$ and then $\rho \to 0$, and using the weak convergence, we obtain the expected identity of energy.

2.4.2. *H^{-1} convergence of conformal factors.*

Claim 2.9. *We assume that $\lambda_1(e^{2u_\varepsilon} g)$ is uniformly lower bounded by a positive constant. Then, Φ_ε converges strongly in $H^1(\Sigma)$ and e^{2u_ε} converges strongly to e^{2u} in $H^{-1}(\Sigma)$.*

Proof. Notice that thanks to the assumption that $\lambda_1^\varepsilon := \lambda_1(e^{2u_\varepsilon} g)$ is uniformly lower bounded, there is not any bubble tree, and Φ_ε converges in H^1. Let $\Psi \in H^1(\Sigma)$, we have

$$\int_\Sigma \nabla\Phi\nabla\Psi = \int_\Sigma e^{2u} \langle \Lambda\Phi, \Psi\rangle \text{ and } \int_\Sigma \nabla\Phi_\varepsilon\nabla\Psi = \int_\Sigma e^{2u_\varepsilon} \langle \Lambda_\varepsilon\Phi_\varepsilon, \Psi\rangle$$

so that

$$\left|\int_\Sigma e^{2u} \langle \Lambda\Phi, \Psi\rangle - \int_\Sigma e^{2u_\varepsilon} \langle \Lambda_\varepsilon\Phi_\varepsilon, \Psi\rangle\right| \leq \|\nabla(\Phi - \Phi_\varepsilon)\|_2 \|\nabla\Psi\|_2 .$$

Knowing that

$$\begin{aligned}&\left|\int_\Sigma \left\langle \Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon} - \Phi_\varepsilon\right), \Psi\right\rangle e^{2u_\varepsilon}\right| \\ =&\left|\int_\Sigma \left\langle \Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon} - \Phi_\varepsilon\right), \left(\Psi - \int_\Sigma \Psi e^{2u_\varepsilon}\right)\right\rangle e^{2u_\varepsilon}\right| \leq \frac{1}{\lambda_1^\varepsilon} \|\nabla(\Phi - \Phi_\varepsilon)\|_2 \|\nabla\Psi\|_2\end{aligned}$$

we obtain that

$$\left|\int_\Sigma e^{2u} \langle \Lambda\Phi, \Psi\rangle - \int_\Sigma e^{2u_\varepsilon} \left\langle \Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon}\right), \Psi\right\rangle\right| \leq C \|\nabla(\Phi - \Phi_\varepsilon)\|_2 \|\nabla\Psi\|_2$$

It is clear that

$$\int_\Sigma \Phi e^{2u_\varepsilon} = \int_\Sigma \Phi\left(e^{2u_\varepsilon} - e^{2u}\right) \to 0$$

as $\varepsilon \to 0$ by weak convergence of $e^{2u_\varepsilon} dA_g$ to $e^{2u} dA_g$. We choose ε small enough so that $\left|\int_\Sigma \Phi e^{2u_\varepsilon}\right|_\Lambda \leq \frac{1}{4}$ and $|\Lambda_\varepsilon - \Lambda| \leq \frac{1}{4}$. Now, setting $\Psi = \frac{\Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon}\right)}{\left|\Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon}\right)\right|^2} f$, for some function $f \in H^1$, we obtain

$$\left|\int_\Sigma e^{2u} \left\langle \Lambda\Phi, \frac{\Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon}\right)}{\left|\Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon}\right)\right|^2}\right\rangle f - \int_\Sigma f e^{2u_\varepsilon}\right| \leq C \|\nabla(\Phi - \Phi_\varepsilon)\|_2 \|\nabla\Psi\|_2$$

so that

$$\begin{aligned}\left|\int_\Sigma f\left(e^{2u} - e^{2u_\varepsilon}\right)\right| \leq& C \|\nabla(\Phi - \Phi_\varepsilon)\|_2 \|\nabla\Psi\|_2 \\ &+ \left|\int_\Sigma \frac{|\nabla\Phi|_\Lambda^2}{|\Lambda\Phi|^2} \frac{1}{\left|\Lambda_\varepsilon\left(\Phi - \int_\Sigma \Phi e^{2u_\varepsilon}\right)\right|} \left(\left|\Lambda_\varepsilon \int_\Sigma \Phi e^{2u_\varepsilon}\right| + |(\Lambda_\varepsilon - \Lambda)\Phi|\right) f\right| \\ \leq& C \|\nabla(\Phi - \Phi_\varepsilon)\|_2 \|f\|_{H^1} + C'\left(\left|\int_\Sigma \Phi e^{2u_\varepsilon}\right| + |\Lambda - \Lambda_\varepsilon|\right) \|f\|_1\end{aligned}$$

$\diamondsuit$

2.4.3. *Convergence of spectral indices.*

Claim 2.10. *Up to a subsequence, we can assume that $k_i^\varepsilon = k_i$ is constant and we have the following upper semi-continuity of eigenvalues*

$$\lim_{\varepsilon\to 0}\lambda_{k_i}^\varepsilon \leq \lambda_{k_i}\left((\Sigma, e^{2u_0}g)\sqcup\bigsqcup_{j=1}^{l}(\mathbb{S}^2, e^{2u_j}h)\right).$$

Proof. We use $\theta_0,\theta_1,\cdots,\theta_{k_m}$ an orthonormal family of first (k_m+1) eigenfunctions for $(\Sigma,\tilde{\mu}_0)\sqcup\bigsqcup_{j=1}^{l}(\mathbb{R}^2,\tilde{\mu}_j)$ as k_m+1 test functions for the variational characterization of $\lambda_1(\Sigma,e^{2u_\varepsilon}g),\cdots,\lambda_{k_m}\left(\Sigma,e^{2u_\varepsilon}g\right)$. For that, we fix $\rho>0$ and we let

$$\eta_0\in\mathcal{C}_c^\infty\left(\Sigma(\rho)\right)\ \text{with}\ \eta_0=1\ \text{on}\ \Sigma\left(\sqrt{\rho}\right)\ \text{and}\ \int_\Sigma|\nabla\eta_0|_g^2\,dA_g\leq\frac{C}{\ln\frac{1}{\rho}},$$

and for $1\leq j\leq l$

$$\eta_j\in\mathcal{C}_\infty\left(\Omega_i(\rho)\right)\ \text{with}\ \eta_j=1\ \text{on}\ \Omega_i(\sqrt{\rho})\ \text{and}\ \int_{\mathbb{R}^2}|\nabla\eta_j|_g^2\,dA_g\leq\frac{C}{\ln\frac{1}{\rho}},$$

where

$$\Sigma(\rho)=\Sigma\setminus\bigcup_{i=1}^{L_0}\mathbb{D}_\rho(q_i^0)\ \text{and}\ \Omega_j(\rho)=\mathbb{D}_{\frac{1}{\rho}}\setminus\bigcup_{i=1}^{L_j}\mathbb{D}_\rho(q_i^j)$$

where $\{q_1^j,\cdots,q_{L_j}^j\}$ is the set of atoms of μ_j, and we set

$$\varphi_a^\varepsilon(z)=\eta_0(z)\theta_a(z)+\eta_1\left(\frac{z-q_1^\varepsilon}{\alpha_1^\varepsilon}\right)\theta_a\left(\frac{z-q_i^\varepsilon}{\alpha_i^\varepsilon}\right)+\cdots+\eta_j\left(\frac{z-q_l^\varepsilon}{\alpha_l^\varepsilon}\right)\theta_a\left(\frac{z-q_l^\varepsilon}{\alpha_l^\varepsilon}\right)$$

for $0\leq a\leq k_m+1$ where all the involved functions in the sum have disjoint support for ε small enough. We obtain

$$\begin{aligned}\lambda_a(\Sigma,e^{2u_\varepsilon}g)\leq&\max_{\varphi\in\langle\varphi_0^\varepsilon,\cdots,\varphi_a^\varepsilon\rangle}\frac{\int_\Sigma|\nabla\varphi|_g^2\,dA_g}{\int_\Sigma\varphi^2e^{2u_\varepsilon}dA_g}\\ \leq&\max_{\theta\in\langle\theta_0,\cdots,\theta_a\rangle}\frac{\int_\Sigma|\nabla\theta|_g^2\,dA_g+\sum_{i=1}^L\int_{(\mathbb{R}^2)_i}|\nabla\theta|^2}{\int_{\Sigma(\sqrt{\rho})}\theta^2d\tilde{\mu}_0+\sum_{i=1}^L\int_{\Omega_i(\sqrt{\rho})}\theta^2d\tilde{\mu}_i}\frac{1}{1-o(1)}+\frac{\tilde{C}}{\sqrt{\ln\left(\frac{1}{\rho}\right)}}\\ \leq&\lambda_a\left((\Sigma,e^{2u}g)\sqcup\bigsqcup_{j=1}^{l}(\mathbb{S}^2,e^{2u_j}h)\right)\frac{1}{1-c\rho}\frac{1}{1-o(1)}+\frac{\tilde{C}}{\sqrt{\ln\left(\frac{1}{\rho}\right)}}\end{aligned}$$

as $\varepsilon\to 0$. Letting $\varepsilon\to 0$ and then $\rho\to 0$ gives the expected inequality. ♢

Claim 2.11. *We assume that the Palais-Smale sequence is a minimizing sequence for $e^{2u}\mapsto E(e^{2u}):=F(\bar{\lambda}_{l_1}(\Sigma,e^{2u}g),\cdots,\bar{\lambda}_{l_p}(\Sigma,e^{2u}g))$, where $l_j=k_{I_j}=\cdots=k_{I_{j+1}-1}$ and $1=I_1<\cdots<I_p<I_{p+1}=m+1$. Then, we have that*

$$\lim_{\varepsilon\to 0}\lambda_{k_i}^\varepsilon=\lambda_{k_i}\left((\Sigma,e^{2u}g)\sqcup\bigsqcup_{j=1}^{l}(\mathbb{S}^2,e^{2u_j}h)\right).$$

Proof. From Claim 2.10, we have

$$\lim_{\varepsilon\to 0}\lambda_{k_i}^{\varepsilon}=\lim_{\varepsilon\to 0}\lambda_{k_i}(\Sigma,e^{2u_\varepsilon}g)\leq\lambda_{k_i}\left((\Sigma,e^{2u}g)\sqcup\bigsqcup_{j=1}^{l}(\mathbb{S}^2,e^{2u_j}h)\right).$$

Now, if the inequality is strict for one of the p inequalities corresponding to $i=l_1,\cdots,l_p$, this would mean that the sequence is not a minimizing sequence. Indeed, the infimum of the functional E over the conformal metrics to $(\Sigma,[g])$ is smaller than the infimum of E over $(\tilde{\Sigma},[\tilde{g}])=(\Sigma,[g])\sqcup\bigsqcup_{j=1}^{l}(\mathbb{S}^2,[h])$. Let's see how to prove that. It suffices to take a metric $\tilde{g}_\delta$ on $(\tilde{\Sigma},[\tilde{g}])=(\Sigma,[g])\sqcup\bigsqcup_{j=1}^{l}(\mathbb{S}^2,[h])$ such that $E(\tilde{g}_\delta)\leq\inf E_{(\tilde{\Sigma},[\tilde{g}])}+\delta$. We denote $\tilde{g}_\delta=e^{2u_\delta}g$ in Σ and $e^{2u_{i,\delta}}dx=\pi^\star\tilde{g}_\delta$ in $\left(\mathbb{R}^2\cup\{\infty\}\right)_i$ where π is a stereographic projection for $i=1,\cdots l$. Taking disjoint points $p_1,\cdots,p_l\in\Sigma$ and a sequence of scale $\alpha\to 0$ we set

$$g_{\delta,\alpha}(z)=\left(e^{2u_\delta(z)}+\sum_{i=1}^{l}\frac{1}{\alpha^2}\eta\left(\frac{z-p_i}{\alpha}\right)e^{2u_{i,\delta}\left(\frac{z-p_i}{\alpha}\right)}\right)g(z).$$

where $\eta\in\mathcal{C}_c^\infty\left(\mathbb{D}_{\frac{1}{\delta}}\setminus\mathbb{D}_\delta\right)$ is a cut-off function with $\eta=1$ on $\mathbb{D}_{\frac{1}{\sqrt{\delta}}}\setminus\mathbb{D}_{\sqrt{\delta}}$ and $\int_\Sigma|\nabla\eta|^2\leq\frac{C}{\ln\left(\frac{1}{\delta}\right)}$. We obtain by asymptotic computations left to the reader (see [CES03] for an analogous asymptotic computation)

$$\inf E_{\Sigma,[g]}\leq E(g_{\delta,\alpha})\to E_{\tilde{\Sigma},[\tilde{g}]}(\tilde{g}_\delta)+A(\delta)\text{ as }\alpha\to 0.$$

where $A(\delta)\to 0$ as $\delta\to 0$. Letting then $\delta\to 0$ gives the expected result. ♢

Claim 2.12. *We assume that the Palais-Smale sequence satisfies that $\lambda_1(\Sigma,e^{2u_\varepsilon})$ is uniformly lower bounded by a positive constant, that it is associated to a functional $e^{2u}\mapsto E(e^{2u}):=F(\bar{\lambda}_1(\Sigma,e^{2u}g),\cdots,\bar{\lambda}_m(\Sigma,e^{2u}g))$ and that the sequences of non negative numbers $|\partial_iF(\lambda_i^\varepsilon,\cdots,\lambda_m^\varepsilon)|$ are uniformly lower bounded by a positive constant. Then, there is no bubbling and we have that for any $1\leq i\leq m$,*

$$\lim_{\varepsilon\to 0}\lambda_i^\varepsilon=\lambda_i\left(\Sigma,e^{2u_0}g\right).$$

Proof. From Claim 2.9 we know that there is no bubbling and that e^{2u_ε} converges to e^{2u_0} in H^{-1}. We extract from the map $\Phi_\varepsilon:\Sigma\to\mathcal{E}_{\Lambda_\varepsilon}$ in the Palais-Smale sequence, an idependant family $(\phi_0^\varepsilon,\cdots,\phi_m^\varepsilon)$ of eigenfunctions associated to the eigenvalues $(0,\lambda_1^\varepsilon,\cdots,\lambda_m^\varepsilon)$ with respect to the metric e^{2u_ε}. We use them as test functions for $\lambda_k(\Sigma,e^{2u_0}g)$ (with $0\leq k\leq m$) because they converge in H^1. Since we know that $\frac{\phi_k^\varepsilon}{\omega_\varepsilon}$ are sequences of bounded functions in L^∞, and that $\Phi_\varepsilon-\frac{\Phi_\varepsilon}{\omega_\varepsilon}$ converges to 0 in H^1 , we have for any $k,l\geq 1$,

$$\left(\frac{\phi_k^\varepsilon}{\omega_\varepsilon}\frac{\phi_l^\varepsilon}{\omega_\varepsilon}-\phi_k^\varepsilon\phi_l^\varepsilon\right)e^{2u_\varepsilon}\to 0\text{ and }\left(\frac{\phi_k^\varepsilon}{\omega_\varepsilon}\frac{\phi_l^\varepsilon}{\omega_\varepsilon}-\phi_k^\varepsilon\phi_l^\varepsilon\right)e^{2u_0}\to 0\text{ in }L^1(g)\text{ as }\varepsilon\to 0.$$

$$\frac{\phi_k^\varepsilon}{\omega_\varepsilon}\frac{\phi_l^\varepsilon}{\omega_\varepsilon}\left(e^{2u_\varepsilon}-e^{2u_0}\right)\to 0\text{ and }\nabla\frac{\phi_k^\varepsilon}{\omega_\varepsilon}\nabla\frac{\phi_k^\varepsilon}{\omega_\varepsilon}-\nabla\phi_k^\varepsilon\nabla\phi_l^\varepsilon\to 0\text{ in }L^1(g)\text{ as }\varepsilon\to 0.$$

Then,

$$\lambda_k\left(\Sigma,e^{2u_0}g\right)\leq\max_{\varphi\in\left\langle 1,\phi_1^\varepsilon,\cdots,\phi_k^\varepsilon\right\rangle}\frac{\int_\Sigma|\nabla\varphi|_g^2dA_g}{\int_\Sigma\varphi^2e^{2u_0}dA_g}=\max_{\varphi\in\left\langle 1,\phi_1^\varepsilon,\cdots,\phi_k^\varepsilon\right\rangle}\frac{\int_\Sigma|\nabla\varphi|_g^2dA_g}{\int_\Sigma\varphi^2e^{2u_\varepsilon}dA_g}+o(1)=\lambda_k^\varepsilon$$

and with Claim 2.10 we have the expected result. ♢

Remark 2.1. *It would be interesting to get more general conditions such that the spectral indices of eigenvalues associated to a Palais-Smale sequence converge to the spectral indices of the limit, in particular when there is bubbling. It is also linked to the question of strong H^1 convergence of all the eigenfunctions associated to the Palais-Smale sequence.*

3. Convergence of Palais-Smale sequences for finite combination of Steklov eigenvalues in dimension 2

In the current section, we prove the new ideas we need to prove the convergence of Palais-Smale sequences associated to Steklov eigenvalues. Therefore, in order to simplify the presentation of the proof, we focus on the case without bubbling since the bubbling phenomenon is the same as for Laplace eigenvalues. More generally, we refer to the analogous case of Laplace eigenvalues in the previous section in order to complete all the arguments of the following proposition that are skiped.

Proposition 3.1. *Let e^{u_ε}, σ_ε, $\Phi_\varepsilon : \Sigma \to \mathbb{R}^m$ be a smooth sequence of maps satisfying the Palais-Smale assumption (PS) as $\varepsilon \to 0$, that is*

- *$\sigma_\varepsilon = diag(\sigma^\varepsilon_{k^\varepsilon_1}, \cdots, \sigma^\varepsilon_{k^\varepsilon_m})$ where the diagonal terms are eigenvalues associated to e^{u_ε} with uniformly bounded spectral indices k^ε_i, and $\sigma_\varepsilon \to \sigma$ as $\varepsilon \to 0$.*
- *$\Delta_g \Phi_\varepsilon = 0$ in Σ and $\partial_\nu \Phi_\varepsilon = e^{u_\varepsilon} \sigma_\varepsilon \Phi_\varepsilon$ on $\partial\Sigma$*
- *$\int_{\partial\Sigma} e^{u_\varepsilon} dL_g = \int_{\partial\Sigma} |\Phi_\varepsilon|^2_{\sigma_\varepsilon} e^{u_\varepsilon} dL_g = \int_\Sigma |\nabla \Phi_\varepsilon|^2_g dA_g = 1$*
- *$|\Phi|^2_{\sigma_\varepsilon} \geq 1 - \delta_\varepsilon$ uniformly on $\partial\Sigma$, where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Then, up to a subsequence Φ_ε bubble tree converges in $W^{1,2}$ to $\Phi_0 : \Sigma \to co(\mathcal{E}_\sigma)$, and $\Phi_j : \mathbb{D} \to co(\mathcal{E}_\sigma)$ for $j = 1, \cdots, l$ $(l \geq 0)$ with an energy identity:

$$1 = \int_\Sigma |\nabla \Phi_0|^2_g dA_g + \sum_{j=1}^{l} \int_{\mathbb{D}} |\nabla \Phi_j|^2_h dA_h$$

Moreover, Φ_j are smooth harmonic maps with free boundary for $j = 0, \cdots, l$ and their i-th coordinates are eigenfunctions associated to λ_{k_i} on the surface $\Sigma \cup \bigcup_{1 \leq i \leq l} \mathbb{D}$ with respect to the metrics $e^{2u} g$ on Σ such that $e^u = \partial_\nu \Phi_0 . \Phi_0$ on $\partial\Sigma$ and $e^{2v_j} \xi$ on $\mathbb{D}$ such that $e^{v_j} = \partial_\nu \Phi_j . \Phi_j$ on $\mathbb{S}^1$. Moreover $e^{u_\varepsilon} dL_g$ bubble tree converges in $W^{-\frac{1}{2},2}$ to $\partial_\nu \Phi_0 . \Phi_0 dL_g$ in Σ and $\partial_\nu \Phi_j . \Phi_j dA_h$ in $\mathbb{S}^2$.

3.1. **Some convergence of ω_ε to 1 and first replacement of Φ_ε.** We let ω_ε be the harmonic extension on Σ of $|\Phi_\varepsilon|_{\sigma_\varepsilon}$. We set $\widetilde{\Phi_\varepsilon} = \frac{\Phi_\varepsilon}{\omega_\varepsilon}$

We integrate $\Delta_g \Phi_\varepsilon = 0$ in Σ and $\partial_\nu \Phi_\varepsilon = \sigma_\varepsilon e^{u_\varepsilon} \Phi_\varepsilon$ on $\partial\Sigma$ against $\sigma\Phi$ and $\frac{\sigma_\varepsilon \Phi_\varepsilon}{\omega_\varepsilon}$. We obtain

$$\int_{\partial\Sigma} |\sigma_\varepsilon \Phi_\varepsilon|^2 e^{u_\varepsilon} dL_g = \int_\Sigma |\nabla \Phi_\varepsilon|^2_{\sigma_\varepsilon} dA_g \tag{3.1}$$

and

$$\int_{\partial\Sigma} \frac{|\sigma_\varepsilon \Phi_\varepsilon|^2}{\omega_\varepsilon} e^{u_\varepsilon} = \int_\Sigma \frac{\sigma_\varepsilon \Phi_\varepsilon}{\omega_\varepsilon} \Delta_g \Phi_\varepsilon = \int_\Sigma \left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \Phi_\varepsilon \right\rangle_{\sigma_\varepsilon} \tag{3.2}$$

We have that

$$\left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \Phi_\varepsilon \right\rangle_{\sigma_\varepsilon} = \frac{\left\langle \left(\nabla \Phi_\varepsilon - \widetilde{\Phi_\varepsilon} \nabla \omega_\varepsilon \right), \nabla \Phi_\varepsilon \right\rangle_{\sigma_\varepsilon}}{\omega_\varepsilon} = \frac{|\nabla \Phi_\varepsilon|^2_{\sigma_\varepsilon}}{\omega_\varepsilon} + \left\langle \nabla \frac{1}{\omega_\varepsilon}, \sigma_\varepsilon \Phi_\varepsilon . \nabla \Phi_\varepsilon \right\rangle , \tag{3.3}$$

noticing that $\sigma_\varepsilon\Phi_\varepsilon.\nabla\Phi_\varepsilon = \langle\Phi_\varepsilon,\nabla\Phi_\varepsilon\rangle_{\mathcal{E}_\varepsilon} = \nabla\left(\frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2}\right)$, we compute the first left-hand side term of (3.2), by considering first the last-hand term in (3.3):

$$\begin{aligned}\int_\Sigma \left\langle \frac{1}{\omega_\varepsilon}, \sigma_\varepsilon\Phi_\varepsilon.\nabla\Phi_\varepsilon\right\rangle dA_g &= \int_\Sigma \left\langle \frac{1}{\omega_\varepsilon}, \nabla\frac{|\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{2}\right\rangle dA_g\\ &= \int_\Sigma \frac{|\Phi|^2_{\sigma_\varepsilon}}{2}\Delta_g\left(\frac{1}{\omega_\varepsilon}\right)dA_g + \int_{\partial\Sigma}\frac{|\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{2}\partial_\nu\left(\frac{1}{\omega_\varepsilon}\right)dL_g\\ &= \int_\Sigma \frac{|\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{\omega_\varepsilon^3}\left(\frac{\omega_\varepsilon}{2}\Delta_g\omega_\varepsilon - |\nabla\omega_\varepsilon|^2_g\right)dA_g - \int_{\partial\Sigma}\frac{|\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{2\omega_\varepsilon^2}\partial_\nu\omega_\varepsilon dL_g\\ &= -\int_\Sigma \frac{|\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{\omega_\varepsilon^3}|\nabla\omega_\varepsilon|^2_g\, dA_g\ ,\end{aligned}$$

where we noticed in the last step that that $\Delta_g\omega_\varepsilon = 0$ and $\omega_\varepsilon = |\Phi_\varepsilon|_{\sigma_\varepsilon}$ on $\partial\Sigma$. Therefore, using (3.2) and (3.3),

$$\int_\Sigma |\nabla\omega_\varepsilon|^2\frac{|\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{\omega_\varepsilon^3}dA_g = \int_\Sigma \frac{|\nabla\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{\omega_\varepsilon}dA_g - \int_{\partial\Sigma}\frac{|\sigma_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon}e^{u_\varepsilon}dL_g\ ,$$

We also have since $\omega_\varepsilon^2 \geq 1-\delta_\varepsilon$ by the maximum principle and (3.1) that

$$\int_\Sigma \frac{|\nabla\Phi_\varepsilon|^2_{\sigma_\varepsilon}}{\omega_\varepsilon}dA_g \leq \frac{1}{\sqrt{1-\delta_\varepsilon}}\int_\Sigma |\nabla\Phi_\varepsilon|^2_{\sigma_\varepsilon}\, dA_g = \frac{1}{\sqrt{1-\delta_\varepsilon}}\int_{\partial\Sigma}|\sigma_\varepsilon\Phi_\varepsilon|^2 e^{u_\varepsilon}dL_g\ ,$$

so that since $\omega_\varepsilon^2 - \omega_\varepsilon = \omega_\varepsilon(\omega_\varepsilon - 1) \geq 1-\delta_\varepsilon - \sqrt{1-\delta_\varepsilon}$

$$\begin{aligned}\int_\Sigma |\nabla\omega_\varepsilon|^2\frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon^3}dA_g &\leq \int_{\partial\Sigma}(\omega_\varepsilon - 1)\frac{|\sigma_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon}e^{u_\varepsilon}dL_g + \left(\frac{1}{\sqrt{1-\delta_\varepsilon}} - 1\right)\int_{\partial\Sigma}|\sigma_\varepsilon\Phi_\varepsilon|^2 e^{u_\varepsilon}dL_g\\ &\leq \max_\Sigma \frac{|\sigma_\varepsilon\Phi_\varepsilon|^2}{\omega_\varepsilon^2}\left(\int_{\partial\Sigma}\left(\omega_\varepsilon^2 - \omega_\varepsilon\right)e^{u_\varepsilon}dL_g + \sqrt{1-\delta_\varepsilon} + \delta_\varepsilon - 1\right)\\ &\quad + \max_j \sigma^\varepsilon_{k_j}\left(\frac{1}{\sqrt{1-\delta_\varepsilon}} - 1\right)\\ &\leq \max_j \sigma_{k^\varepsilon_j}\left(\delta_\varepsilon + \frac{1}{\sqrt{1-\delta_\varepsilon}} - 1\right) = O\left(\delta_\varepsilon\right)\end{aligned}$$

as $\varepsilon \to 0$ where we crucially used the global equality $\int_{\partial\Sigma} e^{u_\varepsilon}\omega_\varepsilon^2 = \int_{\partial\Sigma} e^{u_\varepsilon} = 1$

We also have that

$$\nabla\left(\Phi_\varepsilon - \widetilde{\Phi_\varepsilon}\right) = \left(1 - \frac{1}{\omega_\varepsilon}\right)\nabla\Phi_\varepsilon + \frac{\nabla\omega_\varepsilon}{\omega_\varepsilon^2}\Phi_\varepsilon$$

and then

$$\left|\nabla\left(\Phi_\varepsilon - \widetilde{\Phi_\varepsilon}\right)\right|^2_{\sigma_\varepsilon} = |\nabla\Phi_\varepsilon|^2_{\sigma_\varepsilon}\left(1 - \frac{1}{\omega_\varepsilon}\right)^2 + \frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon^2}|\Phi_\varepsilon|^2_{\sigma_\varepsilon} + \frac{\left\langle\nabla\omega_\varepsilon, \nabla|\Phi_\varepsilon|^2_{\sigma_\varepsilon}\right\rangle}{\omega_\varepsilon^2}\left(1 - \frac{1}{\omega_\varepsilon}\right)\ .$$

We have that

$$\int_\Sigma |\nabla\Phi_\varepsilon|^2_{\sigma_\varepsilon}\left(1-\frac{1}{\omega_\varepsilon}\right)^2 = -\int_\Sigma div\left(\nabla\Phi_\varepsilon\left(1-\frac{1}{\omega_\varepsilon}\right)^2\right)\sigma_\varepsilon\Phi_\varepsilon + \int_{\partial\Sigma}|\sigma_\varepsilon\Phi_\varepsilon|^2\left(1-\frac{1}{\omega_\varepsilon}\right)^2 e^{u_\varepsilon}$$

$$= \int_\Sigma \left\langle \nabla|\Phi_\varepsilon|^2_{\sigma_\varepsilon}, \nabla\frac{1}{\omega_\varepsilon}\right\rangle\left(1-\frac{1}{\omega_\varepsilon}\right) + \int_{\partial\Sigma}|\sigma_\varepsilon\Phi_\varepsilon|^2\left(1-\frac{1}{\omega_\varepsilon}\right)^2 e^{u_\varepsilon}$$

so that

$$\int_\Sigma \left|\nabla\left(\Phi_\varepsilon - \frac{\Phi_\varepsilon}{\omega_\varepsilon}\right)\right|^2_{\sigma_\varepsilon} = \int_{\partial\Sigma}|\sigma_\varepsilon\Phi_\varepsilon|^2\left(1-\frac{1}{\omega_\varepsilon}\right)^2 e^{u_\varepsilon} + \int_\Sigma \frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon^2}|\Phi_\varepsilon|^2_{\sigma_\varepsilon}$$

$$\leq \left(\int_{\partial\Sigma}\frac{|\sigma_\varepsilon\Phi_\varepsilon|^4}{\omega_\varepsilon^2}e^{u_\varepsilon}\right)^{\frac{1}{2}}\left(\int_{\partial\Sigma}e^{u_\varepsilon}\left(\omega_\varepsilon - \frac{1}{\omega_\varepsilon}\right)^2\right)^{\frac{1}{2}} + O(\delta_\varepsilon)$$

$$\leq \max_j \sigma_{k_j}\left(\int_{\partial\Sigma}e^{u_\varepsilon}\left(\frac{1}{\omega_\varepsilon^2}-1\right)\right)^{\frac{1}{2}} + O(\delta_\varepsilon)$$

as $\varepsilon\to 0$, where we crucially used again $\int_{\partial\Sigma}e^{u_\varepsilon}\omega_\varepsilon^2 = \int_{\partial\Sigma}e^{u_\varepsilon} = 1$. We then obtain that

$$\int_\Sigma \left|\nabla\left(\Phi_\varepsilon - \widetilde{\Phi_\varepsilon}\right)\right|^2_{\sigma_\varepsilon} = O\left(\delta_\varepsilon^{\frac{1}{2}}\right) \text{ as } \varepsilon\to 0. \tag{3.4}$$

3.2. **H^1-convergence of eigenfunctions by harmonic replacement (simple case).** Let $p\in\partial\Sigma\setminus\{p_1,\cdots,p_s\}$, where $\{p_1,\cdots,p_s\}$ are bad points (see Claim 2.3 in the Laplace case). Refering to the Laplacian case, we also have an analogous property to (2.5) far from bad points. Let $r>0$ be such that for any $\varepsilon>0$,

$$\int_{\mathbb{D}_r^+(p)}\left|\nabla\widetilde{\Phi_\varepsilon}\right|^2_g < \varepsilon_0$$

Then, let $\Psi_\varepsilon : \mathbb{D}_r^+(p)\to\mathcal{E}_{\Lambda_\varepsilon}$ be the harmonic replacement of $\widetilde{\Phi_\varepsilon}$ and we obtain by [LP19], Theorem 1.2

$$\int_{\mathbb{D}_r^+(p)}\left|\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right)\right|^2_g \leq \frac{1}{2}\left(\int_{\mathbb{D}_r^+(p)}\left|\nabla\widetilde{\Phi_\varepsilon}\right|^2_g - \int_{\mathbb{D}_r^+(p)}|\nabla\Psi_\varepsilon|^2_g\right). \tag{3.5}$$

Let's prove that the right-hand term converges to 0 as $\varepsilon\to 0$. We test the function $\widetilde{\Phi_\varepsilon}^i - \Psi_\varepsilon^i$ in the variational characterization of $\sigma_\star := \sigma_\star(\mathbb{D}_r^+(p), e^{u_\varepsilon})$:

$$\sigma_{k_i^\varepsilon}\int_{I_r(p)}\left(\widetilde{\Phi_\varepsilon}^i - \Psi_\varepsilon^i\right)^2 e^{u_\varepsilon} \leq \sigma_\star\int_{\mathbb{D}_r^+(p)}\left(\widetilde{\Phi_\varepsilon}^i - \Psi_\varepsilon^i\right)^2 e^{2u_\varepsilon} \leq \int_{\mathbb{D}_r^+(p)}\left|\nabla\left(\widetilde{\Phi_\varepsilon}^i - \Psi_\varepsilon^i\right)\right|^2$$

and we sum on i to get

$$\int_{I_r(p)}\left|\widetilde{\Phi_\varepsilon} - \Psi_\varepsilon\right|^2_{\sigma_\varepsilon} e^{u_\varepsilon} \leq \int_{\mathbb{D}_r^+(p)}\left|\nabla\widetilde{\Phi_\varepsilon}\right|^2 + \int_{\mathbb{D}_r(p)}|\nabla\Psi_\varepsilon|^2 - 2\int_{\mathbb{D}_r(p)}\nabla\widetilde{\Phi_\varepsilon}\nabla\Psi_\varepsilon \tag{3.6}$$

Now we test the equation $\Delta\Phi_\varepsilon = 0$ and $\partial_\nu\Phi_\varepsilon = \sigma_\varepsilon e^{u_\varepsilon}\Phi_\varepsilon$ against $\frac{\Phi_\varepsilon}{\omega_\varepsilon^2} - \frac{\Psi_\varepsilon}{\omega_\varepsilon}$ and we get after an integration by part knowing that $\frac{\Phi_\varepsilon}{\omega_\varepsilon^2} - \frac{\Psi_\varepsilon}{\omega_\varepsilon} = 0$ on $\partial\mathbb{D}_r^+(p)\setminus I_r(p)$

$$\int_{\mathbb{D}_r^+(p)}\left(\frac{1}{\omega_\varepsilon}\nabla\Phi_\varepsilon\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right) + \nabla\frac{1}{\omega_\varepsilon}\nabla\Phi_\varepsilon\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right)\right) = \int_{I_r(p)}\left\langle\widetilde{\Phi_\varepsilon}, \widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right\rangle_{\sigma_\varepsilon} e^{u_\varepsilon}$$

so that,

(3.7)
$$\int_{\mathbb{D}_r^+(p)} \nabla\widetilde{\Phi_\varepsilon}\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right) = \int_{I_r(p)} \left\langle \widetilde{\Phi_\varepsilon}, \widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right\rangle_{\sigma_\varepsilon} e^{u_\varepsilon} - \int_{\mathbb{D}_r^+(p)} \nabla\frac{1}{\omega_\varepsilon}\nabla\Phi_\varepsilon\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right) + \int_{\mathbb{D}_r(p)} \Phi_\varepsilon\nabla\frac{1}{\omega_\varepsilon}\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right)$$

Knowing that $\left|\widetilde{\Phi_\varepsilon}\right|_{\sigma_\varepsilon} = |\Psi_\varepsilon|_{\sigma_\varepsilon}$ on $\partial\Sigma$, it is clear that $2\left\langle \widetilde{\Phi_\varepsilon}, \widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right\rangle_{\sigma_\varepsilon} = \left|\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right|^2_{\sigma_\varepsilon}$ and multiplying (3.7) by 2, we obtain

(3.8)
$$\begin{aligned} &2\int_{\mathbb{D}_r^+(p)} \left|\nabla\widetilde{\Phi_\varepsilon}\right|^2 - 2\int_{\mathbb{D}_r^+(p)} \nabla\widetilde{\Phi_\varepsilon}\nabla\Psi_\varepsilon \\ &= \int_{I_r(p)} \left|\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right|^2_{\sigma_\varepsilon} e^{u_\varepsilon} + 2\int_{\mathbb{D}_r^+(p)} \frac{\nabla\omega_\varepsilon}{\omega_\varepsilon}\left(\frac{\nabla\Phi_\varepsilon}{\omega_\varepsilon}\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right) - \widetilde{\Phi_\varepsilon}\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right)\right) \end{aligned}$$

Summing (3.6) and (3.8), we get

$$\int_{\mathbb{D}_r^+(p)} \left|\nabla\frac{\Phi_\varepsilon}{\omega_\varepsilon}\right|^2_g - \int_{\mathbb{D}_r(p)} |\nabla\Psi_\varepsilon|^2_g \leq 2\int_{\mathbb{D}_r^+(p)} \frac{\nabla\omega_\varepsilon}{\omega_\varepsilon}\left(\frac{\nabla\Phi_\varepsilon}{\omega_\varepsilon}\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right) - \widetilde{\Phi_\varepsilon}\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right)\right)$$

Up to reduce ε_0, we assume that $\left|\widetilde{\Phi_\varepsilon}\right|^2_{\sigma_\varepsilon} \geq \frac{1}{2}$ in $\mathbb{D}_r^+(p)$ and we obtain that

$$\begin{aligned} 2\left|\int_{\mathbb{D}_r^+(p)} \frac{\nabla\omega_\varepsilon}{\omega_\varepsilon}\left(\frac{\nabla\Phi_\varepsilon}{\omega_\varepsilon}\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right) - \widetilde{\Phi_\varepsilon}\nabla\left(\widetilde{\Phi_\varepsilon}-\Psi_\varepsilon\right)\right)\right| &\leq O\left(\frac{\left(\int_\Sigma \frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon^4}|\Phi_\varepsilon|^2_{\sigma_\varepsilon}\right)^{\frac{1}{2}}}{\sigma_{k_1^\varepsilon}}\right) \\ &= O\left(\frac{\delta_\varepsilon^{\frac{1}{2}}}{\sigma_{k_1^\varepsilon}}\right) \end{aligned}$$

By (3.5), we obtain that

$$\widetilde{\Phi_\varepsilon} - \Psi_\varepsilon \to 0 \text{ in } H^1\left(\mathbb{D}_r^+(p)\right)$$

Now since Ψ_ε converges in $\mathcal{C}^k\left(\mathbb{D}^+_{\frac{r}{2}}(p)\right)$ for any $k\in\mathbb{N}$ to some free boundary harmonic map Ψ, and because the property is true for all $p\in\partial\Sigma\setminus\{p_1,\cdots,p_s\}$, we have that $\widetilde{\Phi_\varepsilon}$ converges to some free boundary harmonic map Φ in $H^1_{loc}(\{x\in\Sigma; d(x,\partial\Sigma)\leq r_0\}\setminus\{p_1,\cdots,p_s\})$ for some $r_0>0$. Then by (3.4) and since Φ_ε is a harmonic map in an Euclidean space, Φ_ε converges to some free boundary harmonic map Φ in $H^1_{loc}(\Sigma\setminus\{p_1,\cdots,p_s\})$.

3.3. **The limiting measure has a smooth density.** Let $\zeta\in\mathcal{C}^\infty_c\left(\mathbb{D}_r^+(p)\right)$ where $p\in\partial\Sigma\setminus\{p_1,\cdots,p_s\}$ and $\{p_1,\cdots,p_s\}$ are bad points. We have that

$$\begin{aligned} \int_{\partial\Sigma}\zeta\left(\sigma_\varepsilon\Phi_\varepsilon e^{u_\varepsilon}dL_g - \sigma\Phi d\nu\right) &= \int_{\partial\Sigma}\zeta\left(\sigma_\varepsilon\Phi_\varepsilon - \sigma\Phi\right)e^{u_\varepsilon}dL_g \\ &\quad + \int_{\partial\Sigma}\zeta\sigma\Phi\left(e^{u_\varepsilon}dL_g - d\nu\right) . \end{aligned}$$

Then on the first right-hand term, we have that

$$\begin{aligned}\int_{\partial\Sigma}\zeta\left(\Lambda_\varepsilon\Phi_\varepsilon-\Lambda\Phi\right)e^{u_\varepsilon}dL_g &\leq \left(\int_{I_r(p)}\zeta^2\left|\sigma_\varepsilon\Phi_\varepsilon-\sigma\Phi\right|^2e^{u_\varepsilon}dL_g\right)^{\frac{1}{2}}\\ &\leq \left(\frac{1}{\sigma_\star(\mathbb{D}_r^+(p),e^{u_\varepsilon})}\int_{\mathbb{D}_r^+(p)}\left|\nabla\left(\zeta\left|\sigma_\varepsilon\Phi_\varepsilon-\sigma\Phi\right|\right)\right|_g^2dA_g\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\mathbb{D}_r^+(p)}\left|\nabla\left(\Phi_\varepsilon-\Phi\right)\right|_g^2dA_g\right)^{\frac{1}{2}}\end{aligned}$$

for some constant C independent of ε. Letting $\varepsilon\to 0$ in a weak sense to the eigenvalue equation $\Delta_g\Phi_\varepsilon=0$ in Σ and $\partial_\nu\Phi_\varepsilon=\sigma_\varepsilon\Phi_\varepsilon e^{u_\varepsilon}$ on $\partial\Sigma$, we get

$$\Delta\Phi=0 \text{ in } \Sigma \text{ and } \partial_\nu\Phi=\sigma\Phi\nu \text{ on } \partial\Sigma$$

and since Φ is harmonic, we obtain that $\nu=\Phi.\partial_\nu\Phi dL_g$. Knowing that Φ is smooth (see [JLZ19]), we obtain a smooth density for ν.

References

[CES03] B. Colbois, A. El Soufi, *Extremal eigenvalues of the Laplacian in a conformal class of metrics: the 'conformal spectrum'*, Ann. Global Anal. Geom. **24** (2003), no.4, 337–349.

[Cla13] F. Clarke, *Functional analysis, calculus of variations and optimal control*, Graduate Texts in Mathematics , **264**, Springer, London, 2013, xiv+591,

[CM08] T.H. Colding, W.P. Minicozzi II, *Width and finite extinction time of Ricci flow.*, Geom. Topol., **12**, 2008, 5, 2537–2586.

[ESI86] A. El Soufi, S. Ilias, *Immersions minimales, première valeur propre du laplacien et volume conforme*, Mathematische Annalen, 1986, **275**, 257-267

[ESI03] A. El Soufi and S. Ilias, *Extremal metrics for the first eigenvalue of the Laplacian in a conformal class*, Proc. Amer. Math. Soc. **131**, 2003, 1611-1618.

[ESI08] A. El Soufi, S. Ilias, *Laplacian eigenvalue functionals and metric deformations on compact manifolds*, Journal of Geometry and Physics, **58**, Issue 1, January 2008, 89-104

[Fab23] G. Faber, *Beweis, dass unter allen homogenen membranen von gleicher flüche und gleicher spannung die kreisfürmige den tiefsten grundton gibt*, Sitz. ber. bayer. Akad. Wiss., 1923, 169–172,

[FS13] A. Fraser, R. Schoen, *Minimal surfaces and eigenvalue problems*, Contemporary Mathematics, **599**, 2013, 105–121

[FS16] A. Fraser, R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math. **203**, 2016, 823–890.

[FN99] L. Friedlander, N. Nadirashvili, *A differential invariant related to the first eigenvalue of the Laplacian*, Internat. Math. Res. Notices, 1999, 17, 939–952,

[GP22] M.J. Gursky, S. Pérez-Ayala, *Variational properties of the second eigenvalue of the conformal Laplacian*, J. Funct. Anal., **282**, 2022, no.8, Paper No. 109371, 60

[Has11] A. Hassannezhad, *Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem*, J. Funct. Anal., **261**, 2011, no 12, 3419–3436

[Hel96] F. Hélein, *Harmonic maps, conservation laws and moving frames*, Cambridge Tracts in Mathematics, **150**, Second Edition, Cambridge University Press, 2002, xxvi+264.

[Her70] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C.R. Acad. Sci. Paris Sér. A-B **270**, 1970, A1645–A1648.

[JLZ19] J. Jost, L. Liu and M. Zhu, *The qualitative behavior at the free boundary for approximate harmonic maps from surfaces*, Math. Ann., **374**, 2019, 1-2, 133–177

[Kar21] M. Karpukhin, *Index of minimal spheres and isoperimetric eigenvalue inequalities*, Invent. Math., **223**, 2021, no. 1, 335–377

[KNPP17] M.A. Karpukhin, N. Nadirashvili, A. Penskoi, I. Polterovich, *An isoperimetric inequality for Laplace eigenvalues on the sphere*. arXiv: 1706.05713. To appear in J. Differential Geom..

[KNPP20] M. Karpukhin, N. Nadirashvili, A. Penskoi, I. Polterovich, *Conformally maximal metrics for Laplace eigenvalues on surfaces*, arXiv:2003.02871v2

[KNPS21] M. Karpukhin, M. Nahon, I. Polterovich, D.Stern *Stability of isoperimetric inequalities for Laplace eigenvalues on surfaces*, arXiv:2106.15043

[KS20] M. Karpukhin, D. L. Stern *Min-max harmonic maps and a new characterization of conformal eigenvalues* arXiv preprint (2020), arXiv:2004.04086, 69 pp

[KS22] M. Karpukhin, D. L. Stern *Existence of harmonic maps and eigenvalue optimization in higher dimensions* arXiv preprint (2022), arXiv:2207.13635, 60 pp

[Kok14] G. Kokarev, *Variational aspects of Laplace eigenvalues on Riemannian surfaces*, Adv. Math. **258**, 2014, 191–239.

[Kor93] N. Korevaar, *Upper bounds for eigenvalues of conformal metrics*, J. Differential Geom., **37**, 1993, 1, 73–93

[Kra25] Krahn, E., *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann., **94**, 1925, 1, 97–100

[LP19] P. Laurain, R. Petrides, *Existence of min-max free boundary disks realizing the width of a manifold*, Advances in Mathematics **352**, 2019, 326–371.

[LY82] P. Li, S.T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent.. Math. **69**, 1982, 269–291.

[Nad96] N. Nadirashvili, *Berger's isoperimetric problem and minimal immersions of surfaces*, Geom. Func. Anal. **6**, 1996, 877-897.

[NS15] Nikolai Nadirashvili and Yannick Sire. Conformal spectrum and harmonic maps. Moscow J. of maths. Volume **15**, 2014, 1, 123–140

[Pet14] R. Petrides, *Maximization of the second conformal eigenvalue of spheres*, Proc. Amer. Math. Soc., **142**, 2014, 7 , 2385–2394

[Pet15] R. Petrides, *On a rigidity result for the first conformal eigenvalue of the Laplacian*, J. Spectr. Theory, **5**, 2015, no.1, 227–234

[Pet14a] R. Petrides, *Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces*, Geom. Funct. Anal. **24**, 2014, 1336–1376.

[Pet18] R. Petrides, *On the existence of metrics which maximize Laplace eigenvalues on surfaces*, Int. Math. Res. Not., **14**, 2018, 4261–4355.

[Pet19] R. Petrides, *Maximizing Steklov eigenvalues on surfaces*, J. Differential Geom. Volume **113**, 2019, no.1, 95-188.

[Pet21] R. Petrides, *Extremal metrices for combinations of Laplace eigenvalues and minimal surfaces into ellipsoids*, submitted

[Pet22] R. Petrides, *Shape optimization for combinations of Steklov eigenvalues on Riemannian surfaces*, submitted

[Pet22a] R. Petrides, *Maximizing one Laplace eigenvalue on n-dimensional manifolds*, submitted

[PT22] R. Petrides, D. Tewodrose, *Subdifferentials and critical points of eigenvalue functionals*

[Riv08] T. Rivière, *Conservation laws for conformally invariant variational problems*, Inventiones Mathematicae, **168**, 2007, 1–22.

[Sze54] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal., **3**, 1954, 343–356

[Str08] M. Struwe, *Variational Methods*: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer Berlin, Heidelberg, Edition 4, 2008, XX, 302

[Wei56] H.F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Rational Mech. Anal., **5**,1956, 633–636

[YY80] P.C. Yang, S.T. Yau, *Eigenvalues of the Laplacian of compact Riemannian surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7**, 1980, no.1, 53–63.

Romain Petrides, Université de Paris, Institut de Mathématiques de Jussieu - Paris Rive Gauche, bâtiment Sophie Germain, 75205 PARIS Cedex 13, France

Email address: romain.petrides@imj-prg.fr