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Abstract. We prove existence and regularity of metrics which minimize combinations of Steklov eigenvalues over metrics of unit perimeter on a surface with boundary. We show that there are free boundary minimal immersions into ellipsoids parametrized by eigenvalues, such that the coordinate functions are eigenfunctions with respect to the minimal metrics. This work generalizes Fraser-Schoen's and the author's maximization for one eigenvalue among metrics of unit perimeter on a surface giving free boundary minimal immersions into balls. We also generalize the previous results of critical metrics for one eigenvalue to any combination of eigenvalues from target balls to target ellipsoids.

1. Introduction

Let Σ be a compact connected surface with boundary. We denote by

$$0 = \sigma_0(g) < \sigma_1(g) \leq \sigma_2(g) \leq \cdots \leq \sigma_m(g) \to +\infty$$

the eigenvalues with respect to the Dirichlet to Neumann operator $T_g : \mathcal{C}^\infty(\partial\Sigma) \to \mathcal{C}^\infty(\partial\Sigma)$ associated to the laplacian $\Delta_g = -div_g(\nabla)$ for a Riemannian metric g on Σ. This means that for one function $u \in \mathcal{C}^\infty(\partial\Sigma)$, $T_g(u) = \partial_\nu \hat{u}$, where ∂_ν is the outward pointing normal derivative and $\hat{u}$ is the harmonic extention of u on Σ. The eigenvalues $\sigma_m(g)$ are called Steklov eigenvalues on Σ.

In [FS16], Fraser and Schoen studied the maximization problem for $g \mapsto \sigma_m(g) L_g(\partial\Sigma)$, where $L_g(\partial\Sigma)$ stands for the length or the boundary $\partial\Sigma$ of Σ. The motivation was the construction of free boundary minimal surfaces into Euclidean balls, whose induced metrics are critical metrics of Steklov eigenvalues on this surface among metrics with boundary of unit length. This problem is analogous to the seminal work by Nadirashvili [Nad96] and El-Soufi and Ilias [ESI00], who proved the one to one link between critical metrics for Laplace eigenvalues on closed surfaces among metrics with unit area and minimal immersions into a round sphere. This is a beautiful example of a connection between two apparently different fields : spectral geometry and the theory of minimal surfaces.

Concerning spectral geometry, the study of Steklov eigenvalues gained a lot of interest in the past decades. We refer to the nice initiating survey by Girouard and Polterovich [GP17], setting the typical questions of spectral geometry in the context of Steklov eigenvalues (upper bounds of the eigenvalues and their multiplicity with respect to the topology, extremal domains, study of the nodal sets, asymptotics of eigenvalues, isospectrality...) and all the answers given afterwards. In the current paper, we will focus on geometric upper bounds and on the existence question of maximizers for combinations of eigenvalues. Since the paper by Fraser and Schoen [FS16], many advances where given for the maximisation question for one eigenvalue, mainly for the first one. The existence a smooth maximal metric for the first nonzero Steklov eigenvalue for any finite topology of Σ was recently

solved in [Pet19] and [MP20] (see also [KS20] in the particular case of high topology). Up to a conformal factor equal to one on the boundary of Σ, these metrics are the pull-back metrics of the Euclidean one for some free boundary minimal immersions into a ball. In particular, for surfaces of genus zero these immersions are embeddings and the target ball has dimension 3 (see [FS16]). As recent results for genus zero surfaces, in [GL20], the authors study more carefully the shape of these surfaces as the number of boundary components goes to $+\infty$, while in [KOO21], the authors perform a numerical method for maximization of eigenvalues in order to make beautiful pictures of the maximal shapes.

For the maximization problem of higher eigenvalues σ_m, in the main result in [Pet19], we prove the following alternative: either their is a maximal metric for the isoperimetric problem on a fixed surface, or the sharp bound for this problem can be computed as a sum of sharp bounds for previous eigenvalues (σ_k, $k \leq m$) on surfaces of previous topologies. Both alternatives may occur: for $m = 1$ and any topologies we have maximal metrics (see [MP20]), while on the disk, for instance, we never have maximizers for $m \geq 2$. Indeed, testing a metric on the disk converging in some sense to a disjoint union of m flat disks of same perimeter, we obtain that $\sup_g \sigma_m(g) L_g(\mathbb{S}^1) \geq 2\pi m$ (see e.g [FS19]), and the only possible critical metrics for Steklov eigenvalues are flat disks (see [FS15]), so that if there is a maximal metric, $\sup_g \sigma_m(g) L_g(\mathbb{S}^1) = 2\pi \left[\frac{m+1}{2}\right]$, which is false for $m \geq 2$. We can deduce from [Pet19] that the previous large inequality is an equality. By the way, this is a classical result by [HPS75] since they prove by test function methods on the disk that $\sigma_m(g) L_g(\mathbb{S}^1) \leq 2\pi m$ for any metric on the disk and any m. However, this technique to compute sharp bounds was used and fruitful in the analogous context of the Laplacian on spheres and projective planes (see [Pet18],[KNPP19],[Kar19]). By the example of the disk, it seems that maximizing higher eigenvalues do not often give a connected maximizer, so that we cannot build new minimal surfaces. In the current paper, we propose a more flexible way to build free boundary minimal surfaces, working on a combination of eigenvalues instead of a single one.

Concerning the theory of minimal surfaces with free boundary, a very first example is the classical Plateau problem: are there area-minimizing disks whose boundary is a closed curve in $\mathbb{R}^3$? This problem was independently solved by Douglas and Radó. Later, Courant [Cou40] asked for disks minimizing the area where the boundary lies in a constrained surface of $\mathbb{R}^3$. This started an active research on the so-called free-boundary minimal surfaces (see e.g Hildebrandt [Hil85]). Their construction are somewhat analogous to classical constructions of closed minimal surfaces but require subtle techniques due to the boundary. Some techniques follow the natural generalization of the Plateau problem, by a minmax on the Dirichlet energy: see [Fra00], [LP19], [LSZ20]. Other Techniques follow Algrem Pitt's min-max theory on varifolds [Li15], [GLWZ21]. Other ones are somewhat intermediate by the so-called viscosity method [Pi20]. In order to build examples with more elaborate topology, many authors focused on the particular case of target balls. Using perturbation techniques, Folha, Pacard, and Zolotareva [FPZ17] obtained the existence of examples in $\mathbb{B}^3$ with genus 0 and 1 and k boundary components for k large. Using an equivariant version of min-max theory, Ketover obtained the existence of free boundary minimal surfaces in $\mathbb{B}^3$ of unbounded genus and three boundary components [Ke17, Ke17a]. Examples of the same topological type using desingularization techniques were found by Kapouleas and Li [KL17]. Examples with high genus and connected boundary were constructed by Kapouleas and Wiygul [KW17]. Another recent result by Carlotto, Franz

and Schulz [CFS20] gives existence of free boundary minimal surfaces with arbitrary genus, connected boundary, and dihedral symmetry. As already said, existence of free boundary minimal immersions for any topology was solved in [Pet19] and [MP20]. As explained below, in the current paper, we focus on free boundary minimal surfaces into Euclidean ellipsoids arising by maximization of finite combinations of Steklov eigenvalues.

Let $F : \left(\mathbb{R}_+^\star\right)^m \to \mathbb{R}_+$ be a $\mathcal{C}^1$ function with $m \geq 1$, depending on eigenvalues. We set

$$S_F(\Sigma) = \inf_g F\left(L_g(\partial\Sigma)\sigma_1(g), \cdots, L_g(\partial\Sigma)\sigma_m(g)\right) ,$$

where the infimum is taken over all smooth metrics on Σ. We changed the "maximization" point of view into a "minimization" since it is more natural for all the examples we give below, like $\frac{1}{\sigma_1} + \cdots + \frac{1}{\sigma_m}$. For sums of eigenvalues, we minimize $\frac{1}{\sigma_1+\cdots+\sigma_m}$. We only assume the following condition on F, satisfied for all the functionals we consider in the introduction:

(H) $\quad$ F is a nonincreasing function with respect to all the coordinates

As we notice below, for $1 \leq k \leq m$, we naturally have to introduce

$$S_{F,k}(\Sigma) = \inf_g F\left(0, \cdots, 0, L_g(\partial\Sigma)\sigma_{k+1}(g), \cdots, L_g(\partial\Sigma)\sigma_m(g)\right) .$$

If Σ is oriented of genus γ, we denote $S_F(\gamma) := S_F(\Sigma)$ and $S_{F,k}(\gamma) := S_{F,k}(\Sigma)$. We have

Theorem 1.1. *Assume that Σ is a compact oriented surface of genus $\gamma \geq 0$ with $b > 1$ boundary components. Assume that F satisfies (H). If the two following assumptions*

$$S_F(\gamma, b) < S_{F,1}(\gamma, b) \tag{1.1}$$

and

$$(\gamma, b) = (0, 1) \text{ or } \begin{cases} S_F(\gamma, b) < S_F(\gamma - 1, b + 1) \\ S_F(\gamma, b) < S_F(\gamma, b - 1) \end{cases} \tag{1.2}$$

hold, then there is a smooth metric g on Σ, realizing the minimum $S_F(\gamma)$. Moreover, there is a (possibly branched) free boundary minimal immersion $\Phi : \Sigma \to \mathbb{R}^d$, into some ellipsoid

$$\mathcal{E} = \left\{(x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \sigma_1 |x_1|^2 + \cdots + \sigma_m |x_m|^2 = 1\right\}$$

endowed with the Euclidean metric ξ on $\mathbb{R}^d$ with $d = i_1 + \cdots + i_m$. The coordinate functions of Φ_k are eigenfunctions with respect to $\sigma_k := \sigma_k(g)$ for $1 \leq k \leq m$. Moreover, i_k is a positive integer if and only if $\partial_k F(\sigma_1, \cdots, \sigma_m) < 0$.

This is a generalization of the author's result about maximization of one single eigenvalue $g \mapsto L_g(\partial\Sigma)\sigma_m(g)$ for $m \geq 1$ [Pet19]. The first assumption (1.1) just prevents the first eigenvalue of minimizing sequences from converging to 0. With this assumption, we always obtain a connected surface at the limit. In this case, Theorem (1.1) always gives a minimizer on the disk, while $S_F(\gamma, b) < S_F(\gamma - 1, b + 1)$ and $S_F(\gamma, b) < S_F(\gamma, b - 1)$ are natural conditions to prevent from possible degeneration of minimizing sequences to lower topologies. As already said, for $F(\sigma_1) = \frac{1}{\sigma_1}$, strict inequalities occur for any topology.

We also prove in Theorem 2.1 below, that critical metrics g for combinations of eigenvalues are metrics conformal to the induced metrics of free boundary minimal immersions Φ into (Pseudo)-Euclidean ellipsoids, with a conformal factor determined in terms of the coordinate functions of Φ on $\partial\Sigma$ (which are also eigenfunctions with respect to g) (see section 2). With assumption (H) for F, the target manifold is a Euclidean ellipsoid. If

only one eigenvalue appears, the target manifold is a ball. Therefore, we also generalize the characterization of critical metrics for Steklov eigenvalues on surfaces with boundary by Fraser and Schoen [FS16].

Notice that the eigenvalues appearing in minimal immersion may not be all the eigenvalues appearing in the functional. One might just assume in addition that $(\sigma_1, \cdots, \sigma_m)$ is not a critical point of F to have a non-empty conclusion.

Following the same strategy as in [Pet19], we split our minimization into two minimizations:

$$S_F(\Sigma) = \inf_{[g]} S_F(\Sigma, [g]) \, ,$$

where we denote $[g]$ the conformal class of some metric g on Σ and

$$S_F(\Sigma, [g]) = \inf_{\tilde{g} \in [g]} F\left(L_g(\partial\Sigma)\sigma_1(g), \cdots, L_g(\partial\Sigma)\sigma_m(g)\right)$$

is defined as a conformal constrained minimization. As previously, we define

$$S_{F,k}(\Sigma, [g]) = \inf_{\tilde{g} \in [g]} F\left(0, \cdots, 0, L_g(\partial\Sigma)\sigma_{k+1}(g), \cdots, L_g(\partial\Sigma)\sigma_m(g)\right) \, ,$$

and we have the conformal minimization result:

Theorem 1.2. *Let (Σ, g) be a Riemannian surface without boundary. Let $1 \leq k \leq m$. We assume that F satisfies (H), and that*

$$S_F(\Sigma, [g]) < S_{F,k}(\Sigma, [g]) \tag{1.3}$$

and

$$S_F(\Sigma, [g]) < S_F(\widetilde{\Sigma}, [\tilde{g}]) \, , \tag{1.4}$$

for any compact Riemanian surface $(\widetilde{\Sigma}, \tilde{g})$ obtained as a disjoint union of $(\Sigma, [g])$ and at most $k-1$ copies of spheres, or the disjoint union of at most k copies of spheres.

Then there is a minimal smooth metric $\tilde{g}$ for $S_F(\Sigma, [g])$. Moreover, there is a free boundary harmonic map $\Phi : (\Sigma, g) \to \mathbb{R}^d$ into

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \lambda_1 |x_1|^2 + \cdots + \lambda_m |x_m|^2 = 1 \right\} \, ,$$

a Euclidean ellipsoid in $\mathbb{R}^d$, where $d = i_1 + \cdots + i_m$, such that the coordinate functions of Φ_k are eigenfunctions with respect to $\sigma_k := \sigma_k(\tilde{g})$ for $1 \leq k \leq m$ and $\tilde{g} = e^{2u} g$ is given from Φ by

$$e^u = \Phi . \partial_\nu \Phi \text{ on } \partial\Sigma \, .$$

Again, this is a generalization of the author's result about maximization of one single eigenvalue $g \mapsto L_g(\partial\Sigma)\sigma_m(g)$ for $m \geq 1$ in the conformal case [Pet19]. The first assumption (1.3) just gives that the sequence of metrics have at most k connected components at the limit. Assuming in addition (1.4), just gives a connected surface at the limit.

We also have a characterization of critical metrics in the conformal case. By Theorem 2.2 below, the conformal factors of critical metrics $\tilde{g} = e^{2u} g \in [g]$ for combinations of eigenvalues arise as some densities of energy of some harmonic maps Φ into (Pseudo)-Euclidean ellipsoids (whose coordinate functions are also eigenfunctions with respect to g) (see section 2). We loose conformality of Φ because the set of variations to compute the Euler-Lagrange equation for $\tilde{g}$ is constrained to conformal classes. With assumption (H) for F, the target manifold is a Euclidean ellipsoid. If only one eigenvalue appears,

the target manifold is a ball. Therefore, we also generalize the characterization of critical metrics for Steklov eigenvalues on surfaces with boundary in the conformal case by Fraser and Schoen [FS16].

Theorem 1.1 is then a relatively straightforward consequence of Theorem 1.2 as soon as we know the Deligne-Mumford compactification theorem for sequence of surfaces such that the sequence of associated conformal classes degenerates (see [ZHU10]), and the precise description of sequences free boundary harmonic maps into ellipsoids on these surfaces (bubble tree convergence, see [JLZ19]). We chose to skip this proof in the current paper while it relatively follows the proof of the same result for one eigenvalue written in [Pet19]. Then, in the current paper we prove Theorem 1.2 which needs stronger results than in [Pet19] because the target manifold is not as symmetric as a sphere anymore and because the eigenfunctions we have to deal with are not associated to the same eigenvalue anymore. The main novelty is in section 3, while we explain in section 4 all the new ideas we need to improve the previous results in [Pet19].

As fruitful as the variational methods on one eigenvalue developped in [Pet18] and [Pet19] were for the construction of new minimal surfaces and/or the computations of sharp eigenvalue bounds, Theorem 1.1 and Theorem 1.2 are opening ways to pursue a deeper understanding of these questions for combination of eigenvalues. We would like to emphasize that Theorem 1.1 already gives minimizers for a wild number of combinations F on the disk. For instance, for F satisfying $F(\sigma_1, \cdots, \sigma_m) \to +\infty$ as $\sigma_1 \to 0$ uniformly with respect to the other variables, (1.1) is automatic.

In a forecoming paper we aim at constructing new free boundary minimal disks into some Euclidean ellipsoids by maximization of linear combinations of σ_1 and σ_2 as an application of Theorem 1.1. In this case, to have existence of a maximizer, we need to prove (1.1) which is not automatic. What's interesting in this case is that we will prove that there are linar combinations such that the minimizer is not a branched cover over an ellipse but a non flat free boundary minimal surface into an ellipsoid.

There are also interesting recent developments on minimization of infinite combinations of Steklov eigenvalues via the Steklov zeta function $\sum_k \sigma_k(g)^{-s}$ for $s > 1$ (and its meromorphic extensions for $s < 1$) in [JS18], [JS20]. The authors prove in these papers that the flat disk minimizes this functional among metrics on the disk having perimeter 2π. It would be interesting to understand how the minimizers of the partial sum $\sum_{k=1}^{N} \sigma_k(g)^{-s}$ behave as $N \to +\infty$ on the disk, and if minimizing partial sums give a new method to prove existence of a minimizer for other topologies of the surface.

The paper is organized as follows:

In section 2, we make simple remarks about the link between Steklov eigenvalues and free boundary minimal surfaces into ellipsoids. Then we state and prove Theorem 2.1 and 2.2 characterizing critical metrics for very general finite combination of eigenvalues.

In section 3, we recall the harmonic replacement procedure and prove a quantitative ε-regularity estimate for eigenmaps. It is based on a deep understanding of the structure of the equation of harmonic-like maps, initiated by the celebrated paper by Riviere [Riv08], and energy convexity results initiated by Colding and Minicozzi [CM08] we explain more in details in section 5. This is a key result for $W^{1,2}$-convergence of eigenfunctions in section 4.

In section 4, we prove Theorem 1.2. The techniques are based on [Pet19] but stronger and simpler intermediate lemma are introduced.

2. Critical metrics for combinations of eigenvalues

2.1. **Free boundary minimal immersions into ellipsoids.** We show link between free boundary minimal immersions into ellipsoids and the Steklov eigenvalues of some associated metric. Let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid of parameters $\sigma = diag\,(\sigma_1, \cdots, \sigma_n)$, with $\sigma_i > 0$, defined by

$$\mathcal{E} = \{(x_1, \cdots, x_n) \in \mathcal{E}; \sigma_1 x_1^2 + \cdots + \sigma_n x_n^2 = 1\}\,,$$

endowed with the induced metric of the Euclidean metric ξ. We know that x is a harmonic map. We compute the outward normal derivative of x on $\mathcal{E}$:

$$\partial_\nu x = \nu\,,$$

where the outward normal of the ellipsoid is denoted by $\nu = \frac{\sigma x}{|\sigma x|}$ where $|\sigma x| = \left(\sum_{i=1}^n \sigma_i^2 x_i^2\right)^{\frac{1}{2}}$.

Now, let $\Phi : (\Sigma, h) \to \mathbb{R}^n$ be such that $\Phi(\partial\Sigma) \in \mathcal{E}$, a $n-1$ dimentional ellipsoid of parameter $\sigma = (\sigma_1, \cdots, \sigma_n)$. A well-known characterisation of $\Phi : (\Sigma, h) \to \mathbb{R}^n$ to be minimal with free boundary in $\mathcal{E}$ is free boundary harmonicity in $\mathcal{E}$ and conformality. Φ is harmonic in $\mathcal{E}$ with free boundary if it is a critical point of the energy

$$E(\Phi) = \frac{1}{2}\int_\Sigma |\nabla\Phi|_h^2\, dA_h$$

under the constraint $\Phi(\partial\Sigma) \subset \mathcal{E}$. The Euler-Lagrange characterization is

$$\Delta_h \Phi = 0 \text{ in } \Sigma \text{ and } \partial_\nu \Phi \in (T_\Phi \mathcal{E})^\perp \text{ on } \partial\Sigma$$

Then $\partial_\nu \Phi = f\nu$ for some function $f = \Phi.\partial_\nu\Phi$. Conformality is characterized by the vanishing of

$$0 = |\nabla\Phi|_g^2 \frac{g}{2} - d\Phi \otimes d\Phi := \sum_{i=1}^n \left(|\nabla\Phi_i|_g^2 \frac{g}{2} - d\Phi_i \otimes d\Phi_i\right)\,.$$

For a smooth positive function e^{2u}, such that $g = e^{2u}h$, we have

$$\Delta_g f = e^{-2u}\Delta_h f \text{ and } \partial_{\nu_g} f = e^{-u}\partial_{\nu_h} f\,,$$

and if $\Phi : (\Sigma, h) \to \mathbb{R}^n$ is a minimal isometric immersion with free boundary in $\mathcal{E}$, setting $g = e^{2u}h$ for any function u extending the following formula on the boundary,

$$e^u = \Phi.\partial_\nu\Phi = \frac{1}{|\sigma\Phi|} \text{ on } \partial\Sigma\,,$$

the coordinates of Φ are Steklov eigenfunctions on (Σ, g) with eigenvalues $\sigma_1, \cdots, \sigma_n$.

2.2. **Critical metrics for combinations of eigenvalues.** We prove in this part general properties for critical metrics of functionals depending on Steklov eigenvalues. The notion of critical metrics in the context of eigenvalues depending on metrics was introduced by [ESI00] who generalized a result by [Nad96] (Laplace eigenvalues in the closed case). They used that in variational theory, there is a classical way to generalize the notion of derivative of a locally Lipschitz functional by sub-gradients (see for instance [Cla13]). Here, we say that g is critical with respect to $F(L_g(\partial\Sigma)\sigma_1(g), \cdots, L_g(\partial\Sigma)\sigma_m(g))$ if the product of the left and right derivative of $t \mapsto F(L_{g+th}(\partial\Sigma)\sigma_1(g+th), \cdots, L_{g+th}(\partial\Sigma)\sigma_m(g+th))$ is non positive for any 2-symmetric variation h. Of course, an extremal metric satisfies this condition.

Theorem 2.1. *Let Σ be a compact surface with a non-empty boundary and $F : \left(\mathbb{R}_+^\star\right)^m \to \mathbb{R}$ be a smooth function with $m \geq 1$. Let g be a critical metric for the functional*

$$g \mapsto F\left(L_g(\partial\Sigma)\sigma_1(g), \cdots, L_g(\partial\Sigma)\sigma_m(g)\right)$$

and we assume that $L_g(\partial\Sigma) = 1$. Then, there are m non-negative integers $i_1, \cdots, i_m$ and there is a map $\Phi : \Sigma \to \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$ which is conformal and harmonic with free boundary into the space

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \epsilon_1\sigma_1 |x_1|^2 + \cdots + \epsilon_m\sigma_m |x_m|^2 = \sum_{k=1}^m \epsilon_k\sigma_k t_k \right\},$$

where for $1 \leq k \leq m$, $\sigma_k = \sigma_k(g)$, $t_k = |\partial_k F(\sigma_1, \cdots, \sigma_m)|$, ϵ_k is the sign of $\partial_k F(\sigma_1, \cdots, \sigma_m)$ and $\mathcal{E}$ is endowed with the pseudo-Euclidean metric defined for $(x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$ by $\epsilon_1 |dx_1|^2 + \cdots + \epsilon_m |dx_m|^2$.

Notice that if $\partial_k F(\sigma_1, \cdots, \sigma_m)$ all have the same sign, then the target manifold is the ellipsoid of a Euclidean space. t_k is nothing but the meanvalue of $\left|\Phi^k\right|^2$. This result is a generalization of the result by Fraser and Schoen in [FS13], when only one eigenvalue appears in the functional, while they noticed that critical metrics for Steklov eigenvalues on surfaces with boundary arise as the induced metric of a free boundary minimal immersion into a ball. They were inspired by the seminal paper by Nadirashvili [Nad96] and El Soufi and Ilias [ESI00].

Proof. We aim at using the oposite signs of the left and right derivative at $t = 0$ for

$$t \mapsto F\left(L_{g+th}(\Sigma)\sigma_1(g + th), \cdots, L_{g+th}(\Sigma)\sigma_m(g + th)\right)$$

for h a smooth symmetric 2-form. For $h \in L^2(S^2(T\Sigma))$ a L^2 symmetric 2-form on $T\Sigma$ and $f \in L^2(\partial\Sigma)$. We denote by

$$\begin{aligned} Q_{(h,f)}(\Phi) = \sum_{k=1}^m \epsilon_k t_k \int_\Sigma \left(|\nabla\Phi_k|_g^2 \frac{g}{2} - d\Phi_k \otimes d\Phi_k, h \right)_g dA_g \\ + \sum_{k=1}^m \epsilon_k t_k \int_{\partial\Sigma} \frac{\sigma_k}{2} \left(1 - |\Phi_k|^2\right) f dL_g \end{aligned} \tag{2.1}$$

for $\Phi \in E_1(g) \times \cdots \times E_m(g) \in \mathcal{C}^\infty(\Sigma)$. $E_k(g)$ denotes the set of all eigenfunctions associated to the eigenvalue $\sigma_k(g)$. In the following, we denote by τ and ν a unit tangent vector-field and normal vector-field to $\partial\Sigma$ with respect to the metric g. The goal of the following 4 steps is to prove that for any L^2 symmetric 2-form h on $T\Sigma$ and any function $f \in L^2(\partial\Sigma)$, there is a map $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_{(h,f)}(\Phi) = 0$. Here, $S(E_i(g))$ is the unit sphere in $E_i(g)$ endowed with the $L^2(\partial\Sigma, g)$-norm with respect to g on the boundary.

In the following, let $h \in \mathcal{C}^\infty(T\Sigma)$ be a smooth symmetric 2-form on $T\Sigma$.

STEP 1 :

If ϕ^t is an eigenfunction associated to some eigenvalue $\sigma_t = \sigma_k(g+th)$ with $\int_\Sigma \left(\phi^t\right)^2 dL_{g+th} = 1$, then up to a subsequence as $t \to 0$, we have $\sigma > 0$ such that

$$\sigma_t \to \sigma \text{ and } \phi_t \to \phi \text{ in } \mathcal{C}^2(M) \text{ as } t \to 0$$

where $\int_{\partial\Sigma} \phi^2 dL_g = 1$, $\Delta_g \phi = 0$ and $\partial_\nu \phi = \sigma\phi$.

Proof of STEP 1 :

We have $\Delta_{g+th}\phi^t = 0$ and $\partial_{\nu_{g+th}}\phi^t = \sigma^t\phi^t$. In a chart $\theta : U \to V$ such that $\theta\left(\partial\Sigma \cap U\right) = \mathbb{R} \times \{0\} \cap V$, the equation is given by

$$\begin{cases} -\frac{1}{\sqrt{|g+th|}}\partial_i\left((g+th)^{ij}\sqrt{|g+th|}\partial_j\phi^t\right) = 0 \\ \partial_{\nu_t}\phi^t = \sigma^t\phi^t \ , \end{cases} \tag{2.2}$$

where ν_t is the unit normal with respect to $g+th$. Locally, we have a constant $C_0 > 1$ and $t_0 > 0$ such that for $|t| < t_0$,

$$\frac{1}{C_0}\xi \leq g + th \leq C_0\xi$$

where ξ is the Euclidean metric, so that we have constants $C_1 > 1$ and $C_2 > 1$ with

$$\frac{1}{C_1}\left\|X\right\|^2 \leq \sqrt{|g+th|}\left(g+th\right)^{ij}X_iX_j \leq C_1\left\|X\right\|^2 \ ,$$

$$\frac{1}{C_2}\int_\Sigma \left|\nabla\phi^t\right|^2 dx \leq \int_\Sigma \left|\nabla\phi^t\right|^2_{g+th} dA_{g+th} \leq C_2\int_\Sigma \left|\nabla\phi^t\right|^2 dx \ ,$$

$$\frac{1}{C_2}\int_{\partial\Sigma}\left(\phi^t\right)^2 dl \leq \int_{\partial\Sigma}\left(\phi^t\right)^2 dL_{g+th} \leq C_2\int_{\partial\Sigma}\left(\phi^t\right)^2 dl \ .$$

Since $L_{g+th} = 1+o(1)$, σ_t is bounded as $t \to 0$. Up to a subsequence, we assume that $\sigma_t \to \sigma$ as $t \to 0$. Therefore, by standard elliptic theory, since $\nu_t = \nu + o(1)$ as $t \to 0$ is uniformly transversal to the boundary and since $\sigma_t\phi^t$ is bounded in L^2, up to a subsequence, ϕ_t converges in $\mathcal{C}^2(M)$ to some function ϕ. When we let $t \to 0$, λ and ϕ satisfy the conclusion of STEP 1.

STEP 2 :

For any $k \in \{0, 1, \cdots, m\}$, $\sigma_k(g+th) \to \sigma_k(g)$ as $t \to 0$.

Proof of STEP 2 :

We argue by induction. We have that $\sigma_0(g+th) = 0 = \sigma_0(g)$. We now assume for $k \geq 1$ that for any $l \leq k-1$ we have $\sigma_l(g+th) \to \sigma_l(g)$ as $t \to 0$.

Let $(\phi_0^t, \phi_1^t, \cdots, \phi_{k-1}^t, \phi_k^t)$ an orthonormal basis of eigenfunctions associated to $\sigma_0(g+th), \sigma_1(g+th), \cdots, \sigma_{k-1}(g+th), \sigma_k(g+th)$. By the STEP 1, we have up to a subsequence,

$$\sigma_l(g+th) \to \sigma_l \text{ and } \phi_l^t \to \phi_l \text{ in } \mathcal{C}^2(\Sigma) \text{ as } t \to 0 \tag{2.3}$$

for any $1 \leq l \leq k$ for some functions ϕ_l and values $\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_k$ such that

$$\begin{cases} \Delta_g \phi_l = 0 \text{ for any } 1 \leq l \leq k \\ \partial_\nu \phi_l = \sigma_l \phi_l \text{ for any } 1 \leq l \leq k \\ \int_{\partial\Sigma} \phi_l \phi_{l'} dL_g = \delta_{ll'} \text{ for any } 1 \leq l, l' \leq k \, . \end{cases}$$

By this equation, we already have $\sigma_l \geq \sigma_l(g)$ for any $1 \leq l \leq k$. We also know by the assumption that $\sigma_l = \sigma_l(g)$ for $1 \leq l \leq k-1$. We now prove that $\sigma_k \leq \sigma_k(g)$. Let ψ be a k-th eigenfunction with respect to $\sigma_k(g)$ such that

$$\int_{\partial\Sigma} \psi^2 dL_g = 1 \text{ and } \int_{\partial\Sigma} \psi \phi_l dL_g = 0 \tag{2.4}$$

for $l \leq k-1$. Then

$$\sigma_k(g+th) \leq \frac{\int_\Sigma |\nabla (\psi - \pi_t(\psi))|_{g+th} \, dA_{g+th}}{\int_{\partial\Sigma} (\psi - \pi_t(\psi))^2 \, dL_{g+th}}$$

where $\pi_t : \mathcal{C}^\infty(\Sigma) \to E_0(g+th) \oplus E_1(g+th) \oplus \cdots \oplus E_{k-1}(g+th)$ the orthonormal projection in $L^2(\partial\Sigma)$ with respect to the metric $g+th$, extended by a harmonic function on Σ with respect to g. By (2.3) and (2.4), we get that

$$\int_{\partial\Sigma} \psi \phi_l^t dL_{g+th} = \int_{\partial\Sigma} \psi \phi_l dL_g + o(1) = o(1) \text{ as } t \to 0$$

and by (2.3) and what is just before,

$$\pi_t(\psi) = \sum_{l=1}^{k-1} \left(\int_{\partial\Sigma} \psi \phi_l^t dL_{g+th} \right) \phi_l^t \to 0 \text{ in } \mathcal{C}^2(\partial\Sigma) \text{ as } t \to 0 \, .$$

on the boundary. Then $\pi_t(\psi) \to 0$ in $\mathcal{C}^2(\Sigma)$ as a harmonic function. Then,

$$\sigma_k(g+th) \leq \frac{\int_\Sigma |\nabla \psi|_g^2 \, dA_g}{\int_{\partial\Sigma} (\psi)^2 \, dL_g} + o(1) \to \sigma_k(g) \text{ as } t \to 0$$

and $\sigma_k \leq \sigma_k(g)$. As already said, $\sigma_k \geq \sigma_k(g)$ and we obtain STEP 2.

STEP 3 :

Up to a subsequence as $t \to 0$, we have the existence of eigenfunctions ϕ_k^+ and ϕ_k^- in $S(E_k(g))$ such that

$$\begin{aligned} \frac{A_t \lambda_k(g+th) - \lambda_k}{t} \to & \int_\Sigma \left(-d\phi_k^\pm \otimes d\phi_k^\pm + \frac{\left|\nabla \phi_k^\pm\right|_g^2}{2} g, h \right)_g dA_g \\ & - \int_{\partial\Sigma} \frac{\sigma_k}{2} (1 - \left(\phi_k^\pm\right)^2) h(\tau,\tau) dL_g \end{aligned} \tag{2.5}$$

where ϕ_k^+ is the limit as $t \to 0$ and $t > 0$ and ϕ_k^- is the limit as $t \to 0$ and $t < 0$.

Proof of STEP 3 :

By connexity of $S(E_1(g)) \times \cdots \times S(E_m(g))$, it is equivalent to prove that there are Φ^+ and Φ^- in $S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_h(\Phi^+)Q_h(\Phi^-) \leq 0$. We use in this step that g is a critical metric for the functional. $Q_h(\Phi^+)$ and $Q_h(\Phi^-)$ correspond to the opposite signs of the right and left derivatives at 0 with respect to t for the variation $g+th$ of $t \mapsto F\left(L_t\sigma_1(g+th), \cdots, L_t\sigma_m(g+th)\right)$. Here, we denoted $L_t = L_{g+th}(\partial\Sigma) = 1 + \frac{t}{2}\int_{\partial\Sigma} h(\tau,\tau)dL_g + o(t)$.

Let's compute the left and right derivatives of an eigenvalue $\sigma_t = \sigma_k(g+th)$ with its associated eigenfunction ϕ^t in $E_k(g+th)$ with unit $L^2(g+th)$-norm. By STEP 2, we know that σ_t converges to $\sigma = \sigma_k(g)$ and ϕ^t to $\phi \in S(E_k(g))$ in $\mathcal{C}^2(\Sigma)$ as $t \to 0$. We denote by π_k the projection in $L^2(\partial\Sigma, g)$ on the whole eigenspace associated to $\sigma_k(g)$, extended by a harmonic function in Σ, and by $R^t = \phi^t - \pi_k(\phi^t)$. In some chart, we have the equation on Σ

$$\begin{aligned}
\Delta_g R^t &= \Delta_g \phi^t \\
&= (\Delta_g - \Delta_{g+th})\phi^t \\
&= -\left(\frac{1}{\sqrt{|g|}} - \frac{1}{\sqrt{|g+th|}}\right)\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi^t\right) \\
&\quad -\frac{1}{\sqrt{|g+th|}}\partial_i\left(\left(\sqrt{|g|} - \sqrt{|g+th|}\right)g^{ij}\partial_j\phi^t\right) \\
&\quad -\frac{1}{\sqrt{|g+th|}}\partial_i\left(\sqrt{|g+th|}\left(g^{ij} - (g+th)^{ij}\right)\partial_j\phi^t\right)
\end{aligned}$$

and the equation on $\partial\Sigma$

$$\partial_\nu R^t - \sigma R^t = (\sigma_t - \sigma)\,\phi^t - g(\nu_t - \nu, \nabla\phi^t)$$

We set

$$\alpha_t = |\sigma - \sigma_t| + \left\|R^t\right\|_\infty + |t| \tag{2.6}$$

and dividing by α_t we write the previous equation as

$$\begin{cases} \Delta_g\left(\frac{R^t}{\alpha_t}\right) = \frac{f^t}{\alpha_t} & \text{on } \Sigma \\ \partial_\nu\left(\frac{R^t}{\alpha_t}\right) - \sigma\frac{R^t}{\alpha_t} = \frac{(\sigma_t-\sigma)}{\alpha_t}\phi^t - \frac{t}{\alpha_t}g\left(\frac{\nu_t-\nu}{t}, \nabla\phi^t\right) \end{cases}$$

where $\frac{f^t}{t}$ converges to

$$f_0 = \frac{1}{2}(g,h)_g\Delta_g\phi + \frac{1}{2\sqrt{|g|}}\partial_i\left(\sqrt{|g|}(g,h)_g g^{ij}\partial_j\phi\right) - \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ik}h_{kl}g^{lj}\partial_j\phi\right)$$

as $t \to 0$ and

$$g\left(\frac{\nu_t - \nu}{t}, \nabla\phi^t\right) \to g(\dot\nu, \nabla\phi)$$

as $t \to 0$ in $\mathcal{C}^2(\partial\Sigma)$. where $\nu := \nu_0$ is the unit normal with respect to g and $\dot\nu := \left(\frac{d}{dt}\nu_t\right)_{t=0}$ is computed knowing that

$$(g+th)(\nu_t, \nu_t) = 1 \text{ and } (g+th)(\nu_t, \tau) = 0\,.$$

Indeed, taking the derivative at $t = 0$ gives

$$h(\nu,\nu) + 2g(\nu,\dot\nu) = 0 \text{ and } h(\nu,\tau) + g(\dot\nu,\tau) = 0\,,$$

so that in the orthonormal basis (τ,ν), we have

$$\dot\nu = -\left(h(\nu,\tau)\tau + \frac{h(\nu,\nu)}{2}\nu\right)\ . \tag{2.7}$$

Up to a subsequence, by elliptic regularity of the equation we let

$$\frac{t}{\alpha_t}\to t_0\ ,\ \frac{R^t}{\alpha_t}\to R_0\ ,\ \frac{\sigma^t-\sigma}{\alpha_t}\to\delta_0 \text{ as } t\to 0\ .$$

Then, the previous equation becomes as $t\to 0$

$$\begin{cases}\Delta_g R_0 = t_0 f_0 & \text{in } \Sigma\\ \partial_\nu R_0 - \sigma R_0 = \delta_0\phi - t_0 g(\dot\nu,\nabla\phi) & \text{on } \partial\Sigma\ .\end{cases}$$

We integrate it against ϕ so that by integration by part and since $\Delta_g\phi=0$, $\partial_\nu\phi=\sigma\phi$ and $\int_{\partial\Sigma}\phi^2 dL_g=1$,

$$0 = t_0\left(\int_\Sigma f_0\phi dA_g - \int_{\partial\Sigma} g(\dot\nu,\nabla\phi)\phi dL_g\right)+\delta_0\ .$$

If t_0, then $\delta_0=0$ and ΔR_0 and $\partial_\nu R_0-\sigma R_0=0$, which means that $R_0=0$ since $R_0\in E_k(g)^\perp$ on $\partial\Sigma$ as a limit of $\frac{1}{\alpha_t}\left(\phi^t-\pi_k(\phi^t)\right)$. However, by (2.6), the convergence

$$1=\frac{|\lambda-\lambda_t|+\left\|R^t\right\|_\infty+|t|}{\alpha_t}\to\delta_0+\left\|R^0\right\|_\infty+|t_0|=0 \text{ as } t\to 0$$

is absurd. Therefore, $t_0\neq 0$ and

$$\begin{aligned}\frac{\delta_0}{t_0} &= -\int_\Sigma f_0\phi dA_g + \int_{\partial\Sigma} g(\dot\nu,\nabla\phi)\phi dL_g\\ &= \int_\Sigma\left(\frac{\phi}{2}(g,h)_g\left(-\Delta_g\phi\right) - \frac{\phi}{2\sqrt{|g|}}\partial_i\left(\sqrt{|g|}(g,h)_g g^{ij}\partial_j\phi\right)\right)dA_g\\ &\quad+\int_\Sigma\left(\frac{\phi}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ik}h_{kl}g^{lj}\partial_j\phi\right)\right)dA_g+\int_{\partial\Sigma}g(\dot\nu,\nabla\phi)\phi dL_g\\ &= \int_\Sigma\left(\frac{1}{2}(g,h)_g\left(-\phi\Delta_g\phi+|\nabla\phi|_g^2\right)-(d\phi\otimes d\phi,h)_g\right)dA_g\\ &\quad+\int_{\partial\Sigma}\left(g(\dot\nu,\nabla\phi)-\frac{1}{2}(g,h)_g\,\partial_\nu\phi+h\left(\nu,\nabla\phi\right)\right)\phi dL_g\\ &= \int_\Sigma\left(-d\phi\otimes d\phi+\frac{|\nabla\phi|_g^2}{2}g,h\right)_g dA_g-\int_{\partial\Sigma}\frac{\sigma}{2}\phi^2 h(\tau,\tau)dL_g\end{aligned}$$

where the second equality has to be read in a suitable atlas with partitions of unity and the third one is got with an integration by parts. For the last inequality, we easily compute that

$$g(\dot\nu,\nabla\phi)=-h(\nu,\tau)\partial_\tau\phi-\frac{1}{2}h(\nu,\nu)\partial_\nu\phi$$

by (2.7), that

$$(g,h)_g=h(\tau,\tau)+h(\nu,\nu)$$

since (τ,ν) is an orthonormal basis and that

$$h(\nu,\nabla\phi)=h(\tau,\nu)\partial_\tau\phi+h(\nu,\nu)\partial_\nu\phi$$

so that

$$g(\dot{\nu}, \nabla\phi) - \frac{1}{2}(g,h)_g \partial_\nu \phi + h(\nu, \nabla\phi) = -\partial_\nu \phi h(\tau,\tau)\phi = -\frac{\sigma}{2}\phi^2 h(\tau,\tau) .$$

Up to a subsequence, noting that $L_t = 1 + \frac{t}{2}\int_{\partial\Sigma} h(\tau,\tau) dL_g + o(t)$ we have the existence of eigenfunctions ϕ_k^+ and ϕ_k^- in $S(E_k(g))$ such that (2.5) holds true as $t \to 0$ with $t > 0$ and $t \to 0$ with $t < 0$. This ends the proof of STEP 3.

STEP 4 :

For any $(h,f) \in L^2(S^2(T\Sigma) \times L^2(\partial\Sigma))$, there is a map $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_{(h,f)}(\Phi) = 0$ where, $S(E_i(g))$ is the unit sphere in $E_i(g)$ endowed with the $L^2(\partial\Sigma)$ norm with respect to g.

From the following STEP, if $h \in \mathcal{C}^\infty(T\Sigma)$ we have that

$$\frac{d}{dt}_{|t=0^\pm} F(A_t\lambda_1(g+th), \cdots, A_t\lambda_k(g+th)) = Q_{(h,h(\tau,\tau))}(\phi_1^\pm, \cdots, \phi_m^\pm) .$$

Therefore, the maps $\Phi^\pm = (\phi_1^\pm, \cdots, \phi_m^\pm) \in S(E_1(g)) \times \cdots \times S(E_m(g))$ satisfy

$$Q_{(h,h(\tau,\tau))}(\Phi^+) Q_{(h,h(\tau,\tau))}(\Phi^-) \leq 0 .$$

As a product of connected spaces $S(E_1(g)) \times \cdots \times S(E_m(g))$ is a connected space we have that

$$\forall h \in \mathcal{C}^\infty(T\Sigma), \exists \Phi \in S(E_1(g)) \times \cdots \times S(E_m(g)) \text{ s.t } Q_{(h,h(\tau,\tau))}(\Phi) = 0 \tag{2.8}$$

Now, let $(h,f) \in L^2(S^2(T\Sigma) \times L^2(\partial\Sigma))$. There is a sequence $(h_n, f_n) \in \mathcal{C}^\infty(S^2(T\Sigma) \times \mathcal{C}^\infty(\partial\Sigma))$ such that $(h_n, f_n) \to (h,f)$ in $L^2(S^2(T\Sigma)) \times L^2(\partial\Sigma)$ as $n \to +\infty$. Up to a smooth perturbation close to $\partial\Sigma$, one can find $\tilde{h}_n \in \mathcal{C}^\infty(S^2(T\Sigma))$ such that

$$\left\| \tilde{h}_n - h_n \right\|_{L^2(S^2(T\Sigma))} \to 0 \text{ and } \tilde{h}_n(\tau,\tau) = f_n .$$

Therefore, there is $\Phi_n \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_{(\tilde{h}_n, f_n)}(\Phi_n) = 0$. Since $S(E_1(g)) \times \cdots \times S(E_m(g))$ is a compact subspace of a finite-dimensional space, up to a subsequence, Φ_n converges to Φ in $S(E_1(g)) \times \cdots \times S(E_m(g))$ endowed with the $\mathcal{C}^2$ norm.

Passing to the limit as $n \to +\infty$, we get $Q_{(h,f)}(\Phi) = 0$ with $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$. This ends the proof of STEP 4.

Proof of STEP 4 :

End of the Proof of Theorem 2.1 :

Denoting $S^2(T\Sigma)$ the set of symmetric 2-forms on Σ, We now let C be the convex hull of the following set included in $L^2\left(S^2(T\Sigma)\right) \times L^2(\partial\Sigma)$:

$$\left\{ \left(\sum_{k=1}^m \epsilon_k t_k \left(-d\phi_k \otimes d\phi_k + \frac{|\nabla\phi_k|_g^2}{2} g \right), - \left(\sum_{k=1}^m \epsilon_k t_k \frac{\sigma_k}{2}(1-\phi_k^2) \right) \right) ; (\phi_1, \cdots, \phi_k) \in E \right\} ,$$

where $E = S(E_1(g)) \times \cdots \times S(E_m(g))$. If $(0,0) \notin C$, by the Hahn-Banach theorem, there is a symmetric 2-form h and a function f such that

$$\forall (\omega, \theta) \in C; \int_\Sigma (\omega, h)_g dA_g + \int_{\partial\Sigma} \theta f dL_g > 0 \, .$$

This means that for any $\Phi \in E$, $Q_h(\Phi) > 0$ and this contradicts the STEP 4. Therefore, $(0,0) \in C$ and one can define

$$\Phi = (\Phi_1, \cdots, \Phi_m) = (\phi_1^1, \cdots, \phi_1^{i_1}, \cdots, \phi_m^1, \cdots, \phi_m^{i_m})$$

such that $\phi_k^j \in E_k(g)$, $\int_\Sigma |\Phi_k|^2 \, dL_g = 1$ and

$$\sum_{k=1}^m \epsilon_k t_k \left(- \sum_{j=1}^{i_k} d\phi_k^j \otimes d\phi_k^j + \frac{|\nabla \Phi_k|_g^2}{2} g \right) = 0 \tag{2.9}$$

on Σ and

$$\sum_{k=1}^m \epsilon_k t_k \sigma_k (1 - |\Phi_k|^2) = 0 \tag{2.10}$$

on $\partial\Sigma$, where we denote by $|\Phi_k|^2 = \sum_{j=1}^{i_k} \left(\phi_k^j\right)^2$ and $|\nabla \Phi_k|_g^2 = \sum_{j=1}^{i_k} \left|\nabla \phi_k^j\right|_g^2$. The trace with respect to g gives that

$$\sum_{k=1}^m \sigma_k \epsilon_k t_k \, |\Phi_k|^2 = \sum_{k=1}^m \sigma_k \epsilon_k t_k \text{ on } \partial\Sigma \, .$$

We now choose $\widetilde{\Phi_k} = \sqrt{t_k} \Phi_k$ instead of Φ_k so that the new map $\widetilde{\Phi}$ lives in the ellipsoid

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \epsilon_1 \sigma_1 \, |x_1|^2 + \cdots + \epsilon_m \sigma_m \, |x_m|^2 = \sum_{k=1}^m \epsilon_k \sigma_k t_k \right\} \, ,$$

on $\partial\Sigma$. By the Steklov eigenvalue equations on the maps, we deduce that $\Delta_g \widetilde{\Phi} = 0$ on Σ and $\partial_\nu \widetilde{\Phi} \in (T_\Phi \mathcal{E})^\perp$ on $\partial\Sigma$, since the vector $(\sigma_1 \Phi_1, \cdots, \sigma_m \Phi_m)$ is normal to Φ with respect to the pseudo-Riemannian metric $G = \epsilon_1 \, |dx_1|^2 + \cdots + \epsilon_m \, |dx_m|^2$. This is the equation satisfied by the critical maps of the energy $\int_\Sigma G(\nabla \Phi, \nabla \Phi) dA_g$ for maps Φ satisfying the constraint $\Phi \in \mathcal{E}$.

We now get from (2.9) that

$$\sum_{k=1}^m \epsilon_k \left(- \sum_{j=1}^{i_k} d\widetilde{\phi_k^j} \otimes d\widetilde{\phi_k^j} + \frac{\left|\nabla \widetilde{\Phi_k}\right|_g^2}{2} g \right) = 0$$

or equivalently that

$$\widetilde{\Phi}^*(G) = \left(\sum_{k=1}^m \epsilon_k \left|\nabla \widetilde{\Phi_k}\right|_g^2 \right) \frac{g}{2} = G(\nabla \widetilde{\Phi}, \nabla \widetilde{\Phi}) \frac{g}{2}$$

and we obtain that the map $\tilde{\Phi}$ is conformal. This ends the proof of Theorem 2.1.

$\diamondsuit$

We now state the analogous theorem concerning the critical metric for combination of eigenvalues in the conformally constraint case, generalizing previous results in [FS13], inspired by previous results in [ESI03] and again, perfectly adaptable in the conformally constraint Steklov case.

Theorem 2.2. *Let (Σ, g) be a compact Riemannian surface with a smooth boundary and $F : \left(\mathbb{R}_+^\star\right)^m \to \mathbb{R}$ be a smooth function with $m \geq 1$. Let $\tilde{g} \in [g]$ be a critical metric for the functional*

$$\tilde{g} \mapsto F\left(L_{\tilde{g}}(\Sigma)\sigma_1(\tilde{g}), \cdots, L_{\tilde{g}}(\Sigma)\sigma_m(\tilde{g})\right)$$

defined on the conformal class $[g]$ of g and we assume that $L_{\tilde{g}}(\Sigma) = 1$. Then there are m non-negative integers $i_1, \cdots, i_m$ and there is a harmonic map with free boundary $\Phi : \Sigma \to \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$ into the space

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \epsilon_1\sigma_1 |x_1|^2 + \cdots + \epsilon_m\sigma_m |x_m|^2 = \sum_{k=1}^m \epsilon_k\sigma_k t_k \right\}$$

endowed with the pseudo-Euclidean metric on $\mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$

$$\epsilon_1 |dx_1|^2 + \cdots + \epsilon_m |dx_m|^2$$

where for $1 \leq k \leq m$, $\sigma_k = \sigma_k(g)$, $t_k = |\partial_k F(\sigma_1, \cdots, \sigma_m)|$ and ϵ_k is the sign of $\partial_k F(\sigma_1, \cdots, \sigma_m)$.

The proof of this result follows the lines of the proof of Theorem 2.1, but is even simpler since the variations we use stay in the conformal class of g. Of course, the conclusion has to be weaker than in Theorem 2.1 and we exatly loose that the obtained harmonic maps are conformal. More precisely, we can follow step 1, step 2 and step 3 with the symmetric 2-forms $h = fg$, where f is a smooth function to compute the left and right derivative at $t = 0$ for

$$t \mapsto F\left(A_{(1+tf)g}(\Sigma)\lambda_1((1+tf)g), \cdots, A_{(1+tf)g}(\Sigma)\lambda_m((1+tf)g)\right) .$$

Notice than the Dirichlet energy is conformally invariant. Then, denoting by

$$Q_f(\Phi) = \int_{\partial\Sigma} \left(\sum_{k=1}^m \epsilon_k t_k \lambda_k \left(1 - |\Phi_k|^2\right) \frac{g}{2} \right) f dL_g$$

for $\Phi \in E_1(g) \times \cdots \times E_m(g) \in \mathcal{C}^\infty(\Sigma)$, where $E_k(g)$ denotes the set of all eigenfunctions associated to the eigenvalue $\sigma_k(g)$, we prove that for any f, there is a map $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_f(\Phi) = 0$. Here again, $S(E_i(g))$ is the unit sphere in $E_i(g)$ endowed with the L^2 norm with respect to g. Again, by a standard Hahn-Banach argument we complete the proof. These arguments are performed again in the next section to compute the Euler Lagrange equation of a perturbated functional.

3. Harmonic replacement of Steklov eigenmaps

Harmonic replacement is a canonical way to replace a map by a smooth map, decreasing the energy locally, keeping same Dirichlet boundary conditions and a constraint target manifolds. However, in the positively curved target manifolds, such a map is not globally unique. As an analogue to the geodesic problem, one has to restrict such result in a domain with small energy. Thanks to the so-called "energy convexity", proved by Colding and Minicozzi [CM08], harmonic maps are local minimizers of the energy in a quantitative

setting. Notice that it is proved in [LP19] a free-boundary version of this energy-convexity result with a very simple proof, based on on two ingredients: using a ε-regularity result on harmonic maps and a classical Hardy inequality. This technique is well adapted and generalizable to a large class of conformally invariant variational problems.

In this section, we adapt such a result in our context when $N := \mathcal{E}$ is the ellipsoid of parameter $\sigma = diag\,(\sigma_1, \cdots, \sigma_d)$. In the following, we set $\Phi : \mathbb{D}_+ \to \mathbb{R}^d$ such that the coordinates satisfy a Steklov equation on the half-disk $\mathbb{D}_+ := \mathbb{D} \cap \mathbb{R}_+$:

$$\begin{cases} \Delta\Phi = 0 & \text{in } \mathbb{D}_+ \\ \partial_\nu \Phi = \sigma\Phi & \text{on } \mathbb{I}\,, \end{cases} \tag{3.1}$$

where $\mathbb{I} = [-1, 1] \times \{0\}$. We also denore $A = \partial\mathbb{D}_+ \setminus \mathbb{I}$. Notice that now the target manifold is no more an ellipsoid $\mathcal{E}$ on the boundary I. We aim at replacing such a map by a free boundary harmonic-like map into $\mathcal{E}$ up to a weight ω, satisfying $\omega \in L^\infty$ and $\frac{1}{\omega} \in L^\infty$. For this weight, let C_ω be a constant such that

$$\forall x \in \Sigma, \frac{1}{C_\omega} \leq \omega(x) \leq C_\omega\,.$$

We say that $u : \mathbb{D}_+ \to \mathbb{R}^d$ is ω^2-harmonic with free boundary into the half-disk if it is a critical point of the following energy

$$E_\omega(v) = \int_{\mathbb{D}_+} \omega^2 \left|\nabla v\right|^2 dz\,, \tag{3.2}$$

among all $v : \mathbb{D}_+ \to \mathbb{R}^d$ such that $v(\mathbb{I}) \subset \mathcal{E}$. In a $W^{1,2}$ setting, we say that $u \in W^{1,2}\left(\mathbb{D}_+, \mathbb{R}^d\right)$ is a ω^2-harmonic map with free boundary into the half-disk if

$$\int_{\mathbb{D}_+} \omega^2 \left\langle \nabla u, \nabla v \right\rangle dz = 0\,, \tag{3.3}$$

for any $v \in W^{1,2}\left(\mathbb{D}_+, \mathbb{R}^d\right)$ such that $v(x) \in T_{u(x)}\mathcal{E}$ a.e.

From $\Phi : \mathbb{D}_+ \to \mathbb{R}^d$, we define $\widetilde{\Phi} = \frac{\Phi}{\omega}$, where ω satisfies the following equation

$$\begin{cases} \Delta\omega = 0 & \text{in } \mathbb{D}_+ \\ \omega = \left|\Phi\right|_{\mathcal{E}} & \text{on } \mathbb{I}\,. \end{cases} \tag{3.4}$$

We also define

$$\sigma_\star\left(\mathbb{D}_+, e^{2u}dx\right) = \inf_{\varphi \in W^{1,2}_A(\mathbb{D}_+)} \frac{\int_{\mathbb{D}_+} \left|\nabla\varphi\right|^2 dz}{\int_{\mathbb{I}} \varphi^2 e^u dx}\,,$$

where the infimum holds for $W^{1,2}(\mathbb{D}_+)$ functions satisfying $\varphi = 0$ on $A = \mathbb{D}_+ \setminus \mathbb{I}$. Then we have the following result

Proposition 3.1. *Let $\delta > 1$ and $\sigma = diag\,(\sigma_1, \cdots, \sigma_d)$ such that $|\sigma| + \left|\sigma^{-1}\right| \leq \delta$. We assume for instance that $\sigma_1 \leq \cdots \leq \sigma_d$. Then, there is $\varepsilon_0 := \varepsilon(\delta)$ and $C := C(\delta)$ uniform in δ, such that, for any map $\Phi = (\phi_1, \cdots, \phi_d) \in W^{1,2}\left(\mathbb{D}_+, \mathbb{R}^d\right) \cap \mathcal{C}^0(\mathbb{D}_+)$, satisfying*

- *$C_\omega \leq \delta$, where ω satisfies (3.4),*
- *$\int_{\mathbb{D}_+} \left|\nabla\Phi\right|^2 \leq \varepsilon_0$*
- *Φ satisfies (3.1), ie $\Delta\Phi = 0$ and $\partial_\nu\Phi = \sigma e^u \Phi$*
- *$\sigma_\star\left(\mathbb{D}, e^{2u}dx\right) \geq \sigma_d$*

Let $\widetilde{\Psi}$ be the free boundary ω^2-harmonic replacement of $\widetilde{\Phi} := \frac{\Phi}{\omega}$ and we denote by $\Psi = \omega\widetilde{\Psi}$. Then

$$\frac{1}{2}\int_{\mathbb{D}_+} |\nabla(\Phi-\Psi)|^2\,dx \leq C\int_{\mathbb{D}_+} |\nabla\omega|^2\,dx\,. \tag{3.5}$$

Proof. Since $\Phi = \Psi$ on the arc boundary of $\mathbb{D}_+$, we start with the following formula

$$\int_{\mathbb{D}_+} |\nabla(\Phi-\Psi)|^2\,dx = \int_{\mathbb{D}_+} |\nabla\Phi|^2\,dx - \int_{\mathbb{D}_+} |\nabla\Psi|^2\,dx - 2\int_{\mathbb{D}_+} \langle\nabla(\Phi-\Psi),\nabla\Psi\rangle\,dx \tag{3.6}$$

and by integration by parts on the last term, we get

$$\int_{\mathbb{D}_+} |\nabla(\Phi-\Psi)|^2\,dx = \int_{\mathbb{D}_+} |\nabla\Phi|^2\,dx - \int_{\mathbb{D}_+} |\nabla\Psi|^2\,dx - 2\int_{\mathbb{I}} (\Phi-\Psi).\partial_\nu\Psi dx \tag{3.7}$$

We have that $\Delta\omega = 0$ and that $-\omega^{-1}div\left(\omega^2\nabla\widetilde{\Psi}\right) = 0$ because it is ω^2-harmonic. Ψ is then harmonic:

$$\Delta\Psi = -div\left(\nabla\left(\omega\widetilde{\Psi}\right)\right) = -div\left(\omega^{-1}\omega^2\nabla\widetilde{\Psi} + \widetilde{\Psi}\nabla\omega\right) = -\omega^{-1}div\left(\omega^2\nabla\widetilde{\Psi}\right) - (\Delta\omega)\,\widetilde{\Psi} = 0\,.$$

We also have the boundary equation $\partial_\nu\Psi = \partial_\nu\left(\omega\widetilde{\Psi}\right) = \omega\partial_\nu\widetilde{\Psi} + (\partial_\nu\omega)\,\widetilde{\Psi}$. Therefore

$$-2\int_{\mathbb{I}} \langle\Phi-\Psi,\partial_\nu\Psi\rangle = -2\int_{\mathbb{I}} \left\langle\Phi-\Psi,\omega\partial_\nu\widetilde{\Psi}\right\rangle - 2\int_{\mathbb{I}} (\Phi-\Psi).\widetilde{\Psi}\partial_\nu\omega := I_1 + I_2\,. \tag{3.8}$$

Then, the second right-hand term I_2 satisfies

$$\begin{aligned} |I_2| &= 2\left|\int_{\mathbb{D}_+} \left\langle\nabla(\Phi-\Psi),\widetilde{\Psi}\nabla\omega\right\rangle + \int_{\mathbb{D}_+} (\Phi-\Psi).\left\langle\nabla\widetilde{\Psi},\nabla\omega\right\rangle\right| \\ &\leq 2\left(\frac{1}{\sigma_1}\int_{\mathbb{D}_+} |\nabla(\Phi-\Psi)|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{D}_+} |\nabla\omega|^2\right)^{\frac{1}{2}} + \int_{\mathbb{D}_+} |\Phi-\Psi|^2\left|\nabla\widetilde{\Psi}\right|^2 + \int_{\mathbb{D}_+} |\nabla\omega|^2\,, \end{aligned}$$

where for the inequality, we used that

$$\sigma_1\left|\widetilde{\Psi}\right|^2 = \left(\min_{1\leq i\leq d}\sigma_i\right)\left|\widetilde{\Psi}\right|^2 \leq \left|\widetilde{\Psi}\right|_{\mathcal{E}}^2 = \left|\widetilde{\Phi}\right|_{\mathcal{E}}^2 \leq 1 \text{ in } \mathbb{D}_+\,.$$

To estimate the first right-hand term of (3.8), I_1, we use that $\widetilde{\Psi}$ is a critical point for the energy E_ω (see (3.2)) on the half disk under the boundary constraint $\widetilde{\Psi}\in\mathcal{E}$. This means that $-div\left(\omega^2\nabla\widetilde{\Psi}\right) = 0$ and that $\partial_\nu\widetilde{\Psi}\in\left(T_{\widetilde{\Psi}}\mathcal{E}\right)^\perp$. Then $\partial_\nu\widetilde{\Psi}$ is parallel to the outward normal $\tilde{n} = \frac{\sigma\widetilde{\Psi}}{|\sigma\widetilde{\Psi}|}$ of the ellipsoid and we have

$$I_1 := -2\int_{\mathbb{I}} (\Phi-\Psi)\,.\left(\omega\partial_\nu\widetilde{\Psi}\right) = -2\int_{\mathbb{I}} ((\Phi-\Psi).\tilde{n}\omega)\,\tilde{n}.\partial_\nu\widetilde{\Psi} = \int_{\mathbb{I}} |\Phi-\Psi|_{\mathcal{E}}^2\,\tilde{n}.\partial_\nu\widetilde{\Psi} \tag{3.9}$$

noticing again that $|\Psi|_{\mathcal{E}}^2 = |\Phi|_{\mathcal{E}}^2$ on I. Notice that up to reduce ε_0 (see proposition 5.1), we can assume that $\left|\widetilde{\Psi}\right|_{\mathcal{E}} \geq \frac{1}{2}$ in $\mathbb{D}_+$ (in particular $\widetilde{\Psi} \neq 0$) and we obtain:

$$\begin{aligned} I_1 &= \int_{\mathbb{I}} |\Phi - \Psi|_{\mathcal{E}}^2 \, \partial_\nu \left(\left|\sigma\widetilde{\Psi}\right|\right) \\ &= \int_{\mathbb{D}_+} \left\langle \nabla\left(\left|\widetilde{\Phi} - \widetilde{\Psi}\right|_{\mathcal{E}}^2\right), \omega^2 \nabla\left(\left|\sigma\widetilde{\Psi}\right|\right)\right\rangle + \int_{\mathbb{D}_+} \left(\left|\widetilde{\Phi} - \widetilde{\Psi}\right|_{\mathcal{E}}^2\right) . div\left(\omega^2 \nabla\left(\left|\sigma\widetilde{\Psi}\right|\right)\right) . \end{aligned}$$

Simply computing $\frac{1}{2} div\left(\omega^2 \nabla \left|\sigma\widetilde{\Psi}\right|^2\right)$ in two ways, we have that

$$-div\left(\omega^2 \nabla\left(\left|\sigma\widetilde{\Psi}\right|\right)\right) = \omega^2 \frac{\left|\nabla\left|\sigma\widetilde{\Psi}\right|\right|^2 - \left|\nabla\left(\sigma\widetilde{\Psi}\right)\right|^2}{\left|\sigma\widetilde{\Psi}\right|} = -\omega^2 \left|\sigma\widetilde{\Psi}\right| |\nabla\tilde{n}|^2$$

and the equation (3.9) becomes

$$\begin{aligned} I_1 &= 2\int_{\mathbb{D}_+} \left\langle \nabla(\Phi - \Psi) . \sigma(\Phi - \Psi) \nabla\left(\left|\sigma\widetilde{\Psi}\right|\right)\right\rangle \\ &\quad - \int_{\mathbb{D}_+} |\Phi - \Psi|_{\mathcal{E}}^2 \left\langle \frac{\nabla\omega^2}{\omega^2}, \nabla\left|\sigma\widetilde{\Psi}\right|\right\rangle + \int_{\mathbb{D}_+} |\Phi - \Psi|_{\mathcal{E}}^2 \, \omega^2 \left|\sigma\widetilde{\Psi}\right| |\nabla\tilde{n}|^2 . \end{aligned}$$

And we have that for any $0 < \eta < 1$

$$\begin{aligned} |I_1| \leq & \eta \int_{\mathbb{D}_+} |\nabla(\Phi - \Psi)|^2 + \frac{C}{\eta} \int_{\mathbb{D}_+} |\Phi - \Psi|^2 \left|\nabla\left|\sigma\widetilde{\Psi}\right|\right|^2 \\ & + C\int_{\mathbb{D}_+} |\Phi - \Psi|^2 \left(\left|\nabla\left|\sigma\widetilde{\Psi}\right|\right|^2 + \frac{|\nabla\omega|^2}{\omega^2} + \omega^2 \left|\sigma\widetilde{\Psi}\right| |\nabla\tilde{n}|^2\right) . \end{aligned}$$

Gathering all the previous computations, and setting

$$X^2 = \int_{\mathbb{D}_+} |\nabla(\Phi - \Psi)|^2 \, dx \text{ and } Y^2 = \int_{\mathbb{D}_+} |\nabla\omega|^2 ,$$

we can write (3.7) as

$$\begin{aligned} (1-\eta) X^2 \leq & \int_{\mathbb{D}_+} |\nabla\Phi|^2 \, dx - \int_{\mathbb{D}_+} |\nabla\Psi|^2 \, dx + \frac{C}{\eta} \int_{\mathbb{D}_+} |\Phi - \Psi|^2 \left|\nabla\widetilde{\Psi}\right|^2 \\ & + C\left(\int_{\mathbb{D}_+} |\nabla(\Phi - \Psi)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}_+} |\nabla\omega|^2\right)^{\frac{1}{2}} + C\int_{\mathbb{D}_+} |\nabla\omega|^2 . \end{aligned} \tag{3.10}$$

for a constant C depending on σ_d, $\frac{1}{\sigma_1}$ and C_ω. Thanks to $\varepsilon_0 > 0$ given by (5.11), we have

$$\int_{\mathbb{D}_+} |\Phi - \Psi|^2 \left|\nabla\widetilde{\Psi}\right|^2 \leq D\varepsilon_0 \int_{\mathbb{D}_+} |\nabla(\Phi - \Psi)|^2 \, dx$$

for a universal constant $D > 0$. Then, we have a constant $C > 0$ depending on σ_d, $\frac{1}{\sigma_1}$ and C_ω such that

$$\left(1 - \eta - \frac{CD\varepsilon_0}{\eta}\right) X^2 \leq \int_{\mathbb{D}_+} |\nabla\Phi|^2 \, dx - \int_{\mathbb{D}_+} |\nabla\Psi|^2 \, dx + 2CXY + CY^2 . \tag{3.11}$$

Now, let's prove that the difference of energies of Φ and Ψ is non positive. Let $1 \leq i \leq d$. We test $(\phi_i - \psi_i)$ in the variational characterization of $\sigma_\star\left(\mathbb{D}, e^{2u}\right)$, we have

$$\sigma_i \leq \sigma_\star\left(\mathbb{D}_+, e^{2u}dx\right) \leq \frac{\int_{\mathbb{D}_+} |\nabla(\phi_i - \psi_i)|^2 dx}{\int_{\mathbb{I}} (\phi_i - \psi_i)^2 e^u dl}$$

so that we get setting $\sigma_\star := \sigma_\star\left(\mathbb{D}_+, e^{2u}dx\right)$ and we sum on i

$$\sigma_\star \int_{\mathbb{I}} |\Phi - \Psi|^2 e^u dl \leq \int_{\mathbb{D}_+} |\nabla\Phi|^2 dx + \int_{\mathbb{D}_+} |\nabla\Phi|^2 dx - 2\int_{\mathbb{D}_+} \langle \nabla\Phi, \nabla\Psi\rangle dx \,. \tag{3.12}$$

Moreover, we integrate the σ_i-Steklov eigenfunction equation $\Delta\psi_i = 0$ and $\partial_\nu\psi_i = \sigma_i\psi_i e^u$ against $2(\psi_i - \phi_i)$ to get

$$2\int_{\mathbb{D}_+} |\nabla\phi_i|^2 dx - 2\int_{\mathbb{D}_+} \langle \nabla\phi_i, \nabla\psi_i\rangle dx = 2\sigma_i \int_{\mathbb{I}} (\phi_i)^2 e^u dl - 2\lambda_i \int_{\mathbb{I}} \phi_i\psi_i e^u dl \,.$$

We sum on i and we get that

$$2\int_{\mathbb{D}} |\nabla\Phi|^2 dx - 2\int_{\mathbb{D}} \langle \nabla\Phi, \nabla\Psi\rangle dx = 2\int_{\mathbb{I}} \langle \Phi, \Phi - \Psi\rangle_{\mathcal{E}} e^u dx \,. \tag{3.13}$$

Now suming on i and remembering that $|\Psi|^2_{\mathcal{E}} = |\Phi|^2_{\mathcal{E}}$ on I, we get have that

$$2\langle \Phi, \Phi - \Psi\rangle_{\mathcal{E}} = |\Phi - \Psi|^2_{\mathcal{E}} \text{ on } I \,.$$

We sum (3.12) and (3.13) to get

$$\int_{\mathbb{D}_+} |\nabla\Phi|^2 dx - \int_{\mathbb{D}_+} |\nabla\Psi|^2 dx \leq \int_{\mathbb{I}} \left(|\Phi - \Psi|^2_{\mathcal{E}} - \sigma_\star |\Phi - \Psi|^2\right) e^u dx \leq 0 \,, \tag{3.14}$$

because $\sigma_1 \leq \cdots \leq \sigma_d \leq \sigma_\star$.

Therefore, using (3.11), choosing first $\eta = \frac{1}{2}$ and then ε_0 such that $8CD\varepsilon_0 < 1$, we have that $s := \sqrt{1 - \eta - \frac{CD\varepsilon_0}{\eta}} \geq \frac{1}{2}$ and we get

$$\left(sX - \frac{CY}{s}\right)^2 \quad \leq \quad Y^2\left(C + \frac{C^2}{s}\right) \,, \tag{3.15}$$

so that we get the proposition up to increase C. ♢

Notice that if the target manifold on the boundary is a sphere, we already have that $\widetilde{\Phi} = \widetilde{\Psi}$ and that $\widetilde{\Phi} = \frac{\Phi}{\omega}$, where ω is the harmonic extension of $|\Phi|$ is already a free boundary $|\Phi|^2$-harmonic map. In the previous proposition we only find a local non-exact result. Notice then that is the target manifold is a sphere, we do not need this section to prove Claim 4.8 and Claim 4.13 for sequences of eigenfunctions given by Claim 4.1.

4. Existence of a maximal metric in the conformal constraint case

4.1. **Selection of a maximizing sequence.** Let (Σ, g) be a smooth Riemannian manifold without boundary. We denote respectively by $\mathcal{M}(\partial\Sigma)$ and $\mathcal{M}_1(\partial\Sigma)$ the set of Radon measures and probability measures on Σ, endowed with the topology of the weak-$\star$ convergence. We let $\epsilon > 0$. Let K_ϵ be the heat operator associated to g, so that for any positive Radon measure, $\nu \in \mathcal{M}(\partial\Sigma)$, $K_\epsilon[\nu]dL_g$ is the solution at time $\epsilon > 0$ of the heat

equation on $(\partial\Sigma, g)$ which converges to ν as $\epsilon \to 0$ for the weak-$\star$ convergence in $\mathcal{M}(\partial\Sigma)$. We set

$$\Lambda_\varepsilon = \inf_{\nu \in \mathcal{M}_1(\Sigma)} F\left(\lambda_1(K_\epsilon[\nu]g), \cdots, \lambda_m(K_\epsilon[\nu]g)\right) > -\infty \, . \tag{4.1}$$

Λ_ε is finite since F is decreasing with respect to all coordinates and because all the eigenvalues are bounded by the result by [KKP14]. We know that $\nu \mapsto K_\epsilon[\nu]$ is continuous from $\mathcal{M}(\partial\Sigma)$ to $\mathcal{C}^0(\partial\Sigma)$. Therefore, by continuity of the functional and compactness of $\mathcal{M}_1(\partial\Sigma)$, up to the extraction of a subsequence, a minimizing sequence for the variational problem (4.1) converges in $\mathcal{M}_1(M)$ to a measure $\nu_\epsilon \in \mathcal{M}_1(\partial\Sigma)$. We have

$$\Lambda_\varepsilon = F\left(\lambda_1(K_\epsilon[\nu_\epsilon]g), \cdots, \lambda_m(K_\epsilon[\nu_\epsilon]g)\right) \, . \tag{4.2}$$

We set $e^{u_\epsilon} = K_\epsilon[\nu_\epsilon]$. Then, the sequence of smooth positive functions e^{u_ϵ} defines a minimizing sequence for the variational problem $S_F(\Sigma, [g])$ as $\varepsilon \to 0$. Indeed, by definition $\Lambda_\epsilon \geq S_F(\Sigma, [g])$. Now, if $\tilde{g} = e^{2u}g \in [g]$ is such that $A_{\tilde{g}}(\Sigma) = 1$ and

$$F\left(\lambda_1(\tilde{g}), \cdots, \lambda_m(\tilde{g})\right) \leq S_F(\Sigma, [g]) + \frac{\eta}{2} \tag{4.3}$$

for some small $\eta > 0$. Then knowing that $K_\epsilon[e^u] \to e^u$ in $\mathcal{C}^0(\Sigma)$ as $\epsilon \to 0$, there is $\epsilon_0 > 0$ such that

$$\Lambda_\epsilon \leq F\left(\lambda_1(K_\epsilon[e^u]g), \cdots, \lambda_m(K_\epsilon[e^u]g)\right) \leq F\left(\lambda_1(\tilde{g}), \cdots, \lambda_m(\tilde{g})\right) + \frac{\eta}{2}$$

for any $\epsilon < \epsilon_0$. This, with (4.3), we get that $\Lambda_\epsilon \leq S_F(\Sigma, [g]) + \eta$. This means that $\Lambda_\epsilon \to S_F(\Sigma, [g])$ as $\varepsilon \to 0$ as required.

Now, for a given ϵ, $e^{u_\epsilon} = K_\epsilon[\nu_\epsilon]$ corresponds to the minimum of a variational problem (4.1). We let σ_k be the limit of $\sigma_k^\epsilon = \sigma_k(e^{2u_\epsilon}g)$ as $\varepsilon \to 0$.

We obtain by the following claim a system of equations corresponding to this critical point of a regularized functional depending on ϵ.

Claim 4.1. *Let $\epsilon > 0$. Then, there are non-negative integers $i_1^\epsilon, \cdots, i_m^\epsilon$ and a map $\Phi_\epsilon : \Sigma \to \mathbb{R}^{i_1^\epsilon} \times \cdots \times \mathbb{R}^{i_m^\epsilon}$ such that*

- *The family of coordinate functions $\left(\Phi_\epsilon^{k,j}\right)_{1 \leq k \leq m, 1 \leq j \leq i_k^\epsilon}$ is independent.*
- *$\forall k \in \{1, \cdots, m\}, \Delta_g \Phi_\epsilon^k = 0$ and $\partial_\nu \Phi_\epsilon^k = \sigma_\epsilon^k e^{u_\epsilon} \Phi_\epsilon^k$*
- *$\int_{\partial\Sigma} |\nabla \Phi_\varepsilon|_g^2 \, dA_g = \int_\Sigma e^{u_\epsilon} \, |\Phi_\epsilon|_{\mathcal{E}_\varepsilon}^2 \, dL_g = 1$*
- *$|\Phi_\epsilon|_{\mathcal{E}_\varepsilon}^2 \geq 1$ in $\partial\Sigma$ and $|\Phi_\epsilon|_{\mathcal{E}_\varepsilon}^2 = 1$ on $supp(\nu_\varepsilon)$*

where $\Phi_\epsilon^k : \Sigma \to \mathbb{R}^{i_k^\epsilon}$ is a coordinate map of Φ_ϵ and $\Phi_\epsilon^{k,j} : \Sigma \to \mathbb{R}$ is a coordinate function of Φ_ϵ^k, $|\Phi_\epsilon|_{\mathcal{E}_\varepsilon}^2 = \sum_{k=1}^m \sigma_\epsilon^k \left|\Phi_\epsilon^k\right|^2$ is the norm of Φ_ϵ with respect to the quadratic form associated to the ellipsoid $\mathcal{E}_\varepsilon$ and $\left|\Phi_\epsilon^k\right|^2$ is the Euclidean norm of Φ_ϵ^k in $\mathbb{R}^{i_k^\epsilon}$ and $\sigma_\epsilon^k = \sigma_k(e^{2u_\epsilon}g)$.

Proof. Notice that the proof is written in the same spirit as the proof of Theorem 2.1 or the proof of proposition 1 in [Pet19]. Since $\epsilon > 0$ is set, we omit the ϵ index of σ_ϵ^k, Φ_ϵ, i_k^ϵ, ν_ϵ, e^{u_ϵ} and $\mathcal{E}_\varepsilon$.

Let $\mu \in \mathcal{M}(\partial\Sigma)$ be a positive radon measure and $t \geq 0$. We now denote by $\sigma_t^k = \sigma_k(e^{2u_t}g)$ where $e^{u_t} = K_\epsilon[\nu + t\mu]$. Let $\phi_t \in S(E_k(e^{2u_t}g))$ (ie ϕ_t is an eigenfunction associated to σ_t^k such that $\int_{\partial\Sigma} \phi_t^2 e^{u_t} dL_g = 1$). As we did in the proof of Theorem 1.2, we can easily

prove that $\sigma_t^k \to \sigma_k := \sigma_k(e^{2u}g)$ as $t \to 0^+$ and ϕ_t converges to $\phi \in S(E_k(e^{2u}g))$ (ie ϕ is an eigenfunction associated to σ_k such that $\int_{\partial\Sigma} \phi^2 e^u dL_g = 1$ in $\mathcal{C}^2(\Sigma)$).

Now we focus on the equation satisfied by $R_t = \phi_t - \pi_k(\phi_t)$ where π_k is the orthonormal projection in $L^2(\partial\Sigma, e^u)$ on $E_k(e^{2u}g)$, extended by a harmonic function on Σ.

$$\begin{cases} \Delta_g \left(\frac{R_t}{\alpha_t}\right) = 0 \\ \partial_\nu \left(\frac{R_t}{\alpha_t}\right) - \sigma_k e^u \frac{R_t}{\alpha_t} = \frac{\sigma_t^k - \sigma_k}{\alpha_t} e^u \phi_t + \frac{t}{\alpha_t} \sigma_t^k K_\epsilon[\mu]\phi_t \end{cases} \tag{4.4}$$

where

$$\alpha_t = \|R_t\|_\infty + t + \left|\sigma_t^k - \sigma_k\right| . \tag{4.5}$$

Up to the extraction of a subsequence, we have

$$\frac{t}{\alpha_t} \to t_0 \text{ and } \frac{\sigma_t^k - \sigma_k}{\alpha_t} \to \delta_0 \text{ and } \frac{R_t}{\alpha_t} \to R_0 \text{ in } \mathcal{C}^2(\Sigma)$$

as $t \to 0^+$. The last limit follows from the standard elliptic theory applied to the equation (4.4). We pass to the limit in this equation (4.4) and 4.5 and get

$$\begin{cases} \Delta_g R_0 = 0 \\ \partial_\nu R_0 - \sigma_k e^u R_0 = \delta_0 e^u \phi + t_0 \sigma_k K_\epsilon[\mu]\phi , \end{cases} \tag{4.6}$$

and

$$1 = \|R_0\|_\infty + t_0 + |\delta_0| . \tag{4.7}$$

We integrate (4.6) against ϕ and we get that

$$\delta_0 + t_0 \sigma_k \int_{\partial\Sigma} K_\epsilon[\mu]\phi^2 dL_g = 0 .$$

If $t_0 = 0$, then, $\delta_0 = 0$ and (4.6) becomes $\Delta_g R_0 = 0$ and $\partial_\nu R_0 - \sigma_k e^u R_0 = 0$. As a limit of functions in $E_k(e^{2u}g)^\perp$, the orthononormal space to $E_k(e^{2u}g)$ in $L^2(\partial\Sigma)$, we have that $R_0 \in E_k(e^{2u}g) \cap E_k(e^{2u}g)^\perp$ on $\partial\Sigma$ so that $R_0 = 0$, contradicting (4.7). Therefore, $t_0 \neq 0$ and

$$\frac{\sigma_t^k - \sigma_t}{t} \to \frac{\delta_0}{t_0} = -\sigma_k \int_{\partial\Sigma} K_\epsilon[\mu]\phi^2 dL_g \text{ as } t \to 0^+ . \tag{4.8}$$

Since $F\left(\left(1 + t\int_{\partial\Sigma} d\mu\right)\sigma_t^1, \cdots, \left(1 + t\int_{\partial\Sigma} d\mu\right)\sigma_t^m\right) \geq \Lambda_\epsilon$ for any $t > 0$, we deduce from (4.8) that

$$\begin{aligned} \forall \mu \in \mathcal{M}(\partial\Sigma), \exists (\phi_1, \cdots, \phi_m) \in & S\left(E_1(e^{2u}g)\right) \times \cdots \times S\left(E_m(e^{2u}g)\right), \\ & \int_{\partial\Sigma} \left(\sum_{k=1}^m \sigma_k t_k \left(1 - K_\epsilon[\phi_i^2]\right)\right) d\mu \leq 0 \end{aligned} \tag{4.9}$$

where $t_k = \partial_k F(\lambda_1, \cdots, \lambda_m)$. We define the following subset of $\mathcal{C}^0(\partial\Sigma)$

$$\begin{aligned} K = \{\Psi \in \mathcal{C}^0(\partial\Sigma); \exists (\Phi_1, \cdots, \Phi_m) \in & E_1(e^{2u}g)^{i_1} \times \cdots \times E_m(e^{2u}g)^{i_m} \\ & \text{s.t } \Psi = \sum_{k=1}^m \sigma_k t_k \left(K_\epsilon[|\Phi_k|^2] - 1\right) \text{ and } \int_{\partial\Sigma} |\Phi_k|^2 e^u = 1\} \end{aligned} \tag{4.10}$$

and

$$F = \{f \in \mathcal{C}^0(\partial\Sigma); f \geq 0\} .$$

The set F is closed and convex. The set K is convex as the convex hull of

$$\left\{\sum_{k=1}^{m}\sigma_k t_k\left(K_\epsilon[\phi_k^2]-1\right);(\phi_1,\cdots,\phi_m)\in S(E_1(e^{2u}g))^{i_1}\times\cdots\times S(E_m(e^{2u}g))^{i_m}\right\}.$$

Since $E_1(e^{2u}g)\times\cdots\times E_m(e^{2u}g)$ is finite dimensional, the vector space spanned by this set is finite dimensional and is a compact set since it is bounded. Therefore K is also a compact set. If $F\cap K=\emptyset$, by the Hahn-Banach theorem, there is $\mu\in\mathcal{M}(\partial\Sigma)$ such that

$$\forall f\in F,\int_M f d\mu\geq 0 \text{ and } \forall\Psi\in K,\int_M\Psi d\mu<0\,. \tag{4.11}$$

We deduce that $\mu\neq 0$, that μ is positive but (4.11) contradicts (4.9). We proved that $F\cap K\neq\emptyset$. It gives $(\Phi_1,\cdots,\Phi_m)\in E_1(e^{2u}g)^{i_1}\times\cdots\times E_m(e^{2u}g)^{i_m}$ such that

$$\forall 1\leq k\leq m,\int_{\partial\Sigma}|\Phi_k|^2e^u=1 \text{ and } K_\epsilon\left[\sum_{k=1}^{m}\sigma_k t_k|\Phi_k|^2\right]\geq\sum_{k=1}^{m}\sigma_k t_k\,.$$

By Gaussian decomposition of some non-negative quadratic forms defined, we can assume that $(\Phi_{1,1}\cdots,\Phi_{1,i_1},\cdots,\Phi_{m,1},\cdots,\Phi_{m,i_m})$ is a family of independent eigenfunctions in $L^2(\partial\Sigma,g)$ and satisfies up to a renormalization

$$\int_\Sigma\left(\sum_{k=1}^{m}\sigma_k|\Phi_k|^2\right)e^u dL_g=1 \text{ and } K_\epsilon\left[\sum_{k=1}^{m}\sigma_k|\Phi_k|^2\right]\geq 1\,. \tag{4.12}$$

We can write that

$$1=\int_{\partial\Sigma}|\Phi|^2_{\mathcal{E}}e^u dL_g=\int_{\partial\Sigma}K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]d\nu\geq\int_{\partial\Sigma}d\nu=1$$

Therefore, $K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]=1$ ν-a.e and since $K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]$ is continuous, $K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]=1$ on $supp(\nu)$. This ends the proof of Claim 4.12.

$\diamondsuit$

We know by [KKP14] that the multiplicity of σ_ϵ^k is bounded by k and the topology of Σ. Therefore since the family of coordinate functions $\left(\Phi_\epsilon^{k,j}\right)_{1\leq k\leq m,1\leq j\leq i_k^{\epsilon,\alpha}}$ is independent the sum $i_1^\epsilon+\cdots+i_m^\epsilon$, is bounded with respect to ϵ and up to the extraction of a subsequence, we assume that the indices i_k^ϵ do not depend on ϵ.

4.2. **Notations for local analysis and rescalings.** Let (Σ,g) be a smooth Riemannian surface with $L_g(\partial\Sigma)=1$.

We recall that $\mathcal{M}(\partial\Sigma)$ is the set of positive Radon measures provided with the weak* topology and $\mathcal{M}_1(\partial\Sigma)$ the subset of probability measures on $\partial\Sigma$.

For an open set $\Omega\subset\Sigma$ we denote by $\sigma_\star(\Omega,g)$ the solution of the following minimization problem

$$\sigma_\star(\Omega,g)=\inf_{\phi\in W_0^{1,2}(\Omega)}\frac{\int_\Omega|\nabla\varphi|_g^2dA_g}{\int_{\partial\Omega\cap\partial\Sigma}\varphi^2dL_g},$$

where $W_0^{1,2}(\Omega)$ denotes the set of $W^{1,2}(\Omega)$ functions φ such that $\varphi=0$ on $\partial\Omega\setminus\partial\Sigma$.

For all the paper, we fix $\delta_0>0$, a constant $C_0>1$ and a family $(x_l)_{l=1,\cdots,L}$ of points in Σ and smooth functions $v_l:\Sigma\mapsto\mathbb{R}$ such that

- for any $l \in \{1, \dots, L\}$, the metric $g_l = e^{-2v_l} g$ is a flat metric in the ball $B_{g_l}(x_l, 2\delta) = \Omega_l$.
- $\Sigma = \bigcup_{l=1}^{L} \omega_l$ where $\omega_l = B_{g_l}(x_l, \delta)$.
- For any $1 \leq l \leq L$, $C_0^{-2} \leq e^{2v_l} \leq C_0^2$.
- For any $x \in \omega_l$ and $0 < r < \delta$, $B_g(x, C_0^{-1} r) \subset B_{g_i}(x, r) \subset B_g(x, C_0 r)$

For $1 \leq l \leq L$ and a point $z \in \mathbb{D}_{2\delta}(0)$, we let

$$e^{\tilde{v}_l(z)} = e^{v_l(\exp_{g_l, x_l}(z))} \text{ and } \bar{z}^l = \exp_{g_l, x_l}(z)$$

and for $x \in \Omega_l$ and a set $\Omega \subset \Omega_l$,

$$\tilde{x}^l = \exp^{-1}_{g_l, x_l}(x) \text{ and } \widetilde{\Omega}^l = \exp^{-1}_{g_l, x_l}(\Omega) \, .$$

For a smooth density e^u with $e^{2u} g \in [g]$, we let

$$e^{\tilde{u}^l(z)} = e^{\tilde{v}_l(z)} e^{2u(\exp_{g_l, x_l}(z))}$$

so that for $\Omega \subset \Omega_l$, identifying $\mathbb{R} \times \{0\}$ with $\mathbb{R}$,

$$\int_{\partial\Omega \cap \partial\Sigma} e^u dL_g = \int_{\widetilde{\Omega}^l \cap \mathbb{R}} e^{\tilde{u}^l} dz \, .$$

For other functions $\phi \in L^1(M)$ or measures $\nu \in \mathcal{M}(\Sigma)$, we let

$$\tilde{\phi}^l(z) = \phi(\exp_{g_l, x_l}(z)) \text{ and } \tilde{\nu}^l = \exp^{\star}_{g_l, x_l}(\nu) \, .$$

Let $p_\varepsilon(x, y)$ be the heat kernel of (M, g) at time $\varepsilon > 0$. Then, for $y, z \in \Omega_l$, we let

$$\tilde{p}^l_\varepsilon(z, y) = e^{\tilde{v}_l(z)} p_\varepsilon(\exp_{g_l, x_l}(z), \exp_{g_l, x_l}(y))$$

so that for a density $e^{u(x)} = \int_\Omega p_\varepsilon(x, y) d\nu(y)$ for $\Omega \subset \Omega_l$ and some measure ν, we have

$$e^{\tilde{u}^l(z)} = \int_{\widetilde{\Omega}^l \cap \mathbb{R}} \tilde{p}^l_\varepsilon(z, y) d\tilde{\nu}(y) \text{ and } \int_{\widetilde{\Omega}^l \cap \mathbb{R}} \tilde{\phi}^l(z) \tilde{p}^l_\varepsilon(z, \tilde{y}^l) dz = \int_{\Omega \cap \partial\Sigma} \phi(x) p_\varepsilon(x, y) dL_g(x) \, .$$

and for $\phi \in L^1(\partial\Sigma)$. When the context is clear, we drop the exponent l in all the notations.

One fundamental remark for all the following analysis is some scale invariance properties for the heat kernel $p_\varepsilon(x, y)$ and $\sigma_\star\left(\Omega, e^{2u} g\right)$. We shall give convenient notations to handle this scale invariance. For parameters $a \in \mathbb{R}^2_+$ and $\alpha > 0$, we denote the following rescaled objects by

$$\hat{x} = \frac{\tilde{x} - a}{\alpha} \text{ and } \hat{\Omega} = \frac{\widetilde{\Omega} - a}{\alpha} \, ,$$

$$e^{\hat{u}(z)} = \alpha e^{\tilde{u}(\alpha z + a)}, \hat{\phi}(z) = \tilde{\phi}(\alpha z + a) \text{ and } \hat{\nu} = H^{\star}_{a, \alpha}(\tilde{\nu}) \, ,$$

$$\hat{p}_\varepsilon(z, y) = \alpha \tilde{p}^l_\varepsilon(\alpha z + a, \alpha y + a) \, ,$$

where $H_{a,\alpha}(x) = \alpha x + a$, so that if $e^{u(x)} = \int_{\partial\Omega \cap \partial\Sigma} p_\varepsilon(x, y) d\nu(y)$, we have

$$e^{\hat{u}(z)} = \int_{\hat{\Omega} \cap \mathbb{R}} \hat{p}_\varepsilon(z, y) d\hat{\nu}(y) \text{ and } \int_{\hat{\Omega} \cap \mathbb{R}} \phi(z) \hat{p}_\varepsilon(z, \hat{y}) dz = \int_\Omega \phi(x) p_\varepsilon(x, y) dL_g(y) \, .$$

We also let for $z \in \mathbb{R}^2$

$$\breve{z} = \exp_{g_l, x_l}(\alpha z + a) \text{ and } \breve{\Omega} = \exp_{g_l, x_l}(\alpha \Omega + a) \, ,$$

so that $\hat{\check{z}} = z$. Then we also have that

$$\sigma_\star\left(\Omega, e^{2u}g\right) = \sigma_\star\left(\hat{\Omega}, e^{2\hat{u}}dx\right) .$$

4.3. **A bubble tree.** Up to the extraction of a subsequence, we denote by ν the weak-$\star$ limit of $\{e^{u_\epsilon}dL_g\}_{\epsilon>0}$ as $\epsilon \to 0$. Then ν is also the weak-$\star$ limit of $\{\nu_\epsilon\}_{\epsilon>0}$ as $\epsilon \to 0$. Indeed, let $\zeta \in \mathcal{C}^0(\partial\Sigma)$,

$$\begin{aligned}\left|\int_{\partial\Sigma}\zeta\left(d\nu_\epsilon - e^{u_\epsilon}dL_g\right)\right| &= \left|\int_{\partial\Sigma}\zeta\left(d\nu_\epsilon - K_\epsilon[\nu_\epsilon]dL_g\right)\right| \\ &= \left|\int_{\partial\Sigma}\left(\zeta - K_\epsilon[\zeta]\right)d\nu_\epsilon\right| \\ &\leq \left\|\zeta - K_\epsilon[\zeta]\right\|_\infty \to 0 \text{ as } \epsilon \to 0 .\end{aligned}$$

One can perform a bubble-tree to capture the scales of concentration of the sequence of measure $e^{u_\varepsilon}dL_g$ on the boundary. We refer to the work in [Pet19]:

Claim 4.2. *There are points $a_1^\varepsilon, \cdots, a_N^\varepsilon \in \mathbb{R}\times\{0\}$ and scales $0 < \alpha_N^\varepsilon < \alpha_{N-1}^\varepsilon < \cdots < \alpha_1^\varepsilon$ satisfying for $1 \leq i \leq N$, $\alpha_i^\varepsilon \to 0$ as $\varepsilon \to 0$ and there are points $q_1, \cdots, q_{c_i}$ such that $e^{\hat{u}_i^\varepsilon}dz \to \hat{\nu}_i$ with respect to the weak-star convergence in $\mathcal{M}(I \setminus \{q_1, \cdots, q_{c_i}\})$, where $\hat{\nu}_i$ is a measure without atom on $\mathbb{R}\times\{0\}$ and*

$$\int_{\partial\Sigma} d\nu + \sum_{i=1}^N \int_{\mathbb{R}\times\{0\}} d\hat{\nu}_i = 1 . \tag{4.13}$$

Setting $F_i = \left\{ j > i; \frac{d_g(\bar{a}_i^\varepsilon, \bar{a}_j^\varepsilon)}{\alpha_i^\varepsilon} \text{ is bounded } \right\}$, we also have for $j \neq i$ that

$$j \in F_i \Rightarrow \frac{\alpha_j^\varepsilon}{\alpha_i^\varepsilon} \to_{\varepsilon\to 0} 0 \text{ and } j \notin F_i \Rightarrow \frac{d_g(\bar{a}_i^\varepsilon, \bar{a}_j^\varepsilon)}{\alpha_i^\varepsilon} \to_{\varepsilon\to 0} +\infty . \tag{4.14}$$

The main technical work in the paper is to prove that the measures ν or $\hat{\nu}_i$ for $1 \leq i \leq N$ are regular (i.e absolutely continuous with respect to dL_g or the Lebesgue measure with a smooth density). Once we proved it, we refer to section 6.2 in [Pet19].

From now on, we prove that ν and $\hat{\nu}_i$ are regular measures. We assume that $\int_{\mathbb{R}\times\{0\}} d\hat{\nu}_i > 0$ for any i (one may have $N = 0$, but as already said, we prove it a posteriori). We will also have estimates on the surface's scaling as soon as $\int_{\partial\Sigma} d\nu > 0$.

4.4. **Regularity estimates at the scale $\sqrt{\epsilon}$.** The proof of the regularity of a limiting measure $\hat{\nu} := \hat{\nu}_i$ is immediate if we assume that the sequence $\{\frac{\alpha_\epsilon}{\sqrt{\epsilon}}\}$ is bounded ($\alpha_\epsilon := \alpha_\varepsilon^i$ is the scale defined in in claim 4.2). The very suitable scale $\sqrt{\epsilon}$ arises naturally from the choice of the heat kernel. Indeed, in this case, we let $\theta_0 = \lim_{\varepsilon\to 0}\frac{\epsilon}{e^{2\tilde{v}(x_0)}\alpha_\epsilon^2}$ and we denote by $\hat{\nu}$ the weak* limit of $\hat{\nu}_\epsilon$ in $\mathcal{M}(\mathbb{R}^2)$. Let $R_0 > 0$ and $z \in \mathbb{D}_{R_0}$. We have by (5.13) that

$$e^{\hat{u}_\varepsilon}(z) = e^{v_l(\check{x})}\alpha_\varepsilon \int_{\partial\Sigma} p_\varepsilon(\check{z}, y)d\nu_\varepsilon(y) \leq \frac{A_0 e^{v_l(\check{z})}\alpha_\varepsilon}{\sqrt{4\pi\varepsilon}}\int_{\partial\Sigma} d\nu_\varepsilon \leq \frac{A_0}{\sqrt{4\pi\theta_0}}(1 + o(1)) .$$

Since $\int_{\mathbb{I}_R} e^{\hat{u}_\varepsilon}(z)dl \geq 1+\alpha_{\epsilon,R}$, where $\lim_{R\to+\infty}\lim_{\epsilon\to 0}\alpha_{\epsilon,R}=0$, we get that $\theta_0<+\infty$. Now, we let $e^{\hat{u}}$ be a smooth function on $\mathbb{R}^2$ defined by

$$e^{\hat{u}(z)}=\int_{\mathbb{R}^2}\frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}}d\hat{\nu}(y)\ . \tag{4.15}$$

Let $R_0>0$, $R>R_0$ and $z\in\mathbb{D}_{R_0}$. We have that

$$\begin{aligned}\left|e^{\hat{u}_\varepsilon(z)}-e^{\hat{u}(z)}\right| &= \left|\int_{\partial\Sigma}\alpha_\varepsilon p_\varepsilon(\breve{z},y)d\nu_\varepsilon(y)-e^{\hat{u}(z)}\right| \\ &\leq \int_{\partial\Sigma\setminus\breve{\mathbb{I}}_R}\alpha_\varepsilon p_\varepsilon(\breve{z},y)d\nu_\varepsilon(y)+\left|\int_{\mathbb{I}_R}\hat{p}_\varepsilon(z,y)d\hat{\nu}_\varepsilon(y)-\int_{\mathbb{R}}\frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}}d\hat{\nu}(y)\right|\end{aligned}$$

so that

$$\begin{aligned}\left|e^{\hat{u}_\varepsilon(z)}-e^{\hat{u}(z)}\right| &\leq \frac{A_0}{\sqrt{4\pi\theta_0}}(1+o(1))e^{-\frac{(R-R_0)^2}{8\theta_0}}+\left|\int_{\mathbb{D}_R}\left(\hat{p}_\varepsilon(z,y)-\frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}}\right)d\hat{\nu}_\varepsilon\right| \\ &\quad+\left|\int_{\mathbb{I}_R}\frac{e^{-\frac{|z-y|^2}{8\theta_0}}}{\sqrt{4\pi\theta_0}}(d\hat{\nu}_\varepsilon-d\hat{\nu})\right|+\int_{\mathbb{R}\setminus\mathbb{I}_R}\frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}}d\hat{\nu}+o(1) \\ &\to \frac{A_0}{\sqrt{4\pi\theta_0}}e^{-\frac{(R-R_0)^2}{8\theta_0}}+\int_{\mathbb{R}\setminus\mathbb{I}_R}\frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}}d\hat{\nu}\ \text{as}\ \varepsilon\to 0\ .\end{aligned}$$

Letting $R\to+\infty$, we get for any $R_0>0$ that

$$e^{\hat{u}_\varepsilon}\to e^{\hat{u}}\ \text{in}\ \mathcal{C}^0(\mathbb{D}_{R_0})\ \text{as}\ \varepsilon\to 0\ . \tag{4.16}$$

Therefore, the limit $e^{\hat{u}}dz$ of the sequence of measures $\{e^{\hat{u}_\epsilon}dz\}$ is absolutely continuous with respect to the Lebesgue measure, with a smooth density and it is a probability measure.

Up to the end of the proof, we assume that $\frac{\alpha_\epsilon}{\sqrt{\epsilon}}\to+\infty$ for the remaining measures $\hat{\nu}_i$.

We cannot expect to get good estimates on the potential e^{u_ϵ} since $\{\alpha_\epsilon e^{u_\epsilon}\}$ is not uniformly bounded anymore. However, the eigenfunctions which are coordinates of Φ_ϵ, given by Claim 4.1 satisfy lots of conditions. The purpose is now to prove that thanks to finer and finer estimates, they converge in suitable function spaces up to the extraction of a subsequence. We still have an immediate very partial result arising from a look at the scale $\sqrt{\epsilon}$. This weaker but fundamental result is the consequence of Claim 4.1. It states that at the scale $\sqrt{\epsilon}$, the sequence of eigenfunctions is bounded at the neighborhood of the support of ν_ϵ. As already said, this scale comes naturally from the choice of the heat kernel.

Claim 4.3. *For any $R>0$, there is a constant $C_R>0$ such that for any sequence (z_ϵ) of points of $\partial\Sigma$ with $d_g(z_\epsilon,supp(\nu_\epsilon))\leq R\sqrt{\epsilon}$, we have*

$$\left|\Phi^k_\epsilon(z_\epsilon)\right|\leq C_R\ \textit{for all}\ \epsilon>0\ \textit{and}\ k\in\{1,\cdots,m\}$$

Proof. We refer the reader to Section 4.2 for the notations used during this proof. We can assume that $x_\varepsilon \in \omega_l$ for $1 \leq l \leq L$ fixed and we set

$$\hat{\Phi}_\varepsilon(x) = \tilde{\Phi}_\varepsilon\left(\sqrt{\varepsilon}x + \tilde{x}_\varepsilon\right)$$

for $x \in \mathbb{D}_{\frac{\delta}{\sqrt{\varepsilon}}}$. Then

$$\Delta\hat{\Phi}^k_\varepsilon = 0 \text{ in } \mathbb{D}_{\delta\sqrt{\varepsilon}}$$

$$\partial_\nu\hat{\Phi}^k_\varepsilon = \sqrt{\varepsilon}\sigma^\varepsilon_k e^{\tilde{u}_\varepsilon(\sqrt{\varepsilon}x+\tilde{x}_\varepsilon)}\hat{\Phi}^k_\varepsilon \text{ on } \mathbb{I}_{\delta\sqrt{\varepsilon}}$$

for $1 \leq k \leq m$. By estimate (5.12) of Section 5.3, $(\sqrt{\epsilon}p_\epsilon)$ is uniformly bounded so that $\left(\sqrt{\varepsilon}e^{\tilde{u}_\varepsilon(\sqrt{\varepsilon}x+\tilde{x}_\varepsilon)}\right)$ is uniformly bounded.

Now, we let $y_\varepsilon \in supp(\nu_\varepsilon)$ be such that $d_g(x_\varepsilon, y_\varepsilon) \leq R\sqrt{\varepsilon}$. By Claim 4.1, we have that $K_\epsilon\left[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}\right](y_\epsilon) = 1$. Let us write then with (5.12), in Section 5.3, that for $\rho > 0$,

$$\begin{aligned}
1 = K_\epsilon\left[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}\right](y_\epsilon) &\geq \sum_{k=1}^m \sigma_k K_\epsilon\left[\left|\Phi^k_\epsilon\right|^2\right](y_\epsilon) \\
&= \sum_{k=1}^m \sigma_k \int_{\partial\Sigma} p_\varepsilon(y, y_\varepsilon)\left|\Phi^k_\varepsilon(y)\right|^2 dL_g(y) \\
&\geq \sum_{k=1}^m \sigma_k \frac{1}{\sqrt{4\pi\varepsilon}A_0} e^{-\rho^2C_0^2}\int_{B_g(y_\varepsilon, 2\rho C_0\sqrt{\varepsilon})\cap\partial\Sigma}\left|\Phi^k_\varepsilon(y)\right|^2 dL_g(y) \\
&\geq \sum_{k=1}^m \sigma_k \frac{1}{\sqrt{4\pi}A_0C_0^2} e^{-\rho^2C_0^2}\int_{\mathbb{D}^+_{2\rho}(\hat{z}_\varepsilon)}\left|\hat{\Phi}^k_\varepsilon(z)\right|^2 dl\ .
\end{aligned}$$

We set $\hat{z}_\varepsilon = \frac{1}{\sqrt{\varepsilon}}(\tilde{y}_\varepsilon - \tilde{x}_\varepsilon)$ so that, up to a subsequence, $\hat{z}_\varepsilon \to z_0$ as $\varepsilon \to 0$ and we deduce from the previous inequality that, for any $\rho > 0$, $\{\hat{\Phi}^k_\varepsilon\}$ is bounded in $L^2(\mathbb{I}_\rho(z_0))$. By elliptic regularity of the Dirichlet-to-Neumann operator (see Taylor [Tay11], Chapter 7.11, page 37), we get that $\{\hat{\phi}^i_\varepsilon\}$ is uniformly bounded in I_ρ by some constant D_ρ. Setting $C_R = D_{2C_0R}$ gives the claim.

♢

4.5. **Singularity points.** The eigenfunctions given by Claim 4.1 has uniformly bounded Dirichlet energy, but we cannot a priori say anything for instance about the sequence of their $L^2(\partial\Sigma, g)$-norm on the boundary, except in the scale $\sqrt{\varepsilon}$, as noticed in Claim 4.3. We need two ingredients to perform asymptotic analysis on eigenfunctions given by Claim 4.1:

- L^2-estimates on the sequence of eigenfunctions at every intermediate scale $\alpha_\varepsilon > \sqrt{\varepsilon}$. By conformal invariance of the L^2-norm of the gradient, this is given by Poincaré inequalities. They hold on functions vanishing on curves connecting two points whose distance is uniformly lower bounded by a positive constant.
- $W^{1,2}$ approximations by a ω^2-harmonic replacement. As noticed in Proposition 3.1 this is possible if there is a local smallness assumption on $\frac{1}{\sigma^\star(\mathbb{D}_r, e^{2u_\varepsilon}g)}$.

We define points such that at their neighbourhood, we cannot have the previous properties. Fortunately, as noticed in [Pet19], there is just a final number of singularity points. In the following Claim, we give their precise scale of appearance.

Claim 4.4. *Up to the extraction of a subsequence of $\{e^{u_\varepsilon}\}_{\varepsilon>0}$ there are at most s sequences of points $p_1^\varepsilon, \cdots, p_s^\varepsilon \in \partial\Sigma$ with $0 \leq s \leq m$ such that $p_i^\varepsilon \to p_i$ as $\varepsilon \to 0$ and positive scales $r_1^\varepsilon \leq \cdots \leq r_s^\varepsilon$ such that for any $1 \leq i \leq s$, setting A_i as*

$$A_i = \left\{ r > 0, \overline{\mathbb{D}_p^+(r)} \subset \Sigma \setminus \left(\bigcup_{j=1}^{i} \overline{\mathbb{D}_{p_j^\varepsilon}^+\left(r_j^\varepsilon\right)} \right); \sigma_\star\left(\mathbb{D}_p^+(r), e^{2\tilde{u}_\varepsilon} dx\right) \leq \sigma_m(e^{u_\varepsilon}) \right\}$$

$$\lim_{\varepsilon\to 0} (\inf A_s) > 0 \,,$$

$$r_i^\varepsilon := \min A_i \to 0 \text{ as } \varepsilon \to 0 \,,$$

$$\overline{\mathbb{D}_{p_i^\varepsilon}^+(r)} \subset \Sigma \setminus \left(\bigcup_{j=1}^{i-1} \overline{\mathbb{D}_{p_j^\varepsilon}^+\left(r_j^\varepsilon\right)} \right) \,,$$

$$\sigma_\star\left(\mathbb{D}_{p_i^\varepsilon}^+(r_i^\varepsilon), e^{2\tilde{u}_\varepsilon} dx\right) = \sigma_m(e^{u_\varepsilon}) \,.$$

Proof. We easily build this sequence by induction on i, as soon as $\inf A_i = 0$. This sequence has to stop because if $\inf A_{m+1} = 0$, there are at least $m+2$ disjoint domains D_i in Σ such that $\sigma_\star\left(D_i, e^{2u_\varepsilon} dx\right) = \sigma_m(e^{u_\varepsilon} g)$. The associated first eigenfunctions extended by 0 on Σ are test functions for the variational problem of $\sigma_m(e^{u_\varepsilon})$. Their Rayleigh quotient is equal to $\sigma_m(e^{u_\varepsilon})$ for the min-max characterization of $\sigma_m(e^{u_\varepsilon})$. Therefore the case of equality gives that one of them is an eigenfunction and this contradicts the maximum principle. ♢

Far from the singularity points defined in Claim 4.4, the nodal domains of eigenfunctions cannot shrink inside balls whose radii converge to 0 outside $\bigcup_{j=1}^{i} \overline{\mathbb{D}_{p_j^\varepsilon}\left(10 r_j^\varepsilon\right)}$. Thanks to the following Poincaré inequality result (proposition 4.1), we will have uniform bounds of the L^2 norm at every scale when the eigenfunction vanishes and the nodal set of the sequence of eigenfunction has length uniformly lower bounded. They hold on the following domains we use during the proof:

$$\Omega_K = \mathbb{D}^+_{\frac{1}{K\rho}} \setminus \bigcup_{i=1}^{s} \mathbb{D}_{K\rho}$$

for some fixed number $1 < K < 10$ chosen independent of the problem we consider, $\rho > 0$, and $x_i \in \mathbb{D}_{\frac{1}{\rho}}$ such that if $i \neq j$, then $x_i \neq x_j$ and

$$10\rho < \min\left(\min_i d(x_i, \partial\mathbb{D}_{\frac{1}{10\rho}}); \min_{i\neq j} \frac{|x_i - x_j|}{2} \right) \,.$$

We have the following proposition, coming from capacity estimates:

Proposition 4.1 ([AH96], Corollary 8.2.2 and [Hen05], pages 95-97)**.** *Let $r > 0$ fixed. Then, we have a constant $C_r > 0$ such that for every $f \in \mathcal{C}^\infty(\Omega_1)$ which vanishes on a smooth piecewise curve $\Gamma \subset\subset \Omega_K$ which connects two points of distance $r > 0$,*

$$\|f\|_{L^2(\Omega_1)} \leq C_r \|\nabla f\|_{L^2(\Omega_1)} \,.$$

4.6. **Regularity estimates in the non concentrated case.** All the cases of Claim 4.2: the ambiant scale or the scales satisfying $\frac{\alpha_\varepsilon}{\sqrt{\varepsilon}} \to +\infty$ are very similar to handle. In both cases, the sequence of maps Φ_ϵ arising from Claim 4.1 are "almost-weakly-harmonic" maps into an ellipsoid.

Indeed, since $K_\varepsilon[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}] = 1$ on $supp(\nu_\epsilon)$, if one can prove that $supp(\nu_\epsilon) = \partial\Sigma$, we get that $|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon} = 1$ on $\partial\Sigma$, and thanks to the eigenfunction equation, Φ_ϵ is a free boundary harmonic map into an ellipsoid. By compactness results on sequences of free boundary harmonic maps into some manifold [Sch06, LP17, JLZ19], we conclude that, Φ_ϵ converges in $\mathcal{C}^k_{loc}(\Sigma \setminus C)$ as $\epsilon \to 0$, where C is the set of concentration points of ν. Therefore, knowing that $|\Phi|^2_{\mathcal{E}} = 1$ on $\partial\Sigma$ and $\partial_\nu\Phi_\varepsilon = \sigma\Phi_\varepsilon$, the sequence of densities $e^{u_\epsilon} = \Phi_\varepsilon.\partial_\nu\Phi_\varepsilon$ also converges and we get the expected regularity result.

However $supp(\nu_\epsilon)$ can be far from being equal to $\partial\Sigma$. In this case we will prove that we still have convergence as $\epsilon \to 0$ of the sequence of maps Φ_ϵ given by Claim 4.1, to a map Φ which satisfies the weak equation of free boundary harmonic maps into an ellipsoid. Thanks to the regularity results of free boundary harmonic maps into some manifold, we will get that Φ is smooth and that the limiting measure $\nu = (\Phi.\partial_\nu\Phi)\, dL_g$ is absolutely continous with respect to dL_g with a smooth density. By the maximum principle, $\Phi.\partial_\nu\Phi$ is a positive density.

We now assume that $\int_{\partial\Sigma} d\nu > 0$. We aim at proving a pointwise bound on eigenfunctions and then thanks to strong pointwise estimates deduce energy estimates and the $W^{1,2}$ convergence of eigenfunctions.

4.6.1. *Pointwise estimates on eigenfunctions.* We aim at proving that Φ_ε is bounded far from the sequence of singularity points p_i^ε given by Claim 4.4. We set for $\rho > 0$

$$\Sigma(\rho) = \Sigma \setminus \left(\bigcup_{i=1}^{s} B_g^+ (p_i, \rho) \right) .$$

Notice that the novelty compared to [Pet19] is that in the following Claim, the bounds of the eigenfunctions in $W^{1,2}(\Sigma(\rho))$ and $\mathcal{C}^0(\Sigma(\rho))$ do not depend on ρ.

Claim 4.5. *We assume that $\nu \neq 0$. Then there are constants C_1 and C_2 such that*

$$\forall \rho > 0, \limsup_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{W^{1,2}(\Sigma(\rho))} \leq C_1 , \tag{4.17}$$

$$\forall \rho > 0, \limsup_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{C^0(\Sigma(\rho))} \leq C_2 , \tag{4.18}$$

Proof. Notice that (4.17) is implied by (4.18) since $\|\nabla\Phi_\varepsilon\|_{L^2(\Sigma)}$ is bounded by 1.

Now, let us prove (4.18), the strategy proof is the same as in [Pet19], but we aim at getting here constants C_1 and C_2, not depending on ρ. Since the eigenfunctions are harmonic functions, by harnack inequalities far from the boundary, we just have to get uniform bounds at the neighbourhood of the boundary $\partial\Sigma$. We set

$$\delta_i^\varepsilon = d\left(supp(\nu_\varepsilon) \setminus \overline{\mathbb{I}_{p_i^\varepsilon}(10 r_i^\varepsilon)}, p_i^\varepsilon \right) , \tag{4.19}$$

for $i \in \{1, \cdots, s\}$. Up to a subsequence, we have a constant $0 < \tilde{\delta} < \delta_0$ that $\{1, \cdots, s\} = I_1 \cup I_2$, where

$$I_1 = \{i \in \{1, \cdots, s\}; \delta_i^\varepsilon \to 0 \text{ as } \varepsilon \to 0\} \text{ and } I_2 = \{i \in \{1, \cdots, s\}; \delta_i^\varepsilon \geq \tilde{\delta}\} \, . \tag{4.20}$$

Up to reduce $\tilde{\delta}$, we also assume that

$$supp(\nu_\varepsilon) \cap \left(\partial\Sigma \setminus \left(\bigcup_{i=1}^s \overline{\mathbb{I}_{\tilde{\delta}}(p_i^\varepsilon)} \right) \right) \neq \emptyset \, . \tag{4.21}$$

This is possible because the limiting measure of ν_ε has a non-empty support on $\Sigma \setminus \{p_1, \cdots, p_s\}$.

We have the following covering of $\partial\Sigma \setminus \bigcup_{i=1}^s \overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{10}}(p_i^\varepsilon)}$ by intervals of radius $\frac{\tilde{\delta}}{100}$:

$$\partial\Sigma \setminus \bigcup_{i=1}^s \overline{\mathbb{I}_{\frac{\tilde{\delta}}{10}}(p_i^\varepsilon)} \subset \bigcup_{l=1}^L \overline{\mathbb{I}_{\frac{\tilde{\delta}}{100}}(q_k)} \, ,$$

satisfying also

$$\{x; d(x, \partial\Sigma) \leq \frac{\tilde{\delta}}{200}\} \setminus \bigcup_{i=1}^s \overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{10}}(p_i^\varepsilon)} \subset \bigcup_{l=1}^L \overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{100}}(q_k)} \, .$$

Let's handle estimates on the balls $\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{100}}(q_k)}$ and then estimates at the neighbourhood of the singularity points p_i^ε in the following steps:

STEP 1 :

There is a constant C_2 such that for any $l \in \{1, \cdots, L\}$ and any coordinate $\Phi_\varepsilon^{i,j}$,

$$\text{either } \forall \varepsilon > 0, \forall x \in \overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{50}}(q_l)}, \left| \Phi_\varepsilon^{k,j}(x) \right| \leq C_2 \, ,$$

$$\text{or } \forall \varepsilon > 0, \forall x, y \in \overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{50}}(q_l)}, \frac{1}{C_2} \left| \Phi_\varepsilon^{k,j}(y) \right| \leq \left| \Phi_\varepsilon^{k,j}(x) \right| \leq C_2 \left| \Phi_\varepsilon^{k,j}(y) \right| \, .$$

Proof of STEP 1 :

Let $l \in \{1, \cdots, L\}$. Let $1 \leq k \leq m$, $1 \leq j \leq i_k$ and up to change $\Phi_\varepsilon^{k,j}$ into $-\Phi_\varepsilon^{k,j}$, let (x_ε) be a sequence of points such that

$$\Phi_\varepsilon^{k,j}(x_\varepsilon) = \sup_{x \in \overline{\mathbb{I}_{\frac{\tilde{\delta}}{50}}(q_l)}} \left| \Phi_\varepsilon^{k,j}(x) \right| \, .$$

In particular $x_\varepsilon \in \partial\Sigma$. We set

$$\delta_\varepsilon = d(\tilde{x}_\varepsilon, supp(\tilde{\nu}_\varepsilon) \cap \overline{\mathbb{I}_{\frac{\tilde{\delta}}{20}}(q_l)}) \, .$$

We divide the proof of (4.18) into three cases.

Case 1 - We assume that $\delta_\varepsilon = O(\sqrt{\varepsilon})$. Then, $\{\phi_\varepsilon^{k,j}(x_\varepsilon)\}$ is bounded by Claim 4.3.

Case 2 - We assume that $\delta_\varepsilon \to 0$ and $\frac{\sqrt{\varepsilon}}{\delta_\varepsilon} \to 0$ as $\varepsilon \to 0$. We let

$$\psi_\varepsilon = \tilde{\Phi}_\varepsilon^{k,j}(\delta_\varepsilon x + \tilde{x}_\varepsilon) \text{ and } e^{w_\varepsilon} = \delta_\varepsilon e^{\tilde{u}_\varepsilon(\delta_\varepsilon x + x_\varepsilon)}$$

for $x \in \mathbb{D}^+_{\delta\delta_\varepsilon^{-1}}$, so that ψ_ε is still harmonic and

$$\partial_\nu \psi_\varepsilon = \sigma_\varepsilon e^{w_\varepsilon} \psi_\varepsilon \text{ on } \mathbb{I}_{\delta\delta_\varepsilon^{-1}} .$$

Let $y_\varepsilon \in supp(\nu_\varepsilon)$ be such that $|x_\varepsilon - y_\varepsilon| = \delta_\varepsilon$ and set $z_\varepsilon = \frac{\tilde{y}_\varepsilon - \tilde{x}_\varepsilon}{\delta_\varepsilon}$ so that $z_\varepsilon \to z_0$ as $\varepsilon \to 0$ up to the extraction of a subsequence. Thanks to Claim 4.3, we know that $\psi_\varepsilon(z_\varepsilon) = \Phi^{k,j}_\varepsilon(y_\varepsilon) = O(1)$. Thanks to estimates (5.14) on the heat kernel, there exists $D_1 > 0$ such that

$$e^{w_\varepsilon} \leq D_1 \text{ on } \mathbb{I}_{\frac{1}{2}} .$$

We first assume that ψ_ε does not vanish in $\mathbb{D}^+_3$. Then, we can apply Harnack's inequality and get some constant $D_2 > 0$ such that

$$\psi_\varepsilon \geq D_2 \psi_\varepsilon(0) \text{ on } \mathbb{D}^+_{\frac{1}{4}}$$

for all $\varepsilon > 0$. Since ψ_ε is positive on $\mathbb{D}^+_{|z_\varepsilon|}(z_\varepsilon) \subset \mathbb{D}_3$, by the equation, it is also superharmonic and we can write that

$$\psi_\varepsilon(z_\varepsilon) \geq \frac{1}{2\pi\,|z_\varepsilon|} \int_{\partial \mathbb{D}^+_{|z_\varepsilon|}(z_\varepsilon)} \psi_\varepsilon d\sigma .$$

Taking only the part of the integral which lies in $\mathbb{D}^+_{\frac{1}{4}}$, we get the existence of some constant $D_3 > 0$ such that

$$\psi_\varepsilon(z_\varepsilon) \geq D_3 \psi_\varepsilon(0)$$

and this concludes the proof of (4.18) in this case since $\Phi^{k,j}_\varepsilon(x_\varepsilon) = \psi_\varepsilon(0) = O(1)$.

We now assume that ψ_ε vanishes on $\mathbb{D}^+_3$. Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, and $x_\varepsilon \in \Sigma_\varepsilon(\rho)$, by definition (4.19) and Claim 4.4, ψ_ε vanishes on a piecewise smooth curve in $\mathbb{D}^+_4$ which connects two points of distance greater than 1. By proposition 4.1 for $\Omega = \mathbb{D}^+_5$, we get some constant $C > 0$ such that

$$\int_{\mathbb{D}^+_4} \psi_\varepsilon^2 \leq C \int_{\mathbb{D}^+_5} |\nabla \psi_\varepsilon|^2 \, dx$$

which proves by trace Sobolev embeddings that $\{\psi_\varepsilon\}$ is bounded in $L^2(\mathbb{I}_4)$. By elliptic regularity, ψ_ε is bounded in $L^\infty(\mathbb{D}^+_{\frac{1}{4}})$ which gives that $\{\Phi^{k,j}_\varepsilon(x_\varepsilon)\}$ is bounded. The study of these three cases completes the proof of (4.18) with a constant $C_2(\rho)$.

Case 3 - We assume that $\delta_\varepsilon^{-1} = O(1)$ and that, $\{\Phi^{k,j}_\varepsilon\}$ vanishes in $\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{20}}(q_l)}$. Then $\{e^{u_\varepsilon}\}$ is uniformly bounded in $\overline{\mathbb{I}^+_{\frac{\tilde{\delta}}{10}}(x)}$ by (5.13) and $\{\Phi^{k,j}_\varepsilon\}$ is bounded in $L^2\left(\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{40}(q_l)}}\right)$ by proposition 4.1. Then $\{\Phi^{k,j}_\varepsilon\}$ is bounded by some constant C_2 in $L^\infty\left(\overline{\mathbb{I}_{\frac{\tilde{\delta}}{50}(q_l)}}\right)$ by standard elliptic theory on the eigenvalue equation. So is $\{\Phi^{k,j}_\varepsilon(x_\varepsilon)\}$.

Case 4 - We assume that $\delta_\varepsilon^{-1} = O(1)$ and that $\{\Phi^{k,j}_\varepsilon\}$ does not vanishes in $\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{20}}(q_l)}$. Then $\{e^{u_\varepsilon}\}$ is uniformly bounded in $\overline{\mathbb{I}_{\frac{\tilde{\delta}}{10}}(x)}$ by (5.13). Moreover $\{\Phi^{k,j}_\varepsilon\}$ has a constant sign and up to take $-\Phi^{k,j}_\varepsilon$, we assume that $\Phi^{k,j}_\varepsilon > 0$ in $\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{20}}(q_l)}$. Therefore $\Phi^{k,j}_\varepsilon$ is a positive harmonic function with $\partial_\nu \Phi^{k,j}_\varepsilon = e^{u_\varepsilon} \Phi^{k,j}_\varepsilon$ and a sequence of potentials $\{e^{u_\varepsilon}\}$ uniformly

bounded in $\mathbb{D}^+_{\frac{\delta}{R_0}}(x)$. Then $\Phi^{k,j}_\varepsilon$ satisfies a Harnack inequality on $\mathbb{D}^+_{\frac{\delta}{50}}(q_l)$. This conclude the proof of STEP 1.

STEP 2 :

There is $l = 1, \cdots, L$ such that

$$\forall \varepsilon > 0, \forall x \in \overline{\mathbb{I}_{\frac{\tilde\delta}{50}}(q_l)}, |\Phi_\varepsilon(x)| \leq C_2 \, .$$

Proof of STEP 2 :

By (4.21), let $l = 1, \cdots, L$ be such that $\mathbb{I}_{\frac{\tilde\delta}{100}}(q_l) \cap supp(\nu_\varepsilon) \neq \emptyset$. The proof is the same as in the previous STEP, except in Case 4. Instead of getting a Harnack inequality if $\{\Phi^{k,j}_\varepsilon\}$ does not vanishes in $\overline{\mathbb{D}^+_{\frac{\tilde\delta}{20}}(q_l)}$, we proceed as in Case 2 of STEP 1 and we get a uniform bound. This concludes the proof of STEP 2.

STEP 3 :

Up to increase C_2,

$$\forall \varepsilon > 0, \forall x \in \Sigma \setminus \left(\bigcup_{i \in I_1} \overline{\mathbb{D}^+_{\delta^\varepsilon_i}(p^\varepsilon_i)} \cup \bigcup_{i \in I_2} \overline{\mathbb{D}^+_{\frac{\tilde\delta}{10}}(p^\varepsilon_i)} \right), |\Phi_\varepsilon(x)| \leq C_2 \, ,$$

where the definition of I_1 and I_2 is given at the beginning of the proof in (4.20)

Proof of STEP 3 :

We first have that, up to increase C_2, we use the previous STEP 1 and STEP 2 with local L^∞ bounds at the neighbourhood of the boundary. We also use a global Harnack inequality for harmonic functions on $\{x \in \Sigma; d(x, \partial\Sigma) > \frac{\tilde\delta}{200}\}$. By the definition of the covering before Step 1 and since Σ is connected, we get

$$\forall \varepsilon > 0, \forall x \in \Sigma \setminus \left(\bigcup_{i=1}^{s} \overline{\mathbb{D}^+_{\frac{\tilde\delta}{10}}(p^\varepsilon_i)} \right), |\Phi_\varepsilon(x)| \leq C_2 \, . \tag{4.22}$$

Let's now prove the uniform bound in the neighbourhood of points p^ε_i for $i \in I_1$. Let $1 \leq k \leq m$, $1 \leq j \leq i_k$ and up to change $\Phi^{k,j}_\varepsilon$ into $-\Phi^{k,j}_\varepsilon$, let (x_ε) be a sequence of points such that

$$\Phi^{k,j}_\varepsilon(x_\varepsilon) = \sup_{x \in \Sigma \setminus \left(\bigcup_{i \in I_1} \overline{\mathbb{D}^+_{\delta^\varepsilon_i}(p^\varepsilon_i)} \cup \bigcup_{i \in I_2} \overline{\mathbb{D}^+_{\frac{\tilde\delta}{10}}(p^\varepsilon_i)} \right)} \left| \Phi^{k,j}_\varepsilon(x) \right| \, .$$

We set

$$\delta_\varepsilon = d(\tilde{x}_\varepsilon, supp(\tilde\nu_\varepsilon)) \, .$$

If $x_\varepsilon \in \overline{\mathbb{D}^+_{\delta^\varepsilon_i}(p^\varepsilon_i)} \setminus \overline{\mathbb{D}^+_{\frac{\tilde\delta}{10}}(p^\varepsilon_i)}$ for some $i \in I_1$, then by the maximum principle, either $x_\varepsilon \notin \partial\Sigma$ and we get the STEP 3 by (4.22) or $x_\varepsilon \in \partial\Sigma$. We then assume that $x_\varepsilon \in \overline{\mathbb{I}_{\delta^\varepsilon_i}(p^\varepsilon_i)} \setminus \overline{\mathbb{I}_{\frac{\tilde\delta}{10}}(p^\varepsilon_i)}$. We divide the proof of (4.18) into four cases as in the proof of STEP 1. The proof is the same in cases 1, 2 and 3. The key point is that we can always apply proposition 4.1 by the

choice of δ_ε^i in (4.19). To adapt case 4, we assume that $x_\varepsilon \to x$. Then $x \notin \{p_1, \cdots, p_s\}$. Indeed, Because of Claim 4.4, we have

$$\forall i \in \{1, \cdots, s\}, supp(\nu_\varepsilon) \cap \overline{\mathbb{I}_{10r_i^\varepsilon}(p_i^\varepsilon)} \neq \emptyset \, .$$

Indeed, if not, by uniform estimates on the heat kernel, $r_i^\varepsilon e^{\tilde{u}_\varepsilon(r_i^\varepsilon x + p_i^\varepsilon)} \to 0$ uniformly on $x \in \mathbb{D}$. We would then get

$$\sigma_\star \left(\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon), e^{2\tilde{u}_\varepsilon} \right) \to +\infty$$

as $\varepsilon \to 0$ which is not possible by Claim 4.4. Now, since $x \notin \{p_1, \cdots, p_s\}$, up to reduce $\tilde{\delta}$ in all the previous steps, and thanks to (4.22), we conclude the proof of STEP 3.

STEP 4 :

We prove the remaining estimate for indices in I_2. Up to increase C_2,

$$\forall i \in I_2, \forall \rho \in (0, \frac{\tilde{\delta}}{2}), \limsup_{\varepsilon \to 0} \sup_{x \in \Sigma \setminus \overline{\mathbb{D}_\rho(p_i)}} |\Phi_\varepsilon|(x) \leq C_2 \, .$$

Proof of STEP 4 :

For $i \in I_2$, $\delta_\varepsilon^i \geq \tilde{\delta}$. By convergence properties of the heat kernel, we know that for any $0 < \rho < \tilde{\delta}$, $e^{u_\varepsilon} = K_\varepsilon(\nu_\varepsilon)$ converges uniformly to 0 on $\overline{\mathbb{I}_{\tilde{\delta}}(p_i)} \setminus \overline{\mathbb{I}_{\frac{\rho}{2}}(p_i)}$. Therefore, by standard elliptic estimates, up to a subsequence, Φ_ε converges in $\mathcal{C}^2(\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{2}}(p_i)} \setminus \overline{\mathbb{D}^+_\rho(p_i)})$ for any $\rho > 0$ to a harmonic map Φ, with Neumann boundary conditions.

Notice that at this stage, Φ is defined on $\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i) \setminus \{p_i\}$ and may be unbounded since the $\mathcal{C}^0$ bound of $\Phi(x)$ depend on $|x - p_i|$ as $|x - p_i| \to 0$. Since Φ is harmonic and $\nabla\Phi$ belongs to $L^2\left(\mathbb{D}^+_{\frac{\tilde{\delta}}{2}}(p_i)\right)$ as a weak limit of $\nabla\Phi_\varepsilon$ in L^2, by point removability, Φ is in fact harmonic on $\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i)$. By STEP 3, Φ is bounded by C_2 on $\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{2}}(p_i)} \setminus \overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{10}}(p_i)}$. By the maximum principle and strong convergence on $\mathcal{C}^2(\overline{\mathbb{D}^+_{\frac{\tilde{\delta}}{2}}(p_i)} \setminus \overline{\mathbb{D}^+_\rho(p_i)})$ for any ρ and $i \in I_2$, the maximum of Φ_ε on $\Sigma(\rho)$ does not depend on ρ. Then, we get STEP 4.

Gathering STEP 4 and STEP 3 proves (4.18). As already said this also proves (4.17). The proof of the Claim is complete.

♢

4.6.2. $W^{1,2}$ *convergence of eigenfunctions.* In this section, we set the following weight function ω_ε defined as the harmonic extension on Σ of the function $|\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$ defined on $\partial\Sigma$:

$$\begin{cases} \Delta_g \omega_\varepsilon = 0 & \text{in } \Sigma \\ \partial_\nu \omega_\varepsilon = |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon} & \text{on } \partial\Sigma \, . \end{cases} \tag{4.23}$$

After pointwise estimates, we want to prove energy estimates on Φ_ε. Then, we aim at proving that $\nabla\omega_\varepsilon$ converges to 0 in L^2 by the global structure of the equation on Φ_ε given by Claim 4.1, in order to use the quantitative estimate of section 3 for ω_ε-harmonic maps.

We first give quantitative non-concentration estimates:

Claim 4.6. *We assume that $\nu \neq 0$. Then, we have the following: quantitative non-concentration estimates on e^{u_ε} and $|\nabla\Phi_\varepsilon|_g^2$, there are constants $D_1 > 0$ and $D_2 > 0$ such that*

$$\forall \rho > 0, \forall r > 0, \limsup_{\varepsilon\to 0} \sup_{x\in\Sigma(\rho)\cap\partial\Sigma} \int_{B_g^+(x,r)} e^{u_\varepsilon} dL_g \leq \frac{D_1}{\ln(\frac{1}{r})} , \tag{4.24}$$

$$\forall \rho > 0, \forall r > 0, \limsup_{\varepsilon\to 0} \sup_{x\in\Sigma(\rho)} \int_{B_g(x,r)} |\nabla\Phi_\varepsilon|_g^2 dA_g \leq \frac{D_2}{\sqrt{\ln(\frac{1}{r})}} . \tag{4.25}$$

We skip the proof since we follow here the same lines as in [Pet19] but again, here, D_1 and D_2 do not depend on ρ.

Now we aim at getting better uniform estimates than in Claim 4.1. This is a necessary claim to prove that $|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}$ is uniformly lower bounded by a positive constant on $\partial\Sigma$ (so that we can divide by ω_ε in the key Claim 4.8). We also prove a uniform convergence of $|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}$ to 1 close to the support of ν_ε.

Claim 4.7. *We assume that $\nu \neq 0$. Then for any $\rho > 0$, there exists $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that*

$$\forall x \in \Sigma(\rho)\cap\partial\Sigma, |\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}(x) \geq 1 - \beta_\varepsilon , \tag{4.26}$$

$$\forall x \in \Sigma(\rho)\cap supp(\nu_\varepsilon), \left||\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}(x) - 1\right| \leq 1 - \beta_\varepsilon . \tag{4.27}$$

Again, we skip the proof of this Claim since its proof follows the same lines as in [Pet19]. It is a consequence of Claim 4.5 and Claim 4.6.

We are now able to pass to the limit in $W^{1,2}$ and on the equation. The key point is to work on the energy of $\omega_\varepsilon = |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$.

We notice first that $\int_{\Sigma(\rho)} |\nabla\omega_\varepsilon|_g^2$ has to converge to 0 as $\varepsilon \to 0$ and then $\rho \to 0$. This is forced because cut-off functions around the singularity points have a small energy controlled by $\frac{1}{\ln\frac{1}{\rho}}$ (independent of ε), and because of the bounds $\omega_\varepsilon \geq 1-\beta_\varepsilon$ on $\Sigma(\rho)$ and $|\omega_\varepsilon - 1| \leq \beta_\varepsilon$ on $supp(\nu_\varepsilon)$ and the structure of the global equation on Φ_ε.

Then, by the harmonic replacement estimates (Claim (3.1)), the free boundary $(\omega_\varepsilon)^2$-harmonic replacements $\widetilde{\Psi_\varepsilon}$ of the radial projection $\widetilde{\Phi_\varepsilon} = \frac{\Phi_\varepsilon}{\omega_\varepsilon}$ of Φ_ε are then $W^{1,2}$ close to $\widetilde{\Phi_\varepsilon}$. Thanks to the estimates in the appendix, free boundary $(\omega_\varepsilon)^2$-harmonic maps with small energy are bounded in $W^{2,p}$ for any $p \in (1,2)$, so that they converge in $W^{1,2}\cap\mathcal{C}^{0,\alpha}$ by standard compact Sobolev embeddings.

Claim 4.8. *We assume that $\nu \neq 0$. Then up to the extraction of a subsequence of $\{\Phi_\varepsilon\}$ there is a function $\Phi \in W^{1,2}(\Sigma)$ such that for any $\rho > 0$,*

$$\Phi_\epsilon \to \Phi \text{ in } W^{1,2}(\Sigma(\rho)) \text{ as } \varepsilon \to 0 . \tag{4.28}$$

Moreover, $\Phi \in \mathcal{C}^{0,\alpha}(\Sigma_\rho)$ and satisfies

$$|\Phi|^2_{\mathcal{E}} = 1 \text{ on } \partial\Sigma \setminus \{p_1, \cdots, p_s\} \tag{4.29}$$

and for $1 \leq k \leq m$, Φ satisfies

$$\begin{cases} \Delta_g \Phi^k = 0 & \text{in } \Sigma \\ \partial_\nu \Phi^k = \sigma_k \Phi^k d\nu & \text{on } \partial\Sigma \end{cases} \tag{4.30}$$

in a weak sense.

Proof. In the first step, we prove (4.28), in the second one we prove (4.29) and finally we prove (4.30).

STEP 1 :

We have the following energy convergence on ω_ε, the harmonic extension of $|\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$ on Σ: for any $\rho > 0$, there is $\delta > 0$ small enough such that on $\Sigma_{\delta_\rho}(\rho) = \{x \in \Sigma(\rho), d(x, \partial\Sigma) \leq \delta\}$

$$\lim_{\rho \to 0} \lim_{\varepsilon \to 0} \int_{\Sigma_{\delta_\rho}(\rho)} |\nabla \omega_\varepsilon|^2 \, dA_g = 0 \, . \tag{4.31}$$

Notice first that by Claim 4.7, and standard elliptic estimates on harmonic functions (see also proposition 5.1), ω_ε is uniformly lower bounded by a positive constant on $\Sigma(\rho)$ for any ρ, so that we can divide by ω_ε all along the proof. More precisely, we have that that for any $\rho > 0$,

$$\omega_\varepsilon \geq 1 - \beta_\varepsilon \text{ as } \varepsilon \to 0 \text{ on } \Sigma(\rho) \cap \partial\Sigma \, . \tag{4.32}$$

First, we define a cut-off function $\eta \in \mathcal{C}_c^\infty\left(\Sigma(\sqrt{\rho})\right)$ such that $\eta = 1$ on $\Sigma(\rho)$ and

$$\int_\Sigma |\nabla \eta|^2 \leq \frac{C}{\ln \frac{1}{\rho}} \, .$$

In fact we first prove that

$$\lim_{\rho \to 0} \lim_{\varepsilon \to 0} \int_\Sigma \eta \, |\nabla \omega_\varepsilon|^2 \, \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon^3} dA_g = 0 \, . \tag{4.33}$$

We integrate the eigenvalue equation $\Delta_g \Phi_\varepsilon = 0$ and $\partial_\nu \Phi_\varepsilon = \sigma_\varepsilon \Phi_\varepsilon e^{u_\varepsilon}$, against $\sigma_\varepsilon \eta \Phi_\varepsilon$ and $\sigma_\varepsilon \eta \widetilde{\Phi}_\varepsilon$, where $\widetilde{\Phi_\varepsilon} = \frac{\Phi_\varepsilon}{\omega_\varepsilon}$ and we get, remembering that $\omega_\varepsilon = |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$ on $\partial\Sigma$,

$$\int_\Sigma \eta \, |\nabla \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon} \, dA_g + \int_\Sigma \langle \nabla \eta, \sigma_\varepsilon \Phi_\varepsilon . \nabla \Phi_\varepsilon \rangle_g \, dA_g = \int_{\partial\Sigma} \eta \, |\sigma_\varepsilon \Phi_\varepsilon|^2 \, e^{u_\varepsilon} dL_g \, , \tag{4.34}$$

where we consider the norm of the gradient with respect to the ellipsoid and that

$$\int_\Sigma \eta \left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \Phi_\varepsilon \right\rangle_{\mathcal{E}_\varepsilon} dA_g + \int_\Sigma \left\langle \nabla \eta, \frac{\sigma_\varepsilon \Phi_\varepsilon}{\omega_\varepsilon} . \nabla \Phi_\varepsilon \right\rangle_g dA_g = \int_{\partial\Sigma} \eta \frac{|\sigma_\varepsilon \Phi_\varepsilon|^2}{\omega_\varepsilon} e^{u_\varepsilon} dL_g \, , \tag{4.35}$$

Since $\widetilde{\Phi_\varepsilon} \in \mathcal{E}_\varepsilon$. We have that

$$\left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \Phi \right\rangle_{\mathcal{E}_\varepsilon} = \frac{\left\langle \left(\nabla \Phi_\varepsilon - \widetilde{\Phi_\varepsilon} \nabla \omega_\varepsilon \right), \nabla \Phi_\varepsilon \right\rangle_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} = \frac{|\nabla \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} + \left\langle \nabla \frac{1}{\omega_\varepsilon}, \sigma_\varepsilon \Phi_\varepsilon . \nabla \Phi_\varepsilon \right\rangle \, , \tag{4.36}$$

noticing that $\sigma_\varepsilon \Phi_\varepsilon . \nabla \Phi_\varepsilon = \langle \Phi_\varepsilon, \nabla \Phi_\varepsilon \rangle_{\mathcal{E}_\varepsilon} = \nabla \left(\frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2} \right)$, we compute the first left-hand side term of (4.35), by considering first the last-hand term in (4.36):

$$\begin{aligned}
\int_\Sigma \eta \left\langle \nabla \frac{1}{\omega_\varepsilon}, \sigma_\varepsilon \Phi_\varepsilon . \nabla \Phi_\varepsilon \right\rangle dA_g &= \int_\Sigma \eta \left\langle \nabla \frac{1}{\omega_\varepsilon}, \nabla \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2} \right\rangle dA_g \\
&= - \int_\Sigma \left\langle \nabla \eta, \nabla \frac{1}{\omega_\varepsilon} \right\rangle_g \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2} dA_g + \int_\Sigma \eta \frac{|\Phi|^2_{\mathcal{E}_\varepsilon}}{2} \Delta_g \left(\frac{1}{\omega_\varepsilon} \right) dA_g + \int_{\partial \Sigma} \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2} \eta \partial_\nu \left(\frac{1}{\omega_\varepsilon} \right) dL_g \\
&= \int_\Sigma \langle \nabla \eta, \nabla \omega_\varepsilon \rangle_g \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2\omega_\varepsilon^2} dA_g - \int_\Sigma \eta \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon^3} |\nabla \omega_\varepsilon|^2_g \, dA_g - \frac{1}{2} \int_\Sigma \langle \nabla \eta, \nabla \omega_\varepsilon \rangle_g \, dA_g \, ,
\end{aligned}$$

where we notice at the last step that $\Delta_g \omega_\varepsilon = 0$. Therefore, using (4.35) and (4.36),

$$\begin{aligned}
\int_\Sigma \eta \, |\nabla \omega_\varepsilon|^2 \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon^3} dA_g &= \int_\Sigma \frac{|\nabla \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} dA_g + \int_\Sigma \left\langle \nabla \eta, \frac{\sigma_\varepsilon \Phi_\varepsilon}{\omega_\varepsilon} . \nabla \Phi_\varepsilon \right\rangle_g dA_g \\
&\quad - \int_{\partial \Sigma} \eta \frac{|\sigma_\varepsilon \Phi_\varepsilon|^2}{\omega_\varepsilon} e^{u_\varepsilon} dL_g + \int_\Sigma \langle \nabla \eta, \nabla \omega_\varepsilon \rangle_g \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{2\omega_\varepsilon^2} dA_g - \frac{1}{2} \int_\Sigma \langle \nabla \eta, \nabla \omega_\varepsilon \rangle_g \, dA_g \\
&\leq \int_\Sigma \frac{|\nabla \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} dA_g - \int_{\partial \Sigma} \eta \frac{|\sigma_\varepsilon \Phi_\varepsilon|^2}{\omega_\varepsilon} e^{u_\varepsilon} dL_g + \frac{C C_2}{\sqrt{\ln \left(\frac{1}{\rho} \right)}} \, ,
\end{aligned}$$

for a some constant $C > 0$ indepent of C and ρ. We also have by (4.32) and (4.34) that

$$\int_\Sigma \eta \frac{|\nabla \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} dA_g \leq \int_\Sigma \eta \, |\nabla \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon} \, dA_g \leq \int_{\partial \Sigma} \eta \, |\sigma_\varepsilon \Phi_\varepsilon|^2 \, e^{u_\varepsilon} dL_g - \int_\Sigma \langle \nabla \eta, \sigma_\varepsilon \Phi_\varepsilon . \nabla \Phi_\varepsilon \rangle_g \, dA_g \, ,$$

so that

$$\int_\Sigma \eta \, |\nabla \omega_\varepsilon|^2 \frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon^3} dA_g \leq \int_{\partial \Sigma} \eta \, (\omega_\varepsilon - 1) \frac{|\sigma_\varepsilon \Phi_\varepsilon|^2}{\omega_\varepsilon} e^{u_\varepsilon} dL_g + \frac{C' C_2}{\sqrt{\ln \left(\frac{1}{\rho} \right)}}$$

for a some constant $C' > 0$ indepent of C and ρ. We have that

$$\int_{\partial \Sigma} \eta \, (\omega_\varepsilon - 1) \frac{|\sigma_\varepsilon \Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} e^{u_\varepsilon} dL_g \to 0 \text{ as } \varepsilon \to 0 \, . \tag{4.37}$$

Indeed, remember that $e^{u_\varepsilon} = K_\varepsilon(\nu_\varepsilon)$ and that $|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon} = 1$ on $supp(\nu_\varepsilon)$ by (4.27). Then

$$\begin{aligned}
\int_{\Sigma(\rho) \cap \partial \Sigma} \left| |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon} - 1 \right| e^{u_\varepsilon} dL_g &= \int_{\Sigma(\rho) \cap \partial \Sigma} \left(\int_{\partial \Sigma} p_\varepsilon(x, y) \left| |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon} - 1 \right| d\nu_\varepsilon(y) \right) dL_g(x) \\
&= \int_{\Sigma(\rho) \cap \partial \Sigma} \left| \int_{\partial \Sigma} \left| |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon} - 1 \right| d\nu_\varepsilon(y) \right) dL_g(x) \\
&\leq \beta_\varepsilon + D \sum_{i=1}^s \int_{\Sigma(\rho) \cap \partial \Sigma} \int_{B_g(p_i, \frac{\rho}{10})} p_\varepsilon(x, y) dL_g(y) dL_g(x)
\end{aligned}$$

for some constant D, and using uniform estimates on the heat kernel, and remembering that $\frac{|\sigma_\varepsilon \Phi_\varepsilon|^2}{\omega_\varepsilon}$ is bounded, we conclude the proof of (4.37). Then we have that

$$\lim_{\varepsilon\to 0}\int_\Sigma \eta\,|\nabla\omega_\varepsilon|^2\,\frac{|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon^3}dA_g \leq \frac{C'C_2}{\sqrt{\ln\frac{1}{\rho}}}$$

and letting $\rho\to 0$ completes the proof of (4.33). The STEP 1 of the current Claim easily follows from uniform lower boundedness by a positive constant of $|\Phi|_{\mathcal{E}_\varepsilon}$ in the set $\{x\in\Sigma(\rho), d(x,\partial\Sigma)\leq\delta_\rho\}$ for some δ_ρ small enough.

STEP 2 :

We prove now (4.28)*, the strong* $W^{1,2}(\Sigma(\rho))$*-convergence of* $\{\Phi_\varepsilon\}$ *for any* ρ*.*

This result is just a consequence of the quantative estimate (3.5) in Claim (3.1) and the first Step. Indeed, by Claim 4.4 and the boundedness of the energy, up to partitions of unity, one can focus on disks $\mathbb{D}_r^+$ for r small enough to satisfy the assumptions of Claim (3.1). To apply STEP 1, we also assume $r<\delta_\rho$. We define the free boundary $(\omega_\varepsilon)^2$-harmonic replacement $\widetilde{\Psi_\varepsilon}$ of $\widetilde{\Phi_\varepsilon}=\frac{\Phi_\varepsilon}{\omega_\varepsilon}$. Then, by ε-regularity results on these maps (see proposition 5.2 in the appendix), we know that $\widetilde{\Psi_\varepsilon}$ is bounded in $W^{2,q}(\mathbb{D}^+_{\frac{r}{2}})$ for any $q\in(0,2)$. In particular, by compactness embeddings, up to a subsequence $\widetilde{\Psi_\varepsilon}$ converges strongly in $W^{1,2}(\mathbb{D}^+_{\frac{r}{2}})$ and in $\mathcal{C}^{0,\alpha}(\mathbb{D}^+_{\frac{r}{2}})$. Since $\Phi_\varepsilon-\Psi_\varepsilon$ converges to 0 in $W^{1,2}(\mathbb{D}_r^+)$, and $\Psi_\varepsilon=\omega_\varepsilon\widetilde{\Psi_\varepsilon}$ converges strongly in $W^{1,2}(\mathbb{D}^+_{\frac{r}{2}})$, we obtain that Φ_ε converges strongly in $W^{1,2}(\mathbb{D}^+_{\frac{r}{2}})$.

This completes the proof of STEP 2 in the current Claim.

STEP 3 : We prove now the weak eigenvalue equation (4.30)*.*

Let Φ be the strong limit in $W^{1,2}(\Sigma(\rho))$ for any ρ. By the previous convergence results, $\Phi\in\mathcal{C}^{0,\alpha}(\Sigma(\rho))$. Again, we can argue locally up to partitions of unity and take a half-disk $\mathbb{D}_r^+$ for r small enough to satisfy $\sigma_\star(\mathbb{D}_r, e^{2\tilde{u}_\varepsilon})\geq\sigma_m(e^{u_\varepsilon})$. Let $\zeta\in\mathcal{C}_c^\infty(\mathbb{D}_r)$. We have that

$$\begin{aligned}\int_{\partial\Sigma}\overline{\zeta}\left(\sigma_\varepsilon\Phi_\varepsilon e^{u_\varepsilon}dL_g-\sigma\Phi d\nu\right) &= \int_{\partial\Sigma}\overline{\zeta}\left(\sigma_\varepsilon\Phi_\varepsilon-\sigma\Phi\right)e^{u_\varepsilon}dL_g\\ &\quad+\int_{\partial\Sigma}\overline{\zeta}\sigma\Phi\left(e^{u_\varepsilon}dA_g-d\nu\right)\ .\end{aligned}$$

Then on the first right-hand term, we have that

$$\begin{aligned}\int_{\partial\Sigma}\overline{\zeta}\left(\sigma_\varepsilon\Phi_\varepsilon-\sigma\Phi\right)e^{u_\varepsilon}dL_g &\leq \left(\int_{\mathbb{I}_r}\overline{\zeta}^2\left|\sigma_\varepsilon\Phi_\varepsilon-\sigma\Phi\right|^2e^{u_\varepsilon}dL_g\right)^{\frac{1}{2}}\\ &\leq \left(\frac{1}{\sigma_\star(\mathbb{D}_r^+,e^{2\tilde{u}_\varepsilon})}\int_{\mathbb{D}_r^+}\left|\nabla\left(\overline{\zeta}\left|\sigma_\varepsilon\Phi_\varepsilon-\sigma\Phi\right|\right)\right|_g^2dA_g\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\mathbb{D}_r^+}\left|\nabla\left(\Phi_\varepsilon-\Phi\right)\right|_g^2dA_g\right)^{\frac{1}{2}}\end{aligned}$$

for some constant C independent of ε. Letting $\varepsilon\to 0$ in a weak sense to the eigenvalue equation $\Delta_g\Phi_\varepsilon=0$ and $\partial_\nu\Phi_\varepsilon=\sigma_\varepsilon\Phi_\varepsilon e^{2u_\varepsilon}$, we get the equation (4.30) in a weak sense on $\Sigma(\rho)$ for any ρ, and then on $\Sigma\setminus\{p_1,\cdots,p_s\}$. Since Φ belongs to $W^{1,2}(\Sigma)$ as a strong limit of Φ_ε, equation (4.30) occurs on Σ in a weak sense and STEP 3 is proved.

By strong convergence in any L^p of Φ_ε, up to a subsequence again, we have that $|\Phi|^2_{\mathcal{E}} \geq 1$ dA_g-a.e on $\partial\Sigma$. By continuity of Φ on $\Sigma(\rho)$, $|\Phi|^2_{\mathcal{E}} \geq 1$ holds everywhere on $\partial\Sigma$ except maybe on $\{p_1, \cdots p_s\}$. Since $\nabla\omega_\varepsilon$ converges to 0, we obtain that ω is a constant. Then $|\Phi|^2_{\mathcal{E}}$ is a constant. Since $\int_\Sigma \left(|\Phi|^2_{\mathcal{E}} - 1\right) d\nu = 0$ by the previous convergences, $\int_{\partial\Sigma} d\nu \neq 0$ and ν does not have any atom, we conclude that $|\Phi|^2_{\mathcal{E}} = 1$ on $\partial\Sigma \setminus \{p_1, \cdots, p_s\}$ and we get (4.29). This concludes the proof of the Claim.

$\diamondsuit$

Notice that this Claim can be adapted to prove precisely Theorem 1.2 for $k > 1$, if there is $j \geq 1$ is such that $\sigma^\varepsilon_j \to 0$ as $\varepsilon \to 0$ and $\sigma^\varepsilon_{j+1} \to 0$ is uniformly lower bounded by a positive constant. In this case, notice that we do use directly proposition 3.1. We just delete the coordinates of Φ_ε associated to $\sigma^\varepsilon_1, \cdots, \sigma^\varepsilon_j$ because they necessarily converge to a constant in $W^{1,2}$. We then apply proposition 3.1 to the map Φ_ε restricted to coordiantes associated to eigenvalues $\sigma^\varepsilon_{j+1}, \cdots, \sigma^\varepsilon_m$.

Now, we are able to prove Theorem 1.2 with the asumption $\int_{\partial\Sigma} d\nu > 0$. We have that $|\Phi|^2_{\mathcal{E}} = 1$ on $\partial\Sigma \setminus \{p_1, \cdots, p_s\}$. Moreover, ϕ is harmonic in Σ and $\partial_\nu\Phi \in (T_\Phi\mathcal{E})^\perp$ on $\partial\Sigma$. It is exactly the asumption of being a weak free boundary harmonic map into the ellipsoid $\mathcal{E}$. Therefore Φ is a smooth map and satisfies the free boundary harmonic map equation in $\mathcal{E}$ in a strong sense. Now, we have that

$$\nu = (\partial_\nu\Phi.\Phi)\, dL_g \, ,$$

which means that ν is absolutely continous with respect to dL_g with a smooth density.

4.7. **Regularity estimates in the concentrated case.**

4.7.1. *Reduction to a similar problem as in the non-concentrated case.* We now assume $\frac{\alpha_\varepsilon}{\sqrt{\varepsilon}} \to +\infty$, where $\alpha_\varepsilon := \alpha^i_\varepsilon$ for some i is given by Claim 4.2. We set $\theta_\varepsilon = \frac{\varepsilon}{e^{2\tilde{v}_l(a)}\alpha^2_\varepsilon}$, where $a_\varepsilon \to a \in \mathbb{R}^2$ as $\varepsilon \to 0$. Then

$$\theta_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \, . \tag{4.38}$$

As noticed in the beginning of Section 4.6 we handle this case similarly to the non-concentrated case. For all the section, we refer to 4.2 for all the notations we use at the scale of concentration: we use the "hat" notations for all the objects which play the same role as the previous objects up to translation and dilatation.

First, notice that

$$e^{\hat{u}_\varepsilon} dx - d\hat{\nu}_\epsilon \rightharpoonup_\star 0 \text{ in } \mathcal{M}(\mathbb{R}) \text{ as } \varepsilon \to 0 \, . \tag{4.39}$$

Indeed, for $\zeta \in \mathcal{C}^0_c(\mathbb{I}_{R_0})$ for some $R_0 > 0$, and $R > R_0$, we can write that

$$\begin{aligned} \int_{\mathbb{R}} \zeta(z) \left(e^{2\hat{u}_\varepsilon(z)} dz - d\hat{\nu}_\varepsilon(z)\right) \;=\; & \int_{\Sigma\setminus\breve{\mathbb{I}}_R} \left(\int_{\breve{\mathbb{I}}_{R_0}} p_\varepsilon(y,x)\zeta(\hat{y}) dL_g(y) \right) d\nu_\varepsilon(x) \\ & + \int_{\mathbb{I}_R} \left(\int_{\mathbb{I}_R} (\zeta(z) - \zeta(x))\, \hat{p}_\varepsilon(z,x) dz \right) d\hat{\nu}_\varepsilon(x) \\ & + \int_{\mathbb{I}_{R_0}} \left(\int_{\mathbb{I}_R} \hat{p}_\varepsilon(z,x) dz - 1 \right) \zeta(x) d\hat{\nu}_\varepsilon(x) \, . \end{aligned}$$

By estimates (5.15) on the heat kernel, we have that

$$\begin{aligned}\int_{\Sigma\setminus\breve{\mathbb{I}}_R}\left(\int_{\breve{\mathbb{I}}_{R_0}} p_\varepsilon(x,y)\,|\zeta(\hat{y})|\,dL_g(y)\right)d\nu_\varepsilon(x) &\leq \|\zeta\|_\infty \sup_{x\in\Sigma\setminus\breve{\mathbb{I}}_R}\int_{\breve{\mathbb{I}}_{R_0}} p_\varepsilon(x,y)dL_g(y)\\ &\leq O\left(\frac{e^{-\frac{(R-R_0)^2}{8\theta_\varepsilon}}}{\sqrt{\theta_\varepsilon}}\right)\to 0 \text{ as } \varepsilon\to 0\,.\end{aligned}$$

By estimates (5.13) on the heat kernel, we have that

$$\begin{aligned}\int_{\mathbb{I}_R}\left(\int_{\mathbb{I}_R}|\zeta(z)-\zeta(x)|\,\hat{p}_\varepsilon(z,x)dz\right)d\hat{\nu}_\varepsilon(x) &\leq \sup_{x\in\mathbb{I}_R}\int_{\mathbb{R}}|\zeta(x)-\zeta(z)|\,\frac{e^{-\frac{|x-z|^2}{8\theta_\varepsilon}}}{2\pi\theta_\varepsilon}dz\\ &\to 0 \text{ as } \varepsilon\to 0\end{aligned}$$

since ζ is uniformly continuous on $\mathbb{R}$. Finally, we have by the heat kernel estimate (5.16) that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{I}_{R_0}}\left|\int_{\mathbb{I}_R}\hat{p}_\varepsilon(z,x)dz-1\right|=0\,,$$

so that we get (4.39). We denote by $\hat{\nu}$ the weak star limit of both $\{e^{\hat{u}_\varepsilon}dx\}$ and $\{\hat{\nu}_\varepsilon\}$ in $\mathcal{M}(\mathbb{R})$.

Now, in order to perform the same pointwise estimates as previously we have to be far from singularity points. By rescalings of the points p_i^ε and radii r_i^ε, we only keeping indices $1\leq i\leq s$ such that

$$\frac{r_\varepsilon^i}{\alpha_\varepsilon}\to 0 \text{ and } |a_\varepsilon-p_i^\varepsilon|=O(\alpha_\varepsilon) \text{ as } \varepsilon\to 0\,, \tag{4.40}$$

Denoting I this set of indices of cardinal $\sharp I=t\leq s$, and $\sigma:\{1,\cdots,t\}\to I$ a reordering, we denote

$$\hat{p}_j^\varepsilon=\frac{a_\varepsilon-p_{\sigma(j)}^\varepsilon}{\alpha_\varepsilon} \text{ and } \hat{r}_j^\varepsilon=\frac{r_{\sigma(j)}^\varepsilon}{\alpha_\varepsilon}\,,$$

and assume that up to a subsequence, $\hat{p}_j^\varepsilon$ converges to $\hat{p}_j$ for any j. These points $\hat{p}_j^\varepsilon$ with radius $\hat{r}_i^\varepsilon$ play the same role as p_j^ε and r_j^ε in the previous section. We set for $\rho>0$,

$$\Omega(\rho)=\mathbb{D}^+_{\frac{1}{\rho}}\setminus\bigcup_{j=1}^t\mathbb{D}^+_\rho(\hat{p}_j)\,,$$

which plays the same role as $\Sigma(\rho)$ in the previous section.

We have by Claim 4.2

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{\Omega(\rho)\cap\mathbb{R}\times\{0\}}e^{\hat{u}_\varepsilon}=\int_{\mathbb{R}\times\{0\}}d\hat{\nu}_i\,. \tag{4.41}$$

We aim at getting regularity estimates on $\hat{\Phi}_\varepsilon$ and $e^{\hat{u}_\varepsilon}$ in $\Omega(\rho)$, thanks to (4.41), and following the proof of Claim 4.5 and Claim 4.6. We get:

Claim 4.9. *We assume that $\hat{\nu}_i\neq 0$ in Claim 4.2. We have the following*

- *Estimates on $\hat{\Phi}_\varepsilon$, there are $C_1 > 0$ and $C_2 > 0$ such that*

$$\forall \rho > 0, \limsup_{\varepsilon \to 0} \left\| \hat{\Phi}_\varepsilon \right\|_{W^{1,2}(\Omega(\rho))} \leq C_1 \, , \tag{4.42}$$

$$\forall \rho > 0, \limsup_{\varepsilon \to 0} \left\| \hat{\Phi}_\varepsilon \right\|_{C^0(\Omega(\rho))} \leq C_2 \, . \tag{4.43}$$

- *Quantitative non-concentration estimates on $e^{\hat{u}_\varepsilon}$ and $|\nabla \hat{\Phi}_\varepsilon|^2$: there are $D_2 > 0$ and $D_1 > 0$ such that*

$$\forall \rho > 0, \forall r > 0, \limsup_{\varepsilon \to 0} \sup_{x \in \Omega(\rho)} \int_{\mathbb{I}_r(x)} e^{\hat{u}_\varepsilon} \leq \frac{D_1}{\ln(\frac{1}{r})} \, , \tag{4.44}$$

$$\forall \rho > 0, \forall r > 0, \limsup_{\varepsilon \to 0} \sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_r(x)} |\nabla \hat{\Phi}_\varepsilon|^2 \leq \frac{D_2}{\sqrt{\ln(\frac{1}{r})}} \, . \tag{4.45}$$

4.7.2. *Global pointwise bound of eigenfunctions.* What is slightly different to the non-concentrated case is that we need a pointwise estimate of $\{\Phi_\varepsilon\}$ on the whole surface in order to perform in Claim 4.12, the same estimates as in the proof of Claim 4.7. Indeed, since the heat kernel is globally defined on the whole surface, we have to be sure that the integral estimates on the thin parts stay bounded in order to get the expected sharp pointwise inequality on the eigenfunctions in the bubble, we proved in Claim 4.7 on the surface.

In this section, we aim at proving that up to a subsequence, we have a uniform bound for the sequence of eigenfunctions on a large global surface $\Sigma_{\rho,\varepsilon}$ defined below in (4.46), far from singularity points.

Claim 4.10. *For any $\rho > 0$, there is a constant $C_0(\rho) > 0$ such that*

$$\forall x \in \Sigma_{\varepsilon,\rho}, |\Phi_\varepsilon|(x) \leq C_0(\rho) \left| \ln \left(1 + \left| \frac{\alpha_\varepsilon}{d_g(x, a_\varepsilon)} \right| \right) \right| \, .$$

The proof is based on the estimates of Claim 4.5. We formulate in Claim 4.11 below what we need. One can also follow the lines of Claim 11 in the original paper [Pet19].

First, we will only be able to give a uniform pointwise bound far from singularity points on all the dyadic annuli centered at a_ε at the intermediate scales between α_ε and δ_0. We denote by $\{\omega_k^\varepsilon\}_{k=0,\cdots,k_0,k_0+1}$ (with $k_0 \geq 0$) a family of sequences such that $\omega_0^\varepsilon = \alpha_\varepsilon$, $\omega_{k_0+1}^\varepsilon = \delta_0$ and for any $k \in \{0, \cdots, k_0\}$,

$$\frac{\omega_k^\varepsilon}{\omega_{k+1}^\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \, ,$$

where the set

$$I_k = \left\{ i \in \{1, \cdots s\}; \frac{|p_\varepsilon^i - a_\varepsilon|}{\omega_k^\varepsilon} = O(1) \text{ and } \frac{r_\varepsilon^i}{\omega_k^\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \right\}$$

is non empty for any $k \in \{1, \cdots, k_0\}$, and the indices $i \in I \setminus \bigcup_{k=1}^{k_0} I_k$ satisfy

- either $|p_i^\varepsilon - a_\varepsilon| \to 0$, $\frac{\alpha_\varepsilon}{|p_i^\varepsilon - a_\varepsilon|} \to 0$ and $\frac{|p_i^\varepsilon - a_\varepsilon|}{r_i^\varepsilon}$ as $\varepsilon \to 0$,
- or $|p_i^\varepsilon - a_\varepsilon| = O(\alpha_\varepsilon)$ as $\varepsilon \to 0$,
- or $d_g(\bar{a}, \bar{p}_i)$ is lower bounded as $\varepsilon \to 0$.

It is easy to prove by induction that such a family $\{\omega_k^\varepsilon\}$ exists. Up to reorder with a bijection $\sigma_k : \{1, \cdots, l_k\} \to I_k$, we denote by $\overline{p_\varepsilon}^{k,l} = \frac{p^\varepsilon_{\sigma_k(l)} - a_\varepsilon}{\omega_k^\varepsilon}$. Let $R_0 > 0$ be such that for any ε, k, l, $\overline{p_\varepsilon}^{k,l} = \frac{p^\varepsilon_{\sigma_k(l)} - a_\varepsilon}{\omega_k^\varepsilon} \leq R_0$. Up to a subsequence we assume that $\overline{p_\varepsilon}^{k,l} \to \overline{p}^{k,l}$ as $\varepsilon \to 0$.

For $\rho > 0$ and $\varepsilon > 0$, we set

$$\Sigma_{\rho,\varepsilon} = \Sigma \setminus S_{\rho,\varepsilon} \, , \tag{4.46}$$

where S_ε is the singularity set

$$S_{\rho,\varepsilon} = \left(\bigcup_{k=0}^{k_0} \bigcup_{l=1}^{l_k} \overline{\omega_\varepsilon^k \mathbb{D}_\rho^+ \left(\overline{p}^{k,l} \right) + a_\varepsilon} \right) \cup \bigcup_{i \in I_{k_0+1}} B_g^+(\bar{p}_i, \rho) \, ,$$

where we denote by $I_0 := I$ defined as indices i satisfying (4.40) and I_{k_0+1} are indices i satisfying that $d_g(\bar{p}_i, \bar{a})$ is lower bounded by a positive constant.

We fix $0 < \rho < 1$. In order to prove Claim (4.10), we need a key pointwise estimate on annuli $\gamma_\varepsilon \mathbb{A}^+_{\frac{R_0}{\rho}} = \mathbb{D}^+_{\frac{\gamma_\varepsilon R_0}{\rho}} \setminus \mathbb{D}^+_{\rho \frac{\gamma_\varepsilon}{R_0}}$ around $\bar{p}_i$ (where R_0 is defined just before the definition (4.46) of $\Sigma_{\varepsilon,\rho}$): either the sequence (Φ_ε) is bounded in $\mathbb{U}_\rho = \mathbb{A}^+_{\frac{R_0}{\rho\gamma_\varepsilon}} \setminus \overline{S_{\rho,\varepsilon}}^{\gamma_\varepsilon}$ or it satisfies a Harnack inequality on this set. Notice that we very often have $\overline{S_{\rho,\varepsilon}}^{\gamma_\varepsilon} = \emptyset$. It is only non empty at the scales $\omega_\varepsilon^{i,j}$.

Claim 4.11. *We have that for $1 \leq i \leq s$, $1 \leq j \leq t_i$, there is $B(\rho) > 0$ such that for all eigenfunction ϕ_ε as a coordinate of $\overline{\Phi_\varepsilon}^{\gamma_\varepsilon}$ (Φ_ε is the map in Claim 4.1), and for all sequence $2R_0\sqrt{\varepsilon} < \gamma_\varepsilon < \frac{\delta_0}{2R_0}$, either*

$$\forall x \in \mathbb{U}_\rho, |\phi_\varepsilon(x)| \leq B(\rho)$$

or

$$\forall x, y \in \mathbb{U}_\rho, \frac{|\phi_\varepsilon(y)|}{B(\rho)} \leq |\phi_\varepsilon(x)| \leq B(\rho) \, |\phi_\varepsilon(y)| \, .$$

Now we set

$$m_\varepsilon(t) = \frac{1}{\pi t} \int_{\partial \mathbb{D}_t^+(a_\varepsilon)} \Phi_\varepsilon(x) dl(x)$$

the mean value on half circles centered at a_ε of Φ_ε. We classically have

$$m_\varepsilon(t) = f_\varepsilon(\alpha_\varepsilon) - \int_{\alpha_\varepsilon}^t \frac{\int_{\partial \mathbb{D}_s} \partial_\nu \Phi_\varepsilon(u) du}{2\pi s} ds \, .$$

Therefore, since $\partial_\nu \Phi_\varepsilon$ is bounded in L^1, we have a classical log estimate on m_ε

$$|m_\varepsilon(t)| \leq |f_\varepsilon(\alpha_\varepsilon)| + \ln \left(\frac{t}{\alpha_\varepsilon} \right) \, . \tag{4.47}$$

Since by Claim 4.9, $|f_\varepsilon(\alpha_\varepsilon)|$ is uniformly bounded, by Claim 4.11 and (4.47), we get Claim 4.10.

4.7.3. $W^{1,2}$-*estimates on eigenfunctions.* Now, using the estimates by Claim 4.10, we can adapt the proof of Claim 4.7 on the bubble scale (follow the lines of Claim 12 in [Pet19])

Claim 4.12. *We have for any $\rho > 0$, there exists $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that*

$$\forall x \in \Omega(\rho) \cap (\mathbb{R} \times \{0\}) \, , \left| \hat{\Phi}_\varepsilon \right|^2_{\mathcal{E}_\varepsilon} (x) \geq 1 - \beta_\varepsilon \tag{4.48}$$

and

$$\forall x \in \Omega(\rho) \cap supp\,(\hat{\nu}_\varepsilon) \, , \left| \left| \hat{\Phi}_\varepsilon \right|^2_{\mathcal{E}_\varepsilon} (x) - 1 \right| \leq \beta_\varepsilon \, . \tag{4.49}$$

Then one following the lines of Claim 4.7 can pass to the limit in $W^{1,2}(\Omega(\rho))$ for any $\rho > 0$.

Claim 4.13. *Up to the extraction of a subsequence of $\{\hat{\Phi}_\varepsilon\}$, there is a map $\hat{\Phi} \in W^{1,2}(\mathbb{R}^2_+)$ such that for any $\rho > 0$*

$$\hat{\Phi}_\epsilon \to \hat{\Phi} \text{ in } W^{1,2}(\Omega(\rho)) \, . \tag{4.50}$$

Moreover the limiting function satisfies $\hat{\Phi} \in \mathcal{C}^{0,\alpha}(\Omega(\rho))$ and $\left|\hat{\Phi}\right|^2 = 1$ in $\mathbb{R} \times \{0\} \backslash \{\hat{p}_1, \cdots, \hat{p}_t\}$ and the equation $\Delta \hat{\Phi} = 0$ and $\partial_\nu \Phi = \sigma \hat{\Phi} d\hat{\nu}$ on $\mathbb{R} \times \{0\}$ in a weak sense on $\mathbb{R}^2_+$.

Finally, let's prove Theorem 1.2 with $\hat{\nu}_i \neq 0$ in Claim 4.2. We have that $\left|\hat{\Phi}\right|^2 = 1$ everywhere except maybe on a finite number of points. Since we have that $\Delta \hat{\Phi} = 0$ and $\partial_\nu \hat{\Phi} = \sigma \hat{\Phi} \nu$ in a weak sense, then $\partial_\nu \hat{\Phi} \in \left(T_{\hat{\Phi}} \mathcal{E}\right)^\perp$ which is exactly the asumption of being a weakly free boundary harmonic map into the ellipsoid $\mathcal{E}$. Therefore $\hat{\Phi}$ is a smooth map and satisfies the free boundary harmonic map equation in $\mathcal{E}$ in a strong sense. Now, with $\partial_\nu \hat{\Phi} = \sigma \hat{\Phi} \hat{\nu}$ again, we have that

$$\hat{\nu} = (\Phi . \partial_\nu \Phi) \, Leb \, ,$$

which means that $\hat{\nu}$ is absolutely continous with respect to *Leb* with a smooth density. Up to conformal diffeomorphism of the half plane to the disk and point removability of finite energy free boundary harmonic maps, we complete the proof of Theorem 1.2.

5. Appendix

5.1. **Classical oscillation estimates for harmonic functions.** We have (see e.g [Sch06, JLZ19] for a more general result)

Proposition 5.1. *Let M be a compact submanifold of $\mathbb{R}^m$. If $u \in W^{1,2}(\mathbb{D}_+, \mathbb{R}^m)$ is a harmonic map such that $u(I) \subset M$, we have,*

$$\sup_{x \in \mathbb{D}^+_{\frac{1}{2}}} dist(u(x), M) \leq C \left(\int_{\mathbb{D}_+} |\nabla u|^2 \right)^{\frac{1}{2}} \, ,$$

$$\sup_{x,y \in \mathbb{D}^+_{\frac{1}{4}}} |u(x) - u(y)| \leq C \left(\int_{\mathbb{D}_+} |\nabla u|^2 \right)^{\frac{1}{2}} \, .$$

5.2. **Estimates on free boundary ω^2-harmonic maps.** We give a ε-regularity result on free boundary ω^2-harmonic maps into an ellipsoid $\mathcal{E} \subset \mathbb{R}^m$, and we explain how to prove (5.11). Let ω be a $W^{1,2}$-function which satisfies the inequalities

$$\frac{1}{C_\omega} \leq \omega \leq C_\omega \, . \tag{5.1}$$

Let M be a compact submanifold of $\mathbb{R}^m$. We assume for simplicity that $dim(M) = m-1$, and that we have a global normal vector $\vec{N}$. This is the case for ellipsoids. For spheres, we refer to [LP17]. For more general results, see [JLZ19]. At a neighbourhood U of M, we can define the orthonormal projection $p : U \to M$, and the symmetry with respect to M, $s = 2p - id$. Let $u \in W^{1,2}(\mathbb{D}_+, M)$ a free boundary ω^2-harmonic map as defined in section 3. It satisfies the equation

$$\begin{cases} -div(\omega^2 \nabla u) = 0 & \text{in } \mathbb{D}_+ \\ \partial_2 u = (\partial_2 u . N(u))\, N(u) & \text{on } [-1,1] \times \{0\} \, . \end{cases} \tag{5.2}$$

in a weak sense. We assume that $u(x) \in U$ a.e for x in $\mathbb{D}_+$, up to reduce the energy thanks to Claim (5.1) and we set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{D}_+ \\ s \circ u \circ \rho(x) & \text{if } x \in \mathbb{D}_- \, , \end{cases} \tag{5.3}$$

where $\rho(x_1, x_2) = (x_1, -x_2)$. Then we easily check that $\tilde{u}$ satisfies in a weak sense the following equation in $\mathbb{D}$ thanks to (5.2)

$$-div(A . \nabla \tilde{u}) = 0 \text{ in } \mathbb{D} \, , \tag{5.4}$$

where $A \in W^{1,2}(\mathbb{D}, Gl_n(\mathbb{R}))$ such that $A \in L^\infty$ and $A^{-1} \in L^\infty$ is defined by

$$A(x) = \begin{cases} \omega^2 id & \text{in } \mathbb{D}_+ \\ \omega^2 Ds(s^{-1}(x)) & \text{in } \mathbb{D}_- \, . \end{cases} \tag{5.5}$$

Working with $\tilde{u}$, we easily prove from classical Wente-type equations the following ε-regularity result on u:

Proposition 5.2. *Let $\delta > 0$, $p < +\infty$. There is $\varepsilon_0 := \varepsilon_0(M, \delta) > 0$, and a constant $C := C(p, M, \delta)$ such that if ω satisfies (5.1) with $C_\omega \leq \delta$, then any weak ω^2-harmonic map $u \in W^{1,2}(\mathbb{D}^+, M)$ with*

$$\int_{\mathbb{D}^+} |\nabla u|^2 + \int_{\mathbb{D}^+} \left|\nabla \left(\omega^2\right)\right|^2 \leq \varepsilon_0 \, ,$$

satisfies that for any $q \in (1,2)$, $u \in W^{2,q}(\mathbb{D}^+_{\frac{1}{2}})$ and for any $1 \leq p < +\infty$

$$\|\nabla u\|_{L^p(\mathbb{D}^+_{\frac{1}{2}})} \leq C \, \|\nabla u\|_{L^2(\mathbb{D}^+)} \, . \tag{5.6}$$

For more results, see theorem V.3 of [Riv12]. The bootstrap stops at $W^{2,q}$ with $q < 2$ since we do not have more assumption about the regularity of ω^2. In this case, then there is not a priori ε-regularity for the L^∞-norm of the gradient of u. Therefore, we cannot use

directly the techniques in [LP19] to short-cut the proof of Proposition 3.1. More precisely with a L^∞ version of (5.6) for $u = \widetilde{\Psi}$, we would prove directly that

(5.7)
$$I_1 = \int_I |\Phi - \Psi|^2_{\mathcal{E}} \, \tilde{n}.\partial_\nu \widetilde{\Psi} \leq \int_I |\Phi - \Psi|^2_{\mathcal{E}} \left|\partial_\nu \widetilde{\Psi}\right| \leq C \left(\int_{\mathbb{D}_+} |\nabla (\Phi - \Psi)|^2 \right) \left(\int_{\mathbb{D}_+} \left|\nabla \widetilde{\Psi}\right|^2 \right)^{\frac{1}{2}} .$$

from (3.9) thanks to a classical Hardy inequality.

In our context, we have that $|\nabla u|^2$ belongs to the local Hardy space $h^1(\mathbb{D})$ since it is controlled by a Jacobian:

$$|\nabla \tilde{u}|^2 \leq CC_\omega \nabla \tilde{u}.A\nabla \tilde{u} = CC_\omega \nabla \tilde{u}.\nabla^\perp \eta \, , \tag{5.8}$$

where $A\nabla \tilde{u} = \nabla^\perp \eta$ is the Hodge decomposition of $A\nabla \tilde{u}$ and we get:

$$\left\| |\nabla \tilde{u}|^2 \right\|_{h^1(\mathbb{D})} \leq CC_\omega \left\| \nabla \tilde{u} \right\|_{L^2(\mathbb{D})} \left\| \nabla \eta \right\|_{L^2(\mathbb{D})} \, , \tag{5.9}$$

For more general results see Lamm-Lin [LL13]. We also use the following Claim in [LL13]

Claim 5.1. *$Q \in h^1(\mathbb{D})$ such that $Q \geq 0$ a.e. Then there is a solution $f \in W^{1,2}_0(\mathbb{D}) \cap L^\infty(\mathbb{D})$ such that*

$$\begin{cases} \Delta f = Q & \text{in } \mathbb{D} \\ f = 0 & \text{on } \partial \mathbb{D} \, , \end{cases} \tag{5.10}$$

satisfying

$$\|f\|_{L^\infty} + \|\nabla f\|_{L^2} \leq C \|Q\|_{h^1(\mathbb{D})} \, .$$

Then we set $Q = \left|\nabla \widetilde{\Psi}\right|^2 \in h^1(\mathbb{D})$ in order to use the Claim with the formula (5.9). We extend the free-boundary ω^2-harmonic map $\widetilde{\Psi}$ on $\mathbb{D}$ by (5.3) and we get thanks to [CM08] that if there is a map $\widetilde{\Phi}$ such that $\widetilde{\Phi} = \widetilde{\Psi}$ on $\mathbb{I}$, extended by $\widetilde{\Phi}$ on $\mathbb{D}_-$, and if we have $\int_{\mathbb{D}} \left|\nabla \widetilde{\Psi}\right|^2 \leq C \int_{\mathbb{D}_+} \left|\nabla \widetilde{\Psi}\right|^2 \leq \varepsilon_0$ for ε_0 sufficiently small,

$$\int_{\mathbb{D}} \left|\widetilde{\Psi} - \widetilde{\Phi}\right|^2 \left|\nabla \widetilde{\Psi}\right|^2 \leq C\sqrt{\varepsilon_0} \int_{\mathbb{D}} \left|\nabla \left(\widetilde{\Psi} - \widetilde{\Phi}\right)\right|^2 \, . \tag{5.11}$$

5.3. **Uniform estimates on the heat kernel.** The heat kernel $p_\varepsilon(x,y)$ at time $\varepsilon > 0$ of the boundary of a compact Riemannian Σ, g is defined on a union of circles $(\partial\Sigma, g)$. At time $\varepsilon > 0$ it satisfies uniform bounds: there is $A_0 > 1$ and $\rho > 0$ such that for any $\varepsilon > 0$,

$$\forall x, y \in \Sigma, \frac{1}{A_0\sqrt{4\pi\varepsilon}} e^{-\frac{d_g(x,y)^2}{4\varepsilon}(1-\rho)} \leq p_\varepsilon(x,y) \leq \frac{A_0}{\sqrt{4\pi\varepsilon}} e^{-\frac{d_g(x,y)^2}{4\varepsilon}(1+\rho)} \, . \tag{5.12}$$

We also have uniform bounds on the local rescaled heat kernel $\hat{p}_\varepsilon(x,y)$ by some parameters $a_\varepsilon \in \mathbb{R}$ and $\alpha_\varepsilon > 0$ such that $a_\varepsilon \to a \in \mathbb{R}$ and $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. We refer to the notations used in the section 4.2. We have for any $R > 0$, for any fixed $0 < \rho < 1$

$$\frac{e^{-\frac{|y-z|^2}{4\theta_\varepsilon}(1+\rho)}}{\sqrt{4\pi\theta_\varepsilon}}(1-\rho) \leq \hat{p}_\varepsilon(z,y) \leq \frac{e^{-\frac{|y-z|^2}{4\theta_\varepsilon}(1-\rho)}}{\sqrt{4\pi\theta_\varepsilon}}(1+\rho) \tag{5.13}$$

uniformly on $\mathbb{I}_R \times \mathbb{I}_R$, where $\theta_\varepsilon = \frac{\varepsilon}{e^{2\check{v}_l(a)}\alpha_\varepsilon^2}$, for all $\varepsilon > 0$ small enough. For a sequence of measures $\nu_\varepsilon \in \mathcal{M}(\Sigma)$, we also have uniform bounds for $R > r > 0$ and $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$:

$$\sup_{x \in \mathbb{D}_{R-r}} \int_{\Sigma \setminus \breve{\mathbb{D}}_R} \alpha_\varepsilon^2 p_\varepsilon(\breve{x}, y) d\nu_\varepsilon(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}}{\sqrt{\theta_\varepsilon}} \right) . \tag{5.14}$$

We prove it thanks to (5.12) and (5.13). Let $x \in \mathbb{D}_{R-r}$ and let us write that

$$\begin{aligned}
\alpha_\varepsilon \int_{\Sigma \setminus \breve{\mathbb{I}}_R} p_\varepsilon(\breve{x}, y) d\nu_\varepsilon(y) &= e^{-v_l(\breve{x})} \int_{\mathbb{I}_{C_0^2 R} \setminus \mathbb{I}_R} \hat{p}_\varepsilon(x, z) d\hat{\nu}_\varepsilon(z) + \int_{\partial\Sigma \setminus \breve{\mathbb{I}}_{C_0^2 R}} \alpha_\varepsilon p_\varepsilon(\breve{x}, y) d\nu_\varepsilon(y) \\
&\leq C_0^2 \int_{\mathbb{I}_{C_0^2 R} \setminus \mathbb{D}_R} \frac{e^{-\frac{|x-z|^2}{8\theta_\varepsilon}}}{\sqrt{\pi\theta_\varepsilon}} d\hat{\nu}_\varepsilon(z) \\
&\quad + \int_{\partial\Sigma \setminus I_g(\bar{a}_\varepsilon, \frac{\alpha_\varepsilon C_0^2 R}{C_0})} \frac{\alpha_\varepsilon A_0}{\sqrt{4\pi\varepsilon}} e^{-\frac{d_g(\breve{x}, y)^2}{4\varepsilon}} d\nu_\varepsilon(y) \\
&\leq O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}}{\sqrt{\theta_\varepsilon}} \right) + \frac{A_0 \alpha_\varepsilon}{\sqrt{4\pi\varepsilon}} e^{-\frac{\alpha_\varepsilon^2 (R-r)^2}{4\varepsilon}} ,
\end{aligned}$$

where $\breve{\mathbb{I}}_r \subset I_g(\bar{a}_\varepsilon, \alpha_\varepsilon C_0 r) \subset I_g(\bar{a}_\varepsilon, \alpha_\varepsilon C_0 R)$. This proves (5.14). We also have that

$$\sup_{x \in \Sigma \setminus \breve{\mathbb{I}}_R} \int_{\breve{\mathbb{I}}_r} p_\varepsilon(x, y) dA_g(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}}{\theta_\varepsilon} \right) . \tag{5.15}$$

Let $x \in \Sigma \setminus \breve{\mathbb{I}}_R$. We assume that $x \in \mathbb{I}_{C_0^2 R} \setminus \mathbb{I}_R$. We write that

$$\int_{\breve{\mathbb{I}}_r} p_\varepsilon(x, y) dL_g(y) = \int_{\mathbb{I}_r} \hat{p}_\varepsilon(z, \breve{x}) dl \leq \frac{1}{\sqrt{\pi\theta_\varepsilon}} \int_{\mathbb{I}_r} e^{-\frac{|x-z|^2}{8\theta_\varepsilon}} dl \leq \frac{r}{\sqrt{\theta_\varepsilon}} e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}$$

if ε is small enough. If $x \in \Sigma \setminus \breve{\mathbb{D}}_{C_0^2 R} \subset \Sigma \setminus B_g(\bar{a}_\varepsilon, \alpha_\varepsilon R C_0)$, we write that

$$\begin{aligned}
\int_{\breve{\mathbb{I}}_r} p_\varepsilon(x, y) dL_g(y) &\leq \int_{I_g(\bar{a}_\varepsilon, \alpha_\varepsilon C_0 r)} p_\varepsilon(x, y) dL_g(y) \\
&\leq \frac{A_0}{\sqrt{4\pi\varepsilon}} \int_{I_g(\bar{a}_\varepsilon, \alpha_\varepsilon C_0 r)} e^{-\frac{d_g(x,y)^2}{4\varepsilon}} dL_g(y) \\
&\leq O\left(\frac{e^{-\frac{\alpha_\varepsilon^2 (R-r)^2}{4\varepsilon}}}{\sqrt{\theta_\varepsilon}} \right) .
\end{aligned}$$

This proves 5.15. Now let's prove that

$$\lim_{R \to +\infty} \limsup_{\varepsilon \to 0} \sup_{x \in \mathbb{D}_r} \left| \int_{\mathbb{D}_R} \hat{p}_\varepsilon(z, x) dz - 1 \right| = 0 . \tag{5.16}$$

We fix $0 < \rho < \frac{1}{2}$ and $R > 0$. Then for ε small enough, we have by (5.13) that

$$\int_{\mathbb{I}_R} \hat{p}_\varepsilon(z,x)dz \leq \int_{\mathbb{R}} \frac{e^{-\frac{|x-z|^2(1-\rho)}{4\theta_\varepsilon}}}{\sqrt{4\pi\theta_\varepsilon}}(1+\rho)dz = \frac{1+\rho}{\sqrt{1-\rho}}$$

for any $x \in \mathbb{I}_r$ and that

$$\begin{aligned}\int_{\mathbb{I}_R} \hat{p_\varepsilon}(z,x)dz &\geq \int_{\mathbb{I}_R} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\varepsilon}}}{\sqrt{4\pi\theta_\varepsilon}}(1-\rho)dz \\ &\geq \int_{\mathbb{R}} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\varepsilon}}}{\sqrt{4\pi\theta_\varepsilon}}(1-\rho)dz - \int_{\mathbb{R}\setminus\mathbb{I}_R} \frac{e^{\frac{-|x-z|^2}{8\theta_\varepsilon}}}{\sqrt{2\pi\varepsilon}}dz \\ &\geq \frac{1-\rho}{\sqrt{1+\rho}} + o(1) \text{ as } \varepsilon \to 0\end{aligned}$$

uniformly on $\mathbb{I}_r$. Letting $\varepsilon \to 0$, then $R \to +\infty$ and then $\rho \to 0$ gives (5.16).

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