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EXTREMAL METRICS FOR COMBINATIONS OF LAPLACE EIGENVALUES AND MINIMAL SURFACES INTO ELLIPSOIDS

ROMAIN PETRIDES

Abstract. We give a variational method for existence and regularity of metrics which minimize combinations of eigenvalues of the Laplacian among metrics of unit area on a surface. We show that there are minimal immersions into ellipsoids parametrized by eigenvalues, such that the coordinate functions are eigenfunctions with respect to the minimal metrics. This work generalizes the author's maximization for one eigenvalue among metrics of unit area on a surface related to minimal surfaces into spheres. This work also generalizes the previous characterizations of critical metrics for one eigenvalue to any combination of eigenvalues from target spheres to target ellipsoids.

1. Introduction

Let Σ be a compact connected surface without boundary. We denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_m(g) \to +\infty$$

the eigenvalues with respect to the laplacian $\Delta_g = -div_g(\nabla)$ for a Riemannian metric g on Σ. Obtaining sharp bounds on Riemannian quantities depending on the eigenvalues λ_m with respect to the metric g has been the subject of intensive studies in various fields over the past decades. Because of the scaling properties of the eigenvalues $\lambda_m(\alpha g) = \alpha^{-1}\lambda_m(g)$, one can hope for relevant bounds fixing the area $A_g(\Sigma) = 1$ of the surface. With this assumption, the sharp lower bound for any eigenvalue is 0, using the classical "Cheeger dumbbells" as minimizing sequences - creating as many "disconnections" of the surface as needed. The classical Hersch result [Her70] gave the upper bound 8π for the first eigenvalue on the sphere, only realized by the round sphere. Then Berger [Ber73, Remark 2.6] asked the question whether the supremum of the first eigenvalue among metrics of area 1 is finite and if on tori, this supremum is realized by the flat equilateral torus. In their celebrated paper [YY80], Yang and Yau gave a bound depending only on the topology of the surface on the first eigenvalue. In particular, the supremum is finite. After that, Yau [Yau82, problem 71] raised the natural question about the finiteness of the supremum of the following eigenvalues, $\lambda_m(g)$ among metrics of area 1, later proved by Korevaar [Kor93].

Thanks to these results, one can expect by variational methods to maximize eigenvalues among surfaces of fixed area. Nadirashvili [Nad96] answered the question by Berger proving that the flat equilateral torus maximizes the first eigenvalue among tori. Moreover, the Euler-Lagrange equation associated to this variational problem reveals that the maximal metrics are induced metrics of a minimal immersion into a sphere. This makes connection to many problems about minimal surfaces into spheres (see for instance [Cal67, LY82, MR86]...), while it gives regularity of critical configurations for eigenvalues. One might naturally ask for existence of maximizers for any topology and any m. After many investigations for low topology (see discussions below), a general variational method was stated by the author

[Pet14a, Pet18, Pet19]. His work was inspired by [FS16] who noticed the link between Steklov eigenvalues and free boundary minimal surfaces into Euclidean balls. For instance, combining gap results in [MS19a] and the main result in [Pet14a] gives existence of minimal immersions by first eigenfunctions for any topology by maximization of the first eigenvalue.

Now, working with higher eigenvalues, one would prefer to optimize a combination of them instead of maximizing just one for several reasons. As discussed below, the natural eigenvalue functionals initially raised by Berger [Ber73, Proposition 4.22] and Yang and Yau [YY80] are $\lambda_1+\cdots+\lambda_m$, or $\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_m}$. Moreover, many studies ask for interaction properties between lots of eigenvalues and eigenfunctions, even with infinite combinations to capture properties on global Riemannian invariants (see [Sel56, OPS88, BBG94]...). We can also simply remark that maximizing one single eigenvalue may not give a connected maximal configuration if it is not the first one (see [Nad02, Pet14, KNPP19, Kar19b]...). In this case, the maximizers just correspond to maximizers of lower eigenvalues in the connected components, preventing from building new minimal surfaces. This appears to be the general picture: maximizing just one eigenvalue forces some previous eigenvalues to shrink.

In the current paper, we give a much more general and flexible way to obtain sharp bounds and extremal metrics for combination of eigenvalues. We perform a variational procedure on a $\mathcal{C}^1$ function $F:\left(\mathbb{R}_+^\star\right)^m\to\mathbb{R}_+$ with $m\geq 1$, depending on eigenvalues. We set

$$\Lambda_F(\Sigma)=\inf_g F\left(A_g(\Sigma)\lambda_1(g),\cdots,A_g(\Sigma)\lambda_m(g)\right)\ ,$$

where the infimum is taken over all smooth metrics on Σ. We changed the "maximization" point of view into a "minimization" since it is more natural for all the examples we give below, like $\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_m}$. For sums of eigenvalues, we minimize $\frac{1}{\lambda_1+\cdots+\lambda_m}$. We only assume the following condition on F, satisfied for all the functionals we consider in the introduction:

(H) $\quad F$ is a nonincreasing function with respect to all the coordinates

As we notice below, for $1\leq k\leq m$, we naturally introduce

$$\Lambda_{F,k}(\Sigma)=\inf_g F\left(0,\cdots,0,A_g(\Sigma)\lambda_{k+1}(g),\cdots,A_g(\Sigma)\lambda_m(g)\right)\ .$$

If Σ is oriented of genus γ, we denote $\Lambda_F(\gamma):=\Lambda_F(\Sigma)$ and $\Lambda_{F,k}(\gamma):=\Lambda_{F,k}(\Sigma)$. We state

Theorem 1.1. *Assume that Σ is a closed oriented surface of genus $\gamma\geq 0$. Assume that F satisfies (H). If the two following assumptions*

$$\Lambda_F(\gamma)<\Lambda_{F,1}(\gamma) \tag{1.1}$$

and

$$\gamma=0 \text{ or } \Lambda_F(\gamma)<\Lambda_F(\gamma-1) \tag{1.2}$$

hold, then there is a metric g on Σ, realizing the minimum $\Lambda_F(\gamma)$, and g is smooth up to possible conical singularities. Moreover, there is a (possibly branched) conformal minimal immersion $\Phi:(\Sigma,g)\to\mathcal{E}$ into

$$\mathcal{E}=\left\{(x_1,\cdots,x_m)\in\mathbb{R}^{i_1}\times\cdots\times\mathbb{R}^{i_m};\lambda_1\left|x_1\right|^2+\cdots+\lambda_m\left|x_m\right|^2=1\right\}\ ,$$

a Euclidean ellipsoid in $\mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$*, such that the coordinate functions of* Φ_k *are eigenfunctions with respect to* $\lambda_k := \lambda_k(g)$ *for* $1 \leq k \leq m$*. Moreover, denoting for some map* $\Psi : \Sigma \to \mathbb{R}^i$*:* $E_g(\Psi) = \frac{1}{2}\int_\Sigma |\nabla\Psi|_g^2$ *where* $|\nabla\Psi|_g^2 = \sum_{j=1}^i |\nabla\psi_j|_g^2$ *we have*

$$2E_g(\Phi_k) = \lambda_k \int_\Sigma |\Phi_k|^2 \, dA_g = A_g(\Sigma) \frac{\lambda_k \left|\partial_k F(\lambda_1 A_g(\Sigma), \cdots, \lambda_m A_g(\Sigma))\right|}{\sum_{i=1}^m \lambda_i \left|\partial_i F(\lambda_1 A_g(\Sigma), \cdots, \lambda_m A_g(\Sigma))\right|}.$$

This is a generalization of the author's result about maximization of one single eigenvalue $g \mapsto A_g(\Sigma)\lambda_m(g)$ for $m = 1$ [Pet14a] and $m > 1$ [Pet18]. The first assumption (1.1) just prevents the first eigenvalue of minimizing sequences from converging to 0. With this assumption, we always obtain a connected surface at the limit. In this case, Theorem (1.1) always gives a minimizer on the sphere, while $\Lambda_F(\gamma) < \Lambda_F(\gamma - 1)$ is a natural condition to prevent from possible degeneration of minimizing sequences to lower topologies. As already said, for $F(\lambda_1) = \frac{1}{\lambda_1}$, strict inequalities occur for any topology.

We also prove in Theorem 3.1 below, that critical metrics g for combinations of eigenvalues are metrics conformal to the induced metrics of minimal immersions Φ into (Pseudo)-Euclidean ellipsoids, with a conformal factor determined in terms of the coordinate functions of Φ (which are also eigenfunctions with respect to g) (see section 3). With assumption (H) for F, the target manifold is a Euclidean ellipsoid. If only one eigenvalue appears, the target manifold is a sphere. Therefore, we also generalize the characterization of critical metrics for eigenvalues settled by Nadirashvili [Nad96] and then El Soufi-Ilias [ESI00] for one Laplace eigenvalue on closed surfaces, and by Fraser and Schoen [FS16] for critical metrics on one Steklov eigenvalue on surfaces with boundary.

Notice that the eigenvalues appearing in minimal immersion may not be all the eigenvalues appearing in the functional. One might just assume in addition that $(\lambda_1, \cdots, \lambda_m)$ is not a critical point of F to have a non-empty conclusion. More precisely, up to a rearrangement of coordinates functions of Φ associated to the same eigenvalue, we can modify the target ellipsoid

$$\widetilde{\mathcal{E}} = \{(x_1, \cdots, x_s) \in \mathbb{R}^{j_1} \times \cdots \times \mathbb{R}^{j_s}; \mu_1 |x_1|^2 + \cdots + \mu_s |x_s|^2 = 1\}$$

where

$$\{\mu_1, \cdots, \mu_s\} = \{\lambda_i; i \in \{1, \cdots, m\} \text{ and } \partial_i F(\lambda_1 A_g(\Sigma), \cdots, \lambda_m A_g(\Sigma)) \neq 0\}$$

and $j_k \leq mult(\mu_k)$. In particular, the dimension of the target ellipsoid satisfies

$$\max\{2, s-1\} \leq j_1 + \cdots + j_s - 1 \leq \sum_{k=1}^s mult(\mu_k) - 1 \leq m - 1 + mult(\lambda_m) - 1,$$

where the lower bound 2 is necessary because Φ is a (possibly branched) immersion.

A generalization of Theorem 1.1 is given in section 2 (Theorem 2.1). We presented Theorem 1.1 first because it is sufficient for the applications of the introduction and the general Theorem 2.1 gives minor improvements for the existence question. In the following, we discuss the motivations and applications for these theorems.

Remark 1. *Notice that in the current paper, we favor the minimization of functionals* $g \mapsto F(\lambda_1(g)A_g(\Sigma), \cdots, \lambda_m(g)A_g(\Sigma))$ *under all the metrics rather than the minimization of* $g \mapsto F(\lambda_1(g), \cdots, \lambda_m(g))$ *under the metrics of fixed area. These problems are exactly the same when the map* F *is homogeneous while it does not a priori give the same extremal metrics otherwise. All the techniques used in the current paper are also adaptable for the*

second family of problem: thanks to the use of Lagrange multipliers associated to the area constraint on admissible metrics $A_g(\Sigma) = A_0$, we can notice that the minimal metrics are still associated to harmonic/minimal maps into ellipsoids. The second family of problems seems to be more convenient in the context of minimization of the trace of the heat kernel we raise in Section 1.4, while the first family of problems is convenient for all the following examples of applications given in the introduction.

1.1. **Maximization of one eigenvalue.** In the previous papers [Pet14a] and [Pet18], the author stated a general program for the following maximization problem

$$\Lambda_m(\gamma) = \sup_g A_g(\Sigma)\lambda_m(g)$$

for $m \geq 1$, where the supremum is taken among all Riemannian metrics on Σ (oriented of genus γ), computing by induction $\Lambda_m(\gamma)$ and the associated extremal metrics by induction, because one of the following case has to occur

- There is a maximal metric on Σ. In this case, the metric arises as the induced metric of a possibly branched minimal immersion into a Euclidean sphere or
- $\Lambda_m(\gamma)$ can be computed as a sum of previous values $\Lambda_{m_i}(\gamma_i)$ for $\sum_{i=1}^s m_i = m$ and $\sum_{i=1}^s \gamma_i \leq \gamma$ (and $\gamma_1 < \gamma$ if $s = 1$) naturally arising as the maximal m-th eigenvalue of a new surface without boundary (not necessarily connected), obtained by cutting Σ along a finite number of closed curves.

It turns out that both alternatives can occur thanks to these typical examples

- For $k = 1$, [MS19a] proved that if a surface Σ_γ of genus γ is realized by a maximal metric g, then by a very clever and beautiful glueing around two points on Σ_γ with a thin cylinder, one can build a perturbated metric g_ε on a surface $\Sigma_{\gamma+1}$ of genus $\gamma + 1$, such that $A_{g_\varepsilon}(\Sigma_{\gamma+1})\lambda_1(g_\varepsilon) > A_g(\Sigma_\gamma)\lambda_1(g)$. By induction, this result combined with the author's result, and Hersch's result for the initialization, the proof of existence of a maximal metric for the first eigenvalue is complete for any γ. Another recent similar result is given in the Steklov case in [MP20].
- For $\gamma = 0$, on the sphere, $\Lambda_m(\mathbb{S}^2)$ is never realized by a smooth metric for $m \geq 2$ and we deduce by the other alternative that $\Lambda_m(\mathbb{S}^2) = 8\pi m$ for any $m \geq 1$. Indeed, we know by [CES03] that $\Lambda_m(\mathbb{S}^2) \geq \Lambda_{m-1}(\mathbb{S}^2) + 8\pi$ and by induction that $\Lambda_m(\mathbb{S}^2) \geq 8\pi k$. Moreover, if there is a maximal metric g associated to $\Lambda_m(\mathbb{S}^2)$, then g is the induced metric of a minimal immersion into a Euclidean unit sphere, where the coordinate functions are associated to the m-th eigenvalue $\lambda_m(g) = 2$, and we must have $\lambda_{m-1}(g) < 2$. It is the case by Hersch theorem [Her70] for $m = 1$. In [KNPP19] the four authors remarked that it is not possible when $m \geq 2$ thanks to previous results by Ejiri [Eji98]: any minimal 2-sphere into a Euclidean sphere has a degree d such that $A_g(\mathbb{S}^2) = 4\pi d$ and $m \geq d + 1$ as soon as $d > 1$, so that since $8\pi d = A_g(\mathbb{S}^2)\lambda_m(g) \geq 8\pi m$, we have $d > 1$ and

  $$8\pi d = A_g(\mathbb{S}^2)\lambda_m(g) = \Lambda_m(0) \geq 8\pi m \geq 8\pi(d + 1)$$

  leads to a contradiction. By beautiful refined estimates on the so-called spectral index of harmonic maps, the same kind of result occurs on the projective plane $\Lambda_m(\mathbb{RP}^2) = 4\pi(2m + 1)$ (see [Kar19b])

Both applications led to significative improvements in a long story of investigations. For $k = 1$, Hersch [Her70] proved that $\Lambda_1(0) = 8\pi$ is only reached on the sphere. Nadirashvili

solved the Berger problem for the first eigenvalue on tori, proving that $\Lambda_1(1) = \frac{8\pi^2}{\sqrt{3}}$ only realized by the flat equilateral torus [Nad96]. It is also proved $\Lambda_1(2) = 16\pi$ is realized by an infinite set of well described metrics [JLNNP05, NS19]. We do not know the explicit maximal surfaces for $\gamma \geq 3$ but it is proved in [Kar19a] that they cannot arise from the upper bound on the first eigenvalue by Yang and Yau (see below), countrary to genus 0 and 2. Similar results were proved in the non-orientable case on the projective space [LY82] and on the Klein bottle [JNP06, ESGJ06, CKM19]. For $\gamma = 0$, beyond the result by Hersch for $m = 1$, there is a test function method to prove that $\Lambda_2(0) = 16\pi$ is not realized (see [Nad02, Pet14]). The following results for the sphere and projective plane [NS17, NP18] were completed in [KNPP19, Kar19b] for any m. The use of [Pet18], [MS17] and [MS19a] completes the proof of existence of a maximal metric for the first eigenvalue for any topology of Σ.

The question whether there is a maximal metric on Σ for $m \geq 2$ and $\gamma \geq 1$ or not is still open, but it is likely that it is not the case at least for a large m when γ is fixed, so that we should be able to deduce lots of sharp bounds for eigenvalues. This maximizing method for $m \geq 2$ is not the best way to build other new minimal surfaces, because of the topological degenerations (disconnection, loss of genus) of maximal configurations. Thanks to the new approach in the current paper, we relax the target manifold from a Euclidean sphere to a (Pseudo-)Euclidean ellipsoid, which gives much more possibilities to describe extremal behaviours on eigenvalues.

1.2. **Non-planar minimal spheres into 3-ellipsoids.** A problem raised by Yau [Yau87] in section 4, asked for existence of non-planar embedded spheres into 3-ellipsoids. This result is a priori surprising since a classical result by Morse stated that some ellipsoids contain at most three embedded closed geodesics: the equatorial ones. However, in higher dimension, fruitful works initiated by Marques and Neves [MN17] lead to the general resolution of the following famous Yau's conjecture by min-max methods (see Song [Son18]): any manifold of dimension 3 admits infinitely many smoothly embedded minimal surfaces.

In this context, for ellipsoids, we look for many numbers of embedded minimal *spheres*. Of course, ellipsoids contain at least 4 planar embedded spheres (equatorial spheres) but one might expect more ones, which are non-planar. This question, left open for years was recently solved by Haslhofer and Ketover [HK19]: sufficiently elongated ellipsoids contain a non-planar minimal sphere. With other methods, Bettiol and Piccione [BP21] proved the existence of infinitely many non-planar minimal spheres into rotationally symmetric sufficiently elongated ellipsoids.

In our setting, we focus on the minimisation of combinations of first and second eigenvalues on spheres: $F(x_1, x_2)$. We might obtain new minimal spheres into rotationally symmetric 3-ellipsoids. Indeed, since the multiplicity of the second eigenvalue is bounded by 3 on spheres (see [HON99]), the minimal immersions given by Theorem 1.1 have at most 4 independent coordinates (which are eigenfunctions associated to first and second eigenvalues of some metric on the sphere, see section 3). In Theorem 1.1, the second assumption is automatic on spheres. The first one is immediate for many choices of F. For instance, for the combination $F(x_1, x_2) = \frac{1}{x_1} + \frac{2}{x_2}$ (see section 1.3). However, for some other choices like $F(x_1, x_2) = \frac{1}{x_1 + t x_2}$ for $t > 0$, one needs to prove the gap assumption $\Lambda_{F,1}(0) > \Lambda_F(0)$.

In our construction, some combinations do not give non-planar minimal surfaces into a 3-ellipsoid (e.g for F depending on one eigenvalue only or for $F(x_1, x_2) = \frac{1}{x_1} + \frac{2}{x_2}$). Moreover, we also have to rule out 2-ellipsoids and branched covers over 2-ellipsoids. Therefore, we have to choose carefully the combinations. In [Pet21a], we build non-planar minimal spheres into ellipsoids by this method, for instance by the maximization of $\lambda_1 + t\lambda_2$:

Theorem 1.2 ([Pet21a])**.** *For any $t > 0$, there are maximal metrics for the functional $g \mapsto (\lambda_1(g) + t\lambda_2(g))\, A_g\left(\mathbb{S}^2\right)$. Moreover, for any $t > t_\star$ for some $t_\star < 1$, first and second eigenfunctions associated to maximal metrics g_t of $g \mapsto (\lambda_1(g) + t\lambda_2(g))\, A_g\left(\mathbb{S}^2\right)$ are coordinate functions of non-planar minimal spheres Σ_t into the 3-dimensional rotationally symmetric ellipsoid parametrized by $\Lambda := (\lambda_1(g_t), \lambda_2(g_t), \lambda_2(g_t), \lambda_2(g_t))$. Moreover, for $t_1 \neq t_2$, Σ_{t_1} is not isometric to Σ_{t_2} and as $t \to +\infty$, Σ_t converges to the disjoint union of two round spheres with same area.*

Such a result is not specific to Laplace eigenvalues. In the Steklov case, with analogous methods, we can build non-planar free boundary minimal disks into rotationally symmetric ellipsoids in $\mathbb{R}^3$ as proved in a forecoming paper.

1.3. **The Berger problem, the Hersch-Yang-Yau inequality.** We aim at studying the minimizing problem corresponding to Λ_{f_m}, by applying Theorem 1.1 with the family of functions

$$f_m(x_1, \cdots, x_m) = \sum_{i=1}^{m} \frac{1}{x_i}$$

for $m \geq 1$. One might ask:

Q1: Are the minimizers of $\Lambda_{f_m}(\gamma)$ for $m \geq 1$, if they exist (which is the case on the sphere by Theorem 1.1), the same as the minimizers of $\Lambda_{f_1}(\gamma)$, corresponding to the maximizer of $\Lambda_1(\gamma)$?

The answer is sometimes yes and sometimes no. Let's give examples: By [Pet14a] and [MS19a], there is a metric, maximizer for the first eigenvalue $\Lambda_1(\gamma)$, such that the multiplicity m_γ is at least $m_\gamma \geq 3$ since it arises from a minimal immersion into a Euclidean sphere, so that

$$\Lambda_{f_m}(\gamma) \leq \frac{m}{\Lambda_1(\gamma)} \tag{1.3}$$

holds true for any $1 \leq m \leq m_\gamma$. We would like to know whether this inequality is an equality or not. The inequality (1.3) is strict if and only if other metrics than the maximizer of the first eigenvalue arise by minimizing $\Lambda_{f_m}(\gamma)$.

The very first example was given by Hersch [Her70] on the sphere:

$$\frac{1}{\lambda_1(g)} + \frac{1}{\lambda_2(g)} + \frac{1}{\lambda_3(g)} \geq \frac{3A_g(\mathbb{S}^2)}{8\pi}\ . \tag{1.4}$$

with equality if and only if g is a round metric. This gives a positive answer to Q1 for $\Lambda_{f_3}(\mathbb{S}^2)$

Berger proved in [Ber73] that for $\gamma = 1$, we have flat tori with metric g_ε at the neighbourhood of the flat equilateral torus such that

$$\sum_{i=1}^{6} \frac{1}{\lambda_i(g_\varepsilon)} < \frac{6A_{g_\varepsilon}(\mathbb{T})}{\Lambda_1(1)} = \frac{3\sqrt{3}A_{g_\varepsilon}(\mathbb{T})}{4\pi^2}\ ,$$

which implies that the minimiser of $\Lambda_{f_6}(1)$ given by Theorem 1.1 cannot be a flat equilateral torus, the maximizer of $\Lambda_1(1)$. Notice that with this inequality, 1.4, [KNPP19] and Theorem 1.1 we obtain the needed gap assumption to prove existence of a minimal metric for Λ_{f_6} on the torus:

$$\Lambda_{f_6}(1) < \frac{3\sqrt{3}}{4\pi^2} < \frac{3}{8\pi} + \frac{1}{32\pi} + \frac{1}{40\pi} + \frac{1}{48\pi} \leq \Lambda_{f_6}(0)$$

Studying now $\Lambda_{f_3}(\gamma)$: Yang and Yau [YY80] proved in their celebrated seminal paper that on Σ one has

$$\frac{1}{\lambda_1(g)} + \frac{1}{\lambda_2(g)} + \frac{1}{\lambda_3(g)} \geq \frac{3A_g(\Sigma_\gamma)}{8\pi d}$$

as soon as there is a meromorphic map $\pi : \Sigma_\gamma \to \mathbb{S}^2$ of degree d. One can choose such a map with $d \leq \gamma + 1$ (see [YY80]) and even $d \leq \left[\frac{\gamma+3}{2}\right]$, see [ESI84], which gave in particular

$$\frac{1}{\lambda_1(g)} + \frac{1}{\lambda_2(g)} + \frac{1}{\lambda_3(g)} \geq \frac{3A_g(\Sigma_\gamma)}{8\pi\left[\frac{\gamma+3}{2}\right]} . \tag{1.5}$$

For the non-orientable case, we refer to [Kar16]. In the case $\gamma = 0$, the first inequality (1.3) is an equality because thanks to Hersch, the infimum $\Lambda_{f_3}(0)$ is realized only for the round sphere. It could be interesting to know whether the round sphere is still a minimizer for $\Lambda_{f_m}(0)$ for $m \geq 4$.

A particular case of the Yang-Yau inequality (1.5) gives the very used upper bound on the first eigenvalue

$$A_g(\Sigma_\gamma)\lambda_1(g) \leq 8\pi\left[\frac{\gamma+3}{2}\right] .$$

This upper bound was also reproved in [LY82]. We know that this bound for the first eigenvalue is sharp for $\gamma = 0$ by Hersch, and $\gamma = 2$ by [NS19]. However, it is not sharp for $\gamma = 1$ [Nad96] and for $\gamma > 2$ [Kar19a]. Notice that the inequality (1.5) is sharp again for $\gamma = 0$ and $\gamma = 2$ but it would be interesting to know whether one of these inequalities is an equality:

$$\frac{3}{8\pi\left[\frac{\gamma+3}{2}\right]} \leq \Lambda_{f_3}(\gamma) \leq \frac{3}{\Lambda_1(\gamma)} .$$

1.4. **Asymptotic behaviour as $m \to +\infty$.** Many Riemannian invariants depend on the whole spectrum of Riemannian manifolds. Variational problems for these invariants can be represented as a variational problem for infinite combination of eigenvalues. For instance one can think about the ζ function, the trace of the heat kernel Z^t or the determinant of the Laplacian. We refer for instance to the starting papers by [Sel56] and [CdV73a, CdV73b] for trace formula on ζ and Z^t. A striking example for variational questions is the celebrated result by B. Osgood R. Phillips P. Sarnak [OPS88, OPS88]: proving the uniformization theorem by extremizing the determinant with respect to metrics in a fixed conformal class (see also [Cha04, Gur09] for nice developments on the determinant functional).

Meanwhile, the works about embeddings by eigenfunctions via the heat kernel initiated in [Ber85, BBG94] (see also [Por16] for extended results and state of art) suggest to increase our knowledge on the critical configurations for these embeddings with respect to the metric.

On Σ endowed with a smooth metric g, we set for instance

$$\zeta_m^s(x) = \sum_{i=1}^m x_i^{-s}$$

for $s > 0$, where $\zeta_m^1 := f_m$ was previously introduced. Noticing that this is the partial sum of the formal series $\zeta_\infty^s(x) = \sum_{i=1}^\infty x_i^{-s}$, one can evaluate for the sequence $x_i = \lambda_i(g)$, which perfectly makes sense by the Weyl law when $s > 1$, and we get

$$\zeta^s(g) = \sum_{i=1}^{+\infty} \frac{1}{\lambda_i(g)^s}$$

for $s > 1$, the ζ function on (Σ_γ, g). We also set,

$$D_m(x) := \left(\frac{d}{ds}\zeta_m^s\right)_{s=0}(x) = \sum_{i=1}^m \ln\left(\frac{1}{x_i}\right)$$

one can eveluate for the sequence $x_i = \lambda_i(g)$ to get

$$\exp\left(-D_m(x)\right) = \prod_{i=1}^m \lambda_i(g) \ ,$$

formally related to the determinant of the Laplacian as $m \to +\infty$, defined as

$$\det\left(\Delta_g\right) = \exp\left(-\zeta'^0(g)\right) \ ,$$

where ζ'^0 is the derivative at $s = 0$ of the holomorphic extention of ζ. We finally set

$$Z_m^t(x) = \sum_{i=1}^m e^{-x_i t}$$

for $t > 0$, as the partial sum of $Z_\infty^t(x) = \sum_{i=1}^{+\infty} \exp(-x_i t)$, one can evaluate for the sequence $x_i = \lambda_i(g)$ to get

$$Z^t(g) = \sum_{i=1}^{+\infty} e^{-\lambda_i(g)t} \ ,$$

the trace of the heat Kernel on (Σ_γ, g). One natural question is

Q2: If they exist, which is the case on the sphere, are the minimizers of $\Lambda_{\zeta_m^s}(\gamma)$, $\Lambda_{D_m}(\gamma)$, $\Lambda_{Z_m^t}(\gamma)$ converging to the minimizers of the ζ function, $\zeta'(0)$ or the trace of the Heat-kernel among metrics of unit area, as $m \to +\infty$? What are these minimizers ?

Notice that in the case of $\Lambda_{\zeta_m^s}(\gamma)$, the question is still relevant in the case $s \leq 1$, where for any metric, the sum does not converge. What is the asymptotic of the sequence $\Lambda_{\zeta_m^s}(\gamma)$ as $m \to +\infty$? We also have the link between question Q1 and question Q2 in the case $s = 1$.

It is conjectured in [ESI02, M96] that the round sphere is the only minimizer of the trace of the heat kernel for any $t > 0$, while we know by [OPS88] that it is a global minimizer of the determinant. By Berger computations [Ber73] the flat equilateral torus is not always the minimizer of all of these functionals among flat tori whereas by computations in [Mon88], it is a minimum for the trace of the heat kernel among flat tori.

1.5. **Plan of the paper.** In section 2 we state a more general theorem than Theorem 1.1, (Theorem 2.1 below). We also discuss strategy of proof for these theorems, and reduce the proof to the minimization problem of combinations of eigenvalues in the conformally constraint case (see Theorem 2.2 below).

In section 3, we make simple remarks about the link between eigenvalues and minimal surfaces into ellipsoids. Then we state and prove Theorem 3.1 and 3.2 characterizing critical metrics for very general finite combination of eigenvalues.

In section 4, we recall the harmonic replacement procedure and prove a quantitative ε-regularity estimate for eigenmaps. It is based on a deep understanding of the structure of the equation of harmonic-like maps, initiated by the celebrated paper by Riviere [Riv08], and energy convexity results initiated by Colding and Minicozzi [CM08] that we explain in details in section 6. This is a key result for $W^{1,2}$-convergence of eigenfunctions in section 5.

In section 5, we prove Theorem 2.2. The techniques are based on [Pet18], but stronger and simpler intermediate lemma are introduced.

2. Generalizations and strategies on the proof

2.1. **Generalization of the main theorem.** We give a generalization of Theorem 1.1.

Theorem 2.1. *Let Σ be a compact connected surface. We assume that F satisfies (H). Then there is a compact surface $\hat{\Sigma}$ obtained from Σ by cutting Σ along a finite number of closed curves and glueing disks along these curves, and there is a smooth metric (up to possible conical singularities) g on $\hat{\Sigma}$, "realizing" the minimum $\Lambda_F(\hat{\Sigma}) = \Lambda_F(\Sigma)$. There is also a (possibly branched) conformal minimal immersion $\Phi : (\hat{\Sigma}, g) \to \mathcal{E}$ into*

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \lambda_1 |x_1|^2 + \cdots + \lambda_m |x_m|^2 = 1 \right\} ,$$

a Euclidean ellipsoid on $\mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$, where $i_1, \cdots, i_m$ are non-negative integers, such that the coordinate functions of Φ_k are eigenfunctions with respect to $\lambda_k := \lambda_k(g)$ for $1 \leq k \leq m$. Moreover,

$$2E_g(\Phi_k) = \lambda_k \int_\Sigma |\Phi_k|^2 \, dA_g = A_g(\Sigma) \frac{\lambda_k |\partial_k F(\lambda_1 A_g(\Sigma), \cdots, \lambda_m A_g(\Sigma))|}{\sum_{i=1}^m \lambda_i |\partial_i F(\lambda_1 A_g(\Sigma), \cdots, \lambda_m A_g(\Sigma))|}.$$

As stated in Theorem 3.1, the critical metrics for combinations of eigenvalues are conformal to the induced metrics of a minimal immersion into a Pseudo-Euclidean ellipsoid, with a determined conformal factor with respect to the coordinates. Again, with assumption (H), we have a Euclidean ellipsoid in the conclusion of Theorem 2.1.

Let's give two kinds of gap assuptions: let $1 \leq k \leq m$. Assume for instance that

$$\Lambda_F(\Sigma) < \Lambda_{F,k}(\Sigma) \tag{2.1}$$

Then, we only allow the appearance of at most k thick parts in the asymptotic analysis because the eigenvalue λ_k must not converge to 0: then, $\hat{\Sigma}$ has at most k connected components. For $k = 1$ again (assumption (1.1)), $\hat{\Sigma}$ is a connected surface.

For instance, assume in addition that

$$\Lambda_F(\Sigma) < \Lambda_F(\widetilde{\Sigma}) , \tag{2.2}$$

for any compact surface $\widetilde{\Sigma}$ obtained from Σ by cutting Σ along a finite number of closed curves and glue disks along these curves, so that the new surface has at most k connected

components. Then $\hat{\Sigma} = \Sigma$: there is no topological degeneration (disconnection, loss of genus) for the maximizing sequence.

2.2. **Minimization in the conformal constraint case.** Let (Σ, g) be a compact Riemannian surface without boundary. We set

$$\Lambda_F(\Sigma, [g]) = \inf_{\tilde{g} \in [g]} F\left(A_{\tilde{g}}(\Sigma)\lambda_1(\tilde{g}), \cdots, A_{\tilde{g}}(\Sigma)\lambda_m(\tilde{g})\right)$$

where the infimum is taken over all metrics $\tilde{g}$ on Σ in the conformal class $[g]$ of g in Σ. Again, if we also set for $1 \leq k \leq m$

$$\Lambda_{F,k}(\Sigma, [g]) = \inf_{\tilde{g} \in [g]} F\left(0, \cdots, 0, A_{\tilde{g}}(\Sigma)\lambda_{k+1}(\tilde{g}), \cdots, A_{\tilde{g}}(\Sigma)\lambda_m(\tilde{g})\right) ,$$

we have the same theorem as Theorem 2.1 in the conformal constraint case (with gap assumptions):

Theorem 2.2. *Let (Σ, g) be a Riemannian surface without boundary. Let $1 \leq k \leq m$. We assume that F satisfies (H), and that*

$$\Lambda_F(\Sigma, [g]) < \Lambda_{F,k}(\Sigma, [g]) \tag{2.3}$$

and

$$\Lambda_F(\Sigma, [g]) < \Lambda_F(\widetilde{\Sigma}, [\tilde{g}]) , \tag{2.4}$$

for any compact Riemanian surface $(\widetilde{\Sigma}, \tilde{g})$ obtained as a disjoint union of $(\Sigma, [g])$ and at most $k-1$ copies of spheres, or a disjoint union of at most k copies of spheres.

Then, there is a minimal metric $\tilde{g}$ for $\Lambda_F(\Sigma, [g])$, which is smooth up to possible conical singularities. Moreover, there is a harmonic map $\Phi : (\Sigma, g) \to \mathcal{E}$ into

$$\mathcal{E} = \left\{(x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \lambda_1 |x_1|^2 + \cdots + \lambda_m |x_m|^2 = 1\right\} ,$$

a Euclidean ellipsoid in $\mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$, where $i_1, \cdots, i_m$ are non-negative integers, such that the coordinate functions of Φ_k are eigenfunctions with respect to $\lambda_k := \lambda_k(\tilde{g})$ for $1 \leq k \leq m$ and $\tilde{g}$ is given from Φ by

$$\tilde{g} = \frac{\sum_{k=1}^m \lambda_k |\nabla \Phi_k|_g^2}{\sum_{k=1}^m \lambda_k^2 |\Phi_k|^2} g .$$

Moreover,

$$2E_{\tilde{g}}(\Phi_k) = \lambda_k \int_\Sigma |\Phi_k|^2 \, dA_{\tilde{g}} = A_{\tilde{g}}(\Sigma) \frac{\lambda_k |\partial_k F(\lambda_1 A_{\tilde{g}}(\Sigma), \cdots, \lambda_m A_{\tilde{g}}(\Sigma))|}{\sum_{i=1}^m \lambda_i |\partial_i F(\lambda_1 A_{\tilde{g}}(\Sigma), \cdots, \lambda_m A_{\tilde{g}}(\Sigma))|}.$$

As stated in Theorem 3.2 below, the conformal factor of critical metrics for combinations of eigenvalues arises as a density of some energy of a harmonic map into a Pseudo-Euclidean ellipsoid. With assumption (1), we have a Euclidean ellipsoid in the conclusion of Theorem 2.2. This characterization of critical metrics is also a generalization of previous results for one eigenvalue (see [ESI03, FS16]), when the target manifold is a sphere, or a ball in the Steklov case.

While theorem 2.2 is a step to prove the main result of this paper, questions Q1 and Q2 stated before are still very interesting in the conformally constraint case. The known results on the ζ function, the trace of the heat kernel and the determinant of the Laplacian suggest that we could find sequences of critical metrics as $m \to +\infty$ converging to a constant

curvature metric in the conformal class by a homogeneisation process. It is proved in [OPS88] for the determinant of the Laplacian.

This theorem is a generalization in the context of combination of eigenvalues of the maximization of one eigenvalue in a constraint conformal class proved by the author [Pet18]:

$$\Lambda_m(\Sigma,[g]) = \sup_{\tilde{g}\in[g]} A_{\tilde{g}}(\Sigma)\lambda_m(\tilde{g}) \ .$$

Recently, Karpukhin, Nadirashvili, Penskoi and Polterovich wrote another proof of the maximization of one eigenvalue in a conformal class $\Lambda_m(\Sigma,[g])$ in [KNPP20]. As in [Pet18], they look for a suitable maximizing sequence for the variational problem $\Lambda_m(\Sigma,[g])$, of conformal factors for g viewed as the density of a probability measure. However their sequence is constructed in a radically different way: they started with a previous construction in [NS15]. Recently again, Karpukhin and Stern developped another interesting min-max approach in [KS20] for existence question of a maximizer of $\Lambda_1(\Sigma,[g])$ and $\Lambda_2(\Sigma,[g])$, leading to a deeper understanding of the maximum. In the current paper, we use again the original author's construction in [Pet14a], inspired by the work by Fraser and Schoen in the Steklov context for $m=1$ [FS16]. This construction is more geometric since it uses a global regularizer for the conformal factor, the heat flow, so that we obtain a canonical maximizing sequence of smooth metrics $e^{2u_\varepsilon}g$. It is also more natural for problems involving combination of eigenvalues.

2.3. **Proving the main Theorem with the conformal constraint result.** As in [Pet14a] (and [Pet18] in case of multibubbling and disconnections), we use Theorem 2.2 to prove Theorem 2.1 by a maximization of $\Lambda_F(\gamma)$ among all the conformal classes $[g]$ of Σ_γ. Indeed, if the gap assumptions (2.1) and (2.2) hold true, one has a maximizing sequence of conformal classes $[h_n]$, such that (2.3) and (2.4) hold for any conformal class of the sequence. For simplicity, we assume that Σ is orientable of genus γ. For non-orientable questions, see [MS17]. We choose here a metric of constant curvature 0 (with fixed area if $\gamma=1$) or -1 (if $\gamma\geq 2$) as a representative. Then using Theorem (2.2), one has a maximal smooth metric $g_n\in[h_n]$ in each conformal class, whose conformal factor with respect to h_n is some density of energy of a harmonic map into a Euclidean ellipsoid. Then we just have to perform an asymptotic analysis on a sequence of harmonic maps. Following [Pet14a] or [Pet18], (based on the work from [ZHU10]) one has to prevent from bubbling on harmonic maps and degeneration of the sequence of conformal classes $[h_n]$ (closed geodesics which shrink as $n\to+\infty$). In both cases, the surface asymptotically disconnects into at most k thick parts, where k is the minimal eigenvalue that does not shrink as $n\to+\infty$. In Theorem 1.1, $k=1$. Moreover, cutting along the closed geodesics whose length go to 0, and along the necks of bubbles give a surface $\widetilde{\Sigma}$ defined in Theorem 2.1. In Theorem 1.1, we even have one thick part of genus $\tilde{\gamma}\leq\gamma-1$. Notice that the harmonic map perfectly converges in the thick parts. So does the metric. One can transplant the eigenfunctions of the limit metric on the limit surface of genus $\tilde{\gamma}$ on Σ_γ as test functions for the maximizing sequence g_n to prove that $\Lambda_F(\Sigma)\leq\Lambda_F(\widetilde{\Sigma})$. The gap assumption (2.2) leads to a contradiction.

2.4. **Summary for the proof of the conformal constraint result.** Let's discuss about the proof of Theorem 2.2 in section 5.

2.4.1. *Section 5.1.* We replace the minimization over all the conformal factors by a minimization on a regularizing functional built with the heat kernel. This gives a canonical minimizing sequence of conformal metrics $e^{2u_\varepsilon}g$ for $\Lambda_F(\Sigma)$ as $\varepsilon \to 0$.

2.4.2. *Section 5.2 and 5.4.* The main idea in [Pet14a, Pet18] was to perform an asymptotic analysis on the sequence of eigenfunctions Φ_ε coming from the Euler-Lagrange equation for the regularized problem stated in Claim 5.1 as $\varepsilon \to 0$. We introduce in section 5.2, the notations adapted to the scale invariance of the eigenvalue equation and the scale invariance of the heat kernel. In our construction, the natural scaling for good estimates is $\sqrt{\varepsilon}$, the scaling of the heat kernel (see section 5.4)

2.4.3. *Section 5.3.* In [Pet18], one added difficulty for maximization of $\Lambda_m(\Sigma, [g])$ for $m > 1$ compared to the case $m = 1$ ([Pet14a]) is bubbling. If $m > 1$, one cannot prevent from concentration phenomena on the sequence of measures $e^{2u_\varepsilon}dA_g$ and the appearance of k thick parts (the added thick parts are spherical, so-called bubbles). We only use a posteriori the gap assumptions to prevent from bubbles. Of course, this is handled the same way in the current paper, where the maximal number of thick parts is the smallest k such that λ_k does not shrink. As explained below, we assume $k = 1$ for the proof as in Theorem 1.1. We then have one thick part.

2.4.4. *Section 5.5.* For deep reasons, the author defined in [Pet18], some "singularity" points to perform the analysis in the thick parts (the same as in the case $m = 1$) and to perform a delicate neck analysis in the thin parts. These points have a nature very different to the concentration points appearing at each scale of the bubbling. This is more visible now since the maximal number of thick parts in the bubbling is the smallest k such that λ_k does not shrink ($k = 1$ in theorem (1.1)), whereas the number of "singularity" points is m to gather all the possible degenerations for eigenfunctions appearing in the analysis.

If the functional only depends on the first eigenvalue ($m = 1$) and if Σ is not diffeomorphic to a sphere, the thick part cannot be a sphere (by a Hersch argument specific to the first eigenvalue, see [Kok14] and a gap result [Pet14a]) and we do not have any concentration point nor singularity points. In the other cases $m > 1$, we *a priori* still have singularity points. Even in the case $k = 1$ (the case of our proof), the thick part can *a priori* be a sphere so that the main difficulties raised in [Pet18] still hold. The case $k > 1$ works the same way up to a classical bubble tree construction. In the current paper, one of the novelties is a careful selection of the scales of these singularity points by a simplified proof.

2.4.5. *Section 5.6.1.* We prove that the sequence of eigenfunctions is uniformly bounded far from singularity points. The technique in [Pet14a] is improved for Section 5.6.2.

2.4.6. *Section 5.6.2.* We perform a local replacement procedure by the use of $|\Phi|^2_{\mathcal{E}}$-harmonic maps. In the case where only one eigenvalue appears, by the symmetries of the target manifold (a Euclidean sphere), the projection $\frac{\Phi}{|\Phi|}$ is already $|\Phi|^2$-harmonic: the replacement is global. One of the difficulties in [Pet18] was to prove that the radial coordinate $|\Phi_\varepsilon|$ is lower bounded by a uniform positive constant and converges to 1 as strongly as possible. In the current paper, in addition to these, a new difficulty appears since $\frac{\Phi}{|\Phi|_{\mathcal{E}}}$ is not $|\Phi|^2_{\mathcal{E}}$-harmonic anymore. Fortunately, the radial projection of eigenmaps on $\mathcal{E}$ is still locally close in $W^{1,2}$ to their $|\Phi|^2_{\mathcal{E}}$-harmonic replacement with a control by the L^2-norm of $\nabla |\Phi|_{\mathcal{E}}$ (proved in section 4). By a use of the singularity points they renamed "bad points" in

[KNPP20], the four authors proved with their construction that the eigenfunctions converge in $W^{1,2}$ in the neighbourhood of the other points, the "good points". This comes from the use of a nice construction of local quasi-continuous replacement of $W^{1,2}$ functions. In the current paper, the $|\Phi|^2_{\mathcal{E}}$-harmonic replacement has strong $W^{1,p}$ convergence properties for any $p < +\infty$ (see section 6.1).

2.4.7. *Section 5.7.* We adapt the previous analysis for convergence of eigenfunctions in bubbles. We need to add a neck analysis. The technique in [Pet18] is improved.

3. Critical metrics and minimal ellipsoids

In this section, we prove that any critical metric for combinations of eigenvalue arise from a nice geometric configuration. The work on this section can also be done in the Steklov eigenvalues context. While previous computations have been done to compute the critical metrics for eigenvalues, no one noticed this general picture before.

3.1. **Minimal immersions into ellipsoids.** In this part, we explain the link between minimal immersions into ellipsoids and the Laplace eigenvalue of the induced metric times one function of the coordinates of the minimal immersion. Let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid of parameters $\Lambda = diag\,(\lambda_1, \cdots, \lambda_n)$, with $\lambda_i > 0$, defined by

$$\mathcal{E} = \{(x_1, \cdots, x_n) \in \mathcal{E}; \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 = 1\} \ ,$$

endowed with the induced metric of the Euclidean metric ξ. The outward normal of the ellipsoid is denoted by

$$\nu = \frac{\Lambda x}{|\Lambda x|}$$

where

$$|\Lambda x| = \left(\sum_{i=1}^n \lambda_i^2 x_i^2\right)^{\frac{1}{2}} \ .$$

Now, let $\Phi : \Sigma \to \mathcal{E}$ be a conformal immersion of a Riemannian compact surface without boundary (Σ, g) into $\mathcal{E}$, a $n-1$ dimentional ellipsoid of parameter $\Lambda = (\lambda_1, \cdots, \lambda_n)$. Then there is a smooth positive function e^{2u}, such that $g = e^{2u}h$, where the $h = \phi^\star(\xi)$ and we have

$$\Delta_g f = e^{-2u}\Delta_h f \ ,$$

Notice that the mean curvature vector of the isometric immersion $\Phi : (\Sigma, h) \to \mathbb{R}^n$ is $H = \Delta_h \Phi$ and is orthogonal to Σ.

If in addition, the isometric immersion $\Phi : (\Sigma, h) \to \mathcal{E}$ satisfies

$$\Delta_h \Phi = \frac{|H|}{|\Lambda\Phi|}\Lambda\Phi \ ,$$

which is exactly the assumption of being minimal into $\mathcal{E}$, then setting

$$g = e^{2u}h \qquad e^{2u} = \frac{|H|}{|\Lambda\Phi|} \ ,$$

the coordinates of Φ are eigenfunctions on (Σ, g) with eigenvalues $\lambda_1, \cdots, \lambda_n$:

$$\Delta_g \Phi = \Lambda\Phi \ .$$

Another well-known characterisation of an immersion $\Phi : (\Sigma, g) \to \mathbb{R}^n$ to be minimal in $\mathcal{E}$ is harmonicity in $\mathcal{E}$ and conformality. Φ is harmonic in $\mathcal{E}$ if it is a critical point of the energy

$$E(\Phi) = \frac{1}{2}\int_\Sigma |\nabla\Phi|_g^2 \, dA_g$$

under the constraint $\Phi(\Sigma) \subset \mathcal{E}$. The Euler-Lagrange characterization is

$$\Delta_g \Phi \in (T_\Phi \mathcal{E})^\perp$$

Then $\Delta_g \Phi = f\nu \circ \Phi$ for some function f and simply computing f with $0 = \frac{1}{2}\Delta_g |\Phi|_{\mathcal{E}}^2$, we obtain the equation

$$\Delta_g \Phi = \frac{|\nabla\Phi|_g^2}{|\Lambda\Phi|^2}\Lambda\Phi = \langle \nabla\Phi, \nabla(\nu\circ\Phi)\rangle \, \nu\circ\Phi \, .$$

We obtain another formula for the conformal factor of g with respect to the metric h

$$e^{2u} = \frac{|H|}{|\Lambda\Phi|} = \frac{|\nabla\Phi|_{\mathcal{E}}^2}{|\Lambda\Phi|^2} \, .$$

Conformality is characterized by the vanishing of

$$0 = |\nabla\Phi|_g^2 \frac{g}{2} - d\Phi\otimes d\Phi := \sum_{i=1}^n \left(|\nabla\Phi_i|_g^2 \frac{g}{2} - d\Phi_i \otimes d\Phi_i \right) \, .$$

3.2. **Critical metrics for combinations of eigenvalues.** This part proves the general properties for critical metrics of functionals depending on Laplace eigenvalues. The notion of critical metrics was introduced by [ESI00] who generalized a result by [Nad96]. They used that in variational theory, there is a classical way to generalize the notion of derivative of a locally Lipschitz functional by sub-gradients (see e.g. [Cla13]). Here, we say that g is critical with respect to $F(A_g(\Sigma)\lambda_1(g), \cdots, A_g(\Sigma)\lambda_m(g))$ if the product of the left and right derivative of $t \mapsto F(A_{g+th}(\Sigma)\lambda_1(g+th), \cdots, A_{g+th}(\Sigma)\lambda_m(g+th))$ is non positive for any symmetric 2-tensor variation h. Of course, a maximal metric satisfies this condition.

Theorem 3.1. *Let Σ be a compact surface and $F : \left(\mathbb{R}_+^\star\right)^m \to \mathbb{R}$ be a smooth function with $m \geq 1$. Let g be a critical metric for the functional*

$$g \mapsto F\left(A_g(\Sigma)\lambda_1(g), \cdots, A_g(\Sigma)\lambda_m(g)\right)$$

and we assume that $A_g(\Sigma) = 1$. Then, there are m non-negative integers $i_1, \cdots, i_m$ and there is a map $\Phi : \Sigma \to \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$ which is conformal and harmonic into the space

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \epsilon_1\lambda_1 |x_1|^2 + \cdots + \epsilon_m \lambda_m |x_m|^2 = \sum_{k=1}^m \epsilon_k \lambda_k t_k \right\}$$

where for $1 \leq k \leq m$, $\lambda_k = \lambda_k(g)$, $t_k = |\partial_k F(\lambda_1, \cdots, \lambda_m)| = A_g(\Sigma)^{-1}\int_\Sigma |\Phi_k|^2 \, dA_g$, ϵ_k is the sign of $\partial_k F(\lambda_1, \cdots, \lambda_m)$ and $\mathcal{E}$ is endowed with the pseudo-Euclidean metric defined for $(x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$ by $\epsilon_1 |dx_1|^2 + \cdots + \epsilon_m |dx_m|^2$.

Notice that if $\partial_k F(\lambda_1, \cdots, \lambda_m)$ all have the same sign, then the target manifold is the ellipsoid of a Euclidean space. This result is a generalization of the result by El Soufi and Ilias [ESI00, ESI03, ESI08] when only one eigenvalue appears in the functional, proving that they arise as the induced metric of a (possibly branched) minimal immersion into a

sphere. They were inspired by the seminal paper by Nadirashvili [Nad96]. This result was reproved by Fraser and Schoen in [FS13], while they noticed that critical metrics for Steklov eigenvalues on surfaces with boundary arise as the induced metric of a free boundary minimal immersion into a ball. The same work can be done for Steklov eigenvalues: critical metric of general functionals are conformal to induced metrics of a free boundary minimal surface into an ellipsoid (possibly of a pseudo-Euclidean space).

Proof. We aim at using the oposite signs of the left and right derivative at $t = 0$ for

$$t \mapsto F\left(A_{g+th}(\Sigma)\lambda_1(g+th), \cdots, A_{g+th}(\Sigma)\lambda_m(g+th)\right)$$

for h a symmetric 2-tensor on $T\Sigma$. We denote by

$$Q_h(\Phi) = \int_\Sigma \sum_{k=1}^m \epsilon_k t_k \left(|\nabla\Phi_k|_g^2 \frac{g}{2} - d\Phi_k \otimes d\Phi_k + \lambda_k \left(1 - |\Phi_k|^2\right)\frac{g}{2}, h\right)_g dA_g$$

for $\Phi \in E_1(g) \times \cdots \times E_m(g) \in \mathcal{C}^\infty(\Sigma)$. $E_k(g)$ denotes the set of all eigenfunctions associated to the eigenvalue $\lambda_k(g)$. The goal of the following 3 steps is to prove that for any symmetric 2-tensor h on $T\Sigma$, there is a map $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_h(\Phi) = 0$. Here, $S(E_i(g))$ is the unit sphere in $E_i(g)$ endowed with the L^2 norm with respect to g. Let h be a symmetric 2-tensor on $T\Sigma$.

STEP 1 :

If ϕ^t is an eigenfunction associated to some eigenvalue $\lambda_t = \lambda_k(g+th)$ with $\int_\Sigma \left(\phi^t\right)^2 dA_{g+th} = 1$, then up to a subsequence as $t \to 0$, we have

$$\lambda_t \to \lambda \text{ and } \phi_t \to \phi \text{ in } \mathcal{C}^2(\Sigma) \text{ as } t \to 0\ ,$$

where $\int_\Sigma \phi^2 dA_g = 1$ and $\Delta_g \phi = \lambda\phi$.

Proof of STEP 1 :

We have $\Delta_{g+th}\phi^t = \lambda_t\phi^t$. In a chart, the equation is given by

$$-\frac{1}{\sqrt{|g+th|}}\partial_i\left((g+th)^{ij}\sqrt{|g+th|}\partial_j\phi^t\right) = \lambda_t\phi^t\ .$$

Locally, we have a constant $C_0 > 1$ and $t_0 > 0$ such that for $|t| < t_0$,

$$\frac{1}{C_0}\xi \leq g + th \leq C_0\xi$$

where ξ is the Euclidean metric, so that we have constants $C_1 > 1$ and $C_2 > 1$ with

$$\frac{1}{C_1}\|X\|^2 \leq \sqrt{|g+th|}\,(g+th)^{ij}X_iX_j \leq C_1\|X\|^2$$

$$\frac{1}{C_2}\int_\Sigma \left(\phi^t\right)^2 dA_g \leq \int_\Sigma \left(\phi^t\right)^2 dA_{g+th} \leq C_2\int_\Sigma \left(\phi^t\right)^2 dA_g\ .$$

Since $A_{g+th} = 1 + o(1)$, λ_t is bounded as $t \to 0$. Up to a subsequence, we assume that $\lambda_t \to \lambda$ as $t \to 0$. Therefore, by standard elliptic theory, since $\lambda_t\phi^t$ is bounded in L^2, up to

a subsequence, ϕ_t converges in $\mathcal{C}^3(\Sigma)$ to some function ϕ. When we let $t \to 0$, λ and ϕ satisfy the conclusion of STEP 1.

STEP 2 :

For any $k \in \{0, 1, \cdots, m\}$, $\lambda_k(g + th) \to \lambda_k(g)$ as $t \to 0$.

Proof of STEP 2 :

Looking at the min-max definition of eigenvalues :

$$\lambda_k(g) = \inf_{E_{k+1}} \max_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_\Sigma |\nabla \phi|_g^2 dA_g}{\int_\Sigma \phi^2 dA_g}$$

where the infimum is taken on spaces of smooth functions of dimension $k + 1$, and the smooth dependance of the Rayleigh quotient on the metric g, λ_k has to be continuous.

STEP 3 :

There is a map $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_h(\Phi) = 0$ where, $S(E_i(g))$ is the unit sphere in $E_i(g)$ endowed with the L^2 norm with respect to g.

Proof of STEP 3 :

By connectedness of $S(E_1(g)) \times \cdots \times S(E_m(g))$, it is equivalent to prove that there are Φ^+ and Φ^- in $S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_h(\Phi^+)Q_h(\Phi^-) \leq 0$. We use in this step that g is a critical metric for the functional. $Q_h(\Phi^+)$ and $Q_h(\Phi^-)$ correspond to the opposite signs of the right and left derivatives at 0 with respect to t for the variation $g + th$ of $t \mapsto F\left(A_t\lambda_1(g + th), \cdots, A_t\lambda_m(g + th)\right)$. Here, we denoted $A_t = A_{g+th}(\Sigma) = 1 + t\int_\Sigma (g, h)_g dA_g + o(t)$.

Let's compute the left and right derivatives of an eigenvalue $\lambda_t = \lambda_k(g + th)$ with its associated eigenfunction ϕ^t in $E_k(g + th)$ with unit $L^2(g + th)$-norm. By STEP 2, we know that λ_t converges to $\lambda = \lambda_k(g)$. In fact we will only need to compute the variation $\frac{\lambda_{t_j^\pm} - \lambda}{t_j^\pm}$ for one subsequence $t_j^- \to 0$ such that $t_j^- < 0$ and for one subsequence $t_j^+ \to 0$ such that $t_j^+ > 0$.

By STEP 1 we have subsequences $t_j^\pm \to 0$ with $t_j^- < 0$ and $t_j^+ > 0$, $\phi^{t_j^\pm}$ to $\phi^\pm \in S(E_k(g))$ in $\mathcal{C}^3(\Sigma)$ as $j \to +\infty$. We denote by π_k the projection in $L^2(g)$ on the whole eigenspace

associated to $\lambda_k(g)$, and by $R^t = \phi^t - \pi_k(\phi^t)$. In some chart, we have for any t the equation

$$\begin{aligned}
\Delta_g R^t - \lambda R^t &= \Delta_g \phi^t - \lambda \phi^t \\
&= (\Delta_g - \Delta_{g+th})\phi^t - (\lambda - \lambda_t)\,\phi^t \\
&= -\left(\frac{1}{\sqrt{|g|}} - \frac{1}{\sqrt{|g+th|}}\right)\partial_i(\sqrt{|g|}g^{ij}\partial_j\phi^t) \\
&\quad -\frac{1}{\sqrt{|g+th|}}\partial_i\left(\left(\sqrt{|g|} - \sqrt{|g+th|}\right)g^{ij}\partial_j\phi^t\right) \\
&\quad -\frac{1}{\sqrt{|g+th|}}\partial_i\left(\sqrt{|g+th|}\left(g^{ij} - (g+th)^{ij}\right)\partial_j\phi^t\right) \\
&\quad -(\lambda - \lambda_t)\phi^t
\end{aligned}$$

We set

$$\alpha_t = |\lambda - \lambda_t| + \left\|R^t\right\|_\infty + |t| \tag{3.1}$$

and dividing by α_t we write the previous equation as

$$\Delta_g\left(\frac{R^t}{\alpha_t}\right) - \lambda\frac{R^t}{\alpha_t} = \frac{f^t}{\alpha_t} + \frac{\lambda_t - \lambda}{\alpha_t}\phi^t$$

where $\frac{f^{t_j^\pm}}{t_j^\pm}$ converges to

$$f_0^\pm = \frac{1}{2}(g,h)_g\Delta_g\phi^\pm + \frac{1}{2\sqrt{|g|}}\partial_i\left(\sqrt{|g|}(g,h)_g g^{ij}\partial_j\phi^\pm\right) - \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ik}h_{kl}g^{lj}\partial_j\phi^\pm\right)$$

in $\mathcal{C}^1$ as $j \to +\infty$.

By elliptic regularity of the equation, there is a subsequence of (t_j) we still denote $t_j \to 0$ as $j \to +\infty$ such that

$$\frac{t_j^\pm}{\alpha_{t_j^\pm}} \to t_0^\pm \ , \ \frac{\lambda^{t_j^\pm} - \lambda}{\alpha_{t_j^\pm}} \to \delta_0^\pm \text{ as } j \to +\infty$$

and

$$\phi^{t_j^\pm} \to \phi^\pm \text{ in } \mathcal{C}^3(\Sigma) \ , \ \frac{R^{t_j^\pm}}{\alpha_{t_j^\pm}} \to R_0^\pm \text{ in } \mathcal{C}^2(\Sigma) \text{ as } j \to +\infty \ .$$

Then, the previous equation becomes as $j \to +\infty$

$$\Delta_g R_0^\pm - \lambda R_0^\pm = t_0^\pm f_0^\pm + \delta_0^\pm\phi^\pm \ .$$

We integrate it against $\phi^\pm$ so that by integration by part on the left-hand term and since $\Delta_g\phi^\pm = \lambda\phi^\pm$ and $\int_\Sigma (\phi^\pm)^2\,dA_g = 1$,

$$0 = t_0^\pm\int_\Sigma f_0^\pm\phi^\pm + \delta_0^\pm \ .$$

If $t_0^\pm = 0$, then $\delta_0^\pm = 0$ and $\Delta_g R_0^\pm - \lambda R_0^\pm = 0$, which means that $R_0^\pm = 0$ since $R_0^\pm \in E_k(g)^\perp$ as a limit of $\frac{1}{\alpha_{t_j^\pm}}\left(\phi^{t_j^\pm} - \pi_k(\phi^{t_j^\pm})\right)$. However, by (3.1), the convergence

$$1 = \frac{\left|\lambda - \lambda_{t_j^\pm}\right| + \left\|R^{t_j^\pm}\right\|_\infty + |t_j|}{\alpha_{t_j^\pm}} \to \delta_0^\pm + \left\|R_0^\pm\right\|_\infty + \left|t_0^\pm\right| = 0 \text{ as } l \to +\infty$$

is absurd. Therefore, $t_0^\pm \neq 0$ and

$$\begin{aligned}
\lim_{j\to+\infty} \frac{\lambda_k(g + t_j^\pm h) - \lambda_k}{t_j^\pm} &= \lim_{j\to+\infty} \frac{\lambda_k(g + t_j^\pm h) - \lambda_k}{\alpha_{t_j^\pm}} \frac{\alpha_{t_j^\pm}}{t_j^\pm} \\
&= \frac{\delta_0^\pm}{t_0^\pm} \\
&= -\int_\Sigma f_0^\pm \phi^\pm \\
&= \int_\Sigma \left(\frac{\phi}{2}(g,h)_g \left(-\Delta_g \phi^\pm\right) - \frac{\phi}{2\sqrt{|g|}} \partial_i \left(\sqrt{|g|}(g,h)_g g^{ij} \partial_j \phi^\pm \right) \right) dA_g \\
&\quad + \int_\Sigma \left(\frac{\phi^\pm}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ik} h_{kl} g^{lj} \partial_j \phi^\pm \right) \right) dA_g \\
&= \int_\Sigma \left(\frac{1}{2}(g,h)_g \left(-\phi^\pm \Delta_g \phi^\pm + \left|\nabla \phi^\pm\right|_g^2 \right) - (d\phi^\pm \otimes d\phi^\pm, h)_g \right) dA_g \\
&= \int_\Sigma \left(-d\phi^\pm \otimes d\phi^\pm + \frac{|\nabla \phi^\pm|_g^2}{2} g - \lambda \left(\phi^\pm\right)^2 \frac{g}{2}, h \right)_g dA_g
\end{aligned}$$

where the second equality has to be read in a suitable atlas with partitions of unity and the third one is got with an integration by parts. Noting that $A_t = 1 + t\int_\Sigma (g,h)_g dA_g + o(t)$, for any k, we have the existence of eigenfunctions ϕ_k^+ and ϕ_k^- in $S(E_k(g))$ such that

$$\frac{A_{t_j^\pm}\lambda_k(g + t_j^\pm h) - \lambda_k}{t_j^\pm} \to \int_\Sigma \left(-d\phi_k^\pm \otimes d\phi_k^\pm + \frac{|\nabla \phi^\pm|_g^2}{2} g - \lambda_k (1 - \left(\phi_k^\pm\right)^2) \frac{g}{2}, h \right)_g dA_g$$

as $t \to 0$ when $t > 0$ for ϕ_k^+ or $t < 0$ for ϕ_k^-. Then, we have that

$$\frac{F(A_{t_j^\pm}\lambda_1(g + t_j^\pm h), \cdots, A_{t_j^\pm}\lambda_k(g + t_j^\pm h)) - F(\lambda_1(g), \cdots, A\lambda_k(g))}{t_j^\pm} \to Q_h(\phi_1^\pm, \cdots, \phi_m^\pm)\,,$$

as $j \to +\infty$. Therefore, the maps $\Phi^\pm = (\phi_1^\pm, \cdots, \phi_m^\pm) \in S(E_1(g)) \times \cdots \times S(E_m(g))$ satisfy

$$Q_h(\Phi^+) Q_h(\Phi^-) \leq 0\,.$$

As a product of connected spaces $S(E_1(g)) \times \cdots \times S(E_m(g))$ is a connected space and there is $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_h(\Phi) = 0$. This ends the proof of STEP 3.

End of the Proof of Theorem 3.1 :

We now let C be the convex hull of the following set of symmetric 2-tensors :

$$\left\{\sum_{k=1}^m \epsilon_k t_k \left(-d\phi_k\otimes d\phi_k+\frac{|\nabla\phi_k|_g^2}{2}g-\lambda_k(1-\phi_k^2)\frac{g}{2}\right);(\phi_1,\cdots,\phi_k)\in E\right\}\ ,$$

where $E=S(E_1(g))\times\cdots\times S(E_m(g))$. If $0\notin C$, by the Hahn-Banach theorem, there is a symmetric 2-tensors h such that

$$\forall\omega\in C;\int_\Sigma(\omega,h)_g dA_g>0\ .$$

This means that for any $\Phi\in E$, $Q_h(\Phi)>0$ and this contradicts the STEP 3. Therefore, $0\in C$ and one can define

$$\Phi=(\Phi_1,\cdots,\Phi_m)=(\phi_1^1,\cdots,\phi_1^{i_1},\cdots,\phi_m^1,\cdots,\phi_m^{i_m})$$

such that $\phi_k^j\in E_k(g)$, $\int_\Sigma|\Phi_k|^2dA_g=1$ and

$$\sum_{k=1}^m\epsilon_k t_k\left(-\sum_{j=1}^{i_k}d\phi_k^j\otimes d\phi_k^j+\frac{|\nabla\Phi_k|_g^2}{2}g-\lambda_k(1-|\Phi_k|^2)\frac{g}{2}\right)=0\ , \tag{3.2}$$

where we denote by $|\Phi_k|^2=\sum_{j=1}^{i_k}\left(\phi_k^j\right)^2$ and $|\nabla\Phi_k|_g^2=\sum_{j=1}^{i_k}\left|\nabla\phi_k^j\right|_g^2$. The trace with respect to g gives that

$$\sum_{k=1}^m\lambda_k\epsilon_k t_k|\Phi_k|^2=\sum_{k=1}^m\lambda_k\epsilon_k t_k\ .$$

We now choose $\widetilde{\Phi_k}=\sqrt{t_k}\Phi_k$ instead of Φ_k so that the new map $\widetilde{\Phi}$ lives in the ellipsoid

$$\mathcal{E}=\left\{(x_1,\cdots,x_m)\in\mathbb{R}^{i_1+\cdots+i_m};\epsilon_1\lambda_1|x_1|^2+\cdots+\epsilon_m\lambda_m|x_m|^2=\sum_{k=1}^m\epsilon_k\lambda_k t_k\right\}\ .$$

By the eigenvalue equations on the maps, we deduce that $\Delta_g\widetilde{\Phi}\in(T_\Phi\mathcal{E})^\perp$ since the vector $(\lambda_1\Phi_1,\cdots,\lambda_m\Phi_m)$ is normal to Φ with respect to the pseudo-Riemannian metric $G=\epsilon_1|dx_1|^2+\cdots+\epsilon_m|dx_m|^2$. This is the equation satisfied by the critical maps of the energy $\int_\Sigma G(\nabla\Phi,\nabla\Phi)dA_g$ for maps Φ satisfying the constraint $\Phi\in\mathcal{E}$.

We now get from (3.2) that

$$\sum_{k=1}^m\epsilon_k\left(-\sum_{j=1}^{i_k}d\widetilde{\phi_k^j}\otimes d\widetilde{\phi_k^j}+\frac{\left|\nabla\widetilde{\Phi_k}\right|_g^2}{2}g\right)=0$$

or equivalently that

$$\widetilde{\Phi}^*(G)=\left(\sum_{k=1}^m\epsilon_k\left|\nabla\widetilde{\Phi_k}\right|_g^2\right)\frac{g}{2}=G(\nabla\widetilde{\Phi},\nabla\widetilde{\Phi})\frac{g}{2}$$

and we obtain that the map $\tilde{\Phi}$ is conformal. This ends the proof of Theorem 3.1.

$\diamondsuit$

We now state the analogous theorem concerning the critical metric for combination of eigenvalues in the conformally constraint case, generalizing previous results [ESI03] and again, perfectly adaptable in the conformally constraint Steklov case (proof for one eigenvalue: [FS13])

Theorem 3.2. *Let (Σ, g) be a compact Riemannian surface and $F : \left(\mathbb{R}_+^\star\right)^m \to \mathbb{R}$ be a smooth function with $m \geq 1$. Let $\tilde{g} \in [g]$ be a critical metric for the functional*

$$\tilde{g} \mapsto F\left(A_{\tilde{g}}(\Sigma)\lambda_1(\tilde{g}), \cdots, A_{\tilde{g}}(\Sigma)\lambda_m(\tilde{g})\right)$$

defined on the conformal class $[g]$ of g and we assume that $A_{\tilde{g}}(\Sigma) = 1$. Then, there are m non-negative integers $i_1, \cdots, i_m$ and there is a map $\Phi : \Sigma \to \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$ which is harmonic into the space

$$\mathcal{E} = \left\{ (x_1, \cdots, x_m) \in \mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}; \epsilon_1 \lambda_1 |x_1|^2 + \cdots + \epsilon_m \lambda_m |x_m|^2 = \sum_{k=1}^m \epsilon_k \lambda_k t_k \right\}$$

endowed with the pseudo-Euclidean metric on $\mathbb{R}^{i_1} \times \cdots \times \mathbb{R}^{i_m}$

$$\epsilon_1 |dx_1|^2 + \cdots + \epsilon_m |dx_m|^2$$

where for $1 \leq k \leq m$, $\lambda_k = \lambda_k(g)$, $t_k = |\partial_k F(\lambda_1, \cdots, \lambda_m)| = A_{\tilde{g}}(\Sigma)^{-1} \int_\Sigma |\Phi_k|^2 dA_{\tilde{g}}$ and ϵ_k is the sign of $\partial_k F(\lambda_1, \cdots, \lambda_m)$.

The proof of this result follows the lines of the proof of Theorem 3.1, but is even simpler since the variations we use stay in the conformal class of g. Of course, the conclusion has to be weaker than in Theorem 3.1 and we exatly loose that the obtained harmonic maps are conformal. More precisely, we can follow step 1, step 2 and step 3 with the symmetric 2-tensors $h = fg$, where f is a smooth function to compute the left and right derivative at $t = 0$ for

$$t \mapsto F\left(A_{(1+tf)g}(\Sigma)\lambda_1((1+tf)g), \cdots, A_{(1+tf)g}(\Sigma)\lambda_m((1+tf)g)\right) .$$

Notice than the Dirichlet energy is conformally invariant. Then, denoting by

$$Q_f(\Phi) = \int_\Sigma \left(\sum_{k=1}^m \epsilon_k t_k \lambda_k \left(1 - |\Phi_k|^2\right) \frac{g}{2} \right) f dA_g$$

for $\Phi \in E_1(g) \times \cdots \times E_m(g) \in \mathcal{C}^\infty(\Sigma)$, where $E_k(g)$ denotes the set of all eigenfunctions associated to the eigenvalue $\lambda_k(g)$, we prove that for any f, there is a map $\Phi \in S(E_1(g)) \times \cdots \times S(E_m(g))$ such that $Q_f(\Phi) = 0$. Here again, $S(E_i(g))$ is the unit sphere in $E_i(g)$ endowed with the L^2 norm with respect to g. Again, by a standard Hahn-Banach argument we complete the proof. These arguments are performed again in the next section to compute the Euler Lagrange equation of a perturbated functional.

4. A quantitative approximation by harmonic replacement

We prove in this section the tool we use to prove Claim 5.8 and Claim 5.13 in the key estimates on sequences of eigenfunctions given by Claim 5.1. Since it is an interesting quantitative estimate, we state it here independently to the proof of the main theorem of the current paper.

Harmonic replacement is a canonical way to replace a map by a smooth map, decreasing the energy locally, keeping the same Dirichlet boundary conditions and a constraint target

manifold. However, in the positively curved target manifolds, such a map is not globally unique. As an analogue to the geodesic problem, one has to restrict such result in a domain with small energy. The following result, so-called "energy convexity" was proved by Colding and Minicozzi in [CM08] and it is also proved, simplified and generalized in [LP19] using the ε-regularity result on harmonic maps and a classical Hardy inequality.

Theorem 4.1. *Let M be a compact submanifold of $\mathbb{R}^n$. Then there is a constant $\epsilon_0 := \varepsilon(M) > 0$ such that if $u, v \in W^{1,2}(\mathbb{D}, M)$ with $u|_{\partial\mathbb{D}} = v|_{\partial\mathbb{D}}$, $E(u) \leq \epsilon_0$ and u is weakly harmonic, then we have the energy convexity*

$$\frac{1}{2}\int_{\mathbb{D}} |\nabla(v-u)|^2 \leq \int_{\mathbb{D}} |\nabla v|^2 - \int_{\mathbb{D}} |\nabla u|^2 \ . \tag{4.1}$$

Notice that thanks to the techniques developped in [LL13] and in [LP19], one can generalize this energy convexity to a large class of conformally invariant elliptic variational problems.

We apply such techniques in our context, when $M := \mathcal{E}$ is the ellipsoid of parameter $\Lambda = diag\,(\lambda_1, \cdots, \lambda_d)$. We still have maps $\Phi : \Sigma \to \mathbb{R}^d$, satisfying that $\Delta\Phi$ is parallel to $\Lambda\Phi$ (like for the harmonic map equation into $\mathcal{E}$) but the novelty is that they do not lie into the ellipsoid $\mathcal{E}$. Therefore, we first apply the natural radial projection on $\mathcal{E}$. Notice that this projection is also the orthonormal projection on the ellipsoid, with respect to the Ellipsoid scalar product $\langle .,. \rangle_{\mathcal{E}} := \langle \Lambda ., . \rangle$. Now because of the projection, we shall adapt Theorem (4.1). We will compare the map $\Phi = (\phi_1, \cdots, \phi_d) \in W^{1,2} \cap L^\infty \left(\mathbb{D}, \mathbb{R}^d \setminus \{0\}\right)$ to a more clever harmonic-like function. Let ω be a weight satisfying $\omega \in L^\infty$ and $\frac{1}{\omega} \in L^\infty$. We denote by C_ω a constant such that

$$\forall x \in \Sigma, \frac{1}{C_\omega} \leq \omega(x) \leq C_\omega$$

On a surface Σ, we say that $u : \Sigma \to \mathcal{E}$ is ω^2-harmonic if it is a critical point of the following energy

$$E_\omega(v) = \int_\Sigma \omega^2 \left|\nabla v\right|_g^2 dA_g \ , \tag{4.2}$$

among all $v \in W^{1,2}(\Sigma, \mathcal{E})$. For a map v on a disk $\mathbb{D}$, with small energy, we say that u is the ω^2-harmonic replacement of v if it is ω^2-harmonic on $\mathbb{D}$ and if $u = v$ on $\partial\mathbb{D}$.

We have the following quantitative result on the projection of eigenmaps on the ellipsoid, with equation $\Delta\Phi = e^{2u}\Lambda\Phi$, comparing this projection to the ω^2-harmonic replacement of the radial projection $\widetilde{\Phi} = \frac{\Phi}{\omega}$ of Φ, where $\omega = |\Phi|_{\mathcal{E}}$. Moreover, a quantitative estimate occurs, with a smallness assumption on $\frac{1}{\lambda_\star(\mathbb{D}, e^{2u}dx)}$, the inverse of the first non-zero Dirichlet eigenvalue on the disk with respect to e^{2u}.

Proposition 4.1. *Let $\Lambda = diag\,(\lambda_1, \cdots, \lambda_d)$ and a constant $C > 0$. We assume that $\lambda_1 \leq \cdots \leq \lambda_d$. Then there is $\varepsilon_1 := \varepsilon_1(\Lambda, C)$, such that for any map $\Phi = (\phi_1, \cdots, \phi_d) \in W^{1,2} \cap L^\infty \left(\mathbb{D}, \mathbb{R}^d\right)$, satisfying*

- *$C_\omega \leq C$, where $\omega = |\Phi|_{\mathcal{E}}$*
- *$\int_{\mathbb{D}} |\nabla\Phi|^2 \leq \varepsilon_0$*
- *$\Delta\Phi = e^{2u}\Lambda\Phi$.*
- *$\lambda_\star\left(\mathbb{D}, e^{2u}dx\right) \geq \lambda_d$*

We denote $\widetilde{\Psi}$ *the* ω^2*-harmonic replacement of* $\widetilde{\Phi} := \frac{\Phi}{\omega}$ *and we denote by* $\Psi = \omega\widetilde{\Psi}$*. Then*

$$\frac{1}{4}\int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\,dx \leq \left(\frac{2}{\lambda_1}+1\right)\int_{\mathbb{D}} |\nabla\omega|^2\,dx\ . \tag{4.3}$$

Proof. Since $\Phi = \Psi$ on the boundary of $\mathbb{D}$, we start with the following formula

$$\int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\,dx = \int_{\mathbb{D}} |\nabla\Phi|^2\,dx - \int_{\mathbb{D}} |\nabla\Psi|^2\,dx - 2\int_{\mathbb{D}} \langle \Phi-\Psi, \Delta\Psi\rangle \tag{4.4}$$

Now, we have the following equation on Ψ:

$$\begin{aligned}\Delta\Psi &= -div\left(\nabla\left(\omega\widetilde{\Psi}\right)\right)\\ &= -div\left(\omega^{-1}\omega^2\nabla\widetilde{\Psi} + \widetilde{\Psi}\nabla\omega\right)\\ &= -\omega^{-1}div\left(\omega^2\nabla\widetilde{\Psi}\right) + \widetilde{\Psi}\Delta\omega\ .\end{aligned}$$

We then have that

$$-2\int_{\mathbb{D}} \langle \Phi-\Psi, \Delta\Psi\rangle = -2\int_{\mathbb{D}} \left\langle \Phi-\Psi, \left(-\omega^{-1}div\left(\omega^2\nabla\widetilde{\Psi}\right)\right)\right\rangle - 2\int_{\mathbb{D}} \left\langle \Phi-\Psi, \widetilde{\Psi}\Delta\omega\right\rangle \tag{4.5}$$

and that the second right-hand term satisfies

$$\begin{aligned}2\left|\int_{\mathbb{D}} \left\langle \Phi-\Psi, \widetilde{\Psi}\Delta\omega\right\rangle\right| &= 2\left|\int_{\mathbb{D}} \left\langle \nabla(\Phi-\Psi), \widetilde{\Psi}\nabla\omega\right\rangle + \int_{\mathbb{D}} (\Phi-\Psi).\left\langle \nabla\widetilde{\Psi}, \nabla\omega\right\rangle\right|\\ &\leq 2\left(\frac{1}{\lambda_1}\int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{D}} |\nabla\omega|^2\right)^{\frac{1}{2}}\\ &\quad + \int_{\mathbb{D}} |\Phi-\Psi|^2\left|\nabla\widetilde{\Psi}\right|^2 + \int_{\mathbb{D}} |\nabla\omega|^2\ ,\end{aligned}$$

where for the inequality, we used that

$$\lambda_1\left|\widetilde{\Psi}\right|^2 = \left(\min_{1\leq i\leq d}\lambda_i\right)\left|\widetilde{\Psi}\right|^2 \leq \left|\widetilde{\Psi}\right|_{\mathcal{E}}^2 = \left|\widetilde{\Phi}\right|_{\mathcal{E}}^2 = 1\ ,$$

To estimate the first right-hand term of (4.5), we use that $\widetilde{\Psi}$ is a critical point for the energy E_ω (see (4.2)) on the disk. We have the Euler-Lagrange equation $-div\left(\omega^2\nabla\widetilde{\Psi}\right) \in \left(T_{\widetilde{\Psi}}\mathcal{E}\right)^{\perp}$. More precisely, $-div\left(\omega^2\nabla\widetilde{\Psi}\right)$ is parallel to $\Lambda\widetilde{\Psi}$ and computing $0 = \frac{1}{2}\Delta\left(\left|\widetilde{\Psi}\right|_{\mathcal{E}}^2\right)$ we deduce the equation

$$-div\left(\omega^2\nabla\widetilde{\Psi}\right) = \omega^2 Q\Lambda\widetilde{\Psi}\ ,\ \text{where } Q = \frac{\left|\nabla\widetilde{\Psi}\right|_{\mathcal{E}}^2}{\left|\Lambda\widetilde{\Psi}\right|^2}\ . \tag{4.6}$$

Therefore,

$$-2\int_{\mathbb{D}} \left\langle \Phi-\Psi, \left(-\omega^{-1}div\left(\omega^2\nabla\widetilde{\Psi}\right)\right)\right\rangle = -2\int_{\mathbb{D}} \langle \Phi-\Psi, \Lambda\Psi\rangle\, Q = \int_{\mathbb{D}} |\Phi-\Psi|_{\mathcal{E}}^2\, Q$$

noticing again that $|\Psi|^2_{\mathcal{E}} = |\Phi|^2_{\mathcal{E}}$ so that $-2\langle \Psi, \Phi - \Psi\rangle_{\mathcal{E}} = |\Phi - \Psi|^2_{\mathcal{E}}$. We can write (4.4) as

$$\begin{aligned}\int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\,dx \le \int_{\mathbb{D}} |\nabla\Phi|^2\,dx - \int_{\mathbb{D}} |\nabla\Psi|^2\,dx + \left(1+\frac{\lambda_d^2}{\lambda_1}\right)\int_{\mathbb{D}} |\Phi-\Psi|^2 \left|\nabla\widetilde{\Psi}\right|^2 \\ +2\left(\frac{1}{\lambda_1}\int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\right)^{\frac12}\left(\int_{\mathbb{D}} |\nabla\omega|^2\right)^{\frac12} + \int_{\mathbb{D}} |\nabla\omega|^2\ .\end{aligned} \tag{4.7}$$

We choose $\varepsilon_1 < \varepsilon_0$ where ϵ_0 is given by (6.9) in the appendix, we have that

$$\int_{\mathbb{D}} |\Phi-\Psi|^2 \left|\nabla\widetilde{\Psi}\right|^2 \le D\varepsilon_1 \int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\,dx$$

for a universal constant $D > 0$. Let $\eta > 0$. Letting $\left(1+\frac{\lambda_d^2}{\lambda_1}\right) D\epsilon_1 \le \eta$, we have that

$$(1-\eta)\,X^2 \le \int_{\mathbb{D}} |\nabla\Phi|^2\,dx - \int_{\mathbb{D}} |\nabla\Psi|^2\,dx + \frac{2}{\sqrt{\lambda_1}}XY + Y^2\ , \tag{4.8}$$

where $X^2 = \int_{\mathbb{D}} |\nabla(\Phi-\Psi)|^2\,dx$ and $Y^2 = \int_{\mathbb{D}} |\nabla\omega|^2$.

Now, let's prove that the difference of energies of Φ and Ψ is non positive. Let $1 \le i \le d$. We test $(\phi_i - \psi_i)$ in the variational characterization of $\lambda_\star\left(\mathbb{D}, e^{2u}\right)$, we have

$$\lambda_i \le \lambda_\star\left(\mathbb{D}, e^{2u}dx\right) \le \frac{\int_{\mathbb{D}} |\nabla(\phi_i-\psi_i)|^2\,dx}{\int_{\mathbb{D}} (\phi_i-\psi_i)^2\,e^{2u}dx}$$

so that we get setting $\lambda_\star := \lambda_\star\left(\mathbb{D}, e^{2u}dx\right)$ and summing on i

$$\lambda_\star \int_{\mathbb{D}} |\Phi-\Psi|^2\,e^{2u}dx \le \int_{\mathbb{D}} |\nabla\Phi|^2\,dx + \int_{\mathbb{D}} |\nabla\Phi|^2\,dx - 2\int_{\mathbb{D}} \langle\nabla\Phi, \nabla\Psi\rangle\,dx\ . \tag{4.9}$$

Moreover, we integrate the λ_i-eigenfunction equation $\Delta\psi_i = \lambda_i\psi_i e^{2u}$ against $2(\psi_i - \phi_i)$ to get

$$2\int_{\mathbb{D}} |\nabla\phi_i|^2\,dx - 2\int_{\mathbb{D}} \langle\nabla\phi_i, \nabla\psi_i\rangle\,dx = 2\lambda_i\int_{\mathbb{D}} (\phi_i)^2\,e^{2u}dx - 2\lambda_i\int_{\mathbb{D}} \phi_i\psi_i e^{2u}dx\ .$$

We sum on i and we get that

$$2\int_{\mathbb{D}} |\nabla\Phi|^2\,dx - 2\int_{\mathbb{D}} \langle\nabla\Phi, \nabla\Psi\rangle\,dx = 2\int_{\mathbb{D}} \langle\Phi, \Phi-\Psi\rangle_{\mathcal{E}}\,e^{2u}dx\ . \tag{4.10}$$

Now suming on i and remembering that $|\Psi|^2_{\mathcal{E}} = |\Phi|^2_{\mathcal{E}}$, we have that

$$2\langle\Phi, \Phi-\Psi\rangle_{\mathcal{E}} = |\Phi-\Psi|^2_{\mathcal{E}}\ .$$

We sum (4.9) and (4.10) to get

$$\int_{\mathbb{D}} |\nabla\Phi|^2\,dx - \int_{\mathbb{D}} |\nabla\Psi|^2\,dx \le \int_{\mathbb{D}} \left(|\Phi-\Psi|^2_{\mathcal{E}} - \lambda_\star|\Phi-\Psi|^2\right)e^{2u}dx \le 0\ , \tag{4.11}$$

because $\lambda_1 \le \cdots \le \lambda_d \le \lambda_\star$. Therefore, using (4.8), we get

$$\left(\sqrt{1-\eta}X - \frac{Y}{\sqrt{\lambda_1(1-\eta)}}\right)^2 \ \le\ Y^2\left(1+\frac{1}{\lambda_1(1-\eta)}\right)\ , \tag{4.12}$$

so that choosing $(1-\eta)^2 = \frac12$ gives the claim. $\diamondsuit$

Notice that if the target manifold is a sphere, we already have that $\widetilde{\Phi} = \widetilde{\Psi}$ and that $\widetilde{\Phi} = \frac{\Phi}{|\Phi|}$ is already a $|\Phi|^2$-harmonic map. In the previous proposition we only find a local non-exact result. Notice then that if the target manifold is a sphere, we do not need this section to prove Claim 5.8 and Claim 5.13 for sequences of eigenfunctions given by Claim 5.1. It would be interesting to know whether there is a global projection of eigenmaps on ellipsoids that are harmonic-like maps.

5. Existence of a maximal metric in the conformal constraint case

5.1. **Selection of a maximizing sequence.** Let (Σ, g) be a smooth Riemannian manifold without boundary. We denote respectively by $\mathcal{M}(\Sigma)$ and $\mathcal{M}_1(\Sigma)$ the set of Radon measures and probability measures on Σ, endowed with the topology of the weak-$\star$ convergence. We let $\epsilon > 0$. Let K_ϵ be the heat operator associated to g, so that for any positive Radon measure, $\nu \in \mathcal{M}(\Sigma)$, $K_\epsilon[\nu]dA_g$ is the solution at time $\epsilon > 0$ of the heat equation on (Σ, g) which converges to ν as $\epsilon \to 0$ for the weak-$\star$ convergence in $\mathcal{M}(\Sigma)$. We set

$$\Lambda_\varepsilon = \inf_{\nu \in \mathcal{M}_1(\Sigma)} F\left(\lambda_1(K_\epsilon[\nu]g), \cdots, \lambda_m(K_\epsilon[\nu]g)\right) > -\infty \ . \tag{5.1}$$

Λ_ε is finite since F is decreasing with respect to all coordinates and because all the eigenvalues are bounded by the result by Korevaar [Kor93]. We know that $\nu \mapsto K_\epsilon[\nu]$ is continuous from $\mathcal{M}(\Sigma)$ to $\mathcal{C}^0(\Sigma)$. Therefore, by continuity of the functional and compactness of $\mathcal{M}_1(\Sigma)$, up to the extraction of a subsequence, a minimizing sequence for the variational problem (5.1) converges in $\mathcal{M}_1(\Sigma)$ to a measure $\nu_\epsilon \in \mathcal{M}_1(\Sigma)$. We have

$$\Lambda_\varepsilon = F\left(\lambda_1(K_\epsilon[\nu_\epsilon]g), \cdots, \lambda_m(K_\epsilon[\nu_\epsilon]g)\right) \ . \tag{5.2}$$

We set $e^{2u_\epsilon} = K_\epsilon[\nu_\epsilon]$. Then, the sequence of smooth positive functions e^{2u_ϵ} defines a minimizing sequence for the variational problem $\Lambda_F(\Sigma, [g])$. Indeed, by definition $\Lambda_\epsilon \geq \Lambda_F(\Sigma, [g])$. Now, if $\tilde{g} = e^{2u}g \in [g]$ is such that $A_{\tilde{g}}(\Sigma) = 1$ and

$$F\left(\lambda_1(\tilde{g}), \cdots, \lambda_m(\tilde{g})\right) \leq \Lambda_F(\Sigma, [g]) + \frac{\eta}{2} \tag{5.3}$$

for some small $\eta > 0$, then, knowing that $K_\epsilon[e^{2u}] \to e^{2u}$ in $\mathcal{C}^0(\Sigma)$ as $\epsilon \to 0$, there is $\epsilon_0 > 0$ such that

$$\Lambda_\epsilon \leq F\left(\lambda_1(K_\epsilon[e^{2u}]g), \cdots, \lambda_m(K_\epsilon[e^{2u}]g)\right) \leq F\left(\lambda_1(\tilde{g}), \cdots, \lambda_m(\tilde{g})\right) + \frac{\eta}{2}$$

for any $\epsilon < \epsilon_0$. This, with (5.3), we get that $\Lambda_\epsilon \leq \Lambda_F(\Sigma, [g]) + \eta$. This means that $\Lambda_\epsilon \to \Lambda_F(\Sigma, [g])$ as $\epsilon \to 0$ as required.

Now, for a given ϵ, $e^{2u_\epsilon} = K_\epsilon[\nu_\epsilon]$ corresponds to the minimum of a variational problem (5.1). We let λ_k be the limit of $\lambda_k^\epsilon = \lambda_k(e^{2u_\epsilon}g)$ as $\varepsilon \to 0$. Notice that by the asumpsions on F, $\lambda_1 \neq 0$.

We obtain by the following claim a system of equations corresponding to this critical point of a regularized functional depending on ϵ.

Claim 5.1. *Let $\epsilon > 0$. Then, there are non-negative integers $i_1^\epsilon, \cdots, i_m^\epsilon$ and a map $\Phi_\epsilon : \Sigma \to \mathbb{R}^{i_1^\epsilon} \times \cdots \times \mathbb{R}^{i_m^\epsilon}$ such that*

- *The family of coordinate functions $\left(\Phi_\epsilon^{k,j}\right)_{1 \leq k \leq m, 1 \leq j \leq i_k^\epsilon}$ is independent.*
- $\forall k \in \{1, \cdots, m\}, \Delta_g \Phi_\epsilon^k = \lambda_\epsilon^k e^{2u_\epsilon} \Phi_\epsilon^k$
- $\int_\Sigma |\nabla \Phi_\varepsilon|_g^2 \, dA_g = \int_\Sigma e^{2u_\epsilon} |\Phi_\epsilon|_{\mathcal{E}_\varepsilon}^2 \, dA_g = 1$

- $K_\epsilon[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}] \geq 1$ *in* Σ
- $K_\epsilon[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}] = 1$ *on* $supp(\nu_\epsilon)$,

where $\Phi^k_\epsilon : \Sigma \to \mathbb{R}^{i^\epsilon_k}$ *is a coordinate map of* Φ_ϵ *and* $\Phi^{k,j}_\epsilon : \Sigma \to \mathbb{R}$ *is a coordinate function of* Φ^k_ϵ, $|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon} = \sum_{k=1}^m \lambda^k_\epsilon \left|\Phi^k_\epsilon\right|^2$ *is the norm of* Φ_ϵ *with respect to the quadratic form associated to the ellipsoid* $\mathcal{E}_\varepsilon$ *and* $\left|\Phi^k_\epsilon\right|^2$ *is the Euclidean norm of* Φ^k_ϵ *in* $\mathbb{R}^{i^\epsilon_k}$ *and* $\lambda^k_\epsilon = \lambda_k(e^{2u_\epsilon}g)$.

Proof. Notice that the proof is written in the same spirit as the proof of Theorem 3.1 or the proof of Claim 3.1 in [Pet18]. Since $\epsilon > 0$ is set, we omit the ϵ indices of λ^k_ϵ, Φ_ϵ, i^ϵ_k, ν_ϵ, e^{2u_ϵ} and $\mathcal{E}_\varepsilon$.

Let $\mu \in \mathcal{M}(\Sigma)$ be a positive radon measure and $t \geq 0$. We now denote by $\lambda^k_t = \lambda_k(e^{2u_t}g)$ where $e^{2u_t} = K_\epsilon[\nu + t\mu]$. Let $\phi_t \in S(E_k(e^{2u_t}g))$ (ie ϕ_t is an eigenfunction associated to λ^k_t such that $\int_\Sigma \phi_t^2 e^{2u_t} dA_g = 1$). As we did in the proof of Theorem 3.1, we can easily prove that $\lambda^k_t \to \lambda_k := \lambda_k(e^{2u}g)$ as $t \to 0^+$ and ϕ_t converges to $\phi \in S(E_k(e^{2u}g))$ up to the extraction of a subsequence $t_j \to 0^+$ (ie ϕ is an eigenfunction associated to λ_k such that $\int_\Sigma \phi^2 e^{2u} dA_g = 1$ in $\mathcal{C}^2(\Sigma)$).

Now we focus on the equation satisfied by $R_t = \phi_t - \pi_k(\phi_t)$ where π_k is the orthonormal projection on $E_k(e^{2u}g)$

$$\Delta_g\left(\frac{R_t}{\alpha_t}\right) - \lambda_k e^{2u}\frac{R_t}{\alpha_t} = \frac{\lambda^k_t - \lambda_k}{\alpha_t} e^{2u}\phi_t + \frac{t}{\alpha_t}\lambda^k_t K_\epsilon[\mu]\phi_t \tag{5.4}$$

where

$$\alpha_t = \|R_t\|_\infty + t + \left|\lambda^k_t - \lambda_k\right| . \tag{5.5}$$

Up to the extraction of a subsequence, we have

$$\frac{t}{\alpha_t} \to t_0 \text{ and } \frac{\lambda^k_t - \lambda_k}{\alpha_t} \to \delta_0 \text{ and } \frac{R_t}{\alpha_t} \to R_0 \text{ in } \mathcal{C}^2(\Sigma)$$

as $t \to 0^+$. The last limit follows from the standard elliptic theory applied to the equation (5.4). We pass to the limit in this equation (5.4) and (5.5) and get

$$\Delta_g(R_0) - \lambda_k e^{2u} R_0 = \delta_0 e^{2u}\phi_t + t_0\lambda_k K_\epsilon[\mu]\phi \tag{5.6}$$

and

$$1 = \|R_0\|_\infty + t_0 + |\delta_0| . \tag{5.7}$$

We integrate (5.6) against ϕ and we get that

$$\delta_0 + t_0\lambda_k \int_\Sigma K_\epsilon[\mu]\phi^2 dA_g = 0 .$$

If $t_0 = 0$, then, $\delta_0 = 0$ and (5.6) becomes $\Delta_g(R_0) - \lambda_k e^{2u} R_0 = 0$. As a limit of functions in $E_k(e^{2u}g)^\perp$, we have that $R_0 \in E_k(e^{2u}g) \cap E_k(e^{2u}g)^\perp$ so that $R_0 = 0$, contradicting (5.7). Therefore, $t_0 \neq 0$ and

$$\frac{\lambda^k_t - \lambda_t}{t} \to \frac{\delta_0}{t_0} = -\lambda_k \int_\Sigma K_\epsilon[\mu]\phi^2 dA_g \text{ as } t \to 0^+ . \tag{5.8}$$

Since $F\left(\left(1+t\int_\Sigma d\mu\right)\lambda_t^1,\cdots,\left(1+t\int_\Sigma d\mu\right)\lambda_t^m\right)\geq\Lambda_\epsilon$ for any $t>0$, we deduce from (5.8) that

$$\begin{aligned}\forall\mu\in\mathcal{M}(\Sigma),\exists(\phi_1,\cdots,\phi_m)\in&S\left(E_1(e^{2u}g)\right)\times\cdots\times S\left(E_m(e^{2u}g)\right),\\ &\int_\Sigma\left(\sum_{k=1}^m\lambda_k t_k\left(1-K_\epsilon[\phi_i^2]\right)\right)d\mu\leq 0\end{aligned}\tag{5.9}$$

where $t_k=\partial_k F(\lambda_1,\cdots,\lambda_m)$. We define the following subset of $\mathcal{C}^0(\Sigma)$

$$\begin{aligned}K=\{\Psi\in\mathcal{C}^0(\Sigma);\exists(\Phi_1,\cdots,\Phi_m)\in&E_1(e^{2u}g)^{i_1}\times\cdots\times E_m(e^{2u}g)^{i_m}\\ \text{s.t }\Psi=\sum_{k=1}^m\lambda_k t_k\left(K_\epsilon[|\Phi_k|^2]-1\right)&\text{ and }\int_\Sigma|\Phi_k|^2e^{2u}=1\}\end{aligned}\tag{5.10}$$

and

$$F=\{f\in\mathcal{C}^0(\Sigma);f\geq 0\}\ .$$

The set F is closed and convex. The set K is convex as the convex hull of

$$\left\{\sum_{k=1}^m\lambda_k t_k\left(K_\epsilon[\phi_k^2]-1\right);(\phi_1,\cdots,\phi_m)\in S(E_1(e^{2u}g))\times\cdots\times S(E_m(e^{2u}g))\right\}\ .$$

Since $E_1(e^{2u}g)\times\cdots\times E_m(e^{2u}g)$ is finite dimensional, the vector space spanned by this set is finite dimensional and is a compact set since it is bounded. Therefore K is also a compact set. If $F\cap K=\emptyset$, by the Hahn-Banach theorem, there is $\mu\in\mathcal{M}(\Sigma)$ such that

$$\forall f\in F,\int_M fd\mu\geq 0\text{ and }\forall\Psi\in K,\int_M\Psi d\mu<0\ .\tag{5.11}$$

We deduce that $\mu\neq 0$, that μ is positive but (5.11) contradicts (5.9). We proved that $F\cap K\neq\emptyset$. It gives $(\Phi_1,\cdots,\Phi_m)\in E_1(e^{2u}g)^{i_1}\times\cdots\times E_m(e^{2u}g)^{i_m}$ such that

$$\forall 1\leq k\leq m,\int_\Sigma|\Phi_k|^2e^{2u}=1\text{ and }K_\epsilon\left[\sum_{k=1}^m\lambda_k t_k|\Phi_k|^2\right]\geq\sum_{k=1}^m\lambda_k t_k\ .$$

Up to a renormalization of the family $(\Phi_1,\cdots,\Phi_m)$, we obtain that

$$\int_\Sigma\left(\sum_{k=1}^m\lambda_k|\Phi_k|^2\right)e^{2u}dA_g=1\text{ and }K_\epsilon\left[\sum_{k=1}^m\lambda_k|\Phi_k|^2\right]\geq 1\ .\tag{5.12}$$

and we can write that

$$1=\int_\Sigma|\Phi|^2_{\mathcal{E}}e^{2u}dA_g=\int_\Sigma K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]d\nu\geq\int_\Sigma d\nu=1$$

Therefore, $K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]=1$ ν-a.e and since $K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]$ is continuous, $K_\epsilon\left[|\Phi|^2_{\mathcal{E}}\right]=1$ on $supp(\nu)$.

In order to complete the claim, let's prove that up to change the family $(\Phi_{k,j})_{1\leq k\leq m,1\leq j\leq i_k}$, we can assume that it is a family of independent eigenfunctions in $L^2(g)$. It suffices to prove that for any $k\in\{1,\cdots,m\}$ such that $\lambda_k\neq\lambda_{k-1}$, the family $(\Phi_{l,j})_{k\leq i\leq k+\kappa,1\leq j\leq i_l}$ can be assumed to be a family of independent functions where $\kappa\geq 0$ is chosen such that $k+\kappa\leq m$,

$\lambda_{k+\kappa} = \lambda_k$ and $\lambda_{k+\kappa+1} > \lambda_k$. We relabel this family $(\varphi_1, \cdots, \varphi_\theta)$. Let $(u_1, \cdots, u_{\nu_k})$ be a basis of $E_k(e^{2u}g)$. Then we can write

$$\varphi_j = \sum_{l=1}^{\nu_k} \alpha_{j,l} u_l \text{ for } 1 \leq j \leq \theta, \alpha_{j,l} \in \mathbb{R} \ .$$

Now we define the quadratic form $q : \mathbb{R}^{\nu_k} \to \mathbb{R}$ by

$$q(x) = \sum_{j=1}^{\theta} \left(\sum_{l=1}^{\nu_k} \alpha_{j,l} x_l \right)^2$$

By the Gaussian decomposition of quadratic forms, we have independent linear forms $L_{\alpha,1}, \cdots, L_{\alpha,\nu_k}$ and $\varepsilon_{\alpha,1}, \cdots, \varepsilon_{\alpha,\nu_k} \in \{-1, 0, 1\}$ depending on $\alpha = (\alpha_{j,l})$ such that

$$q(x) = \sum_{l=1}^{\nu_k} \varepsilon_{\alpha,l} \left(L_{\alpha,l}(x) \right)^2$$

Of course, since $q(x) \geq 0$, we have that $\varepsilon_{\alpha,l} \in \{0, 1\}$, and we replace the family $(\varphi_1, \cdots, \varphi_\theta)$ by the family $(L_{\alpha,l}(u_1, \cdots, u_{\nu_k}))_{1 \leq l \leq \nu_k;\ \text{s.t. } \varepsilon_{\alpha,l}=1}$.

This ends the proof of Claim 5.12.

$\diamondsuit$

We know by [Che76] that the multiplicity of λ_ϵ^k is bounded by k and the topology of Σ. Therefore since the family of coordinate functions $\left(\Phi_\epsilon^{k,j}\right)_{1 \leq k \leq m, 1 \leq j \leq i_k^\epsilon}$ is independent the sum $i_1^\epsilon + \cdots + i_m^\epsilon$, is bounded with respect to ϵ and up to the extraction of a subsequence, we assume that the indices i_k^ϵ do not depend on ϵ.

5.2. **Notations for local analysis and rescalings.** Let (Σ, g) be a smooth Riemannian surface with $A_g(\Sigma) = 1$.

We recall that $\mathcal{M}(\Sigma)$ is the set of positive Radon measures provided with the weak* topology and $\mathcal{M}_1(\Sigma)$ the subset of probability measures.

For an open set $\Omega \subset \Sigma$ we denote by $\lambda_\star(\Omega, g)$ the first Dirichlet eigenvalue in (Ω, g).

For all the paper, we fix $\delta_0 > 0$, a constant $C_0 > 1$ and a family $(x_l)_{l=1,\ldots,L}$ of points in Σ and smooth functions $v_l : \Sigma \mapsto \mathbb{R}$ such that

- for any $l \in \{1, \ldots, L\}$, the metric $g_l = e^{-2v_l} g$ is a flat metric in the ball $B_{g_l}(x_l, 2\delta_0) = \Omega_l$.
- $\Sigma = \bigcup_{l=1}^{L} \omega_l$ where $\omega_l = B_{g_l}(x_l, \delta_0)$.
- For any $1 \leq l \leq L$, $C_0^{-2} \leq e^{2v_l} \leq C_0^2$.
- For any $x \in \omega_l$ and $0 < r < \delta_0$, $B_g(x, C_0^{-1} r) \subset B_{g_i}(x, r) \subset B_g(x, C_0 r)$

For $1 \leq l \leq L$ and a point $z \in \mathbb{D}_{2\delta_0}(0)$, we let

$$e^{2\tilde{v}_l(z)} = e^{2v_l(\exp_{g_l,x_l}(z))} \text{ and } \bar{z}^l = \exp_{g_l,x_l}(z)$$

and for $x \in \Omega_l$ and a set $\Omega \subset \Omega_l$,

$$\tilde{x}^l = \exp_{g_l,x_l}^{-1}(x) \text{ and } \widetilde{\Omega}^l = \exp_{g_l,x_l}^{-1}(\Omega) \ .$$

For a smooth density e^{2u} with $e^{2u}g \in [g]$, we let

$$e^{2\tilde{u}^l(z)} = e^{2\tilde{v}_l(z)} e^{2u(\exp_{g_l,x_l}(z))}$$

so that for $\Omega \subset \Omega_l$,

$$\int_\Omega e^{2u} dA_g = \int_{\widetilde{\Omega}^l} e^{2\tilde{u}^l} dx \ .$$

For other functions $\phi \in L^1(M)$ or measures $\nu \in \mathcal{M}(\Sigma)$, we let

$$\tilde{\phi}^l(z) = \phi(\exp_{g_l,x_l}(z)) \text{ and } \tilde{\nu}^l = \exp^\star_{g_l,x_l}(\nu) \ .$$

Let $p_\varepsilon(x,y)$ be the heat kernel of (M,g) at time $\varepsilon > 0$. Then, for $y, z \in \tilde{\Omega}_l$, we let

$$\tilde{p}^l_\varepsilon(z,y) = e^{2\tilde{v}_l(z)} p_\varepsilon(\exp_{g_l,x_l}(z), \exp_{g_l,x_l}(y))$$

so that for a density $e^{2u(x)} = \int_\Omega p_\varepsilon(x,y) d\nu(y)$ for $\Omega \subset \Omega_l$ and some measure ν, we have

$$e^{2\tilde{u}^l(z)} = \int_{\tilde{\Omega}^l} \tilde{p}^l_\varepsilon(z,y) d\tilde{\nu}(y) \text{ and } \int_{\widetilde{\Omega}^l} \tilde{\phi}^l(z) \tilde{p}^l_\varepsilon(z,\tilde{y}^l) dz = \int_\Omega \phi(x) p_\varepsilon(x,y) dA_g(x) \ .$$

for $\phi \in L^1(\Sigma)$. When the context is clear, we drop the exponent l in all the notations.

One fundamental remark for all the following analysis is some scale invariance properties for the heat kernel $p_\varepsilon(x,y)$ and $\lambda_\star\left(\Omega, e^{2u}g\right)$. We shall give convenient notations to handle this scale invariance. For parameters $a \in \mathbb{R}^2$ and $\alpha > 0$, we denote the following rescaled objects by

$$\hat{x} = \frac{\tilde{x} - a}{\alpha} \text{ and } \hat{\Omega} = \frac{\widetilde{\Omega} - a}{\alpha} \ ,$$

$$e^{2\hat{u}(z)} = \alpha^2 e^{2\tilde{u}(\alpha z + a)}, \hat{\phi}(z) = \tilde{\phi}(\alpha z + a) \text{ and } \hat{\nu} = H^\star_{a,\alpha}(\tilde{\nu}) \ ,$$

$$\hat{p}_\varepsilon(z,y) = \alpha^2 \tilde{p}^l_\varepsilon(\alpha z + a, \alpha y + a) \ ,$$

where $H_{a,\alpha}(x) = \alpha x + a$, so that if $e^{2u(x)} = \int_\Omega p_\varepsilon(x,y) d\nu(y)$, we have

$$e^{2\hat{u}(z)} = \int_{\hat{\Omega}} \hat{p}_\varepsilon(z,y) d\hat{\nu}(y)$$

and

$$\int_{\hat{\Omega}} \phi(z) \hat{p}_\varepsilon(z,\hat{y}) dz = \int_\Omega \phi(x) p_\varepsilon(x,y) dA_g(y) \ .$$

We also let for $z \in \mathbb{R}^2$,

$$\check{z} = \exp_{g_l,x_l}(\alpha z + a) \text{ and } \check{\Omega} = \exp_{g_l,x_l}(\alpha \Omega + a) \ ,$$

so that $\hat{\check{z}} = z$. Then we also have that

$$\lambda_\star\left(\Omega, e^{2u}g\right) = \lambda_\star\left(\hat{\Omega}, e^{2\hat{u}} dx\right) \ .$$

5.3. **No concentration or one bubble.** From here to the end of the paper, we assume a stronger assumption on F than (2.3) and (2.4):

(H') The maximizing sequence $e^{2u_\varepsilon}dA_g$ of $\Lambda_F(\Sigma,[g])$ given by Claim 5.1 satisfies that $\lambda_1(e^{2u_\varepsilon})$ does not converge to 0.

It slightly simplifies the presentation of the proof. In the end of this subsection, we explain and leave to the reader how to prove Therorem 2.2 with the general assumptions

Up to the extraction of a subsequence, we denote by ν the weak-$\star$ limit of $\{e^{2u_\epsilon}dA_g\}_{\epsilon>0}$ as $\epsilon\to 0$. Then ν is also the weak-$\star$ limit of $\{\nu_\epsilon\}_{\epsilon>0}$ as $\epsilon\to 0$. Indeed, let $\zeta\in\mathcal{C}^0(\Sigma)$,

$$\begin{aligned}\left|\int_\Sigma \zeta\left(d\nu_\epsilon - e^{2u_\epsilon}dA_g\right)\right| &= \left|\int_\Sigma \zeta\left(d\nu_\epsilon - K_\epsilon[\nu_\epsilon]dA_g\right)\right| \\ &= \left|\int_\Sigma \left(\zeta - K_\epsilon[\zeta]\right)d\nu_\epsilon\right| \\ &\le \|\zeta - K_\epsilon[\zeta]\|_\infty \to 0 \text{ as } \epsilon\to 0\,.\end{aligned}$$

In the following claim, we aim at proving that if the sequence $\{e^{2u_\epsilon}dA_g\}_{\epsilon>0}$ concentrates, then there is only one bubble.

Claim 5.2. *With assumption (H'), one of these situations occur*

(i) *The limiting measure of $\{e^{2u_\epsilon}dA_g\}_{\epsilon>0}$ does not have any atoms.*
(ii) *There are points $x_\epsilon\in\Sigma$, scales $\alpha_\epsilon>0$ and a probability measure $\hat\nu\in\mathcal{M}_1(\mathbb{R}^2)$ without atoms such that $\alpha_\epsilon\to 0$ and the measure*

$$e^{2\hat u_\varepsilon}(z)dz := \alpha_\epsilon^2 e^{2\tilde u_\epsilon(a_\epsilon+\alpha_\epsilon z)}dz$$

converges to $\hat\nu$ with respect to the weak-$\star$ convergence in $\mathbb{R}^2$ as $\epsilon\to 0$, where $a_\varepsilon=\tilde x_\varepsilon$.

Proof. Let us assume that $e^{2u_\epsilon}dA_g$ concentrates as $\epsilon\to 0$. We assume that $\nu = m\delta_x+\mu$ where δ_x is the Dirac measure at $x\in\Sigma$, $m\neq 0$ and $\mu\neq 0$. Then we let η_1 and η_2 be smooth functions such that

$$\begin{cases}\eta_1 = 1 & \text{in } B_g(x,r)\\ \eta_1 = 0 & \text{in } \Sigma\setminus B_g(x,r^{\frac12})\\ \int_\Sigma |\nabla\eta_1|_g^2\,dA_g \le \frac{C}{\ln(\frac1r)} & \end{cases}$$

and

$$\begin{cases}\eta_2 = 1 & \text{in } \Sigma\setminus B_g(x,r^{\frac14})\\ \eta_2 = 0 & \text{in } B_g(x,r^{\frac12})\\ \int_\Sigma |\nabla\eta_2|_g^2\,dA_g \le \frac{C}{\ln(\frac1r)} & \end{cases}.$$

Such functions exist with a constant independent of ϵ and r because the capacity of a point is zero. Since η_1 and η_2 are orthogonal functions,

$$\begin{aligned}\lambda_1(e^{2u_\epsilon}g) &\leq \max\left\{\frac{\int_\Sigma |\nabla\eta_1|_g^2\,dA_g}{\int_\Sigma (\eta_1)^2\,e^{2u_\epsilon}dA_g}, \frac{\int_\Sigma |\nabla\eta_2|_g^2\,dA_g}{\int_\Sigma (\eta_2)^2\,e^{2u_\epsilon}dA_g}\right\}\\ &\leq \frac{C}{\ln\left(\frac{1}{r}\right)}\max\left\{\frac{1}{\int_{B_g(x,r)} e^{2u_\epsilon}dA_g}, \frac{1}{\int_{\Sigma\setminus B_g(x,r^{\frac{1}{4}})} e^{2u_\epsilon}dA_g}\right\}\\ &\leq \frac{C}{\ln\left(\frac{1}{r}\right)}\max\left\{\frac{1}{m+o(1)}, \frac{1}{\int_{\Sigma\setminus B_g(x,r^{\frac{1}{4}})} d\mu+o(1)}\right\} \text{ as } \epsilon\to 0\end{aligned}$$

We let $\epsilon\to 0$ and then, $r\to 0$ and we obtain that $\lambda_1(e^{2u_\epsilon}g)\to 0$ as $\epsilon\to 0$, which is not possible. Therefore, $\nu=\delta_x$. We now aim at proving that there is only one bubble at the neighborhood of x in order to prove (ii).

We let some points a_ϵ^1, a_ϵ^2 and some scales α_ϵ^1 and α_ϵ^2 such that $\alpha_\epsilon^1>\alpha_\epsilon^2$ and for $i\in\{1,2\}$,

$$\left(\alpha_\epsilon^i\right)^2 e^{2\tilde{u}_\epsilon(a_\epsilon^i+\alpha_\epsilon^i z)}dz \rightharpoonup \hat{\nu}_i \text{ as } \epsilon\to 0$$

for $\hat{\nu}_1$ and $\hat{\nu}_2$ two non-zero measures on $\mathbb{R}^2$ and such that if $\frac{a_\epsilon^2-a_\epsilon^1}{\alpha_\epsilon^1}$ converges to a_0 as $\epsilon\to 0$, then $\int_{\mathbb{R}^2\setminus z_0} d\hat{\nu}_1>0$. We then aim at proving that $\frac{\left|a_\epsilon^2-a_\epsilon^1\right|}{\alpha_\epsilon^1}+\frac{\alpha_\epsilon^1}{\alpha_\epsilon^2}$ is bounded. Assume by contradiction that this quantity goes to $+\infty$ as $\epsilon\to 0$.

We choose for $R>1$ a function η_R such that

$$\begin{cases}\eta_R=1 & \text{in } \mathbb{D}_R\\ \eta_R=0 & \text{in } \mathbb{R}^2\setminus\mathbb{D}_{R^2}\\ \int_{\mathbb{R}^2}|\nabla\eta_R|^2\,dz\leq\frac{C}{\ln(R)} & \end{cases}.$$

We define η_1 and η_2 by $\widetilde{\eta_1}(z)=\eta_R\left(\frac{z-a_\epsilon^1}{\alpha_\epsilon^1}\right)\left(1-\eta_{R^2}\left(\frac{z-a_\epsilon^2}{\alpha_\epsilon^2}\right)\right)$ and $\widetilde{\eta_2}(z)=\eta_R\left(\frac{z-a_\epsilon^2}{\alpha_\epsilon^2}\right)$ on the chart ω_l at the neighborhood of x and extended by zero elsewhere. They have disjoint supports on Σ and

$$\begin{aligned}\lambda_1(e^{2u_\epsilon}g) &\leq \max\left\{\frac{\int_\Sigma |\nabla\eta_1|_g^2\,dA_g}{\int_\Sigma (\eta_1)^2\,e^{2u_\epsilon}dA_g}, \frac{\int_\Sigma |\nabla\eta_2|_g^2\,dA_g}{\int_\Sigma (\eta_2)^2\,e^{2u_\epsilon}dA_g}\right\}\\ &\leq \max\left\{\frac{2\left(\int_{\mathbb{R}^2}|\nabla\eta_R|^2+\int_{\mathbb{R}^2}|\nabla\eta_{R^2}|^2\right)}{\int_{\mathbb{D}_R\setminus\mathbb{D}_{R^4\left(\frac{\alpha_\epsilon^2}{\alpha_\epsilon^1}\right)^2(a_0)}} d\hat{\nu}_1+o(1)}, \frac{\int_{\mathbb{R}^2}|\nabla\eta_R|^2}{\int_{\mathbb{D}_R} d\hat{\nu}_2+o(1)}\right\}\end{aligned}$$

as $\epsilon\to 0$, where a_0 is the limit of $\frac{a_\epsilon^2-a_\epsilon^1}{\alpha_\epsilon^1}$ as $\epsilon\to 0$ if it exists. If not, we can replace $\mathbb{D}_R\setminus\mathbb{D}_{R^4\left(\frac{\alpha_\epsilon^2}{\alpha_\epsilon^1}\right)^2(a_0)}$ by $\mathbb{D}_R$ in the formula. Letting $\epsilon\to 0$ and then $R\to+\infty$, we get that $\lambda_1(e^{2u_\epsilon}g)\to 0$ as $\epsilon\to 0$, which is not possible. Thus, as expected $\frac{\left|a_\epsilon^2-a_\epsilon^1\right|}{\alpha_\epsilon^1}+\frac{\alpha_\epsilon^1}{\alpha_\epsilon^2}$, is bounded and up to a translation and a dilatation in $\mathbb{R}^2$, we have that $\hat{\nu}_1=\hat{\nu}_2$, and there is never dichotomy. In order to detect the bubble, one can pullback the measure $\eta e^{2\tilde{u}_\epsilon}dz$ on $\mathbb{S}^2$ by the stereographic projection σ (we denote this measure μ_ϵ), and by Hersch theorem

[Her70], we have that $\int_{\mathbb{S}^2} x \circ \theta_\varepsilon d\mu_\varepsilon = 0$ up to a conformal diffeomorphism θ_ε. We can define points a_ϵ and scales α_ϵ such that if $\alpha_\epsilon^2 e^{2\tilde{u}_\epsilon(a_\epsilon+\alpha_\epsilon z)}dz$ converges to $\hat{\nu}$ and $\sigma_\star\theta_\star\mu_\epsilon$ converges to μ, then $\mu = \hat{\nu}$ up to an isometry of $\mathbb{R}^2$. By the Hersch condition, and because of the concentration, μ is a non-zero measure. If μ is not a probability measure or if μ has an atom, then by the Hersch condition one can find a dichotomy which is a contradiction. This ends the proof of claim 5.2.

$\diamondsuit$

Thanks to this dichotomy, we just have to deal with two cases of analysis :

- (i) occurs and we prove that the limiting probability measure ν is absolutely continuous with respect to dA_g with a smooth density e^{2u} on Σ, and that the metric $e^{2u}g$ is smooth up to conical singularities.
- (ii) occurs and we prove that the limiting probability measure $\hat{\nu}$ on $\mathbb{R}^2$ is absolutely continous with respect to the Lebesgue measure with a smooth density $e^{2\hat{u}}$ and that the metric $e^{2\hat{u}}\xi$ is smooth up to conical singularities.

If case (i) occurs, and once we prove the expected regularity, thanks to upper semi-continuity of the eigenvalues in the space of probability measures $\mathcal{M}_1(\Sigma)$ endowed with the weak-$\star$ topology, and the monotonicity assumptions we made on F,

$$F\left(\lambda_1(e^{2u}g), \cdots, \lambda_m(e^{2u}g)\right) \leq \Lambda_F(\Sigma, [g])$$

which proves that $e^{2u}g$ is a minimal metric for the functional

If case (ii) occurs and once we prove the expected regularity, using this regularity result and the gap assumption (2.4), we obtain a contradiction. Indeed, we denote by

$$e^{2\check{u}}dA_h = \sigma^\star\left(e^{2\hat{u}}dz\right)$$

the pullback metric on the sphere $\mathbb{S}^2$ by the stereographic projection σ with respect to the north pole $p = (0,0,1)$. We let $(\psi_0, \psi_1, \cdots, \psi_m)$ be eigenfunctions on the sphere associated to $0 = \lambda_0(e^{2\check{u}}h) < \lambda_1(e^{2\check{u}}h) \leq \cdots \leq \lambda_m(e^{2\check{u}}h)$ and η be a cutoff function on the sphere such that $\eta = 1$ on $\mathbb{S}^2 \setminus B_h(p,r)$, $\eta = 0$ on $B_h(p,r)$ and $\int_{\mathbb{S}^2} |\nabla\eta|^2 \leq \frac{C}{\ln\left(\frac{1}{r}\right)}$. Then, we test the transplantation of the functions $\widetilde{\theta_k} = (\eta\psi_k)\circ\sigma^{-1}$ extended by zero on Σ for $1 \leq k \leq m$. This gives as $\epsilon \to 0$ that

$$\lambda_k^\epsilon \leq \frac{\int_\Sigma |\nabla\theta_k|_g^2 dA_g}{\int_\Sigma \theta_k^2 e^{2u_\epsilon}dA_g} + o(1) \leq \frac{\int_{\mathbb{S}^2\setminus B_h(p,r^2)} |\nabla\psi_k|_g^2 dA_h}{\int_{\mathbb{S}^2\setminus B_h(p,r)} \psi_k^2 e^{2u_\epsilon}dA_h} + o(1)\,.$$

Letting $\epsilon \to 0$ and then $r \to 0$, we get that for any $1 \leq k \leq m$,

$$\limsup_{\epsilon\to 0} \lambda_k^\varepsilon \leq \lambda_k(e^{2u}g)$$

Thanks to this property and the monotonicity assumption we made on F, we get that $\Lambda_F(\mathbb{S}^2) \leq \Lambda_F(\Sigma, [g])$. This contradicts (2.4).

Therefore, in order to prove Theorem 2.2, we prove the regularity of the limiting measures. It is the purpose of the remaining proof.

To prove Theorem 2.2 with assumptions (2.3) (2.4), one has to perform the same dichotomy argument as in Claim 5.2 to build by induction a so-called bubble tree (see proposition 2 in [Pet18]). Notice that the number of thick parts (surface+bubbles) is controlled by the least k such that the k-th eigenvalue does not go to 0. Then we

can conclude as soon as we prove the regularity of the limiting measures in every thick part. Again, all the following regularity estimates in the current paper on eigenfunctions associated to eigenvalues not srinking to 0 work perfectly well for that.

5.4. **Regularity estimates at the scale $\sqrt{\epsilon}$.** In case (ii) of Claim 5.2, the proof of the regularity of the limiting measure is immediate when we assume that the sequence $\{\frac{\alpha_\epsilon}{\sqrt{\epsilon}}\}$ is bounded, where α_ϵ is the scale defined in (ii), in claim 5.2. The very suitable scale $\sqrt{\epsilon}$ arises naturally from the choice of the heat kernel. Indeed, in this case, we let $\theta_0 = \lim_{\varepsilon\to 0} \frac{\epsilon}{e^{2\check{v}(x_0)}\alpha_\epsilon^2}$ and we denote by $\hat{\nu}$ the weak* limit of $\hat{\nu}_\epsilon$ in $\mathcal{M}(\mathbb{R}^2)$. Let $R_0 > 0$ and $z \in \mathbb{D}_{R_0}$. We have by (6.11) that

$$e^{2\hat{u}_\varepsilon}(z) = e^{2v_l(\check{x})}\alpha_\varepsilon^2 \int_\Sigma p_\varepsilon(\check{z}, y) d\nu_\varepsilon(y) \leq \frac{A_0 e^{2v_l(\check{z})}\alpha_\varepsilon^2}{4\pi\varepsilon} \int_\Sigma d\nu_\varepsilon \leq \frac{A_0}{4\pi\theta_0}(1 + o(1)) \ .$$

Since $\int_{\mathbb{D}_R} e^{2\hat{u}_\varepsilon}(z)dz \geq 1 + \alpha_{\epsilon,R}$, where $\lim_{R\to+\infty}\lim_{\epsilon\to 0}\alpha_{\epsilon,R} = 0$, we get that $\theta_0 < +\infty$. Now, we let $e^{2\hat{u}}$ be a smooth function on $\mathbb{R}^2$ defined by

$$e^{2\hat{u}(z)} = \int_{\mathbb{R}^2} \frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{\nu}(y) \ . \tag{5.13}$$

Let $R_0 > 0$, $R > R_0$ and $z \in \mathbb{D}_{R_0}$. We have that

$$\begin{aligned} \left|e^{2\hat{u}_\varepsilon(z)} - e^{2\hat{u}(z)}\right| &= \left|\int_\Sigma \alpha_\varepsilon^2 p_\varepsilon(\check{z}, y) d\nu_\varepsilon(y) - e^{2\hat{u}(z)}\right| \\ &\leq \int_{\Sigma\setminus\check{\mathbb{D}}_R} \alpha_\varepsilon^2 p_\varepsilon(\check{z}, y) d\nu_\varepsilon(y) + \left|\int_{\mathbb{D}_R} \hat{p}_\varepsilon(z, y) d\hat{\nu}_\varepsilon(y) - \int_{\mathbb{R}^2} \frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{\nu}(y)\right| \end{aligned}$$

so that

$$\begin{aligned} \left|e^{2\hat{u}_\varepsilon(z)} - e^{2\hat{u}(z)}\right| &\leq \frac{A_0}{4\pi\theta_0}(1 + o(1))e^{-\frac{(R-R_0)^2}{8\theta_0}} + \left|\int_{\mathbb{D}_R} \left(\hat{p}_\varepsilon(z, y) - \frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{4\pi\theta_0}\right) d\hat{\nu}_\varepsilon\right| \\ &\quad + \left|\int_{\mathbb{D}_R} \frac{e^{-\frac{|z-y|^2}{8\theta_0}}}{4\pi\theta_0}(d\hat{\nu}_\varepsilon - d\hat{\nu})\right| + \int_{\mathbb{R}^2\setminus\mathbb{D}_R} \frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{\nu} + o(1) \\ &\to \frac{A_0}{4\pi\theta_0} e^{-\frac{(R-R_0)^2}{8\theta_0}} + \int_{\mathbb{R}^2\setminus\mathbb{D}_R} \frac{e^{-\frac{|z-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{\nu} \text{ as } \varepsilon \to 0 \ . \end{aligned}$$

Letting $R \to +\infty$, we get for any $R_0 > 0$ that

$$e^{2\hat{u}_\varepsilon} \to e^{2\hat{u}} \text{ in } \mathcal{C}^0(\mathbb{D}_{R_0}) \text{ as } \varepsilon \to 0 \ . \tag{5.14}$$

Therefore, the limit $e^{2\hat{u}}dz$ of the sequence of measures $\{e^{2\hat{u}_\epsilon}dz\}$ is absolutely continuous with respect to the Lebesgue measure, with a smooth density and it is a probability measure.

Now up to the end of the proof we assume that $\frac{\alpha_\epsilon}{\sqrt{\epsilon}} \to +\infty$.

We cannot expect to get good estimates on the potential e^{2u_ϵ} since $\{\alpha_\epsilon^2 e^{2u_\epsilon}\}$ is not uniformly bounded anymore. However, the eigenfunctions which are coordinates of Φ_ϵ, given by Claim 5.1 satisfy lots of conditions. The purpose is now to prove that thanks to finer and finer estimates, they converge in suitable function spaces up to the extraction of a subsequence. We still have an immediate very partial result arising from a look at the scale $\sqrt{\epsilon}$. This weaker but fundamental result is the consequence of Claim 5.1. It states that at the scale $\sqrt{\epsilon}$, the sequence of eigenfunctions is bounded at the neighborhood of the support of ν_ϵ. As already said, this scale comes naturally from the choice of the heat kernel.

Claim 5.3. *For any $R > 0$, there is a constant $C_R > 0$ such that for any sequence (z_ϵ) of points of Σ with $d_g(z_\epsilon, supp(\nu_\epsilon)) \leq R\sqrt{\epsilon}$, we have*

$$\left|\Phi_\epsilon^k(z_\epsilon)\right| \leq C_R \text{ for all } \epsilon > 0 \text{ and } k \in \{1, \cdots, m\}$$

Proof. We refer the reader to Section 5.2 for the notations used during this proof. We can assume that $x_\varepsilon \in \omega_l$ for $1 \leq l \leq L$ fixed and we set

$$\hat{\Phi}_\varepsilon(x) = \tilde{\Phi}_\varepsilon\left(\sqrt{\varepsilon}x + \tilde{x}_\varepsilon\right)$$

for $x \in \mathbb{D}_{\frac{\delta}{\sqrt{\varepsilon}}}$. Then

$$\Delta_\xi \hat{\Phi}_\varepsilon^k = \varepsilon \lambda_k^\varepsilon e^{2\tilde{u}_\varepsilon\left(\sqrt{\varepsilon}x + \tilde{x}_\varepsilon\right)} \hat{\Phi}_\varepsilon^k$$

in $\mathbb{D}_{\delta\sqrt{\varepsilon}}$ for $1 \leq k \leq m$. By estimate (6.10) of Section 6.2, (ϵp_ϵ) is uniformly bounded so that $\left(\varepsilon e^{2\tilde{u}_\varepsilon\left(\sqrt{\varepsilon}x + \tilde{x}_\varepsilon\right)}\right)$ is uniformly bounded.

Now, we let $y_\varepsilon \in supp(\nu_\varepsilon)$ be such that $d_g(x_\varepsilon, y_\varepsilon) \leq R\sqrt{\varepsilon}$. By Claim 5.1, we have that $K_\epsilon\left[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}\right](y_\epsilon) = 1$. Let us write then with (6.10), in Section 6.2, that for $\rho > 0$,

$$\begin{aligned}
1 = K_\epsilon\left[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}\right](y_\epsilon) &\geq \sum_{k=1}^m \lambda_k K_\epsilon\left[\left|\Phi_\epsilon^k\right|^2\right](y_\epsilon) \\
&= \sum_{k=1}^m \lambda_k \int_\Sigma p_\varepsilon(y, y_\varepsilon)\left|\Phi_\varepsilon^k(y)\right|^2 dA_g(y) \\
&\geq \sum_{k=1}^m \lambda_k \frac{1}{4\pi A_0 \varepsilon} e^{-\rho^2 C_0^2} \int_{B_g\left(y_\varepsilon, 2\rho C_0\sqrt{\varepsilon}\right)} \left|\Phi_\varepsilon^k(y)\right|^2 dA_g(y) \\
&\geq \sum_{k=1}^m \lambda_k \frac{1}{4\pi A_0 C_0^2} e^{-\rho^2 C_0^2} \int_{\mathbb{D}_{2\rho}(\hat{z}_\varepsilon)} \left|\hat{\Phi}_\varepsilon^k(z)\right|^2 dz\ .
\end{aligned}$$

We set $\hat{z}_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\left(\tilde{y}_\varepsilon - \tilde{x}_\varepsilon\right)$ so that, up to a subsequence, $\hat{z}_\varepsilon \to z_0$ as $\varepsilon \to 0$ and we deduce from the previous inequality that, for any $\rho > 0$, $\{\hat{\Phi}_\varepsilon^k\}$ is bounded in $L^2(\mathbb{D}_\rho(z_0))$. Thus, by the Sobolev embedding $W^{2,2} \subset \mathcal{C}^0$ (see [GT01], Corollary 7.11, page 158) and the L^2 elliptic estimate (see [GT01], Theorem 9.11, page 235), it is clear that $\left\{\hat{\phi}_\varepsilon\right\}$ is uniformly bounded in $\mathbb{D}_\rho$ by some constant D_ρ. Setting $C_R = D_{2C_0R}$ gives the claim.

$\diamondsuit$

5.5. **Singularity points.** In order to prove finer and finer estimates on the sequence of eigenfunctions given by Claim 5.1, we have to work far from the neighbourhood of points having bad properties to pass to the limit. As in [Pet18], we call them singularity points. Indeed, for instance, this sequence of eigenfunctions has uniformly bounded Dirichlet energy, but for instance, we cannot a priori tell anything about the sequence of their $L^2(g)$ norm (except in the scale $\sqrt{\varepsilon}$ as noticed in Claim 5.3).

We prove in a first part of this section that there are at most m singularity points. Because the bad properties are concentrated around these points, they will not affect the regularity properties of the limit of the minimizing sequence nor the energy estimates. Notice that the terminology "good point"-"bad point" is rather used in [KNPP20]. "Bad points" correspond to the "singularity points" in the originated proof of the author for maximization of the m-th eigenvalue in a conformal class [Pet18].

We recall in a second part the tools we need to have L^2 estimates on the sequence of eigenfunctions, using Poincaré inequalities if they vanish on curves connecting two points whose distance is uniformly lower bounded by a positive constant. All eigenfunctions vanishing outside the neighbourhood of singularity points satisfy this property.

5.5.1. *Scales of singularity points.* Following [Pet18], we would like to localize these singularity points. At their neighbourhood, we cannot use any quantitative estimates coming from Section 4 nor the Poincaré inequalities (Proposition 5.1 below). In the current paper, we give a more precise and simpler Claim than in [Pet18], giving the scales of appearance of these singularity points.

Claim 5.4. *Up to the extraction of a subsequence of $\{e^{2u_\varepsilon}g\}_{\varepsilon>0}$ we can find a maximal collection of points $p_1^\varepsilon, \cdots, p_s^\varepsilon \in \Sigma$ with $0 \leq s \leq m-1$ such that $p_i^\varepsilon \to p_i$ as $\varepsilon \to 0$ and positive scales $r_1^\varepsilon \leq \cdots \leq r_s^\varepsilon$ such that for any $1 \leq i \leq s$, setting A_i^ε as*

$$A_i^\varepsilon = \left\{ r > 0; \overline{\mathbb{D}_r(p)} \subset \Sigma \setminus \left(\bigcup_{j=1}^{i} \overline{\mathbb{D}_{r_j^\varepsilon}\left(p_j^\varepsilon\right)} \right) \text{ and } \lambda_\star\left(\mathbb{D}_r(p), e^{2\tilde{u}_\varepsilon}dx\right) \leq \lambda_m(e^{2u_\varepsilon}g) \right\}$$

$$\lim_{\varepsilon\to 0}\left(\inf A_s^\varepsilon\right) > 0\,,$$

$$r_i^\varepsilon := \min A_i^\varepsilon \to 0 \text{ as } \varepsilon \to 0\,,$$

$$\overline{\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon)} \subset \Sigma \setminus \left(\bigcup_{j=1}^{i-1} \overline{\mathbb{D}_{r_j^\varepsilon}\left(p_j^\varepsilon\right)} \right)\,,$$

$$\lambda_\star\left(\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon), e^{2\tilde{u}_\varepsilon}dx\right) = \lambda_m(e^{2u_\varepsilon}g)\,.$$

Proof. We first define

$$A_0^\varepsilon = \left\{ r > 0; \overline{\mathbb{D}_r(p)} \subset \Sigma \text{ and } \lambda_\star\left(\mathbb{D}_r(p), e^{2\tilde{u}_\varepsilon}dx\right) \leq \lambda_m(e^{2u_\varepsilon}g) \right\}$$

Notice first that if $\lim_{\varepsilon\to 0}\inf A_0^\varepsilon > 0$ then $s = 0$ and there is not such a sequence and the claim is proved. Otherwise $\lim_{\varepsilon\to 0}\inf A_0^\varepsilon = 0$. We set $r_1^\varepsilon = \min A_0^\varepsilon$ (notice that the infimum is a minimum) and we chose p_1^ε such that $\overline{\mathbb{D}_{r_1^\varepsilon}(p_1^\varepsilon)} \subset \Sigma$ and $\lambda_\star\left(\mathbb{D}_{r_1^\varepsilon}(p_1^\varepsilon), e^{2\tilde{u}_\varepsilon}dx\right) \leq \lambda_m(e^{2u_\varepsilon}g)$. Since r_1^ε is a minimum, the previous inequality has to be an equality.

If for a given $i \geq 1$ the sequences $r_1^\varepsilon, \cdots, r_i^\varepsilon, p_1^\varepsilon, \cdots, p_i^\varepsilon$ are built, then if $\lim_{\varepsilon\to 0}\inf A_i^\varepsilon > 0$, the construction terminates and $s = i$. Otherwise $\lim_{\varepsilon\to 0}\inf A_i^\varepsilon = 0$ and we set $r_{i+1}^\varepsilon =$

$\min A_i^\varepsilon$, and we chose p_1^ε such that $\overline{\mathbb{D}_{r_{i+1}^\varepsilon}\left(p_{i+1}^\varepsilon\right)} \subset \Sigma$ and $\lambda_\star\left(\mathbb{D}_{r_{i+1}^\varepsilon}\left(p_{i+1}^\varepsilon\right), e^{2\tilde{u}_\varepsilon}dx\right) \leq \lambda_m(e^{2u_\varepsilon}g)$. Since r_{i+1}^ε is a minimum, the previous inequality has to be an equality.

Finally we prove that this sequence terminates after $m-1$ steps. Indeed, if not, we set $D_i^\varepsilon = \mathbb{D}_{r_i^\varepsilon}\left(p_i^\varepsilon\right)$ for $i \in \{1, \cdots, m\}$. The domains D_i^ε are disjoint in Σ and we have $\lambda_\star\left(D_i^\varepsilon, e^{2u_\varepsilon}\right) = \lambda_m(e^{2u_\varepsilon}g)$. Let φ_i^ε be first Dirichlet eigenfunctions on D_i^ε extended by 0 on Σ. We also take $\varphi_0^\varepsilon \in \mathcal{C}_c^\infty\left(\Sigma \setminus \bigcup D_i^\varepsilon\right)$ such that $\varphi_0^\varepsilon = 1$ on $\Sigma \setminus \bigcup_{i=1}^m \mathbb{D}_{\sqrt{r_i^\varepsilon}}\left(p_i^\varepsilon\right)$ and

$$\int_\Sigma |\nabla \varphi_i^\varepsilon|_g^2 \, dA_g \leq \frac{C}{\ln\left(\frac{1}{r_m^\varepsilon}\right)}$$

We use these functions as test functions for the variational characterization of $\lambda_m(e^{2u_\varepsilon}, g)$:

$$\lambda_m(e^{2u_\varepsilon}, g) = \inf_{E_{m+1}} \max_{\varphi \in E_{m+1}} \frac{\int_\Sigma |\nabla \varphi|_g^2 \, dA_g}{\int_\Sigma (\varphi)^2 \, dA_g} \leq \max_{0 \leq i \leq m} \left\{ \frac{\int_\Sigma |\nabla \varphi_i^\varepsilon|_g^2 \, dA_g}{\int_\Sigma (\varphi_i^\varepsilon)^2 \, dA_g} \right\} = \lambda_m(e^{2u_\varepsilon}, g)$$

where the inequality holds because the test functions have support in disjoint sets.

Therefore, we obtain the case of equality in the variational characterization of $\lambda_m(e^{2u_\varepsilon}, g)$ and deduce that there is a linear combination of the φ_i which is an eigenfunction associated to $\lambda_m(e^{2u_\varepsilon}, g)$. Such a function vanishes on an open set : this is absurd. $\diamondsuit$

Notice that for $m = 1$, such singularities never occur. That's why we do not deal with this problem in [Pet14a].

5.5.2. *Poincaré inequalities and nodal lines.* We aim at stating a specific Poincaré inequality in Proposition 5.1. Thanks to this, we have uniform bounds of the L^2 norm at every scale when the nodal set of the sequence of eigenfunction has length uniformly lower bounded. Far from the singularity points defined in Claim 5.4, this is exactly the property we have since nodal domains of eigenfunctions cannot shrink inside balls whose radii converge to 0 outside $\bigcup_{j=1}^i \overline{\mathbb{D}_{p_j^\varepsilon}\left(10 r_j^\varepsilon\right)}$. Let us state the Poincaré inequalities on the domains we consider during the proof: we denote by

$$\Omega_K = \mathbb{D}_{\frac{1}{K\rho}} \setminus \bigcup_{i=1}^s \mathbb{D}_{K\rho}(x_i) \ ,$$

where $\rho > 0$, $1 \leq K < 10$, $x_i \in \mathbb{D}_{\frac{1}{\rho}}$ such that if $i \neq j$, then $x_i \neq x_j$ and

$$10\rho < \min\left(\min_i d(x_i, \partial \mathbb{D}_{\frac{1}{10\rho}}); \min_{i \neq j} \frac{|x_i - x_j|}{2}\right) \ .$$

We state the following proposition, coming from capacity estimates

Proposition 5.1 ([AH96], Corollary 8.2.2 and [Hen05], pages 95-97)**.** *Let $r > 0$ and $1 < K < 10$ fixed. Then, we have a constant $C_r > 0$ such that for every $f \in \mathcal{C}^\infty(\Omega_1)$ which vanishes on a smooth piecewise curve $\Gamma \subset\subset \Omega_K$ which connects two points of distance $r > 0$,*

$$\|f\|_{L^2(\Omega_K)} \leq C_r \|\nabla f\|_{L^2(\Omega_1)} \ .$$

5.6. **Regularity estimates in the non concentrated case.** The cases of Claim 5.2: (i), and (ii) when $\frac{\alpha_\varepsilon}{\sqrt{\varepsilon}} \to +\infty$ are very similar to handle. In both cases, the sequence of maps Φ_ϵ arising from Claim 5.1 are "almost-weakly-harmonic" maps into an ellipsoid.

Indeed, since $K_\varepsilon[|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}] = 1$ on $supp(\nu_\epsilon)$, if one can prove that $supp(\nu_\epsilon) = \Sigma$, we get that $|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon} = 1$. Using the eigenfunction equation, and that $\Delta_g(|\Phi_\epsilon|^2_{\mathcal{E}_\varepsilon}) = 0$, we obtain that Φ_ϵ is a harmonic map into an ellipsoid and a formula for the densities $e^{2u_\epsilon} = \frac{\sum_{k=1}^m \lambda_k^\varepsilon \left|\nabla \Phi_\epsilon^k\right|_g^2}{\sum_{k=1}^m \left(\lambda_k^\varepsilon\right)^2 \left|\Phi_\epsilon^k\right|^2}$. By compactness results on sequences of harmonic maps into some manifold [Hel96, Riv08], we conclude that without concentration, Φ_ϵ converges in $\mathcal{C}^2(\Sigma)$ as $\epsilon \to 0$. Therefore, the sequence of densities e^{2u_ϵ} also converges and we get a regularity result.

However $supp(\nu_\epsilon)$ can be far from being equal to Σ. In this case we will prove that we still have convergence as $\epsilon \to 0$ of the sequence of maps Φ_ϵ given by Claim 5.1, to a map Φ which satisfies the weak equation of harmonic maps into an ellipsoid. Thanks to the regularity results by Hélein [Hel96] of weak harmonic maps into some manifold, we will get that Φ is smooth and that the limiting measure $\nu = \frac{\sum_{k=1}^m \lambda_k \left|\nabla \Phi^k\right|_g^2}{\sum_{k=1}^m \lambda_k^2 \left|\Phi^k\right|^2} dA_g$ is absolutely continous with respect to dA_g with a smooth density.

Notice also that the zeros of this density are isolated and give conical singularities to the metric we get.

We now assume that (i) of Claim 5.2 occurs. We aim at proving a pointwise bound on eigenfunctions and then thanks to strong pointwise estimates deduce energy estimates and the $W^{1,2}$ convergence of eigenfunctions.

5.6.1. *Pointwise estimates on eigenfunctions.* We aim at proving that Φ_ε is bounded far from the sequence of singularity points p_i^ε given by Claim 5.4. We set for $\rho > 0$

$$\Sigma(\rho) = \Sigma \setminus \left(\bigcup_{i=1}^s B_g\left(p_i, \rho\right) \right) .$$

Notice that the novelty compared to [Pet18] is that in the following Claim, the bounds of the eigenfunctions in $W^{1,2}\left(\Sigma(\rho)\right)$ and $\mathcal{C}^0\left(\Sigma(\rho)\right)$ do not depend on ρ.

Claim 5.5. *We assume that (i) of Claim 5.2 holds. Then there are constants C_1 and C_2 such that*

$$\forall \rho > 0, \limsup_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{W^{1,2}(\Sigma(\rho))} \leq C_1 , \tag{5.15}$$

$$\forall \rho > 0, \limsup_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{C^0(\Sigma(\rho))} \leq C_2 , \tag{5.16}$$

Proof. Notice that (5.15) is implied by (5.16) since $\|\nabla \Phi_\varepsilon\|_{L^2(\Sigma)}$ is bounded by 1.

Now, let us prove (5.16), the strategy proof is the same as in [Pet18], but we aim at getting here constants C_1 and C_2, not depending on ρ. We set

$$\delta_i^\varepsilon = d\left(supp(\nu_\varepsilon) \setminus \overline{\mathbb{D}_{10 r_i^\varepsilon}\left(p_i^\varepsilon\right)}, p_i^\varepsilon\right) , \tag{5.17}$$

for $i \in \{1, \cdots, s\}$. Up to a subsequence, we have a constant $0 < \tilde{\delta} < \delta_0$ that $\{1, \cdots, s\} = I_1 \cup I_2$, where

$$I_1 = \{i \in \{1, \cdots, s\}; \delta_i^\varepsilon \to 0 \text{ as } \varepsilon \to 0\} \text{ and } I_2 = \{i \in \{1, \cdots, s\}; \delta_i^\varepsilon \geq \tilde{\delta}\} \ . \tag{5.18}$$

Up to reduce $\tilde{\delta}$, we also assume that

$$supp(\nu_\varepsilon) \cap \left(\Sigma \setminus \left(\bigcup_{i=1}^{s} \overline{\mathbb{D}_{\tilde{\delta}}(p_i^\varepsilon)} \right) \right) \neq \emptyset \ . \tag{5.19}$$

This is possible because the limiting measure of ν_ε has a non-empty support on $\Sigma \setminus \{p_1, \cdots, p_s\}$.

We have the following finite covering of $\Sigma \setminus \bigcup_{i=1}^{s} \overline{\mathbb{D}_{\frac{\tilde{\delta}}{10}}(p_i^\varepsilon)}$ by balls of radius $\frac{\tilde{\delta}}{100}$:

$$\Sigma \setminus \bigcup_{i=1}^{s} \overline{\mathbb{D}_{\frac{\tilde{\delta}}{10}}(p_i^\varepsilon)} \subset \bigcup_{l=1}^{L} \overline{\mathbb{D}_{\frac{\tilde{\delta}}{100}}(q_l)} \ ,$$

where for $l = 1, \cdots, L$, $q_l \in \Sigma \setminus \bigcup_{i=1}^{s} \overline{\mathbb{D}_{\frac{\tilde{\delta}}{10}}(p_i^\varepsilon)}$. Let's handle estimates on the balls $\overline{\mathbb{D}_{\frac{\tilde{\delta}}{100}}(q_k)}$ and then estimates at the neighbourhood of the singularity points p_i^ε in the following steps:

STEP 1 :

There is a constant C_2 such that for any $l \in \{1, \cdots, L\}$ and any coordinate $\Phi_\varepsilon^{i,j}$,

$$\text{either } \forall \varepsilon > 0, \forall x \in \overline{\mathbb{D}_{\frac{\tilde{\delta}}{50}}(q_l)}, \left| \Phi_\varepsilon^{k,j}(x) \right| \leq C_2 \ ,$$

$$\text{or } \forall \varepsilon > 0, \forall x, y \in \overline{\mathbb{D}_{\frac{\tilde{\delta}}{50}}(q_l)}, \frac{1}{C_2} \left| \Phi_\varepsilon^{k,j}(y) \right| \leq \left| \Phi_\varepsilon^{k,j}(x) \right| \leq C_2 \left| \Phi_\varepsilon^{k,j}(y) \right| \ .$$

Proof of STEP 1 :

Let $l \in \{1, \cdots, L\}$. Let $1 \leq k \leq m$, $1 \leq j \leq i_k$ and up to change $\Phi_\varepsilon^{k,j}$ into $-\Phi_\varepsilon^{k,j}$, let $x_\varepsilon \in \overline{\mathbb{D}_{\frac{\tilde{\delta}}{50}}(q_l)}$ be a sequence of points such that

$$\Phi_\varepsilon^{k,j}(x_\varepsilon) = \sup_{x \in \overline{\mathbb{D}_{\frac{\tilde{\delta}}{50}}(q_l)}} \left| \Phi_\varepsilon^{k,j}(x) \right| \ .$$

We set

$$\delta_\varepsilon = d(\tilde{x}_\varepsilon, supp(\tilde{\nu}_\varepsilon) \cap \overline{\mathbb{D}_{\frac{\tilde{\delta}}{20}}(q_l)}) \ .$$

We divide the proof of (5.16) into three cases.

Case 1 - We assume that $\delta_\varepsilon = O(\sqrt{\varepsilon})$. Then, $\{\phi_\varepsilon^{k,j}(x_\varepsilon)\}$ is bounded by Claim 5.3.

Case 2 - We assume that $\delta_\varepsilon \to 0$ and $\frac{\sqrt{\varepsilon}}{\delta_\varepsilon} \to 0$ as $\varepsilon \to 0$. We let

$$\psi_\varepsilon = \tilde{\Phi}_\varepsilon^{k,j}(\delta_\varepsilon x + \tilde{x}_\varepsilon) \text{ and } e^{2w_\varepsilon} = \delta_\varepsilon^2 e^{2\tilde{u}_\varepsilon(\delta_\varepsilon x + x_\varepsilon)}$$

for $x \in \mathbb{D}_{\delta\delta_\varepsilon^{-1}}$ so that

$$\Delta \psi_\varepsilon = \lambda_\varepsilon e^{2w_\varepsilon} \psi_\varepsilon \text{ in } \mathbb{D}_{\delta\delta_\varepsilon^{-1}} \ .$$

Let $y_\varepsilon \in supp(\nu_\varepsilon)$ be such that $|x_\varepsilon - y_\varepsilon| = \delta_\varepsilon$ and set $z_\varepsilon = \frac{\tilde{y}_\varepsilon - \tilde{x}_\varepsilon}{\delta_\varepsilon}$ so that $z_\varepsilon \to z_0$ as $\varepsilon \to 0$ up to the extraction of a subsequence. Thanks to Claim 5.3, we know that

$\psi_\varepsilon(z_\varepsilon) = \Phi_\varepsilon^{k,j}(y_\varepsilon) = O(1)$. Thanks to estimates (6.12) on the heat kernel, there exists $D_1 > 0$ such that

$$e^{2w_\varepsilon} \leq D_1 \text{ on } \mathbb{D}_{\frac{1}{2}} .$$

We first assume that ψ_ε does not vanish in $\mathbb{D}_3$. Then, we can apply Harnack's inequality and get some constant $D_2 > 0$ such that

$$\psi_\varepsilon \geq D_2 \psi_\varepsilon(0) \text{ on } \mathbb{D}_{\frac{1}{4}}$$

for all $\varepsilon > 0$. Since ψ_ε is positive on $\mathbb{D}_{|z_\varepsilon|}(z_\varepsilon) \subset \mathbb{D}_3$, by the equation, it is also superharmonic and we can write that

$$\psi_\varepsilon(z_\varepsilon) \geq \frac{1}{2\pi |z_\varepsilon|} \int_{\partial \mathbb{D}_{|z_\varepsilon|}(z_\varepsilon)} \psi_\varepsilon d\sigma .$$

Taking only the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}$, we get the existence of some constant $D_3 > 0$ such that

$$\psi_\varepsilon(z_\varepsilon) \geq D_3 \psi_\varepsilon(0)$$

and this concludes the proof of (5.16) in this case since $\Phi_\varepsilon^{k,j}(x_\varepsilon) = \psi_\varepsilon(0) = O(1)$.

We now assume that ψ_ε vanishes somewhere on $\mathbb{D}_3$: $\psi_\varepsilon(z_\varepsilon) = 0$. We claim that ψ_ε vanishes on a piecewise smooth curve in $\mathbb{D}_4$ which connects two points of distance greater than 1. By contradiction, we assume the contrary. It means that the connected component of nodal set of $\tilde{\Phi}_\varepsilon^{k,j}$ containing $\delta_\varepsilon z_\varepsilon + \tilde{x}_\varepsilon$ is a subset of $\mathbb{D}_{\delta_\varepsilon}(\delta_\varepsilon z_\varepsilon + \tilde{x}_\varepsilon)$. Then, one nodal domain D_ε of $\tilde{\Phi}_\varepsilon^{k,j}$ is a subset of $\mathbb{D}_{\delta_\varepsilon}(\delta_\varepsilon z_\varepsilon + \tilde{x}_\varepsilon)$. This gives that

$$\lambda_\star \left(\overline{\mathbb{D}_{\delta_\varepsilon}(\delta_\varepsilon z_\varepsilon + \tilde{x}_\varepsilon)}, e^{2u_\varepsilon} g \right) \leq \lambda_\star \left(\overline{D_\varepsilon}, e^{2u_\varepsilon} g \right) = \lambda_m \left(\Sigma, e^{2u_\varepsilon} g \right) . \tag{5.20}$$

Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, and $x_\varepsilon \in \Sigma_\varepsilon(\rho)$ for ρ small enough, we obtain a contradiction with Claim 5.4. By proposition 5.1 for $\Omega = \mathbb{D}_5$, we get some constant $C > 0$ such that

$$\int_{\mathbb{D}_4} \psi_\varepsilon^2 \leq C \int_{\mathbb{D}_5} |\nabla \psi_\varepsilon|^2 \, dx$$

which proves that $\{\psi_\varepsilon\}$ is bounded in $L^2(\mathbb{D}_4)$. By elliptic regularity, ψ_ε is bounded in $L^\infty(\mathbb{D}_{\frac{1}{4}})$ which gives that $\{\Phi_\varepsilon^{k,j}(x_\varepsilon)\}$ is bounded. The study of these three cases completes the proof of (5.16) with a constant $C_2(\rho)$.

Case 3 - We assume that $\delta_\varepsilon^{-1} = O(1)$ and that, $\{\Phi_\varepsilon^{k,j}\}$ vanishes somewhere in $\overline{\mathbb{D}_{\frac{\tilde{\delta}}{20}}(q_l)}$.

Up to the extraction of a subsequence, we assume that $\delta_\varepsilon \to \hat{\delta} > 0$ and $x_\varepsilon \to x$ as $\varepsilon \to 0$. Then $\{e^{2u_\varepsilon}\}$ is uniformly bounded in $\overline{\mathbb{D}_{\frac{\hat{\delta}}{2}}(\tilde{x})}$ by (6.11).

In a same argument as previously in CASE 2 (see (5.20) and the arguments around), $\widetilde{\Phi}_\varepsilon^{k,j}$ has to vanish on a piecewise smooth curve which connects to points of distance uniformly lower bounded as $\varepsilon \to 0$. Indeed, if not, one would build a new sequence of balls of radius converging to 0 having first Dirichlet eigenvalue less than $\lambda_m \left(\Sigma, e^{2u_\varepsilon} g \right)$. This contradicts Claim 5.4. Therefore, $\{\Phi_\varepsilon^{k,j}\}$ is bounded in $L^2 \left(\overline{\mathbb{D}_{\frac{\tilde{\delta}}{40}}(q_l)} \right)$ by proposition 5.1.

Then $\{\Phi_\varepsilon^{k,j}\}$ is bounded by some constant C_2 in $L^\infty \left(\overline{\mathbb{D}_{\frac{\tilde{\delta}}{50}}(q_l)} \cap \overline{\mathbb{D}_{\frac{\hat{\delta}}{4}}(\tilde{x})} \right)$ by standard elliptic theory on the eigenvalue equation. So is $\{\Phi_\varepsilon^{k,j}(x_\varepsilon)\}$.

Case 4 - We assume that $\delta_\varepsilon^{-1} = O(1)$ and that $\{\Phi_\varepsilon^{k,j}\}$ does not vanishes in $\overline{\mathbb{D}_{\frac{\tilde\delta}{20}}(q_l)}$. Then $\{e^{2u_\varepsilon}\}$ is uniformly bounded in $\overline{\mathbb{D}_{\frac{\tilde\delta}{10}}(x)}$ by (6.11). Moreover $\{\Phi_\varepsilon^{k,j}\}$ has a constant sign and up to take $-\Phi_\varepsilon^{k,j}$, we assume that $\Phi_\varepsilon^{k,j} > 0$ in $\overline{\mathbb{D}_{\frac{\tilde\delta}{20}}(q_l)}$. Therefore $\Phi_\varepsilon^{k,j}$ is a positive sub-harmonic function with potential $\{e^{2u_\varepsilon}\}$ uniformly bounded in $\mathbb{D}_{\frac{\delta}{R_0}}(x)$. Then $\Phi_\varepsilon^{k,j}$ satisfies a Harnack inequality on $\mathbb{D}_{\frac{\delta}{50}}(q_l)$. This conclude the proof of STEP 1.

STEP 2 :

There is $l = 1, \cdots, L$ such that

$$\forall \varepsilon > 0, \forall x \in \overline{\mathbb{D}_{\frac{\tilde\delta}{50}}(q_l)}, |\Phi_\varepsilon(x)| \leq C_2 \ .$$

Proof of STEP 2 :

By (5.19), let $l = 1, \cdots, L$ be such that $\overline{\mathbb{D}_{\frac{\tilde\delta}{100}}(q_l)} \cap supp(\nu_\varepsilon) \neq \emptyset$. The proof is the same as in the previous STEP, except in Case 4. Instead of getting a Harnack inequality if $\{\Phi_\varepsilon^{k,j}\}$ does not vanishes in $\overline{\mathbb{D}_{\frac{\tilde\delta}{20}}(q_l)}$, we proceed as in Case 2 of STEP 1 and we get a uniform bound. This concludes the proof of STEP 2.

STEP 3 :

Up to increase C_2,

$$\forall \varepsilon > 0, \forall x \in \Sigma \setminus \left(\bigcup_{i \in I_1} \overline{\mathbb{D}_{\delta_i^\varepsilon}(p_i^\varepsilon)} \cup \bigcup_{i \in I_2} \overline{\mathbb{D}_{\frac{\tilde\delta}{10}}(p_i^\varepsilon)} \right), |\Phi_\varepsilon(x)| \leq C_2 \ ,$$

where the definition of I_1 and I_2 is given at the beginning of the proof in (5.18)

Proof of STEP 3 :

We first have that, up to increase C_2, by the previous STEP 1 and STEP 2, since Σ is connected,

$$\forall \varepsilon > 0, \forall x \in \Sigma \setminus \left(\bigcup_{i=1}^{s} \overline{\mathbb{D}_{\frac{\tilde\delta}{10}}(p_i^\varepsilon)} \right), |\Phi_\varepsilon(x)| \leq C_2 \ . \tag{5.21}$$

Let's now prove the uniform bound in the neighbourhood of points p_i^ε for $i \in I_1$. Let $1 \leq k \leq m$, $1 \leq j \leq i_k$ and up to change $\Phi_\varepsilon^{k,j}$ into $-\Phi_\varepsilon^{k,j}$, let (x_ε) be a sequence of points such that

$$\Phi_\varepsilon^{k,j}(x_\varepsilon) = \sup_{x \in \Sigma \setminus \left(\bigcup_{i \in I_1} \overline{\mathbb{D}_{\delta_i^\varepsilon}(p_i^\varepsilon)} \cup \bigcup_{i \in I_2} \overline{\mathbb{D}_{\frac{\tilde\delta}{10}}(p_i^\varepsilon)} \right)} \left| \Phi_\varepsilon^{k,j}(x) \right| \ .$$

We set

$$\delta_\varepsilon = d(\tilde x_\varepsilon, supp(\tilde\nu_\varepsilon)) \ .$$

If $x_\varepsilon \in \overline{\mathbb{D}_{\frac{\tilde\delta}{10}}(p_i^\varepsilon)} \setminus \overline{\mathbb{D}_{\delta_i^\varepsilon}(p_i^\varepsilon)}$ for some $i \in I_1$, we divide the proof of (5.16) into four cases as in the proof of STEP 1. The proof is the same in cases 1, 2 and 3. The key point is that we

can always apply proposition 5.1 by the choice of δ_ε^i in (5.17). To adapt case 4, we assume that $x_\varepsilon \to x$. Then $x \notin \{p_1, \cdots, p_s\}$. Indeed, Because of Claim 5.4, we have

$$\forall i \in \{1, \cdots, s\}, supp(\nu_\varepsilon) \cap \overline{\mathbb{D}_{10r_i^\varepsilon}(p_i^\varepsilon)} \neq \emptyset \ .$$

Indeed, if not, by uniform estimates on the heat kernel, $(r_i^\varepsilon)^2 \, e^{2\tilde{u}_\varepsilon(r_i^\varepsilon x + p_i^\varepsilon)} \to 0$ uniformly on $x \in \mathbb{D}$. We would then get

$$\lambda_\star \left(\mathbb{D}_{r_i^\varepsilon}(p_i^\varepsilon), e^{2\tilde{u}_\varepsilon}\right) \to +\infty$$

as $\varepsilon \to 0$ which is not possible by Claim 5.4. Now, since $x \notin \{p_1, \cdots, p_s\}$, up to reduce $\tilde{\delta}$ in all the previous steps, and thanks to (5.21), we conclude the proof of STEP 3.

STEP 4 :

We prove the remaining estimate for indices in I_2. Up to increase C_2,

$$\forall i \in I_2, \forall \rho \in (0, \frac{\tilde{\delta}}{2}), \limsup_{\varepsilon \to 0} \sup_{x \in \Sigma \setminus \overline{\mathbb{D}_\rho(p_i)}} |\Phi_\varepsilon| \, (x) \leq C_2 \ .$$

Proof of STEP 4 :

For $i \in I_2$, $\delta_\varepsilon^i \geq \tilde{\delta}$. By convergence properties of the heat kernel, we know that for any $0 < \rho < \tilde{\delta}$, $e^{2u_\varepsilon} = K_\varepsilon(\nu_\varepsilon)$ converges uniformly to 0 on $\overline{\mathbb{D}_{\tilde{\delta}}(p_i)} \setminus \overline{\mathbb{D}_{\frac{\rho}{2}}(p_i)}$. Therefore, by standard elliptic estimates, up to a subsequence, Φ_ε converges in $\mathcal{C}^2(\overline{\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i)} \setminus \overline{\mathbb{D}_\rho(p_i)})$ for any $\rho > 0$ to a Euclidean harmonic map Φ.

Notice that at this stage, Φ is defined on $\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i) \setminus \{p_i\}$ and may be unbounded since the $\mathcal{C}^0$ bound of $\Phi(x)$ depend on $|x - p_i|$ as $|x - p_i| \to 0$. Since Φ is harmonic and $\nabla\Phi$ belongs to $L^2\left(\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i)\right)$ as a weak limit of $\nabla\Phi_\varepsilon$ in L^2, by point removability, Φ is in fact harmonic on $\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i)$. By STEP 3, Φ is bounded by C_2 on $\overline{\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i)} \setminus \overline{\mathbb{D}_{\frac{\tilde{\delta}}{10}}(p_i)}$. By the maximum principle and strong convergence on $\mathcal{C}^2(\overline{\mathbb{D}_{\frac{\tilde{\delta}}{2}}(p_i)} \setminus \overline{\mathbb{D}_\rho(p_i)})$ for any ρ and $i \in I_2$, the maximum of Φ_ε on $\Sigma(\rho)$ does not depend on ρ. Then, we get STEP 4.

Gathering STEP 4 and STEP 3 proves (5.16). As already said this also proves (5.15). The proof of the Claim is complete.

$\diamondsuit$

5.6.2. $W^{1,2}$ *convergence of eigenfunctions.* In this section, we set $\omega_\varepsilon = |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$. After pointwise estimates, we want to prove energy estimates on Φ_ε. Then, we aim at proving that $\nabla\omega_\varepsilon$ converges to 0 in L^2 by the global structure of the equation on Φ_ε given by Claim 5.1, in order to use the quantitative estimate of section 4 for ω_ε-harmonic maps.

We first give quantitative non-concentration estimates. We follow here the same lines as in [Pet18] but again, here, D_1 and D_2 do not depend on ρ.

Claim 5.6. *We assume that (i) of Claim 5.2 holds. Then, we have the following: quantitative non-concentration estimates on e^{2u_ε} and $|\nabla\Phi_\varepsilon|_g^2$, there are constants $D_1 > 0$ and*

$D_2 > 0$ *such that*

$$\forall \rho > 0, \forall r > 0, \limsup_{\varepsilon \to 0} \sup_{x \in \Sigma(\rho)} \int_{B_g(x,r)} e^{2u_\varepsilon} dA_g \leq \frac{D_1}{\ln(\frac{1}{r})} , \tag{5.22}$$

$$\forall \rho > 0, \forall r > 0, \limsup_{\varepsilon \to 0} \sup_{x \in \Sigma(\rho)} \int_{B_g(x,r)} |\nabla \Phi_\varepsilon|_g^2 dA_g \leq \frac{D_2}{\sqrt{\ln(\frac{1}{r})}} . \tag{5.23}$$

Proof. By isocapacity estimates (see [M11], section 2.3.3, corollary of Theorem 2.3.2)

$$\int_{B_g(x,r)} e^{2u_\varepsilon} dA_g \leq \frac{Cap_2(B_g(x,r), B_g(x,\delta_0))}{\lambda_\star(B_g(x,r), e^{2u_\varepsilon} g)} \leq \frac{Cap_2(\mathbb{D}_{\frac{r}{C_0}}, \mathbb{D}_{C_0\delta_0})}{\lambda_m} \leq \frac{2\pi}{\lambda_m \ln\left(\frac{C_0^2 \delta_0}{r}\right)} ,$$

where δ_0 is defined in section 5.2 and the second inequality comes from Claim 5.4. We obtain (5.22).

Finally, we prove (5.23). We set $z = \tilde{x}$ for $x \in \Sigma(\rho)$ and for $0 < r \leq \delta$

$$F_\varepsilon(r) = \int_{\mathbb{D}_r(z)} \left|\nabla \tilde{\Phi}_\varepsilon\right|^2 dx .$$

We just aim at proving that

$$F_\varepsilon(r) \leq \frac{D_0}{\sqrt{\ln\left(\frac{1}{r}\right)}} .$$

We know that $\tilde{\Phi}_\varepsilon$ satisfies the equations

$$\Delta \tilde{\Phi}_\varepsilon^k = \lambda_k^\varepsilon e^{2u_\varepsilon} \tilde{\Phi}_\varepsilon^k \text{ in } \mathbb{D}_\delta$$

and we deduce that

$$F_\varepsilon(r) = \sum_{k=1}^m \left(\lambda_k^\varepsilon \int_{\mathbb{D}_r(z)} e^{2\tilde{u}_\varepsilon} \left|\tilde{\Phi}_\varepsilon^k\right|^2 dx + \int_{\partial \mathbb{D}_r(z)} \tilde{\Phi}_\varepsilon^k . \partial_\nu \tilde{\Phi}_\varepsilon^k d\sigma_\xi \right) .$$

Using (5.16) and (5.22), there exist some constants K_1, K_2 and K independent of ε, r, ρ and z, such that

$$F_\varepsilon(r)^2 \leq \frac{K_1}{\ln\left(\frac{1}{r}\right)^2} + K_2 \left(\int_{\partial \mathbb{D}_r(x)} \left|\nabla \tilde{\Phi}_\varepsilon\right| d\sigma_\xi \right)^2 \leq K \left(\frac{1}{\ln\left(\frac{1}{r}\right)^2} + r F_\varepsilon'(r) \right)$$

for any $0 < r < \delta$. We divide by $r F_\varepsilon(r)^2$ and integrate between r and δ and get

$$\ln\left(\frac{\delta}{r}\right) - K \left(\frac{1}{F_\varepsilon(r)} - \frac{1}{F_\varepsilon(\delta)} \right) \leq K \left(\int_r^\delta \frac{1}{r (\ln r)^2 F_\varepsilon(r)} \right) \leq \frac{K}{F_\varepsilon(r)^2} \left(\frac{1}{\ln \frac{1}{\delta}} - \frac{1}{\ln \frac{1}{r}} \right) .$$

By the conformal invariance of the L^2-norm of the gradient, $F_\varepsilon(\delta)$ is bounded. Multiplying by $F_\varepsilon(r)^2$, we get a constant $C > 0$ such that

$$F_\varepsilon(r)^2 \ln\left(\frac{\delta}{r}\right) \leq K F_\varepsilon(r) + C .$$

Therefore,

$$\left(F_\varepsilon(r)\left(\ln\left(\frac{\delta}{r}\right)\right)^{\frac{1}{2}}-\frac{K}{2\left(\ln\left(\frac{\delta}{r}\right)\right)^{\frac{1}{2}}}\right)^2\leq C+\frac{K^2}{4\left(\ln\left(\frac{\delta}{r}\right)\right)}\ .$$

We get (5.23) and this ends the proof of the claim. ♢

Now we aim at getting better uniform estimates than in Claim 5.1. This is a necessary claim to prove that $|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}$ is uniformly lower bounded by a positive constant (so that we can divide by $|\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$ in the key Claim 5.8). We also prove a uniform convergence of $|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}$ to 1 close to the support of ν_ε.

Claim 5.7. *We assume that (i) of Claim 5.2 holds. Then for any $\rho>0$, there exists $\beta_\varepsilon\to 0$ as $\varepsilon\to 0$ such that*

$$\forall x\in\Sigma(\rho),|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}(x)\geq 1-\beta_\varepsilon\ , \tag{5.24}$$

$$\forall x\in\Sigma(\rho)\cap supp(\nu_\varepsilon),\left||\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}(x)-1\right|\leq\beta_\varepsilon\ . \tag{5.25}$$

Moreover, for any smooth function $G:\mathbb{R}^{i_1+\cdots+i_m}\to\mathbb{R}$ such that there is a constant C_G such that for any X, $|G(X)|\leq C_G\left(|X|+|X|^2\right)$, we have

$$\forall x\in\Sigma(\rho),|G(\Phi_\varepsilon)(x)-K_\varepsilon[G(\Phi_\varepsilon)](x)|\leq\left(C_G+\|G\|_{\mathcal{C}^1\left(B_{2C_0}\setminus B_{\frac{1}{2\lambda_m}}\right)}\right)\beta_\varepsilon\ . \tag{5.26}$$

Proof. Notice that (5.24) and (5.25) are just consequences of the Euler-Lagrange equation Claim 5.12 combined with

$$\forall x\in\Sigma(\rho),\left||\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}(x)-K_\varepsilon[|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}](x)\right|\leq\beta_\varepsilon\ . \tag{5.27}$$

Notice that (5.27) is just (5.50) in the case $G(X)=|X|^2_{\mathcal{E}_\varepsilon}$ Let's prove (5.50) in 3 STEPS:

STEP 1 :

Let $1\leq i\leq s$. We prove that at the neighbourhood of the singular points defined in Claim 5.4,

$$\sup_{x\in\Sigma(\rho)}\int_{B_g(p_i,\frac{\rho}{10})}|\Phi_\varepsilon(y)|^2_{\mathcal{E}_\varepsilon}p_\varepsilon(x,y)dA_g(y)=O(e^{-\frac{\rho^2}{8\varepsilon}})\ .$$

Proof of STEP 1 :

Let $x\in\Sigma(\rho)$. Then, by estimates (6.10) of Section 6.2

$$\begin{aligned}e^{\frac{\rho^2}{8\varepsilon}}\int_{B_g(p_i,\frac{\rho}{10})}|\Phi_\varepsilon(y)|^2_{\mathcal{E}_\varepsilon}p_\varepsilon(x,y)dA_g(y)&\leq\frac{A_0}{4\pi\varepsilon}e^{-\frac{31\rho^2}{400\varepsilon}}\frac{\int_{B_g(p_i,\frac{\rho}{10})}|\Phi_\varepsilon|^2e^{2u_\varepsilon}dA_g}{\inf_{B_g(p_i,\frac{\rho}{10})}e^{2u_\varepsilon}}\\&\leq\frac{A_0}{4\pi\varepsilon}\frac{e^{-\frac{31\rho^2}{400\varepsilon}}}{\inf_{B_g(p_i,\frac{\rho}{10})}e^{2u_\varepsilon}}\end{aligned}$$

since by Claim 5.1,

$$\int_\Sigma |\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}\, e^{2u_\varepsilon} dA_g = \int_\Sigma K_\varepsilon[|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}] d\nu_\varepsilon = 1\ .$$

We assume by contradiction that

$$\inf_{B_g(p_i,\frac{\rho}{10})} e^{2u_\varepsilon} \leq \frac{e^{-\frac{31\rho^2}{400\varepsilon}}}{\varepsilon}\ .$$

Let $y \in \overline{B_g(p_i,\frac{\rho}{10})}$ be such that $e^{2u_\varepsilon(y)} = \inf_{B_g(p_i,\frac{\rho}{10})} e^{2u_\varepsilon}$. Then, by (6.10) of Section 6.2,

$$e^{2u_\varepsilon(y)} = \int_\Sigma p_\varepsilon(y,x) d\nu_\varepsilon(x) \geq \frac{e^{-\left(\frac{2\rho}{10}\right)^2\frac{1}{4\varepsilon}}}{4\pi A_0\varepsilon}\int_{B_g(p_i,\frac{\rho}{10})} d\nu_\varepsilon\ .$$

We deduce from this and the previous inequality that

$$\int_{B_g(p_i,\frac{\rho}{10})} d\nu_\varepsilon \leq 4\pi A_0 e^{-\frac{27\rho^2}{400\varepsilon}}\ .$$

Let $z \in B_g(p_i,\frac{\rho}{20})$, and let us write thanks again to (6.10) of Section 6.2 that

$$e^{2u_\varepsilon(z)} \leq A_0\frac{\int_{B_g(p_i,\frac{\rho}{10})} d\nu_\varepsilon + e^{-\frac{\rho^2}{4\varepsilon}\frac{1}{20^2}}}{4\pi\varepsilon} \leq \frac{A_0^2}{\varepsilon}e^{-\frac{27\rho^2}{400\varepsilon}} + \frac{A_0}{4\pi\varepsilon}e^{-\frac{\rho^2}{1600\varepsilon}}\ .$$

Then, $\left\|e^{2u_\varepsilon}\right\|_{\mathcal{C}^0(B_g(p_i,\frac{\rho}{20}))} \to 0$ as $\varepsilon \to 0$. Then $\lambda_\star\left(B_g(p_i,\frac{\rho}{20}), e^{2u_\varepsilon}g\right) \to +\infty$ as $\varepsilon \to 0$ which contradicts Claim 5.4 since for any ε, there is a domain included in $B_g(p_i,\frac{\rho}{20})$ such that the first eigenvalue on this domain for $e^{2u_\varepsilon}g$ is bounded. This completes the proof of Step 1.

STEP 2 :

There exists $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$\forall x,y \in \Sigma(\rho),\quad d_g(x,y) \leq \frac{\sqrt{\varepsilon}}{\beta_\varepsilon} \Rightarrow |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|_{\mathcal{E}_\varepsilon} \leq \beta_\varepsilon\ . \tag{5.28}$$

Proof of STEP 2 :

We set $\gamma_\varepsilon = \left\|\varepsilon e^{2u_\varepsilon}\right\|^{\frac{1}{3}}_{L^\infty(\Sigma(\rho))}$. We have $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, for $r > 0$, and $x \in \Sigma(\rho)$ such that $\gamma_\varepsilon^3 = \varepsilon e^{2u_\varepsilon(x)}$,

$$\varepsilon e^{2u_\varepsilon(x)} \leq \frac{A_0}{\sqrt{4\pi}}\int_{B_g(x,r)} d\nu_\varepsilon + o(1) = \frac{A_0}{\sqrt{4\pi}}\nu(B_g(x,r)) + o(1) \leq \frac{A_0 D_1}{\sqrt{4\pi}\ln\left(\frac{1}{r}\right)} + o(1)$$

by the estimate (6.10), since $\nu_\varepsilon \rightharpoonup_\star \nu$ as $\varepsilon \to 0$ and by (5.22) of Claim 5.6. Letting $\varepsilon \to 0$ and then $r \to 0$, we get $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$. We also have that $\frac{\gamma_\varepsilon}{\sqrt{\varepsilon}} \to +\infty$ as $\varepsilon \to 0$, since $\gamma_\varepsilon \geq 2^{-\frac{1}{3}}\varepsilon^{\frac{1}{3}}$ (indeed $\frac{1}{2} \leq \|e^{2u_\varepsilon}\|_{L^1(\Sigma(\rho))} \leq \|e^{2u_\varepsilon}\|_{L^\infty(\Sigma(\rho))}$). Let now let $x_\varepsilon, y_\varepsilon \in \Sigma(\rho)$ with

$d_g(x_\varepsilon, y_\varepsilon) \leq \frac{\sqrt{\varepsilon}}{\gamma_\varepsilon}$. Up to the extraction of a subsequence, $x_\varepsilon \in \Omega_l$ for some l fixed, and we set

$$\begin{cases} \hat{\Phi}_\varepsilon(x) = \tilde{\Phi}(\tilde{x}_\varepsilon + \frac{\sqrt{\varepsilon}}{\gamma_\varepsilon} x) \\ e^{2\hat{u}_\varepsilon(x)} = \frac{\varepsilon}{\gamma_\varepsilon^2} e^{2\tilde{u}_\varepsilon(\tilde{x}_\varepsilon + \frac{\sqrt{\varepsilon}}{\gamma_\varepsilon} x)} \end{cases} .$$

Notice from the definition of γ_ε that $\left\| e^{2\hat{u}_\varepsilon} \right\|_{L^\infty} \leq \gamma_\varepsilon$ and that the following equation holds

$$\Delta_\xi \hat{\Phi}_\varepsilon^k = \lambda_k^\varepsilon e^{2\hat{u}_\varepsilon} \hat{\Phi}_\varepsilon^k \text{ in } \mathbb{D}_{3C_0} . \tag{5.29}$$

Let α_ε be the mean value of $\hat{\Phi}_\varepsilon$ in $\mathbb{D}_{3C_0}$. Then

$$\begin{aligned} \left\| \hat{\Phi}_\varepsilon - \alpha_\varepsilon \right\|_{L^\infty(\mathbb{D}_{2C_0}(0))} &\leq D \left\| \Delta \hat{\Phi}_\varepsilon \right\|_{L^2(\mathbb{D}_{3C_0}(0))} + D \left\| \hat{\Phi}_\varepsilon - \alpha_\varepsilon \right\|_{L^2(\mathbb{D}_{3C_0}(0))} \\ &\leq 2D\lambda_m \left\| \Phi_\varepsilon \right\|_{L^\infty(\Sigma(\rho))} \left\| e^{2\hat{u}_\varepsilon} \right\|_{L^\infty(\mathbb{D}_{3C_0}(0))} + D' \left\| \nabla \hat{\Phi}_\varepsilon \right\|_{L^2(\mathbb{D}_{3C_0}(0))} \\ &\leq D'' C_2 \gamma_\varepsilon + \frac{D'\sqrt{D_2}}{\ln\left(\frac{\gamma_\varepsilon}{3C_0^2\sqrt{\varepsilon}}\right)^{\frac{1}{4}}} . \end{aligned}$$

The first inequality comes from standard elliptic regularity theory. The second inequality comes from the classical Poincaré inequality on $\mathbb{D}_{3C_0}$, and finally we use (5.16) and (5.23) in Claim 5.5 and Claim 5.6. Setting

$$\beta_\varepsilon = 2D'' C_2 \gamma_\varepsilon + \frac{2D'\sqrt{D_2}}{\ln\left(\frac{\gamma_\varepsilon}{3C_0^2\sqrt{\varepsilon}}\right)^{\frac{1}{4}}} ,$$

we have that $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ and that

$$|\Phi_\varepsilon(x_\varepsilon) - \Phi_\varepsilon(y_\varepsilon)| \leq \beta_\varepsilon .$$

Up to increasing β_ε so that $\frac{\sqrt{\varepsilon}}{\beta_\varepsilon} \leq \frac{\sqrt{\varepsilon}}{\gamma_\varepsilon}$ we proved Step 2.

STEP 3 :

We prove now (5.50) *and complete the proof of the Claim.*

Proof of STEP 3 :

For $x \in \Sigma(\rho)$, we write thanks to (5.16) that

$$\begin{aligned} \left| F(\Phi_\varepsilon) - K_\epsilon[G(\Phi_\varepsilon)] \right|(x) \leq & \int_{B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} \left| G(\Phi_\varepsilon)(x) - G(\Phi_\varepsilon)(y) \right| p_\epsilon(x,y) dA_g(y) \\ & + 2 \left\| G(\Phi_\varepsilon) \right\|_{L^\infty\left(\Sigma(\frac{\rho}{10})\right)} \int_{\Sigma \setminus B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x,y) dA_g(y) \\ & + \sum_{i=1}^{s} \int_{B_g(p_i, \frac{\rho}{10})} \left| G(\Phi_\varepsilon)(y) \right| p_\varepsilon(x,y) dA_g(y) . \end{aligned}$$

We can estimate the first right-hand side term thanks to (5.28), the second RHS term thanks to (5.16) and (6.10) of Section 6.2 and the third RHS term thanks to Step 1, the assumption $G(X) \leq C_G \left(|X| + |X|^2\right)$ on G and a Hölder inequality to obtain

$$|G(\Phi_\varepsilon) - K_\epsilon[G(\Phi_\varepsilon)]|(x) \leq \left(C_G + \|G\|_{\mathcal{C}^1\left(B_{2C_0}\setminus B_{\frac{1}{2\lambda_m}}\right)}\right)\left(\beta_\varepsilon + 2e^{-\frac{1}{4C_0^4\beta_\varepsilon^2}} + 2se^{-\frac{\rho^2}{8\varepsilon}\times\frac{1}{2}}\right) .$$

Up to increase β_ε (independently from G) we get (5.50).

$\diamondsuit$

We are now able to pass to the limit in $W^{1,2}$ and on the equation. The key point is to work on the energy of $\omega_\varepsilon = |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$.

We notice first that $\int_{\Sigma(\rho)} |\nabla\omega_\varepsilon|_g^2$ has to converge to 0 as $\varepsilon \to 0$ and then $\rho \to 0$. This is forced because cut-off functions around the singularity points have a small energy controlled by $\frac{1}{\ln\frac{1}{\rho}}$ (independent of ε), and because of the bounds $\omega_\varepsilon \geq 1-\beta_\varepsilon$ on $\Sigma(\rho)$ and $|\omega_\varepsilon - 1| \leq \beta_\varepsilon$ on $supp(\nu_\varepsilon)$ and the structure of the global equation on Φ_ε.

Then, by the harmonic replacement estimates (Claim (4.1)), the $(\omega_\varepsilon)^2$-harmonic replacements $\widetilde{\Psi_\varepsilon}$ of the radial projection $\widetilde{\Phi_\varepsilon} = \frac{\Phi_\varepsilon}{\omega_\varepsilon}$ of Φ_ε are then $W^{1,2}$ close to $\widetilde{\Phi_\varepsilon}$. Thanks to the estimates in the appendix, $(\omega_\varepsilon)^2$-harmonic maps with small energy are bounded in $W^{2,p}$ for any $p \in (1,2)$, so that they converge in $W^{1,2} \cap \mathcal{C}^{0,\alpha}$ by standard compact Sobolev embeddings.

Claim 5.8. *We assume that (i) of Claim 5.2 holds. Then up to the extraction of a subsequence of $\{\Phi_\varepsilon\}$ there is a function $\Phi \in W^{1,2}(\Sigma)$ such that for any $\rho > 0$,*

$$\Phi_\epsilon \to \Phi \text{ in } W^{1,2}(\Sigma(\rho)) \text{ as } \varepsilon \to 0 . \tag{5.30}$$

Moreover, $\Phi \in \mathcal{C}^{0,\alpha}(\Sigma_\rho)$ and satisfies

$$|\Phi|_{\mathcal{E}}^2 = 1 \text{ on } \Sigma \setminus \{p_1, \cdots, p_s\} \tag{5.31}$$

and for $1 \leq k \leq m$, Φ satisfies

$$\Delta_g \Phi^k = \lambda_k \Phi^k d\nu \tag{5.32}$$

in a weak sense on Σ.

Proof. In the first step, we prove (5.33), in a second step (5.30), and in the third one we prove (5.31). Finally we prove (5.32).

STEP 1 :

We first prove the following energy convergence on $\omega_\varepsilon = |\Phi_\varepsilon|_{\mathcal{E}_\varepsilon}$

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{\Sigma(\rho)} |\nabla\omega_\varepsilon|^2 = 0 \tag{5.33}$$

First, we define a cut-off function $\eta \in \mathcal{C}_c^\infty\left(\Sigma(\sqrt{\rho})\right)$ such that $\eta = 1$ on $\Sigma(\rho)$ and

$$\int_\Sigma |\nabla \eta|^2 \leq \frac{C}{\ln \frac{1}{\rho}} \ .$$

We integrate the eigenvalue equation $\Delta_g \Phi_\varepsilon = \Lambda_\varepsilon \Phi_\varepsilon e^{2u_\varepsilon}$, against $\Lambda_\varepsilon \eta \Phi_\varepsilon$ and $\Lambda_\varepsilon \eta \widetilde{\Phi}_\varepsilon$, where $\widetilde{\Phi_\varepsilon} = \frac{\Phi_\varepsilon}{\omega_\varepsilon}$ and we get

$$\int_\Sigma \eta \left|\nabla \Phi_\varepsilon\right|^2_{\mathcal{E}_\varepsilon} dA_g + \int_\Sigma \left\langle \nabla \eta, \nabla \left(\frac{\omega_\varepsilon^2}{2}\right)\right\rangle_g dA_g = \int_\Sigma \eta \left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2 e^{2u_\varepsilon} dA_g \ , \tag{5.34}$$

where we consider the norm of the gradient with respect to the ellipsoid and that

$$\int_\Sigma \eta \left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \Phi_\varepsilon \right\rangle_{\mathcal{E}_\varepsilon} dA_g + \int_\Sigma \left\langle \nabla \eta, \nabla \omega_\varepsilon \right\rangle_g dA_g = \int_\Sigma \eta \frac{\left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2}{\omega_\varepsilon} e^{2u_\varepsilon} dA_g \ , \tag{5.35}$$

where we used that $\langle \Phi_\varepsilon, \nabla \Phi_\varepsilon \rangle_{\mathcal{E}_\varepsilon} = \nabla \left(\frac{\omega_\varepsilon^2}{2}\right) = \omega_\varepsilon \nabla \omega_\varepsilon$.

Since $\widetilde{\Phi_\varepsilon} \in \mathcal{E}_\varepsilon$. We have that

$$\left\langle \nabla \widetilde{\Phi_\varepsilon}, \nabla \Phi_\varepsilon \right\rangle_{\mathcal{E}_\varepsilon} = \frac{\left\langle \left(\nabla \Phi_\varepsilon - \widetilde{\Phi_\varepsilon} \nabla \omega_\varepsilon\right), \nabla \Phi_\varepsilon \right\rangle_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} = \frac{\left|\nabla \Phi_\varepsilon\right|^2_{\mathcal{E}_\varepsilon} - \left|\nabla \omega_\varepsilon\right|^2}{\omega_\varepsilon} \ , \tag{5.36}$$

Therefore since $\omega_\varepsilon^2 \geq 1 - \beta_\varepsilon$ by (5.24),

$$\begin{aligned}
\int_\Sigma \eta \frac{\left|\nabla \omega_\varepsilon\right|^2}{\omega_\varepsilon} dA_g &= \int_\Sigma \eta \frac{\left|\nabla \Phi_\varepsilon\right|^2_{\mathcal{E}_\varepsilon}}{\omega_\varepsilon} dA_g - \int_\Sigma \eta \frac{\left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2}{\omega_\varepsilon} e^{2u_\varepsilon} dA_g + \int_\Sigma \left\langle \nabla \eta, \nabla \omega_\varepsilon \right\rangle_g dA_g \\
&\leq \frac{1}{\sqrt{1-\beta_\varepsilon}} \int_\Sigma \eta \left|\nabla \Phi_\varepsilon\right|^2_{\mathcal{E}_\varepsilon} dA_g - \int_\Sigma \eta \left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2 e^{2u_\varepsilon} dA_g \\
&\quad + \int_\Sigma \eta \left(1 - \frac{1}{\omega_\varepsilon}\right) \left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2 e^{2u_\varepsilon} dA_g + \int_\Sigma \left\langle \nabla \eta, \nabla \omega_\varepsilon \right\rangle_g dA_g
\end{aligned}$$

and noting (5.34) and that $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$, we deduce

$$\begin{aligned}
\int_\Sigma \eta \frac{\left|\nabla \omega_\varepsilon\right|^2}{\omega_\varepsilon} dA_g &\leq \int_\Sigma (1 - \omega_\varepsilon) \left\langle \nabla \eta, \nabla \omega_\varepsilon \right\rangle_g dA_g + \int_\Sigma \eta \left(\omega_\varepsilon - 1\right) \frac{\left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2}{\omega_\varepsilon} e^{2u_\varepsilon} dA_g + o(1) \\
&\leq \left(\int_\Sigma \left|\nabla \omega_\varepsilon\right|^2_g dA_g\right)^{\frac{1}{2}} \left(\int_\Sigma (1 - \omega_\varepsilon)^2 \left|\nabla \eta\right|^2_g dA_g\right)^{\frac{1}{2}} \\
&\quad + \int_\Sigma \eta \left(\omega_\varepsilon - 1\right) \frac{\left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2}{\omega_\varepsilon} e^{2u_\varepsilon} dA_g + o(1) \\
&\leq \frac{C_2}{\sqrt{\ln\left(\frac{1}{\rho}\right)}} + \int_\Sigma \eta \left(\omega_\varepsilon - 1\right) \frac{\left|\Lambda_\varepsilon \Phi_\varepsilon\right|^2}{\omega_\varepsilon} e^{2u_\varepsilon} dA_g + o(1) \ .
\end{aligned}$$

We set for $X \in \mathbb{R}^{i_1 + \cdots + i_m}$

$$G(X) = \left| |X|_{\mathcal{E}_\varepsilon} - 1 \right| \frac{\left|\Lambda_\varepsilon X\right|^2}{|X|_{\mathcal{E}_\varepsilon}}$$

and we claim that

$$\left|\int_\Sigma \eta\left(\omega_\varepsilon-1\right)\frac{\left|\Lambda_\varepsilon\Phi_\varepsilon\right|^2}{\omega_\varepsilon}e^{2u_\varepsilon}dA_g\right|\leq\int_{\Sigma(\rho)}G\left(\Phi_\varepsilon\right)e^{2u_\varepsilon}dA_g\to 0 \text{ as } \varepsilon\to 0\,. \tag{5.37}$$

Indeed, we have $e^{2u_\varepsilon}=K_\varepsilon[\nu_\varepsilon]$ and we obtain

$$\begin{aligned}
\int_{\Sigma(\rho)}G\left(\Phi_\varepsilon\right)e^{2u_\varepsilon}dA_g &= \int_{\Sigma(\rho)}\left(\int_\Sigma G\left(\Phi_\varepsilon\right)(x)p_\varepsilon(x,y)d\nu_\varepsilon(y)\right)dA_g(x)\\
&= \int_{\Sigma(\frac{\rho}{10})}\left(\left(\int_{\Sigma(\rho)}G\left(\Phi_\varepsilon\right)(x)p_\varepsilon(x,y)dA_g(x)\right)-G(\Phi_\varepsilon)(y)\right)d\nu_\varepsilon(y)\\
&\quad+\int_{\Sigma(\frac{\rho}{10})}G(\Phi_\varepsilon)(y)d\nu_\varepsilon(y)\\
&\quad+\sum_{i=1}^s\int_{B_g(p_i,\frac{\rho}{10})}\left(\int_{\Sigma(\rho)}G\left(\Phi_\varepsilon\right)(x)p_\varepsilon(x,y)dA_g(x)\right)d\nu_\varepsilon(y)\\
&\leq \int_{\Sigma(\frac{\rho}{10})}\left(K_\varepsilon[G\left(\Phi_\varepsilon\right)]-G(\Phi_\varepsilon)\right)d\nu_\varepsilon+\int_{\Sigma(\frac{\rho}{10})}G(\Phi_\varepsilon)d\nu_\varepsilon\\
&\quad+\left\|G\right\|_{\mathcal{C}^0\left(B_{2C_0}\setminus B_{\frac{1}{2\lambda_m}}\right)}\sum_{i=1}^s\int_{B_g(p_i,\frac{\rho}{10})}\left(\int_{\Sigma(\rho)}p_\varepsilon(x,y)dA_g(x)\right)d\nu_\varepsilon(y)
\end{aligned}$$

where the first term after the inequality holds because $G\geq 0$, and the third term holds because of (5.16). We use that G satisfies the assumptions for (5.50) for the first term (notice that the slight dependance on ε of G does not matter since $C_G+\left\|G\right\|_{\mathcal{C}^1\left(B_{2C_0}\setminus B_{\frac{1}{2\lambda_m}}\right)}$ is uniformly bounded as $\varepsilon\to 0$). We use (5.25) for the second term, and (6.10) of Section 6.2 for the third term. We then obtain (5.37). Then we have that

$$\lim_{\varepsilon\to 0}\int_{\Sigma(\sqrt{\rho})}\frac{\left|\nabla\omega_\varepsilon\right|^2}{\omega_\varepsilon}dA_g\leq\frac{C_2}{\sqrt{\ln\frac{1}{\rho}}}$$

and letting $\rho\to 0$ completes the proof of STEP 1 of the current Claim.

STEP 2 :

We prove now (5.30)*, the strong $W^{1,2}(\Sigma(\rho))$-convergence of $\{\Phi_\varepsilon\}$ for any ρ.*

This result is just a consequence of the quantative estimate (4.3) in Claim (4.1) and the first Step. Indeed, by Claim 5.4 and the boundedness of the energy, up to partitions of unity, one can focus on disks $\mathbb{D}_r$ for r small enough to satisfy the assumptions of Claim (4.1). We take the $(\omega_\varepsilon)^2$-harmonic replacement $\widetilde{\Psi_\varepsilon}$ of $\widetilde{\Phi_\varepsilon}=\frac{\Phi_\varepsilon}{\omega_\varepsilon}$. Then, by ε-regularity results on $(\omega_\varepsilon)^2$-harmonic maps (see proposition 6.1 in the appendix), we know that $\widetilde{\Psi_\varepsilon}$ is bounded in $W^{2,p}(\mathbb{D}_{\frac{r}{2}})$ for any $p\in(0,2)$. In particular, by compactness embeddings, up to a subsequence $\widetilde{\Psi_\varepsilon}$ converges strongly in $W^{1,2}(\mathbb{D}_{\frac{r}{2}})$ and in $\mathcal{C}^{0,\alpha}(\mathbb{D}_{\frac{r}{2}})$. Since $\Phi_\varepsilon-\Psi_\varepsilon$

converges to 0 in $W^{1,2}(\mathbb{D}_r)$, and $\Psi_\varepsilon = \omega_\varepsilon \widetilde{\Psi_\varepsilon}$ converges strongly in $W^{1,2}(\mathbb{D}_{\frac{r}{2}})$, we obtain that Φ_ε converges strongly in $W^{1,2}(\mathbb{D}_{\frac{r}{2}})$.

This completes the proof of STEP 2 in the current Claim.

STEP 3 : We prove now the weak eigenvalue equation (5.32).

Let Φ be the strong limit in $W^{1,2}(\Sigma(\rho))$ for any ρ. By the previous convergence results, $\Phi \in \mathcal{C}^{0,\alpha}(\Sigma(\rho))$. Again, we can argue locally up to partitions of unity and take a disk $\mathbb{D}_r$ for r small enough to satisfy $\lambda_\star(\mathbb{D}_r, e^{2\tilde{u}_\varepsilon}) \geq \lambda_m(e^{2u_\varepsilon} g)$. Let $\zeta \in \mathcal{C}_c^\infty(\mathbb{D}_r)$. We have that

$$\begin{aligned}\int_\Sigma \overline{\zeta}\left(\Lambda_\varepsilon \Phi_\varepsilon e^{2u_\varepsilon} dA_g - \Lambda\Phi d\nu\right) &= \int_\Sigma \overline{\zeta}\left(\Lambda_\varepsilon \Phi_\varepsilon - \Lambda\Phi\right) e^{2u_\varepsilon} dA_g \\ &\quad + \int_\Sigma \overline{\zeta}\Lambda\Phi\left(e^{2u_\varepsilon} dA_g - d\nu\right) .\end{aligned}$$

Then on the first right-hand term, we have that

$$\begin{aligned}\int_\Sigma \overline{\zeta}\left(\Lambda_\varepsilon \Phi_\varepsilon - \Lambda\Phi\right) e^{2u_\varepsilon} dA_g &\leq \left(\int_{\mathbb{D}_r} \overline{\zeta}^2 \left|\Lambda_\varepsilon \Phi_\varepsilon - \Lambda\Phi\right|^2 e^{2u_\varepsilon} dA_g\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\lambda_\star(\mathbb{D}_r, e^{2\tilde{u}_\varepsilon})} \int_{\mathbb{D}_r} \left|\nabla\left(\overline{\zeta}\left|\Lambda_\varepsilon \Phi_\varepsilon - \Lambda\Phi\right|\right)\right|_g^2 dA_g\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\mathbb{D}_r} \left|\nabla\left(\Phi_\varepsilon - \Phi\right)\right|_g^2 dA_g\right)^{\frac{1}{2}}\end{aligned}$$

for some constant C independent of ε. Letting $\varepsilon \to 0$ in a weak sense to the eigenvalue equation $\Delta_g \Phi_\varepsilon = \Lambda_\varepsilon \Phi_\varepsilon e^{2u_\varepsilon}$, we get the equation (5.32) in a weak sense on $\Sigma(\rho)$ for any ρ, and then on $\Sigma \setminus \{p_1, \cdots, p_s\}$. Since Φ belongs to $W^{1,2}(\Sigma)$ as a strong limit of Φ_ε, equation (5.32) occurs on Σ in a weak sense and STEP 3 is proved.

By strong convergence in any L^p of Φ_ε, up to a subsequence again, we have that $|\Phi|_{\mathcal{E}}^2 \geq 1$ dA_g-a.e on Σ. By continuity of Φ on $\Sigma(\rho)$, $|\Phi|_{\mathcal{E}}^2 \geq 1$ holds everywhere except maybe on $\{p_1, \cdots p_s\}$. Since $\nabla\omega_\varepsilon$ converges to 0, we obtain that $|\Phi|_{\mathcal{E}}^2$ is a constant. Since $\int_\Sigma \left(|\Phi|_{\mathcal{E}}^2 - 1\right) d\nu = 0$ by the previous convergences, $\int_\Sigma d\nu \neq 0$ and ν does not have any atom, we conclude that $|\Phi|_{\mathcal{E}}^2 = 1$ on $\Sigma \setminus \{p_1, \cdots, p_s\}$ and we get (5.31). This concludes the proof of the Claim.

$\diamondsuit$

Notice that this Claim can be adapted to prove precisely Theorem 2.2 for $k > 1$, if there is $j \geq 1$ is such that $\lambda_j^\varepsilon \to 0$ as $\varepsilon \to 0$ and $\lambda_{j+1}^\varepsilon$ is uniformly lower bounded by a positive constant. In this case, notice that we do use directly proposition 4.1. We just delete the coordinates of Φ_ε associated to $\lambda_1^\varepsilon, \cdots, \lambda_j^\varepsilon$ because they necessarily converge to a constant in $W^{1,2}$. We then apply proposition 4.1 to the map Φ_ε restricted to coordinates associated to eigenvalues $\lambda_{j+1}^\varepsilon, \cdots, \lambda_m^\varepsilon$.

Now, we are able to prove Theorem 2.2 with the assumption (i) of Claim 5.2. We have that $|\Phi|_{\mathcal{E}}^2 = 1$. Moreover, $\Delta\Phi \in (T_\Phi \mathcal{E})^\perp$ which is exactly the assumption of being a weakly harmonic map into the ellipsoid $\mathcal{E}$. Therefore Φ is a smooth map and satisfies the harmonic

map equation in $\mathcal{E}$ in a strong sense. Now, noticing that $\Delta(|\Phi|^2_{\mathcal{E}}) = 0$, it is easy to compute, thanks to the equation $\Delta\Phi = \Lambda\Phi\nu$ that

$$\nu = \frac{\sum_{k=1}^m \lambda_k \left|\nabla\Phi_k\right|_g^2}{\sum_{k=1}^m \lambda_k^2 \left|\Phi_k\right|^2} dA_g \ ,$$

which means that ν is absolutely continous with respect to dA_g with a smooth density. The zeros of this density are exactly the conical singularities one may get for the metric

$$\tilde{g} = \frac{\sum_{k=1}^m \lambda_k \left|\nabla\Phi_k\right|_g^2}{\sum_{k=1}^m \lambda_k^2 \left|\Phi_k\right|^2} g$$

such that Φ is an eigenmap $\Delta_{\tilde{g}}\Phi = \Lambda\Phi$ and $\Phi \in \mathcal{E}$.

5.7. **Regularity estimates in the concentrated case.**

5.7.1. *Reduction to a similar problem as in the non-concentrated case.* We now assume that (ii) of Claim 5.2 occurs and that $\frac{\alpha_\varepsilon}{\sqrt{\varepsilon}} \to +\infty$. We set $\theta_\varepsilon = \frac{\varepsilon}{e^{2\bar{v}_l(a)}\alpha_\varepsilon^2}$, where $a_\varepsilon \to a \in \mathbb{R}^2$ as $\varepsilon \to 0$. Then

$$\theta_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \ . \tag{5.38}$$

As noticed in the beginning of Section 5.6 we handle this case similarly to the non-concentrated case. For all the section, we refer to 5.2 for all the notations we use at the scale of concentration: we use the "hat" notations for all the objects which play the same role as the previous objects in the non-concentrated case up to translation and dilatation.

First, notice that

$$e^{2\hat{u}_\varepsilon}dx - d\hat{\nu}_\epsilon \rightharpoonup_\star 0 \text{ in } \mathcal{M}(\mathbb{R}^2) \text{ as } \varepsilon \to 0 \ . \tag{5.39}$$

Indeed, for $\zeta \in \mathcal{C}^0_c(\mathbb{D}_{R_0})$ for some $R_0 > 0$, and $R > R_0$, we can write that

$$\begin{aligned}
\int_{\mathbb{R}^2} \zeta(z)\left(e^{2\hat{u}_\varepsilon(z)}dz - d\hat{\nu}_\varepsilon(z)\right) &= \int_{\Sigma\setminus\breve{\mathbb{D}}_R}\left(\int_{\breve{\mathbb{D}}_{R_0}} p_\varepsilon(y,x)\zeta(\hat{y})dA_g(y)\right)d\nu_\varepsilon(x) \\
&\quad + \int_{\mathbb{D}_R}\left(\int_{\mathbb{D}_R}\left(\zeta(z)-\zeta(x)\right)\hat{p}_\varepsilon(z,x)dz\right)d\hat{\nu}_\varepsilon(x) \\
&\quad + \int_{\mathbb{D}_{R_0}}\left(\int_{\mathbb{D}_R}\hat{p}_\varepsilon(z,x)dz - 1\right)\zeta(x)d\hat{\nu}_\varepsilon(x) \ .
\end{aligned}$$

By estimates (6.13) on the heat kernel, we have that

$$\begin{aligned}
\int_{\Sigma\setminus\breve{\mathbb{D}}_R}\left(\int_{\breve{\mathbb{D}}_{R_0}} p_\varepsilon(x,y)\left|\zeta(\hat{y})\right|dA_g(y)\right)d\nu_\varepsilon(x) &\leq \|\zeta\|_\infty \sup_{x\in\Sigma\setminus\breve{\mathbb{D}}_R}\int_{\breve{\mathbb{D}}_{R_0}} p_\varepsilon(x,y)dA_g(y) \\
&\leq O\left(\frac{e^{-\frac{(R-R_0)^2}{8\theta_\varepsilon}}}{\theta_\varepsilon}\right) \to 0 \text{ as } \varepsilon \to 0 \ .
\end{aligned}$$

By estimates (6.11) on the heat kernel, we have that

$$\begin{aligned}\int_{\mathbb{D}_R}\left(\int_{\mathbb{D}_R}|\zeta(z)-\zeta(x)|\,\hat{p}_\varepsilon(z,x)dz\right)d\hat{\nu}_\varepsilon(x) &\leq \sup_{x\in\mathbb{D}_R}\int_{\mathbb{R}^2}|\zeta(x)-\zeta(z)|\,\frac{e^{-\frac{|x-z|^2}{8\theta_\varepsilon}}}{2\pi\theta_\varepsilon}dz\\ &\to 0 \text{ as } \varepsilon\to 0\end{aligned}$$

since ζ is uniformly continuous on $\mathbb{R}^2$. Finally, we have by the heat kernel estimate (6.14) that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{D}_{R_0}}\left|\int_{\mathbb{D}_R}\hat{p}_\varepsilon(z,x)dz-1\right|=0\ ,$$

so that we get (5.39). We denote by $\hat{\nu}$ the weak star limit of both $\{e^{2\hat{u}_\varepsilon}dx\}$ and $\{\hat{\nu}_\varepsilon\}$ in $\mathcal{M}(\mathbb{R}^2)$.

Now, in order to perform the same pointwise estimates as previously we have to be far from singularity points. By rescalings of the points p_i^ε and radii r_i^ε given by Claim 5.4, we only keep indices $1\leq i\leq s$ such that

$$\frac{r_i^\varepsilon}{\alpha_\varepsilon}\to 0 \text{ and } |a_\varepsilon-p_i^\varepsilon|=O\left(\alpha_\varepsilon\right) \text{ as } \varepsilon\to 0\ , \tag{5.40}$$

Denoting I this set of indices of cardinal $\sharp I=t\leq s$, and $\sigma:\{1,\cdots,t\}\to I$ a reordering, we denote

$$\hat{p}_j^\varepsilon=\frac{a_\varepsilon-p_{\sigma(j)}^\varepsilon}{\alpha_\varepsilon} \text{ and } \hat{r}_j^\varepsilon=\frac{r_{\sigma(j)}^\varepsilon}{\alpha_\varepsilon}\ ,$$

and assume that up to a subsequence, $\hat{p}_j^\varepsilon$ converges to $\hat{p}_j$ for any j. These points $\hat{p}_j^\varepsilon$ with radius $\hat{r}_i^\varepsilon$ play the same role as p_j^ε and r_j^ε in the previous section. We set for $\rho>0$,

$$\Omega(\rho)=\mathbb{D}_{\frac{1}{\rho}}\setminus\bigcup_{j=1}^t\mathbb{D}_\rho(\hat{p}_j)\ ,$$

which plays the same role as $\Sigma(\rho)$ in the previous section.

We have by Claim 5.2, case (ii),

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{\Omega(\rho)}e^{2\hat{u}_\varepsilon}=1\ . \tag{5.41}$$

We aim at getting regularity estimates on $\hat{\Phi}_\varepsilon$ and $e^{2\hat{u}_\varepsilon}$ in $\Omega(\rho)$, thanks to (5.41), and following the proof of Claim 5.5 and Claim 5.6. We get:

Claim 5.9. *We assume that case (ii) of Claim 5.2 holds. We have the following*

- *Estimates on $\hat{\Phi}_\varepsilon$, there are $C_1>0$ and $C_2>0$ such that*

$$\forall\rho>0,\limsup_{\varepsilon\to 0}\left\|\hat{\Phi}_\varepsilon\right\|_{W^{1,2}(\Omega(\rho))}\leq C_1\ , \tag{5.42}$$

$$\forall\rho>0,\limsup_{\varepsilon\to 0}\left\|\hat{\Phi}_\varepsilon\right\|_{C^0(\Omega(\rho))}\leq C_2\ . \tag{5.43}$$

- *Quantitative non-concentration estimates on $e^{2\hat{u}_\varepsilon}$ and $|\nabla\hat{\Phi}_\varepsilon|^2$: there are $D_2>0$ and $D_1>0$ such that*

$$\forall\rho>0,\forall r>0,\limsup_{\varepsilon\to 0}\sup_{x\in\Omega(\rho)}\int_{\mathbb{D}_r(x)}e^{2\hat{u}_\varepsilon}\leq\frac{D_1}{\ln(\frac{1}{r})}\ , \tag{5.44}$$

$$\forall\rho>0,\forall r>0,\limsup_{\varepsilon\to 0}\sup_{x\in\Omega(\rho)}\int_{\mathbb{D}_r(x)}|\nabla\hat{\Phi}_\varepsilon|^2\leq\frac{D_2}{\sqrt{\ln(\frac{1}{r})}}\ . \tag{5.45}$$

The proof of this claim follows line by line the proof of Claim 5.5 and Claim 5.6, so that we can skip to write it here.

5.7.2. *Global pointwise bound of eigenfunctions.* What is slightly different to the non-concentrated case is that we need a pointwise estimate of $\{\Phi_\varepsilon\}$ on the whole surface in order to perform in Claim 5.12, the same estimates as in the proof of Claim 5.7. Indeed, since the heat kernel is globally defined on the whole surface, we have to be sure that the integral estimates on the thin parts stay bounded in order to get the expected sharp pointwise inequality on the eigenfunctions in the bubble, we proved in Claim 5.7 on the surface.

In this section, we aim at proving that up to a subsequence, we have a uniform bound for the sequence of eigenfunctions on a large global surface $\Sigma_{\rho,\varepsilon}$ defined below in (5.46), far from singularity points.

Claim 5.10. *For any $\rho>0$, there is a constant $C_0(\rho)>0$ such that*

$$\forall x\in\Sigma_{\varepsilon,\rho},|\Phi_\varepsilon|(x)\leq C_0(\rho)\left|\ln\left(1+\left|\frac{\alpha_\varepsilon}{d_g(x,a_\varepsilon)}\right|\right)\right|\ .$$

The proof is based on the estimates of Claim 5.5, reproved in Claim 5.11 below. One can also follow the lines of Claim 11 in the original paper [Pet18].

First, we will only be able to give a uniform pointwise bound far from singularity points on all the dyadic annuli centered at a_ε at the intermediate scales between α_ε and δ_0. We denote by $\{\omega_k^\varepsilon\}_{k=0,\cdots,k_0,k_0+1}$ (with $k_0\geq 0$) a family of sequences such that $\omega_0^\varepsilon=\alpha_\varepsilon$, $\omega_{k_0+1}^\varepsilon=\delta_0$ and for any $k\in\{0,\cdots,k_0\}$,

$$\frac{\omega_k^\varepsilon}{\omega_{k+1}^\varepsilon}\to 0\text{ as }\varepsilon\to 0\ ,$$

where the set

$$I_k=\left\{i\in\{1,\cdots s\};\frac{|p_i^\varepsilon-a_\varepsilon|}{\omega_k^\varepsilon}=O(1)\text{ and }\frac{r_i^\varepsilon}{\omega_k^\varepsilon}\to 0\text{ as }\varepsilon\to 0\right\}$$

is non empty for any $k\in\{1,\cdots,k_0\}$, and such that the indices $i\in I\setminus\bigcup_{k=1}^{k_0}I_k$ satisfy

- either $|p_i^\varepsilon-a_\varepsilon|\to 0$ and $\frac{r_i^\varepsilon}{|p_i^\varepsilon-a_\varepsilon|}$ is uniformly lower bounded by a positive constant as $\varepsilon\to 0$,
- or $|p_i^\varepsilon-a_\varepsilon|=O(\alpha_\varepsilon)$ and $\frac{r_i^\varepsilon}{\alpha_\varepsilon}\to 0$ as $\varepsilon\to 0$,
- or $d_g(\bar{a},\bar{p}_i)$ is lower bounded as $\varepsilon\to 0$.

It is easy to prove by induction that up to a subsequence, such a family $\{\omega_k^\varepsilon\}$ exists. As before (see discussion around (5.40)), notice that indices that satisfy the first condition are not "true" singularity points if we rescale the problem at center a_ε and radius $|p_i^\varepsilon - a_\varepsilon|$ so that we do not need to keep them in the analysis. We denote by I_0 the indices satisfying the second condition. They correspond to indices $I_0 := I$ satisfying (5.40). We denote by I_{k_0+1} the indices satisfying the third condition. They correspond to indices i satisfying that $d_g(\bar{p}_i, \bar{a})$ is lower bounded by a positive constant.

Up to reorder with a bijection $\sigma_k : \{1, \cdots, l_k\} \to I_k$, we denote by $\overline{p_\varepsilon}^{k,l} = \frac{p^\varepsilon_{\sigma_k(l)} - a_\varepsilon}{\omega_k^\varepsilon}$. Let $R_0 > 0$ be such that for any ε, k, l, $\left|\overline{p_\varepsilon}^{k,l}\right| = \frac{p^\varepsilon_{\sigma_k(l)} - a_\varepsilon}{\omega_k^\varepsilon} \leq R_0$. Up to a subsequence we assume that $\overline{p_\varepsilon}^{k,l} \to \overline{p}^{k,l}$ as $\varepsilon \to 0$.

For $\rho > 0$ and $\varepsilon > 0$, we set

$$\Sigma_{\rho,\varepsilon} = \Sigma \setminus S_{\rho,\varepsilon} \ , \tag{5.46}$$

where S_ε is the singularity set

$$S_{\rho,\varepsilon} = \left(\bigcup_{k=0}^{k_0} \bigcup_{l=1}^{l_k} \overline{\omega_\varepsilon^k \mathbb{D}_\rho \left(\overline{p}^{k,l}\right) + a_\varepsilon} \right) \cup \bigcup_{i \in I_{k_0+1}} B_g(\bar{p}_i, \rho) \ .$$

In the following claim, we take a sequence $\gamma_\varepsilon \to 0$. We denote by $\overline{\Omega_\varepsilon}^{\gamma_\varepsilon}$, $\overline{\Phi_\varepsilon}^{\gamma_\varepsilon}$ the rescalings of center a_ε and radii γ_ε of sequences of sets Ω_ε and functions Φ_ε:

$$\overline{\Omega_\varepsilon}^{\gamma_\varepsilon} = \frac{1}{\gamma\varepsilon}\left(\tilde{\Omega} - a_\varepsilon\right) \text{ and } \overline{\Phi_\varepsilon}^{\gamma_\varepsilon}(z) = \widetilde{\Phi_\varepsilon}(\gamma_\varepsilon z + a_\varepsilon)$$

We fix $0 < \rho < 1$. In order to prove Claim (5.10), we need a key pointwise estimate on annuli $\gamma_\varepsilon \mathbb{A}_{\frac{R_0}{\rho}}(a_\varepsilon) := \mathbb{D}_{\frac{\gamma_\varepsilon R_0}{\rho}}(a_\varepsilon) \setminus \mathbb{D}_{\rho\frac{\gamma_\varepsilon}{R_0}}(a_\varepsilon)$ around $\bar{p}_i$ (where R_0 is defined just before the definition (5.46) of $\Sigma_{\varepsilon,\rho}$): either the sequence $\left(\overline{\Phi_\varepsilon}^{\gamma_\varepsilon}\right)$ is bounded in $\mathbb{U}_\rho = \mathbb{A}_{\frac{R_0}{\rho}} \setminus \overline{S_{\rho,\varepsilon}}^{\gamma_\varepsilon}$ or it satisfies a Harnack inequality on this set. Notice that we very often have $\mathbb{A}_{\frac{R_0}{\rho}} \cap \overline{S_{\rho,\varepsilon}}^{\gamma_\varepsilon} = \emptyset$. It is only non empty at the scales $\omega_\varepsilon^{i,j}$.

Claim 5.11. *We have that for $1 \leq i \leq s$, $1 \leq j \leq t_i$, there is $B(\rho) > 0$ such that for all eigenfunction ϕ_ε as a coordinate of $\overline{\Phi_\varepsilon}^{\gamma_\varepsilon}$ (Φ_ε is the map in Claim 5.1), and for all sequence $2R_0\sqrt{\varepsilon} < \gamma_\varepsilon < \frac{\delta_0}{2R_0}$, either*

$$\forall x \in \mathbb{U}_\rho, |\phi_\varepsilon(x)| \leq B(\rho)$$

or

$$\forall x, y \in \mathbb{U}_\rho, \frac{|\phi_\varepsilon(y)|}{B(\rho)} \leq |\phi_\varepsilon(x)| \leq B(\rho)\, |\phi_\varepsilon(y)| \ .$$

Proof. Let a subsequence $\varepsilon_m \to 0$ as $m \to +\infty$ and let $x_m \in \mathbb{U}_\rho$ be such that $|\phi_{\varepsilon_m}(x_m)| = \max_{x \in \mathbb{U}_\rho} |\phi_{\varepsilon_m}(x)|$. We set $\delta_m = d(x_m, supp\left(\overline{\nu_{\varepsilon_m}}^{\gamma^{\varepsilon_m}}\right))$ and take $y_m \in supp\left(\overline{\nu_{\varepsilon_m}}^{\gamma^{\varepsilon_m}}\right)$ such that $|x_m - y_m| = \delta_m$. We consider 3 cases.

Case 1 - $\delta_m = O\left(\frac{\sqrt{\varepsilon_m}}{\gamma_{\varepsilon_m}}\right)$.

We apply Claim 5.3 for the sequence: $\{\exp_{g_l, x_l}(\gamma_{\varepsilon_m} x_m + p_i)\}_m$ of points in M and we get a uniform bound for $|\phi_{\varepsilon_m}(x_m)|$.

Case 2 - $\delta_m \to 0$ and $\frac{\delta_m \gamma_{\varepsilon_m}}{\sqrt{\varepsilon_m}} \to +\infty$ as $m \to +\infty$. We set

$$e^{2w_m} = \delta_m^2 e^{2\overline{u_{\varepsilon_m}}^{\gamma_{\varepsilon_m}}}(x_m + \delta_m x) \ ,$$
$$\psi_m = \phi_{\varepsilon_m}(x_m + \delta_m x) \ ,$$
$$z_m = \frac{1}{\delta_m}(y_m - x_m) \ ,$$

so that

$$\Delta \psi_m = \lambda_{\varepsilon_m} e^{2w_m} \psi_m \ .$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R}^2$ with $|z_0| = 1$ such that $z_m \to z_0$ as $m \to +\infty$. By the heat kernel estimates (6.12), there is $D_1 > 0$ such that

$$e^{2w_m} \leq D_1 \text{ on } \mathbb{D}_{\frac{1}{2}} \ .$$

By Claim 5.3, since $y_m \in supp\,(\overline{\nu_{\varepsilon_m}}^{\gamma_{\varepsilon_m}})$, $\psi_m(z_m) = O(1)$ as $m \to +\infty$.

We first assume that ψ_m does not vanish in $\mathbb{D}_3(0)$. Up to take $-\psi_m$, we may assume that $\psi_m > 0$ on $\mathbb{D}_3(0)$. Then, by Harnack inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}, \psi_m(x) \geq D_2 \psi_m(0) \ .$$

Since ψ_m is positive, ψ_m is superharmonic in $\mathbb{D}_{|z_m|}(z_m) \subset \mathbb{D}_3(0)$. Then,

$$\psi_m(z_m) \geq \frac{1}{2\pi\,|z_m|} \int_{\partial \mathbb{D}_{|z_m|}(z_m)} \psi_m d\sigma$$

and keeping the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}$, we get a constant $D_3 > 0$ such that $\psi_m(z_m) \geq D_3 \psi_m(0)$. We conclude that $\overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon_m}}(x_m) = \psi_m(0) = O(1)$.

We assume now that ψ_m vanishes somewhere in $\mathbb{D}_3(0)$. By Claim 5.4, ψ_m vanishes in $\mathbb{D}_4(0)$ on a piecewise smooth curve between two points of distance greater than 1. By proposition 5.1 for $\Omega = \mathbb{D}_5(0)$, we get a Poincaré inequality

$$\int_{\mathbb{D}_4(0)} \psi_m^2 dx \leq C_1 \int_{\mathbb{D}_5(0)} |\nabla \psi_m|^2 \, dx \ .$$

By elliptic regularity, ψ_m is uniformly bounded on $\mathbb{D}_{\frac{1}{4}}(0)$ and $\overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon_m}}(x_m) = \psi_m(0) = O(1)$.

Case 3 - $\frac{1}{\delta_m} = O(1)$. Up to the extraction of a subsequence, we assume that $x_m \to x$ in $\mathbb{U}_\rho$ as $m \to +\infty$.

We first assume that $\psi_m := \overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon_m}}$ vanishes somewhere in $\mathbb{U}_{\frac{\rho}{2}}$. We get with Claim 5.4 and proposition 5.1 for $\Omega = \mathbb{U}_{\frac{\rho}{4}}$ a constant $C_r > 0$ such that

$$\int_{\mathbb{U}_{\frac{\rho}{3}}} \psi_m^2 dx \leq C_r \int_{\mathbb{U}_{\frac{\rho}{4}}} |\nabla \psi_m|^2 \, dx \ .$$

By (6.12), there are some constants $\tilde{r} > 0$ and $D_1 > 0$ such that

$$e^{2\overline{u_{\varepsilon_m}}^{\omega_i^{\varepsilon_m}}} \leq D_1 \text{ on } \mathbb{D}_{\tilde{r}}(x) \ .$$

By elliptic estimates, $\{\psi_m\}$ is uniformly bounded on $\mathbb{U}_{\frac{\rho}{2}} \cap \mathbb{D}_{\frac{\tilde{r}}{2}}(x)$ so that $\overline{\phi^{\beta}_{\varepsilon_m}}^{\gamma_{\varepsilon_m}}(x_m) = O(1)$.

We assume now that $\psi_m := \overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon_m}}$ does not vanish in $\mathbb{U}_{\frac{\rho}{2}}$. Up to take $-\psi_m$, we may assume that $\psi_m > 0$ on $\mathbb{U}_{\frac{\rho}{2}}$.

Let's assume that $y_m \to y$ as $m \to +\infty$ with $y \in \mathbb{U}_{\frac{7\rho}{10}}$. By Claim 5.1, $\psi_m(y_m) = O(1)$. By (6.12), there exists a constant $D_1 > 0$ such that

$$e^{2\overline{u_{\varepsilon_m}}^{\omega_i^{\varepsilon m}}} \leq D_1 \text{ in } \mathbb{D}_{\delta-\tilde{\delta}}(x) \ ,$$

where $\tilde{\delta} = \min\left(\frac{\delta}{4}, \frac{\rho}{40}\right)$. By Harnack's inequality, there exists $D_2 > 0$ such that

$$\forall z \in \mathbb{U}_{\frac{6\rho}{10}} \cap \mathbb{D}_{\delta-2\tilde{\delta}}(x), \psi_m(x_m) \leq D_2 \psi_m(z) \ .$$

By superharmonicity on $\partial\mathbb{D}_{3\tilde{\delta}}(y_m) \subset \mathbb{U}_{\frac{\rho}{2}}$,

$$\psi_m(y_m) \geq \frac{1}{2\pi \times 3\tilde{\delta}} \int_{\partial\mathbb{D}_{3\tilde{\delta}}(y_m)} \psi_m d\sigma$$

We keep the part of the integral which lies in $\mathbb{U}_{\frac{6\rho}{10}} \cap \mathbb{D}_{\delta-2\tilde{\delta}}$ since the length of $\partial\mathbb{D}_{3\tilde{\delta}}(y_m) \cap \mathbb{U}_{\frac{6\rho}{10}} \cap \mathbb{D}_{\delta-2\tilde{\delta}}$ is uniformly bounded below and we get a constant $D_3 > 0$ such that $\psi_m(y_m) \geq D_3 \psi_m(x_m)$. Then, $\overline{\phi^{\beta}_{\varepsilon_m}}^{\omega_i^{\varepsilon m}}(x_m) = \psi_m(x_m) = O(1)$.

Assume now that $y_m \in \mathbb{R}^2 \setminus \mathbb{U}_{\frac{8\rho}{10}}$. By (6.12), there is a constant $D_1 > 0$ such that

$$e^{2\overline{u_{\varepsilon_m}}^{\gamma_{\varepsilon m}}} \leq D_1 \text{ in } \mathbb{U}_{\frac{9\rho}{10}} \ .$$

By Harnack inequality, there exists a constant $C_1 > 0$ such that

$$\forall z \in \mathbb{U}_\rho, |\psi_m|(x_m) \leq C_1 |\psi_m|(z)$$

By definition of x_m, we get

$$\forall z, \tilde{z} \in \mathbb{U}_\rho, \left|\overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon m}}\right|(\tilde{z}) \leq \left|\overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon m}}\right|(x_m) \leq C_1 \left|\overline{\phi_{\varepsilon_m}}^{\gamma_{\varepsilon m}}\right|(z)$$

which concludes the proof of the Claim. ♢

Now we set

$$m_\varepsilon(t) = \frac{1}{2\pi t} \int_{\partial\mathbb{D}_t(a_\varepsilon)} \Phi_\varepsilon(x) dl(x)$$

the mean value on circles centered at a_ε of Φ_ε. We classically have

$$m_\varepsilon(t) = f_\varepsilon(\alpha_\varepsilon) - \int_{\alpha_\varepsilon}^t \frac{\int_{\mathbb{D}_s} \Delta\Phi_\varepsilon(u) du}{2\pi s} ds \ .$$

Therefore, since $\Delta\Phi_\varepsilon$ is bounded in L^1, we have a classical log estimate on m_ε

$$|m_\varepsilon(t)| \leq |f_\varepsilon(\alpha_\varepsilon)| + \ln\left(\frac{t}{\alpha_\varepsilon}\right) \ . \tag{5.47}$$

Since by Claim 5.9, $|f_\varepsilon(\alpha_\varepsilon)|$ is uniformly bounded, by Claim 5.11 and (5.47), we get Claim 5.10.

5.7.3. *$W^{1,2}$-estimates on eigenfunctions.* Now, using the estimates by Claim 5.10, we can adapt the proof of Claim 5.7 on the bubble scale:

Claim 5.12. *We have for any $\rho > 0$, there exists $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that*

$$\forall x \in \Omega(\rho), \left|\hat{\Phi}_\varepsilon\right|^2_{\mathcal{E}_\varepsilon}(x) \geq 1 - \beta_\varepsilon \tag{5.48}$$

and

$$\forall x \in \Omega(\rho) \cap supp\,(\hat{\nu}_\varepsilon)\,, \left|\left|\hat{\Phi}_\varepsilon\right|^2_{\mathcal{E}_\varepsilon}(x) - 1\right| \leq \beta_\varepsilon\ . \tag{5.49}$$

Moreover, for any smooth function $G : \mathbb{R}^{i_1+\cdots+i_m} \to \mathbb{R}$ such that there is a constant C_G such that for any X, $|G(X)| \leq C_G\left(|X| + |X|^2\right)$, we have that for all $x \in \Sigma$

$$\hat{x} \in \Omega(\rho) \Rightarrow \left|G\left(\hat{\Phi}_\varepsilon\right)(\hat{x}) - K_\varepsilon[G\left(\Phi_\varepsilon\right)](x)\right| \leq \left(C_G + \|G\|_{\mathcal{C}^1\left(B_{2C_2}\setminus B_{\frac{1}{2\lambda_m}}\right)}\right)\beta_\varepsilon\ . \tag{5.50}$$

Proof. As in the proof of Claim 5.7, it suffices to prove that there is a sequence $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that for all $x \in \Sigma$,

$$\hat{x} \in \Omega(\rho) \Rightarrow \left|\left|\hat{\Phi}_\varepsilon(\hat{x})\right|^2_{\mathcal{E}_\varepsilon} - K_\varepsilon[|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}](x)\right| \leq \beta_\varepsilon\ . \tag{5.51}$$

Indeed, (5.51) gives (5.48) and (5.49) for $x \in supp(\nu_\varepsilon)$ by Claim 5.1. Notice that (5.51) is again a specific case of (5.50). We can then follow the lines of the following proof of (5.51) in order to prove (5.50).

Step 1: We recall that $a_\varepsilon \to a$ as $\varepsilon \to 0$ with $\widetilde{z_{i_0}} = a$.
For $1 \leq j \leq s_0$ and $\theta_\varepsilon = \frac{\varepsilon}{e^{2\tilde{v}_l(a)}\alpha_\varepsilon^2}$,

$$\sup_{x\in\Omega(\rho)} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{0,j})} \left|\hat{\Phi}_\varepsilon(z)\right|^2 \hat{p}_\varepsilon(z,x)dz = O(e^{-\frac{\rho^2}{8\theta_\varepsilon}})\ . \tag{5.52}$$

For $0 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\varepsilon = \frac{\varepsilon}{e^{2\tilde{v}_l(a)}\left(\omega_i^\varepsilon\right)^2}$,

$$\sup_{x\in\Omega(\rho)} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} \left|\overline{\Phi_\varepsilon}^{\omega_i^\varepsilon}(z)\right|^2 \overline{p_\varepsilon}^{\omega_i^\varepsilon}\left(z, \frac{\alpha_\varepsilon}{\omega_i^\varepsilon}x\right) dz = O(e^{-\frac{\rho^2}{8\tau_i^\varepsilon}})\ . \tag{5.53}$$

For $1 \leq i \leq s$ and $i \neq i_0$,

$$\sup_{x\in\Omega(\rho)} \int_{B_g(p_i,\frac{\rho}{10})} |\Phi_\varepsilon(z)|^2\, p_\varepsilon(\breve{x}, z)dA_g(z) = O(e^{-\frac{\rho^2}{8\varepsilon}})\ . \tag{5.54}$$

Note that (5.54) was already proved in STEP 1 of Claim 5.7. Note also that the proof of (5.52) reduces to (5.53) for $i = 0$. By scale invariance on properties satisfied by singularity points and the uniform estimates on the heat kernel, the proof of (5.53) follows the lines of the proof of STEP 1 in Claim 5.7, so that we can skip it (see also [Pet18])

Step 2: *There is a sequence $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that*

$$\forall x, y \in \Omega(\rho), |x - y| \leq \frac{\sqrt{\theta_\varepsilon}}{\beta_\varepsilon} \Rightarrow \left|\hat{\Phi}_\varepsilon(x) - \hat{\Phi}_\varepsilon(y)\right| \leq \beta_\varepsilon\ . \tag{5.55}$$

Again, by scale invariance, we follow the same lines as in the proof of STEP 2 in Claim 5.7 (see also [Pet18])

STEP 3: *We prove* (5.51).

Let $x \in M$ be such that $\hat{x} \in \Omega(\rho)$.

$$\begin{aligned}\left|\left|\hat{\Phi}_\varepsilon(\hat{x})\right|^2_{\mathcal{E}_\varepsilon} - K_\varepsilon[|\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}](x)\right| &\leq \int_M p_\varepsilon(x,y)\left||\Phi_\varepsilon(x)|^2_{\mathcal{E}_\varepsilon} - |\Phi_\varepsilon(y)|^2_{\mathcal{E}_\varepsilon}\right| dA_g(y)\\ &\leq \int_{\mathbb{D}_{\frac{\sqrt{\theta_\varepsilon}}{\beta_\varepsilon}}(\hat{x})} \hat{p}_\varepsilon(z,\hat{x})\left|\left|\hat{\Phi}_\varepsilon(\hat{x})\right|^2_{\mathcal{E}_\varepsilon} - \left|\hat{\Phi}_\varepsilon(z)\right|^2_{\mathcal{E}_\varepsilon}\right| dz\\ &\quad + I_\varepsilon\\ &\quad + \sum_{i\neq i_0}\int_{B_g(p_i,\frac{\rho}{10})} |\Phi_\varepsilon|^2_{\mathcal{E}_\varepsilon}\, p_\varepsilon(x,y) dA_g(y)\\ &\quad + \sum_{i=0}^{t}\sum_{j=1}^{s_t}\int_{\mathbb{D}_\rho(p_{i,j})}\left|\overline{\Phi_\varepsilon}^{\omega_i^\varepsilon}\right|^2_{\mathcal{E}_\varepsilon}\overline{p_\varepsilon}^{\omega_i^\varepsilon}\left(z,\frac{\alpha_\varepsilon}{\omega_i^\varepsilon}\hat{x}\right) dz\ ,\end{aligned}$$

where

$$I_\varepsilon = \int_{M\setminus\check{\mathbb{D}}_{\frac{\sqrt{\theta_\varepsilon}}{\beta_\varepsilon}}(\hat{x})} p_\varepsilon(x,y)\left(C_2^2(\rho) + C_0^2(\rho)\left(\ln\left(1+\frac{d_g(y,\bar{a}_\varepsilon)}{\alpha_\varepsilon}\right)+1\right)^2\right) dA_g(y)\ .$$

Here, we used Claim 5.9 and Claim 5.10. By (5.52), (5.53), (5.54) and (5.55),

$$\left|\left|\hat{\Phi}_\varepsilon(\hat{x})\right|^2_{\mathcal{E}_\varepsilon} - K_\varepsilon[|\Phi_\varepsilon|^2](x)\right|_{\mathcal{E}_\varepsilon} \leq 2C_2(\rho)\beta_\varepsilon + O(e^{-\frac{\rho^2}{8\alpha_\varepsilon^2}}) + I_\varepsilon$$

and there exists some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that

$$\begin{aligned} I_\varepsilon &\leq K_0(\rho)\ln\left(\frac{\delta(M)}{\alpha_\varepsilon}\right)^2\int_{M\setminus\Omega_l} p_\varepsilon(x,y) dA_g(y)\\ &\quad + K_1(\rho)\int_{\hat{\Omega}_l\setminus\mathbb{D}_{\frac{\sqrt{\theta_\varepsilon}}{\beta_\varepsilon}}(\hat{x})} \hat{p}_\varepsilon(z,\hat{x})\left(\ln(1+|z|)^2+1\right) dz\ .\end{aligned}$$

Since $\frac{\alpha_\varepsilon^2}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$,

$$\ln\left(\frac{\delta(M)}{\alpha_\varepsilon}\right)^2\int_{M\setminus\Omega_l} p_\varepsilon(x,y) dA_g(y) \leq \ln\left(\frac{\delta(M)}{\alpha_\varepsilon}\right)^2 \times O\left(\frac{e^{-\frac{\delta(M)^2}{4\varepsilon}}}{\varepsilon}\right) = o(1) \text{ as } \varepsilon \to 0$$

and by (6.10),

$$\begin{aligned}\int_{\hat{\Omega}_l\setminus\mathbb{D}_{\frac{\sqrt{\theta_\varepsilon}}{\beta_\varepsilon}}(\hat{x})} \hat{p}_\varepsilon(z,\hat{x})\left(\ln(1+|z|)^2+1\right) dz &\leq \int_{\mathbb{R}^2\setminus\mathbb{D}_{\frac{\sqrt{\theta_\varepsilon}}{\beta_\varepsilon}}} \frac{A_0}{4\pi\theta_\varepsilon}e^{-\frac{|\hat{x}-z|^2}{8\theta_\varepsilon}}\left(\ln(1+|z|)^2+1\right) dz\\ &\leq \int_{\mathbb{R}^2\setminus\mathbb{D}_{\frac{1}{\beta_\varepsilon}}(0)} \frac{A_0}{4\pi}e^{-\frac{|y|^2}{8}}\left(\ln\left(1+\left|\hat{x}+\sqrt{\theta_\varepsilon}y\right|\right)^2+1\right) dy\\ &= o(1) \text{ uniformly for } \hat{x} \in \Omega(\rho)\ .\end{aligned}$$

Up to increase β_ε, we get (5.51).

$\diamondsuit$

Now, one can pass to the limit in $W^{1,2}(\Omega(\rho))$.

Claim 5.13. *Up to the extraction of a subsequence of $\{\hat{\Phi}_\varepsilon\}$, there is a map $\hat{\Phi} \in W^{1,2}(\mathbb{R}^2)$ such that for any $\rho > 0$*

$$\hat{\Phi}_\epsilon \to \hat{\Phi} \text{ in } W^{1,2}(\Omega(\rho)) \ . \tag{5.56}$$

Moreover the limiting function satisfies $\hat{\Phi} \in \mathcal{C}^{0,\alpha}(\Omega(\rho))$ and $\left|\hat{\Phi}\right|^2 = 1$ in $\mathbb{R}^2 \setminus \{\hat{p}_1, \cdots, \hat{p}_t\}$ and the equation $\Delta\hat{\Phi} = \Lambda\hat{\Phi}d\hat{\nu}$ in a weak sense on $\mathbb{R}^2$.

We can follow the lines of Claim 5.7 to prove this claim.

Now, let's prove Theorem 2.2 with the assumption (ii) of Claim 5.2. We have that $\left|\hat{\Phi}\right|^2 = 1$ everywhere except maybe on a finite number of points. Since we have that $\Delta\hat{\Phi} = \Lambda\hat{\Phi}\nu$ in a weak sense, then $\Delta\hat{\Phi} \in \left(T_{\hat{\Phi}}\mathcal{E}\right)^\perp$ which is exactly the assumption of being a weakly harmonic map into the ellipsoid $\mathcal{E}$. Therefore $\hat{\Phi}$ is a smooth map and satisfies the harmonic map equation in $\mathcal{E}$ in a strong sense. Now, noticing that $\Delta\left(\left|\hat{\Phi}\right|^2_{\mathcal{E}}\right) = 0$, thanks to the equation $\Delta\hat{\Phi} = \Lambda\hat{\Phi}\hat{\nu}$, we have that

$$\hat{\nu} = \frac{\sum_{k=1}^m \lambda_k \left|\nabla\hat{\Phi}_k\right|^2_g}{\sum_{k=1}^m \lambda_k^2 \left|\hat{\Phi}_k\right|^2} Leb \ ,$$

which means that $\hat{\nu}$ is absolutely continous with respect to Leb with a smooth density. By energy reasons, after a stereographic projection we get a measure on the sphere without atom. By point-removability theorem, the harmonic map is defined on all the sphere and we get a smooth conformal factor with respect to the round metric h. The zeros of this conformal factor are exactly the conical singularities one may get for the metric

$$\check{g} = \frac{\sum_{k=1}^m \lambda_k \left|\nabla\check{\Phi}_k\right|^2_g}{\sum_{k=1}^m \lambda_k^2 \left|\check{\Phi}_k\right|^2} h$$

such that $\check{\Phi}$ satisfies $\Delta_{\check{g}}\check{\Phi} = \Lambda\check{\Phi}$ and $\check{\Phi} \in \mathcal{E}$. This completes the proof of Theorem 2.2.

6. Appendix

6.1. **Estimates on ω^2-harmonic maps.** Let ω be a $W^{1,2}$-function which satisfies the inequalities

$$\frac{1}{C_\omega} \leq \omega \leq C_\omega \ . \tag{6.1}$$

We have the following ε-regularity result on ω^2-harmonic maps into a compact manifold $M \subset \mathbb{R}^m$, stated with an added smallness assumption on the L^2 norm of $\nabla\left(\omega^2\right)$. For simplicity, we assume that the dimension of M is $m-1$ and that we have a global normal $\vec{N}$ (this is the case in the current paper). Therefore, locally, we can define the normal

$\nu \in W^{1,2}(\mathbb{D}, \mathbb{R}^m)$ to a map $u \in W^{1,2}(\mathbb{D}, M)$ as $\nu = \vec{N} \circ u$. The Euler-Lagrange equation for this map is $-div(\omega^2 \nabla u) \in (T_u M)^\perp$ and we also get that

$$-div(\omega^2 \nabla u) = \omega^2 \langle \nabla u, \nabla \nu \rangle \nu \ . \tag{6.2}$$

This is exactly the harmonic map equation when $\omega = 1$. We have the following ε-regularity on ω^2-harmonic maps adapted from the celebrated paper by Rivière [Riv08].

Proposition 6.1. *There is $\varepsilon_0 > 0$, and a constant C_p such that if a ω^2-harmonic map $u \in W^{1,2}(\mathbb{D}, M)$ satisfies*

$$\int_{\mathbb{D}} |\nabla u|^2 + \int_{\mathbb{D}} \left|\nabla \left(\omega^2\right)\right|^2 \leq \varepsilon_0 \ ,$$

then for any $q \in (1,2)$, $u \in W^{2,q}(\mathbb{D}_{\frac{1}{2}})$ and for any $p < +\infty$,

$$\|\nabla u\|_{L^p(\mathbb{D}_{\frac{1}{2}})} \leq C_p \|\nabla u\|_{L^2(\mathbb{D})} \ . \tag{6.3}$$

Proof. u is a weak solution of

$$div(\omega^2 \nabla u) = \omega^2 \Omega \nabla u, \tag{6.4}$$

where Ω, is defined by

$$\Omega_{i,j} = \nu_j \nabla \nu_i - \nu_i \nabla \nu_j \in L^2(\mathbb{D}) \ ,$$

where $\nu(x)$ is a normal vector to $T_{u(x)}M$. Notice that $\Omega_{i,j} = \Omega_{j,i}$.

Moreover there is $C > 0$ such that

$$\omega^2 |\Omega| \leq C |\nabla u| \text{ almost everywhere} \ .$$

In order to prove the ε-regularity, we follow the strategy of Rivière [Riv08]. Since Ω is anti-symmetric, there is $P \in W^{1,2}(\mathbb{D}, SO(m))$, such that

$$div(\ {}^tP \nabla P - \ {}^tP \Omega P) = 0,$$

and

$$\|\nabla P\|_2 \leq 2 \|\Omega\|_2.$$

Hence we have ,

$$div(Q \nabla u) = \nabla^\perp B (Q \nabla u)$$

with $Q = \ {}^tP \omega^2$ and $\nabla^\perp B = -\ {}^tP \nabla P + \ {}^tP \Omega P$. Then we would like to rewrite the system like a Jacobian on the right hand-side. Let $A \in W^{1,2}(\mathbb{D}, Gl_m(\mathbb{R}))$ and $C \in W^{1,2}(\mathbb{D}, \mathcal{M}_m(\mathbb{R}))$, such that

$$div(AQ \nabla \tilde{u}) = \nabla A \left(Q \nabla u\right) + A \nabla^\perp B (Q \nabla u) = \nabla^\perp C \nabla u.$$

Hence A and C must satisfy

$$\begin{cases} \Delta A & = \nabla^\perp C \nabla (Q^{-1}) - \nabla A \nabla^\perp B \\ \Delta C & = \nabla A \nabla^\perp Q - div(A \left(\nabla B\right) Q) \end{cases} \tag{6.5}$$

This is the same system as in (V.36) of [Riv12]. Hence there exists $A \in W^{1,2}(\mathbb{D}, Gl_m(\mathbb{R}))$ and $C \in W^{1,2}(\mathbb{D}, \mathcal{M}_m)$ such that

$$\begin{cases} \Delta A &= \nabla^\perp C \nabla(Q^{-1}) - \nabla A \nabla^\perp B \text{ on } \mathbb{D} \\ \partial_\nu A &= 0 \text{ on } \partial\mathbb{D} \text{ and } \int_{\mathbb{D}} A = \pi^2 Id \\ \Delta C &= \nabla A \nabla^\perp Q + div(A\,(\nabla B)\,Q) \text{ on } \mathbb{D} \\ C &= 0 \text{ on } \partial\mathbb{D} \end{cases} \tag{6.6}$$

with

$$\int_{\mathbb{D}} |\nabla A|^2\, dx + \|dist(A, SO(m))\|_\infty \leq C \left(\int_{\mathbb{D}} |\Omega|^2 dx + \int_{\mathbb{D}} |\nabla\left(\omega^2\right)|^2\, dx \right)$$

and

$$\int_{\mathbb{D}} |\nabla C|^2\, dx \leq C \left(\int_{\mathbb{D}} |\Omega|^2 dx + \int_{\mathbb{D}} |\nabla\left(\omega^2\right)|^2\, dx \right).$$

Finally setting, $\widetilde{A} = AQ$, we have

$$\Delta(\widetilde{A}\nabla u) = \nabla^\perp C \nabla u\,. \tag{6.7}$$

Then we are reduced to some classical Wente-type equation and the result follows directly from theorem V.3 of [Riv12]. ♢

Notice that when the target manifold is a sphere, this result was already proved in [KhS20], and the analysis to handle (6.7) is done in this case. Of course, as in the spherical case, the bootstrap stops at $W^{2,q}$ with $q < 2$ since we do not have more assumption about the regularity of ω^2. We generalized here the spherical case to the non-symmetric case thanks to the celebrated construction by Rivière [Riv08].

We notice that in this case, there is not a priori ε-regularity for the L^∞-norm of the gradient of u. Therefore, we cannot use directly the very simple proof in [LP19] of the Colding-Minicozzi energy convexity result in [CM08]. We have to go back to the techniques by Lamm and Lin [LL13], to prove that, thanks to a deep use of the structure of the equations (6.2), (6.4) (6.7), $|\nabla u|^2$ belongs to the local Hardy space $h^1(\mathbb{D})$. We skip the proof here since we just have to replace ${}^tP \in SO(m)$ in their proof by ${}^tP\omega^2$ in our case remembering the proof of Proposition 6.1 and assumption (6.1). We also use the Claim proved by Lamm and Lin [LL13]

Claim 6.1. *$Q \in h^1(\mathbb{D})$ such that $Q \geq 0$ a.e. Then there is a solution $f \in W_0^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$ such that*

$$\begin{cases} \Delta f = Q & \text{in } \mathbb{D} \\ f = 0 & \text{on } \partial\mathbb{D}\,, \end{cases} \tag{6.8}$$

satisfying

$$\|f\|_{L^\infty} + \|\nabla f\|_{L^2} \leq C\, \|Q\|_{h^1(\mathbb{D})}\,.$$

We set $Q = \left|\nabla\widetilde{\Psi}\right|^2 \in h^1(\mathbb{D})$ for a ω^2 -harmonic map $\widetilde{\Psi}$ on $\mathbb{D}$ and we get thanks to [CM08] that if there is a map $\widetilde{\Phi}$ such that $\widetilde{\Psi} = \widetilde{\Phi}$ on $\mathbb{D}$ and if $\int_{\mathbb{D}} \left|\nabla\widetilde{\Psi}\right|^2 \leq \varepsilon_0$ for ε_0 sufficiently small,

$$\int_{\mathbb{D}} \left|\widetilde{\Psi} - \widetilde{\Phi}\right|^2 \left|\nabla\widetilde{\Psi}\right|^2 \leq C \int_{\mathbb{D}} \left|\nabla\widetilde{\Psi}\right|^2 \int_{\mathbb{D}} \left|\nabla\left(\widetilde{\Psi} - \widetilde{\Phi}\right)\right|^2\,. \tag{6.9}$$

6.2. **Uniform estimates on the heat kernel.** The heat kernel $p_\varepsilon(x,y)$ of a compact Riemannian surface (Σ,g) at time $\varepsilon>0$ satisfies uniform bounds (see [Dav89, Gri12] for state of art): there is $A_0>1$ and $\rho>0$ such that for any $\varepsilon>0$,

$$\forall x,y\in\Sigma,\ \frac{1}{A_0 4\pi\varepsilon}e^{-\frac{d_g(x,y)^2}{4\varepsilon}(1-\rho)}\leq p_\varepsilon(x,y)\leq\frac{A_0}{4\pi\varepsilon}e^{-\frac{d_g(x,y)^2}{4\varepsilon}(1+\rho)}\ . \tag{6.10}$$

We also have uniform bounds on the local rescaled heat kernel $\hat{p}_\varepsilon(x,y)$ by some parameters $a_\varepsilon\in\mathbb{R}^2$ and $\alpha_\varepsilon>0$ such that $a_\varepsilon\to a\in\mathbb{R}^2$ and $\alpha_\varepsilon\to 0$ as $\varepsilon\to 0$. We refer to the notations used in the section 5.2. We have for any $R>0$, for any fixed $0<\rho<1$

$$\frac{e^{-\frac{|y-z|^2}{4\theta_\varepsilon}(1+\rho)}}{4\pi\theta_\varepsilon}(1-\rho)\leq\hat{p}_\varepsilon(z,y)\leq\frac{e^{-\frac{|y-z|^2}{4\theta_\varepsilon}(1-\rho)}}{4\pi\theta_\varepsilon}(1+\rho) \tag{6.11}$$

uniformly on $\mathbb{D}_R\times\mathbb{D}_R$, where $\theta_\varepsilon=\frac{\varepsilon}{e^{2\check{v}_l(a)}\alpha_\varepsilon^2}$, for all $\varepsilon>0$ small enough.

For a sequence of measures $\nu_\varepsilon\in\mathcal{M}_1(\Sigma)$, we also have uniform bounds for $R>r>0$ and $\theta_\varepsilon\to 0$ as $\varepsilon\to 0$:

$$\sup_{x\in\mathbb{D}_{R-r}}\int_{\Sigma\setminus\breve{\mathbb{D}}_R}\alpha_\varepsilon^2 p_\varepsilon(\breve{x},y)d\nu_\varepsilon(y)=O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}}{\theta_\varepsilon}\right)\ . \tag{6.12}$$

We prove it thanks to (6.10) and (6.11). Let $x\in\mathbb{D}_{R-r}$ and let us write that

$$\begin{aligned}
\alpha_\varepsilon^2\int_{\Sigma\setminus\breve{\mathbb{D}}_R}p_\varepsilon(\breve{x},y)d\nu_\varepsilon(y) &= e^{-2v_l(\breve{x})}\int_{\mathbb{D}_{C_0^2R}\setminus\mathbb{D}_R}\hat{p}_\varepsilon(x,z)d\hat{\nu}_\varepsilon(z)+\int_{\Sigma\setminus\breve{\mathbb{D}}_{C_0^2R}}\alpha_\varepsilon^2 p_\varepsilon(\breve{x},y)d\nu_\varepsilon(y)\\
&\leq C_0^2\int_{\mathbb{D}_{C_0^2R}\setminus\mathbb{D}_R}\frac{e^{-\frac{|x-z|^2}{8\theta_\varepsilon}}}{2\pi\theta_\varepsilon}d\hat{\nu}_\varepsilon(z)\\
&\quad+\int_{\Sigma\setminus B_g(\bar{a}_\varepsilon,\frac{\alpha_\varepsilon C_0^2R}{C_0})}\frac{\alpha_\varepsilon^2A_0}{4\pi\varepsilon}e^{-\frac{d_g(\breve{x},y)^2}{4\varepsilon}}d\nu_\varepsilon(y)\\
&\leq O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}}{\theta_\varepsilon}\right)+\frac{A_0\alpha_\varepsilon^2}{4\pi\varepsilon}e^{-\frac{\alpha_\varepsilon^2(R-r)^2}{4\varepsilon}}\ ,
\end{aligned}$$

where $\breve{\mathbb{D}}_r\subset B_g(\bar{a}_\varepsilon,\alpha_\varepsilon C_0 r)\subset B_g(\bar{a}_\varepsilon,\alpha_\varepsilon C_0 R)$. This proves (6.12). We also have that

$$\sup_{x\in\Sigma\setminus\breve{\mathbb{D}}_R}\int_{\breve{\mathbb{D}}_r}p_\varepsilon(x,y)dA_g(y)=O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}}{\theta_\varepsilon}\right)\ . \tag{6.13}$$

Let $x\in\Sigma\setminus\breve{\mathbb{D}}_R$. We assume that $x\in\mathbb{D}_{C_0^2R}\setminus\mathbb{D}_R$. We write that

$$\int_{\breve{\mathbb{D}}_r}p_\varepsilon(x,y)dA_g(y)=\int_{\mathbb{D}_r}\hat{p}_\varepsilon(z,\breve{x})dz\leq\frac{1}{2\pi\theta_\varepsilon}\int_{\mathbb{D}_r}e^{-\frac{|x-z|^2}{8\theta_\varepsilon}}dz\leq\frac{r^2}{\theta_\varepsilon}e^{-\frac{(R-r)^2}{8\theta_\varepsilon}}$$

if ε is small enough. If $x \in \Sigma \setminus \breve{\mathbb{D}}_{C_0^2 R} \subset \Sigma \setminus B_g(\bar{a}_\varepsilon, \alpha_\varepsilon R C_0)$, we write that

$$\begin{aligned}\int_{\breve{\mathbb{D}}_r} p_\varepsilon(x,y)dA_g(y) &\leq \int_{B_g(\bar{a}_\varepsilon,\alpha_\varepsilon C_0 r)} p_\varepsilon(x,y)dA_g(y)\\ &\leq \frac{A_0}{4\pi\varepsilon}\int_{B_g(\bar{a}_\varepsilon,\alpha_\varepsilon C_0 r)} e^{-\frac{d_g(x,y)^2}{4\varepsilon}}dA_g(y)\\ &\leq O\left(\frac{e^{-\frac{\alpha_\varepsilon^2(R-r)^2}{4\varepsilon}}}{\theta_\varepsilon}\right).\end{aligned}$$

This proves 6.13. Now let's prove that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{D}_r}\left|\int_{\mathbb{D}_R}\hat{p}_\varepsilon(z,x)dz - 1\right| = 0\,. \tag{6.14}$$

We fix $0<\rho<\frac{1}{2}$ and $R>0$. Then for ε small enough, we have by (6.11) that

$$\int_{\mathbb{D}_R}\hat{p}_\varepsilon(z,x)dz \leq \int_{\mathbb{R}^2}\frac{e^{-\frac{|x-z|^2(1-\rho)}{4\theta_\varepsilon}}}{4\pi\theta_\varepsilon}(1+\rho)dz = \frac{1+\rho}{1-\rho}$$

for any $x\in\mathbb{D}_r$ and that

$$\begin{aligned}\int_{\mathbb{D}_R}\hat{p_\varepsilon}(z,x)dz &\geq \int_{\mathbb{D}_R}\frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\varepsilon}}}{4\pi\theta_\varepsilon}(1-\rho)dz\\ &\geq \int_{\mathbb{R}^2}\frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\varepsilon}}}{4\pi\theta_\varepsilon}(1-\rho)dz - \int_{\mathbb{R}^2\setminus\mathbb{D}_R}\frac{e^{\frac{-|x-z|^2}{8\theta_\varepsilon}}}{2\pi\varepsilon}dz\\ &\geq \frac{1-\rho}{1+\rho} + o(1) \text{ as } \varepsilon\to 0\end{aligned}$$

uniformly on $\mathbb{D}_r$. Letting $\varepsilon\to 0$, then $R\to+\infty$ and then $\rho\to 0$ gives (6.14).

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