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Approximate Message Passing for sparse matrices with application to the equilibria of large ecological Lotka-Volterra systems

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Abstract

This paper is divided into two parts. The first part is devoted to the study of a class of Approximate Message Passing (AMP) algorithms which are widely used in the fields of statistical physics, machine learning, or communication theory. The AMP algorithms studied in this part are those where the measurement matrix has independent elements, up to the symmetry constraint when this matrix is symmetric, with a variance profile that can be sparse. The AMP problem is solved by adapting the approach of Bayati, Lelarge, and Montanari (2015) to this matrix model.

The Lotka-Volterra (LV) model is the standard model for studying the dynamical behavior of large dimensional ecological food chains. The second part of this paper is focused on the study of the statistical distribution of the globally stable equilibrium vector of a LV system in the situation where the random symmetric interaction matrix among the living species is sparse, and in the regime of large dimensions. This equilibrium vector is the solution of a Linear Complementarity Problem, which distribution is shown to be characterized through the AMP approach developed in the first part. In the large dimensional regime, this distribution is close to a mixture of a large number of truncated Gaussians.

Keywords: Approximate Message Passing, Equilibria of ecological systems, Lotka-Volterra Ordinary Differential Equations, Sparse random matrices.

1 Introduction

An ecosystem can be seen as a multi-dimensional dynamical system that represents the time evolution of the abundances of the interacting species. The behavior of such systems is governed by the intrinsic population dynamics and by the strengths of the interactions among these species. Given a system model, it is of interest to evaluate the distribution of the species at the equilibrium when this equilibrium exists and is unique. The present paper is motivated by this general problem.

An archetypal model for an ecological dynamical system is provided by the so called Lotka-Volterra (LV) multi-dimensional Ordinary Differential Equation (ODE). The dynamics of a LV ODE with n species take the form

$$\dot{u}(t) = u(t) \odot (r + (\Sigma - I_n) u(t)), \quad t \geq 0,$$

where the vector function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ represents the abundances of the n coexisting species after a proper normalization, \odot is the element-wise product, r is the so-called vector of intrinsic growth rates of the species, and Σ is a $n \times n$ matrix which $(i, j)^{\text{th}}$ element reflects the interaction effect of Species j on the growth of Species i [Tak96].

Denoting as $\|\cdot\|$ the spectral norm, it is known that under a condition such as $\|(\Sigma + \Sigma^\top)/2\| < 1$, the ODE solution is well-defined, and this ODE has a unique globally stable equilibrium $u_\star = [u_{\star,i}]_{i=1}^n$ in the classical sense of the Lyapounov theory [Tak96, LJM09]. It is of interest to study the distribution of the elements of this vector, which quantifies the relative abundances of the species at the equilibrium. It is useful to note here that u_\star frequently lies at the boundary of the first quadrant of \mathbb{R}^n , and therefore, it is of particular interest to evaluate the proportion of surviving species at these equilibria.

Usually, the interaction matrix Σ is difficult to measure or to evaluate, and all the more so as the ecosystem's dimension n gets large. To circumvent this difficulty, a whole line of research in theoretical ecology considers that the matrix Σ is a random matrix, and focuses on the dynamics of the LV system in the regime where $n \rightarrow \infty$ [AT15, ABC+22]. The idea is to predict some essential aspects of the dynamical behavior of the ecosystem on the basis of a few ‘‘phenomenological’’ statistical features of the interaction matrix, rather than on its fine structure. The application of large random matrix tools to the dynamical behavior study of ecological systems dates back to the work of May [May72]. Among the most widely studied statistical models for the interaction matrix from the standpoint of the large random matrix theory are the Gaussian Orthogonal Ensemble (GOE) model, the Gaussian model with i.i.d. elements (sometimes called the Ginibre model), or the so-called elliptical model, which can be seen as an ‘‘interpolation’’ these two with possibly a non-zero mean [AT12].

Given a statistical model for Σ , and assuming the existence of u_\star which is now a random vector, the problem amounts in our context to evaluating the asymptotic behavior of the random probability measure

$$\mu^{u_\star} = \frac{1}{n} \sum_{i \in [n]} \delta_{u_{\star,i}}$$

as $n \rightarrow \infty$. In the literature, this has been mostly done with tools issued from the physics. In [Bun17], Bunin obtained the asymptotic distribution of the equilibrium for the elliptical model by using the so-called dynamical cavity method. A similar result was obtained by Galla in [Gal18] with the help of generating functionals techniques. Older results in the same vein can be found in, *e.g.*, [OD92, Tok04]. Heuristic evaluations of the asymptotic behavior of μ^{u_\star} were proposed in [CEN22, CNM22], most generally in the elliptical non-centered case.

In this paper, we consider a symmetric model for Σ , which can be used to represent the competitive and the mutualistic interactions [BBC18, ABC+22]. In this framework, we assume that the coefficients of Σ are non-necessarily Gaussian, are independent up to the symmetry constraint, and are subjected to a variance profile that can be sparse. The two main features of our model are thus the inhomogeneity of the interaction strengths between the species, and their sparsity. Regarding this last assumption, it is indeed commonly observed that a species interacts with a very small proportion of the other species coexisting within the ecosystem [BSHM17].

Our approach is mathematically rigorous, contrary to the references we just mentioned. To obtain our results, we generalize the technique of our recent preprint [AHMN23] devoted to the GOE case. The idea goes as follows: The vector u_\star can be identified as the solution of a Linear Complementarity Problem (LCP), a class of problems studied in the field of linear programming (see [Tak96, CPS09]). In [AHMN23], it is shown that in the GOE case, the asymptotic behavior of μ^{u_\star} , where u_\star is now the LCP solution, can be evaluated with the help of an Approximate Message Passing (AMP) technique. Such techniques have recently aroused an intense and growing research effort in the fields of statistical physics, communication theory, or statistical Machine Learning [FVRS22]. In a word, given a function $h : \mathbb{R} \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ and a random symmetric $n \times n$ so-called measurement matrix W , a standard AMP algorithm is an iterative algorithm of the form

$$\hat{x}^{t+1} = Wh(\hat{x}^t, \eta, t) + \text{a ‘‘correction’’ term},$$

where $\eta = [\eta_i]_{i=1}^n \in \mathbb{R}^n$ is a parameter vector, and where $h(\hat{x}^t, \eta, t) = [h(\hat{x}_i^t, \eta_i, t)]_{i=1}^n$. By properly designing the correction term, one is able to control the joint distribution of the $(t+1)$ -uple $(\eta, \hat{x}^1, \dots, \hat{x}^t)$ for each $t \geq 0$ and for $n \rightarrow \infty$, as will be developed more precisely below. We shall be

able to make use of such a result to evaluate the large- n distribution of our equilibrium vector, and show that this distribution is close to a mixture of Gaussians.

Thus, from the standpoint of theoretical ecology, the present paper is a generalization of [AHMN23] to the case where Σ is non-necessarily Gaussian and is subjected to a variance profile that can be sparse. To that end, a version of the AMP algorithm well-suited to these kind of matrices is developed below, and might have its own interest due to its potential applications in other fields than in ecology.

Let us provide a quick review of the AMP literature in order to better position our contribution in this respect. Many of the original ideas lying behind the AMP algorithms come from the fields of statistical physics and communication theory. The first rigorous AMP results in a framework close to this paper were developed by Bolthausen [Bol14] and Bayati and Montanari [BM11] both for GOE matrices and for rectangular matrices with i.i.d. Gaussian elements. Since then, the AMP approach has been generalized in many directions. Let us cite the Generalized AMP of [Ran11], the work of [JM13] where block variance profiles are considered, [Fan22] devoted to rotationally invariant matrices, or the graph-based approach of [GB21]. Universality results in terms of the distribution of the elements of W were proposed in the recent papers [CL21], [DLS22], and [WZF22]. The closest to our paper among these is [WZF22], which considers among others the case where W is a centered symmetric matrix with independent elements satisfying $\mathbb{E}W_{ij}^2 \lesssim 1/n$ and $\lim_n \max_i |(\sum_j \mathbb{E}W_{ij}^2) - 1| = 0$, and shows that W can be replaced with a GOE matrix as regards the AMP problem. This constraint on the variance profile is alleviated in our context, leading to a more involved expression of the asymptotic joint distribution of the algorithm iterates.

In [BLM15], Bayati, Lelarge, and Montanari were among the first to establish a universal AMP result. Starting with polynomial activation functions and using a combinatorial approach, these authors propose a moment computation of the elements of the vector iterates, where these elements are expressed as sums of monomials in the matrix entries which are indexed by labelled trees. As regards the AMP part, the present paper is essentially an adaptation of the approach of [BLM15] to the case of a sparse variance profile.

Section 2 is devoted to our general AMP results for symmetric and non-symmetric measurement matrices, independently of the ecological application. Our LV problem is then stated in Section 3, along with the result on the large- n distribution of the equilibrium. The proofs for Sections 2 and 3 are provided in Sections 4 and 5 respectively.

2 Sparse AMP with a variance profile: problem statement and results

We start with our assumptions. Let (n) be a sequence of integers in the set $\{2, 3, \dots\}$ that converges to infinity. For each n , let $\{X_{ij}^{(n)}\}_{1 \leq i < j \leq n}$ be a set of real random variables such that:

Assumption 1. The following facts hold true.

- The $n(n-1)/2$ random variables $X_{ij}^{(n)}$ for $1 \leq i < j \leq n$ are independent.
- $\mathbb{E}X_{i,j}^{(n)} = 0$ and $\mathbb{E}(X_{ij}^{(n)})^2 = 1$.
- For each integer $k > 2$, there exists a constant $C_{\text{mom}}(k) > 0$ such that

$$\sup_n \max_{1 \leq i < j \leq n} \left(\mathbb{E} \left| X_{ij}^{(n)} \right|^k \right)^{1/k} \leq C_{\text{mom}}(k).$$

Let us write $X_{ji}^{(n)} = X_{ij}^{(n)}$ for $1 \leq i < j \leq n$, and $X_{ii}^{(n)} = 0$ for $i \in [n]$, and let us consider the $n \times n$ random symmetric matrix $X^{(n)} = [X_{ij}^{(n)}]_{i,j=1}^n$.

For each n , let $\{s_{ij}^{(n)}\}_{1 \leq i < j \leq n}$ be a set of deterministic non-negative numbers. Write $s_{ji}^{(n)} = s_{ij}^{(n)}$ for $1 \leq i < j \leq n$, and $s_{ii}^{(n)} = 0$ for $i \in [n]$, and consider the $n \times n$ symmetric matrix with non-negative elements $S^{(n)} = [s_{ij}^{(n)}]_{i,j=1}^n$. Define the random symmetric matrix $W^{(n)}$ as

$$W^{(n)} = \left[W_{ij}^{(n)} \right]_{i,j=1}^n = (S^{(n)})^{\odot 1/2} \odot X^{(n)}, \quad (1)$$

where \odot is the Hadamard product, and where $A^{\odot 1/2}$ is the element-wise square root of the matrix A .

Letting (K_n) be a sequence of positive integers indexed by n such that $K_n \leq n$, the variance profile matrix $S^{(n)}$ of $W^{(n)}$ complies with the following assumption:

Assumption 2. The following facts hold true.

- $K_n \rightarrow \infty$.
- There exists a constant $C > 0$ such that

$$\forall n, \forall i \in [n], \left| \left\{ j \in [n] : s_{ij}^{(n)} > 0 \right\} \right| \leq CK_n,$$

where $|\cdot|$ is the cardinality of a set.

- There exists a constant $C > 0$ such that $s_{ij}^{(n)} \leq CK_n^{-1}$ for all n and all $i, j \in [n]$.

Beyond these constraints, we do not put any structural assumption on the variance profile matrix $S^{(n)}$. In this paper, we shall be mostly interested in the situations where $K_n/n \rightarrow_n 0$, making the matrices $W^{(n)}$ sparse.

For each n , let $x^{(n),0} = [x_1^{(n),0}, \dots, x_n^{(n),0}]^\top \in \mathbb{R}^n$ be a deterministic vector that will represent the initial value of our AMP sequence, and let $\eta^{(n)} = [\eta_1^{(n)}, \dots, \eta_n^{(n)}]^\top \in \mathbb{R}^n$ be a deterministic parameter vector. These vectors obey the following assumptions:

Assumption 3. There exists a compact set $\mathcal{Q}_x \subset \mathbb{R}$ such that $x_i^{(n),0} \in \mathcal{Q}_x$ for all n and all $i \in [n]$.

Assumption 4. There exists a compact set $\mathcal{Q}_\eta \subset \mathbb{R}$ such that $\eta_i^{(n)} \in \mathcal{Q}_\eta$ for all n and all $i \in [n]$.

Some comments on this model for the vectors $x^{(n),0}$ and $\eta^{(n)}$ are provided in Remark 3 below. Consider a measurable function $h : \mathbb{R} \times \mathcal{Q}_\eta \times \mathbb{N} \rightarrow \mathbb{R}$ that satisfies the following assumption:

Assumption 5. The following properties hold true:

- For each $t \in \mathbb{N}$, there exists two constants $m, C > 0$ and a continuous non-decreasing function $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\kappa(0) = 0$, such that:

$$|h(\underline{x}, \underline{\eta}, t)| \leq C(1 + |\underline{x}|^m),$$

and

$$\forall \underline{\eta}, \underline{\eta}' \in \mathcal{Q}_\eta, \quad |h(\underline{x}, \underline{\eta}, t) - h(\underline{x}, \underline{\eta}', t)| \leq \kappa(|\underline{\eta} - \underline{\eta}'|)(1 + |\underline{x}|^m).$$

- For each $\underline{\eta} \in \mathcal{Q}_\eta$ and each $t \in \mathbb{N}$, the function $h(\cdot, \underline{\eta}, t)$ is absolutely continuous with a locally integrable derivative.

At the heart of the AMP algorithm that we shall study is a Gaussian \mathbb{R}^n -valued sequence of random vectors that we denote for each n as $(\mathbf{Z}^{(n),t})_{t \in \mathbb{N}_*}$, where $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$. The probability distribution of this sequence is defined recursively in t as follows. Let $\mathbf{Z}^{(n),t} = [Z_1^{(n),t}, \dots, Z_n^{(n),t}]^\top$. Writing $Z_i^{(n)} = (Z_i^{(n),1}, Z_i^{(n),2}, \dots)$, the sequences $\{Z_i^{(n)}\}_{i=1}^n$ are centered, Gaussian, and independent. Denote as $R_i^{(n),t}$ the covariance matrix of the vector $\bar{Z}_i^{(n),t} = [Z_i^{(n),1}, \dots, Z_i^{(n),t}]^\top$.

These matrices are constructed recursively in the parameter t as follows. Defining the variances $\{\Xi_i^{(n),0}\}_{i=1}^n$ as $\Xi_i^{(n),0} = h(x_i^{(n),0}, \eta_i^{(n)}, 0)^2$, we set

$$R_i^{(n),1} = \sum_{l \in [n]} s_{il}^{(n)} \Xi_l^{(n),0}.$$

Given the matrices $\{R_i^{(n),t}\}_{i=1}^n$, the covariance matrices

$$\Xi_i^{(n),t} = \mathbb{E} \begin{bmatrix} h(x_i^{(n),0}, \eta_i^{(n)}, 0) \\ h(Z_i^{(n),1}, \eta_i^{(n)}, 1) \\ \vdots \\ h(Z_i^{(n),t}, \eta_i^{(n)}, t) \end{bmatrix} \begin{bmatrix} h(x_i^{(n),0}, \eta_i^{(n)}, 0) & h(Z_i^{(n),1}, \eta_i^{(n)}, 1) & \cdots & h(Z_i^{(n),t}, \eta_i^{(n)}, t) \end{bmatrix}, \quad (2)$$

are well-defined for all $i \in [n]$. With this at hand, we set

$$R_i^{(n),t+1} = \sum_{l \in [n]} s_{il}^{(n)} \Xi_l^{(n),t}. \quad (3)$$

It is clear that $R_i^{(n),t}$ is the principal matrix of $R_i^{(n),t+1}$ consisting in its first t rows and columns.

The distribution of the sequence $(\mathbf{Z}^{(n),t})_t$ is determined by the initial value $x^{(n),0}$, the parameter vector $\eta^{(n)}$, the variance profile matrix $S^{(n)}$, and the activation function h . We shall say from now on that this distribution is determined by the $(S^{(n)}, h, \eta^{(n)}, x^{(n),0})$ -state evolution equations, which provide the covariance matrices $R_i^{(n),t}$.

The following assumption will be used to ensure that the variances of the random variables $Z_i^{(n),t}$ are bounded away from zero.

Assumption 6. There is a sequence of positive constants $(c_h(0), c_h(1), \dots)$ such that for each set $\mathcal{S}^{(n)} \subset [n]$ with $|\mathcal{S}^{(n)}| = K_n$, it holds that

$$\frac{1}{K_n} \sum_{l \in \mathcal{S}^{(n)}} h(x_l^{(n),0}, \eta_l^{(n)}, 0)^2 \geq c_h(0),$$

and for each $t \geq 1$,

$$\frac{1}{K_n} \sum_{l \in \mathcal{S}^{(n)}} \mathbb{E} h(\xi, \eta_l^{(n)}, t)^2 \geq c_h(t),$$

where $\xi \sim \mathcal{N}(0, 1)$.

Moreover, there exists a constant $c_S > 0$ such that

$$\min_{i \in [n]} \sum_{j \in [n]} s_{ij}^{(n)} \geq c_S.$$

Non-degeneracy conditions similar to this assumption are usually not constraining and are invoked in the vast majority of the contributions dealing with AMP algorithms, see, *e.g.*, [JM13, FVRS22]. If needed, it is certainly possible to lighten this assumption by properly perturbing the matrix $S^{(n)}$ or the function h in the course of the proof of Theorem 1 below. A perturbation argument of this sort was considered in, *e.g.*, [BMN19].

In all this paper, we follow the following notational convention. Given a function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and a k -uple of \mathbb{R}^n -valued vectors $(a^\ell = [a_i^\ell]_{i=1}^n)_{\ell \in [k]}$, we denote as $f(a^1, \dots, a^k)$ the \mathbb{R}^n -valued vector $[f(a_i^1, \dots, a_i^k)]_{i=1}^n$. Similarly, if f is a $\mathbb{R}^k \times N \rightarrow \mathbb{R}$ function, $f(a^1, \dots, a^k, t)$ is the \mathbb{R}^n -valued vector $[f(a_i^1, \dots, a_i^k, t)]_{i=1}^n$.

With Assumption 5 at hand, we denote as $\partial h(\underline{x}, \underline{\eta}, t)$ a measurable function that coincides almost everywhere on \mathbb{R} with the partial derivative $\partial h(\cdot, \underline{\eta}, t)$ at \underline{x} . Here, Stein's lemma says that

$(\mathbb{E}\xi^2)\mathbb{E}\partial h(\xi, \eta, t) = \mathbb{E}\xi h(\xi, \eta, t)$ for each Gaussian centered random variable ξ , see, *e.g.*, [FSW18, Th. 2.1]. From this identity, we can deduce that $\mathbb{E}\partial h(\mathbf{Z}^{(n),t}, \eta^{(n)}, t)$ exists for each $t \in \mathbb{N}_*$.

We are now ready to state the main result of this section. An AMP algorithm built around the matrix $W^{(n)}$ and the activation function h is expected to take the following form, in line with, *e.g.*, [JM13] or [BLM15]. Starting with the vector $\hat{x}^{(n),0} = x^{(n),0}$, the vector $\hat{x}^{(n),t+1} \in \mathbb{R}^n$ delivered by this algorithm at Iteration $t + 1$ would be given as

$$\hat{x}^{(n),t+1} = W^{(n)}h(\hat{x}^{(n),t}, \eta^{(n)}, t) - \text{diag}\left(S^{(n)}\mathbb{E}\partial h(\mathbf{Z}^{(n),t}, \eta^{(n)}, t)\right)h(\hat{x}^{(n),t-1}, \eta^{(n)}, t-1) \quad (4)$$

(variants for this algorithm could be considered by replacing the diagonal matrix

$$\text{diag}(S^{(n)}\mathbb{E}\partial h(\mathbf{Z}^{(n),t}, \eta^{(n)}, t))$$

above with $\text{diag}(S^{(n)}\partial h(\hat{x}^{(n),t}, \eta^{(n)}, t))$, or with $\text{diag}((W^{(n)})^{\odot 2}\partial h(\hat{x}^{(n),t}, \eta^{(n)}, t))$ where $A^{\odot 2}$ is the square of the matrix A in the Hadamard product, as is done in the literature in similar contexts).

Write $\hat{x}^{(n),t} = [\hat{x}_i^{(n),t}]_{i=1}^n$. What is usually done in the AMP literature is to make use of an algorithmic construction of the type of Equation (4) to capture the large- n behavior of the empirical distribution of the particles $\{(\eta_i^{(n)}, \hat{x}_i^{(n),1}, \dots, \hat{x}_i^{(n),t})\}_{i=1}^n$. Our main result for this part of the paper is a bit weaker in that it only concerns an approximation of these iterates. Yet, it suffices for virtually all the applications. Let $\|\cdot\|$ be the operator norm of a matrix or the Euclidean norm of a vector. Given $x \in \mathbb{R}^n$, write $\|x\|_n = \|x\|/\sqrt{n}$.

Theorem 1. Let Assumptions 1–6 hold true. Then for each integer $t > 0$, it holds that

$$\sup_n \max_{i \in [n]} \|R_i^{(n),t}\| < \infty. \quad (5)$$

Let $C_W > 0$ be an arbitrary constant. Fix an arbitrary integer $t_{\max} > 0$. Then, for every small $\varepsilon > 0$, there exists a sequence of matrices $(\mathbf{X}^{(n)})_{n \geq 2}$ with $\mathbf{X}^{(n)} = [x^{(n),1}, \dots, x^{(n),t_{\max}}] \in \mathbb{R}^{n \times t_{\max}}$, and such that

$$\mathbb{P}\left[\left[\|x^{(n),t+1} - \bar{x}^{(n),t+1}\|_n > \varepsilon\right] \cap \left[\|W^{(n)}\| \leq C_W\right]\right] \xrightarrow{n \rightarrow \infty} 0 \quad (6)$$

for each $t = 0, 1, \dots, t_{\max} - 1$, with

$$\bar{x}^{(n),t+1} = W^{(n)}h(x^{(n),t}, \eta^{(n)}, t) - \text{diag}\left(S^{(n)}\mathbb{E}\partial h(\mathbf{Z}^{(n),t}, \eta^{(n)}, t)\right)h(x^{(n),t-1}, \eta^{(n)}, t-1),$$

and with $x^{(n),0}$ being the initial vector introduced before Assumption 3. In the previous equation, the term $\text{diag}(\dots)h(\dots)$ is equal to zero for $t = 0$.

Let $(\beta_1^{(n)}, \dots, \beta_n^{(n)})$ be a n -uple of real numbers such that $\sup_n \max_{i \in [n]} |\beta_i^{(n)}| < \infty$. Let $\varphi : \mathcal{Q}_\eta \times \mathbb{R}^{t_{\max}} \rightarrow \mathbb{R}$ be a continuous function such that $|\varphi(\eta, \mathbf{x}^1, \dots, \mathbf{x}^{t_{\max}})| \leq C(1 + |\mathbf{x}^1|^m + \dots + |\mathbf{x}^{t_{\max}}|^m)$ for some $C, m > 0$. Then, there exists a non-negative function $\delta_\varphi(\varepsilon)$, defined for $\varepsilon > 0$ small enough, that satisfies $\delta_\varphi(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and for which

$$\mathbb{P}\left[\left|\frac{1}{n} \sum_{i \in [n]} \beta_i^{(n)} \varphi(\eta_i^{(n)}, x_i^{(n),1}, \dots, x_i^{(n),t_{\max}}) - \beta_i^{(n)} \mathbb{E}\varphi(\eta_i^{(n)}, Z_i^{(n),1}, \dots, Z_i^{(n),t_{\max}})\right| \geq \delta_\varphi(\varepsilon)\right] \xrightarrow{n \rightarrow \infty} 0. \quad (7)$$

Thanks to this theorem, the following approximation result holds true. Keeping the same assumptions and notations, we have:

Corollary 2. Consider the setting of the previous theorem. Fix the test function φ , and choose a sequence of n -uples $(\beta_i^{(n)})_{i \in [n]}$. Then, there exists a sequence of matrices $(\widetilde{\mathbf{X}}^{(n)})_{n \geq 2}$, with $\widetilde{\mathbf{X}}^{(n)} = [\tilde{x}^{(n),1}, \dots, \tilde{x}^{(n),t_{\max}}] \in \mathbb{R}^{n \times t_{\max}}$, such that for every $\varepsilon > 0$,

$$\mathbb{P}\left[\left[\|\tilde{x}^{(n),t+1} - \bar{x}^{(n),t+1}\|_n > \varepsilon\right] \cap \left[\|W^{(n)}\| \leq C_W\right]\right] \xrightarrow{n \rightarrow \infty} 0$$

for each $t = 0, 1, \dots, t_{\max} - 1$, with

$$\tilde{x}^{(n),t+1} = W^{(n)}h(\tilde{x}^{(n),t}, \eta^{(n)}, t) - \text{diag}\left(S^{(n)}\mathbb{E}\partial h(\mathbf{Z}^{(n),t}, \eta^{(n)}, t)\right)h(\tilde{x}^{(n),t-1}, \eta^{(n)}, t-1),$$

and furthermore,

$$\frac{1}{n} \sum_{i \in [n]} \beta_i^{(n)} \varphi(\eta_i^{(n)}, \tilde{x}_i^{(n),1}, \dots, \tilde{x}_i^{(n),t_{\max}}) - \beta_i^{(n)} \mathbb{E}\varphi(\eta_i^{(n)}, Z_i^{(n),1}, \dots, Z_i^{(n),t_{\max}}) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

This corollary will be proven in Section 4.5.

To complete these results, we need to bound the operator norm $\|W^{(n)}\|$. With the help of the results of Bandeira and van Handel in [BvH16], this can be done in the framework of the following assumption:

Assumption 7. There exists a constant $C > 0$ such that the moment bounds from Assumption 1 satisfy

$$C_{\text{mom}}(2k) \leq Ck^{\rho/2},$$

for some $\rho \geq 0$.

As is well known, the cases $\rho = 1$ and $\rho = 2$ correspond to the sub-Gaussian and the sub-exponential distributions respectively.

Proposition 3. Let Assumptions 2 and 7 hold true, and assume furthermore that $K_n \gtrsim (\log n)^{\rho \vee 1}$. Then, there exists a constant $C > 0$ such that $\limsup_n \|W^{(n)}\| \leq C$ with probability one.

Thus, the heavier the tail of the probability distributions of the $X_{ij}^{(n)}$, the faster we need to make K_n converge to infinity in order to ensure that $\|W^{(n)}\|$ is bounded. This proposition is proven in Appendix A.

We now provide some comments on Theorem 1.

Remark 1. This theorem is weaker than what is frequently found in the AMP literature from two standpoints: The sequences $(x^{(n),1}, x^{(n),2}, \dots)$ described in the statement do not follow exactly the AMP iterative scheme (4), and convergences stated therein hold in probability instead of holding in the almost sure sense. These limitations do not impede the applicability of Theorem 1.

Remark 2. It is useful to interpret the result of Theorem 1 in terms of the empirical distribution of the particles $\{(\eta_i^{(n)}, x_i^{(n),1}, \dots, x_i^{(n),t_{\max}})\}_{i=1}^n$ as an element of a Wasserstein space of probability measures. For two integers $k > 0$ and $r > 0$, denote as $\mathcal{P}(\mathbb{R}^k)$ the space of probability measures on \mathbb{R}^k , and by $\mathcal{P}_r(\mathbb{R}^k)$ the Wasserstein space of probability measures on \mathbb{R}^k with finite r^{th} -moment, equipped with the Wasserstein distance \mathbf{d}_r [Vil09]. It is well-known that the convergence of a sequence of probability measures (ν_n) to a probability measure ν in the space $\mathcal{P}_r(\mathbb{R}^k)$ (in the sense that $\mathbf{d}_r(\nu_n, \nu) \rightarrow_n 0$) is equivalent to the convergence

$$\int \varphi d\nu_n \xrightarrow[n \rightarrow \infty]{} \int \varphi d\nu$$

for each continuous function $\varphi : \mathbb{R}^k \rightarrow \mathbb{R}$ satisfying $|\varphi(\underline{x}^1, \dots, \underline{x}^k)| \leq C(1 + |\underline{x}^1|^r + \dots + |\underline{x}^k|^r)$ for some $C > 0$. Furthermore, if the sequence (ν_n) is random, then, the convergence $\mathbf{d}_r(\nu_n, \nu) \xrightarrow{\mathcal{P}} 0$ is equivalent to the convergence $\int \varphi d\nu_n \xrightarrow{\mathcal{P}} \int \varphi d\nu$ for the same class of test functions φ [FVRS22].

Given an arbitrary integer $r > 0$, we are interested here in the asymptotic behavior of $\mathcal{P}_r(\mathbb{R}^{t+1})$ -valued random probability measure $\mu^{\eta^{(n)}, x^{(n),1}, \dots, x^{(n),t_{\max}}}$ defined as

$$\mu^{\eta^{(n)}, x^{(n),1}, \dots, x^{(n),t_{\max}}} = \frac{1}{n} \sum_{i=1}^n \delta_{(\eta_i^{(n)}, x_i^{(n),1}, \dots, x_i^{(n),t_{\max}})}.$$

In general, this random measure does not converge to any fixed probability measure, because we did not put any assumption on the variance profile of the elements of $W^{(n)}$ beyond Assumption 2. However, let us define the deterministic measure $\boldsymbol{\mu}^{(n),t_{\max}} \in \mathcal{P}(\mathbb{R}^{t_{\max}+1})$ as follows: letting $\theta^{(n)}$ be a discrete random variable uniformly distributed on the set $[n]$ and independent of the sequence $(\mathbf{Z}^{(n),t})_t$, we put $\boldsymbol{\mu}^{(n),t_{\max}} = \mathcal{L}((\eta_{\theta^{(n)}}^{(n)}, \vec{Z}_{\theta^{(n)}}^{(n),t_{\max}}))$. In particular, $\boldsymbol{\mu}^{(n),t_{\max}}(\mathbb{R} \times \cdot)$ is a mixture of n multivariate Gaussians. By Assumption 4 and the bound (5), one can readily show that the sequence $(\boldsymbol{\mu}^{(n),t_{\max}})_{n \geq 2}$ is pre-compact in the space $\mathcal{P}_r(\mathbb{R}^{t_{\max}+1})$. Moreover, the convergence (7) is rewritten as

$$\int \varphi d\boldsymbol{\mu}^{\eta^{(n)}, x^{(n),1}, \dots, x^{(n),t_{\max}}} - \int \varphi d\boldsymbol{\mu}^{(n),t_{\max}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0, \quad (8)$$

which is equivalent to $\mathbf{d}_r(\boldsymbol{\mu}^{\eta^{(n)}, x^{(n),1}, \dots, x^{(n),t_{\max}}}, \boldsymbol{\mu}^{(n),t_{\max}}) \xrightarrow{\mathcal{P}} 0$.

Remark 3. Let us consider the classical special case where $K_n = n$ and where the variances satisfy $s_{ij}^{(n)} = 1/n$ for all $i \neq j$. In this case, it is easy to check that the Gaussian vectors $\{\vec{Z}_i^{(n),t}\}_{i \in [n]}$ are i.i.d., and furthermore, writing $\vec{Z}_i^{(n),t} \stackrel{\mathcal{L}}{=} \vec{Z}^{(n),t} = [Z^{(n),1}, \dots, Z^{(n),t}]^\top$ with the covariance matrix $R^{(n),t}$, it holds that

$$R^{(n),t+1} = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \begin{bmatrix} h(x_i^{(n),0}, \eta_i^{(n)}, 0) \\ h(Z^{(n),1}, \eta_i^{(n)}, 1) \\ \vdots \\ h(Z^{(n),t}, \eta_i^{(n)}, t) \end{bmatrix} \begin{bmatrix} h(x^{(n),0}, \eta_i^{(n)}, 0) & h(Z^{(n),1}, \eta_i^{(n)}, 1) & \dots & h(Z^{(n),t}, \eta_i^{(n)}, t) \end{bmatrix}.$$

Assume now that the probability measure $\nu^{(n)} = n^{-1} \sum_i \delta_{(x_i^{(n),0}, \eta_i^{(n)})}$ converges narrowly to a deterministic probability measure ν^∞ as $n \rightarrow \infty$, as it is frequently done in the literature, usually, under a randomness assumption on $(x^{(n),0}, \eta^{(n)})$ coupled with an independence assumption from $W^{(n)}$. Then, given a couple of real random variables $(\bar{x}, \bar{\eta})$ with the distribution ν^∞ , we can readily replace $R^{(n),t+1}$ with the matrix $R^{\infty,t+1}$ that does not depend on n and that is written as

$$R^{\infty,t+1} = \mathbb{E}_{(\bar{x}, \bar{\eta})} \mathbb{E}_{Z^{\infty,t}} \begin{bmatrix} h(\bar{x}, \bar{\eta}, 0) \\ h(Z^{\infty,1}, \bar{\eta}, 1) \\ \vdots \\ h(Z^{\infty,t}, \bar{\eta}, t) \end{bmatrix} \begin{bmatrix} h(\bar{x}, \bar{\eta}, 0) & h(Z^{\infty,1}, \bar{\eta}, 1) & \dots & h(Z^{\infty,t}, \bar{\eta}, t) \end{bmatrix},$$

with $Z^{\infty,t} \sim \mathcal{N}(0, R^{\infty,t})$ being independent of $(\bar{x}, \bar{\eta})$. One consequence of this replacement is that the measure $\boldsymbol{\mu}^{(n),t_{\max}}$ in (8) can be replaced with the distribution $\boldsymbol{\mu}^{\infty,t_{\max}}$ of $(\bar{\eta}, Z^{\infty,t_{\max}})$, and the convergence (8) is replaced with

$$\int \varphi d\boldsymbol{\mu}^{\eta^{(n)}, x^{(n),1}, \dots, x^{(n),t_{\max}}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \int \varphi d\boldsymbol{\mu}^{\infty,t_{\max}}.$$

We recover here a well-know result given in, *e.g.*, [BM11, FVRS22].

In the general setting of this paper, only the convergence (8) is available. In general, putting the assumptions on $x^{(n),0}, \eta^{(n)}$ mentioned above does not transform this convergence to a convergence to a fixed deterministic measure. Once again, this is due to the fact that we did not put any assumption on the asymptotic behavior of the variance profile matrix $S^{(n)}$ beyond Assumption 2.

The AMP algorithms studied in the literature for symmetric measurement matrices have non-symmetric analogues. For the sake of generality, we now succinctly deal with this case. The results of the next paragraph will not be used elsewhere in this paper.

The non-symmetric variant (sketch)

Let $(p(n))$ be a sequence of positive integers such that $0 < \inf p(n)/n \leq \sup p(n)/n < \infty$. For a given $p = p(n)$, let $G^{(n)} = [G_{ij}^{(n)}]_{i,j=1}^{p,n}$ be a random rectangular $p \times n$ matrix, where the random variables $G_{ij}^{(n)}$ for $(i, j) \in [p] \times [n]$ are independent, centered, and unit-variance random variables that share with the elements of $X^{(n)}$ the moment bounds given by Assumption 1. Let $B^{(n)} = [b_{ij}^{(n)}]_{i,j=1}^{p,n} \in \mathbb{R}^{p \times n}$ be a deterministic matrix with non-negative elements, and assume that the $(n+p) \times (n+p)$ symmetric matrix $\begin{bmatrix} 0 & B^{(n)} \\ (B^{(n)})^\top & 0 \end{bmatrix}$ satisfies Assumption 2 in the role of $S^{(n)}$ there. The rectangular $p \times n$ matrix of interest here is the random matrix $Y^{(n)}$ with independent elements and a variance profile defined as

$$Y^{(n)} = (B^{(n)})^{\odot 1/2} \odot G^{(n)}.$$

Let $\tilde{x}^{(n),0} = [\tilde{x}_j^{(n),0}]_{j \in [n]} \in \mathbb{R}^n$ be a deterministic vector which elements belong to a compact set as in Assumption 3. Let $\eta^{(n)} = [\eta_i^{(n)}]_{i \in [p]} \in \mathbb{R}^p$ and $\tilde{\eta}^{(n)} = [\tilde{\eta}_j^{(n)}]_{j \in [n]} \in \mathbb{R}^n$ be deterministic parameter vectors which elements belong to a compact set as in Assumption 4. Define two functions $h(\underline{x}, \underline{\eta}, t)$ and $\tilde{h}(\tilde{x}, \tilde{\eta}, t)$ that both satisfy Assumption 5. Assume furthermore that these functions satisfy non-degeneracy assumptions of the type of Assumption 6, and that all the row and column sums of $B^{(n)}$ are lower-bounded by a positive bound independent of n .

With these objects, we define a centered Gaussian \mathbb{R}^p -valued process $(\mathbf{Z}^{(n),t})_{t \in \mathbb{N}}$, and a centered Gaussian \mathbb{R}^n -valued process $(\tilde{\mathbf{Z}}^{(n),t})_{t \in \mathbb{N}_*}$ as follows. Write $\mathbf{Z}^{(n),t} = [Z_1^{(n),t}, \dots, Z_p^{(n),t}]^\top$, and $\tilde{\mathbf{Z}}^{(n),t} = [\tilde{Z}_1^{(n),t}, \dots, \tilde{Z}_n^{(n),t}]^\top$, and define the sequences $Z_i^{(n)} = (Z_i^{(n),0}, Z_i^{(n),1}, \dots)$ for $i \in [p]$, and $\tilde{Z}_j^{(n)} = (\tilde{Z}_j^{(n),1}, \tilde{Z}_j^{(n),2}, \dots)$ for $j \in [n]$. Then, the sequences $\{Z_i^{(n)}\}_{i=1}^p$ are independent, and so is the case of the sequences $\{\tilde{Z}_j^{(n)}\}_{j=1}^n$. The covariance matrices $R_i^{(n),t} = \text{Cov}(Z_i^{(n),0}, \dots, Z_i^{(n),t})$ and $\tilde{R}_j^{(n),t} = \text{Cov}(\tilde{Z}_j^{(n),1}, \dots, \tilde{Z}_j^{(n),t})$ are recursively defined in t as follows.

Writing $\tilde{\Xi}_j^{(n),0} = \tilde{f}(\tilde{x}_j^0, \tilde{\eta}_j^{(n)}, 0)^2$ for $j \in [n]$, we set

$$R_i^{(n),0} = \sum_{j \in [n]} b_{ij}^{(n)} \tilde{\Xi}_j^{(n),0}, \quad i \in [p].$$

With these variances at hand, we can construct the variances $\Xi_i^{(n),0} = \mathbb{E}f(Z_i^{(n),0}, \eta_i^{(n)}, 0)^2$ for $i \in [p]$, and then, we set

$$\tilde{R}_j^{(n),1} = \sum_{i \in [p]} b_{ij}^{(n)} \Xi_i^{(n),0}, \quad j \in [n].$$

Let $t > 0$, and assume that the matrices $\tilde{R}_j^{(n),t}$ are available. Then we are able to construct the n covariance matrices

$$\tilde{\Xi}_j^{(n),t} = \mathbb{E} \begin{bmatrix} \tilde{f}(\tilde{x}_j^{(n),0}, \tilde{\eta}_j^{(n)}, 0) \\ \tilde{f}(\tilde{Z}_j^{(n),1}, \tilde{\eta}_j^{(n)}, 1) \\ \vdots \\ \tilde{f}(\tilde{Z}_j^{(n),t}, \tilde{\eta}_j^{(n)}, t) \end{bmatrix} \begin{bmatrix} \tilde{f}(\tilde{x}_j^{(n),0}, \tilde{\eta}_j^{(n)}, 0) & \dots & \tilde{f}(\tilde{Z}_j^{(n),t}, \tilde{\eta}_j^{(n)}, t) \end{bmatrix},$$

and we can then construct the p covariance matrices

$$R_i^{(n),t} = \sum_{j \in [n]} b_{ij}^{(n)} \tilde{\Xi}_j^{(n),t}.$$

With these matrices at hand, we are able to write

$$\Xi_i^{(n),t} = \mathbb{E} \begin{bmatrix} f(Z_i^{(n),0}, \eta_i^{(n)}, 0) \\ \vdots \\ f(Z_i^{(n),t}, \eta_i^{(n)}, t) \end{bmatrix} \begin{bmatrix} f(Z_i^{(n),0}, \eta_i^{(n)}, 0) & \cdots & f(Z_i^{(n),t}, \eta_i^{(n)}, t) \end{bmatrix},$$

and we set

$$\tilde{R}_j^{(n),t+1} = \sum_{i \in [p]} b_{ij}^{(n)} \Xi_i^{(n),t}, \quad j \in [n].$$

With these equations, the AMP non-symmetric equations take the form

$$\begin{aligned} x^{(n),t} &\simeq Y^{(n)} \tilde{f}(\tilde{x}^{(n),t}, \tilde{\eta}^{(n)}, t) - \text{diag}(B^{(n)} \mathbb{E} \partial \tilde{f}(\tilde{Z}^{(n),t}, \tilde{\eta}^{(n)}, t)) f(x^{(n),t-1}, \eta^{(n)}, t-1) \\ \tilde{x}^{(n),t+1} &\simeq (Y^{(n)})^\top f(x^{(n),t}, \eta^{(n)}, t) - \text{diag}((B^{(n)})^\top \mathbb{E} \partial f(Z^{(n),t}, \eta^{(n)}, t)) \tilde{f}(\tilde{x}^{(n),t}, \tilde{\eta}^{(n)}, t), \end{aligned}$$

where the vectors $\tilde{f}(\tilde{x}^t, \tilde{\eta}, t) \in \mathbb{R}^n$ and $f(x^t, \eta, t) \in \mathbb{R}^p$ are defined in the obvious way, and where $f(x^{-1}, \eta, -1)$ is zero. In these expressions, the symbols \simeq are to be interpreted as in (6), and t runs through the set $\{0, \dots, t_{\max}\}$ with $t_{\max} \geq 0$ being arbitrary.

Furthermore, considering continuous test functions φ and $\tilde{\varphi}$ that increase at most polynomially, and considering bounded sequences of weights $(\beta_i^{(n)})_{i \in [p]}$ and $(\tilde{\beta}_j^{(n)})_{j \in [n]}$, we have

$$\begin{aligned} \frac{1}{n} \sum_{i \in [p]} \beta_i^{(n)} \varphi(\eta_i^{(n)}, x_i^{(n),0}, \dots, x_i^{(n),t_{\max}}) - \beta_i^{(n)} \mathbb{E} \varphi(\eta_i^{(n)}, Z_i^{(n),0}, \dots, Z_i^{(n),t_{\max}}) &\simeq 0, \\ \frac{1}{n} \sum_{j \in [n]} \tilde{\beta}_j^{(n)} \tilde{\varphi}(\tilde{\eta}_j^{(n)}, \tilde{x}_j^{(n),1}, \dots, \tilde{x}_j^{(n),t_{\max}+1}) - \tilde{\beta}_j^{(n)} \mathbb{E} \tilde{\varphi}(\tilde{\eta}_j^{(n)}, \tilde{Z}_j^{(n),1}, \dots, \tilde{Z}_j^{(n),t_{\max}+1}) &\simeq 0, \end{aligned}$$

where the meanings of the symbols \simeq are those given by (7).

These expressions can be deduced from the symmetric case by replacing the matrix W there with the $(n+p) \times (n+p)$ symmetric matrix $\begin{bmatrix} 0 & Y^{(n)} \\ (Y^{(n)})^\top & 0 \end{bmatrix}$, and by choosing the activation function adequately. More details on this passage from the symmetric to the non-symmetric case are provided in [JM13].

3 Application to the equilibria of LV systems

We now apply the previous results to the study of the distribution of the globally stable equilibria of large Lotka-Volterra ecological systems.

Let $\mathbb{R}_{*+} = (0, \infty)$. Keeping the sequence (n) of integers introduced in Section 2, let $(r^{(n)})$ be a sequence of vectors such that $r^{(n)} \in \mathbb{R}_{*+}^n$, and let $(\Sigma^{(n)})$ be a sequence of random symmetric matrices such that $\Sigma^{(n)} \in \mathbb{R}^{n \times n}$. For a given n , starting with a vector $u^{(n)}(0) \in \mathbb{R}_{*+}^n$, our LV ODE is given as

$$\dot{u}^{(n)}(t) = u^{(n)}(t) \odot \left(r^{(n)} + \left(\Sigma^{(n)} - I_n \right) u^{(n)}(t) \right), \quad t \geq 0. \quad (9)$$

We recall that $r^{(n)}$ is the so-called vector of intrinsic growth rates of the species, and the matrix $\Sigma^{(n)}$ is the species interaction matrix. When this ODE has a unique solution $u^{(n)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ for each $u^{(n)}(0) \in \mathbb{R}_{*+}^n$, and when the image of this function is pre-compact, we say that the ODE is well-defined. A sufficient condition for well-definiteness is $\|\Sigma^{(n)}\| < 1$ [LJM09]. It is also well-known that under this condition, the ODE (9) has a globally stable equilibrium $u_\star^{(n)} \in \mathbb{R}_+^n$ in the classical sense of the Lyapounov theory [Tak96, Chap. 3]. We shall focus herein on the distribution $\mu^{u_\star^{(n)}} \in \mathcal{P}(\mathbb{R}_+)$ of the elements of $u_\star^{(n)}$ for the large values of n , which reflects the relative abundances of the species at the equilibrium.

We now state our assumptions regarding the vectors $r^{(n)}$ and the matrices $\Sigma^{(n)}$. Let us call ‘‘hypotheses’’ the assumptions relative to our LV system. Our first hypothesis reads as follows:

Hypothesis 1. Let $X^{(n)}$ be the random symmetric $n \times n$ matrix constructed after Assumption 1 above. Let $V^{(n)} = \left[v_{ij}^{(n)} \right]_{i,j=1}^n$ be a deterministic symmetric matrix with the same structure as $S^{(n)}$ in Section 2, and that complies with Assumption 2 above, with the $s_{ij}^{(n)}$ in this assumption being replaced with $v_{ij}^{(n)}$. Furthermore, assume that there is a constant $c_V > 0$ such that

$$\min_{i \in [n]} \sum_{j \in [n]} v_{ij}^{(n)} \geq c_V$$

(see Assumption 6). Then, the matrices $\Sigma^{(n)}$ are written as

$$\Sigma^{(n)} = (V^{(n)})^{\odot 1/2} \odot X^{(n)}.$$

According to our model, Species i interacts only with the species in the set $\mathcal{V}_i^{(n)} = \{j \in [n], v_{ij}^{(n)} > 0\}$. The relative cardinalities of these sets satisfy $\min_i (|\mathcal{V}_i^{(n)}|/n) \sim K_n/n$ and $\max_i (|\mathcal{V}_i^{(n)}|/n) \sim K_n/n$. This ratio is the ‘‘degree of sparsity’’ of our model,

Since $\Sigma^{(n)}$ is random, we need to guarantee that $\limsup_n \|\Sigma^{(n)}\| < 1$ with probability one to ensure that the LV ODE is well-defined and has a global equilibrium. We consider this condition as a hypothesis:

Hypothesis 2. $\limsup_n \|\Sigma^{(n)}\| < 1$ with probability one.

Let us provide some comments on this hypothesis. The spectral norms of random matrices with structures close to W were studied in [BvH16] in the framework of Assumption 7. In this setting, explicit bounds on the spectral norm of $\Sigma^{(n)}$ can be obtained in some cases, among which the Gaussian case plays a prominent role. We note that in general, these bounds are not tight. Let $\|A\|$ and $\|A\|_\infty$ be respectively the maximum row sum norm and the max norm of the matrix A respectively. When $X^{(n)}$ is a Gaussian matrix, we know from [BvH16, Th. 1.1] that

$$\mathbb{E} \|\Sigma^{(n)}\| \leq T_{\text{Gauss}}^{(n)},$$

where

$$T_{\text{Gauss}}^{(n)} = (1 + \varepsilon) \left(2 \|V^{(n)}\|^{1/2} + \frac{6}{\sqrt{\log(1 + \varepsilon)}} (\|V^{(n)}\|_\infty \log n)^{1/2} \right)$$

for an arbitrary $\varepsilon > 0$. Furthermore, by Gaussian concentration,

$$\mathbb{P} \left[\|\Sigma^{(n)}\| \geq T_{\text{Gauss}}^{(n)} + t \right] \leq \exp(-t^2/(2\|V^{(n)}\|_\infty^2)), \quad (10)$$

for $\varepsilon \in (0, 1/2]$, as given by [BvH16, Cor. 3.9]. By consequence, it holds that in the Gaussian case, $\limsup_n \|\Sigma^{(n)}\| < 1$ with probability one if $\limsup_n T_{\text{Gauss}}^{(n)} < 1$ for some choice of $\varepsilon \in (0, 1/2]$. Owing to the constraint on $\|V^{(n)}\|_\infty$ provided in Assumption 2, this requires that the sequence (K_n) satisfies $K_n \gtrsim \log n$. In the Gaussian case, we can therefore admit a degree of sparsity of $\log n/n$. Under our hypothesis 1, this rate turns out to be the optimal rate for Hypothesis 2 to be verified, as a consequence of [BvH16, Cor. 3.15].

Similar explicit bounds on $\limsup_n \|\Sigma^{(n)}\|$ are available in the case where the random variables $X_{ij}^{(n)}$ are bounded, see [BvH16, Cor. 3.12] for more details. Here also, Hypothesis 2 is ensured when $\|V^{(n)}\|$ and $\|V^{(n)}\|_\infty \log n$ are bounded properly.

Under Hypothesis 1 without more specific information on the distributions of the X_{ij} , Hypothesis 2 is verified if both $\limsup_n \|V^{(n)}\|$ and $\limsup_n \|V^{(n)}\|_\infty (\log n)^{\rho \vee 1}$ are small enough, as shown by Proposition 3. We thus need that $K_n \gtrsim (\log n)^{\rho \vee 1}$ with a large enough factor.

In any case, we shall also need

Hypothesis 3. $\limsup_n \|V^{(n)}\| < 1/4$.

In the classical Wigner case where $V^{(n)} = \alpha n^{-1} \mathbf{1}_n \mathbf{1}_n^\top$ for some $\alpha > 0$, this condition is necessary and sufficient to ensure that $\limsup_n \|\Sigma^{(n)}\| < 1$ with probability one, by the well-known result on the almost sure convergence of $\|\Sigma^{(n)}\|$ to the edge of the semi-circle law. Sparse cases that behave similarly to the Wigner case in this respect, and thus, that require Hypothesis 3, were treated in the literature. These include the models dealt with in [BvH16, Sec. 4]. Similar results can be found in [Kho08, Sod10, BGP14].

Regarding the intrinsic growth rate vector, we consider the following hypothesis:

Hypothesis 4. The vector $r^{(n)}$ is a deterministic vector whose elements belong to a compact $\mathcal{Q}_r \subset \mathbb{R}_{*+}$.

In the practical ecological settings where the intrinsic growth rates within a given ecosystem are positive, it is not unrealistic to assume that these rates are lower and upper bounded by positive constants.

Let $x, y \in \mathbb{R}^n$. In the statement and in the proof of the following theorem, we denote as $xy, x/y, \sqrt{x}$, etc., the \mathbb{R}^n -valued vector obtained by performing these operations elementwise. Given a \mathbb{R}^n -valued random vector $U = [U_i]_{i=1}^n$, the notations $\mathbb{E}U$ and $\mathbb{P}[U \geq 0]$ will respectively refer in the remainder to the vectors $[\mathbb{E}U_i]_{i=1}^n$ and $[\mathbb{P}[U_i \geq 0]]_{i=1}^n$.

Theorem 4. Let Hypotheses 1 to 4 hold true. Then, for each n for which $\|\Sigma^{(n)}\| < 1$, the ODE (9) is well-defined, and it has a globally stable equilibrium $u_\star^{(n)}$. For the other values of n , put $u_\star^{(n)} = 0$. The distribution $\mu^{u_\star^{(n)}}$ is a $\mathcal{P}_2(\mathbb{R})$ -valued random variable on the probability space where $\Sigma^{(n)}$ is defined.

Let $\xi^{(n)} \sim \mathcal{N}(0, I_n)$. Then, for each n large enough, the system of equations

$$p = V^{(n)} \text{diag}(1 + \zeta)^2 \mathbb{E} \left(\sqrt{p} \xi^{(n)} + r^{(n)} \right)_+^2 \quad (11a)$$

$$\zeta = \text{diag}(1 + \zeta) V^{(n)} \text{diag}(1 + \zeta) \mathbb{P} \left[\sqrt{p} \xi^{(n)} + r^{(n)} \geq 0 \right]. \quad (11b)$$

admits a unique solution $(p, \zeta) = (p^{(n)}, \zeta^{(n)}) \in \mathbb{R}_+^n \times [0, 1]^n$. This solution satisfies

$$\sup_n \|p^{(n)}\|_\infty < \infty.$$

Let $Y^{(n)}$ be the Gaussian vector

$$Y^{(n)} = \left[Y_i^{(n)} \right]_{i=1}^n = \left(1 + \zeta^{(n)} \right) \left(\sqrt{p^{(n)}} \xi^{(n)} + r^{(n)} \right),$$

and define the deterministic measure $\mu^{(n)} \in \mathcal{P}_2(\mathbb{R})$ as $\mu^{(n)} = \mathcal{L}((Y_{\theta^{(n)}}^{(n)})_+)$, where $\theta^{(n)}$ is a uniformly distributed random variable on the set $[n]$, which is independent of $\xi^{(n)}$. Then

$$d_2 \left(\mu^{u_\star^{(n)}}, \mu^{(n)} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

Remark 4. Since $\sup_n \|p^{(n)}\|_\infty < \infty$, the sequence of measures $(\mu^{(n)})$, which are mixtures of truncated Gaussians with bounded means and variances, is a pre-compact sequence in the space $\mathcal{P}_2(\mathbb{R})$. Therefore, for each sub-sequence of (n) , there is a further sub-sequence (n') and a measure $\nu \in \mathcal{P}_2(\mathbb{R})$ such that $\mu^{u_\star^{(n')}} \rightarrow \nu$ in probability in $\mathcal{P}_2(\mathbb{R})$.

Remark 5. Write $u_\star^{(n)} = [u_{\star,i}^{(n)}]_{i=1}^n$. By Theorem 4,

$$\frac{1}{n} \sum_{i \in [n]} \varphi(u_{\star,i}^{(n)}) - \mathbb{E} \varphi((Y_{\theta^{(n)}}^{(n)})_+) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0$$

for each continuous function $\varphi : \mathbb{R} \rightarrow [0, 1]$ such that $\varphi(0) = 0$. With this at hand, we can consider the positive number $\gamma^{(n)} = \mathbb{P}[Y_{\theta^{(n)}}^{(n)} > 0]$ as an approximation of the proportion of surviving

species at the equilibrium, in the sense that $\gamma^{(n)} \simeq n^{-1} \sum_{i \in [n]} \varphi(u_{\star, i}^{(n)})$ when φ is chosen as a “continuous approximation” of the real function $\psi(x) = \mathbf{1}_{x>0}$. Of course, this does not imply that $\gamma^{(n)} - n^{-1} \sum_i \mathbf{1}_{u_{\star, i}^{(n)} > 0}$ converges to zero in probability.

Note that $\liminf_n \gamma^{(n)} > 1/2$ by Hypothesis 4 and by the bound $\sup \|p^{(n)}\|_\infty < \infty$, which implies that the proportion of species that survive at the equilibrium for large n is lower bounded by $1/2$ with a positive gap.

The following corollary to Theorem 4 shows that when the variance profile matrix is $V^{(n)} = \alpha n^{-1} \mathbf{1}_n \mathbf{1}_n^\top$ with $\alpha \in (0, 1/4)$, then, the large- n behavior of the empirical measure $\mu^{u_\star^{(n)}}$ is the one predicted by [Bun17, Gal18, ABC⁺22]. Remember that we are not restricted in the present paper to the Gaussian case.

Corollary 5. In the setting of the previous theorem, assume that $V^{(n)} = \alpha n^{-1} \mathbf{1}_n \mathbf{1}_n^\top$ with $\alpha \in (0, 1/4)$. Let $\underline{\xi} \sim \mathcal{N}(0, 1)$, and let $\theta^{(n)}$ be independent of $\underline{\xi}$ and uniformly distributed on the set $[n]$. Then, there is a unique couple $(\underline{p}^{(n)}, \underline{\zeta}^{(n)}) \in \mathbb{R}_+ \times [0, 1]$ such that, dropping the superscript $^{(n)}$,

$$\begin{aligned} \underline{p} &= \alpha(1 + \underline{\zeta})^2 \mathbb{E}_\theta \mathbb{E}_\xi [(\sqrt{\underline{p}} \underline{\xi} + r_\theta)_+^2], \\ \underline{\zeta} &= \alpha(1 + \underline{\zeta})^2 \mathbb{E}_\theta \mathbb{P}_\xi [\sqrt{\underline{p}} \underline{\xi} + r_\theta \geq 0]. \end{aligned}$$

Moreover,

$$d_2 \left(\mu^{u_\star}, \mathcal{L} \left((1 + \underline{\zeta}) (\sqrt{\underline{p}} \underline{\xi} + r_\theta)_+ \right) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

To prove this corollary, we notice that due to the structure of V , the solution of the system (11) is of the form $(p^{(n)}, \zeta^{(n)}) = (p^{(n)} \mathbf{1}_n, \zeta^{(n)} \mathbf{1}_n)$ where $(p^{(n)}, \zeta^{(n)})$ is the solution of the system shown in the statement. The result follows.

We now turn to our proofs. In the remainder of this paper, the notations $C, c > 0$ refer to constants that can change from line to line. Some of the statements hold true for n large enough instead of holding for each n . This will not be specified.

4 Proof of Theorem 1 and Corollary 2

This section is essentially devoted towards proving Theorem 1. The first and most important step of the proof is to consider an AMP sequence where the function $h(\cdot, \eta, t)$ is replaced with a polynomial, and to apply to this sequence the combinatorial technique of Bayati, Lelarge and Montanari in [BLM15].

4.1 AMP with polynomial activation functions

Let $d > 0$ be a fixed integer. Consider a function $p^{(n)} : \mathbb{R} \times [n] \times \mathbb{N} \rightarrow \mathbb{R}$ with the property that $p^{(n)}(\cdot, i, t)$ is a polynomial with a degree bounded by d , and is thus written as

$$p^{(n)}(\underline{x}, i, t) = \sum_{\ell=0}^d \alpha_\ell^{(n)}(i, t) \underline{x}^\ell.$$

Starting with the deterministic vector $x^{(n),0} = x^{(n),0}$, our purpose here is to study the AMP recursion in the vectors $x^{(n),1}, x^{(n),2}, \dots$ given as

$$x^{(n),t+1} = W^{(n)} p^{(n)}(x^{(n),t}, \cdot, t) - \text{diag} \left((W^{(n)})^{\odot 2} \partial p^{(n)}(x^{(n),t}, \cdot, t) \right) p^{(n),t-1}(\cdot, t-1), \quad (12)$$

where, writing $x = [x_i]_{i=1}^n$, we set $p^{(n)}(x, \cdot, t) = [p^{(n)}(x_i, i, t)]_{i=1}^n$ and $\partial p^{(n)}(x, \cdot, t) = [\partial p^{(n)}(\cdot, i, t)|_{x_i}]_{i=1}^n$, and we consider that $p^{(n)}(\cdot, \cdot, -1) \equiv 0$.

The general idea goes as follows. With the polynomial activation function $p^{(n)}(\cdot, i, t)$, the elements of the vectors $x^{(n),t}$ are random variables which are sums of monomials in the elements of $W^{(n)}$ with indices indexed by labelled trees. Our purpose will be to compute the moments of these random variables. Due to the presence of the “correction” so-called Onsager term $\text{diag}(\cdot)p(\cdot)$, the effect of the paths with the so-called backtracking edges will be cancelled, rendering these moments close to the Gaussian moments provided by the state evolution equations below. A nice intuitive explanation of the main proof idea is provided in [BLM15, § 2].

Let us provide the expressions of the state evolution equations for Algorithm (12). Consider the Gaussian sequence $(\check{Z}^{(n),t})_{t \in \mathbb{N}_*}$ described as follows. Writing $\check{Z}^{(n),t} = [\check{Z}_1^{(n),t}, \dots, \check{Z}_n^{(n),t}]^\top$ and $\check{Z}_i^{(n)} = (\check{Z}_i^{(n),1}, \check{Z}_i^{(n),2}, \dots)$, the sequences $\{\check{Z}_i^{(n)}\}_{i=1}^n$ are centered, Gaussian, and independent, and the covariance matrix $\check{R}_i^{(n),t}$ of the vector $\check{Z}_i^{(n),t} = [\check{Z}_i^{(n),1}, \dots, \check{Z}_i^{(n),t}]^\top$ is constructed recursively in t as follows.

$$\check{R}_i^{(n),1} = \sum_{l \in [n]} s_{il}^{(n)} p^{(n)}(x_l^{(n),0}, l, 0)^2,$$

and

$$\check{R}_i^{(n),t+1} = \sum_{l \in [n]} s_{il}^{(n)} \mathbb{E} \begin{bmatrix} p^{(n)}(x_l^{(n),0}, l, 0) \\ p^{(n)}(\check{Z}_l^{(n),1}, l, 1) \\ \vdots \\ p^{(n)}(\check{Z}_l^{(n),t}, l, t) \end{bmatrix} \begin{bmatrix} p^{(n)}(x_l^{(n),0}, l, 0) & p^{(n)}(\check{Z}_l^{(n),1}, l, 1) & \dots & p^{(n)}(\check{Z}_l^{(n),t}, l, t) \end{bmatrix}.$$

With this at hand, the main result of this section reads as follows.

Proposition 6. Let Assumptions 1–3 hold true, and consider the AMP algorithm (12). Assume that for each integer $t \geq 0$,

$$\sup_n \max_{\ell \leq d} \max_{i \in [n]} |\alpha_\ell^{(n)}(i, t)| < \infty. \quad (13)$$

Then,

$$\forall t > 0, \quad \sup_n \max_{i \in [n]} \|\check{R}_i^{(n),t}\| < \infty.$$

Moreover,

$$\forall t > 0, \quad \forall m \in \mathbb{N}_*, \quad \sup_n \max_{i \in [n]} \mathbb{E}|x_i^{(n),t}|^m < \infty. \quad (14)$$

Given $t \in \mathbb{N}_*$, let $\psi^{(n)} : \mathbb{R}^t \times [n] \rightarrow \mathbb{R}$ be such that $\psi^{(n)}(\cdot, l)$ is a multivariate polynomial with a bounded degree and bounded coefficients as functions of (l, n) . Let $\mathcal{S}^{(n)} \subset [n]$ be such that $|\mathcal{S}^{(n)}| \leq CK^{(n)}$. Then,

$$\frac{1}{|\mathcal{S}^{(n)}|} \sum_{i \in \mathcal{S}^{(n)}} \psi^{(n)}(x_i^{(n),1}, \dots, x_i^{(n),t}, i) - \mathbb{E} \psi^{(n)}(\check{Z}_i^{(n),1}, \dots, \check{Z}_i^{(n),t}, i) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0, \quad \text{and} \quad (15a)$$

$$\frac{1}{n} \sum_{i \in [n]} \psi^{(n)}(x_i^{(n),1}, \dots, x_i^{(n),t}, i) - \mathbb{E} \psi^{(n)}(\check{Z}_i^{(n),1}, \dots, \check{Z}_i^{(n),t}, i) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (15b)$$

The proof of this proposition will revolve around a version of our AMP sequence where the \mathbb{R}^n -valued vectors $x^{(n),t}$ will be replaced with $\mathbb{R}^{n \times q}$ -valued matrices, with $q > 0$ being a given integer. In all the remainder, a vector $\mathbf{x} \in \mathbb{R}^q$ will be written as $\mathbf{x} = [x(1), \dots, x(q)]^\top$. Consider the function

$$f^{(n)} : \mathbb{R}^q \times [n] \times \mathbb{N} \longrightarrow \begin{matrix} \mathbb{R}^q \\ \left[\begin{array}{c} f_1^{(n)}(\mathbf{x}, l, t) \\ \vdots \\ f_q^{(n)}(\mathbf{x}, l, t) \end{array} \right] \end{matrix},$$

which is a polynomial in \mathbf{x} , with degree bounded by d , written as

$$f_r^{(n)}(\mathbf{x}, l, t) = \sum_{i_1 + \dots + i_q = 0}^d \alpha_{i_1, \dots, i_q}^{(n)}(r, l, t) \prod_{s=1}^q x(s)^{i_s}.$$

Starting with the deterministic n -uple $(\mathbf{x}_1^{(n),0}, \dots, \mathbf{x}_n^{(n),0})$ with $\mathbf{x}_i^{(n),0} \in \mathbb{R}^q$, the AMP recursion in t will provide us at Iteration $t+1$ with the n -uple $(\mathbf{x}_1^{(n),t+1}, \dots, \mathbf{x}_n^{(n),t+1})$, where $\mathbf{x}_i^{(n),t+1} = [x_i^{(n),t+1}(1), \dots, x_i^{(n),t+1}(q)]^\top$ is given as

$$\begin{aligned} x_i^{(n),t+1}(r) &= \sum_{l \in [n]} W_{i,l}^{(n)} f_r^{(n)}(\mathbf{x}_l^{(n),t}, l, t) \\ &\quad - \sum_{s \in [q]} f_s^{(n)}(\mathbf{x}_i^{(n),t-1}, i, t-1) \sum_{l \in [n]} (W_{i,l}^{(n)})^2 \frac{\partial f_r^{(n)}}{\partial x(s)}(\mathbf{x}_l^{(n),t}, l, t), \quad r \in [q], \end{aligned} \quad (16)$$

with $f(\cdot, \cdot, -1) \equiv 0$.

We also introduce the following sequence of Gaussian \mathbb{R}^{nq} -valued random vectors $(U^{(n),t})_{t \in \mathbb{N}_*}$. Writing

$$U^{(n),t} = \begin{bmatrix} U_1^{(n),t} \\ \vdots \\ U_n^{(n),t} \end{bmatrix},$$

the $\{U_i^{(n),t}\}_{i \in [n]}$ are \mathbb{R}^q -valued independent Gaussian random vectors, $U_i^{(n),t} \sim \mathcal{N}(0, Q_i^{(n),t})$, and the covariance matrices $Q_i^{(n),t}$ are recursively defined in t as follows. Starting with $t=1$, we set

$$Q_i^{(n),1} = \sum_{l \in [n]} s_{il}^{(n)} f^{(n)}(\mathbf{x}_l^{(n),0}, l, 0) f^{(n)}(\mathbf{x}_l^{(n),0}, l, 0)^\top \quad \text{for } i \in [n].$$

Given $\{Q_i^{(n),t}\}_{i \in [n]}$, we set

$$Q_i^{(n),t+1} = \sum_{l \in [n]} s_{il}^{(n)} \mathbb{E} f^{(n)}(U_l^{(n),t}, l, t) f^{(n)}(U_l^{(n),t}, l, t)^\top \quad \text{for } i \in [n].$$

The correlations between the elements of $U^{(n),t}$ and $U^{(n),s}$ for $t \neq s$ are irrelevant to our purpose.

Given $\mathbf{x} \in \mathbb{R}^q$ and a multi-index $\mathbf{m} = [m(1), \dots, m(q)] \in \mathbb{N}^q$, we write in the sequel

$$\mathbf{x}^{\mathbf{m}} = \prod_{r \in [q]} x(r)^{m(r)}.$$

Proposition 7. Let Assumptions 1 and 2 hold true, and consider the iterative algorithm(16). Assume that for each $t > 0$, there is a constant $C > 0$ such that

$$\left| \alpha_{i_1, \dots, i_q}^{(n)}(r, l, t) \right| \leq C, \quad (17)$$

and furthermore,

$$\sup_n \max_{i \in [n]} \left\| \mathbf{x}_i^{(n),0} \right\| < \infty. \quad (18)$$

Then,

$$\forall t > 0, \quad \sup_n \max_{i \in [n]} \|Q_i^{(n),t}\| < \infty. \quad (19)$$

Moreover,

$$\forall t > 0, \quad \forall \mathbf{m} \in \mathbb{N}^q, \quad \sup_n \max_{i \in [n]} \mathbb{E} |(\mathbf{x}_i^t)^{\mathbf{m}}| < \infty. \quad (20)$$

Let $\psi^{(n)} : \mathbb{R}^q \times [n] \rightarrow \mathbb{R}$ be such that $\psi^{(n)}(\cdot, l)$ is a multivariate polynomial with a bounded degree and bounded coefficients as functions of (l, n) . Let $\mathcal{S}^{(n)} \subset [n]$ be a non empty set such that $|\mathcal{S}^{(n)}| \leq CK^{(n)}$. Then,

$$\frac{1}{K^{(n)}} \sum_{i \in \mathcal{S}^{(n)}} \psi(\mathbf{x}_i^{(n),t}, i) - \mathbb{E} \psi(U_i^{(n),t}, i) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0, \quad \text{and} \quad (21a)$$

$$\frac{1}{n} \sum_{i \in [n]} \psi(\mathbf{x}_i^{(n),t}, i) - \mathbb{E} \psi(U_i^{(n),t}, i) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (21b)$$

The proof of Proposition 7 is an adaptation of the approach of [BLM15] to the structure of the variance profile at interest in this paper. For self-containedness, we reproduce large parts of the proof of [BLM15], putting the focus on the parts of this proof where these adaptations are necessary.

In the remainder, the superscript (n) will be often omitted for notational simplicity.

4.2 Proof of Proposition 7

Let us quickly prove that $\sup_n \max_{i \in [n]} \|Q_i^t\| < \infty$ for each integer $t > 0$. For $t = 1$, this is a consequence of Assumption 2 and the bounds (17) and (18). Assume the result is true for t . Then, using the bound (17) again, standard Gaussian derivations show that $\sup_n \max_i \|\mathbb{E} f(U_i^t, i, t) f(U_i^t, i, t)^\top\| < \infty$. Then, using Assumption 2, we obtain from the expression of Q_i^{t+1} above that $\sup_n \max_i \|Q_i^{t+1}\| < \infty$.

Of importance in the proof of Proposition 7 are the sequences issued from the so-called ‘‘non-backtracking’’ iterations. Given any $i, j \in [n]$ with $i \neq j$, define the set of \mathbb{R}^q -valued vectors $\{\mathbf{z}_{i \rightarrow j}^0, i, j \in [n], i \neq j\}$ as $\mathbf{z}_{i \rightarrow j}^0 = \mathbf{x}_i^0$. Assuming that the \mathbb{R}^q -valued vectors $\{\mathbf{z}_{i \rightarrow j}^t, i, j \in [n], i \neq j\}$ are defined, the vectors $\mathbf{z}_{i \rightarrow j}^{t+1}$ for $i \neq j$ are given as

$$\mathbf{z}_{i \rightarrow j}^{t+1}(r) = \sum_{l \in [n] \setminus \{j\}} W_{il} f_r(\mathbf{z}_{l \rightarrow i}^t, l, t) \quad (22)$$

(here, we implicitly consider that in the summation over l , the case $l = i$ is excluded because $W_{ii} = 0$). Having the vectors $\{\mathbf{z}_{i \rightarrow j}^t, i, j \in [n], i \neq j\}$ at hand, we define the \mathbb{R}^q -valued vectors $\{\mathbf{z}_i^{t+1}, i \in [n]\}$ as

$$\mathbf{z}_i^{t+1}(r) = \sum_{l \in [n]} W_{il} f_r(\mathbf{z}_{l \rightarrow i}^t, l, t). \quad (23)$$

For each n , let us now consider an i.i.d. sequence $(W^{(n),t})_{t=0,1,\dots}$ of symmetric $n \times n$ matrices such that $W^{(n),t} \stackrel{\mathcal{L}}{=} W^{(n)}$. We define the vectors $\mathbf{y}_{i \rightarrow j}^t$ and \mathbf{y}_i^t recursively in t similarly to what we did for the vectors $\mathbf{z}_{i \rightarrow j}^t$ and \mathbf{z}_i^t , with the difference that we now replace the matrix W with the matrix W^t at step t . More precisely, we set $\mathbf{y}_{i \rightarrow j}^0 = \mathbf{x}_i^0$ for each $i, j \in [n]$ with $i \neq j$; Given $\{\mathbf{y}_{i \rightarrow j}^t, i, j \in [n], i \neq j\}$, we set

$$\mathbf{y}_{i \rightarrow j}^{t+1}(r) = \sum_{l \in [n] \setminus \{j\}} W_{il}^t f_r(\mathbf{y}_{l \rightarrow i}^t, l, t), \quad i \neq j.$$

Also,

$$\mathbf{y}_i^{t+1}(r) = \sum_{l \in [n]} W_{il}^t f_r(\mathbf{y}_{l \rightarrow i}^t, l, t).$$

The tree structure. The pivotal object in the proof is a rooted and labelled tree T with the following structure (we paraphrase here [BLM15, §4.2] with the same notations). Write $T = (V(T), E(T))$ where $V(T)$ and $E(T)$ are the sets of vertices and edges of T respectively. We also denote as $L(T)$ the set of the leaves of T . Denote as $\circ \in V(T)$ the root of the tree, and let $|u|$ be

the distance of a vertex u to \circ . We consider that the edges are oriented towards the root. Thus, considering the edge $(u \rightarrow v) \in E(T)$, we have that $v = \pi(u)$, where $\pi(u)$ the parent of the node u . We assume that the root has only one child (thus, T is a so called planted tree). Each vertex other than the root and the leaves can have up to d children, thus, its degree is bounded by $d + 1$. By definition, the degree of a leaf is one, its only neighbor being its parent.

The tree T is a labelled tree. We now describe this labelling. The label of the root is an integer $\ell(\circ) \in [n]$. The label of a vertex v which is neither the root nor a leaf is a couple $(\ell(v), r(v)) \in [n] \times [q]$. The label of a leaf v is $(\ell(v), r(v), v[1], \dots, v[q]) \in [n] \times [q] \times \{(a_1, \dots, a_q) \in \mathbb{N}^q, a_1 + \dots + a_q \leq d\}$. If $v \in L(T)$ and $|v| \leq t - 1$, then we set $v[1] = \dots = v[q] = 0$.

The integer $\ell(v)$ is denoted as the ‘‘type’’ of the vertex v , and $r(v)$ is the ‘‘mark’’ of this vertex when it exists.

For a vertex $u \in V(T) \setminus L(T)$, we denote as $u[r]$ the number of children of u with the mark r . The children of $u \in V(T) \setminus L(T)$ are ordered with respect to their mark. Specifically, the labels of the children of u are $(\ell^1, 1), \dots, (\ell^{u[1]}, 1), (\ell^{u[1]+1}, 2), \dots, (\ell^{u[1]+\dots+u[q]}, q)$

Specific classes of these trees will be of importance. Following the definitions and notations of [BLM15], we introduce the following families of trees:

- $\overline{\mathcal{T}}^t$ is the set of labelled trees as above, with depth t at most.
- $\mathcal{T}^t \subset \overline{\mathcal{T}}^t$ is the subset that additionally satisfies the following so-called non-backtracking condition: if $v_1 = \circ, v_2, \dots, v_k$ is a path starting from \circ , *i.e.*, $v_i = \pi(v_{i+1})$, then the corresponding sequence of types is non-backtracking. This means that for each $i \in [k - 2]$, the three types $\ell(v_i), \ell(v_{i+1})$ and $\ell(v_{i+2})$ are distinct.
- $\mathcal{T}_{i \rightarrow j}^t(r) \subset \mathcal{T}^t$ is the subset of trees in \mathcal{T}^t for which the type of the root is i , the type of the child v of the root satisfies $\ell(v) \notin \{i, j\}$, and the mark of v is $r(v) = r$.
- $\mathcal{T}_i^t(r) \subset \mathcal{T}^t$ is the subset of trees in \mathcal{T}^t for which the type of the root is i , the type of the child v of the root satisfies $\ell(v) \neq i$, and the mark of v is $r(v) = r$.

Given a tree T , we also set

$$\begin{aligned} W(T) &= \prod_{(u \rightarrow v) \in E(T)} W_{\ell(u)\ell(v)}, \\ \Gamma(T, \boldsymbol{\alpha}, t) &= \prod_{(u \rightarrow v) \in E(T)} \alpha_{u[1], \dots, u[q]}(r(u), \ell(u), t - |u|), \\ x(T) &= \prod_{v \in L(T)} \prod_{s \in [q]} \left(x_{\ell(v)}^0(s) \right)^{v[s]}. \end{aligned}$$

Lemma 8 (Lemma 1 of [BLM15]).

$$\begin{aligned} z_{i \rightarrow j}^t(r) &= \sum_{T \in \mathcal{T}_{i \rightarrow j}^t(r)} W(T) \Gamma(T, \boldsymbol{\alpha}, t) x(T), \\ z_i^t(r) &= \sum_{T \in \mathcal{T}_i^t(r)} W(T) \Gamma(T, \boldsymbol{\alpha}, t) x(T). \end{aligned}$$

This lemma is a structural lemma which proof is not impacted by our specific variance profile. The following notions will be needed below. Define the set

$$\mathcal{K} = \{\{i, j\} \subset [n], s_{ij} > 0\}.$$

For each $i \in [n]$, we define the section

$$\mathcal{K}_i = \{j \in [n], s_{ij} > 0\}.$$

The next proposition shows that in the large dimensional regime, the joint moments of the elements of a vector \mathbf{z}_i^t issued from the non-backtracking iterations depend for large n only on the first two moments of the elements of W .

Proposition 9 (adaptation of Proposition 1 of [BLM15]). Let $\{\tilde{X}_{ij}\}_{1 \leq i < j \leq n}$ be a family of independent random variables satisfying Assumption 1, with distributions not necessarily identical to their analogues X_{ij} . Let \tilde{W} be the symmetric matrix constructed similarly to W , but with the X_{ij} replaced with the \tilde{X}_{ij} . Starting with the set of \mathbb{R}^q -valued vectors $\{\tilde{z}_{i \rightarrow j}^0, i, j \in [n], i \neq j\}$ given as $\tilde{z}_{i \rightarrow j}^0 = \mathbf{x}_i^0$, define the vectors $\tilde{z}_i^t \in \mathbb{R}^q$ by the recursion (22) and the equation (23), where W is replaced with \tilde{W} . Then, for each $t \geq 1$ and each $\mathbf{m} \in \mathbb{N}^q$, there exists C such that for each $i \in [n]$,

$$|\mathbb{E}(z_i^t)^{\mathbf{m}} - \mathbb{E}(\tilde{z}_i^t)^{\mathbf{m}}| \leq CK^{-1/2}.$$

Proof. For simplicity, we restrict the proof to the case where the multi-index \mathbf{m} satisfies $m(r) = m$ for some integer $m > 0$ and $m(s) = 0$ for $s \in [q] \setminus \{r\}$. By Lemma 8, we have

$$\mathbb{E}(z_i^t(r))^{\mathbf{m}} = \sum_{T_1, \dots, T_m \in \mathcal{T}_i^t(r)} \left(\prod_{k=1}^m \Gamma(T_k, \boldsymbol{\alpha}, t) \right) \mathbb{E} \left[\prod_{k=1}^m W(T_k) \right] \prod_{k=1}^m x(T_k).$$

For a tree T and $j, l \in [n]$, define

$$\phi(T)_{jl} = |\{(u \rightarrow v) \in E(T), \{\ell(u), \ell(v)\} = \{j, l\}\}|.$$

There is an integer constant $C_E = C_E(d, t, m)$ that bounds the total number of edges in the trees $T_1, \dots, T_m \in \mathcal{T}_i^t(r)$, leading to

$$\sum_{k \in [m]} \sum_{j < l} \phi(T_k)_{jl} \leq C_E.$$

Given an integer $\mathbf{m} \in [C_E]$, let

$$\begin{aligned} \mathcal{A}_i(\mathbf{m}) = & \left\{ (T_1, \dots, T_m) : T_k \in \mathcal{T}_i^t(r) \text{ for all } k \in [m], \right. \\ & \forall j < l, \sum_{k \in [m]} \phi(T_k)_{jl} \neq 1, \\ & \forall j < l, \sum_{k \in [m]} \phi(T_k)_{jl} > 0 \Rightarrow \{j, l\} \in \mathcal{K}, \\ & \left. \sum_{k \in [m]} \sum_{j < l} \phi(T_k)_{jl} = \mathbf{m} \right\}. \end{aligned}$$

Then, since the elements of W beneath the diagonal are centered and independent,

$$\mathbb{E} z_i^t(r)^{\mathbf{m}} = \sum_{\mathbf{m}=1}^{C_E} \sum_{(T_1, \dots, T_m) \in \mathcal{A}_i(\mathbf{m})} \left(\prod_{k=1}^m \Gamma(T_k, \boldsymbol{\alpha}, t) \right) \left(\prod_{k=1}^m x(T_k) \right) \mathbb{E} \left[\prod_{k=1}^m W(T_k) \right]. \quad (24)$$

Notice that the contributions of the m -uples of trees in $\mathcal{A}_i(\mathbf{m})$ for which

$$\forall j < l, \sum_{k \in [m]} \phi(T_k)_{jl} \in \{0, 2\}$$

are the same for $\mathbb{E} z_i^t(r)^{\mathbf{m}}$ and $\mathbb{E} \tilde{z}_i^t(r)^{\mathbf{m}}$ by the assumptions on the matrices W and \tilde{W} . Thus, defining the set

$$\tilde{\mathcal{A}}_i(\mathbf{m}) = \left\{ (T_1, \dots, T_m) \in \mathcal{A}_i(\mathbf{m}) : \exists j < l, \sum_{k \in [m]} \phi(T_k)_{jl} \geq 3 \right\},$$

the proposition will be proven if we prove that for all $\mathbf{m} \in [C_E]$, the real number

$$\xi_{\mathbf{m}} = \sum_{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m})} \left(\prod_{k=1}^m \Gamma(T_k, \boldsymbol{\alpha}, t) \right) \left(\prod_{k=1}^m x(T_k) \right) \mathbb{E} \left[\prod_{k=1}^m W(T_k) \right]$$

satisfies $|\xi_m| \leq CK^{-1/2}$.

Using the bounds (17) and (18) provided in the statement of Proposition 7, it is obvious that $\prod_{k=1}^m \Gamma(T_k, \alpha, t)$ and $\prod_{k=1}^m x(T_k)$ are bounded.

Since $|\mathbb{E}W_{jl}^s| \leq CK^{-s/2}$ for each integer $s > 0$ by Assumptions 1 and 2, for each $(T_1, \dots, T_m) \in \check{\mathcal{A}}_i(\mathbf{m})$, we have

$$\left| \mathbb{E} \prod_{k=1}^m W(T_k) \right| = \prod_{j < l} \left| \mathbb{E} W_{jl}^{\sum_k \phi(T_k)_{jl}} \right| \leq CK^{-m/2}.$$

To complete the proof, we shall show that

$$|\check{\mathcal{A}}_i(\mathbf{m})| \leq CK^{(m-1)/2}. \quad (25)$$

From a m -uple $(T_1, \dots, T_m) \in \check{\mathcal{A}}_i(\mathbf{m})$, let us construct a graph $G = \mathbf{G}(T_1, \dots, T_m)$ as follows. The graph G is the rooted, undirected, labelled, and unmarked graph obtained by merging the m trees T_1, \dots, T_m and by identifying the nodes that have the same type; This common type will be the label of the obtained node in G . The root node of G will be the node obtained by merging the roots of the trees T_1, \dots, T_m (remember that they all have the same type i). The other nodes are numbered, say, in the increasing order of their labels. The edges of G are furthermore unweighted.

The number of edges of G is

$$|E(G)| = \sum_{j < l} \mathbb{1}_{\sum_k \phi(T_k)_{jl} > 0}.$$

Remember that when $\sum_k \phi(T_k)_{jl} > 0$, this sum is ≥ 2 , and for some $j < l$, it is ≥ 3 . Consequently, $2|E(G)| + 1 \leq \sum_{j < l} \sum_k \phi(T_k)_{jl}$, in other words,

$$|E(G)| \leq \frac{m-1}{2}.$$

Note that since G is connected as being obtained through the merger of trees, $|V(G)| \leq |E(G)| + 1$. Thus, $|\{v \in G, v \neq \circ\}| \leq (m-1)/2$. Also, by construction, G satisfies the following property:

$$\{u, v\} \in E(G) \Rightarrow \ell(u) \in \mathcal{K}_{\ell(v)}.$$

We shall denote as \mathcal{G}_i^m the set of rooted, undirected, labelled and connected graphs such that $\ell(\circ) = i$, $|E(G)| \leq \frac{m-1}{2}$, and the last property is satisfied. We denote as \mathcal{R}^m the set of all the elements of \mathcal{G}_i^m but without the labels. Given a graph $G \in \mathcal{G}_i^m$, let us denote as $\bar{G} = \mathbf{U}(G) \in \mathcal{R}^m$ the unlabelled version of G . With these notations, we have

$$|\check{\mathcal{A}}_i(\mathbf{m})| = \sum_{\bar{G} \in \mathcal{R}^m} \sum_{\substack{G \in \mathcal{G}_i^m \\ \mathbf{U}(G) = \bar{G}}} \left| \left\{ (T_1, \dots, T_m) \in \check{\mathcal{A}}_i(\mathbf{m}) : \mathbf{G}(T_1, \dots, T_m) = G \right\} \right|. \quad (26)$$

For each graph G , we have

$$\left| \left\{ (T_1, \dots, T_m) \in \check{\mathcal{A}}_i(\mathbf{m}) : \mathbf{G}(T_1, \dots, T_m) = G \right\} \right| \leq C, \quad (27)$$

where $C = C(d, t, m)$ is independent of G . Our purpose is now to show that

$$|\{G \in \mathcal{G}_i^m : \mathbf{U}(G) = \bar{G}\}| \leq CK^{(m-1)/2}. \quad (28)$$

Given $\bar{G} \in \mathcal{R}^m$, denote as \circ the root node of \bar{G} , write $M = |V(\bar{G})| - 1 \leq (m-1)/2$, and denote as $\{v_1, \dots, v_M\} = V(\bar{G}) \setminus \{\circ\}$ the set of vertices of \bar{G} without the root. Recalling that \bar{G} is connected, let us consider a spanning tree of this graph rooted in \circ . Denote as $\pi(v)$ the parent of the node v in this tree. Writing $j_\circ = i$, we obtain that

$$|\{G \in \mathcal{G}_i^m : \mathbf{U}(G) = \bar{G}\}| \leq \left| \left\{ (j_1, \dots, j_M) \in [n]^M : \forall k \in [M], j_k \in \mathcal{K}_{j_{\pi(v_k)}} \right\} \right|.$$

Denoting as $L \subset \{v_1, \dots, v_M\}$ the set of the leaves of the spanning tree, we can write

$$\begin{aligned} \left| \left\{ (j_1, \dots, j_M) \in [n]^M : \forall k \in [M], j_k \in \mathcal{K}_{j_{\pi(v_k)}} \right\} \right| &= \sum_{\substack{j_1, \dots, j_M \in [n] : \\ \forall k \in [M], j_k \in \mathcal{K}_{j_{\pi(v_k)}}}} 1 \\ &= \sum_{k: v_k \in \{v_1, \dots, v_M\} \setminus L} \sum_{j_k \in \mathcal{K}_{j_{\pi(v_k)}}} \left(\sum_{p: v_p \in L} \sum_{j_p \in \mathcal{K}_{j_{\pi(v_p)}}} 1 \right) \\ &\leq CK^{|L|} \sum_{k: v_k \in \{v_1, \dots, v_M\} \setminus L} \sum_{j_k \in \mathcal{K}_{j_{\pi(v_k)}}} 1, \end{aligned}$$

due to the fact that $|\mathcal{K}_j| \leq CK$ for all j by Assumption 2. If we prune the leaves of the original spanning tree, what remains is a tree made of the nodes that constitute the first sum above plus the root node. We can apply the pruning operation to the new tree as above, and iterate until exhausting all the set $\{v_1, \dots, v_M\}$. This leads to

$$\left| \left\{ (j_1, \dots, j_M) \in [n]^M : \forall k \in [M], j_k \in \mathcal{K}_{j_{\pi(v_k)}} \right\} \right| \leq CK^M \leq CK^{(m-1)/2}.$$

Inequality (28) follows.

It is furthermore easy to check that

$$|\mathcal{R}^m| \leq C.$$

Getting back to Inequality (26), and using this last inequality along with Inequalities (28) and (27), we obtain Inequality (25), and the proposition is established. \square

Let us keep the notations of the former proof. Consider the subset $\tilde{\mathcal{A}}_i(\mathbf{m})$ of $\mathcal{A}_i(\mathbf{m})$ defined as

$$\tilde{\mathcal{A}}_i(\mathbf{m}) = \left\{ (T_1, \dots, T_m) \in \mathcal{A}_i(\mathbf{m}) : \forall j < l, \sum_{k \in [m]} \phi(T_k)_{jl} \in \{0, 2\} \right\}.$$

For $(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m})$, let us denote as $G = \mathbf{G}(T_1, \dots, T_m) \in \mathcal{G}_i^m$ the graph obtained by merging these trees, and $\bar{G} = \mathbf{U}(G) \in \mathcal{R}^m$ the unlabelled version of G , as we did for $(T_1, \dots, T_m) \in \mathcal{A}_i(\mathbf{m})$. As we did in (24), $\mathbb{E}z_i^t(r)^m$ can be written as

$$\mathbb{E}z_i^t(r)^m = \sum_{\mathbf{m}=1}^{C_E} \chi_{\mathbf{m}} + \sum_{\mathbf{m}=1}^{C_E} \xi_{\mathbf{m}}, \quad (29)$$

where

$$\begin{aligned} \chi_{\mathbf{m}} &= \sum_{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m})} \left(\prod_{k=1}^m \Gamma(T_k, \boldsymbol{\alpha}, t) \right) \left(\prod_{k=1}^m x(T_k) \right) \mathbb{E} \left[\prod_{k=1}^m W(T_k) \right] \\ &= \sum_{\bar{G} \in \mathcal{R}^m} \sum_{\substack{G \in \mathcal{G}_i^m : \\ \mathbf{U}(G) = \bar{G}}} \sum_{\substack{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m}) : \\ \mathbf{G}(T_1, \dots, T_m) = G}} \left(\prod_{k=1}^m \Gamma(T_k, \boldsymbol{\alpha}, t) \right) \left(\prod_{k=1}^m x(T_k) \right) \mathbb{E} \left[\prod_{k=1}^m W(T_k) \right], \end{aligned}$$

and where we recall from the former proof that $\xi_{\mathbf{m}} = \sum_{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m})} \dots$ satisfies $|\xi_{\mathbf{m}}| \leq CK^{-1/2}$.

Let us further decompose $\chi_{\mathbf{m}}$ as

$$\begin{aligned} \chi_{\mathbf{m}} &= \sum_{\substack{\bar{G} \in \mathcal{R}^m : \\ \bar{G} \text{ is a tree}}} \sum_{\substack{G \in \mathcal{G}_i^m : \\ \mathbf{U}(G) = \bar{G}}} \sum_{\substack{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m}) : \\ \mathbf{G}(T_1, \dots, T_m) = G}} \dots + \sum_{\substack{\bar{G} \in \mathcal{R}^m : \\ \bar{G} \text{ not a tree}}} \sum_{\substack{G \in \mathcal{G}_i^m : \\ \mathbf{U}(G) = \bar{G}}} \sum_{\substack{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m}) : \\ \mathbf{G}(T_1, \dots, T_m) = G}} \dots \\ &= \chi_{\mathbf{m}}^T + \chi_{\mathbf{m}}^{\text{NT}}. \end{aligned}$$

Then, the contribution of the term $\chi_{\mathbf{m}}^{\text{NT}}$ is negligible:

Lemma 10 (see Lemma 2 of [BLM15]). $|\chi_m^T| \leq C$, and $|\chi_m^{\text{NT}}| \leq C/K$.

Proof. By repeating the argument of the former proof, the terms $\prod_{k=1}^m \Gamma(T_k, \alpha, t)$ and $\prod_{k=1}^m x(T_k)$ are both bounded, and the term $\mathbb{E} \prod_{k=1}^m W(T_k)$ accounts for a factor of order $K^{-m/2}$ in both χ_m^T and χ_m^{NT} .

Furthermore, when $(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m})$, the graph $G = \mathbf{G}(T_1, \dots, T_m)$ satisfies $|E(G)| = m/2$. We further know that $\bar{G} = \mathbf{U}(G)$ satisfies $|V(\bar{G})| \leq |E(\bar{G})| + 1 = |E(G)| + 1$, with equality if and only if G is a tree. Therefore, when $\bar{G} \in \mathcal{R}^m$ is a tree, $|V(\bar{G})| = m/2 + 1$. Once we set $M = |V(\bar{G})| - 1 = m/2$, the argument for establishing Inequality (28) in the former proof can be reproduced word for word here to show that $|\{G \in \mathcal{G}_i^m : \mathbf{U}(G) = \bar{G}\}| \leq CK^{m/2}$. When $\bar{G} \in \mathcal{R}^m$ is not a tree, $|V(\bar{G})| \leq m/2$, and we obtain that $|\{G \in \mathcal{G}_i^m : \mathbf{U}(G) = \bar{G}\}| \leq CK^{m/2-1}$. Thus,

$$\begin{aligned} |\chi_m^T| &\leq CK^{-m/2} K^{m/2} = C, \\ |\chi_m^{\text{NT}}| &\leq CK^{-m/2} K^{m/2-1} = CK^{-1}, \end{aligned}$$

and the lemma is proven. \square

Getting back to the expression (29) of $\mathbb{E} z_i^t(r)^m$ and using this lemma along with the bound $|\xi_m| \leq CK^{-1/2}$, we obtain that for each $t > 0$ and each multi-index $\mathbf{m} \in \mathbb{N}^q$, there exists a constant $C > 0$ such that

$$\max_{i \in [n]} \mathbb{E} |z_i^t|^m \leq C. \quad (30)$$

This bound will be needed below.

Recall that the samples \mathbf{y}_i^t are obtained by drawing an independent matrix W^t at each iteration in the parameter t . We have:

Proposition 11 (Proposition 2 of [BLM15]). For each $t \geq 1$ and each $\mathbf{m} \in \mathbb{N}^q$, there exists C such that for each $i \in [n]$,

$$|\mathbb{E}(z_i^t)^{\mathbf{m}} - \mathbb{E}(\mathbf{y}_i^t)^{\mathbf{m}}| \leq CK^{-1/2}.$$

Proof. Paralleling the quantities $W(T)$ introduced above for a tree $T \in \bar{\mathcal{T}}^t$, [BLM15] introduced the quantities

$$\bar{W}(T, t) = \prod_{(u \rightarrow v) \in E(T)} W_{\ell(u), \ell(v)}^{t-|u|}.$$

By an easy adaptation of the proof of [BLM15, Lm. 1] (Lemma 8 above), we can show that

$$\begin{aligned} y_{i \rightarrow j}^t(r) &= \sum_{T \in \mathcal{T}_{i \rightarrow j}^t(r)} \bar{W}(T) \Gamma(T, \alpha, t) x(T), \\ y_i^t(r) &= \sum_{T \in \mathcal{T}_i^t(r)} \bar{W}(T) \Gamma(T, \alpha, t) x(T). \end{aligned}$$

Similarly to the proof of Proposition 9, we assume that $m(r) = m$ and $m(s) = 0$ for $s \in [q] \setminus \{r\}$. Similarly to Equation (24), and with the same notations, it holds that

$$\mathbb{E} y_i^t(r)^m = \sum_{\mathbf{m}=1}^{C_E} \sum_{(T_1, \dots, T_m) \in \mathcal{A}_i(\mathbf{m})} \left(\prod_{k=1}^m \Gamma(T_k, \alpha, t) \right) \left(\prod_{k=1}^m x(T_k) \right) \mathbb{E} \left[\prod_{k=1}^m \bar{W}(T_k) \right].$$

As we did for $\mathbb{E} z_i^t(r)^m$ above, we partition $\mathcal{A}_i(\mathbf{m})$ as $\mathcal{A}_i(\mathbf{m}) = \tilde{\mathcal{A}}_i(\mathbf{m}) \cup \check{\mathcal{A}}_i(\mathbf{m})$. We prove with the same arguments that the contribution of $\check{\mathcal{A}}_i(\mathbf{m})$ is of order $K^{-1/2}$. Furthermore, paralleling Lemma 10, we also obtain that within $\tilde{\mathcal{A}}_i(\mathbf{m})$ we can limit ourselves to the terms $\bar{\chi}_m^T$ defined as

$$\bar{\chi}_m^T = \sum_{\substack{\bar{G} \in \mathcal{R}^m : \\ \bar{G} \text{ is a tree}}} \sum_{\substack{G \in \mathcal{G}_i^m : \\ \mathbf{U}(G) = \bar{G}}} \sum_{\substack{(T_1, \dots, T_m) \in \tilde{\mathcal{A}}_i(\mathbf{m}) : \\ \mathbf{G}(T_1, \dots, T_m) = G}} \left(\prod_{k=1}^m \Gamma(T_k, \alpha, t) \right) \left(\prod_{k=1}^m x(T_k) \right) \mathbb{E} \left[\prod_{k=1}^m \bar{W}(T_k) \right],$$

the terms for which \bar{G} is not a tree being of order K^{-1} . With this at hand, the proposition will be established once we show that $\bar{\chi}_m^T = \chi_m^T$, where we recall that χ_m^T , introduced in the last proof, has the same expression as $\bar{\chi}_m^T$ except that the terms $\bar{W}(T_k)$ in the latter are replaced with $W(T_k)$.

Consider an arbitrary m -uple (T_1, \dots, T_m) in the inner sum above. We first notice that if $\mathbb{E} \prod_{k=1}^m \bar{W}(T_k) \neq 0$, then $\mathbb{E} \prod_{k=1}^m \bar{W}(T_k) = \mathbb{E} \prod_{k=1}^m W(T_k)$. This is due to the fact that if $j < l$ is active in $\mathbb{E} \prod_{k=1}^m \bar{W}(T_k)$ (in the sense that $\sum_k \phi(T_k)_{jl} \neq 0$, and thus, is equal to 2), then the corresponding contribution of this $j < l$ to $\mathbb{E} \prod_{k=1}^m W(T_k)$ will be exactly the same.

The proof will then be terminated if we show that if $\mathbb{E} \prod_{k=1}^m W(T_k) \neq 0$ and $\mathbb{E} \prod_{k=1}^m \bar{W}(T_k) = 0$, then necessarily, the graph $G = \mathbf{U}(T_1, \dots, T_m)$ will not be a tree, *i.e.*, it will contain a cycle. Assume that $\mathbb{E} \prod_{k=1}^m W(T_k) \neq 0$ and $\mathbb{E} \prod_{k=1}^m \bar{W}(T_k) = 0$. Then, there will be an edge $\{u, v\}$ in G , with $\{\ell(u), \ell(v)\} = \{j, l\}$, but that will appear at two different distances to \circ in the trees T_k . Let us consider the three possible cases where this could happen:

1. This happens in the same tree, say T_1 , and on the same path to \circ . Then, due to the non-backtracking nature of T_1 , a cycle appears in G .
2. This happens in two different trees, say T_1 and T_2 . Namely, there exists two edges $u \rightarrow v \in E(T_1)$ and $u' \rightarrow v' \in E(T_2)$ such that $\{\ell(u), \ell(v)\} = \{\ell(u'), \ell(v')\}$, and $|u| \neq |u'|$. Then, keeping in mind the backtracking property, it is easy to observe that a cycle is created in G .
3. A similar remark can be made when this happens in the same tree but on two different paths to the root.

Thus, we have a contradiction in the three cases, and we get that $\bar{\chi}_m^T = \chi_m^T$. The proposition is proven. \square

The following proposition links the joint moments of the elements of the vectors z_i^t with those of the vectors x_i^t provided by the AMP algorithm (16).

Proposition 12 (proposition 3 of [BLM15]). For each $t \geq 1$ and each $m \in \mathbb{N}^q$, there exists C such that for each $i \in [n]$,

$$|\mathbb{E}(z_i^t)^m - \mathbb{E}(x_i^t)^m| \leq CK^{-1/2}.$$

Recalling the bound (30), the bound (20) can be deduced from this proposition.

To prove this proposition, new objects need to be introduced. In a directed and labelled graph,

- A backtracking path of length 3 is a path $a \rightarrow b \rightarrow c \rightarrow d$ such that $\ell(a) = \ell(c)$ and $\ell(b) = \ell(d)$.
- A backtracking star is a structure $a, b \rightarrow c \rightarrow d$ where $\ell(a) = \ell(b) = \ell(d)$.

Let $\bar{\mathcal{U}}^t$ be the set of equivalence classes of trees in $\bar{\mathcal{T}}^t$ from which the marks have been removed. Denote as \mathcal{B}^t the set of trees T in $\bar{\mathcal{U}}^t$ that satisfy the following additional conditions:

- If $u \rightarrow v \in E(T)$, then $\ell(u) \neq \ell(v)$.
- There exists in T at least one backtracking path of length 3 or one backtracking star.

Finally, \mathcal{B}_i^t is the subset of trees in \mathcal{B}^t with the root having the type i , and such that the type of the child v of the root satisfies $\ell(v) \neq i$.

The proof of Proposition 12 relies on the following structural lemma:

Lemma 13 (Lemma 3 of [BLM15]).

$$x_i^t(r) = z_i^t(r) + \sum_{T \in \mathcal{B}_i^t} W(T) \tilde{\Gamma}(T, t, r) x(T),$$

where $|\tilde{\Gamma}(T, t, r)| \leq C(d, t)$.

Proof of Proposition 12. Once again, we assume that $m(r) = m$ and $m(s) = 0$ for $s \in [q] \setminus \{r\}$. We write

$$\mathbb{E}(x_i^t(r))^m = \mathbb{E}\left(z_i^t(r) + \sum_{T \in \mathcal{B}_i^t} W(T) \tilde{\Gamma}(T, t, r) x(T)\right)^m$$

where $z_i^t(r)$ is given by Lemma 8. With this at hand, the proposition will be established once we bound the terms of the type

$$\sum_{T_1 \in \mathcal{B}_i^t} \sum_{T_2, \dots, T_m \in \mathcal{B}_i^t \cup \mathcal{T}_i^t(r)} \left| \mathbb{E} \prod_{k=1}^m W(T_k) \right|.$$

The argument is nearly the same as in the proof of Proposition 9. Defining the set

$$\begin{aligned} \mathcal{D}_i(\mathbf{m}) = & \left\{ (T_1, \dots, T_m) : T_1 \in \mathcal{B}_i^t, T_2, \dots, T_m \in \mathcal{B}_i^t \cup \mathcal{T}_i^t(r), \right. \\ & \forall j < l, \sum_{k \in [m]} \phi(T_k)_{jl} \neq 1, \\ & \forall j < l, \sum_{k \in [m]} \phi(T_k)_{jl} > 0 \Rightarrow \{j, l\} \in \mathcal{K}, \\ & \left. \sum_{k \in [m]} \sum_{j < l} \phi(T_k)_{jl} = \mathbf{m} \right\}. \end{aligned}$$

Recalling the notations of the proof of Proposition 9, we need to show that

$$\delta_{\mathbf{m}} = \sum_{\bar{G} \in \mathcal{R}^m} \sum_{\substack{G \in \mathcal{G}_i^m : \\ \mathcal{U}(G) = \bar{G}}} \sum_{\substack{(T_1, \dots, T_m) \in \mathcal{D}_i(\mathbf{m}) : \\ \mathbf{G}(T_1, \dots, T_m) = G}} \left| \mathbb{E} \prod_{k=1}^m W(T_k) \right|$$

satisfies $\delta_{\mathbf{m}} \leq CK^{-1/2}$.

As usual, $|\mathbb{E} \prod_{k=1}^m W(T_k)| \leq CK^{-m/2}$. We need to bound $|\mathcal{D}_i(\mathbf{m})|$. To this end, we observe that since $T_1 \in \mathcal{B}_i^t$, resulting in this tree having a backtracking path or a backtracking star, it is easy to see that the graph \bar{G} has an edge that results from the fusion of three edges at least. This implies that $|E(\bar{G})| \leq (\mathbf{m} - 1)/2$. Reusing the argument of the proof of Proposition 9, we obtain that $|\mathcal{D}_i(\mathbf{m})| \leq CK^{(\mathbf{m}-1)/2}$, which shows that $\delta_{\mathbf{m}} \leq CK^{-1/2}$ as required. Proposition 12 is proven. \square

Making use of the independence of the matrices W^t , we now show that the joint moments of the elements of a sample \mathbf{y}_i^t are close to their analogues for U_i^t , which distribution is provided before Proposition 7. It will be enough to consider that the matrices W^t are Gaussian.

Proposition 14. Assume that the matrix W is Gaussian. Then for each multi-index $\mathbf{m} \in \mathbb{N}^q$, each integer $t > 0$, and each sequence (i_n) valued in $[n]$, it holds that

$$\mathbb{E}(\mathbf{y}_{i_n}^t)^{\mathbf{m}} - \mathbb{E}(U_{i_n}^t)^{\mathbf{m}} \xrightarrow[n \rightarrow \infty]{} 0.$$

Proof. Remember that $\mathbf{y}_{i \rightarrow j}^0 = \mathbf{x}_i^0$ and $\mathbf{y}_{i \rightarrow j}^{t+1} = \sum_{l \in [n] \setminus \{j\}} W_{il}^t f(\mathbf{y}_{l \rightarrow i}^t, l, t)$ for each $i \neq j$. First, using this equation, it is easy to establish by recurrence on t that

$$\forall t \geq 0, \forall \mathbf{m} \in \mathbb{N}^q, \sup_n \max_{i \neq j} \mathbb{E} |(\mathbf{y}_{i \rightarrow j}^t)^{\mathbf{m}}| < \infty. \quad (31)$$

Fixing t , we shall show by recurrence on $u = 1, \dots, t-1$ the following assertion that we denote as $\mathcal{A}(u)$: For each multi-index \mathbf{m} , each sequence (j_n) valued in $[n]$, and each $(n-1)$ -uple $(b_\ell)_{\ell \in [n] \setminus \{j_n\}}$ with bounded elements, it holds that

$$\sum_{\ell \in [n] \setminus \{j\}} s_{i\ell} b_\ell (\mathbf{y}_{\ell \rightarrow j}^u)^{\mathbf{m}} - s_{i\ell} b_\ell \mathbb{E}(U_\ell^u)^{\mathbf{m}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0,$$

where $j = j_n$. In all the proof, we shall need the covariance matrices H_{ij}^u defined for $u \geq 1$ and $i \neq j$ as

$$H_{ij}^u = \sum_{l \in [n] \setminus \{j\}} s_{il} f(\mathbf{y}_{l \rightarrow i}^{u-1}, l, u-1) f(\mathbf{y}_{l \rightarrow i}^{u-1}, l, u-1)^\top.$$

Starting with $\mathcal{A}(1)$, let us assume for notational simplicity and without generality loss that $j = n$. We have

$$\begin{aligned} \sum_{l \in [n-1]} s_{il} b_l ((\mathbf{y}_{l \rightarrow n}^1)^m - \mathbb{E}(U_l^1)^m) &= \sum_{l \in [n-1]} s_{il} b_l ((\mathbf{y}_{l \rightarrow n}^1)^m - \mathbb{E}(\mathbf{y}_{l \rightarrow n}^1)^m) \\ &\quad + \sum_{l \in [n-1]} s_{il} b_l (\mathbb{E}(\mathbf{y}_{l \rightarrow n}^1)^m - \mathbb{E}(U_l^1)^m) \\ &= \chi_1 + \chi_2. \end{aligned}$$

It is obvious that $\mathbf{y}_{l \rightarrow n}^1 \sim \mathcal{N}(0, H_{ln}^1)$, where $H_{ln}^1 = Q_l^1 - E_{l,n}$, and the rank-one matrix

$$E_{l,n} = s_{l,n} f(\mathbf{x}_n^0, n, 0) f(\mathbf{x}_n^0, n, 0)^\top$$

has a spectral norm that converges to zero by Assumptions 2 and 3. It is then easy to deduce that $\chi_2 \rightarrow_n 0$.

To deal with χ_1 , we make use of Poincaré's inequality [PS11, Ch. 2]. For $u \geq 1$, let Σ_n^u be the $q(n-1) \times q(n-1)$ covariance matrix defined as $\Sigma_n^u = [\Sigma_n^u(k, l)]_{k, \ell=1}^{n-1}$ where the $q \times q$ block $\Sigma_n^u(k, l)$ is given as

$$\Sigma_n^u(k, l) = \begin{cases} H_{k,n}^u & \text{if } k = l \\ s_{kl} f(\mathbf{y}_{l \rightarrow k}^{u-1}, l, u-1) f(\mathbf{y}_{k \rightarrow l}^{u-1}, k, u-1)^\top & \text{if not.} \end{cases}$$

Note that Σ_n^1 is deterministic, and the $\mathbb{R}^{q(n-1)}$ -valued vector $\mathbf{y}_{l \rightarrow n}^1 = [\mathbf{y}_{i \rightarrow n}^1]_{i=1}^{n-1}$ has the distribution $\mathcal{N}(0, \Sigma_n^1)$. Define the function $\Gamma(\mathbf{y}_{l \rightarrow n}^1) = \sum_{l \in [n-1]} s_{il} b_l (\mathbf{y}_{l \rightarrow n}^1)^m$, and write $\nabla_{\mathbf{y}} \mathbf{y}^m = p_m(\mathbf{y})$, a \mathbb{R}^q -valued polynomial. Then, we obtain by Poincaré's inequality

$$\mathbb{E} \chi_1^2 = \text{Var}(\Gamma(\mathbf{y}_{l \rightarrow n}^1)) \leq \sum_{k, l=1}^{n-1} s_{ik} s_{il} b_k b_l \mathbb{E} p_m(\mathbf{y}_{k \rightarrow n}^1)^\top \Sigma_n^1(k, l) p_m(\mathbf{y}_{l \rightarrow n}^1).$$

Considering the expression of $\Sigma_n^1(k, l)$, and using Assumption 3 and the bound (31), the right hand side of the previous display satisfies the bounds

$$\left| \sum_{k=l} \dots \right| \leq C \sum_{k=1}^{n-1} s_{ik}^2 \leq \frac{C}{K}, \quad \text{and} \quad \left| \sum_{k \neq l} \dots \right| \leq C \sum_{k, \ell=1}^{n-1} s_{ik} s_{il} s_{kl} \leq \frac{C}{K}.$$

It results that $\chi_1 \xrightarrow{\mathcal{P}}_n 0$, and $\mathcal{A}(1)$ is established.

Assuming that $\mathcal{A}(u)$ is true, let us establish $\mathcal{A}(u+1)$. Define the σ -field $\mathcal{F}^u = \sigma(W^0, \dots, W^u)$. Then, still setting $j = n$, the conditional distribution $\mathcal{L}(\mathbf{y}_{l \rightarrow n}^{u+1} | \mathcal{F}^{u-1})$ of the vector $\mathbf{y}_{l \rightarrow n}^{u+1} = [\mathbf{y}_{i \rightarrow n}^{u+1}]_{i=1}^{n-1}$ given \mathcal{F}^{u-1} is $\mathcal{N}(0, \Sigma_n^{u+1})$. We also have from $\mathcal{A}(u)$ that

$$\forall (i_n) \text{ valued in } [n], H_{i_n n}^{u+1} - Q_{i_n}^{u+1} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (32)$$

With this at hand, we write

$$\begin{aligned} \sum_{l \in [n-1]} s_{il} b_l ((\mathbf{y}_{l \rightarrow n}^{u+1})^m - \mathbb{E}(U_l^{u+1})^m) &= \sum_{l \in [n-1]} s_{il} b_l ((\mathbf{y}_{l \rightarrow n}^{u+1})^m - \mathbb{E}[(\mathbf{y}_{l \rightarrow n}^{u+1})^m | \mathcal{F}^{u-1}]) \\ &\quad + \sum_{l \in [n-1]} s_{il} b_l (\mathbb{E}[(\mathbf{y}_{l \rightarrow n}^{u+1})^m | \mathcal{F}^{u-1}] - \mathbb{E}(U_l^{u+1})^m) \\ &= \chi_1 + \chi_2. \end{aligned}$$

Given a small $\delta > 0$, we write

$$\begin{aligned} |\chi_2| &\leq \sum_{l \in [n-1]} s_{il} |b_l| \sup_{H \geq 0: \|H - Q_l^{u+1}\| \leq \delta} |\varphi(H) - \varphi(Q_l^{u+1})| \\ &\quad + \sum_{l \in [n-1]} s_{il} |b_l| |\varphi(H_{ln}^{u+1})| \mathbb{1}_{\|H_{ln}^{u+1} - Q_l^{u+1}\| > \delta} + \sum_{l \in [n-1]} s_{il} |b_l| |\varphi(Q_l^{u+1})| \mathbb{1}_{\|H_{ln}^{u+1} - Q_l^{u+1}\| > \delta} \\ &= \chi_{2,1} + \chi_{2,2} + \chi_{2,3}, \end{aligned}$$

where $\varphi(H) = \mathbb{E}Y^m$ when $Y \sim \mathcal{N}(0, H)$. Observe that $\sum_\ell s_{il} |b_l|$ is bounded. Furthermore, assuming that H and Q belong to a compact, it holds that $\varphi(H) - \varphi(Q) \rightarrow 0$ when $\|H - Q\| \rightarrow 0$ by the continuity of φ . Thus, using the bound (19), we obtain that $\chi_{2,1} \rightarrow 0$ as $\delta \rightarrow 0$. We also have by the Jensen and the Cauchy-Schwarz inequalities that

$$\mathbb{E}\chi_{2,2} \leq \sum_{l \in [n-1]} s_{il} |b_l| (\mathbb{E}|\mathbf{y}_{l \rightarrow n}^{u+1}|^m)^{1/2} \mathbb{P}[\|H_{ln}^{u+1} - Q_l^{u+1}\| > \delta]^{1/2}$$

which converges to zero as $n \rightarrow \infty$ by (32), the bound (31), and the Dominated Convergence. Thus, $\chi_{2,2} \xrightarrow{\mathcal{P}} 0$. It is easy to show that $\chi_{2,3} \xrightarrow{\mathcal{P}} 0$. In conclusion, $\chi_2 \xrightarrow{\mathcal{P}} 0$.

To deal with χ_1 , we use Poincaré's inequality involving this time the conditional distribution $\mathcal{L}(\mathbf{y}_{\rightarrow n}^{u+1} | \mathcal{F}^{u-1})$. By an argument similar to above, this leads to

$$\mathbb{E}\chi_1^2 = \mathbb{E} \text{Var}(\Gamma(\mathbf{y}_{\rightarrow n}^{u+1}) | \mathcal{F}^{u-1}) \leq \sum_{k,l=1}^{n-1} s_{ik} s_{il} b_k b_l \mathbb{E} [p_m(\mathbf{y}_{k \rightarrow n}^{u+1})^\top \Sigma_n^{u+1}(k, l) p_m(\mathbf{y}_{l \rightarrow n}^{u+1})] \leq \frac{C}{K}$$

with the help of Inequality (31). It results that $\chi_1 \xrightarrow{\mathcal{P}} 0$, and $\mathcal{A}(u+1)$ is established.

We now use $\mathcal{A}(t-1)$ to prove the convergence stated by our proposition. Recall that $\mathbf{y}_i^t = \sum_{l \in [n]} W_{il}^{t-1} f(\mathbf{y}_{l \rightarrow i}^{t-1}, l, t-1)$. Set $i = i_n$ as in the statement. Given $A > 0$, define the real function $\eta_A : \mathbb{R} \rightarrow \mathbb{R}$ as the function that coincides with the identity on $[-A, A]$ and is equal to A on (A, ∞) and to $-A$ on $(-\infty, -A)$. To study $\mathbb{E}(\mathbf{y}_i^t)^m - \mathbb{E}(U_i^t)^m$, we can assume that $\mathbf{y}_i^t = (H_i^t)^{1/2} \xi$ and $U_i^t = (Q_i^t)^{1/2} \xi$, where

$$H_i^t = \sum_{l \in [n]} s_{il} f(\mathbf{y}_{l \rightarrow i}^{t-1}, l, t-1) f(\mathbf{y}_{l \rightarrow i}^{t-1}, l, t-1)^\top,$$

$\xi \sim \mathcal{N}(0, I_q)$ is independent of H_i^t , and $(\cdot)^{1/2}$ is the semidefinite positive square root. Write $Y = (\mathbf{y}_i^t)^m$ and $U = (U_i^t)^m$. With this, we have

$$\begin{aligned} \mathbb{E}Y - \mathbb{E}U &= (\mathbb{E}\eta_A(Y) - \mathbb{E}\eta_A(U)) + (\mathbb{E}Y - \mathbb{E}\eta_A(Y)) + (\mathbb{E}U - \mathbb{E}\eta_A(U)) \\ &= \chi_1 + \chi_2 + \chi_3. \end{aligned}$$

For $\delta > 0$, we have

$$\begin{aligned} |\chi_1| &\leq \sup_{H \geq 0: \|H - Q_i^t\| \leq \delta} \left| \mathbb{E}_\xi \eta_A(H^{1/2} \xi)^m - \mathbb{E}_\xi \eta_A((Q_i^t)^{1/2} \xi)^m \right| + 2A \mathbb{P}[\|H_i^t - Q_i^t\| > \delta] \\ &\leq C_1(A, \delta) + C_2(A, \delta, n), \end{aligned}$$

where $C_1(A, \delta) \rightarrow 0$ as $\delta \rightarrow 0$, and $C_2(A, \delta, n) \rightarrow 0$ as $n \rightarrow \infty$ as in (32). Regarding χ_2 , we have

$$\chi_2 = \mathbb{E}(Y - \eta_A(Y)) \mathbb{1}_{|Y| > A} \leq \mathbb{E}(|Y| + A) \mathbb{1}_{|Y| > A} \leq \sup_n \frac{\mathbb{E}Y^2 + \mathbb{E}|Y|^3}{A^2} \leq \frac{C}{A^2}$$

for some $C > 0$. We have a similar bound for χ_3 . By taking A large enough then δ small enough, we easily obtain that $\mathbb{E}(\mathbf{y}_i^t)^m - \mathbb{E}(U_i^t)^m \rightarrow_n 0$. \square

The results of Propositions 9, 11, 12, and 14 will lead to the convergences (21), which will be the consequences of the two following propositions.

Proposition 15. Let $\psi^{(n)} : \mathbb{R}^q \times [n] \rightarrow \mathbb{R}$ be such that $\psi^{(n)}(\cdot, l)$ is a multivariate polynomial with a bounded degree and bounded coefficients as functions of (l, n) . Then, for each set $\mathcal{S}^{(n)} \subset [n]$ with $|\mathcal{S}^{(n)}| \rightarrow_n \infty$, it holds that

$$\frac{1}{|\mathcal{S}^{(n)}|} \sum_{i \in \mathcal{S}^{(n)}} \mathbb{E} \psi(\mathbf{x}_i^{(n),t}, i) - \mathbb{E} \psi(U_i^{(n),t}, i) \xrightarrow{n \rightarrow \infty} 0.$$

Proof. By Propositions 9, 11 and 12, we obtain that for each $t \geq 1$ and each $\mathbf{m} \in \mathbb{N}^q$, there exists $C > 0$ such that $|\mathbb{E}(\mathbf{x}_i^t)^{\mathbf{m}} - \mathbb{E}(\mathbf{y}_i^t)^{\mathbf{m}}| \leq CK^{-1/2}$ for each $i \in [n]$, and furthermore, W can be assumed Gaussian in the construction of the \mathbf{y}_i^t . By Proposition 14, we obtain that $\mathbb{E}(\mathbf{x}_{i_n}^t)^{\mathbf{m}} - \mathbb{E}(U_{i_n}^t)^{\mathbf{m}} \rightarrow_n 0$, where the sequence (i_n) is arbitrary.

Furthermore, using the moment bound (20) and observing that the mixed moments of the U_i^t are bounded by (19), we obtain the result by Dominated Convergence. \square

Proposition 16 (adaptation of Proposition 5 of [BLM15]). Let $\psi^{(n)} : \mathbb{R}^q \times [n] \rightarrow \mathbb{R}$ be as in the previous proposition. Let $\mathcal{S}^{(n)} \subset [n]$ be a non empty set such that $|\mathcal{S}^{(n)}| \leq CK^{(n)}$. Then, for each $t > 0$,

$$\begin{aligned} \text{Var} \left(\frac{1}{K} \sum_{i \in \mathcal{S}} \psi(\mathbf{x}_i^t, i) \right) &\xrightarrow{n \rightarrow \infty} 0, \quad \text{and} \\ \text{Var} \left(\frac{1}{n} \sum_{i \in [n]} \psi(\mathbf{x}_i^t, i) \right) &\xrightarrow{n \rightarrow \infty} 0. \end{aligned} \tag{33}$$

Proof. We adapt the proof of [BLM15, Prop. 5] to our situation. Let $\{\check{X}_{ij}\}_{1 \leq i < j \leq n}$ be a set of real independent random variables that satisfy the same assumptions as $\{X_{ij}\}_{1 \leq i < j \leq n}$. Assume furthermore that these two sets are independent. Write $\check{X}_{ji} = \check{X}_{ij}$ for $1 \leq i < j \leq n$, and let $\check{X}_{ii} = 0$ for $i \in [n]$. Define the $n \times n$ matrix $\check{X} = [\check{X}_{ij}]_{i,j=1}^n$.

Let $B = [b_{ij}]$ be the $n \times n$ symmetric matrix which first row is defined as $b_{1j} = \mathbb{1}_{j \in \mathcal{S}}$ for $j \in [n]$, and which have zeros outside its first row and first column. To establish the first convergence, we build the matrix $\check{W} = [\check{W}_{ij}]_{i,j=1}^n$ defined as

$$\check{W}_{ij} = \frac{1}{\sqrt{K}} B \odot \check{X}$$

(here we assume without affecting the conclusion of the proposition that $1 \notin \mathcal{S}$). We construct a new AMP sequence around the $2n \times 2n$ matrix $\begin{bmatrix} W & 0 \\ 0 & \check{W} \end{bmatrix}$, which obviously satisfies Assumptions 1 and 2. The new algorithm, which delivers the $2n$ -uple $(\check{\mathbf{y}}_1^k, \dots, \check{\mathbf{y}}_{2n}^k)$ at Iteration k , is written as follows: $\check{\mathbf{y}}_i^0 = \mathbf{x}_i^0$ for $i \in [n]$ and $\check{\mathbf{y}}_i^0 = 0$ otherwise,

$$\begin{bmatrix} (\check{\mathbf{y}}_1^{k+1})^\top \\ \vdots \\ (\check{\mathbf{y}}_n^{k+1})^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & \check{W} \end{bmatrix} \begin{bmatrix} f(\check{\mathbf{y}}_1^k, 1, k)^\top \\ \vdots \\ f(\check{\mathbf{y}}_n^k, n, k)^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} f(\check{\mathbf{y}}_1^{k-1}, 1, k-1)^\top \\ \vdots \\ f(\check{\mathbf{y}}_n^{k-1}, n, k-1)^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left[\sum_{l=1}^q W_{i,l}^2 \frac{\partial f_r}{\partial y^{(s)}}(\check{\mathbf{y}}_l^k, l, k) \right]_{s,r=1}^q,$$

for $k = 0, \dots, t-1$, and

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ (\tilde{\mathbf{y}}_{n+1}^{t+1})^\top \\ \vdots \\ (\tilde{\mathbf{y}}_{2n}^{t+1})^\top \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & \tilde{W} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ [\psi(\tilde{\mathbf{y}}_1^t, 1), 0, \dots, 0] \\ \vdots \\ [\psi(\tilde{\mathbf{y}}_n^t, n), 0, \dots, 0] \end{bmatrix}$$

(next iterations are irrelevant). This sequence enters the framework of Proposition 7.

It is clear that $\tilde{\mathbf{y}}_i^k = \mathbf{x}_i^k$ for $k \in [t]$ and $i \in [n]$, and therefore,

$$\tilde{\mathbf{y}}_{n+i}^{t+1}(1) = \sum_{l \in [n]} \tilde{W}_{il} \psi(\mathbf{x}_i^t, l) \quad \text{for } i \in [n].$$

Set $i = 1$. On the one hand, we have

$$\begin{aligned} \mathbb{E} \tilde{\mathbf{y}}_{n+1}^{t+1}(1)^4 &= \sum_{l_1 l_2 l_3 l_4 \in \mathcal{S}} \mathbb{E} \tilde{W}_{1l_1} \tilde{W}_{1l_2} \tilde{W}_{1l_3} \tilde{W}_{1l_4} \mathbb{E} \psi(\mathbf{x}_{l_1}^t, l_1) \psi(\mathbf{x}_{l_2}^t, l_2) \psi(\mathbf{x}_{l_3}^t, l_3) \psi(\mathbf{x}_{l_4}^t, l_4) \\ &= \frac{3}{K^2} \sum_{l_1, l_2 \in \mathcal{S}} \mathbb{E} \psi(\mathbf{x}_{l_1}^t, l_1)^2 \psi(\mathbf{x}_{l_2}^t, l_2)^2 + \varepsilon \end{aligned} \quad (34)$$

where $|\varepsilon| \leq C/K$. On the other hand, Propositions 9, 11, 12 and 14 applied to our new AMP sequence show that

$$\mathbb{E} \tilde{\mathbf{y}}_{n+1}^{t+1}(1)^4 - \mathbb{E}(\tilde{U}^{t+1})^4 \xrightarrow{n \rightarrow \infty} 0,$$

where $\tilde{U}^{t+1} \sim \mathcal{N}(0, (\sigma^{t+1})^2)$, with

$$(\sigma^{t+1})^2 = \frac{1}{K} \sum_{l \in \mathcal{S}} \mathbb{E} \psi(U_l^t, l)^2,$$

By the Gaussianity and centeredness of \tilde{U}^{t+1} , we thus have

$$\mathbb{E} \tilde{\mathbf{y}}_{n+1}^{t+1}(1)^4 - \frac{3}{K^2} \left(\mathbb{E} \sum_{l \in \mathcal{S}} \psi(U_l^t, l)^2 \right)^2 \xrightarrow{n \rightarrow \infty} 0,$$

and since $\mathbb{E}(\mathbf{x}_i^t)^m - \mathbb{E}(U_i^t)^m \rightarrow 0$, we get by Dominated Convergence that

$$\mathbb{E} \tilde{\mathbf{y}}_{n+1}^{t+1}(1)^4 - \frac{3}{K^2} \left(\mathbb{E} \sum_{l \in \mathcal{S}} \psi(\mathbf{x}_i^t, l)^2 \right)^2 \xrightarrow{n \rightarrow \infty} 0,$$

Combining this convergence with (34), we obtain the first convergence stated by our proposition for polynomials of the type $\psi(\cdot, l)^2$. To obtain this convergence for arbitrary polynomials, write $\Psi(\mathbf{x}, l) = (1 + \varepsilon \psi(\mathbf{x}, l))$ for $\varepsilon > 0$. Since

$$\frac{1}{\varepsilon^2} \text{Var} \left(\frac{1}{K} \sum_{i \in \mathcal{S}} \Psi(\mathbf{x}_i^t, i)^2 \right) = \text{Var} \left(\frac{\varepsilon}{K} \sum_{i \in \mathcal{S}} \psi(\mathbf{x}_i^t, i)^2 + \frac{2}{K} \sum_{i \in \mathcal{S}} \psi(\mathbf{x}_i^t, i) \right)$$

must vanish for all $\varepsilon > 0$, we get the convergence (33).

To establish the other convergence in the statement, let $\mathcal{S}_i^{(n)} = \{i+1, ((i+1) \bmod n) + 1, \dots, ((i+K_n-1) \bmod n) + 1\}$ for $i \in \{0, \dots, n-1\}$. Then, we have that

$$\forall l \in [n], \quad \psi(\mathbf{x}_i^t, l) = \frac{1}{K} \sum_{i=0}^{n-1} \psi(\mathbf{x}_i^t, l) \mathbb{1}_{l \in \mathcal{S}_i}.$$

Therefore, writing

$$\frac{1}{n} \sum_{l \in [n]} \psi(\mathbf{x}_i^t, l) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{K} \sum_{l \in S_i} \psi(\mathbf{x}_i^t, l),$$

we can use Minkowski's inequality, the convergence (33) and the Dominated Convergence theorem to show that the variance of the left hand side converges to zero. \square

The convergences (21) follow at once from Propositions 15 and 16. Proposition 7 is proven.

4.3 Proof of Proposition 6

To establish Proposition 6 for a given fixed $t > 0$, we apply Proposition 7 with $q = t$ and a properly designed sequence of functions $f(\cdot, \cdot, 0), \dots, f(\cdot, \cdot, t-1)$, along the idea used in [BLM15, Proof of Th. 5].

For $k = 0, \dots, t-1$ define the function $f(\mathbf{x}, i, k)$ of the statement of Proposition 7 as follows. Consider the initial vector x^0 in Algorithm (12) as a constant parameter vector. For $\mathbf{x} = [x(1), \dots, x(t)]$, set

$$\begin{aligned} f(\mathbf{x}, i, 0)^\top &= \begin{bmatrix} 0 & \dots & 0 & p(x_i^0, i, 0) \end{bmatrix} \\ f(\mathbf{x}, i, 1)^\top &= \begin{bmatrix} 0 & \dots & p(x(1), i, 1) & p(x_i^0, i, 0) \end{bmatrix} \\ &\vdots \\ f(\mathbf{x}, i, t-1)^\top &= \begin{bmatrix} p(x(t-1), i, t-1) & \dots & p(x(1), i, 1) & p(x_i^0, i, 0) \end{bmatrix}. \end{aligned}$$

(note that $f(\cdot, i, 0)$ is a polynomial with degree zero). With this construction, if we start Algorithm (12) with the initial value x^0 and Algorithm (16) with an arbitrary initial value, then we can easily show by recurrence on the first t iterations running in parallel for both algorithms that

$$\begin{aligned} (\mathbf{x}_i^1)^\top &= \begin{bmatrix} 0 & \dots & 0 & x_i^1 \end{bmatrix} \\ (\mathbf{x}_i^2)^\top &= \begin{bmatrix} 0 & \dots & x_i^2 & x_i^1 \end{bmatrix} \\ &\vdots \\ (\mathbf{x}_i^t)^\top &= \begin{bmatrix} x_i^t & \dots & x_i^2 & x_i^1 \end{bmatrix}. \end{aligned}$$

We also notice that

$$\forall i \in [n], \quad Q_i^t = \check{R}_i^t.$$

With this at hand, Proposition 6 follows from Proposition 7.

4.4 From polynomial to general activation functions. Proof of Theorem 1

The bound (5) on the norms $\|R_i^{(n),t}\|$ can be easily established recursively in t similarly to the bound (19).

Theorem 1 will be deduced from Proposition 6. We mainly need to pass from polynomial activation functions to the function h described in the statement of the theorem.

The proof of the following lemma makes use of the density of the polynomials in the Hilbert space $L^2(\mathbb{R}, \nu)$ when ν is a Gaussian measure. Density arguments of this kind have been used in the AMP literature in, e.g., [DLS22, WZF22].

Lemma 17. Fix t , and fix two positive numbers $0 < \sigma_{\min}^2 \leq \sigma_{\max}^2$. Let $\varepsilon > 0$ be an arbitrarily small number. Then, there exists a function $g_\varepsilon(\cdot, \cdot, t) : \mathbb{R} \times \mathcal{Q}_\eta \rightarrow \mathbb{R}$ that satisfies the following properties: For each $\underline{\eta} \in \mathcal{Q}_\eta$, the function $g_\varepsilon(\cdot, \underline{\eta}, t)$ is a polynomial. Denoting as ∂g_ε the derivative of g_ε with respect to the first parameter, the inequalities

$$\mathbb{E}(h(\underline{\xi}, \underline{\eta}, t) - g_\varepsilon(\underline{\xi}, \underline{\eta}, t))^2 \leq \varepsilon, \quad \text{and} \quad |\mathbb{E}[\partial h(\underline{\xi}, \underline{\eta}, t) - \partial g_\varepsilon(\underline{\xi}, \underline{\eta}, t)]| \leq \varepsilon$$

hold true for each random variable $\underline{\xi} \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 \in [\sigma_{\min}^2, \sigma_{\max}^2]$ and each $\underline{\eta} \in \mathcal{Q}_\eta$. Finally, the function $p^{(n)}(\underline{x}, i, t) = g_\varepsilon(\underline{x}, \eta_i^{(n)}, t)$ satisfies the assumptions of Proposition 6.

Proof. Let $\varepsilon' > 0$ be a small number to be set at the end of the proof. Given the function κ and the integer m from Assumption 5, define $\delta > 0$ as

$$\delta = \max\left\{e \in (0, D_{\mathcal{Q}_\eta}], : \kappa(e)^2 \leq \frac{\varepsilon'}{\mathbb{E}(1 + |\xi|^m)^2}\right\},$$

where $\xi \sim \mathcal{N}(0, \sigma_{\max}^2)$, and where $D_{\mathcal{Q}_\eta}$ is the diameter of \mathcal{Q}_η . Since \mathcal{Q}_η is compact, it contains a δ -net with finite cardinality. Let $\mathcal{S} \subset \mathcal{Q}_\eta$ be a δ -net of \mathcal{Q}_η with the smallest cardinality, and write $M = |\mathcal{S}|$. Let $\phi : \mathcal{Q}_\eta \rightarrow \mathcal{S}$ be such that $\phi(\underline{\eta})$ is the closest element in \mathcal{S} to $\underline{\eta}$ if this closest element is unique, and $\phi(\underline{\eta})$ is the closest element smaller than $\underline{\eta}$ if not. Denoting as $\eta_1 < \eta_2 < \dots < \eta_M$ the elements of \mathcal{S} , define the function $\psi : \mathcal{Q}_\eta \rightarrow [M]$ as $\phi(\underline{\eta}) = \eta_{\psi(\underline{\eta})}$. With these definitions, we have $\mathbb{E}(h(\underline{\xi}, \underline{\eta}, t) - h(\underline{\xi}, \phi(\underline{\eta}), t))^2 \leq \varepsilon'$.

It is now well-known that the polynomials are dense in the space $L^2(\mathbb{R}, \nu)$ when ν is a Gaussian distribution. Therefore, there are M polynomials $P(\cdot, l, t)$ such that $\mathbb{E}(h(\underline{\xi}, \eta_l, t) - P(\underline{\xi}, l, t))^2 \leq \varepsilon'$ for each $l \in [M]$. Fixing l , let $\varphi(\underline{x}) = (h(\underline{x}, \eta_l, t) - P(\underline{x}, l, t))^2$. For $\underline{\xi} \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 \in [\sigma_{\min}^2, \sigma_{\max}^2]$, we have

$$\mathbb{E}(h(\underline{\xi}, \eta_l, t) - P(\underline{\xi}, l, t))^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \int \varphi(\underline{x}) e^{-\frac{x^2}{2\sigma^2}} d\underline{x} \leq \frac{\sigma_{\max}}{\sigma_{\min}} \frac{1}{\sqrt{2\pi\sigma_{\max}^2}} \int \varphi(\underline{x}) e^{-\frac{x^2}{2\sigma_{\max}^2}} d\underline{x} = \frac{\sigma_{\max}}{\sigma_{\min}} \varepsilon'.$$

Putting things together, we obtain that

$$\begin{aligned} \mathbb{E}(h(\underline{\xi}, \underline{\eta}, t) - P(\underline{\xi}, \psi(\underline{\eta}), t))^2 &\leq 2\mathbb{E}(h(\underline{\xi}, \underline{\eta}, t) - h(\underline{\xi}, \phi(\underline{\eta}), t))^2 + 2\mathbb{E}(h(\underline{\xi}, \phi(\underline{\eta}), t) - P(\underline{\xi}, \psi(\underline{\eta}), t))^2 \\ &\leq 2(1 + \sigma_{\max}/\sigma_{\min})\varepsilon'. \end{aligned}$$

We also have by Stein's lemma and Assumption 5

$$\mathbb{E}[\partial h(\underline{\xi}, \underline{\eta}, t) - \partial_\xi P(\underline{\xi}, \psi(\underline{\eta}), t)] = \frac{1}{\sigma^2} \mathbb{E}[\underline{\xi}(h(\underline{\xi}, \underline{\eta}, t) - P(\underline{\xi}, \psi(\underline{\eta}), t))].$$

Thus, we obtain by Cauchy-Schwarz and the previous result that

$$|\mathbb{E}[\partial h(\underline{\xi}, \underline{\eta}, t) - \partial_\xi P(\underline{\xi}, \psi(\underline{\eta}), t)]| \leq \frac{\sigma_{\max}}{\sigma_{\min}^2} \sqrt{2(1 + \sigma_{\max}/\sigma_{\min})} \sqrt{\varepsilon'},$$

By adjusting ε' , we thus obtain that

$$\mathbb{E}(h(\underline{\xi}, \underline{\eta}, t) - P(\underline{\xi}, \psi(\underline{\eta}), t))^2 \leq \varepsilon, \quad \text{and} \quad |\mathbb{E}[\partial h(\underline{\xi}, \underline{\eta}, t) - \partial_\xi P(\underline{\xi}, \psi(\underline{\eta}), t)]| \leq \varepsilon,$$

and it remains to set $g_\varepsilon(\underline{x}, \underline{\eta}, t) = P(\underline{x}, \psi(\underline{\eta}), t)$. We also notice that the polynomials $p^{(n)}(\underline{x}, i, t) = g_\varepsilon(\underline{x}, \eta_i^{(n)}, t)$ have bounded degrees and bounded coefficients, and thus comply with the statement of Proposition 6. \square

Lemma 18. There exists a constant $c > 0$ such that $R_i(t, t) \geq c$ for all $i \in [n]$ and $t \in [t_{\max}]$. Let $e > 0$ be a small number. Then, there exists a set of $\mathbb{R} \times \mathcal{Q}_\eta \rightarrow \mathbb{R}$ functions $\{g_e(\cdot, \cdot, t)\}_{t=0}^{t_{\max}-1}$ such that for each $\eta \in \mathcal{Q}_\eta$, the function $g_e(\cdot, \eta, t)$ is a polynomial, and furthermore,

$$\forall i \in [n], \quad |h(x_i^0, \eta_i, 0) - g_e(x_i^0, \eta_i, 0)| \leq e,$$

and $\forall t \in [t_{\max} - 1], \forall i \in [n],$

$$\mathbb{E}(h(Z_i^t, \eta_i, t) - g_e(Z_i^t, \eta_i, t))^2 \leq e, \quad |\mathbb{E}[\partial h(Z_i^t, \eta_i, t) - \partial g_e(Z_i^t, \eta_i, t)]| \leq e,$$

and the functions $p(x, i, t) = g_e(x, \eta_i, t)$ defined for $0 \leq t \leq t_{\max} - 1$ satisfy the assumptions of Proposition 6.

Furthermore, there exists a non-negative function $\delta(e)$ on a small interval $(0, \varepsilon)$ that converges to zero as $e \rightarrow 0$, and that satisfies the following property. Given $e \in (0, \varepsilon)$, construct the function

g_e as above, and let $(\check{Z}^t = [\check{Z}_i^t]_{i=1}^n)_{t \in [t_{\max}]}$ be the finite sequence of centered Gaussian vectors which distribution is determined by the (S, g_e, η, x^0) -state evolution equations stopped at t_{\max} , and leading to the covariance matrices $\{\check{R}_i^{t_{\max}}\}_{i \in [n]}$. Then, $\|R_i^{t_{\max}} - \check{R}_i^{t_{\max}}\| \leq \delta(e)$ for all $i \in [n]$. In particular,

$$\sup_n \max_{i \in [n]} \|\check{R}_i^{t_{\max}}\| < \infty. \quad (35)$$

Proof. Recalling the structure of the variance profile matrix S and using Assumption 6, we have that $R_i(1, 1) = \sum_l s_{il} h(x_l^0, \eta_l, 0)^2 \geq c > 0$ for each $i \in [n]$. Assuming that $R_i(t, t) \geq c > 0$ for each $i \in [n]$, we also have that $R_i(t+1, t+1) = \sum_l s_{il} \mathbb{E} h(Z_l^t, \eta_l, 0)^2 \geq c > 0$: This can be checked by noticing from Assumption 6 and the properties of h that there exists $c > 0$ such that $K^{-1} \sum_{l: s_{il} > 0} \mathbb{E} h(\xi, \eta_l, t)^2 \geq c_h(t)$ when $\xi \sim \mathcal{N}(0, 1)$, and by making standard Gaussian calculations to modify the variance of ξ and the bound $c_h(t)$. This establishes the first result of the lemma.

With this result, the existence of the functions $\{g_e(\underline{x}, \underline{\eta}, t)\}_{t=1}^{t_{\max}-1}$ follows at once from Lemma 17, with the numbers σ_{\min}^2 and σ_{\max}^2 in the statement of that lemma chosen as $\sigma_{\min}^2 = \min_{i,t} R_i(t, t)/2$ and $\sigma_{\max}^2 = 2 \max_{i,t} R_i(t, t)$. As regards $g(\underline{x}, \underline{\eta}, 0)$, we can use Assumptions 3 and 5 and invoke the Stone-Weierstrass theorem to choose $g(\cdot, \cdot, 0)$ as a multivariate polynomial that satisfies $\max_{(\underline{x}, \underline{\eta}) \in \mathcal{Q}_x \times \mathcal{Q}_\eta} |g(\underline{x}, \underline{\eta}, 0) - h(\underline{x}, \underline{\eta}, 0)| \leq e$. It is readily checked that $p(\underline{x}, l, 0) = g_e(\underline{x}, \eta_l, 0)$ complies with the assumptions of Proposition 6.

We show the last result by recurrence on t . We first have $|R_i^1 - \check{R}_i^1| \leq \sum_l s_{il} (h(x_l^0, \eta_l, 0)^2 - g_e(x_l^0, \eta_l, 0)^2) \leq C e^2$. Assume that there exists a non-negative function $\delta_t(e)$ that converges to zero as $e \rightarrow 0$, and such that $\|R_i^t - \check{R}_i^t\| \leq \delta_t(e)$. We have

$$\begin{aligned} |R_i^{t+1}(t+1, 1) - \check{R}_i^{t+1}(t+1, 1)| &\leq \sum_l s_{il} |\mathbb{E} h(Z_l^t, \eta_l, t) h(x_l^0, \eta_l, 0) - \mathbb{E} g_e(\check{Z}_l^t, \eta_l, t) g_e(x_l^0, \eta_l, 0)| \\ &\leq \sum_l s_{il} |\mathbb{E}[h(Z_l^t, \eta_l, t) - h(\check{Z}_l^t, \eta_l, t)]| |h(x_l^0, \eta_l, 0)| \\ &\quad + \sum_l s_{il} |\mathbb{E} h(\check{Z}_l^t, \eta_l, t)| |h(x_l^0, \eta_l, 0) - g_e(x_l^0, \eta_l, 0)| \\ &\quad + \sum_l s_{il} |\mathbb{E}[h(\check{Z}_l^t, \eta_l, t) - g_e(\check{Z}_l^t, \eta_l, t)]| |g_e(x_l^0, \eta_l, 0)|. \end{aligned}$$

Using Assumptions 3–5, the bound (5), and the recurrence assumption, we obtain that the term at the second line can be bounded by some non-negative function $\delta_{t+1}(e)$ converging to zero with e . Indeed, when the triple $(\eta, \sigma^2, \check{\sigma}^2)$ belongs to a compact set, the function $\varphi(\eta, \sigma^2, \check{\sigma}^2) = \mathbb{E} h(\underline{\xi}, \underline{\eta}, t) - \mathbb{E} h(\check{\underline{\xi}}, \underline{\eta}, t)$ with $\underline{\xi} \sim \mathcal{N}(0, \sigma^2)$ and $\check{\underline{\xi}} \sim \mathcal{N}(0, \check{\sigma}^2)$ is (uniformly) continuous on this compact, and vanishes on the set $\{(\eta, \sigma^2, \check{\sigma}^2) : \sigma^2 = \check{\sigma}^2\}$. One can readily check that the terms at the third and fourth line of the previous display can be bounded by Ce and $C\sqrt{e}$ respectively. We thus obtain that $|R_i^{t+1}(t+1, 1) - \check{R}_i^{t+1}(t+1, 1)| \leq \delta_{t+1}(e)$, by possibly modifying the function δ_{t+1} . A similar treatment can be applied to $R_i^{t+1}(t+1, k) - \check{R}_i^{t+1}(t+1, k)$ for $k = 2, \dots, t+1$, leading to $\|R_i^{t+1} - \check{R}_i^{t+1}\| \leq \delta_{t+1}(e)$ by possibly modifying $\delta_{t+1}(e)$ once again. This leads to the required result. \square

We now use the previous lemma for $e > 0$ small to construct a polynomial AMP sequence. Starting with $x^0 = x^0$, this sequence denoted as $(x^t)_{t \in [t_{\max}]}$ is given as

$$x^{t+1} = W g_e(x^t, \eta, t) - \text{diag}\left(W^{\odot 2} \partial g_e(x^t, \eta, t)\right) g_e(x^{t-1}, \eta, t-1). \quad (36)$$

Proposition 6 can be applied to this sequence, which implies in particular that the convergences (15) hold true for polynomial test functions. We now show that the polynomial test function can be replaced with a continuous function that increases at most polynomially near the infinity.

Lemma 19. Consider the polynomial AMP sequence $(x^t)_{t \in [t_{\max}]}$ provided by Algorithm (36). Let the n -uple $(\beta_1, \dots, \beta_n)$ and the function $\varphi : \mathcal{Q}_\eta \times \mathbb{R}^{t_{\max}} \rightarrow \mathbb{R}$ be as in the statement of Theorem 1. Then,

$$\frac{1}{n} \sum_{i \in [n]} \beta_i \varphi(\eta_i, x_i^1, \dots, x_i^{t_{\max}}) - \beta_i \mathbb{E} \varphi(\eta_i, \check{Z}_i^1, \dots, \check{Z}_i^{t_{\max}}) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

Proof. Define the $\mathcal{P}(\mathbb{R}^{t_{\max}+2})$ -valued random probability measure $\hat{\varrho}^{(n)}$ as

$$\hat{\varrho}^{(n)} = \frac{1}{n} \sum_{i \in [n]} \delta_{(\beta_i, \eta_i, x_i^1, \dots, x_i^{t_{\max}})},$$

and the deterministic measure $\varrho^{(n)}$ valued on the same space as

$$\varrho^{(n)} = \mathcal{L}((\beta_\theta, \eta_\theta, \check{Z}_\theta^1, \dots, \check{Z}_\theta^{t_{\max}})),$$

where the random variable θ is uniformly distributed on $[n]$ and is independent of $\{\check{Z}_i^t\}_{t \in [t_{\max}], i \in [n]}$. Let $P(\eta, \underline{x}^1, \dots, \underline{x}^{t_{\max}})$ be a multivariate polynomial. Since the elements η_i of the vector η are in \mathcal{Q}_η , and since the β_i are bounded by assumption, the function $\psi : \mathbb{R}^{t_{\max}} \times [n] \rightarrow \mathbb{R}$, $(\underline{x}^1, \dots, \underline{x}^{t_{\max}}, l) \mapsto \beta_l P(\eta_l, \underline{x}^1, \dots, \underline{x}^{t_{\max}})$ complies with the assumptions of the statement of Proposition 6. Observing that $\int \underline{\beta} P(\eta, \underline{x}^1, \dots, \underline{x}^{t_{\max}}) \varrho^{(n)}(d\underline{\beta}, d\eta, d\underline{x}^1, \dots, d\underline{x}^{t_{\max}}) = n^{-1} \sum_i \beta_i \mathbb{E} P(\eta_i, \check{Z}_i^1, \dots, \check{Z}_i^{t_{\max}})$, we obtain from Proposition 6 that

$$\int \underline{\beta} P(\eta, \underline{x}^1, \dots, \underline{x}^{t_{\max}}) d\hat{\varrho}^{(n)} - \int \underline{\beta} P(\eta, \underline{x}^1, \dots, \underline{x}^{t_{\max}}) d\varrho^{(n)} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (37)$$

We now show that this convergence remains true when P is replaced with the function φ of the statement. Recalling the bound (35) and the bounds on the β_i and the η_i , we know that for each sequence (n) of integers, there is a subsequence (that will still denote as (n)) and a deterministic measure $\varrho^\infty \in \mathcal{P}(\mathbb{R}^{t_{\max}+2})$ such that $\varrho^{(n)}$ converges narrowly to ϱ^∞ . We shall show that $\hat{\varrho}^{(n)}$ converges narrowly in probability towards ϱ^∞ , or, equivalently, that

$$\forall \omega \in \mathbb{R}^{t_{\max}+2}, \quad \Phi_{\hat{\varrho}^{(n)}}(\omega) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \Phi_{\varrho^\infty}(\omega), \quad (38)$$

where $\Phi_{\hat{\varrho}^{(n)}}$ (respectively Φ_{ϱ^∞}) is the characteristic function of $\hat{\varrho}^{(n)}$ (respectively ϱ^∞). This narrow convergence, coupled with the moment convergence implied by (37) along our subsequence, leads to the result of the lemma.

Define respectively as $\hat{\varrho}_\omega^{(n)}$, $\varrho_\omega^{(n)}$, $\varrho_\omega^\infty \in \mathcal{P}(\mathbb{R})$ the push-forward of $\hat{\varrho}^{(n)}$, $\varrho^{(n)}$, and ϱ^∞ by the linear function $\langle \omega, \cdot \rangle$. Writing $\omega = [\omega_\beta, \omega_\eta, \omega_x^\top]^\top$ where ω_β and ω_η are scalars, the moment generating function $\psi_\omega^{(n)}(z)$ of $\varrho_\omega^{(n)}$ is given as

$$\psi_\omega^{(n)}(z) = \frac{1}{n} \sum_{i=1}^n \exp((\omega_\beta \beta_i + \omega_\eta \eta_i)z - \omega_x^\top \check{R}_i^{t_{\max}} \omega_x z^2 / 2).$$

Using the normal family theorem in conjunction with the boundedness of the β_i , the η_i and the norms $\|\check{R}_i^{t_{\max}}\|$, we obtain that the moment generating function $\psi_\omega^\infty(z)$ of ϱ_ω^∞ is holomorphic in a neighborhood of zero (even entire), thus, ϱ_ω^∞ is determined by its moments. Moreover, we obtain from the convergence (37) that each moment of $\hat{\varrho}_\omega^{(n)}$ converges in probability to its analogue for ϱ_ω^∞ . This implies that $\hat{\varrho}_\omega^{(n)}$ converges narrowly in probability towards ϱ_ω^∞ for each ω , which is equivalent to the convergence (38). \square

In Algorithm (36), the term $\text{diag}(W^{\odot 2} \partial g_e(x^t, \eta, t))$ can be approximated with the more manipulable term $\text{diag}(S \partial g_e(x^t, \eta, t))$. This will be a consequence of the next lemma.

Lemma 20. For each $t \in [t_{\max} - 1]$, there is a constant $C \geq 0$ such that for each $i \in [n]$,

$$\mathbb{E} \left[\left(\sum_{l \in [n]} (W_{il}^2 - s_{il}) \partial g_e(x_l^t, \eta_l, t) \right)^4 \right] \leq C/K^2.$$

The proof of this lemma follows closely the development of [BLM15, Sec. A.2], with an adaptation to our variance profile model very similar to what we did in the proof of Proposition 9 above. We omit the details.

Theorem 1: end of proof

In this paragraph and in the next one, we shall write $g_e(\underline{x}) = g_e(\underline{x}, \underline{\eta}, t)$ and $h(\underline{x}) = h(\underline{x}, \underline{\eta}, t)$ for conciseness; The values of $\underline{\eta}$ and t will be clear from the context.

Let us establish the convergence (6). Given a small $e > 0$, construct the functions $g_e(\cdot, \cdot, t)$ for $t = 0, \dots, t_{\max} - 1$ as specified by Lemma 18. With these functions at hand, let $(x^1, \dots, x^{t_{\max}})$ be the iterates obtained by Algorithm (36). We shall show that there exists a function $\delta(e)$ that converges to zero as $e \rightarrow 0$, and for which

$$\forall t = 0, \dots, t_{\max} - 1, \quad \mathbb{P}\left[\|x^{t+1} - \bar{x}^{t+1}\|_n > \delta(e)\right] \cap [\|W\| \leq C_W] \xrightarrow[n \rightarrow \infty]{} 0, \quad (39)$$

where the vectors \bar{x}^{t+1} are those defined after (6).

In what follows, the function $\delta(e)$ can change from line to line. We first establish

$$\mathbb{P}\left[\|\text{diag}(W^{\odot 2} \partial g_e(x^t))g_e(x^{t-1}) - \text{diag}(S\mathbb{E}\partial h(\mathbf{Z}^t))h(x^{t-1})\|_n \geq \delta(e)\right] \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 \quad (40)$$

for $t \in [t_{\max} - 1]$. To this end, we write

$$\begin{aligned} & \text{diag}(W^{\odot 2} \partial g_e(x^t))g_e(x^{t-1}) - \text{diag}(S\mathbb{E}\partial h(\mathbf{Z}^t))h(x^{t-1}) \\ &= \text{diag}((W^{\odot 2} - S)\partial g_e(x^t))g_e(x^{t-1}) + \text{diag}(S(\partial g_e(x^t) - \mathbb{E}\partial g_e(\check{\mathbf{Z}}^t)))g_e(x^{t-1}) \\ & \quad + \text{diag}(S\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t))(g_e(x^{t-1}) - h(x^{t-1})) + \text{diag}(S(\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t) - \mathbb{E}\partial h(\mathbf{Z}^t)))h(x^{t-1}). \end{aligned}$$

Using Lemma (20) and the bound (14), the term χ_1 defined as

$$\chi_1 = \|\text{diag}((W^{\odot 2} - S)\partial g_e(x^t))g_e(x^{t-1})\|_n^2 = \frac{1}{n} \sum_{i \in [n]} [(W^{\odot 2} - S)\partial g_e(x^t)]_i^2 [g_e(x^{t-1})]_i^2$$

satisfies $\mathbb{E}\chi_1 \rightarrow 0$ by the Cauchy-Schwarz inequality, thus, $\chi_1 \xrightarrow{\mathcal{P}} 0$.

We now deal with the next term $\chi_2 = \|\text{diag}(S(\partial g_e(x^t) - \mathbb{E}\partial g_e(\check{\mathbf{Z}}^t)))g_e(x^{t-1})\|_n^2$. For $i = i_n \in [n]$, write $\xi_i = [S(\partial g_e(x^t) - \mathbb{E}\partial g_e(\check{\mathbf{Z}}^t))]_i$. Applying the convergence (15a) in the statement of Proposition 6 with $\mathcal{S} = \mathcal{K}_i = \{j \in [n], s_{ij} > 0\}$ and $\psi(x_j^1, \dots, x_j^t, j) = K s_{ij} \partial g_e(x_j^t)$, we obtain that $\xi_{i_n} \xrightarrow{\mathcal{P}} 0$. Furthermore, for each integer $m > 0$, we obtain from the bound (14) and Minkowski's inequality that $\max_{j \in [n]} \mathbb{E}|\xi_j|^m$ is bounded. Let $\varepsilon > 0$. Writing $\xi_i = \xi_i \mathbb{1}_{|\xi_i| \geq \varepsilon} + \xi_i \mathbb{1}_{|\xi_i| < \varepsilon}$ and using Cauchy-Schwarz inequality along with the bound (14), we obtain that

$$\mathbb{E}\chi_2 \leq \frac{C}{n} \sum_{i \in [n]} \mathbb{P}[\xi_i \geq \varepsilon]^{1/2} + C\varepsilon^2,$$

which converges to zero. Thus, $\mathbb{E}\chi_2 \rightarrow 0$.

Let us deal with $\chi_4 = \|\text{diag}(S(\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t) - \mathbb{E}\partial h(\mathbf{Z}^t)))h(x^{t-1})\|_n^2$. We have

$$\begin{aligned} & |[S\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t) - S\mathbb{E}\partial h(\mathbf{Z}^t)]_i| \leq |[S\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t) - S\mathbb{E}\partial h(\check{\mathbf{Z}}^t)]_i| + |[S\mathbb{E}\partial h(\check{\mathbf{Z}}^t) - S\mathbb{E}\partial h(\mathbf{Z}^t)]_i| \\ & \leq Ce + \delta(e), \end{aligned} \quad (41)$$

where the bound Ce on the first term comes from the construction of g_e in Lemma 18, and the bound $\delta(e)$ on the second term is due to the inequality $\max_i \|\check{R}_i^{t_{\max}} - R_i^{t_{\max}}\| \leq \delta(e)$ stated by the same lemma. Write $b_i = [S\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t) - S\mathbb{E}\partial h(\mathbf{Z}^t)]_i^2$, and assume that $t \geq 2$. By Lemma 19,

$$\frac{1}{n} \sum_{i \in [n]} b_i (h(x_i^{t-1})^2 - \mathbb{E}h(\check{Z}_i^{t-1})^2) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

Since the numbers $\mathbb{E}h(\check{Z}_i^{t-1})^2$ are bounded, we obtain that $\mathbb{P}[\chi_4 \geq \delta(e)] \xrightarrow{\mathcal{P}} 0$. If $t = 1$, it is easy to see that $\chi_4 \leq \delta(e)$.

We now consider $\chi_3 = \|\text{diag}(S\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t)(g_e(x^{t-1}) - h(x^{t-1}))\|_n^2$. By Inequality (41) and the bound (5), the real numbers $b_i = [S\mathbb{E}\partial g_e(\check{\mathbf{Z}}^t)]_i^2$ are bounded. Thus, if $t \geq 2$,

$$\chi_3 - \frac{1}{n} \sum_{i \in [n]} b_i \mathbb{E}(g_e(\check{Z}_i^{t-1}) - h(\check{Z}_i^{t-1}))^2 \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0$$

by Lemma 19. Since $\mathbb{E}(g_e(\check{Z}_i^{t-1}) - h(\check{Z}_i^{t-1}))^2 \leq e$, we obtain that $\mathbb{P}[\chi_3 \geq Ce] \rightarrow_n 0$. If $t = 1$, then $\chi_3 \leq Ce^2$ by Lemma 18.

Gathering these results on χ_1 to χ_4 , we obtain the convergence (40).

Now, on the set $\|W\| \leq C_W$, we furthermore have $\|W(h(x^t) - g_e(x^t))\| \leq C_W \|h(x^t) - g_e(x^t)\|$ for $t = 0, \dots, t_{\max} - 1$. By making a derivation similar to χ_3 above, we obtain that

$$\mathbb{P} \left[\left[\|W(h(x^t) - g_e(x^t))\|_n \geq C\sqrt{e} \right] \cap \left[\|W\| \leq C_W \right] \right] \xrightarrow[n \rightarrow \infty]{} 0$$

for $t \geq 1$, and $\|W(h(x^0) - g_e(x^0))\|_n \leq C_W e$ on $\left[\|W\| \leq C_W \right]$.

Combining this with the convergence (40), we obtain the convergences (39), which lead to the convergences (6) by choosing e small enough.

The convergence (7) is obtained from Lemma 19 and from the bound

$$\left| \frac{1}{n} \sum_{i \in [n]} \beta_i \mathbb{E} \varphi(\eta_i, \check{Z}_i^1, \dots, \check{Z}_i^t) - \beta_i \mathbb{E} \varphi(\eta_i, Z_i^1, \dots, Z_i^t) \right| \leq \delta(e).$$

Theorem 1 is proven.

4.5 Proof of Corollary 2

We want to show that there exist a sequence of matrices $\widetilde{\mathbf{X}}^{(n)} = [\tilde{x}^{(n),1} \ \dots \ \tilde{x}^{(n),t_{\max}}]$ such that for each $\varepsilon > 0$,

$$\mathbb{P} \left[\left[\left\| \tilde{x}^{(n),t+1} - \left(W^{(n)} h(\tilde{x}^{(n),t}) - \text{diag} \left(S^{(n)} \mathbb{E} \partial h(\mathbf{Z}^{(n),t}) \right) h(\tilde{x}^{(n),t-1}) \right) \right\|_n > \varepsilon \right] \cap \left[\|W^{(n)}\| \leq C_W \right] \right] \xrightarrow[n \rightarrow \infty]{} 0,$$

for each $t = 0, \dots, t_{\max} - 1$, and

$$\mathbb{P} \left[\left| \frac{1}{n} \sum_{i \in [n]} \beta_i^{(n)} \varphi(\eta_i^{(n)}, \tilde{x}_i^{(n),1}, \dots, \tilde{x}_i^{(n),t_{\max}}) - \beta_i^{(n)} \mathbb{E} \varphi(\eta_i^{(n)}, Z_i^{(n),1}, \dots, Z_i^{(n),t_{\max}}) \right| \geq \varepsilon \right] \xrightarrow[n \rightarrow \infty]{} 0.$$

Given a large enough integer k_0 , let $k \geq k_0$, and choose $\varepsilon > 0$ small enough so that $\varepsilon \vee \delta_\varphi(\varepsilon) = 1/k$, where $\delta_\varphi(\varepsilon)$ comes from (7). By Theorem 1, there exists a random matrix ${}^k \mathbf{X}^{(n)} = [{}^k x^{(n),1} \ \dots \ {}^k x^{(n),t_{\max}}] \in \mathbb{R}^{n \times t_{\max}}$ such that

$$\mathbb{P} \left[\left[\left\| {}^k x^{t+1} - \left(W h({}^k x^t) - \text{diag} \left(S \mathbb{E} \partial h(\mathbf{Z}^t) \right) h({}^k x^{t-1}) \right) \right\|_n > 1/k \right] \cap \left[\|W\| \leq C_W \right] \right] \xrightarrow[n \rightarrow \infty]{} 0$$

for each $t = 0, \dots, t_{\max} - 1$, and

$$\mathbb{P} \left[\left| \frac{1}{n} \sum_{i \in [n]} \varphi(\eta_i, [{}^k x^1]_i, \dots, [{}^k x^{t_{\max}}]_i) - \mathbb{E} \varphi(\eta_i, Z_i^1, \dots, Z_i^{t_{\max}}) \right| \geq 1/k \right] \xrightarrow[n \rightarrow \infty]{} 0,$$

where the $[{}^k x^t]_i$ are the elements of the vector ${}^k x^t$. If $k = k_0$, let $N_k \in \mathbb{N}_*$ be the smallest integer such that for all $n \geq N_k$, the probabilities above are upper bounded by $1/k$. If $k > k_0$, do the same thing for N_k with the supplementary condition that $N_k > N_{k-1}$. Construct the sequence of matrices $(\widetilde{\mathbf{X}}^{(n)})_{n \geq 2}$, with $\widetilde{\mathbf{X}}^{(n)} = [\tilde{x}^{(n),1} \ \dots \ \tilde{x}^{(n),t_{\max}}] \in \mathbb{R}^{n \times t_{\max}}$ in such a way that $\widetilde{\mathbf{X}}^{(n)} = {}^k \mathbf{X}^{(n)}$ for $n = N_k, N_k + 1, \dots, N_{k+1} - 1$. Then, this sequence satisfies the properties provided in the statement of the corollary.

5 Proof of Theorem 4

As said in the introduction, $u_\star^{(n)}$ can be identified as the solution of a LCP problem [Tak96]. We recall herein the elements of the LCP theory needed in this paper. As above, we often drop the superscript (n) . We shall also use the well-known notations \geq , \leq , and $<$ to refer to element-wise inequalities between vectors.

5.1 The equilibrium u_\star the solution of a LCP problem

Given a matrix $B \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$, the LCP problem, denoted as $\text{LCP}(B, b)$, consists in finding a couple of vectors $(z, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{aligned} y &= Bz + b \geq 0, \\ z &\geq 0, \\ \langle z, y \rangle &= 0. \end{aligned}$$

When a solution (z, y) exists and is unique, we write $z = \text{LCP}(B, b)$.

Obviously, an equilibrium \bar{u} of our LV dynamical system (9) is defined by the system

$$\begin{aligned} \bar{u} &\geq 0, \\ \bar{u} (r + (\Sigma - I) \bar{u}) &= 0. \end{aligned}$$

Furthermore, the supplementary condition

$$r + (\Sigma - I) \bar{u} \leq 0$$

turns out to be a necessary condition for the equilibrium \bar{u} to be stable in the classical sense of Lyapounov theory (see [Tak96, Chapter 3] to recall the different notions of stability, and [Tak96, Theorem 3.2.5] for this result). These three conditions can be rewritten as

$$\begin{aligned} \bar{w} &= (I - \Sigma) \bar{u} - r \geq 0, \\ \bar{u} &\geq 0, \\ \langle \bar{w}, \bar{u} \rangle &= 0, \end{aligned}$$

in other words, the couple (\bar{u}, \bar{w}) solves the problem $\text{LCP}(I - \Sigma, -r)$.

Theorem 4 is more specific, since it asserts that the ODE (9) has a (unique) *globally stable* equilibrium when $\|\Sigma\| < 1$. Recalling that Σ is a symmetric matrix, this can be obtained from the following result:

Proposition 21 (Lemma 3.2.2 and Theorem 3.2.1 of [Tak96]). Given a symmetric matrix $B \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$, consider the LV ODE

$$\dot{u}(t) = u(t) (b + Bu(t)), \quad t \geq 0, \quad u(0) \in \mathbb{R}_{*+}^n.$$

Then, the LCP problem $\text{LCP}(-B, -b)$ has an unique solution for each $b \in \mathbb{R}^n$ if and only if B is negative definite (notation $B < 0$). On the domain where $B < 0, b \in \mathbb{R}^n$, the function $x = \text{LCP}(-B, -b)$ is measurable. Moreover, if $B < 0$, then for each $b \in \mathbb{R}^n$, the ODE above has a globally stable equilibrium u given as $u = \text{LCP}(-B, -b)$.

Considering our LV ODE (9), the first part of Theorem 4 is true by setting

$$u_{\star}^{(n)} = \begin{cases} \text{LCP}(I_n - \Sigma^{(n)}, -r^{(n)}) & \text{if } \|\Sigma^{(n)}\| < 1, \\ 0 & \text{otherwise.} \end{cases} \quad (42)$$

5.2 Behavior of $\mu^{u_{\star}}$ by an AMP approach: proof outline

Recall that the notation xy refers to the elementwise product of the \mathbb{R}^n -valued vectors x and y . In the remainder, given a $n \times n$ matrix A , we further denote as Axy and xAy the vectors $A(xy)$ and $x(Ay)$ respectively.

Let us outline our approach for studying the asymptotic behavior of the measure $\mu^{u_{\star}^{(n)}}$.

Let $S^{(n)} \in \mathbb{R}_+^{n \times n}$ and $\eta^{(n)} \in \mathbb{R}_+^n$ be respectively a deterministic symmetric matrix and a deterministic vector that comply with Assumptions 2 and 4 respectively. These objects will be specified below. For the moment, we just assume that

$$\limsup_n \|S^{(n)}\| < 1. \quad (43)$$

Given the matrix S , let W be random symmetric $n \times n$ matrix given by Equation (1). In this equation, the matrix X is precisely the one used to construct the interaction matrix Σ of our LV model. Regarding the function $h : \mathbb{R} \times \mathcal{Q}_\eta \times \mathbb{N} \rightarrow \mathbb{R}$ of Section 2, set $h(\underline{x}, \underline{\eta}, t) = (\underline{x} + \underline{\eta})_+$. We can thus consider with a small notational modification that h is a $\mathbb{R} \times \mathcal{Q}_\eta \rightarrow \mathbb{R}$ function, and write

$$h(\underline{x}, \underline{\eta}) = (\underline{x} + \underline{\eta})_+. \quad (44)$$

Let us consider a Gaussian sequence $(\mathbf{Z}^t = [Z_i^t]_{i=1}^n)_t$ which distribution is determined by the $(S, h, \eta, 1_n)$ -state evolution equations. In particular, writing $a_i^t = \mathbb{E}(Z_i^t)^2$, the vector of variances $a^t = [a_i^t]_{i=1}^n$ satisfies the recursion

$$a^1 = S(1 + \eta)^2, \quad \text{and} \quad a^{t+1} = S\mathbb{E}\left(\sqrt{a^t}\xi + \eta\right)_+^2, \quad (45)$$

with $\xi \sim \mathcal{N}(0, I_n)$.

Notice that $\mathbb{E}\partial h(Z^t, \eta) = \mathbb{P}[Z^t + \eta \geq 0] = \mathbb{P}[\sqrt{a^t}\xi + \eta \geq 0]$. Write $\zeta^t = S\mathbb{P}[\sqrt{a^t}\xi + \eta \geq 0]$. Choosing a an integer $t > 0$ and applying Theorem 1, there exists a matrix $[x^{t-1}, x^t, x^{t+1}] \in \mathbb{R}^{n \times 3}$ such that

$$x^{t+1} = W(x^t + \eta)_+ - \zeta^t(x^{t-1} + \eta)_+ + \chi_1, \quad (46)$$

where the random variable $\|\chi_1\|_n$ is “small”. Furthermore, given a random variable θ uniformly distributed on $[n]$ and independent of \mathbf{Z}^t , the distribution μ^{x^t} can be approximated by $\mathcal{L}(Z_\theta^t)$ in the sense of the convergence (7).

In all what follows, when we say that a sequence $(y^t)_{t=1,2,\dots}$ of \mathbb{R}^n -valued vectors converge to the vector y as $t \rightarrow \infty$, we always consider that this convergence occurs in the max-norm $\|\cdot\|_\infty$ and is uniform in n .

Using the condition $\limsup_n \|S\| < 1$, it is not difficult to show that the iterates a^t converge as $t \rightarrow \infty$ to the vector a defined as the unique solution of the identity

$$a = S\mathbb{E}\left(\sqrt{a}\xi + \eta\right)_+^2. \quad (47)$$

On the one hand, this implies that for large values of t , we can replace the vector \mathbf{Z}^t with a vector $\mathbf{Z} = [Z_i]_{i=1}^n \stackrel{\mathcal{L}}{=} \sqrt{a}\xi$, which implies that μ^{x^t} can be approximated by $\mathcal{L}(Z_\theta)$, where the uniformly distributed random variable θ on $[n]$ is independent of \mathbf{Z} . On the other hand, ζ^t converges in t to the vector ζ given as

$$\zeta = S\mathbb{P}[\sqrt{a}\xi + \eta \geq 0]. \quad (48)$$

We can thus write

$$x^{t+1} = W(x^t + \eta)_+ - \zeta(x^{t-1} + \eta)_+ + \chi_1 + \chi_2,$$

where $\chi_2 = (\zeta - \zeta^t)(x^{t-1} + \eta)_+$ is such that $\|\chi_2\|_n$ is small for large t .

Next, we need to show that the vectors x^t and x^{t+1} tend to become aligned as t grows. To that end, building on a result of [MR16, DM16], we show that the correlation vector q^t , defined as

$$q^t = \frac{\mathbb{E} \mathbf{Z}^t \mathbf{Z}^{t-1}}{\sqrt{a^t a^{t-1}}},$$

converges to the vector 1_n as $t \rightarrow \infty$. With these elements, the AMP iteration can be written as

$$x^t = W(x^t + \eta)_+ - \zeta(x^t + \eta)_+ + \chi_1 + \chi_2 + \chi_3$$

where

$$\chi_3 = \zeta [(x^t + \eta)_+ - (x^{t-1} + \eta)_+] + x^t - x^{t+1}$$

has the property that $\|\chi_3\|_n$ is small for large values of t . The next to last equation can be rewritten

$$-(x^t + \eta)_- = W(x^t + \eta)_+ - (1 + \zeta)(x^t + \eta)_+ + \eta + \chi_1 + \chi_2 + \chi_3,$$

which leads to the equivalent equation

$$\begin{aligned} -(1 + \zeta)^{-1/2}(x^t + \eta)_- &= (1 + \zeta)^{-1/2}W(1 + \zeta)^{-1/2}[(1 + \zeta)^{1/2}(x^t + \eta)_+] - (1 + \zeta)^{1/2}(x^t + \eta)_+ \\ &\quad + (1 + \zeta)^{-1/2}\eta + \boldsymbol{\varepsilon}^t, \end{aligned} \quad (49)$$

with $\boldsymbol{\varepsilon}^t = (1 + \zeta)^{-1/2}(\chi_1 + \chi_2 + \chi_3)$.

We know that when $\|\Sigma\| < 1$, the equilibrium vector of the ODE (9) is $u_\star = \text{LCP}(I - \Sigma, -r)$. On the other hand, the previous equation shows that the vector $u^t = (1 + \zeta)^{1/2}(x^t + \eta)_+$ is a solution to the LCP problem $\text{LCP}(I - (1 + \zeta)^{-1/2}W(1 + \zeta)^{-1/2}, -(1 + \zeta)^{-1/2}\eta - \boldsymbol{\varepsilon}^t)$. Thus, if we choose the matrix W as $W = (1 + \zeta)^{1/2}\Sigma(1 + \zeta)^{1/2}$, or, equivalently, if we put

$$S = (1 + \zeta)V(1 + \zeta), \quad (50)$$

and furthermore, if we set

$$\eta = (1 + \zeta)^{1/2}r, \quad (51)$$

then we obtain that

$$u^t = \text{LCP}(I - \Sigma, -r - \boldsymbol{\varepsilon}^t).$$

Thus, we constructed with the help of an AMP approach a ‘‘perturbed’’ version of u_\star which empirical distribution can be evaluated with arbitrary precision for large n , and the perturbation $\boldsymbol{\varepsilon}^t$ can be made ‘‘arbitrarily small’’ in the norm $\|\cdot\|_n$ for large t .

Recalling Equations (47) and (48), the choices made in (50) and (51) result in the following system of $2n$ equations in the unknown vectors a and ζ :

$$\begin{aligned} a &= (1 + \zeta)V(1 + \zeta)\mathbb{E} \left(\sqrt{a}\xi + (1 + \zeta)^{1/2}r \right)_+^2 \\ \zeta &= (1 + \zeta)V(1 + \zeta)\mathbb{P} \left[\sqrt{a}\xi + (1 + \zeta)^{1/2}r \geq 0 \right]. \end{aligned}$$

It is a bit more convenient to write

$$p = \frac{a}{1 + \zeta}, \quad (52)$$

resulting in the equivalent system (11). Remembering that $\mu^{x^t} \simeq \mathcal{L}(Z_\theta)$ and observing that $u^t = (1 + \zeta)^{1/2}(x^t + (1 + \zeta)^{1/2}r)_+$, we obtain from what precedes that $\mu^{u^\star} \simeq \mathcal{L}((Y_\theta)_+)$ where Y_θ is the random variable specified in the statement of Theorem 4.

To prove this theorem rigorously, we thus need to perform the following steps:

1. Prove that under Hypothesis 3, the system (11) admits a unique solution $(p, \eta) \in \mathbb{R}_+^n \times [0, 1]^n$. This is the content of Lemma 22 below. With this solution, construct the matrix S and the vector η as in (50) and (51) respectively. Note that the bound (43) is satisfied thanks to Hypothesis 3.

2. With the help of Theorem 1, establish Equation (49) with a control on the error term ε^t .
3. We just saw that u^t is a perturbed version of u_* . To be able to approximate μ^{u_*} with μ^{u^t} , we need a LCP perturbation result. This will be provided by the work of Chen and Xiang in [CX07], which will let us control the norm $\|u^t - u_*\|$.

5.3 Behavior of μ^{u_*} : detailed proof

Lemma 22. The system (11) admits an unique solution $(p^{(n)}, \zeta^{(n)}) \in \mathbb{R}_+^n \times [0, 1]$, and this solution satisfies $\sup_n \|p^{(n)}\|_\infty < \infty$.

This lemma will be proven in Section 5.4 below.

We now consider the second step of the proof. Given the solution (p, ζ) of System (11), let $S \in \mathbb{R}_+^{n \times n}$ and $\eta \in \mathbb{R}_{*+}^n$ be given by equations (50) and (51) respectively. Note that the bound (43) is satisfied thanks to Hypothesis 3. Define the random matrix $W = S^{\odot 1/2} \odot X$. Define the function h as in (44), and let $x^0 = 1_n$.

We these definitions, let us check that the assumptions leading to Theorem 1 are satisfied. Using Hypothesis 1, it is clear that our matrix W satisfies Assumptions 1 and 2. Trivially, $x^0 = 1_n$ satisfies Assumption 3. Letting $r_{\min}, r_{\max} > 0$ be respectively the minimum and maximum values of the elements of the compact \mathcal{Q}_r of Hypothesis 4, we obtain from Equation (51) that the elements of η belong to the compact interval $\mathcal{Q}_\eta = [\eta_{\min}, \eta_{\max}] \subset \mathbb{R}_{*+}$, where $\eta_{\min} = r_{\min}$ and $\eta_{\max} = \sqrt{2}r_{\max}$, and Assumption 4 is satisfied. Finally, one can readily check that the function h and the matrix S satisfy Assumptions 5 and 6.

Lemma 23. As $t \rightarrow \infty$, the sequence (a^t) converges to the vector $a \in \mathbb{R}_{*+}^n$ given as the unique solution of Equation (47), the sequence (ζ^t) converges to ζ , and the sequence (q^t) of correlation vectors converges to 1_n .

Proof. Let $\xi \sim \mathcal{N}(0, 1)$, and define the function $f : \mathbb{R}_+ \times \mathcal{Q}_\eta \rightarrow \mathbb{R}_+$, $(\underline{a}, \underline{\eta}) \mapsto \mathbb{E}(\sqrt{\underline{a}}\xi + \underline{\eta})_+^2$. A small calculation that we omit reveals that the derivative $\partial f(\underline{a}, \underline{\eta}) = \partial f(\cdot, \underline{\eta})|_{\underline{a}}$ on \mathbb{R}_{*+} is given as $\partial f(\underline{a}, \underline{\eta}) = \mathbb{P}[\sqrt{\underline{a}}\xi + \underline{\eta} \geq 0]$. In particular, $f(\cdot, \underline{\eta})$ is increasing. Therefore, $f(\underline{a}, \underline{\eta}) \geq f(0, \underline{\eta}) = \underline{\eta}^2 \geq \eta_{\min}$. We thus have from Assumption 6 and Equations (45) that

$$a^1 \geq c_S 1 \quad \text{and} \quad \forall t \geq 1, \quad a^{t+1} \geq S\eta^2 \geq c_S \eta_{\min}^2 1.$$

Moreover, since $\|S\| < 1$, and since $f(\underline{a}, \cdot)$ is also increasing, we have

$$a^{t+1} = Sf(a^t, \eta) \leq Sf(a^t, \eta_{\max} 1) < f(\|a^t\|_\infty, \eta_{\max}) 1,$$

thus, $\|a^{t+1}\|_\infty < f(\|a^t\|_\infty, \eta_{\max})$. For \underline{a} large, we have that $f(\underline{a}, \eta_{\max}) = \underline{a} \mathbb{E}(\xi + \eta_{\max}/\sqrt{\underline{a}})_+^2 \sim \underline{a}/2$. Noting from (45) that $\|a^1\|_\infty < (1 + \eta_{\max})^2$, we readily obtain that for each t , the elements of the vector a^t stay in a compact interval $\mathcal{Q}_a \subset \mathbb{R}_{*+}$ which is independent of n .

Let us study the iterations $a^{t+1} = Sf(a^t, \eta)$ given by (45). The vector-valued matrix function $F(a, \eta) = Sf(a, \eta)$ satisfies $|[F(a, \eta) - F(a', \eta)]_i| \leq \|S\| \max_i |f(a_i, \eta_i) - f(a'_i, \eta_i)| \leq \|S\| \|a - a'\|_\infty$, where $a = [a_i]$, $a' = [a'_i]$, and $\eta = [\eta_i]$. Thus $\|F(a, \eta) - F(a', \eta)\|_\infty \leq \|S\| \|a - a'\|_\infty$, and the convergence $a^t \rightarrow_t a$, where a is defined as the unique solution of Equation (47), is obtained by Banach's fixed point theorem. That this convergence is uniform in n in the norm $\|\cdot\|_\infty$ results from the inequality $\|a^t - a\|_\infty \leq \|S\|^{t-1} \|a^1 - a\|_\infty \leq \|S\|^{t-1} D_{\mathcal{Q}_a}$, where $D_{\mathcal{Q}_a}$ is the diameter of the compact \mathcal{Q}_a .

Recall that $\zeta^t = S\mathbb{P}[\sqrt{a^t}\xi + \eta \geq 0]$. The convergence $\zeta^t \rightarrow_t \zeta$ uniformly in n in the norm $\|\cdot\|_\infty$ is due to the boundedness of $(\|S^{(n)}\|)$, the continuity of the function $(\underline{a}, \underline{\eta}) \mapsto \mathbb{P}[\sqrt{\underline{a}}\xi + \underline{\eta} \geq 0]$ on the compact $\mathcal{Q}_a \times \mathcal{Q}_\eta$, and to the convergence of a^t to a as above.

We now establish the convergence $q^t \rightarrow_t 1$. Let $[G_1, G_2]$ be a centered Gaussian vector such that $\mathbb{E}G_1^2 = \mathbb{E}G_2^2 = 1$, and such that $\mathbb{E}G_1G_2 = \underline{q} \in [0, 1]$. Writing $\mathcal{Q}_a = [a_{\min}, a_{\max}]$, define

the compact interval $\mathcal{Q}_H = [\eta_{\min}/\sqrt{a_{\max}}, \eta_{\max}/\sqrt{a_{\min}}]$, and define the continuous function $H : [0, 1] \times \mathcal{Q}_H \rightarrow [0, 1]$ as

$$(\underline{q}, \underline{b}) \mapsto H(\underline{q}, \underline{b}) = \frac{\mathbb{E}(G_1 + \underline{b})_+(G_2 + \underline{b})_+}{\mathbb{E}(G_1 + \underline{b})_+^2}.$$

A consequence of [MR16, Lemma 38 and proof of Lemma 37] is that for each $\underline{b} \in \mathcal{Q}_H$, the function $H(\cdot, \underline{b})$ satisfies $H(q, \underline{b}) > q$ for all $q \in [0, 1)$, and $H(1, \underline{b}) = 1$.

We also define the continuous function $H : [0, 1] \times \mathcal{Q}_H \times \mathcal{Q}_H \rightarrow [0, 1]$ as

$$(\underline{q}, \underline{b}, \underline{d}) \mapsto H(\underline{q}, \underline{b}, \underline{d}) = \frac{\mathbb{E}(G_1 + \underline{b})_+(G_2 + \underline{d})_+}{(\mathbb{E}(G_1 + \underline{b})_+^2 \mathbb{E}(G_2 + \underline{d})_+^2)^{1/2}}.$$

Given $t > 0$ and $k = t - 1, t$, write the Gaussian vectors \mathbf{Z}^k from the $(S, h, \eta, 1)$ -state evolution equations as $\mathbf{Z}^k = \sqrt{a^k} \xi^k$, with $\xi^k \sim \mathcal{N}(0, I_n)$, and with $q^t = \mathbb{E} \xi^t \xi^{t-1}$. By the $(S, h, \eta, 1)$ -state evolution equations, we have

$$\begin{aligned} \mathbb{E} \mathbf{Z}^{t+1} \mathbf{Z}^t &= S \mathbb{E}(\mathbf{Z}^t + \eta)_+(\mathbf{Z}^{t-1} + \eta)_+ \\ &= S (\mathbb{E}(\mathbf{Z}^t + \eta)_+^2 \mathbb{E}(\mathbf{Z}^{t-1} + \eta)_+^2)^{1/2} H(q^t, \eta/\sqrt{a^t}, \eta/\sqrt{a^{t-1}}) \\ &= S (f(a^t, \eta) f(a^{t-1}, \eta))^{1/2} H(q^t, \eta/\sqrt{a^t}, \eta/\sqrt{a^{t-1}}), \end{aligned}$$

and using the state evolution equations again, we obtain that

$$q^{t+1} = (a^{t+1} a^t)^{-1/2} S (f(a^t, \eta) f(a^{t-1}, \eta))^{1/2} H(q^t, \eta/\sqrt{a^t}, \eta/\sqrt{a^{t-1}}).$$

Notice that $H(\underline{q}, \underline{b}, \underline{b}) = H(\underline{q}, \underline{b})$. By the continuity of H on $[0, 1] \times \mathcal{Q}_H \times \mathcal{Q}_H$ and the uniform convergence $\|a^t - a\|_\infty \rightarrow_t 0$, we obtain that $\|H(q^t, \eta/\sqrt{a^t}, \eta/\sqrt{a^{t-1}}) - H(q^t, \eta/\sqrt{a})\|_\infty \rightarrow_t 0$ uniformly in n . We also have that $\|(a^{t+1} a^t)^{-1/2} - a^{-1}\|_\infty \rightarrow_t 0$ uniformly, and by using the continuity of f on $\mathcal{Q}_a \times \mathcal{Q}_\eta$, that $\|(f(a^t, \eta) f(a^{t-1}, \eta))^{1/2} - f(a, \eta)\|_\infty \rightarrow_t 0$ uniformly. Using the boundedness of $\|S\|$ and observing that $a^{-1} S f(a, \eta) = 1$, we obtain that $\|q^{t+1} - H(q^t, \eta/\sqrt{a})\|_\infty \rightarrow_t 0$ uniformly in n .

Since the set of zeros of the continuous function $(\underline{q}, \underline{b}) \mapsto \underline{q} - H(\underline{q}, \underline{b})$ on the compact $[0, 1] \times \mathcal{Q}_H$ is reduced to $\{1\} \times \mathcal{Q}_H$, we deduce from the last convergence that $\|q^{(n), t} - 1_n\|_\infty \rightarrow_t 0$ uniformly in n . \square

In the remainder of the proof, the function $\delta : (0, \epsilon) \rightarrow \mathbb{R}_+$ defined for some $\epsilon > 0$ is a generic function, independent of n , such that $\delta(e) \rightarrow 0$ as $e \rightarrow 0$. This function can change from line to line. Given a sequence $(\epsilon^{(n)})$ of non-negative random variables and a number $e > 0$, the notation $\epsilon^{(n)} \stackrel{\mathcal{P}}{\leq} e$ will stand for $\mathbb{P}[\epsilon^{(n)} \geq e] \rightarrow_n 0$. The relation $\stackrel{\mathcal{P}}{\leq}$ satisfies some obvious calculation rules, such as $\epsilon_1^{(n)} + \epsilon_2^{(n)} \stackrel{\mathcal{P}}{\leq} e_1 + e_2$ when $\epsilon_1^{(n)} \stackrel{\mathcal{P}}{\leq} e_1$ and $\epsilon_2^{(n)} \stackrel{\mathcal{P}}{\leq} e_2$.

Recalling that $W = (1 + \zeta)^{1/2} \Sigma (1 + \zeta)^{1/2}$, we have by Hypothesis 2 that $\limsup_n \|W\| < 2$ with probability one. Let us set $C_W = 2$ in the statement of Theorem 1.

Fix a small number $e > 0$, and let t be large enough so that

$$\begin{aligned} \mathbb{E} \|\mathbf{Z}^{t+1} - \mathbf{Z}^t\|_n^2 + \mathbb{E} \|\mathbf{Z}^t - \mathbf{Z}^{t-1}\|_n^2 &\leq e, \\ \|\zeta^t - \zeta\|_\infty &\leq e, \quad \text{and} \\ \|a^t - a\|_\infty &\leq e, \end{aligned} \tag{53}$$

which is possible by Lemma 23, after noting that $\mathbb{E} \|\mathbf{Z}^t - \mathbf{Z}^{t-1}\|_n^2 = n^{-1} \sum_i a_i^t + a_i^{t-1} - 2(a_i^t a_i^{t-1})^{1/2} q_i^t$. Applying Theorem 1 with $t_{\max} = t + 1$, we obtain that there exists a sequence of matrices $([x^{(n), t-1}, x^{(n), t}, x^{(n), t+1}])_n$ such Equation (46) is true, with $\|\chi_1\|_n \stackrel{\mathcal{P}}{\leq} e$, and thus,

$$\|(1 + \zeta)^{-1/2} \chi_1\|_n \stackrel{\mathcal{P}}{\leq} e.$$

Repeating the derivations that follow this equation, we reach the equation (49), and we obtain the bounds

$$\begin{aligned} \|(1 + \zeta)^{-1/2} \chi_2\|_n^2 &\leq \|(\zeta - \zeta^t)(x^{t-1} + \eta)_+\|_n^2 \\ &\leq \|(\zeta - \zeta^t)\|_\infty^2 (\|x^{t-1} + \eta\|_n^2 - \mathbb{E}\|(\mathbf{Z}^{t-1} + \eta)_+\|_n^2) + \|(\zeta - \zeta^t)\|_\infty^2 \mathbb{E}\|(\mathbf{Z}^{t-1} + \eta)_+\|_n^2 \\ &\stackrel{\mathcal{P}}{\leq} Ce^2 \end{aligned}$$

by Theorem 1 and Lemma 23, and

$$\begin{aligned} \|(1 + \zeta)^{-1/2} \chi_3\|_n^2 &\leq 2\|(x^t + \eta)_+ - (x^{t-1} + \eta)_+\|_n^2 + 2\|x^{t+1} - x^t\|_n^2 \\ &\leq 2\|(x^t + \eta)_+ - (x^{t-1} + \eta)_+\|_n^2 - 2\mathbb{E}\|(\mathbf{Z}^t + \eta)_+ - (\mathbf{Z}^{t-1} + \eta)_+\|_n^2 \\ &\quad + 2\mathbb{E}\|(\mathbf{Z}^t + \eta)_+ - (\mathbf{Z}^{t-1} + \eta)_+\|_n^2 \\ &\quad + 2\|x^{t+1} - x^t\|_n^2 - 2\mathbb{E}\|\mathbf{Z}^{t+1} - \mathbf{Z}^t\|_n^2 \\ &\quad + 2\mathbb{E}\|\mathbf{Z}^{t+1} - \mathbf{Z}^t\|_n^2 \\ &\stackrel{\mathcal{P}}{\leq} \delta(\epsilon), \end{aligned}$$

by Theorem 1 and Lemma 23 again, and by noticing that $\mathbb{E}\|(\mathbf{Z}^t + \eta)_+ - (\mathbf{Z}^{t-1} + \eta)_+\|_n^2 + \mathbb{E}\|\mathbf{Z}^{t+1} - \mathbf{Z}^t\|_n^2 \leq \epsilon$ thanks to the Lipschitz property of the function $\underline{x} \mapsto (\underline{x} + \underline{\eta})_+$.

All in all, we have that the vector $u^t = (1 + \zeta)^{1/2}(x^t + \eta)_+$ satisfies $u^t = \text{LCP}(I - \Sigma, -r - \epsilon^t)$ on the event $[\|\Sigma\| < 1]$, with

$$\|\epsilon^t\|_n \stackrel{\mathcal{P}}{\leq} \delta(\epsilon). \quad (54)$$

We now tackle the third step of the proof, applying the LCP perturbation result of Chen and Xiang in [CX07]. By [CX07, Th. 2.7 and Th. 2.8], we have

$$\|u^t - u_\star\| \leq \|(I - \Sigma)^{-1}\| \|\epsilon^t\|,$$

thus, $\|u^t - u_\star\| \leq C\|\epsilon^t\|$ with probability one for all n large by Hypothesis 2.

To finish the proof, we shall need a convenient characterization of the convergence in probability in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$. A function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is said pseudo-Lipschitz of degree 2 (notation $\varphi \in \text{PL}(2)$) with Lipschitz constant $L > 0$ if $|\varphi(x) - \varphi(y)| \leq L|x - y|(1 + |x| + |y|)$ for each $(x, y) \in \mathbb{R}^2$. By [FVRS22, Cor. 7.21], the convergence $\mathbf{d}_2(\mu^{u_\star}, \mu^{(n)}) \xrightarrow{\mathcal{P}} 0$ is equivalent in our context to

$$\forall \varphi \in \text{PL}(2), \quad \frac{1}{n} \sum_{i \in [n]} \varphi(u_{\star, i}) - \mathbb{E}\varphi((Y_i)_+) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (55)$$

Notice that the random vector $Y = (1 + \zeta)(\sqrt{p}\xi + r)$ in the statement of Theorem 4 satisfies $Y = (1 + \zeta)^{1/2}(\sqrt{a}\xi + \eta) \stackrel{\mathcal{L}}{=} (1 + \zeta)^{1/2}(\mathbf{Z} + \eta)$ by Equations (51) and (52). For $\varphi \in \text{PL}(2)$ with Lipschitz constant L , we write

$$\begin{aligned} &\frac{1}{n} \sum_{i \in [n]} \varphi(u_{\star, i}) - \mathbb{E}\varphi((1 + \zeta_i)^{1/2}(Z_i + \eta_i)_+) \\ &= \frac{1}{n} \sum_{i \in [n]} (\varphi(u_{\star, i}) - \varphi(u_i^t)) + \frac{1}{n} \sum_{i \in [n]} (\varphi(u_i^t) - \mathbb{E}\varphi((1 + \zeta_i)^{1/2}(Z_i^t + \eta_i)_+)) \\ &\quad + \frac{1}{n} \sum_{i \in [n]} \mathbb{E}\varphi((1 + \zeta_i)^{1/2}(Z_i^t + \eta_i)_+) - \mathbb{E}\varphi((1 + \zeta_i)^{1/2}(Z_i + \eta_i)_+) \\ &= \epsilon_1 + \epsilon_2 + \epsilon_3. \end{aligned}$$

We have

$$|\epsilon_1| \leq \frac{L}{n} \sum_{i \in [n]} |u_{\star, i} - u_i^t| (1 + 2|u_i^t| + |u_{\star, i} - u_i^t|) \leq L\|u_\star - u^t\|_n (1 + 2\|u^t\|_n + \|u_\star - u^t\|_n)$$

by Cauchy-Schwarz. Thus, with probability one, $|\epsilon_1| \leq C\|\epsilon^t\|_n(1 + \|u^t\|_n + \|\epsilon^t\|_n)$ for all n large. Applying Theorem 1 to $\|u^t\|_n$ and using the bound $\|\epsilon^t\|_n \stackrel{\mathcal{P}}{\leq} \delta(e)$, we obtain that $|\epsilon_1| \stackrel{\mathcal{P}}{\leq} \delta(e)$.

We also have

$$|\epsilon_2| \stackrel{\mathcal{P}}{\leq} \delta(e) \quad (56)$$

by Theorem 1.

To deal with ϵ_3 , we can write $\mathbf{Z}^t = \sqrt{a^t}\xi$ and $\mathbf{Z} = \sqrt{a}\xi$, with $\xi \sim \mathcal{N}(0, I_n)$, and use the pseudo-Lipschitz property of φ along with the bound (53) to show after a small derivation that $|\epsilon_3| \leq Ce$.

Putting these bounds together, we obtain that

$$\left| \frac{1}{n} \sum_{i \in [n]} \varphi(u_{\star, i}) - \mathbb{E}\varphi((Y_i)_+) \right| \stackrel{\mathcal{P}}{\leq} \delta(e).$$

Since e is arbitrary, the convergence (55) follows. Theorem 4 is proven.

5.4 Proof of Lemma 22

Given $\zeta \in [0, 1]^n$, Equation (11a) is $p = V(1 + \zeta)^2 f(p, r)$, where f is the function introduced in the proof of Lemma 23. We have that $2f(\underline{p}, \underline{r})/\underline{p} \rightarrow 1$ as $\underline{p} \rightarrow \infty$, uniformly in $\underline{r} \in \mathcal{Q}_r$ from Hypothesis 4. Thus, there exists $p_{\max} > 0$ such that $f(\underline{p}, \underline{r}) \leq 3\underline{p}/4$ for each $\underline{p} \geq p_{\max}$ and each $\underline{r} \in \mathcal{Q}_r$. By consequence, if $\|p\|_{\infty} > p_{\max}$, then $\|V(1 + \zeta)^2 f(p, r)\|_{\infty} \leq (3\|p\|_{\infty}/4)\|V(1 + \zeta)^2 \mathbf{1}_n\|_{\infty} < (3\|p\|_{\infty}/4)$ by Hypothesis 3, thus, $V(1 + \zeta)^2 f(p, r)$ cannot be equal to p when $p \notin [0, p_{\max}]^n$. On the other hand, if $p \in [0, p_{\max}]^n$, it holds that $\|V(1 + \zeta)^2 f(p, r)\|_{\infty} \leq f(p_{\max}, r_{\max})\|V(1 + \zeta)^2 \mathbf{1}_n\|_{\infty} \leq 3p_{\max}/4$, thus, $V(1 + \zeta)^2 f(p, r) \in [0, p_{\max}]^n$.

Turning to Equation (11b), for each $\zeta \in [0, 1]^n$, we see that $(1 + \zeta)V(1 + \zeta)\mathbb{P}[\sqrt{p}\xi + r \geq 0] \subset [0, 1]^n$ for each p by Hypothesis 3 again.

Thus, writing the system (11) as $(p, \zeta) = G(p, \zeta)$, we obtain that $G([0, p_{\max}]^n \times [0, 1]^n) \subset [0, p_{\max}]^n \times [0, 1]^n$, and furthermore, G does not have a fixed point outside $[0, p_{\max}]^n \times [0, 1]^n$. Since $[0, p_{\max}]^n \times [0, 1]^n$ is a compact convex set of \mathbb{R}^{2n} , we obtain by Brouwer's fixed point theorem that G has a fixed point in this set. In particular, when (p, ζ) is a fixed point, $\|p\|_{\infty} < p_{\max}$.

To complete the proof, we need to show that this fixed point is unique. To that end, we rely on the construction of the previous paragraph, where we note that this uniqueness is never used.

In all the remainder of this proof, the integer n is fixed. Choose a solution (p, ζ) of the system (11). From this solution, construct the matrix $S^{(n)}$ and the vector $\eta^{(n)}$ according to (50) and (51) respectively. Let (M) be a sequence of integers converging to infinity. For each M , construct the symmetric matrix $\mathbf{S}^{(M)} \in \mathbb{R}_+^{nM \times nM}$ and the vector $\boldsymbol{\eta}^{(M)} \in \mathbb{R}_{*+}^{nM}$ as

$$\mathbf{S}^{(nM)} = S^{(n)} \otimes (M^{-1} \mathbf{1}_M \mathbf{1}_M^{\top}), \quad \text{and} \quad \boldsymbol{\eta}^{(nM)} = \eta^{(n)} \otimes \mathbf{1}_M.$$

Let $\mathbf{X}^{(nM)} = [\mathbf{X}_{ij}^{(nM)}]_{1 \leq i, j \leq nM}$ be a real random symmetric $nM \times nM$ matrix such that the random variables $\{\mathbf{X}_{ij}^{(nM)}\}_{1 \leq i < j \leq nM}$ are independent $\mathcal{N}(0, 1)$ random variables, and such that $\mathbf{X}_{ii}^{(nM)} = 0$ for $i \in [nM]$. Define the random matrix $\mathbf{W}^{(nM)}$ as

$$\mathbf{W}^{(nM)} = (\mathbf{S}^{(nM)})^{\odot 1/2} \odot \mathbf{X}^{(nM)}.$$

We shall consider herein AMP iterations based on the matrix $\mathbf{W}^{(nM)}$ and on the $(\mathbf{S}^{(nM)}, h, \boldsymbol{\eta}^{(nM)}, \mathbf{1}_{nM})$ -state evolution equations. In this context, one can check that Assumptions 1–6 are satisfied with n and K_n there replaced with nM and $K_n M$ respectively. Write

$$\mathbf{V}^{(nM)} = V^{(n)} \otimes (M^{-1} \mathbf{1}_M \mathbf{1}_M^{\top}) \quad \text{and} \quad \mathbf{r}^{(nM)} = r^{(n)} \otimes \mathbf{1}_M$$

(so that $\mathbf{S} = (\mathbf{1}_{nM} + \boldsymbol{\zeta})\mathbf{V}(\mathbf{1}_{nM} + \boldsymbol{\zeta})$ with $\boldsymbol{\zeta} = \zeta \otimes \mathbf{1}_M$ and $\boldsymbol{\eta} = (\mathbf{1}_{nM} + \boldsymbol{\zeta})^{1/2} \mathbf{r}$), and let

$$\boldsymbol{\Sigma}^{(nM)} = (\mathbf{V}^{(nM)})^{\odot 1/2} \odot \mathbf{X}^{(nM)}.$$

Observe that since n is now fixed, our matrices $\mathbf{V}^{(nM)}$ are no more sparse, and $\|\mathbf{V}^{(nM)}\|_\infty \lesssim 1/M$. Thus, recalling the spectral norm controls made above in the Gaussian case, the positive number

$$\mathbf{T}_{\text{Gauss}}^{(nM)} = (1 + \varepsilon) \left(2\|\mathbf{V}^{(nM)}\|^{1/2} + \frac{6}{\sqrt{\log(1 + \varepsilon)}} (\|\mathbf{V}^{(nM)}\|_\infty \log(nM))^{1/2} \right)$$

satisfies $\limsup_M \mathbf{T}_{\text{Gauss}}^{(nM)} < 1$ by choosing ε small enough. Thus, by the Gaussian concentration such as in (10), we obtain that $\limsup_M \|\Sigma^{(nM)}\| < 1$ with the probability one. By consequence, on this event, the equation $\mathbf{u}_\star^{(nM)} = \text{LCP}(I - \Sigma^{(nM)}, -\mathbf{r}^{(nM)})$ is well-defined for all M large. It is important to note that this vector does not depend on the chosen solution (p, ζ) of the system (11).

Write $\mathbf{u}_\star^{(nM)} = [\mathbf{u}_\star^{(nM)}(1)^\top, \dots, \mathbf{u}_\star^{(nM)}(n)^\top]^\top$ where $\mathbf{u}_\star^{(nM)}(i) = [\mathbf{u}_{\star,1}^{(nM)}(i), \dots, \mathbf{u}_{\star,M}^{(nM)}(i)]^\top$. Recall the definition of the \mathbb{R}^n -valued Gaussian vector $Y^{(n)}$ as provided in the statement of Theorem 4. By repeating the argument of the previous paragraph, this time with the AMP algorithm based on the matrix $\mathbf{W}^{(nM)}$ and on the $(\mathbf{S}^{(nM)}, h, \boldsymbol{\eta}^{(nM)}, 1_{nM})$ -state evolution equations, we are able to show that

$$\forall \varphi \in \text{PL}(2), \quad \frac{1}{nM} \sum_{i \in [n]} \sum_{j \in [M]} \varphi(\mathbf{u}_{\star,j}^{(nM)}(i)) \xrightarrow{M \rightarrow \infty} \frac{1}{n} \sum_{i \in [n]} \mathbb{E} \varphi((Y_i)_+).$$

However, we need here a bit more than this convergence, which requires a slight modification of the approach of the previous paragraph. By relying on our new AMP construction stopped at a certain iteration t , we have that for each $\varepsilon > 0$, there exists a random \mathbb{R}^{nM} -valued vector $\mathbf{u}^{(nM),t}$, and an error vector $\boldsymbol{\varepsilon}^{(nM),t}$ such that

$$\|\mathbf{u}_\star^{(nM)} - \mathbf{u}^{(nM),t}\| \leq \|(I - \Sigma^{(nM)})^{-1}\| \|\boldsymbol{\varepsilon}^{(nM),t}\|$$

with

$$\mathbb{P}[\|\boldsymbol{\varepsilon}^{(nM),t}\|_{nM}^2 \geq \varepsilon] \xrightarrow{M \rightarrow \infty} 0$$

Writing $\mathbf{u}^{(nM),t} = [\mathbf{u}^{(nM),t}(1)^\top, \dots, \mathbf{u}^{(nM),t}(n)^\top]^\top$ with $\mathbf{u}^{(nM),t}(i) = [\mathbf{u}_1^{(nM),t}(i), \dots, \mathbf{u}_M^{(nM),t}(i)]^\top$ and remembering that n is fixed, this implies that there is $C > 0$ such that

$$\forall i \in [n], \quad \mathbb{P}[\|\mathbf{u}_\star^{(nM)}(i) - \mathbf{u}^{(nM),t}(i)\|_M^2 \geq Cne] \xrightarrow{M \rightarrow \infty} 0.$$

This is the analogue of the bound (54).

Furthermore, let $\varphi \in \text{PL}(2)$, let $i \in [n]$, and define the nM -uple

$$(\beta_1^{(nM)}, \dots, \beta_{nM}^{(nM)}) = (0, \dots, 0, \underbrace{1, 1, \dots, 1}_{\text{length } M}, 0, \dots, 0),$$

where the first element of the M -uple $(1, \dots, 1)$ is at the $((i-1)M + 1)^{\text{th}}$ place. By applying Theorem 1 with these weights, we obtain

$$\forall i \in [n], \quad \mathbb{P}\left[\left|\frac{1}{M} \sum_{l=1}^M \varphi(\mathbf{u}_l^{(nM),t}(i)) - \mathbb{E} \varphi((1 + \zeta_i)^{1/2} (\sqrt{a_i^t \xi} + \eta_i)_+)\right| \geq \delta(e)\right] \xrightarrow{M \rightarrow \infty} 0,$$

where $\xi \sim \mathcal{N}(0, 1)$, and where the vector $a^t = [a_i^t]_{i=1}^n$ is precisely the one given by the recursion (45). This is the analogue of the bound (56).

Completing the argument of the previous paragraph with these new bounds, we obtain that

$$\forall i \in [n], \quad \mu_{\mathbf{u}_\star^{(nM)}(i)} \xrightarrow{M \rightarrow \infty} \mathcal{L}((1 + \zeta_i)(\sqrt{p_i} \xi + r_i)_+) \quad \text{in probability.}$$

From the uniqueness of these limits, we deduce that the solution $(p = [p_i]_{i=1}^n, \zeta = [\zeta_i]_{i=1}^n)$ of the system (11) is unique.

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A Proof of Proposition 3

Assume that $\rho > 0$. Following the notations of [BvH16], write $\sigma = \max_{i \in [n]} (\sum_{j \in [n]} s_{ij})^{1/2}$ and $\sigma_* = \max_{ij} \sqrt{s_{ij}}$. Then, the proof of [BvH16, Cor. 3.5] shows that

$$\left(\mathbb{E}\|W\|^{2\lceil \log n \rceil}\right)^{1/(2\lceil \log n \rceil)} \leq C \left(\sigma + \sigma_*(\log n)^{(\rho \vee 1)/2}\right) \leq C \left(1 + \sqrt{\frac{(\log n)^{\rho \vee 1}}{K_n}}\right),$$

where the second inequality is due to our Assumption 2. Using Markov's inequality and the hypothesis $K_n \gtrsim (\log n)^{\rho \vee 1}$, we obtain that

$$\mathbb{P}[\|W\| \geq \eta]^{1/(2\lceil \log n \rceil)} \leq C/\eta$$

for any $\eta > 0$. Choosing η large enough, the result follows from the Borel-Cantelli lemma.

If $\rho = 0$, we can just apply the concentration results provided by [BvH16, Cor 3.12 and Rem. 3.13].

References

- [ABC⁺22] I. Akjouj, M. Barbier, M. Clenet, W. Hachem, M. Maïda, F. Massol, J. Najim, and V. C. Tran. Complex systems in ecology: A guided tour with large Lotka-Volterra models and random matrices. *arXiv preprint arXiv:2212.06136*, 2022.
- [AHMN23] I. Akjouj, W. Hachem, M. Maïda, and J. Najim. Equilibria of large random Lotka-Volterra systems with vanishing species: a mathematical approach. *arXiv preprint arXiv:2302.07820*, 2023.
- [AT12] S. Allesina and S. Tang. Stability criteria for complex ecosystems. *Nature*, 483(7388):205–208, 2012.
- [AT15] S. Allesina and S. Tang. The stability–complexity relationship at age 40: a random matrix perspective. *Population Ecology*, 57(1):63–75, 2015.
- [BBC18] G. Biroli, G. Bunin, and C. Cammarota. Marginally stable equilibria in critical ecosystems. *New Journal of Physics*, 20(8):083051, 2018.
- [BGP14] F. Benaych-Georges and S. Péché. Largest eigenvalues and eigenvectors of band or sparse random matrices. *Electron. Commun. Probab.*, 19:no. 4, 9, 2014.
- [BLM15] M. Bayati, M. Lelarge, and A. Montanari. Universality in polytope phase transitions and message passing algorithms. *Ann. Appl. Probab.*, 25(2):753–822, 2015.
- [BM11] M. Bayati and A. Montanari. The dynamics of message passing on dense graphs, with applications to compressed sensing. *IEEE Transactions on Information Theory*, 57(2):764–785, 2011.
- [BMN19] R. Berthier, A. Montanari, and P.-M. Nguyen. State evolution for approximate message passing with non-separable functions. *Information and Inference: A Journal of the IMA*, 9(1):33–79, 01 2019.

- [Bol14] E. Bolthausen. An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model. *Communications in Mathematical Physics*, 325(1):333–366, 2014.
- [BSHM17] D. M. Busiello, S. Suweis, J. Hidalgo, and A. Maritan. Explorability and the origin of network sparsity in living systems. *Scientific reports*, 7(1):12323, 2017.
- [Bun17] G. Bunin. Ecological communities with Lotka-Volterra dynamics. *Phys. Rev. E*, 95:042414, Apr 2017.
- [BvH16] A. S. Bandeira and R. van Handel. Sharp nonasymptotic bounds on the norm of random matrices with independent entries. *The Annals of Probability*, 44(4):2479 – 2506, 2016.
- [CEN22] M. Clenet, H. El Ferchichi, and J. Najim. Equilibrium in a large Lotka–Volterra system with pairwise correlated interactions. *Stochastic Processes and their Applications*, 153:423–444, 2022.
- [CL21] W.-K. Chen and W.-K. Lam. Universality of approximate message passing algorithms. *Electronic Journal of Probability*, 26(none):1 – 44, 2021.
- [CNM22] M. Clenet, J. Najim, and F. Massol. Equilibrium and surviving species in a large Lotka-Volterra system of differential equations. *ArXiv preprint*, arXiv:2205.00735, 2022.
- [CPS09] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The linear complementarity problem*, volume 60 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009. Corrected reprint of the 1992 original [MR1150683].
- [CX07] X. Chen and S. Xiang. Perturbation bounds of P -matrix linear complementarity problems. *SIAM J. Optim.*, 18(4):1250–1265, 2007.
- [DLS22] R. Dudeja, Y. M. Lu, and S. Sen. Universality of approximate message passing with semi-random matrices. *arXiv preprint arXiv:2204.04281*, 2022.
- [DM16] D. Donoho and A. Montanari. High dimensional robust M-estimation: asymptotic variance via approximate message passing. *Probab. Theory Related Fields*, 166(3-4):935–969, 2016.
- [Fan22] Z. Fan. Approximate message passing algorithms for rotationally invariant matrices. *Ann. Statist.*, 50(1):197–224, 2022.
- [FSW18] D. Fourdrinier, W. E. Strawderman, and M. T. Wells. *Shrinkage estimation*. Springer Series in Statistics. Springer, Cham, 2018.
- [FVRS22] O. Y. Feng, R. Venkataramanan, C. Rush, and R. J. Samworth. A unifying tutorial on approximate message passing. *Foundations and Trends® in Machine Learning*, 15(4):335–536, 2022.
- [Gal18] T. Galla. Dynamically evolved community size and stability of random Lotka-Volterra ecosystems. *Europhysics Letters*, 123(4):48004, sep 2018.
- [GB21] C. Gerbelot and R. Berthier. Graph-based Approximate Message Passing iterations. *ArXiv preprint*, arXiv:2109.11905, 2021.
- [JM13] A. Javanmard and A. Montanari. State evolution for general approximate message passing algorithms, with applications to spatial coupling. *Information and Inference: A Journal of the IMA*, 2(2):115–144, 2013.

- [Kho08] O. Khorunzhiy. Estimates for moments of random matrices with Gaussian elements. In *Séminaire de probabilités XLI*, volume 1934 of *Lecture Notes in Math.*, pages 51–92. Springer, Berlin, 2008.
- [LJM09] X. Li, D. Jiang, and X. Mao. Population dynamical behavior of Lotka-Volterra system under regime switching. *J. Comput. Appl. Math.*, 232(2):427–448, 2009.
- [May72] R. M. May. Will a large complex system be stable? *Nature*, 238(5364):413–414, 1972.
- [MR16] A. Montanari and E. Richard. Non-negative principal component analysis: message passing algorithms and sharp asymptotics. *IEEE Trans. Inform. Theory*, 62(3):1458–1484, 2016.
- [OD92] M. Opper and S. Diederich. Phase transition and $1/f$ noise in a game dynamical model. *Phys. Rev. Lett.*, 69:1616–1619, Sep 1992.
- [PS11] L. A. Pastur and M. Shcherbina. *Eigenvalue Distribution of Large Random Matrices*, volume 171 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.
- [Ran11] S. Rangan. Generalized approximate message passing for estimation with random linear mixing. In *2011 IEEE International Symposium on Information Theory Proceedings*, pages 2168–2172. IEEE, 2011.
- [Sod10] S. Sodin. The spectral edge of some random band matrices. *Ann. of Math. (2)*, 172(3):2223–2251, 2010.
- [Tak96] Y. Takeuchi. *Global dynamical properties of Lotka-Volterra systems*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [Tok04] K. Tokita. Species abundance patterns in complex evolutionary dynamics. *Phys. Rev. Lett.*, 93:178102, Oct 2004.
- [Vil09] C. Villani. *Optimal transport*, volume 338 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.
- [WZF22] T. Wang, X. Zhong, and Z. Fan. Universality of approximate message passing algorithms and tensor networks. *arXiv preprint arXiv:2206.13037*, 2022.