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Parabolic Anderson model on critical Galton-Watson trees in a Pareto environment

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Abstract

The parabolic Anderson model is the heat equation with some extra spatial randomness. In this paper we consider the parabolic Anderson model with i.i.d. Pareto potential on a critical Galton-Watson tree conditioned to survive. We prove that the solution at time t is concentrated at a single site with high probability and at two sites almost surely as $t \rightarrow \infty$. Moreover, we identify asymptotics for the localisation sites and the total mass, and show that the solution $u(t, v)$ at a vertex v can be well-approximated by a certain functional of v . The main difference with earlier results on \mathbb{Z}^d is that we have to incorporate the effect of variable vertex degrees within the tree, and make the role of the degrees precise.

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1 Introduction

In this paper we study the parabolic Anderson model (PAM) on a critical Galton-Watson tree conditioned to survive. Given an infinite tree T_∞ , with root O , the PAM on T_∞ is the heat equation on T_∞ with a random potential:

$$\begin{aligned} \partial_t u(t, v) &= \Delta u(t, v) + \xi(v)u(t, v), & (t, v) &\in (0, \infty) \times V(T_\infty), \\ u(0, v) &= \mathbb{1}\{v = O\}, & v &\in V(T_\infty), \end{aligned} \tag{1}$$

where $V(T_\infty)$ denotes the set of vertices of T_∞ . Here, Δ is the discrete Laplacian on T_∞ , defined by

$$(\Delta f)(v) = \sum_{y \sim v} [f(y) - f(v)], \quad v \in V(T_\infty), \quad f : V(T_\infty) \rightarrow \mathbb{R}.$$

One can also work under other initial conditions, such as $u(0, v) = 1$ for all $v \in V(T_\infty)$, but we are interested here in studying spread of particles from a single initial seed. We work under the *quenched* law of T_∞ , whereby we sample the tree T_∞ first, and then independently sample $(\xi(v) : v \in V(T_\infty))$ as a collection of independent identically distributed random variables, static in time.

The random operator $H := \Delta + \xi$ is known as the Anderson Hamiltonian and was originally analysed by Anderson in 1958 in the context of random Schroedinger equations that model electrons moving through a crystal with impurities [And58]. The parabolic analogue, the PAM, is now a classical model for mean-field spread of particles in an inhomogeneous random environment, with applications including population genetics, reaction kinetics and magnetism, to name but a few. See the book [Kĭ6] or the surveys [And58, Mör09, GK05] for more background.

The solution to the PAM is well-known to exhibit an *intermittency* effect, whereby almost all the solution is asymptotically concentrated in a small number of disjoint regions, known as *islands*. This effect has been

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observed by physicists for some time, but since the early 1990s there has been a concerted mathematical effort to prove this quantitatively, and on \mathbb{Z}^d this effect is now well-understood. The size and number of islands depend on the tail decay of the potential, which can give rise to several regimes on the behaviour of the islands.

In this paper we will be exclusively focused on the case where the potential is Pareto distributed, i.e. with polynomially decaying tails, for which the intermittency effect is strongly pronounced. In this case, it was shown in [KLMS09] that the solution to the PAM on \mathbb{Z}^d almost surely localises at two sites as time runs to infinity. In fact, the authors show that the solution localises at a single site with high probability, and identify the localisation site as the maximiser of a certain functional; this localisation site evolves in time, so that there will be a sequence of transition times when the solution “jumps” from one site to another, meaning that both sites contain non-negligible mass, and in particular that two-site localisation is the best result one can hope for in the almost sure setting.

The solution to the PAM on \mathbb{Z}^d in this regime was also studied by Ortgiese and Roberts, who showed in [OR16] that the solution can be well approximated by a mean-field “lilypad model” in which particles spread out from sites of high potential in a sequence of growing lilypads. Their main motivation was to study the behaviour of a branching random walk, for which the PAM describes the expected number of particles, but they show that the solution to the PAM at any given site corresponds roughly to the number of particles reaching that site from the centre of a nearby lilypad.

In this paper we prove similar results for the solution to the PAM on a critical Galton-Watson tree conditioned to survive. This extends the current literature since the underlying graph is now random, with fractal properties, and has non-uniform volume growth. One consequence of this is that the random walk appearing in the Feynman-Kac solution to (1) on T_∞ is known to be subdiffusive [Kes86, CK08], in contrast to the \mathbb{Z}^d case. Moreover, in contrast to most of the existing literature, we consider the PAM on a graph with unbounded degrees and we make the role of the degrees explicit.

The PAM has also been studied on supercritical Galton-Watson trees of unbounded degree in the recent article [dHW21]. Here the authors consider the (very different) regime where the potential has double-exponential tails, and show that results of [dKd20] on the PAM profile in the bounded degree case can be extended to the case where the offspring distribution decays super-double-exponentially. (The PAM was also considered on sequences of complete graphs in [FM90] and hypercubes in [AGH20], which both have unbounded degrees in the limit, but in these cases the Laplace operator was multiplied by an appropriate prefactor to cancel out the effect, which is not the case in the present paper). In our case, due to the heavier tails on the potential ξ , the localisation effect of the PAM is more pronounced and we will prove our results under weaker assumptions on the tail of the offspring distribution.

Before stating our main results, we give the setup. We suppose that T_∞ is an infinite Galton-Watson tree defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let d denote the fractal dimension of T_∞ (we will state d explicitly in Corollary 2.4; in particular $d = 2$ a.s. when the offspring distribution of T_∞ has finite variance). Conditionally on T_∞ , we fix some $\alpha > d$ (this restriction is both necessary and sufficient for a finite solution to (1) to exist, almost surely), and let $(\xi(v) : v \in V(T_\infty))$ denote a sequence of i.i.d. random variables with law \mathcal{P} , satisfying

$$\mathcal{P}(\xi(v) \geq x) = x^{-\alpha}$$

for all $x \geq 1$. This is known as the Pareto distribution with parameter α . If $v \in V(T_\infty)$, we call $\xi(v)$ the *potential* at v . We let P denote the law of the product measure $\mathbf{P} \times \mathcal{P}$.

Given vertices $u, v \in V(T_\infty)$, we let $|u - v|$ denote the graph distance between them in T_∞ , and $|v| = |v - O|$.

Given T_∞ and $(\xi(v) : v \in V(T_\infty))$, we let $u(t, v)$ denote the solution to (1), and let $U(t) = \sum_{v \in V(T_\infty)} u(t, v)$ denote the total mass in the system at time t . Following the notation of [KLMS09], we also set $q := \frac{d}{\alpha - d}$, and define the two functions

$$r(t) := \left(\frac{t}{\log t} \right)^{q+1}, \quad a(t) := \left(\frac{t}{\log t} \right)^q.$$

We will see later that, roughly speaking, the bulk of the solution at time t is supported on a ball of radius $r(t)$, and the highest potential seen on this ball will be of order $a(t)$.

We first present some high probability bounds on the solution, starting with the total mass. We prove these under the following assumption.

Assumption 1.1. *Assume the offspring distribution is critical and there exists $\beta \in (1, 2]$ such that the offspring distribution is in the domain of normal attraction of a β -stable law.*

Recall that a critical offspring distribution μ is in the normal domain of attraction of a β -stable law if there exists a constant c such that

$$\frac{S_n - n}{cn^{\frac{1}{\beta}}} \xrightarrow{(d)} X$$

as $n \rightarrow \infty$, where $S_n = \sum_{i=1}^n Z_i$ with Z_i i.i.d. with law μ , and X is a β -stable random variable, i.e. such that $\mathbb{E}[\exp(-\lambda X)] = \exp(-\lambda^\beta)$ for all $\lambda > 0$. In the finite variance case we have $\beta = 2$. One could also incorporate slowly-varying functions, but for simplicity we have not done this.

Theorem 1.2. *Under Assumption 1.1,*

$$\lim_{\lambda \rightarrow \infty} \liminf_{t \rightarrow \infty} P(\log U(t) \in [\lambda^{-1}ta(t), \lambda ta(t)]) = 1.$$

This result follows by similar considerations to those in [KLMS09].

In light of the behaviour of the Pareto-distributed PAM on \mathbb{Z}^d , the main focus of this paper is to prove a localisation result. In order to prove this we have to make slightly stronger assumptions on the offspring distribution of T_∞ .

Assumption 1.3. *Assume the offspring distribution of T_∞ is critical and decays super-polynomially, i.e. for every $K < \infty$ it satisfies $\mathbf{P}(\#\{\text{offspring}\} > x) = o(x^{-K})$ as $x \rightarrow \infty$.*

Remark 1.4. 1. *In particular, Assumption 1.3 implies that the distribution has finite variance and is thus in the domain of attraction of a $\beta = 2$ stable law (via the classical central limit theorem).*

2. *We use Assumption 1.3 when summing over paths that avoid favourable sites in order to show that their contribution to the total mass is small (Propositions 7.4 and 7.7), for which we require that the number of such paths within a ball is not too large. Although we do not believe that the results are false when Assumption 1.3 is not satisfied, it seems that Assumption 1.3 is required to implement a proof strategy that involves counting paths. Otherwise, it seems theoretically possible that at exceptional times, a non-negligible contribution to the mass can be spread over many paths, though not necessarily concentrating on a single group of vertices.*
3. *The assumption on the offspring distribution in Assumption 1.3 is not too restrictive since it is satisfied by most offspring distributions for which T_∞ is used as a local proxy for structures in statistical physics (for example the $\text{Poi}(1)$ offspring distribution which gives a good approximation to critical percolation clusters [Ald97] and uniform spanning trees [NP20] in high dimensions).*
4. *Assumption 1.3 is somewhat reminiscent of [dHW21, Assumption 1.3], in that we require super-polynomial tail decay to deal straightforwardly with polynomial tails for the potential, and in [dHW21] the authors require super-double-exponential tail decay in their main result to deal straightforwardly with double-exponential tails for the potential. In fact, we could slightly weaken our assumption to the requirement that the tails are $O(x^{-K_{\alpha,\beta}})$ for some deterministic $K_{\alpha,\beta} < \infty$, but it was not possible to bound $K_{\alpha,\beta}$ uniformly in α and β .*

The main localisation result is the following.

Theorem 1.5. *Let Assumption 1.3 hold. There exist processes $(\hat{Z}_t^{(1)})_{t \geq 0}$ and $(\hat{Z}_t^{(2)})_{t \geq 0}$ taking values in $V(T_\infty)$ such that, P -almost surely as $t \rightarrow \infty$,*

$$\frac{u(t, \hat{Z}_t^{(1)}) + u(t, \hat{Z}_t^{(2)})}{U(t)} \rightarrow 1. \tag{2}$$

As remarked above, two-site localisation is the best result we can hope for in the almost sure setting, due to the existence of transition times between the best sites. However, we actually have one-site localisation with high probability, as outlined in the following theorem.

Theorem 1.6. *Let Assumption 1.1 hold. There exists a process $(\hat{Z}_t^{(1)})_{t \geq 0}$ taking values in $V(T_\infty)$ such that*

$$\frac{u(t, \hat{Z}_t^{(1)})}{U(t)} \rightarrow 1 \text{ in } P\text{-probability as } t \rightarrow \infty. \quad (3)$$

Under Assumption 1.3, $\hat{Z}_t^{(1)}$ can be chosen to be the same as that appearing in Theorem 1.5.

The main tool in our analysis is the Feynman-Kac formula, which allows us to represent the solution to (1) as

$$u(t, v) = \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) \, ds \right\} \mathbb{1}\{X_t = v\} \right] \quad (4)$$

for all $v \in V(T_\infty)$, where $(X_s)_{s \geq 0}$ is a continuous time variable speed random walk on T_∞ with law \mathbb{P} and we denote the corresponding expectation as \mathbb{E} . It is clear from the formula that random walk trajectories spending time at sites of higher potential will contribute more to the total mass, so to find the best trajectory we have to balance the high potential enjoyed at good sites against the cost of forcing the random walk to visit and then stay at that site.

Our strategy is broadly the same as that used to prove similar results for the PAM on \mathbb{Z}^d in [KLMS09], but the non-uniformity of the underlying graph presents extra challenges which take some effort to overcome. In particular, due to variable degree on T_∞ the random walk estimates are less precise, and the effect of the degree cannot be neglected when comparing the solution at different sites, since this quantifies the cost of forcing a random walk to remain at a given site. An important part of the localisation proof will be to prove an analogous localisation result for the principal eigenfunction of the Anderson Hamiltonian operator, and here in particular we have to work a lot harder to obtain the desired result. In addition, fluctuations in the tree T_∞ will lead to extra fluctuations in the solution of the PAM and the location of the concentration site, whereas on \mathbb{Z}^d the behaviour of these quantities is determined solely by the realisation of the random potential.

In order to balance the potential at a site and the cost of reaching and then staying at it, we define the functional (cf [KLMS09, Equation (1.9)]):

$$\begin{aligned} \psi_t(v) &= \sup_{\rho \in [0,1]} \left\{ (1-\rho)(\xi(v) - \deg(v)) - \frac{|v|}{t} \log \left(\frac{|v|}{\rho e t} \right) \right\} \mathbb{1}\{t(\xi(v) - \deg(v)) \geq |v|\} \\ &= \left[\xi(v) - \deg(v) - \frac{|v|}{t} \log(\xi(v) - \deg(v)) \right] \mathbb{1}\{t(\xi(v) - \deg(v)) \geq |v|\}. \end{aligned} \quad (5)$$

This can be understood as follows: $e^{t(1-\rho)\xi(v)}$ is the contribution to the exponential term in the Feynman-Kac formula (4) if the random walk remains at site v for a time duration $(1-\rho)t$, then $e^{t(1-\rho)\deg(v)}$ is the cost of forcing the random walk to remain at site v for a time of $(1-\rho)t$ after reaching it, and we will see in Proposition 2.10 that $e^{-|v| \log(\frac{|v|}{\rho e t})}$ is roughly the probability that the random walk hits v by time ρt . At the best sites, it turns out that $u(t, v)$ is well-approximated by $e^{t\psi_t(v)}$.

The condition $t(\xi(v) - \deg(v)) \geq |v|$ ensures that the optimal value of $\rho = \frac{|v|}{t(\xi(v) - \deg(v))}$ in (5) is indeed in $[0, 1]$. A similar functional for the \mathbb{Z}^d case is defined in [KLMS09, Equation (1.9)], though they do not need to incorporate the degree term there. We define the processes $\hat{Z}_t^{(1)}$ and $\hat{Z}_t^{(2)}$ by

$$\hat{Z}_t^{(1)} := \arg \max_{v \in V(T_\infty)} \psi_t(v), \quad \hat{Z}_t^{(2)} := \arg \max_{v \in V(T_\infty) \setminus \{\hat{Z}_t^{(1)}\}} \psi_t(v). \quad (6)$$

We will see in Section 5 that $\hat{Z}_t^{(1)}$ and $\hat{Z}_t^{(2)}$ are well-defined and in particular enjoy the following asymptotic.

Theorem 1.7. *Let Assumption 1.1 hold. For each $i = 1, 2$, we have that*

$$\lim_{\lambda \rightarrow \infty} \liminf_{t \rightarrow \infty} P \left(|\hat{Z}_t^{(i)}| \in [\lambda^{-1}r(t), \lambda r(t)] \right) = 1.$$

Our final result approximates the solution and follows by similar arguments to [OR16, Theorem 1.4], though extra care is needed to deal with the unbounded degrees in our case. Analogously to [OR16], to state the result, for $v \in V(T_\infty)$ we define

$$\lambda(t, v) := \sup_{y \in V(T_\infty)} \left\{ t(\xi(y) - \deg y) - |y| \log \left(\frac{|y|}{te} \right) - |v - y| \log \left(\frac{|v - y|}{te} \right) \right\}, \quad (7)$$

and $\lambda_+(t, v) = \lambda(t, v) \vee 0$. Note the similarity with the terms in ψ_t in (5).

Theorem 1.8. *Let Assumption 1.1 hold. With high P -probability,*

$$(ta(t))^{-1} \sup_{v \in V(T_\infty)} |\log_+(u(t, v)) - \lambda_+(t, v)| \rightarrow 0$$

as $t \rightarrow \infty$.

The intuition here is that the vertex y achieving the supremum in (7) represents the “best” vertex that has the right balance between being close to v , and having high potential. The main contribution to the solution at v comes from particles that spend most of the time interval $[0, t]$ at site y where they reproduce at rate $\xi(y)$, and then dart to v just before time t . The total number of particles obtained in this way is roughly equal to $e^{t\xi(y)}$, and the cost of darting quickly from O to y and then quickly from y to v is roughly equal to the exponential of the subtracted terms in (7).

We note that in the case of the PAM on \mathbb{Z}^d , the authors in [KLMS09] obtain stronger results than those in Theorems 1.2 and 1.7 by identifying the scaling limit of the relevant quantities. This is achieved by showing that the scaling limit of the random potential field $(\xi(z) : z \in \mathbb{Z}^d)$ can be represented by a Poisson point process in \mathbb{R}^d and then interpreting the other scaling limits in terms of this point process. Since Gromov-Hausdorff-Prokhorov scaling limits of critical Galton-Watson trees are well understood, we anticipate that a similar approach could be employed in our setting, and in this case the limiting Poisson point process would have a similar density but with respect to the canonical mass measure on the continuum random sin-trees of Aldous [Ald91] (in the finite variance case) and Duquesne [Duq09] (in the β -stable case). Due to space considerations, we have not attempted to prove this result in this article, but we believe this makes our proofs more robust since we have not relied on any information about the limiting random variables and their densities, which is not the case in [KLMS09].

In light of these results regarding the PAM on critical Galton-Watson trees, we anticipate that the PAM should exhibit similar behaviour on similar graphs such as critical Erdős-Rényi graphs and the critical configuration model. Moreover, critical Galton-Watson trees are conjectured and in many cases known to be good toy models for a range of stochastic structures such as high-dimensional critical percolation clusters and uniform spanning trees, and some classes of random planar maps, so we also expect that the PAM on these more complicated models will display similar behaviour.

Finally, most of our proofs do not specifically take advantage of the tree structure. Instead, in order to upgrade the almost sure localisation result of [KLMS09] on \mathbb{Z}^d to random graphs, it seems that it is mainly important to have tight control on the degrees of typical vertices, which can be used to give bounds on the number of random walk trajectories that stay inside a ball, and volumes of balls, which allow us to prove the localisation of appropriate eigenfunctions on these balls. In particular we require that the volumes of balls around typical vertices exhibit at most logarithmic fluctuations around their typical values. We therefore expect that the results of [KLMS09] can also be extended to many other other random graphs of polynomial volume growth provided there are sufficiently good tails on vertex degrees and volumes. One example is the Uniform Infinite Planar Triangulation, which has exponential tails on typical vertex degrees and only logarithmic volume fluctuations [AS03].

This paper is organised as follows. In Section 2 we visit the definition of a critical Galton-Watson tree conditioned to survive and we derive several estimates regarding its volume growth. We also give some

bounds for the random walk defined on T_∞ which will be needed later on. In Section 3 we analyse extremal values of the potential on certain subsets of T_∞ . In Section 4 we prove the asymptotic behaviour stated in Theorem 1.2 and 1.7, and the approximation for the solution as given in Theorem 1.8. Subsequently, we are concerned with proving the localisation result. In particular, in Section 5 we derive several estimates for the first maximisers of the functional defined in (5) and the gap between the largest values of this functional. Section 6 contains the spectral analysis that we mentioned above, that is, we prove that the support of the principal eigenfunction of the Anderson Hamiltonian restricted to certain time-dependent subsets is strongly concentrated as time goes to infinity. Finally, these auxiliary results are used to prove the almost sure two-site localisation of the PAM on T_∞ (Theorem 1.5) in Section 7 and the high probability one-site localisation (Theorem 1.6) in Section 8.

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Notation. To facilitate reading we summarise notation in the following table.

Notation	Definition
α	Pareto parameter for the potential ξ
β	parameter for the GW offspring distribution, in $(1, 2]$
d	$\frac{\beta}{\beta-1}$
\mathbb{T}_∞	set of rooted trees with a single (one-ended) infinite path to infinity
$O = s_0$	root of T_∞
s_0, s_1, s_2, \dots	ordered vertices of the backbone of T_∞
$u \prec v$	u is an ancestor of v
$[[u, v]]$	(unique) direct path from u to v , inclusive of endpoints
$ u - v , v $	$ u - v $ is the graph distance between u and v , $ v := O - v $
$B(v, r), B_r$	$B(v, r)$ closed ball in T_∞ of radius r around v , $B_r := B(O, r)$
$B_r^{T_v}$	$B_r^{T_v} = B(v, r) \cap T_v$ for subtree T_v of T_∞
A_r	connected component containing O obtained after removing the vertex s_r from T_∞
$V(\mathcal{T})$	set of vertices of a tree/subtree \mathcal{T}
$\#S$	number of vertices in S
$\text{Height}(\mathcal{T})$	$\text{Height}(\mathcal{T}) := \max_{v \in V(\mathcal{T})} \tilde{O} - v $ for a finite tree \mathcal{T} with root \tilde{O}
\tilde{T}	$\{v \in V(T_\infty) : \deg(v) \leq 4\}$
\tilde{B}_r	$\{v \in B_r : \deg(v) \leq 4\}$
D_r	$\sup_{v \in B_r} \deg(v)$
\mathcal{L}	$\{u \in T_\infty : \deg u = 1\}$
$p(u)$	parent of u in T_∞
$\ell(u)$	leftmost child of u that is also a leaf
$N_r(n, v)$	number of distinct paths of length n starting at O , passing through v , fully contained in B_r
$\eta_r(n, v)$	$\log(N_r(n, v))$
ξ	the potential
$\xi_r^{(i)}$	the i^{th} highest value of ξ on B_r
$\tilde{\xi}_r^{(i)}$	the i^{th} highest value of ξ on \tilde{B}_r
q	$\frac{d}{\alpha-d}$
$a(t)$	$\left(\frac{t}{\log t}\right)^q$
$r(t)$	$\left(\frac{t}{\log t}\right)^{q+1}$
z	$\frac{d\alpha}{\alpha-d}$
p	$q + 2$
$R(t)$	$r(t)(\log t)^p$
C	$\frac{pd}{\alpha} + z + 1$
F_t	$\{v \in V(B_{R(t)}) : \xi(v) \geq R(t)^{\frac{d}{\alpha}} (\log t)^{-C}\}$
E_t	$\{v \in V(B_{R(t)}) \setminus F_t : \exists z \in F_t \text{ such that } \xi(z) - \deg(z) \leq \xi(v) - \deg(v)\}$
$\psi_t(v)$	$\left[(\xi(v) - \deg(v)) - \frac{ v }{t} \log(\xi(v) - \deg(v))\right] \mathbb{1}\{t(\xi(v) - \deg(v)) \geq v \}$
$\lambda(t, v, y)$	$t(\xi(y) - \deg y) - y [\log y - \log t - 1] - v - y [\log v - y - \log t - 1]$
$\lambda(t, v)$	$\sup_{y \in T_\infty} \lambda(t, v, y)$

Notation	Definition
$Y(t, v)$	$\arg \max_{y \in B_{r(t)(\log t)^m}} \{\lambda(t, v, y)\}$, where m is a constant in (q, ∞)
$\tilde{Y}(t, v)$	$\arg \max_{y \in B_{r(t)(\log t)^m}} \{\lambda(t, v, y) + t \deg y\}$
$\hat{Z}_t^{(1)}$	$\arg \max_{v \in V(T_\infty)} \psi_t(v)$
$\hat{Z}_t^{(2)}$	$\arg \max_{v \in V(T_\infty) \setminus \{\hat{Z}_t^{(1)}\}} \psi_t(v)$
Z_{B_r}	$\arg \max_{z \in B_r} \{\xi(z)\}$
\tilde{Z}_{B_r}	$\arg \max_{z \in B_r} \{\xi(z) - \deg(z)\}$
g_{B_r}	$\xi(Z_{B_r}) - \max_{z \in B_r, z \neq Z_{B_r}} \{\xi(z)\}$
\tilde{g}_{B_r}	$\xi(\tilde{Z}_{B_r}) - \deg(\tilde{Z}_{B_r}) - \max_{z \in B_r, z \neq \tilde{Z}_{B_r}} \{\xi(z) - \deg(z)\}$
j_t	$t^{\frac{d+\varepsilon}{\beta}+1} r(t)$
$\Gamma_t^{(i)}$	$\left\{ v \in V(T_\infty) : v - \hat{Z}_t^{(i)} + \min\{ v , \hat{Z}_t^{(i)} \} \leq (1 + (\log t)^{-z}) \hat{Z}_t^{(i)} \right\}$
Λ_t	$\Gamma_t^{(1)} \cup \Gamma_t^{(2)}$
Ω_t	$\{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$
$\Gamma_{x,t}^{(i)}$	the set of all paths from x to $\hat{Z}_t^{(i)}$
$\gamma_{x,t}^{(i)}$	the direct path from x to $\hat{Z}_t^{(i)}$
$ \gamma $	the length (number of vertices) of the path γ
$\pi(X_{[0,t]})$	the path that consists of all the steps taken by the random walk $(X_s)_{s \geq 0}$ between times 0 and t .
$\bar{\psi}_t(v)$	$\xi(v) - \frac{ v }{t} \log \left(\frac{ v }{t} \right)$

2 Critical Galton-Watson trees conditioned to survive

We assume that the reader is familiar with the definition of a Galton-Watson tree with offspring distribution μ , and simply recall that such a Galton-Watson tree can be formed from a single individual who reproduces with offspring number distributed according to μ , and that individuals in subsequent generations continue to reproduce independently with offspring distribution μ , ad infinitum (or until the process dies out). We refer to [AD15] for further background and definitions.

In this paper, we will restrict to the case of *critical* GW trees where $\sum_{k=0}^{\infty} k\mu_k = 1$. We let T denote such a GW tree, and O its root. For a given vertex $v \in T$, its offspring number k_v is equal to the number of children under the natural genealogical structure (i.e. $k_v = \deg v - 1$). Additionally, we will denote by T_v the subtree containing v and all its descendants under this genealogical structure, rooted at v .

In order to consider the long-time behaviour of the PAM on a Galton-Watson tree, it is natural to restrict to the case of infinite trees. A natural candidate for such a tree would be a supercritical Galton-Watson tree; however, in the case of a Pareto potential, the exponential volume growth in a supercritical Galton-Watson tree causes the solution of the PAM to blow up, almost surely; hence the restriction to critical Galton-Watson trees. Although a critical Galton-Watson tree is known to have zero probability of survival, Kesten [Kes86] gave a construction of a critical Galton-Watson tree conditioned to survive, and showed that it arises as the local limit of finite Galton-Watson trees conditioned to have large height [Kes86, Lemma 1.14]. We give his construction below.

Definition 2.1. Let $\mu = (\mu_n)_{n \geq 0}$ be a critical offspring distribution. Define its size-biased version μ^* by

$$\mu_n^* = n\mu_n$$

for all $n \geq 0$. The Kesten's tree T_∞ associated to the probability distribution μ is a two-type Galton-Watson tree with the following properties:

- Each individual is of one of two types: normal or special.
- The root of T_∞ is special.
- A normal individual produces normal individuals according to μ .

- A special individual produces individuals according to the size-biased distribution μ^* . One of them is chosen uniformly at random. This individual is of type special, the rest of the produced individuals are of type normal.

We let \mathbf{P} denote the law of T_∞ .

We let \mathbb{T}_∞ denote the set of rooted trees with a single (one-ended) infinite path to infinity. Note that, \mathbf{P} -almost surely, $T_\infty \in \mathbb{T}_\infty$, since the special vertices form a unique one-ended infinite backbone of T_∞ . We let s_0, s_1, s_2, \dots denote the ordered vertices of the backbone, so that s_0 is the root, and s_n is at distance n from the root. Moreover, each of the subtrees emanating from normal children of the backbone vertices are independent unconditioned Galton-Watson trees with offspring distribution μ .

2.1 Properties of T_∞

In this section we give some properties of T_∞ that we will use later in the paper, mainly regarding its volume growth. Since T_∞ (\mathbf{P} -almost surely) consists of a collection of finite Galton-Watson trees grafted to its infinite backbone, it will also be useful to state some bounds for an unconditioned Galton-Watson tree with the same offspring distribution μ , which we denote by T .

Lemma 2.2 ([Kor12, Lemma 1.11] and [Sla68, Theorem 2]). *Under Assumption 1.1, there exist constants $c, c' \in (0, \infty)$ such that, as $n \rightarrow \infty$,*

$$\begin{aligned}\mathbf{P}(\#T \geq n) &\sim cn^{\frac{-1}{\beta}} \\ \mathbf{P}(\text{Height}(T) \geq n) &\sim c'n^{\frac{-1}{\beta-1}}.\end{aligned}$$

Proposition 2.3. (i) [CK08, Proposition 2.5]. *Under Assumption 1.1, for any $\varepsilon > 0$ there exists $C < \infty$ such that $\mathbf{P}(\#B_r \geq \lambda r^{\frac{\beta}{\beta-1}}) \leq C\lambda^{-(\beta-1-\varepsilon)}$ for all $r, \lambda \geq 1$.*

(ii) *Under Assumption 1.3, for any $K < \infty$ there exists $C < \infty$ such that $\mathbf{P}(\#B_r \geq \lambda r^2) \leq C\lambda^{-K}$ for all $r, \lambda \geq 1$.*

(iii) [Arc20, Section 8.1]. *Under Assumption 1.1 there exist $c, C \in (0, \infty)$ such that*

$$\mathbf{P}(\#B_r \leq \lambda^{-1} r^{\frac{\beta}{\beta-1}}) \leq Ce^{-c\lambda^{\frac{\beta-1}{\beta}}} \text{ for all } r, \lambda \geq 1.$$

Proof. We just prove (ii). Take some $\varepsilon > 0$ and set $a = \frac{1}{2} - 3\varepsilon$ and $b = \frac{1}{2} - 2\varepsilon$. We bound B_r by considering all subtrees attached to the backbone within distance r of the root. We say that a subtree is “tall” if it has height exceeding $r\lambda^b$. We start by identifying all tall subtrees attached to the backbone. If a subtree is tall, we can similarly decompose this subtree into subsubtrees by decomposing along its (leftmost) path of maximal height, which we call its spine, and again identify all tall subsubtrees that are attached somewhere within distance r of the root of the subtree. We can then keep repeating this process on tall subsubtrees in an inductive way until we eventually end with something as in Figure 1.

This collection of tall subtrees can be indexed by a different branching process whereby the offspring of a given tall subtree correspond to the tall subtrees attached to its spine (so in Figure 1, different colours represent subsequent generations). We let the total number of tall subtrees (including the original tree T_∞) be N . Then N is an $O(1)$ random variable and moreover we can control $\#B_r$ by controlling the masses of all the non-tall subtrees in B_r .

Given a tall subtree T , we define its r -spine to be the collection of vertices in the spine of T that also lie within distance r of the root of T , and let $Sp_T(r)$ denote the set offspring of vertices in the r -spine of T . By Definition 2.1 and [GK98, Lemma 2.1] the offspring tails on the r -spine are at most size-biased, so under Assumption 1.3 they still decay super-polynomially. Consequently, for any $z > 0$ we have that $\mathbf{P}(\#Sp_T(r) \geq r\lambda^a) \leq c_z\lambda^{-az}$. In particular, it follows from a union bound that for the first λ^ε tall subtrees present in Figure 1 (say ordered in a breadth-first order), the probability that one of them has $\#Sp_T(r) \geq r\lambda^a$ is at most $c_z\lambda^{\varepsilon-az}$.

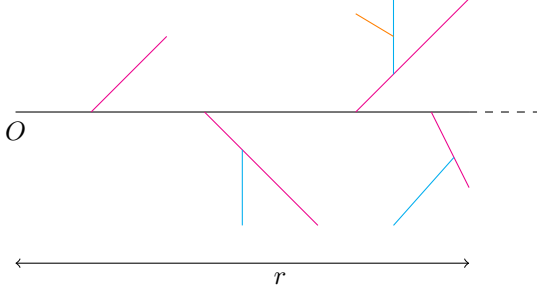


Figure 1: Spines of tall subtrees

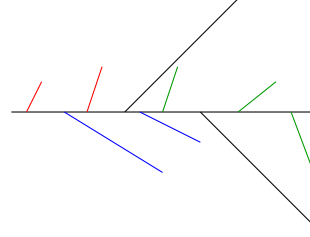


Figure 2: One tall subtree T

Additionally, if $\#Sp_T(r) < r\lambda^a$, then the number of tall subtrees attached to the r -spine of T is stochastically dominated by a $\text{Binomial}(r\lambda^a, cr^{-1}\lambda^{-b})$ random variable by Lemma 2.2. Consequently, it follows from the main theorem of [Dwa69] applied to the indexing branching process that, letting $GW_{\text{Binomial}(n, \tilde{p})}$ denote the total progeny of a Galton-Watson tree with $\text{Binomial}(n, \tilde{p})$ offspring distribution,

$$\begin{aligned} \mathbf{P}(N \geq \lambda^\varepsilon) &\leq \mathbf{P}(\text{one of the first } \lambda^\varepsilon \text{ tall subtrees has } \#Sp_T(r) \geq r\lambda^a) \\ &\quad + \mathbf{P}(N \geq \lambda^\varepsilon, \text{none of the first } \lambda^\varepsilon \text{ tall subtrees has } \#Sp_T(r) \geq r\lambda^a) \\ &\leq c_z \lambda^{\varepsilon-az} + \mathbf{P}(GW_{\text{Binomial}(r\lambda^a, cr^{-1}\lambda^{-b})} \geq \lambda^\varepsilon) \\ &\leq c_z \lambda^{\varepsilon-az} + e^{-c\lambda^\varepsilon}. \end{aligned} \quad (8)$$

(e.g. see [Arc21, Proposition 5.1] for a similar proof). Also let M denote the number of subtrees attached to the r -spine of some tall subtree, with volume exceeding $r^2\lambda^{2b+2\varepsilon}$ but with height less than $r\lambda^b$. On the event $N < \lambda^\varepsilon$ and $\#Sp_T(r) < r\lambda^a$ for all tall subtrees, we have from [Kor17, page 5] that M is stochastically dominated by a $\text{Binomial}(r\lambda^{a+\varepsilon}, cr^{-1}\lambda^{-b-\varepsilon}e^{-c\lambda^\varepsilon})$ random variable, so that by Markov's inequality,

$$\mathbf{P}(M > 0) \leq Ce^{-c\lambda^\varepsilon}. \quad (9)$$

We now look at a single tall subtree T . Its contribution to $\#B_r$ is equal to the sum of the volumes of the masses of small subtrees attached to its r -spine, plus r (which is the volume of its r -spine). Each of these small subtrees can be sorted into groups whereby two small subtrees are in the same group precisely when there are no tall subtrees between them in the contour exploration of T . These groups are indicated by colours in Figure 2. The number of groups is equal to $n+1$, where n is the number of tall subtrees attached to T . Therefore, if we sum over all tall subtrees T , the total number of such groups is at most $2N$. Label the groups as X_1, X_2, \dots, X_{2N} . Moreover, by Lemma 2.2 the total volume of one single group falls into the framework of [Arc20, Lemma A.1] with $\beta = \frac{1}{2}$ there. We deduce that, on the event $N < \lambda^\varepsilon$,

$$\mathbf{P}\left(\sum_{i \leq 2N} \#X_i \geq 2r^2\lambda^{2b+4\varepsilon}\right) \leq \mathbf{P}(\exists i \leq 2N : \#X_i \geq r^2\lambda^{2b+3\varepsilon}) \leq 2Ne^{-\lambda^\varepsilon} \leq 2\lambda^\varepsilon e^{-\lambda^\varepsilon}. \quad (10)$$

Therefore, since $\#B_r \leq \sum_{i \leq 2N} \#X_i + Nr$, we have from a union bound on all the events described in (8), (9) and (10) above that

$$\mathbf{P}(\#B_r \geq 3r^2\lambda) = \mathbf{P}(\#B_r \geq 3r^2\lambda^{2b+4\varepsilon}) \leq 3c_z \lambda^{\varepsilon - (\frac{1}{2} - 3\varepsilon)z}.$$

Therefore, choosing $z > 2(K+1)$ and $\varepsilon < \frac{1}{4(z \vee 1)}$ gives the stated result. \square

By applying Borel-Cantelli along the sequence $r_n = 2^n$, taking $\lambda_n = \delta(\log r_n)^{\frac{1+\varepsilon}{\beta-1}}$ (and similarly for the other bounds), using monotonicity of volumes and then taking $\delta \downarrow 0$, we deduce the following result.

Corollary 2.4. *Set $d := \frac{\beta}{\beta-1}$. Under Assumption 1.1, \mathbf{P} -almost surely for any $\varepsilon > 0$ it holds that:*

$$\limsup_{r \rightarrow \infty} \frac{\#B_r}{r^d (\log r)^{\frac{1+\varepsilon}{\beta-1}}} = 0, \quad \liminf_{r \rightarrow \infty} \frac{\#B_r}{r^d (\log \log r)^{-(d+\varepsilon)}} = \infty.$$

Under Assumption 1.3, \mathbf{P} -almost surely for any $\varepsilon > 0$ it holds that:

$$\limsup_{r \rightarrow \infty} \frac{\#B_r}{r^2(\log r)^\varepsilon} = 0, \quad \liminf_{r \rightarrow \infty} \frac{\#B_r}{r^2(\log \log r)^{-(2+\varepsilon)}} = \infty.$$

We will also need the following result on the largest degree in a ball.

Lemma 2.5. (i) For any $\varepsilon > 0$, under Assumption 1.1 it holds eventually \mathbf{P} -almost surely that

$$D_r := \sup_{v \in B_r} \deg(v) \leq r^{\frac{d(1+\varepsilon)}{\beta-1}} = r^{\frac{1+\varepsilon}{\beta-1}}.$$

(ii) Under Assumption 1.3 we have for any $\varepsilon > 0$ that eventually \mathbf{P} -almost surely

$$D_r \leq r^\varepsilon.$$

Proof. (i) Note that there are at most r backbone vertices in B_r with the size-biased offspring tail bound, and all other vertices have the original offspring tail bounds, independently of each other (by Definition 2.1). By Corollary 2.4, \mathbf{P} -almost surely for all sufficiently large r we can compute

$$\begin{aligned} \mathbf{P}\left(D_r > 2^{-\frac{d(1+\varepsilon)}{\beta}} r^{\frac{d(1+\varepsilon)}{\beta}}\right) &\leq r^d (\log r)^{\frac{1+\varepsilon}{\beta-1}} \sup_{v \in B_r \setminus \{s_0, \dots, s_r\}} \mathbf{P}(\deg v > 2^{-\frac{d(1+\varepsilon)}{\beta}} r^{\frac{d(1+\varepsilon)}{\beta}}) \\ &\quad + r \sup_{v \in \{s_0, \dots, s_r\}} \mathbf{P}(\deg v > 2^{-\frac{d(1+\varepsilon)}{\beta}} r^{\frac{d(1+\varepsilon)}{\beta}}) \\ &\leq cr^d (\log r)^{\frac{1+\varepsilon}{\beta-1}} r^{-d-d\varepsilon} + crr^{-\frac{d(1+\varepsilon)(\beta-1)}{\beta}} \\ &\leq r^{-\varepsilon/2}. \end{aligned}$$

This probability is summable along the sequence $r_n = 2^n$ and thus by Borel-Cantelli along this sequence it holds eventually \mathbf{P} -almost surely that $\sup_{v \in B_{r_n}} \deg(v) \leq 2^{-\frac{d(1+\varepsilon)}{\beta}} r_n^{\frac{d(1+\varepsilon)}{\beta}}$. Then note for $r \in [r_n, r_{n+1}]$ we have by monotonicity that $\sup_{v \in B_r} \deg(v) \leq 2^{-\frac{d(1+\varepsilon)}{\beta}} r^{\frac{d(1+\varepsilon)}{\beta}} \leq r^{\frac{d(1+\varepsilon)n}{\beta}}$.

(ii) Fix $\varepsilon \in (0, 1)$. Since the offspring distribution tails decay super-polynomially under Assumption 1.3, the same holds for the size-biased offspring tails, so we certainly have that $\sup_{v \in B_r} \mathbf{P}(\deg v > x) = o(x^{-(2/\varepsilon+1)})$ as $x \rightarrow \infty$. The proof then proceeds exactly as in part (i). \square

Given $v \in B_r, n \geq |v|$, we let $N_r(n, v)$ denote the number of distinct paths starting at O and passing through v that are fully contained in B_r and have length exactly n , and set

$$\eta_r(n, v) := \log N_r(n, v).$$

The previous lemma allows us to bound $\eta_r(n, v)$.

Lemma 2.6. Under Assumption 1.1, for all $\varepsilon > 0$, eventually almost surely, for all $v \in B_r$ we have that

$$\eta_r(n, v) \leq (n - |v|) \left(\frac{d + \varepsilon}{2\beta} (\log r) + \log \left(\frac{n}{n - |v|} \right) \right).$$

Under Assumption 1.3, for all $\varepsilon > 0$, eventually almost surely, for all $v \in B_r$ we have that

$$\eta_r(n, v) \leq (n - |v|) \left(\varepsilon \log r + \log \left(\frac{n}{n - |v|} \right) \right).$$

Proof. By counting deviations from the direct path from O to v , and letting $D_r = \sup_{v \in B_r} \deg v$ we have that

$$N_r(n, v) \leq \binom{n}{n-|v|} D_r^{\frac{n-|v|}{2}}.$$

The factor of $\frac{1}{2}$ appears since if we cross one edge not on the direct path we must return to the direct path by the same edge, so this reduces our number of choices by a factor of 2. Applying Stirling's formula and the bound on D_r from Lemma 2.5(i) or 2.5(ii) then gives the results. For example, under Assumption 1.3 we get

$$\eta_r(n, v) \leq \log \left(\binom{n}{n-|v|} (r^\varepsilon)^{\frac{n-|v|}{2}} \right) \leq \log \left(\left(\frac{ne}{n-|v|} \right)^{n-|v|} (r^\varepsilon)^{\frac{n-|v|}{2}} \right) \leq (n-|v|) \left(\varepsilon \log r + \log \left(\frac{n}{n-|v|} \right) \right).$$

□

Remark 2.7. As $n \rightarrow \infty$, it is not possible to do much better than the result in Lemma 2.6 under Assumption 1.1, since the maximal degree of a vertex in B_r grows polynomially in r , and is specifically of order $r^{\frac{d}{\beta}}$ in the infinite variance case. Therefore one can really count paths that dart directly to this vertex of high degree, then alternate between this vertex and its neighbours for as long as possible, and finally dart directly to v at the last possible moment. Counting these paths gives a lower bound asymptotic to $\frac{d-\varepsilon}{2\beta} (n-|v|)(\log r)$ (and we will mainly use the result with $\frac{n}{n-|v|} \ll r$).

Lemma 2.8. Let $\tilde{B}_r = \{v \in B_r : \deg v \leq 4\}$. Then, under Assumption 1.1, for any $\varepsilon > 0$ there exist $C, c \in (0, \infty)$ such that

$$\mathbf{P}(\#\tilde{B}_r \leq r^d \lambda^{-1}) \leq C e^{-c\lambda^{\frac{\beta-1}{\beta}}} + C(r\lambda^{-1})^{-(\beta-1-\varepsilon)}.$$

Consequently, \mathbf{P} -almost surely,

$$\liminf_{r \rightarrow \infty} \frac{\#\tilde{B}_r}{r^d (\log \log r)^{-(d+\varepsilon)}} > 0.$$

Proof. In any finite tree at least half of the vertices have at most degree 4. Taking care of the boundary vertices in B_r this yields

$$\#\tilde{B}_r \geq \frac{1}{2} \#B_r - Z_r^*,$$

where Z_r^* is the generation size at level r . Thus, invoking Proposition 2.3(iii) and [CK08, Proposition 2.2] we get that

$$\begin{aligned} \mathbf{P}(\#\tilde{B}_r \leq r^d \lambda^{-1}) &\leq \mathbf{P}\left(\frac{1}{2} \#B_r - Z_r^* \leq r^d \lambda^{-1}\right) \leq \mathbf{P}(\#B_r \leq 4r^d \lambda^{-1}) + \mathbf{P}(Z_r^* \geq r^d \lambda^{-1}) \\ &\leq C e^{-c\lambda^{\frac{\beta-1}{\beta}}} + C(r\lambda^{-1})^{-(\beta-1-\varepsilon)}. \end{aligned}$$

The second claim follows by Borel-Cantelli exactly as in Corollary 2.4. □

We will also use the following result in the random walk estimates.

Lemma 2.9. Let Γ_r be the set of all direct paths with both endpoints in B_r , i.e. $\Gamma_r = \{[[u, v]] : u, v \in B_r\}$. Under Assumption 1.1 there exist deterministic constants $B, \tilde{B} < \infty$ such that \mathbf{P} -almost surely,

$$\sup_{\gamma \in \Gamma_r} \left\{ \sum_{v \in \gamma} \log(\deg v) - \tilde{B}|\gamma| - B \log r \right\} \leq 0$$

for all sufficiently large r , where $|\gamma|$ denotes the number of vertices contained in γ .

Proof. First note that, by breaking paths at the most recent common ancestor, it is sufficient to prove the result for the set $\Gamma'_r = \{[u, v] : u, v \in B_r, u \preceq v\}$.

Now let $\varepsilon \in (0, \beta - 1)$, define $c := \beta - 1 - \varepsilon > 0$ and choose $B > \frac{1}{c} \left(2 \frac{\beta + \varepsilon}{\beta - 1} + \varepsilon \right)$. Let $\gamma \in \Gamma'_r$, with endpoints u and v such that $u \preceq v$. Note that the degrees of the ancestors of v can be stochastically dominated by an independent set of size-biased random variables by the same logic as in [GK98, Equation (1)], and such a size-biased random variable X satisfies $\mathbf{E}[X^c] < \infty$. We set $A := \log \mathbf{E}[X^c] < \infty$. Given γ , we also define $\lambda_{\gamma, r} := \frac{B \log r}{|\gamma|} + \frac{A}{c}$ and $v_r := r^{\frac{\beta + \varepsilon}{\beta - 1}}$. With this choice we calculate using a Chernoff bound that

$$\mathbf{P} \left(\sum_{v \in \gamma} \log(\deg v) \geq \lambda_{\gamma, r} |\gamma| \right) \leq \mathbf{E} \left[\exp \left(c \sum_{v \in \gamma} \log(\deg v) \right) \right] \exp(-c \lambda_{\gamma, r} |\gamma|) \leq \exp(A |\gamma| - c \lambda_{\gamma, r} |\gamma|) = r^{-cB}.$$

Note also that $|\Gamma'_r| \leq (\#B_r)^2$ since we are only bounding direct paths. Applying a union bound over all $\gamma \in \Gamma_r$ and Proposition 2.3(i) therefore yields (for all sufficiently large r) that

$$\begin{aligned} \mathbf{P} \left(\sup_{\gamma \in \Gamma'_r} \left\{ \sum_{v \in \gamma} \log(\deg v) - \lambda_{\gamma, r} |\gamma| \right\} \geq 0 \right) &\leq \mathbf{P}(\#B_r \geq v_r) + v_r^2 \mathbf{P} \left(\sup_{\gamma \in \Gamma_r} \left\{ \sum_{v \in \gamma} \log(\deg v) - \lambda_{\gamma, r} |\gamma| \right\} \geq 0 \right) \\ &\leq r^{\frac{\varepsilon}{2}} + v_r^2 r^{-cB} \leq 2r^{\frac{\varepsilon}{2}}. \end{aligned}$$

The result therefore follows by applying Borel-Cantelli along the subsequence $r_n = 2^n$, and then applying monotonicity if $r \in [2^n, 2^{n+1}]$, since $\log r = o(r)$. \square

2.2 Random walk on T_∞

Given a particular realisation of T_∞ , we let $X = ((X_t)_{t \geq 0}, \mathbb{P}_x^{T_\infty}, x \in V(T_\infty))$ be a continuous time variable speed random walk on T_∞ started at $x \in V(T_\infty)$. This means that for $v, w \in V(T_\infty)$, X jumps from v to w at rate $\mathbb{1}\{v \sim w\}$, so the holding times at vertex v are exponentially distributed random variables with parameter $\deg v$. In particular, the mean holding time at vertex v is $\frac{1}{\deg v}$.

In the case where X is started from the root O of T_∞ , we just write \mathbb{P} for the law of X .

Let $A \subset V(T_\infty)$. We denote the first exit time from A as τ_A , i.e. $\tau_A = \inf\{t \geq 0 : X_t \notin A\}$. Given a vertex $v \in V(T_\infty)$, we also let $H_v = \inf\{s \geq 0 : X_s = v\}$ denote the first hitting time of v .

The next proposition is actually a result about random walks on deterministic trees.

Proposition 2.10. *For \mathbf{P} -almost every realisation of T_∞ , the following holds.*

(i) *For all $v \in V(T_\infty)$ with $|v| \geq 3$,*

$$\mathbb{P}(H_v \leq t) \leq \exp\{-|v|([\log |v| - \log t - 1] \vee 0)\}.$$

(ii) *For all $v \in V(T_\infty), t > 0$,*

$$\mathbb{P}(H_v \leq t) \geq \left(\prod_{u \prec v} \frac{1}{\deg(u)} \right) e^{-|v|([\log |v| - \log t - 1] \vee 0) - O(t + \frac{|v|}{t})}.$$

Proof. (i) Set $r = |v|$. Let $O = v_0 \prec v_1 \prec \dots \prec v_r = v$ denote the ordered ancestors of v . To reach v from v_0 , the random walk must first pass through each of the points v_1, \dots, v_{r-1} . When it reaches the point v_i , the time to jump to v_{i+1} will stochastically dominate an $\text{Exp}(1)$ random variable (since it can only reach v_{i+1} through one edge which rings at rate 1). The time delay for the random walk to reach v will thus be greater than $\sum_{u \prec v} E_u$, where $(E_u)_{u \prec v}$ is a sequence of i.i.d. $\text{Exp}(1)$ random variables. Hence, using Stirling's formula we obtain

$$\begin{aligned}
\mathbb{P}(H_v \leq t) &\leq \mathbb{P}\left(\sum_{u \prec v} E_u \leq t\right) = \mathbb{P}(\text{Poi}(t) \geq r) = e^{-t} \left(\frac{t^r}{r!} + \sum_{i=r+1}^{\infty} \frac{t^i}{i!}\right) \leq e^{-t} \frac{t^r}{r!} \left(1 + \sum_{i=1}^{\infty} \frac{t^i}{i!}\right) \\
&\leq \frac{t^r e^{-t} e}{\sqrt{2\pi} r^{r+1/2} e^{-r}} \\
&\leq \exp\{-r[\log r - \log t - 1]\}.
\end{aligned}$$

(ii) We lower bound this by the probability of going directly to v in time t . Accordingly, again let $O = v_0 \prec v_1 \prec \dots \prec v_r = v$ denote the ordered ancestors of v , but this time let $(E_i)_{i=0}^{r-1}$ be independent random variables, where $E_i \sim \text{Exp}(\deg(v_i))$ for each $i = 0, \dots, r-1$. Also, recall that $e^{-x} + e^{-x^{-1}} \leq \frac{1}{1+x} + \frac{1}{1+x^{-1}} = 1$ for all $x > 0$.

Then, if $r(\log 2) > 3t$, since all vertices except perhaps v_0 have degree at least 2, we have (by Stirling's formula) that for all sufficiently large r

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=0}^{r-1} E_i \leq t\right) &\geq \mathbb{P}(E_0 \leq 1) \mathbb{P}\left(\sum_{i=1}^{r-1} E_i \leq t-1\right) \\
&\geq (1 - e^{-1}) \mathbb{P}(\text{Poi}(2(t-1)) \geq r-1) \geq (1 - e^{-1}) \frac{e^{-2(t-1)} (2t-2)^{(r-1)}}{(r-1)!} \\
&\geq \exp\{-r(\log r - \log t - 1) - O\left(\log(r \vee t) + \frac{r}{t}\right) + (r+1) \log 2 - (2t+1)\}.
\end{aligned} \tag{11}$$

□

We first use this to obtain the following almost sure bound on exit times on T_∞ .

Corollary 2.11. *Under Assumption 1.1, \mathbf{P} -almost surely, we have for all sufficiently large r that*

$$\mathbb{P}(\tau_{B_{r-1}} \leq t) \leq \exp\left\{-\frac{r}{5} \log\left(\frac{r}{et}\right)\right\}$$

for all $t > 0$.

Proof. Let us define for $r > 0$ the set

$$C_r := \left\{v \in V(T_\infty) : |v| = \frac{r}{2}, \exists u \in V(T_\infty) : |u| = r, v \prec u\right\}, \tag{12}$$

that is, the set of vertices in generation $\frac{r}{2}$ that have a descendant in generation r . We will first derive an upper bound for the volume of this set. Let Z_n^* denote the size of the n^{th} generation of T_∞ and let v_i , $i = 1, \dots, Z_n^*$ denote the vertices of the n^{th} generation. Furthermore, let T_{v_i} denote the subtree emanating from v_i . Then

$$\#C_r = \sum_{w \in \{v_1, \dots, v_{Z_{r/2}^*}\}} \mathbf{1}\{\text{Height}(T_w) \geq r/2\} = 1 + \sum_{w \in \{v_1, \dots, v_{Z_{r/2}^*}\} \setminus \{s_{r/2}\}} \mathbf{1}\{\text{Height}(T_w) \geq r/2\}.$$

Recall from Lemma 2.2 that there exists $c < \infty$ such that for all $v_i \neq s_{r/2}$, $\mathbf{P}(\text{Height}(T_{v_i}) \geq x) \sim cx^{\frac{-1}{\beta-1}}$. Consequently $\#C_r - 1$ is stochastically dominated by a $\text{Binomial}(Z_{r/2}^* - 1, c'r^{-\frac{1}{\beta-1}})$ random variable. Hence, for any $\varepsilon > 0$, setting $c_2 = \frac{1+\varepsilon}{\beta-1-\varepsilon}$ and $c_1 = c_2 + \varepsilon$ we have that

$$\mathbf{P}(\#C_r \geq (\log r)^{c_1}) \leq \mathbf{P}(Z_{r/2}^* \geq r^{\frac{1}{\beta-1}} (\log r)^{c_2}) + \mathbf{P}(X \geq (\log r)^{c_1} - 1), \tag{13}$$

where $X \sim \text{Binom}(r^{\frac{1}{\beta-1}} (\log r)^{c_2} - 1, c'r^{-\frac{1}{\beta-1}})$. We have by [CK08, Proposition 2.2] that there exists $c_3 < \infty$, depending on ε , such that

$$\mathbf{P}(Z_{r/2}^* \geq r^{\frac{1}{\beta-1}} (\log r)^{c_2}) \leq c_3 (\log r)^{-c_2(\beta-1-\varepsilon)} = c_3 (\log r)^{-(1+\varepsilon)}, \tag{14}$$

for some constant $c_3 > 0$. Furthermore, since $c_1 > c_2$, by a Chernoff bound we have

$$\begin{aligned} \mathbf{P}(X \geq (\log r)^{c_1} - 1) &\leq \frac{\mathbf{E}[e^X]}{e^{((\log r)^{c_1} - 1)}} \leq \frac{\exp((r^{\frac{1}{\beta-1}}(\log r)^{c_2} - 1)c'r^{-\frac{1}{\beta-1}}(e-1))}{\exp(((\log r)^{c_1} - 1))} \\ &\leq \exp(c'(\log r)^{c_2}e - (\log r)^{c_1} + 1). \end{aligned} \quad (15)$$

Combining (14) and (15) with (13), we deduce that for any $\varepsilon > 0$, there exists $c_4 < \infty$ such that

$$\mathbf{P}(\#C_r \geq (\log r)^{\frac{1+\varepsilon}{\beta-1}}) \leq c_4(\log r)^{-(1+\varepsilon)}.$$

For $r_n = 2^n$ this probability is summable over n and thus by Borel-Cantelli \mathbf{P} -almost surely for n large enough $\#C_{r_n} \leq (\log r_n)^{c_1}$.

Now, since the random walk needs to visit a vertex in C_r before it can exit the ball B_{r-1} , we have by a union bound and Proposition 2.10(i) that

$$\mathbb{P}(\tau_{B_{r-1}} \leq t) \leq \mathbb{P}(\exists v \in C_r : H_v \leq t) \leq \sum_{v \in C_r} \mathbb{P}(H_v \leq t) \leq \#C_r \exp\left\{-\frac{r}{2} \log\left(\frac{r}{2et}\right)\right\}.$$

Therefore, \mathbf{P} -almost surely along the subsequence r_n , for all n sufficiently large it holds that

$$\mathbb{P}(\tau_{B_{r_n-1}} \leq t) \leq (\log r_n)^{c_1} \exp\left\{-\frac{r_n}{2} \log\left(\frac{r_n}{2et}\right)\right\} = \exp\left\{-\frac{r_n}{2} \log\left(\frac{r_n}{2et}\right) + c_1 \log(\log(r_n))\right\}.$$

Consequently, for $r \in [r_n, r_{n+1}]$ sufficiently large:

$$\mathbb{P}(\tau_{B_{r-1}} \leq t) \leq \mathbb{P}(\tau_{B_{r_n}} \leq t) \leq \exp\left\{-\frac{r_n}{2} \log\left(\frac{r_n}{2et}\right) + c_1 \log(\log(r_n))\right\} \leq \exp\left\{-\frac{r}{5} \log\left(\frac{r}{et}\right)\right\}.$$

□

We now give a high probability version of the preceding result.

Lemma 2.12. *Take any $\tilde{f}, \tilde{p} > 0$. Under Assumption 1.1, we have with high \mathbf{P} -probability as $t \rightarrow \infty$, that for all $r \in [r(t)(\log t)^{-\tilde{f}}, r(t)(\log t)^{\tilde{p}}]$,*

$$\mathbb{P}(\tau_{B_{r-1}} \leq t) \leq \exp\left\{-r \log\left(\frac{r}{et}\right) + o(r(t))\right\}.$$

Proof. We follow a refined version of the previous proof, this time with several steps.

STEP 1. Choose some $\varepsilon > 0$ and $0 < \delta < 1$. Let $m > \tilde{p} + 1$, $k > \frac{m+\delta}{\beta-1-\varepsilon} + 2\varepsilon$ and set $K := k + \frac{m}{\beta-1}$. Given $r > 0$, this time let Z_r^* denote the number of vertices in generation $\left\lceil \left(1 - \frac{1}{(\log r)^m}\right)r \right\rceil$, and let C_r denote the set of vertices in this generation that have a descendant at level r . Then, similarly to the calculations in (13)-(15) we compute

$$\begin{aligned} &\mathbf{P}(\#C_r \geq (\log r)^K) \\ &\leq \mathbf{P}\left(Z_r^* \geq r^{\frac{1}{\beta-1}}(\log r)^{k-\varepsilon}\right) + \mathbf{P}\left(\text{Binomial}\left(\left\lceil r^{\frac{1}{\beta-1}}(\log r)^{k-\varepsilon} \right\rceil, c'r^{\frac{-1}{\beta-1}}(\log r)^{\frac{m}{\beta-1}}\right) \geq (\log r)^K\right) \\ &\leq c(\log r)^{-(k-\varepsilon)(\beta-1-\varepsilon)} + \exp\{-e(\log r)^\varepsilon\} \leq c'(\log r)^{-(k-\varepsilon)(\beta-1-\varepsilon)}. \end{aligned} \quad (16)$$

Since the random walk needs to visit a vertex in C_r before it can exit the ball B_{r-1} , we can lower bound the exit time of the set B_{r-1} by the hitting time of the set C_r . For all sufficiently large r , on the event $\{\#C_r \leq (\log r)^K\}$, we can thus compute using Proposition 2.10(i) that

$$\begin{aligned} \mathbb{P}(\tau_{B_{r-1}} \leq t) &\leq \sum_{v \in C_r} \mathbb{P}(H_v \leq t) \leq \#C_r \exp\left\{-\left(1 - \frac{1}{(\log r)^m}\right)r \left[\log\left(\frac{r}{et}\right) + \log\left(1 - \frac{1}{(\log r)^m}\right)\right]\right\} \\ &\leq \exp\left\{K \log \log r - \left(1 - \frac{3}{(\log r)^m}\right)r \log\left(\frac{r}{et}\right)\right\}. \end{aligned} \quad (17)$$

STEP 2. We now define $\tilde{r}(t) := r(t)(\log t)^{-\tilde{f}}$, $\tilde{R}(t) := r(t)(\log t)^{\tilde{p}}$ and for $0 \leq n \leq N_t := \lceil \log(\tilde{R}(t))^{m+\delta} \rceil$ set

$$r_n := \left(1 - \frac{1}{(\log \tilde{R}(t))^m}\right)^n \tilde{R}(t).$$

Then $r_0 = \tilde{R}(t)$ and for t large enough we have

$$r_{N_t} \leq \exp\left(-\frac{\lceil \log(\tilde{R}(t))^{m+\delta} \rceil}{\log(\tilde{R}(t))^m}\right) \tilde{R}(t) \leq \exp\left\{\frac{-1}{2}(\log(\tilde{R}(t)))^\delta\right\} \tilde{R}(t) \leq \tilde{r}(t).$$

Therefore, using (16), (17) and a union bound, we have that

$$\begin{aligned} \mathbf{P}\left(\exists n \leq N_t : \mathbb{P}(\tau_{B_{r_n-1}} \leq t) \geq \exp\left\{K(\log \log r_n) - \left(1 - \frac{3}{(\log r_n)^m}\right) r_n \log\left(\frac{r_n}{et}\right)\right\}\right) \\ \leq \mathbf{P}(\exists n \leq N_t : \#C_{r_n} \geq (\log r_n)^K) \\ \leq \sum_{n \leq N_t} c(\log r_n)^{-(k-\varepsilon)(\beta-1-\varepsilon)} \end{aligned}$$

Applying the definition of r_n and the inequality $\log(1-x) \geq -2x$ for all sufficiently small x , this is upper bounded by

$$\begin{aligned} \sum_{n \leq N_t} c \left[\log(\tilde{R}(t)) + n \log\left(1 - \frac{1}{(\log \tilde{R}(t))^m}\right) \right]^{-(k-\varepsilon)(\beta-1-\varepsilon)} &\leq \sum_{n \leq N_t} c \left[\log(\tilde{R}(t)) - n \frac{2}{(\log \tilde{R}(t))^m} \right]^{-(k-\varepsilon)(\beta-1-\varepsilon)} \\ &\leq cN_t \left[\log \tilde{R}(t) \left(1 - \frac{4}{(\log \tilde{R}(t))^{1-\delta}}\right) \right]^{-(k-\varepsilon)(\beta-1-\varepsilon)} \\ &\leq C \left[\log(\tilde{R}(t)) \right]^{m+\delta-(k-\varepsilon)(\beta-1-\varepsilon)}. \end{aligned}$$

Since $k > \frac{m+\delta}{\beta-1-\varepsilon} + 2\varepsilon$ this probability goes to zero as $t \rightarrow \infty$, i.e. with high \mathbf{P} -probability

$$\mathbb{P}(\tau_{B_{r_n-1}} \leq t) \leq \exp\left\{K(\log \log r_n) - \left(1 - \frac{3}{(\log r_n)^m}\right) r_n \log\left(\frac{r_n}{et}\right)\right\} \quad \text{for all } n \leq N_t. \quad (18)$$

STEP 3. Moreover, on the event in (18), again using that $\log(1-x) \geq -2x$ for small x we have for all $r \in [\tilde{r}(t), \tilde{R}(t)]$ with $r \in [r_{n+1}, r_n]$ that

$$\begin{aligned} \mathbb{P}(\tau_{B_{r-1}} < t) &\leq \mathbb{P}(\tau_{B_{r_{n+1}-1}} < t) \leq \exp\left\{K(\log \log r_{n+1}) - \left(1 - \frac{3}{(\log r_{n+1})^m}\right) r_{n+1} \log\left(\frac{r_{n+1}}{et}\right)\right\} \\ &\leq \exp\left\{K \log \log r - r_n \left(1 - \frac{1}{(\log \tilde{R}(t))^m}\right) \log\left(\frac{r_{n+1}}{et}\right) + \frac{3r \log\left(\frac{r}{et}\right)}{(\log r_{n+1})^m}\right\} \\ &\leq \exp\left\{-r \log\left(\frac{r}{et}\right) + K(\log \log r) + \frac{2r \log\left(\frac{r}{et}\right)}{(\log \tilde{R}(t))^m} + \frac{6r \log\left(\frac{r}{et}\right)}{(\log r)^m}\right\}. \end{aligned}$$

Then, since $m > \tilde{p} + 1$, $r \in [\tilde{r}(t), \tilde{R}(t)]$ and $r_{n+1} \in [\frac{1}{2}\tilde{r}(t), \tilde{R}(t)]$ we have that $\log r_{n+1} \geq \frac{1}{2} \log r(t) \geq \frac{1}{2} \log t \forall n$ provided that t is sufficiently large, so it follows that

$$K(\log \log r) + \frac{2r \log\left(\frac{r}{et}\right)}{(\log \tilde{R}(t))^m} + \frac{6r \log\left(\frac{r}{et}\right)}{(\log r_{n+1})^m} = O\left(\frac{r(t)(\log t)^{\tilde{p}+1}}{(\log t)^m}\right) = o(r(t)),$$

which establishes the result. \square

2.3 Time reversal for Feynman-Kac formula on T_∞

Here we record the following well-known straightforward proposition which will be useful for manipulating the Feynman-Kac formula of (4).

Proposition 2.13 (Time reversal). *For almost every realisation of T_∞ , it holds for all $v \in V(T_\infty)$ that*

$$\mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{X_t = v\} \right] = \mathbb{E}_v \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{X_t = 0\} \right].$$

Proof. The proof is just a computation and applies to the Feynman-Kac formula on any locally finite graph. Let $\pi(X_{[0,t]})$ denote the path that consists of all the steps taken by the random walk $(X_s)_{s \geq 0}$ between times 0 and t . Note that

$$\mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{X_t = v\} \right] = \sum_{\gamma: 0 \rightarrow v} f(\gamma, t) \quad (19)$$

where, if γ denotes the path $0 = v_0, \dots, v_n = v$, then (in what follows we cancelled $\mathbb{P}(\pi(X_{[0,t]}) = \gamma) = \prod_{i=0}^{n-1} \frac{1}{\deg v_i}$ with the terms $\deg v_i$ in the density of each $(s_i)_{i=0}^{n-1}$):

$$\begin{aligned} f(\gamma, t) &= \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\pi(X_{[0,t]}) = \gamma\} \right] \\ &= \underbrace{\int_0^\infty \dots \int_0^\infty}_{\text{from 1 to } n-1} e^{(\xi(v_i) - \deg v_i) s_i} \left[\int_0^{t - \sum_{i=1}^{n-1} s_i} e^{(\xi(v_0) - \deg v_0) s_0} \mathbb{1} \left\{ s_0 > 0, \sum_{i=0}^{n-1} s_i < t \right\} ds_0 \cdot e^{(\xi(v_n) - \deg v_n)(t - \sum_{i=0}^{n-1} s_i)} \right] \prod_{i=1}^{n-1} ds_i. \end{aligned}$$

We apply the substitution $s_n = t - \sum_{i=0}^{n-1} s_i = t - \sum_{i=1}^{n-1} s_i - s_0$ to the integral over s_0 to deduce that the expression in the square brackets is equal to

$$\int_0^{t - \sum_{i=1}^n s_i} e^{(\xi(v_n) - \deg v_n) s_n} \mathbb{1} \left\{ s_n > 0, \sum_{i=1}^n s_i < t \right\} ds_n \cdot e^{(\xi(v_0) - \deg v_0)(t - \sum_{i=1}^n s_i)}.$$

In other words, we can perfectly exchange the roles played by (v_n, s_n) and (v_0, s_0) to deduce that, if $\overleftarrow{\gamma}$ is the reversal of the path γ , then $f(\gamma, t) = f(\overleftarrow{\gamma}, t)$, from which the result follows by (19). \square

3 Potential estimates

3.1 Extremal values of the potential

The following lemma bounds the maximal potential on a ball.

Lemma 3.1. *For any $\varepsilon > 0$, there exists $C < \infty$ such that for all $r > 1$ and $\lambda > 1$,*

$$\begin{aligned} P \left(\sup_{v \in B_r} \xi(v) \geq r^{d/\alpha} \lambda \right) &\leq C \lambda^{-\frac{\alpha-\varepsilon}{d}} \text{ under Assumption 1.1,} \\ P \left(\sup_{v \in B_r} \xi(v) \geq r^{2/\alpha} \lambda \right) &\leq C \lambda^{-(\alpha-\varepsilon)} \text{ under Assumption 1.3.} \end{aligned}$$

Proof. Set $p = \frac{\alpha}{\beta}$. Using Proposition 2.3(i), under Assumption 1.1 we compute

$$P \left(\exists v \in B_r : \xi(v) > r^{\frac{1}{\alpha} \frac{\beta}{\beta-1}} \lambda \right) \leq r^d \lambda^p P \left(\xi(O) > r^{\frac{d}{\alpha}} \lambda \right) + P \left(\#B_r \geq r^d \lambda^p \right) < \lambda^{p-\alpha} + c \lambda^{-p(\beta-1)(1-\varepsilon)} \leq C \lambda^{-\frac{\alpha(1-\varepsilon)}{d}}.$$

Under Assumption 1.3, we instead take $p = \varepsilon$ and choose $K > \alpha \varepsilon^{-1}$ in Proposition 2.3(ii) to get

$$P \left(\exists v \in B_r : \xi(v) > r^{\frac{2}{\alpha}} \lambda \right) \leq r^2 \lambda^p P \left(\xi(O) > r^{\frac{2}{\alpha}} \lambda \right) + P \left(\#B_r \geq r^2 \lambda^p \right) < \lambda^{p-\alpha} + c \lambda^{-pK} \leq C \lambda^{-(\alpha-\varepsilon)}.$$

\square

Lemma 3.2. *P-almost surely, we have for any $\varepsilon > 0$ that*

$$\lim_{r \rightarrow \infty} \frac{\sup_{v \in B_r} \xi(v)}{r^{\frac{d}{\alpha}} (\log r)^{\frac{d+\varepsilon}{\alpha}}} = 0 \text{ under Assumption 1.1,} \quad \lim_{r \rightarrow \infty} \frac{\sup_{v \in B_r} \xi(v)}{r^{\frac{2}{\alpha}} (\log r)^{\frac{1+\varepsilon}{\alpha}}} = 0 \text{ under Assumption 1.3.}$$

Proof. We give the proof under Assumption 1.1; the proof under Assumption 1.3 is identical. Let $\varepsilon > 0$ be arbitrary and set $\varepsilon' = \frac{\varepsilon}{1+\beta^2/(\beta-1)} > 0$. Using Lemma 3.1 we deduce

$$P\left(\exists v \in B_r : \xi(v) > r^{\frac{d}{\alpha}} (\log r)^{\frac{d+\varepsilon'}{\alpha}}\right) \leq (\log r)^{-1-\frac{\varepsilon'}{2d}}$$

This probability is summable along $r_n = 2^n$, and thus by Borel-Cantelli and monotonicity if $r \in [r_n, r_{n+1}]$, we deduce that *P*-almost surely

$$\lim_{n \rightarrow \infty} \frac{\sup_{v \in B_{r_n}} \xi(v)}{r_n^{\frac{d}{\alpha}} (\log r_n)^{\frac{d+\varepsilon'}{\alpha}}} \leq 1, \quad \lim_{r \rightarrow \infty} \frac{\sup_{v \in B_r} \xi(v)}{r^{\frac{d}{\alpha}} (\log r)^{\frac{d+\varepsilon'}{\alpha}}} \leq 2^{\frac{d+1}{\alpha}}.$$

Since $\varepsilon' < \varepsilon$ this gives the result. \square

Recall that $\tilde{B}_r = \{v \in B_r : \deg(v) \leq 4\}$. Given $i > 0$, let $\xi_r^{(i)}$ denote the i^{th} highest value of ξ on B_r and $\tilde{\xi}_r^{(i)}$ denote the i^{th} highest value of ξ on \tilde{B}_r .

Lemma 3.3. *P-almost surely under Assumption 1.1, for $i = 1, 2, 3$ and any $\varepsilon > 0$,*

$$\liminf_{r \rightarrow \infty} \left(\frac{\xi_r^{(i)}}{r^{\frac{d}{\alpha}} (\log \log r)^{\frac{-(2\beta-1+\varepsilon)}{\alpha(\beta-1)}}} \right) = \infty, \quad \liminf_{r \rightarrow \infty} \left(\frac{\tilde{\xi}_r^{(i)}}{r^{\frac{d}{\alpha}} (\log \log r)^{\frac{-(2\beta-1+\varepsilon)}{\alpha(\beta-1)}}} \right) = \infty.$$

Proof. It is sufficient to prove the result for $\tilde{\xi}^{(3)}$ only. By Lemma 2.8, $\#\tilde{B}_r \geq r^d (\log \log r)^{\frac{-(\beta+\varepsilon)}{\beta-1}}$ for all sufficiently large r , **P**-almost surely. When this happens, we can stochastically dominate by the case $\#\tilde{B}_r = r^d (\log \log r)^{\frac{-(\beta+\varepsilon)}{\beta-1}}$ to deduce that

$$P\left(\tilde{\xi}_r^{(3)} \leq r^{\frac{d}{\alpha}} \lambda^{-1}\right) \leq \binom{r^d (\log \log r)^{\frac{-(\beta+\varepsilon)}{\beta-1}}}{3} (1 - r^{-d} \lambda^\alpha)^{r^d (\log \log r)^{\frac{-(\beta+\varepsilon)}{\beta-1}} - 3} (r^{-d} \lambda^\alpha)^3 \leq C e^{-c(\log \log r)^{\frac{-(\beta+\varepsilon)}{\beta-1}} \lambda^\alpha}. \quad (20)$$

Setting $\lambda_r = 2c^{-1}(\log \log r)^{\frac{2\beta-1+\varepsilon}{\alpha(\beta-1)}}$ gives the result using Borel-Cantelli along the sequence $r_n = 2^n$ and using monotonicity as in Lemma 3.2. \square

3.2 Sites of high potential

It will also be useful to control some properties of the sites with atypically high potential.

Recall from the introduction that $q = \frac{d}{\alpha-d}$. Now we also set $p = q + 2, z = \frac{d\alpha}{\alpha-d}$ and $C = \frac{pd}{\alpha} + z + 1$. Let $R(t) := r(t)(\log t)^p$. We define the set

$$F_t := \{v \in B_{R(t)} : \xi(v) \geq R(t)^{\frac{d}{\alpha}} (\log t)^{-C} = a(t)(\log t)^{-(z+1)}\}. \quad (21)$$

Also let

$$E_t := \{v \in B_{R(t)} \setminus F_t : \exists z \in F_t \text{ such that } \xi(z) - \deg(z) \leq \xi(v) - \deg(v)\}.$$

We now prove several estimates regarding these sets.

Proposition 3.4. *Under Assumption 1.1 it holds that:*

- (i) *There exists $D < \infty$ such that $\#F_t \leq (\log t)^D$ eventually *P*-almost surely.*

(ii) There exists $B < \infty$ such that $\sup_{v \in E_t \cup F_t} \deg v \leq (\log t)^B$ eventually P -almost surely.

Proof. (i) Define a related set

$$\tilde{F}_t := \left\{ v \in V(B_{2R(t)}) : \xi(v) \geq \frac{1}{2} R(t)^{\frac{d}{\alpha}} (\log t)^{-C} \right\}.$$

We have

$$\#\tilde{F}_t = \sum_{v \in V(B_{2R(t)})} \mathbb{1} \left\{ \xi(v) \geq \frac{1}{2} R(t)^{d/\alpha} (\log t)^{-C} \right\}.$$

Set

$$p_t := \mathcal{P} \left(\xi(O) \geq \frac{1}{2} R(t)^{d/\alpha} (\log t)^{-C} \right) = 2^\alpha R(t)^{-d} (\log t)^{C\alpha}.$$

Since $\#V(B_{2R(t)}) \leq R(t)^d (\log R(t))^{\frac{1+\varepsilon}{\beta-1}}$ eventually \mathbf{P} -almost surely by Corollary 2.4, we (\mathbf{P} -almost surely) have that $\#\tilde{F}_t \stackrel{s.d.}{\preceq} \text{Binomial}(R(t)^d (\log R(t))^{\frac{1+\varepsilon}{\beta-1}}, p_t)$ for all sufficiently large t . In this case, using a Chernoff bound we compute

$$P(\#\tilde{F}_t \geq (\log t)^D) \leq \frac{E[\exp(\#\tilde{F}_t)]}{\exp((\log t)^D)} \leq \frac{\exp(R(t)^d (\log R(t))^{\frac{1+\varepsilon}{\beta-1}} 2^\alpha R(t)^{-d} (\log t)^{C\alpha} (e-1))}{\exp((\log t)^D)},$$

and for $D > \alpha C + \frac{1+\varepsilon}{\beta-1}$ the right hand side is for t sufficiently large upper bounded by $\exp(-C'(\log t)^D)$. Now for $t_n := 2^n$ we have

$$\sum_{n=0}^{\infty} P(\#\tilde{F}_{t_n} \geq (\log t_n)^D) < \infty,$$

and thus by Borel-Cantelli, $\#\tilde{F}_{t_n} \leq (\log t_n)^D$ eventually P -almost surely. Finally, for $t \in [t_n, t_{n+1}]$, note that $F_t \subset \tilde{F}_{t_{n+1}}$ for all sufficiently large t , which implies the result.

(ii) Since the potential is independent of T_∞ , it follows that, given T_∞ , each of the points in F_t is uniform on $B_{R(t)}$. Now note that, if T is a finite tree and U a uniform vertex of T , we have that

$$\sum_{v \in V(T)} \deg v = 2(\#V(T) - 1)$$

so that $\mathbf{E}[\deg U] \leq 2$. Therefore, letting $\deg_t(v) = \sum_{u \in V(B_{R(t)})} \mathbb{1}\{u \sim v\}$ and U_t a uniform vertex of $B_{R(t)}$, we have from Markov's inequality that

$$\mathbf{P}(\deg_t(U_t) \geq x) \leq 2x^{-1}.$$

Note that $\deg v = \deg_t v$ unless $v \in \partial B_{R(t)} \cup B_{R(t)}^c$. Therefore, if Z_r^* denotes the size of generation r , we can choose $\varepsilon > 0$ small enough that we have from [CK08, Proposition 2.2] and Proposition 2.3(iii) that

$$\begin{aligned} \mathbf{P}(\deg(U_t) \geq x) &\leq \mathbf{P}(\deg_t(U_t) \geq x) + \mathbf{P}\left(Z_{R(t)}^* \geq R(t)^{\frac{1}{\beta-1} + \varepsilon}\right) + \mathbf{P}(V(B_{R(t)}) \leq R(t)^{d-\varepsilon}) \\ &\quad + \mathbf{P}\left(U_t \in Z_{R(t)}^* \mid Z_{R(t)}^* < R(t)^{\frac{1}{\beta-1} + \varepsilon}, V(B_{R(t)}) > R(t)^{d-\varepsilon}\right) \\ &\leq 2x^{-1} + C(R(t))^{-\varepsilon(\beta-1-\varepsilon)}. \end{aligned} \tag{22}$$

Recall the set \tilde{F}_t defined in part (i), for which we proved that $\#\tilde{F}_{t_n} \leq (\log t_n)^D$ eventually almost surely for $t_n = 2^n$. Therefore, using this and (22), we can compute using a union bound that almost surely, for all sufficiently large t :

$$P\left(\sup_{v \in \tilde{F}_t} \deg v \geq (\log t)^{B'}\right) \leq (\log t)^D \left(2(\log t)^{-B'} + C(R(t))^{-\varepsilon(\beta-1-\varepsilon)}\right)$$

eventually almost surely. For $B > D + 1 + \varepsilon$, we can choose $B' \in (D + 1 + \varepsilon, B)$ and the right hand side is summable along $t_n = 2^n$; that is, by Borel-Cantelli $\sup_{v \in \tilde{F}_{t_n}} \deg v \leq (\log t_n)^{B'}$ eventually P -almost surely, and for $t \in [t_n, t_{n+1}]$

$$\sup_{v \in F_t} \deg v \leq \sup_{v \in \tilde{F}_{t_{n+1}}} \deg v \leq (\log t_{n+1})^B \leq C(\log t)^{B'} \leq (\log t)^B.$$

If $v \in E_t$, there exists $z \in F_t$ such that $\xi(v) - \deg v \geq \xi(z) - \deg z$. But also $\xi(z) \geq \xi(v)$, so that

$$\deg v \leq \xi(v) - \xi(z) + \deg z \leq \deg z,$$

which proves the claim for $v \in E_t$ as well. □

We will also need to consider the following slightly bigger set. For fixed $0 < \delta < \frac{d}{3\alpha} \frac{q}{q+1}$, we define the following set

$$G_t = G_t(\delta) := \{v \in V(B_{R(t)}) : \xi(v) \geq R(t)^{\frac{d}{\alpha} - \delta}\}. \quad (23)$$

We can prove similar results for G_t as we did for F_t .

Lemma 3.5. *Let Assumption 1.1 hold. Then, eventually P -almost surely,*

- (i) $\#G_t(\delta) \leq t^{2\alpha(q+1)\delta}$.
- (ii) G_t is disconnected, i.e. if $u \in G_t$ then for all $v \in V(B_{R(t)})$ with $v \sim u$ it holds $v \in G_t^c$. (Consequently the same is true for F_t).

Proof. (i) Fix $\delta > 0$ and define a related set

$$\tilde{G}_t := \{v \in V(B_{R(t)}) : \xi(v) \geq \frac{1}{2}R(t)^{\frac{d}{\alpha} - \delta}\}. \quad (24)$$

As in Proposition 3.4(i), we have that (\mathbf{P} -almost surely) $\#\tilde{G}_t \stackrel{s.d.}{\preceq} \text{Binomial}(R(t)^d (\log R(t))^{\frac{1+\varepsilon}{\beta-1}}, p_t)$ for all sufficiently large t , where here

$$p_t := \mathcal{P}\left(\xi(O) \geq \frac{1}{2}R(t)^{\frac{d}{\alpha} - \delta}\right) = 2^\alpha R(t)^{-d+\alpha\delta}.$$

Using a Chernoff bound we compute

$$P(\#\tilde{G}_t \geq \frac{1}{2}t^{2\alpha(q+1)\delta}) \leq \frac{E[\exp(\#\tilde{G}_t)]}{\exp(\frac{1}{2}t^{2\alpha(q+1)\delta})} \leq \frac{\exp(2^\alpha R(t)^d (\log R(t))^{\frac{1+\varepsilon}{\beta-1}} R(t)^{-d+\alpha\delta} (e-1))}{\exp(\frac{1}{2}t^{2\alpha(q+1)\delta})},$$

and the right hand side is for t sufficiently large upper bounded by $\exp(-Ct^{2\alpha(q+1)\delta})$. Now for $t_n := n$ we have

$$\sum_{n=0}^{\infty} P(\#\tilde{G}_{t_n} \geq \frac{1}{2}t_n^{2\alpha(q+1)\delta}) < \infty,$$

and thus by Borel-Cantelli $\#\tilde{G}_{t_n} \leq \frac{1}{2}t_n^{2\alpha(q+1)\delta}$ eventually P -almost surely. Finally, for $t \in [t_n, t_{n+1}]$, note that $G_t \subset \tilde{G}_{\lceil t \rceil}$ for all sufficiently large t , which implies the result.

- (ii) Fix some $v \in V(B_{R(t)})$, suppose that $u \in V(T_\infty)$ satisfies $v \sim u$. Since the potentials at different vertices are independent, we have that

$$\mathcal{P}\left(\xi(v) \wedge \xi(u) \geq \frac{1}{2^q}R(t)^{\frac{d}{\alpha} - \delta}\right) \leq c(R(t))^{-2(d-\alpha\delta)}.$$

The number of pairs of neighbouring vertices in $B_{R(t)}$ is given by the number of edges in the ball, which is upper bounded by the number of vertices, i.e. by $\#V(B_{2R(t)})$. Therefore, since $\#V(B_{2R(t)}) \leq R(t)^d (\log t)^{\frac{1+\varepsilon}{\beta-1}}$ eventually \mathbf{P} -almost surely by Corollary 2.4, we get from a union bound that for all such t , whenever $\delta < \frac{d}{2\alpha}$, we can choose $\varepsilon, \varepsilon' > 0$ small enough that

$$\begin{aligned} P\left(\exists v, u \in V(B_{R(t)}) : v \sim u, [\xi(v) \wedge \xi(u)] \geq \frac{1}{2^q} R(t)^{\frac{d}{\alpha} - \delta}\right) &\leq R(t)^d (\log t)^{\frac{1+\varepsilon}{\beta-1}} c(R(t))^{-2(d-\alpha\delta)} \\ &\leq c(r(t))^{2\alpha\delta + \varepsilon - d} \leq t^{-\varepsilon'}. \end{aligned}$$

Taking $t_n = 2^n$, and \tilde{G}_t as in (i), we obtain from Borel-Cantelli that

$$P\left(\exists (v, u) \in V(B_{2R(t_n)}) : u \sim v, u \in \tilde{G}_{t_n}, v \in \tilde{G}_{t_n} \text{ i.o.}\right) = 0.$$

We can extend to all t with G_t in place of \tilde{G}_t since $G_t \subset \tilde{G}_{2^{\lceil \log_2 t \rceil}}$ for all sufficiently large t , which establishes the result. \square

3.3 Gap between highest values

To prove Theorems 1.5 and 1.6 we will also need to control the gap between the highest values of ξ on a ball of radius r . We therefore define

$$Z_{B_r} := \arg_{z \in B_r} \max \{\xi(z)\}, \quad g_{B_r} := \xi(Z_{B_r}) - \max_{z \in B_r, z \neq Z_{B_r}} \{\xi(z)\},$$

and

$$\tilde{Z}_{B_r} := \arg_{z \in B_r} \max \{\xi(z) - \deg(z)\}, \quad \tilde{g}_{B_r} := \xi(\tilde{Z}_{B_r}) - \deg(\tilde{Z}_{B_r}) - \max_{z \in B_r, z \neq \tilde{Z}_{B_r}} \{\xi(z) - \deg(z)\}.$$

Note that, since the law of ξ is non-atomic, these are well-defined, almost surely.

Lemma 3.6. *Under Assumption 1.1, for any $\varepsilon > 0$ there P -almost surely exists $c > 0$ such that, for all $r \geq 1$,*

$$g_{B_r} \geq cr^{\frac{d}{\alpha}} (\log r)^{\frac{-(1+\varepsilon)}{\alpha}}.$$

Moreover, $g_{B_r} \geq r^{\frac{d}{\alpha}} (\log r)^{-\varepsilon}$ with high P -probability as $r \rightarrow \infty$.

Proof. We will start by proving the following: let $(X_i)_{i=1}^n$ be i.i.d. Pareto random variables with parameter α , and let $m_n = \arg \max_{1 \leq i \leq n} X_i$, $g_n = \sup_{1 \leq i \leq n} X_i - \sup_{1 \leq i \leq n, i \neq m_n} X_i$. Then

$$\mathcal{P}(g_n \leq y) \leq (ny^{-\alpha} + 1)e^{-2^{-\alpha}y^{-\alpha}(n-1)}. \quad (25)$$

Recall that the distribution and density functions for the Pareto distribution are given by $F(x) = 1 - x^{-\alpha}$ and $f(x) = \alpha x^{-(\alpha+1)}$. The maximum of X_1, \dots, X_n has density $nf(x)F(x)^{n-1}$, so that

$$\begin{aligned} \mathcal{P}(g_n \leq y) &\leq F(2y)^n + \int_{2y}^{\infty} nf(x)F(x)^{n-1} \left[1 - \left(\frac{F(x-y)}{F(x)}\right)^{n-1}\right] dx \\ &= (1 - (2y)^{-\alpha})^n + \int_{2y}^{\infty} \alpha n x^{-(\alpha+1)} [(1 - x^{-\alpha})^{n-1} - (1 - (x-y)^{-\alpha})^{n-1}] dx \\ &\leq (1 - (2y)^{-\alpha})^n + ny^{-\alpha}(1 - (2y)^{-\alpha})^{n-1}, \end{aligned}$$

which implies (25). In particular, (25) decays to zero as $y, n \rightarrow \infty$ provided $y = o(n^{\frac{1}{\alpha}})$. Also, if $y = \frac{1}{4}n^{\frac{1}{\alpha}}(\log n)^{\frac{-1}{\alpha}}$, then this probability is upper bounded by $(1 + 4^\alpha \log n)n^{-2^\alpha}$ for all sufficiently large n , which

is summable in n . Therefore, since the volumes of the sets $(B_r)_{r \geq 1}$ are strictly increasing, we deduce from Borel-Cantelli that

$$P\left(g_{B_r} \leq \frac{1}{4}(\#B_r)^{\frac{1}{\alpha}} \log(\#B_r)^{\frac{-1}{\alpha}} \text{ i.o.}\right) = 0.$$

Since it also follows from Corollary 2.4 that there exists $c > 0$ such that

$$\mathbf{P}\left(\#B_r \leq cr^{\frac{\beta}{\beta-1}} (\log \log r)^{-\frac{\beta+\varepsilon}{\beta-1}} \text{ i.o.}\right) = 0,$$

we deduce from Borel-Cantelli that for any $\varepsilon > 0$

$$g_{B_r} \geq c' r^{\frac{\beta}{\alpha(\beta-1)}} (\log r)^{\frac{-(1+\varepsilon)}{\alpha}}$$

eventually P -almost surely. The high probability result follows similarly. \square

Finally, the following lemma shows that eventually the maximisers Z_{B_r} and \tilde{Z}_{B_r} coincide.

Lemma 3.7. *Take any $\varepsilon > 0$. Under Assumption 1.1, it holds eventually P -almost surely that*

$$Z_{B_r} = \tilde{Z}_{B_r}, \quad \tilde{g}_{B_r} \geq r^{\frac{\beta}{\alpha(\beta-1)}} (\log r)^{\frac{-(1+\varepsilon)}{\alpha}}.$$

Proof. First note that it follows from Lemma 3.4(ii) and Lemma 3.6 that $2 \deg Z_{B_r} \leq g_{B_r}$ eventually almost surely. This proves the first statement, since on this event we have for all $z \in B_r$ with $z \neq Z_{B_r}$ that

$$[\xi(Z_{B_r}) - \deg(Z_{B_r})] - [\xi(z) - \deg z] = [\xi(Z_{B_r}) - \xi(z)] - [\deg(Z_{B_r}) - \deg z] \geq \frac{g_{B_r}}{2},$$

from which it follows that $\tilde{Z}_{B_r} = Z_{B_r}$ (uniquely) and $\tilde{g}_{B_r} \geq \frac{g_{B_r}}{2}$. \square

4 PAM on T_∞

It follows exactly as in [GM90, Section 2] for the \mathbb{Z}^d case that (1) has a solution if and only if $\alpha > d$. Roughly speaking, this is because, for each fixed $t > 0$, the rate of growth of the maximum potential attained on the annulus $B_{2r} \setminus B_r$ as $r \rightarrow \infty$ is less than the rate of decay for $(\frac{1}{t}$ times the log of) the probability of reaching the annulus, so the solution given by the Feynman-Kac formula (4) can't blow up. Since the volume fluctuations for B_r are at most logarithmic in r (by Proposition 2.3), and since most vertices have degrees of constant order, this does not affect the existence threshold obtained in the \mathbb{Z}^d case, and the arguments carry through with minor adaptations.

Before proving the main localisation result, we prove some asymptotics for the total mass and the location of the localisation site. We then show that the solution at any single site v can be well-approximated by considering trajectories that spend most of their time at a nearby “good” site, and then jump to v just before time t . The strategy to prove this approximation broadly follows that used to prove analogous results for the \mathbb{Z}^d case in [OR16, Section 7], though we have to work harder to control the extra randomness in T_∞ . We work under Assumption 1.1 for the whole of Section 4.

4.1 Asymptotics: proof of Theorems 1.2 and 1.7

In this subsection we prove Theorems 1.2 and 1.7. Recall ψ_t and $\hat{Z}_t^{(1)}$ as defined in (5) and (6).

Proof of Theorems 1.2 and 1.7. Upper bounds. We start with the upper bound on $U(t)$ and then on $|\hat{Z}_t^{(1)}|$. First note that by Lemma 3.1, we have for all $i \geq -1$, $\lambda \geq 1$, $t > 0$ that

$$\begin{aligned} P\left(\sup_{v \in B_{2^{i+1}r(t)\lambda}} \xi(v) \geq \frac{1 \wedge q}{8} 2^i \frac{r(t)}{5t} (\log t) \lambda\right) &= P\left(\sup_{v \in B_{2^{i+1}r(t)\lambda}} \xi(v) \geq 2^{\frac{(i+1)d}{\alpha}} r(t)^{\frac{d}{\alpha}} (\lambda)^{\frac{d}{\alpha}} \frac{1 \wedge q}{40} 2^{i(1-\frac{d}{\alpha}) - \frac{d}{\alpha}} \lambda^{(1-\frac{d}{\alpha})}\right) \\ &\leq c_q \left(2^{i(1-\frac{d}{\alpha})} \lambda^{(1-\frac{d}{\alpha})}\right)^{\frac{-(\alpha-\varepsilon)}{d}}. \end{aligned} \tag{26}$$

Now for any $\lambda \geq 1, t > 0$ let $A_{\lambda,t} = \{\exists i \geq -1 : \sup_{v \in B_{2^{i+1}r(t)\lambda}} \xi(v) \geq \frac{1 \wedge q}{8} 2^i \frac{r(t)}{5t} (\log t) \lambda\}$ and $B_{\lambda,t} = \{\sup_{v \in B_{r(t)\lambda}} \xi(v) \geq \frac{1}{2} a(t) \lambda\}$. Now take some $\varepsilon \in (0, \alpha)$. By a union bound, $P(A_{\lambda,t}) \leq c\lambda^{-(\frac{\alpha-\varepsilon}{d}-1)}$ for all $t > 0$. Also, $B_{\lambda,t} \subset A_{\lambda,t}$ for all $t > 0$ by Lemma 3.1. Moreover, by Corollary 2.11, it follows that, as $T \rightarrow \infty$,

$$\mathbf{P}\left(\mathbb{P}\left(\tau_{B_{2^{i-1}r(t)\lambda-1}} \leq t\right) \leq \exp\left\{-\frac{2^{i-1}r(t)\lambda}{5} \log\left(\frac{2^{i-1}r(t)\lambda}{et}\right)\right\} \forall t > T, i \geq 1, \lambda \geq 1\right) \geq 1 - o(1).$$

Now take some $T > 0$. For the rest of the proof we work on the high probability event above. By the Feynman-Kac formula (4), we therefore have for all $t > T$ that

$$\begin{aligned} U(t) &\leq \exp\left\{t \sup_{v \in B_{r(t)\lambda}} \xi(v)\right\} + \sum_{i \geq 1} \mathbb{E}\left[\exp\left\{t \sup_{v \in B_{2^i r(t)\lambda}} \xi(v)\right\} \mathbb{1}\left\{2^{i-1}r(t)\lambda \leq \sup_{s \leq t} |X_s| \leq 2^i r(t)\lambda\right\}\right] \\ &\leq \exp\left\{t \sup_{v \in B_{r(t)\lambda}} \xi(v)\right\} + \sum_{i \geq 1} \exp\left\{t \sup_{v \in B_{2^i r(t)\lambda}} \xi(v)\right\} \mathbb{P}\left(\tau_{B_{2^{i-1}r(t)\lambda-1}} \leq t\right). \end{aligned}$$

Therefore, when the events $A_{\lambda,t}^c$ and $B_{\lambda,t}^c$ additionally occur it follows that

$$U(t) \leq \exp\left\{\frac{1}{2}ta(t)\right\} + \sum_{i \geq 1} \exp\left\{\frac{1 \wedge q}{8} \frac{2^{i-1}r(t)\lambda}{5} (\log t)\right\} \exp\left\{-\frac{2^{i-1}r(t)\lambda}{5} \log\left(\frac{2^{i-1}r(t)\lambda}{et}\right)\right\} \leq e^{ta(t)\lambda}.$$

Also, since $\psi_t(v) \leq \xi(v) - \frac{|v|}{t} \left(\log\left(\frac{|v|}{et}\right)\right)$ (by respectively taking $\rho = 0$ and $\rho = 1$ in the two terms in (5)), it additionally follows that on the event $A_{\lambda,t}^c$,

$$\begin{aligned} \sup_{v \notin B_{r(t)\lambda}} \psi_t(v) &\leq \sup_{i \geq 0} \left\{ \frac{1 \wedge q}{8} 2^i \frac{r(t)}{5t} (\log t) \lambda - \frac{2^i r(t) \lambda}{t} \left(\log\left(\frac{2^i r(t) \lambda}{et}\right) \right) \right\} \\ &\leq \sup_{i \geq 0} \left\{ \frac{q 2^i r(t) \lambda}{40t} (\log t) - \frac{q 2^i r(t) \lambda}{2t} (\log(t\lambda)) \right\} \leq 0. \end{aligned}$$

Note that, P -almost surely, there exist at least two leaves $v_1, v_2 \in V(T_\infty)$ with $\xi(v_i) > 1$ for each i . Moreover, if we choose v_1 and v_2 to be the two closest such leaves to the root (breaking ties arbitrarily), then

$$P(v_1, v_2 \in B_{r(t)\lambda}) \rightarrow 1$$

as $t \rightarrow \infty$, uniformly over $\lambda \geq 1$. Note also that by definition, $\psi_t(v_i) > 0$ for all sufficiently large t for each $i = 1, 2$. Combining with the preceding calculation, it therefore follows that $|\hat{Z}_t^{(i)}| \leq r(t)\lambda$ for each $i = 1, 2$ when the high probability events above occur. This establishes the upper bounds in Theorems 1.2 and 1.7.

Lower bounds. Take some $\varepsilon \in (0, 1)$ and $\lambda \geq 1$. We first make two observations.

- (a) We have from Lemma 3.1 that with probability at least $1 - c\lambda^{\frac{-(1-\varepsilon)}{2}}$,

$$\sup_{v \in B_{r(t)\lambda-1}} \psi_t(v) \leq \sup_{v \in B_{r(t)\lambda-1}} \xi(v) \leq \frac{1}{4} (r(t))^{d/\alpha} \lambda^{-d/2\alpha}.$$

- (b) Recall that $\tilde{B}_r = \{v \in B_r : \deg v \leq 4\}$. Choose some $\varepsilon > 0$ small enough that $\frac{(1+\varepsilon)d}{4\alpha} < \frac{1}{4} \wedge \frac{d}{2\alpha}$. By the same calculation as in (20), instead using that $\#\tilde{B}_r \geq r^d \lambda^{\frac{-\varepsilon d}{4}}$ with probability at least $1 - Ce^{-c\lambda^{\frac{\varepsilon}{4}}}$, we obtain that, with probability at least $1 - Ce^{-c\lambda^{\frac{\varepsilon d}{8}}}$,

$$\tilde{\xi}_{r(t)\lambda^{-1/4}}^{(2)} \geq r(t)^{d/\alpha} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}}.$$

Now suppose that the two events in (a) and (b) hold. If $v \in \tilde{B}_{r(t)\lambda^{-1/4}}$ satisfies $\xi(v) > r(t)^{d/\alpha} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}}$, then, provided t is sufficiently large:

$$\begin{aligned} \psi_t(v) &> r(t)^{d/\alpha} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}} - 4 - \frac{r(t)}{t} \lambda^{-1/4} \log(r(t)^{d/\alpha} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}}) > r(t)^{d/\alpha} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}} - 4 - \frac{d(q+1)r(t)}{\alpha t} \lambda^{-1/4} \log t \\ &> \frac{1}{2} r(t)^{\frac{d}{\alpha}} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}} \\ &> \sup_{v \in B_{r(t)\lambda^{-1}}} \psi_t(v). \end{aligned}$$

This shows that, on the events in (a) and (b), $\hat{Z}_t^{(i)}$ as defined by (6) satisfies $|\hat{Z}_t^{(i)}| \geq r(t)\lambda^{-1}$ for each $i = 1, 2$. Finally, for the lower bound on $\log U(t)$, again take such a v and note that from Lemma 2.9 and Proposition 2.10(ii), and taking $\rho = \frac{|v|}{t(\xi(v) - \deg v)} < \frac{r(t)\lambda^{-1/4}}{t(r(t)^{d/\alpha} \lambda^{\frac{-(1+\varepsilon)d}{4\alpha}} - 4)} < 1$ below, there exists $C < \infty$ such that it holds with probability $1 - o(1)$ as $t \rightarrow \infty$ that

$$\begin{aligned} U(t) &\geq \sup_{\rho \in (0,1)} \{ \mathbb{P}(H_v \leq \rho t, X_s = v \ \forall s \in [H_v, H_v + (1-\rho)t]) \exp\{\xi(v)(1-\rho)t\} \} \\ &\geq \sup_{\rho \in (0,1)} \{ \mathbb{P}(H_v \leq \rho t) \exp\{-(1-\rho)t \deg(v) + \xi(v)(1-\rho)t\} \} \\ &\geq \exp \left\{ t\psi_t(v) - C|v| + O\left(t + \frac{|v|}{t}\right) \right\}. \end{aligned}$$

We deduce that $\log U(t) \geq \frac{1}{4}ta(t)\lambda^{\frac{-(1+\varepsilon)d}{4\alpha}}$ on these events, which proves the lower bound (i.e. by replacing $\lambda^{\frac{(1+2\varepsilon)d}{4\alpha}}$ with λ for the latter event). \square

4.2 Solution to the PAM: proof of Theorem 1.8

Let $u(t, v)$ denote the solution to the PAM at vertex $v \in V(T_\infty)$ at time t . In this subsection we prove Theorem 1.8.

Recall from (7) that, analogously to [OR16, Section 1.4], for $v \in V(T_\infty)$ we define

$$\begin{aligned} \lambda(t, v, y) &= t(\xi(y) - \deg y) - |y| \log \left(\frac{|y|}{te} \right) - |v - y| \log \left(\frac{|v - y|}{te} \right), \\ \lambda(t, v) &= \sup_{y \in T_\infty} \lambda(t, v, y), \end{aligned}$$

and set $\lambda_+(t, v) = \lambda(t, v) \vee 0$. We also fix some $m \in (q, \infty)$ and set

$$Y(t, v) = \arg \max_{y \in B_{r(t)(\log t)^m}} \{ \lambda(t, v, y) \}, \quad \tilde{Y}(t, v) = \arg \max_{y \in B_{r(t)(\log t)^m}} \{ \lambda(t, v, y) + t \deg y \}.$$

Since the law of ξ is non-atomic, these are well-defined, P -almost surely.

Recall from Proposition 2.10 that if the random walk $(X_t)_{t \geq 0}$ starts at some point $y \in T_\infty$ and $r = |v - y|$, then

$$\left(\prod_{u \in [[y, v))} \frac{1}{\deg(u)} \right) e^{-r([\log r - \log t - 1] \vee 0) - O(t + \frac{r}{t})} \leq \mathbb{P}_y(H_v < t) \leq e^{-r([\log r - \log t - 1] \vee 0)}. \quad (27)$$

We first state a couple of lemmas and then show how they can be used to prove Theorem 1.8.

The first lemma is similar in flavour to some of the results in Section 3.2, and allows us to control the location and potential of $Y(t, v)$.

Lemma 4.1. (i) $\lambda(t, v, y) + t \deg y \leq 0$ whenever $v \in B_{r(t)(\log t)^\delta}^c$ or $y \in B_{r(t)(\log t)^\delta}^c$. Consequently, $Y(t, v)$ and $\tilde{Y}(t, v)$ are in $B_{r(t)(\log t)^\delta}$ whenever $\lambda(t, v) > 0$.

(ii) $\lambda_+(t, v) = 0 \vee \sup_{y \in B_{r(t)(\log t)^\delta}} \lambda(t, v, y)$ for all $v \in B_{r(t)(\log t)^\delta}$.

(iii) $\sup_{v \in B_{r(t)(\log t)^\delta}} \xi(Y(t, v)) \leq a(t)(\log t)^{\frac{2\delta}{\alpha}}$.

(iv) $\log u(t, v) \leq 0$ for all $v \in B_{r(t)(\log t)^\delta}^c$.

(v) For all $v \in V(T_\infty)$, $\mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \{ \sup_{0 \leq s \leq t} \xi(X_s) \notin B_{r(t)(\log t)^m}, X_t = v \} \right] \leq 1$.

The second lemma allows us to control the degree of good vertices.

Lemma 4.2. Take any $\delta, \eta > 0$. Under Assumption 1.1, it holds with high P -probability as $t \rightarrow \infty$ that

$$\sup_{v \in B_{r(t)(\log t)^\delta}} \left\{ \left[\deg \left(\tilde{Y}(t, v) \right) \right] \mathbb{1} \left\{ \lambda(t, v, \tilde{Y}(t, v)) + t \deg \left(\tilde{Y}(t, v) \right) > 0 \right\} \right\} \leq (\log t)^{\frac{d\delta}{\beta} + \eta}.$$

Remark 4.3. Note that $\deg(Y(t, v)) \leq \deg(\tilde{Y}(t, v))$.

Finally, let $\mathcal{L} = \{u \in T_\infty : \deg u = 1\}$ denote the set of leaves in T_∞ . The third lemma will be useful as we will be able to lower bound the solution at a high degree vertex by considering trajectories that spend a lot of a time at a neighbouring leaf and then jump to the high degree vertex just before time t .

Lemma 4.4. Under Assumption 1.1, it holds with high P -probability as $t \rightarrow \infty$ that

$$\mathbf{P}(\exists v \in B_{r(t)(\log t)^\delta} : \deg v > a(t), \nexists u \sim v \text{ with } u \in \mathcal{L}) \rightarrow 0.$$

On the event that $\deg v > a(t)$ and $\exists u \sim v$ with $p(u) = v$ and $u \in \mathcal{L}$, let $\ell(v)$ denote the leftmost such choice of u amongst the children of v .

Proof of Theorem 1.8, assuming Lemmas 4.1, 4.2 and 4.4. By Lemma 4.1(i) and (iv), it is sufficient to show that

$$(ta(t))^{-1} \sup_{v \in B_{r(t)(\log t)^\delta}} |\log_+(u(t, v)) - \sup_{y \in B_{r(t)(\log t)^\delta}} \{\lambda(t, v, y) \vee 0\}| \rightarrow 0$$

with high P -probability.

Upper bound. For all $v \in B_{r(t)(\log t)^\delta}$, we deduce from Lemma 4.1(v) then (27) then Lemma 4.1(i) that

$$\begin{aligned} u(t, v) &\leq \sum_{y \in V(T_\infty) \setminus B_{r(t)(\log t)^m}} \mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \left\{ \sup_{0 \leq s \leq t} \xi(X_s) = \xi(y), X_t = v \right\} \right] \\ &\quad + \sum_{y \in B_{r(t)(\log t)^m}} \mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \left\{ \sup_{0 \leq s \leq t} \xi(X_s) = \xi(y), X_t = v \right\} \right] \\ &\leq 1 + \sum_{y \in B_{r(t)(\log t)^m}} \exp\{t\xi(y)\} \mathbb{P}(H_y < t) \mathbb{P}_y(H_v < t) \\ &\leq 1 + \sum_{y \in B_{r(t)(\log t)^m}} \exp\{t\xi(y)\} \exp\{-|y|([\log |y| - \log t - 1] \vee 0) - |v - y|([\log |v - y| - \log t - 1] \vee 0)\} \\ &\leq 1 + \#(B_{r(t)(\log t)^m}) \sup_{y \in B_{r(t)(\log t)^\delta}} \exp\{(\lambda(t, v, y) + t \deg y) \vee 0\}. \end{aligned}$$

Since $\#(B_{r(t)(\log t)^m}) \leq (r(t))^{\frac{\beta+1}{\beta-1}}$ with high probability as $t \rightarrow \infty$ by Proposition 2.3(i), taking logarithms we get from Lemma 4.1(ii) that with high P -probability as $t \rightarrow \infty$,

$$\log(u(t, v)) \leq \lambda_+(t, v) + t \deg \tilde{Y}(t, v) + \frac{\beta+1}{\beta-1} \log r(t),$$

so that, by Lemma 4.2,

$$\sup_{v \in B_{r(t)(\log t)^\delta}} (ta(t))^{-1} (\log_+(u(t, v)) - \lambda_+(t, v)) \leq (ta(t))^{-1} \left[t(\log t)^{\frac{d\delta}{\beta} + \eta} + \frac{\beta + 1}{\beta - 1} \log r(t) \right] \rightarrow 0.$$

Finally we can extend this to the supremum over all $v \in V(T_\infty)$ using Lemma 4.1(iv).

Lower bound. We break the proof into two cases, depending on $\deg v$.

a) **Case 1:** $\deg v \leq a(t)$. By Lemma 4.1(i), we can assume wlog that $v \in B_{r(t)(\log t)^\delta}$. We obtain the complementary lower bound from (27) by writing

$$\begin{aligned} u(t, v) &\geq \sup_{y \in B_{r(t)(\log t)^\delta}} \mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \left\{ H_y \leq \frac{t}{\log t}, X_s = y \ \forall s \in \left[\frac{t}{\log t}, t - \frac{t}{\log t} \right], X_t = v \right\} \right] \\ &\geq \sup_{y \in B_{r(t)(\log t)^\delta}} \exp \left\{ \left(t - \frac{2t}{\log t} \right) \xi(y) \right\} \mathbb{P}_O \left(H_y \leq \frac{t}{\log t}, X_s = y \ \forall s \in \left[H_y, t - \frac{t}{\log t} \right], X_t = v \right). \end{aligned} \quad (28)$$

To bound the probability appearing here, we use the Markov property to write

$$\begin{aligned} &\mathbb{P}_O \left(H_y \leq \frac{t}{\log t}, X_s = y \ \forall s \in \left[H_y, t - \frac{t}{\log t} \right], X_t = v \right) \\ &\geq \mathbb{P}_O \left(H_y \leq \frac{t}{\log t} \right) \mathbb{P}_y \left(X_s = y \ \forall s \in \left[0, t - \frac{t}{\log t} - H_y \right] \mid H_y \leq \frac{t}{\log t} \right) \mathbb{P}_y \left(X_{\frac{t}{\log t}} = v \right) \\ &\geq \mathbb{P}_O \left(H_y \leq \frac{t}{\log t} \right) e^{-(t - \frac{t}{\log t}) \deg y} \mathbb{P}_y \left(H_v \leq \frac{t}{\log t} \right) e^{-\frac{t}{\log t} \deg v}. \end{aligned} \quad (29)$$

Combining these and using (27), we deduce that $u(t, v)$ is lower bounded by

$$\begin{aligned} &\sup_{y \in B_{r(t)(\log t)^\delta}} \exp \left\{ \left(t - \frac{2t}{\log t} \right) \xi(y) \right\} \exp \left\{ - \left[\sum_{u \in [[O, y]]} \log(\deg u) \right] - |y| \left(\log_+ \left(\frac{|y|}{te} \right) + \log \log t \right) \right\} e^{-t \deg y} \\ &\times \exp \left\{ - \left[\sum_{u \in [[y, v]]} \log(\deg u) \right] - |v - y| \left(\log_+ \left(\frac{|v - y|}{te} \right) + \log \log t \right) + O \left(t + \frac{r(t)(\log t)^\delta}{t} \right) \right\} e^{-\frac{t}{\log t} \deg v}. \end{aligned}$$

Also recall from Lemma 4.1(i) that on the event $\lambda(t, v) > 0$, $Y(t, v) \in B_{r(t)(\log t)^\delta}$. (If $\lambda(t, v) < 0$ then the lower bound is trivial). In this case we can therefore write, conditioning on the high probability events in Lemma 4.1:

$$\begin{aligned} \log u(t, v) &\geq \sup_{y \in B_{r(t)(\log t)^\delta}} t\xi(y) - |y| \left(\log_+ \left(\frac{|y|}{te} \right) \right) - |v - y| \left(\log_+ \left(\frac{|v - y|}{te} \right) \right) - t \deg y - \frac{2t}{\log t} \xi(y) \\ &\quad - \sum_{u \in [[O, y]]} \log(\deg u) - \sum_{u \in [[y, v]]} \log(\deg u) - \frac{t}{\log t} \deg v - (|y| + |v - y|) \log \log t - O \left(t + \frac{r(t)(\log t)^\delta}{t} \right) \\ &\geq \lambda(t, v) - \frac{2t}{\log t} \xi(Y(t, v)) - \sum_{u \preceq Y(t, v)} \log(\deg u) - \sum_{u \in [[Y(t, v), v]]} \log(\deg u) - \frac{ta(t)}{\log t} - O(r(t)(\log t)^\delta \log \log t) \\ &\geq \lambda(t, v) - 2ta(t)(\log t)^{\frac{2\delta}{\alpha} - 1} - 4\tilde{B}r(t)(\log t)^\delta - 4(q + 1)B \log t - \frac{ta(t)}{\log t} - O(r(t)(\log t)^\delta \log \log t). \end{aligned}$$

Here the penultimate line follows by the assumption that $\deg v \leq a(t)$, and the last line follows by Lemmas 2.9 and 4.1(iii). Since $\delta < 1$, the result then follows since

$$(ta(t))^{-1} \left(2ta(t)(\log t)^{\frac{2\delta}{\alpha} - 1} + 4\tilde{B}r(t)(\log t)^\delta + 4(q + 1)B \log t + \frac{ta(t)}{\log t} + O(r(t)(\log t)^\delta \log \log t) \right) \rightarrow 0$$

as $t \rightarrow \infty$ (note that $ta(t) = r(t) \log t$).

b) **Case 2:** $\deg v > a(t)$. In this second case we lower bound $u(t, v)$ using a slightly different event. Again we can assume wlog that $v \in B_{r(t)(\log t)^\delta}$. We again proceed as in (28) and (29) but this time we observe that, letting $H_v^+ = \inf\{s \geq H_{\ell(v)} : X_s = v\}$, it follows from (27) and Lemma 4.4 that with high \mathbf{P} -probability

$$\begin{aligned} & \mathbb{P}_y \left(X_{\frac{t}{\log t}} = v \right) \\ & \geq \mathbb{P}_y \left(H_{\ell(v)} \leq \frac{t}{2 \log t}, H_v^+ \in \left(\frac{t}{\log t} - \frac{2}{\deg v}, \frac{t}{\log t} - \frac{1}{\deg v} \right), X_s = v \ \forall s \in \left[H_v^+, \frac{t}{\log t} \right] \right) \\ & \geq \mathbb{P}_y \left(H_{\ell(v)} \leq \frac{t}{2 \log t} \right) \left\{ \inf_{T \in (0, \frac{t}{2 \log t})} \mathbb{P} \left(\exp(1) \in \left(\frac{t}{\log t} - \frac{2}{\deg v} - T, \frac{t}{\log t} - \frac{1}{\deg v} - T \right) \right) \right\} \mathbb{P} \left(\exp(\deg v) \geq \frac{2}{\deg v} \right) \\ & \geq \exp \left\{ - \left[\sum_{u \in [[y, v]]} \log(\deg u) \right] - (|v - y| + 1) \left(\log_+ \left(\frac{2(|v - y| + 1)(\log t)}{te} \right) \right) + O \left(t + \frac{r(t)(\log t)^\delta}{t} \right) \right\} e^{-\frac{t}{\log t}} \frac{1}{2 \deg v} e^{-2}. \end{aligned}$$

Substituting back into the second line of (29) and then the second line of (28), and then conditioning on the event in Lemma 4.1(i), we see that

$$\begin{aligned} \log u(t, v) & \geq \sup_{y \in B_{r(t)(\log t)^\delta}} \left(t - \frac{2t}{\log t} \right) \xi(y) - \left\{ \left[\sum_{u \in [[y, v]]} \log(\deg u) \right] + |y| \left(\log_+ \left(\frac{|y|}{te} \right) + \log \log t \right) \right\} - t \deg y \\ & \quad - \left\{ \left[\sum_{u \in [[y, v]]} \log(\deg u) \right] + (|v - y| + 1) \left(\log_+ \left(\frac{2(|v - y| + 1)(\log t)}{te} \right) \right) \right\} - \frac{2t}{\log t} - O \left(t + \frac{r(t)(\log t)^\delta}{t} \right) \\ & \geq \lambda(t, v) - \frac{2t}{\log t} \xi(Y(t, v)) - \sum_{u \in [[O, Y(t, v)]]} \log(\deg u) - \sum_{u \in [[Y(t, v), v]]} \log(\deg u) - O(r(t)(\log t)^\delta \log \log t) \\ & \geq \lambda(t, v) - O(ta(t)(\log t)^{\frac{2\delta}{\alpha}-1}) - O(r(t)(\log t)^\delta \log \log t), \end{aligned}$$

which implies the result as in a), since $\delta < 1$. \square

We therefore just need to prove Lemmas 4.1, 4.2 and 4.4.

Proof of Lemma 4.1. (i) Take some $\varepsilon > 0$. Just as in (26), it follows from Lemma 3.1 that for any $i \geq 1$,

$$P \left(\sup_{v \in B_{2^{i+1}r(t)(\log t)^\delta}} \xi(v) \geq \frac{1 \wedge q}{8} 2^i \frac{r(t)}{t} (\log t)^{1+\delta} \right) \leq c_q \left(2^{i(1-\frac{d}{\alpha})} (\log t)^{\delta(1-\frac{d}{\alpha})} \right)^{\frac{-(\alpha-\varepsilon)}{d}}. \quad (30)$$

Therefore, given $v \in B_{r(t)(\log t)^\delta}$ and wlog assuming that $\varepsilon > 0$ small enough that $\frac{\alpha-\varepsilon}{d}$ is positive, and since $|y| \vee |v - y| \geq \frac{1}{2}|v|$ for all $v, y \in T_\infty$, we have that

$$\begin{aligned} & P(\exists i \geq 0, y \in B_{2^{i+1}r(t)(\log t)^\delta} \setminus B_{2^i r(t)(\log t)^\delta} : \lambda(t, v, y) + t \deg y \geq 0) \\ & \leq \sum_{i \geq 0} P \left(\sup_{v \in B_{2^{i+1}r(t)(\log t)^\delta}} \xi(v) \geq \frac{1}{8} 2^i \frac{r(t)}{t} (\log t)^{1+\delta} \right) \\ & \leq \sum_{i \geq 0} c 2^{\frac{-i(1-\frac{d}{\alpha})(\alpha-\varepsilon)}{d}} (\log t)^{\frac{-\delta(1-\frac{d}{\alpha})(\alpha-\varepsilon)}{d}}, \end{aligned}$$

which vanishes as $t \rightarrow \infty$.

(ii) Follows from (i), since $\lambda(t, v, y) \leq \lambda(t, y)$ for all $v \in V(T_\infty)$.

(iii) Follows from (i) and Lemma 3.1.

(iv) Choose some $\varepsilon \in (0, m(\alpha - d) - d)$ (by our choice of m , such an ε always exists). Note that by Corollary 2.11, Lemma 2.12 and (30), we have whp as $t \rightarrow \infty$ that

$$\begin{aligned} \sum_{v \notin B_{r(t)(\log t)^\delta}} u(t, v) &\leq \mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) \, ds \right\} \mathbb{1}_{\{\tau_{B_{r(t)(\log t)^\delta}} \leq t\}} \right] \\ &\leq \sum_{r \geq r(t)(\log t)^\delta} \mathbb{P}(\tau_{B_r} \leq t) \exp \left\{ t \sup_{v \in B_r} \xi(v) \right\} \\ &\leq \sum_{r(t)(\log t)^\delta \leq r \leq r(t)(\log t)^m} \exp \left\{ -r \log \left(\frac{r}{et} \right) + o(r(t)) \right\} \exp \left\{ \frac{q}{4} r(\log t) \right\} \\ &\quad + \sum_{r \geq r(t)(\log t)^m} \exp \left\{ -\frac{r}{5} ([\log r - \log t - 1] \vee 0) \right\} \exp \left\{ tr^{\frac{d}{\alpha}} (\log r)^{\frac{d+\varepsilon}{\alpha}} \right\}, \end{aligned}$$

The choice of ε implies that $m > \frac{dm}{\alpha} + \frac{d+\varepsilon}{\alpha}$, which is exactly what is required for this sum to vanish as $t \rightarrow \infty$.

(v) The calculation in (iv) similarly gives (v) as well. □

Proof of Lemma 4.2. Recall that

$$\lambda(t, v, y) = \{t(\xi(y) - \deg(y)) - |y|[\log |y| - \log t - 1] - |v - y|[\log |v - y| - \log t - 1]\},$$

and $\lambda(t, v) = \sup_{y \in T_\infty} \lambda(t, v, y)$. We also define the set

$$C_{t,\varepsilon} = \left\{ y \in B_{r(t)(\log t)^\delta} : t\xi(y) > \varepsilon r(t) \log \left(\frac{r(t)}{t} \right) \right\}.$$

We will show that for any $\delta' > 0$, we can choose $\varepsilon > 0$ small enough so that for all sufficiently large t we have with probability at least $1 - \delta'$ that

1. $C_{t,\varepsilon} \neq \emptyset$;
2. $\left\{ \tilde{Y}(t, v) : v \in B_{r(t)}, \lambda(t, v, \tilde{Y}(t, v)) + t \deg(\tilde{Y}(t, v)) > 0 \right\} \subset C_{t,\varepsilon}$;
3. For any $\varepsilon' > 0$, the cardinality of the set $C_{t,\varepsilon}$ is upper bounded by $(\log t)^{d\delta + \varepsilon'}$;
4. For any $\varepsilon' > 0$, $\sup_{v \in C_{t,\varepsilon}} \deg v \leq (\log t)^{\frac{d\delta}{\beta} + \varepsilon'}$.

We deal with each of these four points in four separate steps below.

Step 1. To deal with the first point, choose some $k < \alpha$ and note from Proposition 2.3(iii) that

$$\begin{aligned} P(C_{t,\varepsilon} = \emptyset) &\leq \mathbf{P}(V(B_{r(t)}) \leq \varepsilon^k r(t)^d) + \mathcal{P}(C_{t,\varepsilon} = \emptyset \mid V(B_{r(t)}) > \varepsilon^k r(t)^d) \\ &\leq e^{-\varepsilon^{\frac{-(\beta-1)k}{\beta}}} + \left(1 - t^\alpha r(t)^{-\alpha} \varepsilon^{-\alpha} \left(\log \left(\frac{r(t)}{t} \right) \right)^{-\alpha} \right)^{\varepsilon^k r(t)^d} \leq e^{-\varepsilon^{\frac{-(\beta-1)k}{\beta}}} + \exp \left\{ -\frac{1}{2} \varepsilon^{k-\alpha} q^{-\alpha} \right\}, \end{aligned}$$

which clearly vanishes as $\varepsilon \downarrow 0$.

Step 2. To deal with the second point: firstly, note that if $\tilde{Y}(t, v) \notin B_{\varepsilon r(t)}$ or $v \notin B_{\varepsilon r(t)}$, then it must be that $\tilde{Y}(t, v) \in C_{t,\varepsilon}$, since $\lambda(t, v, \tilde{Y}(t, v)) + t \deg(\tilde{Y}(t, v)) > 0$.

We will now show that, with high probability as $t \rightarrow \infty$, for all $v \in B_{\varepsilon r(t)}$ there exists $y \in B_{2r(t)}$ such that $\lambda(t, v, y) \geq \varepsilon r(t) \log \left(\frac{r(t)}{t} \right)$, which implies the result since $t\xi(\tilde{Y}(t, v)) \geq \sup_{y \in B_{2r(t)}} \lambda(t, v, y)$. To do this, we first let $C_{k,t} = B_{(k+1)\varepsilon r(t)} \setminus A_{k\varepsilon r(t)}$ for $1 \leq k \leq \lfloor \varepsilon^{-1} \rfloor$, where A_r is the connected component containing the

root obtained by deleting s_r from T_∞ (recall that s_r is the r^{th} backbone vertex). Note that the definition of the set A_r ensures that $(C_{k,t})_{k=1}^{\lfloor \varepsilon^{-1} \rfloor}$ is a sequence of independent sets, each with law $B_{\varepsilon r(t)}$. Moreover, by Lemma 2.8, for each $k \geq 4$ we have for all sufficiently large t that

$$\begin{aligned} & P\left(\exists v \in C_{k,t} : t(\xi(v) - \deg v) \geq ((2k+4)\varepsilon)r(t) \log\left(\frac{r(t)}{t}\right)\right) \\ & \geq P\left(\exists v \in C_{k,t} : \deg v \leq 4, \xi(v) \geq ((2k+5)\varepsilon)\frac{r(t)}{t} \log\left(\frac{r(t)}{t}\right)\right) \\ & \geq \mathbf{P}(|\{v \in B_{\varepsilon r(t)} : \deg v \geq 4\}| \geq \varepsilon^d r(t)^d) \left[1 - \left(1 - ((2k+5)\varepsilon)^{-\alpha} r(t)^{-\alpha} t^\alpha \left(\log\left(\frac{r(t)}{t}\right)\right)^{-\alpha}\right)^{\varepsilon^d r(t)^d}\right] \\ & \geq C \left(1 - \exp\left\{-\frac{q}{2}((2k+5)\varepsilon)^{-\alpha} \varepsilon^d\right\}\right). \end{aligned}$$

In particular, for $1 \leq k \leq \varepsilon^{\frac{-(\alpha-d)}{\alpha}}$ this is uniformly bounded below by a positive constant, so that

$$P\left(\exists k \leq \lfloor \varepsilon^{\frac{-(\alpha-d)}{\alpha}} \rfloor, v \in C_{k,t} : t(\xi(v) - \deg v) \geq ((2k+4)\varepsilon)r(t) \log\left(\frac{r(t)}{t}\right)\right) \geq 1 - Ce^{-c\varepsilon^{\frac{-(\alpha-d)}{\alpha}}}.$$

Moreover, if $y \in C_{k,t}$ with $t(\xi(y) - \deg y) \geq ((2k+4)\varepsilon)r(t) \log\left(\frac{r(t)}{t}\right)$ and $v \in B_{\varepsilon r(t)}$ we have that $|y| \leq (k+1)\varepsilon r(t)$ and $|v-y| \leq (k+2)\varepsilon r(t)$ so that $\lambda(t, v, y) \geq \varepsilon r(t) \log\left(\frac{r(t)}{t}\right)$, which proves the claim.

Step 3. The third step is therefore to bound the cardinality of this set. Choose $c_2 > d\delta$ and $c_1 \in (d\delta, c_2)$, set $N_t = \left(\log\left(\frac{r(t)}{t}\right)\right)^{c_1} r(t)^d$ and set $p_t = t^{-\alpha} r(t)^{-\alpha} \varepsilon^{-\alpha} (\log r(t))^{-\alpha}$. We then have from Proposition 2.3(i) and a Chernoff bound that

$$\begin{aligned} P(|C_{t,\varepsilon}| \geq (\log t)^{c_2}) & \leq \mathbf{P}(V(B_{r(t)(\log t)^\delta}) \geq N_t) + P(\text{Binomial}(N_t, p_t) \geq (\log t)^{c_2}) \\ & \leq (\log t)^{-\frac{1}{2}(\beta-1)(c_1-d\delta)} + \exp\{(e-1)N_t p_t - (\log t)^{c_2}\} \\ & = (\log t)^{-\frac{1}{2}(\beta-1)(c_1-d\delta)} + \exp\left\{-(\log t)^{c_2} \left(1 - 2(\log t)^{-(c_2-c_1)}(e-1)\varepsilon^{-\alpha}\right)\right\}, \end{aligned}$$

which clearly vanishes as $t \rightarrow \infty$.

Step 4. For the fourth point above, note that since the potential at each vertex in T_∞ is independent of the underlying tree structure, each vertex v in $C_{t,\varepsilon}$ is a uniform point of $B_{r(t)(\log t)^\delta}$, so whp it satisfies (here we additionally work on the event $\#B_{r(t)(\log t)^\delta} \geq r(t)^d$ which occurs whp as $t \rightarrow \infty$ by Proposition 2.3(i)):

$$P(\deg v \geq x) \leq c \frac{r(t)(\log t)^\delta}{r(t)^d} x^{-(\beta-1)} + cx^{-\beta}$$

for all $x \geq 0$. Therefore, by a union bound, and conditionally on $|C_{t,\varepsilon}| \leq (\log t)^{d\delta+\varepsilon'}$, we have whp that (for any $M > 0$):

$$\mathcal{P}\left(\sup_{v \in C_{t,\varepsilon}} \deg v \geq (\log t)^M\right) \leq c(\log t)^{d\delta+\varepsilon'-M\beta}.$$

Conclusion. Therefore, to tie up, if we choose $M > \frac{d\delta}{\beta}$ this ensures that we can choose ε' small enough that

$$\mathbb{P}\left(\sup_{v \in B_{r(t)(\log t)^\delta}} \deg \tilde{Y}(t, v) \geq (\log t)^M\right) \geq 1 - 2\delta'$$

for all sufficiently large t . Since δ' was arbitrary, this proves the result. \square

Proof of Lemma 4.4. If $\deg v > a(t)$, then $\mathbf{P}(\nexists u \sim v \text{ with } u \in \mathcal{L}) \leq (1 - \mu_0)^{a(t)-1} \leq e^{-ca(t)}$. Therefore, taking a union bound over all vertices in $B_{r(t)(\log t)^\delta}$, we obtain from Proposition 2.3(i) that

$$\mathbf{P}(\exists v \in B_{r(t)(\log t)^\delta} : \deg v > a(t), \nexists u \sim v \text{ with } p(u) = v, u \in \mathcal{L}) \leq \mathbf{P}(V(B_{r(t)(\log t)^\delta}) \geq r(t)^{2d}) + r(t)^{2d} e^{-ca(t)} \rightarrow 0$$

as $t \rightarrow \infty$. \square

5 The concentration sites: bounds on $|\hat{Z}_t^{(i)}|$, $\psi_t(\hat{Z}_t^{(i)})$ and $\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)})$

In this section we derive some asymptotics for the functional ψ_t and the maximisers $\hat{Z}_t^{(1)}$ and $\hat{Z}_t^{(2)}$ introduced in (5) and (6).

5.1 Maximisers of the functional ψ_t

Recall from (5) and (6) the definition of the random functional $\psi_t : V(T_\infty) \rightarrow \mathbb{R}$ by

$$\psi_t(z) := \left[(\xi(z) - \deg(z)) - \frac{|z|}{t} \log(\xi(z) - \deg(z)) \right] \mathbb{1}\{t(\xi(z) - \deg(z)) \geq |z|\},$$

and its maximisers $\hat{Z}_t^{(1)}$, $\hat{Z}_t^{(2)}$ and $\hat{Z}_t^{(3)}$ defined by

$$\psi_t(\hat{Z}_t^{(1)}) = \max_{z \in V(T_\infty)} \psi_t(z), \quad \psi_t(\hat{Z}_t^{(2)}) = \max_{z \in V(T_\infty) \setminus \{\hat{Z}_t^{(1)}\}} \psi_t(z), \quad \psi_t(\hat{Z}_t^{(3)}) = \max_{z \in V(T_\infty) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \psi_t(z). \quad (31)$$

(We will show in Proposition 5.1 that these are almost surely uniquely defined). In this section we establish some elementary properties of these maximisers. We first prove the following.

Proposition 5.1. *P-almost surely under Assumption 1.3, we have for any $\varepsilon > 0$ that for all sufficiently large t , and $i = 1, 2, 3$,*

$$(i) \quad r(t)(\log t)^{-(1/2+\varepsilon)} \leq |\hat{Z}_t^{(i)}|,$$

$$(ii) \quad a(t)(\log t)^{-\varepsilon} \leq \psi_t(\hat{Z}_t^{(i)}),$$

$$(iii) \quad |\hat{Z}_t^{(i)}| \leq r(t)(\log t)^{\frac{1}{\alpha-d}+\varepsilon},$$

$$(iv) \quad \psi_t(\hat{Z}_t^{(i)}) \leq a(t)(\log t)^{\frac{1}{\alpha-d}+\varepsilon}.$$

Under Assumption 1.1 we instead have that for any $\varepsilon > 0$, whp:

$$(i') \quad r(t)(\log t)^{-\varepsilon} \leq |\hat{Z}_t^{(i)}|,$$

$$(ii') \quad a(t)(\log t)^{-\varepsilon} \leq \psi_t(\hat{Z}_t^{(i)}),$$

$$(iii') \quad |\hat{Z}_t^{(i)}| \leq r(t)(\log t)^\varepsilon,$$

$$(iv') \quad \psi_t(\hat{Z}_t^{(i)}) \leq a(t)(\log t)^\varepsilon.$$

Proof. Note that $d = 2$ under Assumption 1.3, but we carry d through the proofs in order to make the high probability extensions clearer.

(i) Let $\varepsilon > 0$ and also let $\varepsilon' \in (0, \frac{\varepsilon}{2})$. Let $f(t) = (\log t)^{-\frac{1+\varepsilon}{2}}$, $g(t) = (\log t)^{-\varepsilon/16}$. We perform two calculations.

(a) We have from Lemma 3.2 that eventually almost surely

$$\sup_{v \in B_{r(t)f(t)}} \psi_t(v) \leq \sup_{v \in B_{r(t)f(t)}} \xi(v) \leq (r(t)f(t))^{d/\alpha} (\log(r(t)f(t)))^{\frac{(2+\varepsilon)}{2\alpha}} \leq C_q a(t)(\log t)^{-\varepsilon/2\alpha}.$$

- (b) We have from Lemma 3.3 that eventually almost surely, for any $\delta > 0$ there exist distinct u_1, u_2, u_3 in $B_{r(t)g(t)}$ such that

$$\xi(u_i) - \deg(u_i) \geq (r(t)g(t))^{d/\alpha} (\log(r(t)g(t)))^{-\varepsilon\delta/8\alpha} \geq c_q a(t) (\log t)^{-\varepsilon(1+\delta)/8\alpha},$$

Consequently, provided t is sufficiently large, and again using Lemma 3.2 for an upper bound on $\xi(u_i)$, and that $\alpha > 2$, we can choose $\delta > 0$ small enough that

$$\begin{aligned} \psi_t(u_i) &= (\xi(u_i) - \deg(u_i)) - \frac{|u_i|}{t} \log(\xi(u_i) - \deg(u_i)) \geq c_q a(t) (\log t)^{-\varepsilon(1+\delta)/8\alpha} - c'_q a(t) (\log t)^{-\varepsilon/16} \\ &> \frac{c_q}{2} a(t) (\log t)^{-\varepsilon(1+\delta)/8\alpha}. \end{aligned}$$

Combining (a) and (b) and assuming that $\delta < 3$ we deduce that $\psi_t(u_i) > \sup_{v \in B_{r(t)f(t)}} \psi_t(v)$ for each $i = 1, 2, 3$. Since u_1, u_2, u_3 are distinct, it follows that $\psi_t(\hat{Z}_t^{(3)})$ enjoys the same lower bound, and that $|\hat{Z}_t^{(i)}| > r(t)f(t)$ eventually almost surely.

- (ii) For the lower bound on ψ_t , note that we can conclude from the proof of part (i) that for $i = 1, 2, 3$

$$\psi_t(\hat{Z}_t^{(i)}) \geq \min_{i=1,2,3} \psi_t(u_i) > a(t) (\log t)^{-\varepsilon d/\alpha}.$$

- (iii) Take $b = \frac{1+2\varepsilon}{\alpha-d}$. If $x > r(t)(\log t)^b$, then

$$x^{\frac{d}{\alpha}} (\log x)^{\frac{1+\varepsilon}{\alpha}} \leq \frac{x}{t} \log t$$

for all sufficiently large t . Therefore, if $v \in V(T_\infty)$ is such that $|v| \geq r(t)(\log t)^b$, then either $\xi(v) - \deg(v) \leq t$, in which case $v \neq \hat{Z}_t^{(i)}$ by part (ii) (since $\xi(\hat{Z}_t^{(i)}) - \deg(\hat{Z}_t^{(i)}) \geq \psi_t(\hat{Z}_t^{(i)})$), or otherwise we have from Lemma 3.2 that

$$\psi_t(v) \leq |v|^{\frac{d}{\alpha}} (\log |v|)^{\frac{1+\varepsilon}{\alpha}} - \frac{|v|}{t} \log(\xi(v) - \deg(v)) \leq |v|^{\frac{d}{\alpha}} (\log |v|)^{\frac{1+\varepsilon}{\alpha}} - \frac{|v|}{t} \log t \leq 0$$

by the computation above, so again $v \neq \hat{Z}_t^{(i)}$ by part (ii).

- (iv) From (iii) and Lemma 3.2, almost surely for all sufficiently large t we have that

$$\psi_t(\hat{Z}_t^{(i)}) \leq t \sup_{v \in B_{r(t)(\log t)^{\frac{1}{\alpha-d} + \varepsilon}}} \xi(v) \leq C_q \left(r(t)(\log t)^{\frac{1}{\alpha-d} + \varepsilon} \right)^{\frac{d}{\alpha}} (\log t)^{\frac{1+\varepsilon}{\alpha}} = r(t)^{\frac{d}{\alpha}} (\log t)^{\frac{1}{\alpha-d} + 2\varepsilon}.$$

(i') This is the same as above but instead taking $f(t) = (\log t)^{-\varepsilon}$, $g(t) = (\log t)^{-\varepsilon/2}$.

(ii') Similarly follows from considering the largest value of ξ on $B_{r(t)f(t)}$ as above.

(iii') This follows from the computation in (30).

(iv') This also follows from the computation in (30). □

Remark 5.2. Proposition 5.1 also shows that $\hat{Z}_t^{(i)}$ is well-defined for each $i = 1, 2, 3$.

The next lemma will be useful for converting bounds on $\psi_{t_n}(\hat{Z}_{t_n}^{(i)})$ along some subsequence to almost sure bounds on $\psi_t(\hat{Z}_t^{(i)})$ for all sufficiently large t .

Lemma 5.3. Let $\varepsilon > 0$ be arbitrary. Under Assumption 1.3, we have for each $i = 1, 2, 3$ that eventually P -almost surely:

$$(i) \quad \psi_u(\hat{Z}_u^{(i)}) - \psi_t(\hat{Z}_t^{(i)}) \leq \frac{u-t}{t} a(u) (\log u)^{\frac{1}{\alpha-d} + \varepsilon}, \text{ for all } u > t,$$

(ii) $t \mapsto \psi_t(\hat{Z}_t^{(i)})$ is increasing.

Proof. This follows exactly as in [KLMS09, Lemma 3.2(iv),(v)], using Proposition 5.1(iii). \square

The third lemma will be useful for later running Borel-Cantelli arguments on the maximisers $\hat{Z}_t^{(i)}$ for each $i = 1, 2, 3$.

Lemma 5.4. *Let $t_n = 2^n$. Under Assumption 1.3, we have for any $\varepsilon > 0$ that P -almost surely for all sufficiently large n and each $i = 1, 2$ that*

$$\#\left\{\hat{Z}_t^{(i)} : t \in [t_n, t_{n+1}]\right\} \leq (\log t_n)^\varepsilon.$$

Proof. Set $r_n = 2^n$, define the set

$$N_n := \left\{v \in B_{r_{n+1}} \setminus B_{r_n} : \xi(v) > 2^{-d/\alpha} r_n^{d/\alpha} (\log r_n)^{-\varepsilon}\right\}$$

and set

$$p_n := \mathcal{P}\left(\xi(O) > 2^{-d/\alpha} r_n^{d/\alpha} (\log r_n)^{-\varepsilon}\right) = 2^d r_n^{-d} (\log r_n)^{\alpha\varepsilon}.$$

Note that, conditionally on T_∞ , $\#N_n \sim \text{Binom}(\#(B_{r_{n+1}} \setminus B_{r_n}), p_n)$. By Corollary 2.4, almost surely for all n large enough it holds that $\#(B_{r_{n+1}} \setminus B_{r_n}) \leq r_n^d (\log r_n)^\varepsilon$. Using a Chernoff bound we thus obtain that with \mathbf{P} -probability $1 - o(1)$ as $N \rightarrow \infty$, it holds for all $n \geq N$ that

$$P(\#N_n \geq (\log r_n)^{2(1+\alpha)\varepsilon}) \leq \frac{E[e^{\#N_n} \mid \#(B_{r_{n+1}} \setminus B_{r_n}) \leq r_n^d (\log r_n)^\varepsilon]}{e^{(\log r_n)^{2(1+\alpha)\varepsilon}}} \leq e^{-c(\log r_n)^{2(1+\alpha)\varepsilon}}.$$

This probability is summable over n and thus by Borel Cantelli $\#N_n \leq (\log r_n)^{2(1+\alpha)\varepsilon}$ eventually almost surely.

Now take $t > 0$ and set $M_t = \lfloor \log_2(r(t)) \rfloor$. We claim that Proposition 5.1 implies that there exists a deterministic $c < \infty$ such that, almost surely for all sufficiently large n ,

$$\left\{\hat{Z}_t^{(i)} : t \in [t_n, t_{n+1}]\right\} \subset \bigcup_{k=M_{t_n}-c(\log \log t_n)}^{M_{t_n}+c(\log \log t_n)} N_k. \quad (32)$$

Combined with the bound on N_n , this readily implies the result since then

$$\#\left\{\hat{Z}_t^{(i)} : t \in [t_n, t_{n+1}]\right\} \leq \sum_{k=M_{t_n}-c(\log \log t_n)}^{M_{t_n}+c(\log \log t_n)} \#N_k \leq 2c(\log \log t_n) [\log(r(t_n)(\log t_n)^c)]^{2(1+\alpha)\varepsilon} \leq (\log t_n)^{3(1+\alpha)\varepsilon}.$$

(Clearly, this proves the result since $\varepsilon > 0$ was arbitrary).

All that remains is thus to prove (32). Take $n \geq 0$ and some $t \in [t_n, t_{n+1}]$. First note that it follows from Proposition 5.1(i) and (ii) that we can choose $c < \infty$ such that we almost surely have for all sufficiently large n and all $t \in [t_n, t_{n+1}]$ that $\hat{Z}_t^{(i)} \in \bigcup_{k=M_{t_n}-c(\log \log t)}^{M_{t_n}+c(\log \log t)} N_k$ whenever $|\hat{Z}_t^{(i)}| \leq r(t)$. If instead $t \in [t_n, t_{n+1}]$ and $|\hat{Z}_t^{(i)}| > r(t)$, then note that almost surely we must also have $|\hat{Z}_t^{(i)}| \leq r(t)(\log t)^{\frac{2}{\alpha-d}}$ provided n is large enough by Proposition 5.1(iii). By Proposition 5.1(ii), we also have that $\xi(\hat{Z}_t^{(i)}) - \deg(\hat{Z}_t^{(i)}) \geq a(t)(\log t)^{-\varepsilon}$. Combining this again with Proposition 5.1(ii) implies that

$$\xi(\hat{Z}_t^{(i)}) \geq \frac{|\hat{Z}_t^{(i)}|}{t} \log(a(t)(\log t)^{-\varepsilon}) \geq \frac{q}{2} \frac{|\hat{Z}_t^{(i)}|}{t} \log(t) \geq |\hat{Z}_t^{(i)}|^{\frac{d}{\alpha}} (\log t)^{-\varepsilon}.$$

Here the final inequality follows since $\frac{y}{t} \log t \geq y^{\frac{d}{\alpha}}$ precisely when $y > r(t)$. Setting $L_t = \lfloor \log_2(|\hat{Z}_t^{(i)}|) \rfloor$, this implies that $\hat{Z}_t^{(i)} \in N_{L_t}$. Provided that $c > \frac{2}{(\alpha-d)\log 2}$, the condition $r(t) \leq |\hat{Z}_t^{(i)}| \leq r(t)(\log t)^{\frac{2}{\alpha-d}}$ then implies that $M_{t_n} \leq L_t \leq M_{t_n} + c \log \log t$. This completes the proof. \square

5.2 The gap between the highest values of ψ_t

To prove the localisation result, it will be important to have a bound between the largest values of ψ_t . We start with a tail bound.

Lemma 5.5. *Under Assumption 1.1, for any $k < \infty$ and $0 < \varepsilon < k$, we have for all t large enough that*

$$P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)}) < a(t)(\log t)^{-k}) \leq 4\alpha(\log t)^{-(k-\varepsilon)}, \quad (33)$$

$$P(\psi_t(\hat{Z}_t^{(2)}) - \psi_t(\hat{Z}_t^{(3)}) < a(t)(\log t)^{-k}) \leq 4\alpha(\log t)^{-(k-\varepsilon)}. \quad (34)$$

Proof. We just prove (33) (the proof of (34) is identical). Fix k, ε as above and note that it follows from the proofs of Proposition 5.1 and Lemma 3.3 that

$$P\left(\psi(\hat{Z}_t^{(2)}) \leq a(t)(\log t)^{-\varepsilon}\right) + P\left(|\hat{Z}_t^{(1)}| \leq et\right) \leq (\log t)^{-(k-\varepsilon)}$$

for all sufficiently large t .

We can therefore write for all sufficiently large t and any $x > 0$ that

$$\begin{aligned} & P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)}) > x) \\ & \geq \inf_{z > a(t)(\log t)^{-\varepsilon} + 1} P\left(\psi_t(\hat{Z}_t^{(1)}) > x + z \mid \psi_t(\hat{Z}_t^{(2)}) \in [z - 1, z], \psi(\hat{Z}_t^{(2)}) \geq a(t)(\log t)^{-\varepsilon}, \hat{Z}_t^{(1)} > et\right) \\ & \quad - P\left(\psi(\hat{Z}_t^{(2)}) \leq a(t)(\log t)^{-\varepsilon}\right) - P\left(|\hat{Z}_t^{(1)}| \leq et\right) \\ & \geq \sup_{\mathbb{S}_\infty \subset \mathbb{T}_\infty: \mathbf{P}(\mathbb{S}_\infty)=1} \inf_{T \in \mathbb{S}_\infty} \inf_{v \in T} \inf_{z > a(t)(\log t)^{-\varepsilon} + 1} P\left(\psi_t(v) > x + z \mid \psi_t(\hat{Z}_t^{(2)}) \in [z - 1, z], \psi_t(v) > z - 1, |v| > et\right) - (\log t)^{-(k-\varepsilon)} \\ & = \left\{ \sup_{\mathbb{S}_\infty \subset \mathbb{T}_\infty: \mathbf{P}(\mathbb{S}_\infty)=1} \inf_{T \in \mathbb{S}_\infty} \inf_{\substack{v \in T: \\ |v| > et}} \inf_{z > a(t)(\log t)^{-\varepsilon} + 1} \frac{\mathcal{P}(\psi_t(v) > x + z)}{\mathcal{P}(\psi_t(v) > z - 1)} \right\} - (\log t)^{-(k-\varepsilon)}. \end{aligned} \quad (35)$$

(Here the condition that $|v| > et$ is just for technical reasons later.) Our aim is therefore to bound the quantity above. For $v \in V(T), t > 0$ we define the function

$$f_{v,t}(y) = y - \deg v - \frac{|v|}{t} \log(y - \deg(v)),$$

so that $\psi_t(v) = f_{v,t}(\xi(v)) \mathbb{1}\{t(\xi(v) - \deg(v)) > |v|\}$. Note that for $y \in I_{v,t} := (|v|/t + \deg(v), \infty)$,

$$\frac{d}{dy} f_{v,t}(y) = 1 - \frac{|v|}{t} \frac{1}{y - \deg(v)} > 0;$$

that is, $f_{v,t}$ has a continuous, increasing inverse on the interval $I_{v,t}$, which we denote by $f_{v,t}^{-1}$. Also note that for $y \in I_{v,t}$ we have that $f_{v,t}(y) \in J_{v,t} := (|v|/t - |v|/t \log(|v|/t), \infty)$, and vice versa. We give two preliminary claims.

- *Claim 1:* Let $w > \deg(v) + e$ and $u > 0$. If $\frac{f_{v,t}(w+u)}{f_{v,t}(w)} < \frac{1}{1 - 2(\log t)^{-(k-\varepsilon)}}$ then

$$\frac{w - \deg(v)}{w + u - \deg(v)} > 1 - 2(\log t)^{-(k-\varepsilon)}. \quad (36)$$

Proof of Claim 1: We prove the contrapositive. Note that for $y > e$ the function $g(y) = \frac{\log(y)}{y}$ is decreasing. Therefore, if (36) does not hold then

$$\begin{aligned} f_{v,t}(w+u) = u + w - \deg(v) - \frac{|v|}{t} \log(u + w - \deg(v)) & \geq \frac{u + w - \deg(v)}{w - \deg(v)} \left[w - \deg(v) - \frac{|v|}{t} \log(w - \deg(v)) \right] \\ & \geq \frac{1}{1 - 2(\log t)^{-(k-\varepsilon)}} f_{v,t}(w), \end{aligned}$$

which proves the claim.

- *Claim 2:* Set $x_t = a(t)(\log t)^{-k}$. For all (deterministically) sufficiently large t and any $z \geq a(t)(\log t)^{-\varepsilon} + 1$,

$$\frac{x_t + z}{z - 1} < \frac{1}{1 - 2(\log t)^{-(k-\varepsilon)}}.$$

Proof of Claim 2: We can compute, provided t is sufficiently large, that

$$\frac{x_t + z}{z - 1} = 1 + \frac{x_t + 1}{z - 1} \leq 1 + \frac{a(t)(\log t)^{-k} + 1}{a(t)(\log t)^{-\varepsilon}} < \frac{1}{1 - 2(\log t)^{-(k-\varepsilon)}},$$

which proves the claim.

Recall from the last line of (35) that we are assuming the following:

- $z > a(t)(\log t)^{-\varepsilon} + 1$,
- $|v| > et$.

Together, these imply that $z - 1 \in J_{v,t}$. Now set $w = f_{v,t}^{-1}(z - 1)$ and $u = f_{v,t}^{-1}(z + x_t) - w$, so that $\frac{x_t + z}{z - 1} = \frac{f_{v,t}(u + w)}{f_{v,t}(w)}$. (Recall that $x_t = a(t)(\log t)^{-k}$). Since $z - 1 \in J_{v,t}$ and $|v| > et$, it follows that $w \in I_{v,t}$ and in particular that $w > \deg(v) + e$. Moreover, since $f_{v,t}^{-1}$ is increasing on $J_{v,t}$ it follows that $u > 0$.

Now note that, by Claim 2 and the assumption that $z > a(t)(\log t)^{-\varepsilon} + 1$,

$$\frac{f_{v,t}(u + w)}{f_{v,t}(w)} = \frac{x_t + z}{z - 1} < \frac{1}{1 - 2(\log t)^{-(k-\varepsilon)}}.$$

It therefore follows from Claim 1 that

$$\frac{f_{v,t}^{-1}(z - 1)}{f_{v,t}^{-1}(x + z)} = \frac{w}{w + u} > \frac{w - \deg v}{w + u - \deg v} > 1 - 2(\log t)^{-(k-\varepsilon)}. \quad (37)$$

Also note that the final expression in (35) can be written as (taking $x = x_t$)

$$\frac{\mathcal{P}(\psi_t(v) > x_t + z)}{\mathcal{P}(\psi_t(v) > z - 1)} = \frac{\mathcal{P}(f_{t,v}(\xi(v)) > x_t + z \text{ and } t(\xi(v) - \deg(v)) > |v|)}{\mathcal{P}(f_{t,v}(\xi(v)) > z - 1 \text{ and } t(\xi(v) - \deg(v)) > |v|)} = \frac{\mathcal{P}(f_{t,v}(\xi(v)) > x_t + z)}{\mathcal{P}(f_{t,v}(\xi(v)) > z - 1)}.$$

Here, the final equality holds because of our restrictions on v and z ; in particular, since $z - 1 \in J_{v,t}$ and $w = f_{v,t}^{-1}(z - 1) > \deg v + e$, the requirement that $\xi(v) > f_{v,t}^{-1}(z - 1)$ already implies that $\xi(v) > \deg v + e$, so we deduce that necessarily

$$\xi(v) - \deg v - \frac{|v|}{t} \geq f_{t,v}(\xi(v)) > 0.$$

From (35) and (37) we thus obtain the bound

$$\begin{aligned} P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)}) > x_t) &\geq \sup_{\mathbb{S}_\infty \subset \mathbb{T}_\infty : \mathbf{P}(\mathbb{S}_\infty) = 1} \inf_{T \in \mathbb{S}_\infty} \inf_{\substack{v \in T \\ |v| > et}} \inf_{z: z-1 > a(t)(\log t)^{-\varepsilon}} \left(\frac{f_{v,t}^{-1}(z - 1)}{f_{v,t}^{-1}(x_t + z)} \right)^\alpha - (\log t)^{-(k-\varepsilon)} \\ &> (1 - 2(\log t)^{-(k-\varepsilon)})^\alpha - (\log t)^{-(k-\varepsilon)}. \end{aligned}$$

Recalling that $1 - e^{-x} \leq 3x/2$ for $x \leq 2/3$ we can conclude that for t large enough

$$\begin{aligned} P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)}) < a(t)(\log t)^{-k}) &\leq 1 - (1 - 2(\log t)^{-(k-\varepsilon)})^\alpha + (\log t)^{-(k-\varepsilon)} \\ &\leq 4\alpha(\log t)^{-(k-\varepsilon)}, \end{aligned}$$

as required. \square

We can now use this to derive an almost sure lower bound on the gap of ψ_t between the maximiser and the third maximiser.

Proposition 5.6. *Take any $k > 1 + \frac{1}{\alpha-d}$. Under Assumption 1.3, eventually P -almost surely:*

$$\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(3)}) \geq a(t)(\log t)^{-k}.$$

Proof. It holds that

$$\begin{aligned} P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(3)}) < x) &\leq P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)}) < x, \psi_t(\hat{Z}_t^{(2)}) - \psi_t(\hat{Z}_t^{(3)}) < x) \\ &= P(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)}) < x | \psi_t(\hat{Z}_t^{(2)}) - \psi_t(\hat{Z}_t^{(3)}) < x) P(\psi_t(\hat{Z}_t^{(2)}) - \psi_t(\hat{Z}_t^{(3)}) < x). \end{aligned}$$

Note from the proof of Lemma 5.5 that the estimate in (33) does not depend on $\psi_t(\hat{Z}_t^{(3)})$. Thus we can use the two estimates (33) and (34) to bound the probability above. That is, choosing some small $\varepsilon > 0$ (to be fixed precisely later), we obtain that

$$P\left(\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(3)}) < 2a(t)(\log t)^{-k}\right) \leq C(\log t)^{-2(k-\varepsilon)}. \quad (38)$$

Now, take $t_n = e^{n^\gamma}$ for some $\frac{1}{2(k-\varepsilon)} < \gamma < \frac{1}{k+1+\frac{1}{\alpha-d}} < 1$ (since $k > 1 + \frac{1}{\alpha-d}$, we can always choose $\varepsilon > 0$ small enough that this is possible). Then

$$\sum_{n \in \mathbb{N}} P\left(\psi_{t_n}(\hat{Z}_{t_n}^{(1)}) - \psi_{t_n}(\hat{Z}_{t_n}^{(3)}) < 2a(t_n)(\log t_n)^{-k}\right) \leq C \sum_{n \in \mathbb{N}} n^{-\gamma 2(k-\varepsilon)} < \infty, \quad (39)$$

so by Borel-Cantelli along the sequence t_n we have that eventually almost surely

$$\psi_{t_n}(\hat{Z}_{t_n}^{(1)}) - \psi_{t_n}(\hat{Z}_{t_n}^{(3)}) \geq 2a(t_n)(\log t_n)^{-k}.$$

Finally, for $t \in [t_n, t_{n+1})$, since $\psi_t(\hat{Z}_t^{(i)})$ is an increasing function for each $i = 1, 2, 3$, we compute invoking Lemma 5.3 that

$$\begin{aligned} \psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(3)}) &\geq [\psi_{t_n}(\hat{Z}_{t_n}^{(1)}) - \psi_{t_n}(\hat{Z}_{t_n}^{(3)})] - [\psi_{t_{n+1}}(\hat{Z}_{t_{n+1}}^{(3)}) - \psi_{t_n}(\hat{Z}_{t_n}^{(3)})] \\ &\geq 2a(t_n)(\log t_n)^{-k} - \frac{t_{n+1} - t_n}{t_n} a(t_{n+1})(\log t_{n+1})^{\frac{1}{\alpha-d} + \frac{\varepsilon}{2}} \\ &\geq 2a(t_n)(\log t_n)^{-k} - (\log t_n)^{-(1-\gamma)/\gamma} a(t_{n+1})(\log t_n)^{\frac{1}{\alpha-d} + \varepsilon} \\ &\geq a(t)(\log t)^{-k}. \end{aligned} \quad \square$$

Remark 5.7. *Note that in the proof above it was essential for (39) to have the 2 in the exponent in (38) coming from multiplying the probabilities for the gaps $\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)})$ and $\psi_t(\hat{Z}_t^{(2)}) - \psi_t(\hat{Z}_t^{(3)})$. For this reason the argument above would not give an almost sure bound for the gap $\psi_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(2)})$ (for which we know the result is not true).*

6 Spectral results

In this section we derive a concentration result for the principal eigenfunction of the Anderson Hamiltonian $H = \Delta + \xi$ restricted to a suitable subset of $V(T_\infty)$. This will be our main tool for proving the concentration result for $u(t, \cdot)$.

In Section 6.1 we explain how eigenfunction localisation implies a similar localisation result for $u(t, \cdot)$, and rephrase localisation of $u(t, \cdot)$ in terms of localisation of the principal eigenfunction (Lemma 6.1). In Sections 6.2 and 6.3 we establish an almost sure version of this criterion under Assumption 1.3, and in Section 6.4 we establish an analogous high probability version under Assumption 1.1.

6.1 Role of the principal eigenfunction

Take some finite, connected set $\Lambda \subset V(T_\infty)$, with $O \in \Lambda$, and consider the equation (1) restricted to Λ with zero boundary condition. As before, there is a Feynman-Kac representation for the solution on Λ , namely

$$u_\Lambda(t, z) = \mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) \, ds \right\} \mathbb{1}\{X_t = z, \tau_\Lambda > t\} \right], \quad t > 0, z \in \Lambda.$$

Let $\lambda_\Lambda^{(1)} \geq \dots \geq \lambda_\Lambda^{(|\Lambda|)}$ and $\phi_\Lambda^{(1)}, \dots, \phi_\Lambda^{(|\Lambda|)}$ be the respective eigenvalues and corresponding orthogonal eigenfunctions of the Anderson Hamiltonian H_Λ restricted to the class of real-valued functions supported on Λ . If $y \in \Lambda$ and $\phi_\Lambda^{(1)}$ is normalised so that $\phi_\Lambda^{(1)}(y) = 1$, then (e.g. see [MP16, Proposition 3.3]; the same proof applies in our setting) we have the representation

$$\phi_\Lambda^{(1)}(x) = \mathbb{E}_x \left[\exp \left\{ \int_0^{H_y} (\xi(X_s) - \lambda_\Lambda^{(1)}) \, ds \right\} \mathbb{1}\{H_y < \tau_\Lambda\} \right]. \quad (40)$$

We are interested in using (40) when y is a “good” site, in order to show that the contribution from other sites is negligible. In particular, suppose that $\Omega \subset \Lambda$ is a small set of good sites, and set

$$u_{\Omega, \Lambda}(t, z) := \mathbb{E}_O \left[\exp \left\{ \int_0^t \xi(X_s) \, ds \right\} \mathbb{1}\{X_t = z, \tau_\Lambda > t, H_\Omega < t\} \right], \quad t > 0, z \in \Lambda. \quad (41)$$

We also define the following gap

$$\tilde{g}_{\Omega, \Lambda} = \min_{z \in \Omega} [\xi(z) - \deg(z)] - \max_{z \in \Lambda \setminus \Omega} [\xi(z) - \deg(z)].$$

For $A \subset \Lambda$ also recall from page 21 that

$$\tilde{Z}_A = \arg_{z \in A} \max \{\xi(z) - \deg(z)\}, \quad \tilde{g}_A = \xi(\tilde{Z}_A) - \deg(\tilde{Z}_A) - \max_{z \in A, z \neq \tilde{Z}_A} \{\xi(z) - \deg(z)\}.$$

We will use (40) to show that the principal eigenfunction localises on Ω , and transfer this result to (41) via the following lemma. Henceforth, the eigenfunction ϕ_A will be normalised so that $\phi_A(\tilde{Z}_A) = 1$.

Lemma 6.1. (cf [KLMS09, Lemma 2.4]). *If $\tilde{g}_{\Omega, \Lambda} > 0$, then for all $v \in V(T_\infty)$ and $t > 0$ we have*

$$(i) \quad u_{\Omega, \Lambda}(t, v) \leq \sum_{y \in \Omega} u_{\Omega, \Lambda}(t, y) \|\phi_{(\Lambda \setminus \Omega) \cup \{y\}}\|_2^2 \phi_{(\Lambda \setminus \Omega) \cup \{y\}}(v),$$

$$(ii) \quad \frac{\sum_{v \in \Lambda \setminus \Omega} u_{\Omega, \Lambda}(t, v)}{\sum_{v \in \Lambda} u_{\Omega, \Lambda}(t, v)} \leq \sum_{y \in \Omega} \|\phi_{(\Lambda \setminus \Omega) \cup \{y\}}\|_2^2 \sum_{v \in \Lambda \setminus \Omega} \phi_{(\Lambda \setminus \Omega) \cup \{y\}}(v).$$

Proof. It follows from an eigenvalue expansion exactly as in the lower bound of [dKd20, Lemma 2.1] that for any $y \in \Omega$,

$$\mathbb{E}_y \left[\exp \left\{ \int_0^u \xi(X_s) \, ds \right\} \mathbb{1}\{X_t = y, \tau_\Lambda > t, H_{\Omega \setminus \{y\}} > t\} \right] \geq \frac{e^{u\lambda_{\Lambda \setminus \Omega \cup \{y\}}^{(1)}} \phi_{(\Lambda \setminus \Omega) \cup \{y\}}(y)^2}{\|\phi_{(\Lambda \setminus \Omega) \cup \{y\}}\|_2^2} = \frac{e^{u\lambda_{\Lambda \setminus \Omega \cup \{y\}}^{(1)}}}{\|\phi_{(\Lambda \setminus \Omega) \cup \{y\}}\|_2^2}.$$

(note that [dKd20, Lemma 2.1] is for the normalised eigenvectors, hence the extra renormalisation constant in our case). Recall that we also established the time reversal property for the Feynman-Kac formula in Proposition 2.13. The result therefore follows by exactly the same proof as in [KLMS09, Lemma 2.4], which gives the corresponding statement on \mathbb{Z}^d . \square

6.2 Choice of Λ_t

To apply Lemma 6.1, for each $t > 0$ we will need to make an appropriate choice of a set Λ . Following [KLMS09, Section 5.1], recall that $z = \frac{2\alpha}{\alpha-2}$ under Assumption 1.3 and let us define for $i = 1, 2$ the sets

$$\Gamma_t^{(i)} := \left\{ v \in V(T_\infty) : |v - \hat{Z}_t^{(i)}| + \min\{|v|, |\hat{Z}_t^{(i)}|\} \leq (1 + (\log t)^{-z}) |\hat{Z}_t^{(i)}| \right\},$$

i.e. the set of all points contained within a path of length $(1 + (\log t)^{-z}) |\hat{Z}_t^{(i)}|$ that also contains $\hat{Z}_t^{(i)}$. We will make the following choices for Λ and Ω at time t :

$$\Lambda_t := \Gamma_t^{(1)} \cup \Gamma_t^{(2)} \text{ and } \Omega_t := \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}.$$

Note that, on the event $\{\tilde{g}_{\Omega_t, \Lambda_t} > 0\}$

$$\tilde{g}_{\Omega_t, \Lambda_t} = \min\{\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}, \tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}}\}.$$

We first show in Lemma 6.3 that for all $\varepsilon > 0$, we have that $\tilde{g}_{\Omega_t, \Lambda_t} \geq a(t)(\log t)^{-\varepsilon}$ eventually almost surely. Then we show in Lemma 6.6 that we can restrict to a direct path of $(X_s)_{s \geq 0}$ to bound (40). Estimating (40) by the contribution by the direct path, we then show that the value of the eigenfunction summed over points not in $\{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$ is negligible in Lemma 6.8. Finally we use Lemma 6.1 to transfer this result back to the solution of the PAM.

It will first be important to bound the volume of Λ_t .

Lemma 6.2. *Under Assumption 1.3, for any $\delta > 0$, eventually P -almost surely, for each $i = 1, 2$:*

$$\#\Gamma_t^{(i)} \leq \frac{1}{2}(\log t)^{-(z-\delta)} |\hat{Z}_t^{(i)}|^2 \leq \frac{1}{2}(\log t)^{-(z-\delta) + \frac{2}{\alpha-2}} r(t)^2.$$

Proof. We just bound $\#\Gamma_t^{(1)}$ (the same argument works for $\#\Gamma_t^{(2)}$). Note that $d = 2$ under Assumption 1.3. Fix $t > 0$, set $R = |\hat{Z}_t^{(1)}|$, set $r = \frac{1}{(\log t)^z} |\hat{Z}_t^{(1)}|$, and let $\gamma_{O,t}^{(1)}$ denote the direct path from O to $\hat{Z}_t^{(1)}$. Note that, if $v \in \gamma_{O,t}^{(1)}$ and $u_1, \dots, u_{\deg v - 1}$ are the offspring of v , then

$$P(\deg v = k + 1 \mid v \in \gamma_{O,t}^{(1)}) = \mu_k \sum_{i=1}^k P(u_i \preceq \hat{Z}_t^{(1)} \mid \deg v = k + 1) \leq k\mu_k. \quad (42)$$

We deduce that the offspring tails along the path $\gamma_{O,t}^{(1)}$ independently satisfy a size-biased tail bound, which is therefore also super-polynomial under Assumption 1.3. Moreover, due to the Galton-Watson structure, the subtrees rooted at each of the u_i are independent Galton-Watson trees, of which one is conditioned to survive to $\hat{Z}_t^{(1)}$, a second of which may be conditioned to survive forever, and the rest of which are unconditioned. Let $M = \sum_{v \in \gamma_{O,t}^{(1)}} (\deg v - 2) + 1$ be the number of such subtrees attached to $\gamma_{O,t}^{(1)}$ (including at the final vertex $\hat{Z}_t^{(1)}$). We deduce the following (working under Assumption 1.3 throughout).

- (a) For any $q < \infty$, there exists $c < \infty$ such that $P\left(\sum_{v \in \gamma_{O,t}^{(1)}} \deg v \geq R\lambda^{1/2}\right) \leq c\lambda^{-q}$ (e.g. by [Arc20, Lemma A.2(iii)]).
- (b) Conditionally on $\sum_{v \in \gamma_{O,t}^{(1)}} \deg v < R\lambda^{1/2}$, the number N of subtrees attached to $\gamma_{O,t}^{(1)}$ with volume exceeding r^2 is stochastically dominated by two more than a $\text{Binomial}(R\lambda^{1/2}, cr^{-1})$ random variable (by Lemma 2.2, since $\beta = 2$ under Assumption 1.3). Therefore $P(N > 2(\log t)^z \lambda^{1/2}) \leq e^{-c(\log t)^z \lambda^{1/2}}$ (by a Chernoff bound).
- (c) Let $(T_i)_{i=1}^M$ be the subtrees attached to the path $\gamma_{O,t}^{(1)}$. Conditionally on $N \leq 2(\log t)^z \lambda^{1/2}$, we have by [Arc20, Lemma A.1], Lemma 2.2 and a union bound that

$$P\left(\sum_{i=1}^M \#V(T_i) \mathbb{1}\{\#V(T_i) \leq r^2\} \geq (\log t)^z \lambda r^2\right) \leq 2(\log t)^z \lambda^{1/2} e^{-c\lambda^{1/2}}.$$

(d) Now consider $\#V(T_i)$ for some Galton-Watson tree where $\#V(T_i) \geq r^2$. If ρ_i is the “root” of T_i , we then have for any $q < \infty$ that $P(\#B_{T_i}(\rho_i, r) \geq \lambda r^2) \leq c\lambda^{-3q}$ by Proposition 2.3(ii). (Proposition 2.3(ii) was written for T_∞ but applies to a finite GW tree by the same proof).

(e) It follows from the previous point that $\sum_{i=1}^M \#B_{T_i}(\rho_i, r) \mathbb{1}\{\#V(T_i) \geq r^2\} \stackrel{s.d.}{\preceq} r^2 \left(Y + \sum_{i=1}^N X_i \right)$ where $P(X_i \geq \lambda) \leq c\lambda^{-3q}$ independently for each i , and $P(Y \geq \lambda) \leq c\lambda^{-3q}$ too (Y is the sum of the volumes of the balls of radius r in the two subtrees that may be conditioned to survive). Conditionally on $N \leq 2(\log t)^z \lambda^{1/2}$, we therefore have from (d) and [Arc20, Lemma A.3(iii)] that for any $q < \infty$, there exists $c < \infty$ such that

$$P\left(\sum_{i=1}^M \#B_{T_i}(\rho_i, r) \mathbb{1}\{\#V(T_i) \geq r^2\} \geq r^2(\log t)^z \lambda\right) \leq P\left(Y + \sum_{i=1}^N X_i \geq \frac{1}{2} N \lambda^{1/2}\right) \leq c\lambda^{-q}.$$

We combine the bounds in (a), (b), (c) and (e) above in the following union bound (to include the vertices in $\gamma_{O,t}^{(1)}$ we have simply replaced $2(\log t)^z \lambda r^2$ with $4(\log t)^z \lambda r^2$ below):

$$\begin{aligned} P\left(\#\Gamma_t^{(1)} \geq 4(\log t)^z r^2 \lambda\right) &\leq P\left(\sum_{v \in \gamma_{O,t}^{(1)}} \deg v \geq R\lambda^{1/2}\right) + P\left(N \geq 2(\log t)^z \lambda^{1/2} \mid \sum_{v \in \gamma_{O,t}^{(1)}} \deg v < R\lambda^{1/2}\right) \\ &\quad + P\left(\sum_{i=1}^M \#V(T_i) \mathbb{1}\{V(T_i) \leq r^2\} \geq r^2(\log t)^z \lambda \mid N < 2(\log t)^z \lambda^{1/2}\right) \\ &\quad + P\left(\sum_{i=1}^M \#B_{T_i}(\rho_i, r) \mathbb{1}\{V(T_i) \geq r^2\} r^2(\log t)^z \lambda \mid N < 2(\log t)^z \lambda^{1/2}\right) \\ &\leq 2(\log t)^z \lambda^{1/2} e^{-c\lambda^{1/2}} + C\lambda^{-q}. \end{aligned}$$

Taking $\lambda = (\log t)^\delta$ and recalling that $r = \frac{1}{(\log t)^z} |\hat{Z}_t^{(1)}|$ we get for any $q > 0$ that there exists $C < \infty$ such that

$$P\left(\#\Gamma_t^{(1)} \geq 4(\log t)^{z+\delta-2z} |\hat{Z}_t^{(1)}|^2\right) \leq C(\log t)^{-q}.$$

(Here we can replace δq by q on the right-hand side since q was arbitrary.) Therefore, taking $q > 1$ and $\varepsilon < 1 - q$, applying Markov's inequality and using Lemma 5.4, we deduce that

$$\begin{aligned} P\left(\#\Gamma_t^{(1)} \geq 4(\log t)^{\delta-z} |\hat{Z}_t^{(1)}|^2 \text{ i.o.}\right) &\leq \limsup_{T \rightarrow \infty} \sum_{t \geq T: \hat{Z}_t^{(1)} \neq \hat{Z}_{t-}^{(1)}} P\left(\#\Gamma_t^{(1)} \geq 4(\log t)^{z+\delta-2z} |\hat{Z}_t^{(1)}|^2\right) \\ &\leq \limsup_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{\substack{t \in [2^n, 2^{n+1}]: \\ \hat{Z}_t^{(1)} \neq \hat{Z}_{t-}^{(1)}}} C(\log t)^{-q} \\ &\leq \limsup_{N \rightarrow \infty} \left\{ o(1) + \sum_{n=N}^{\infty} c n^\varepsilon n^{-q} \right\} = 0. \end{aligned}$$

The same bound holds for $\#\Gamma_t^{(2)}$ in place of $\#\Gamma_t^{(1)}$ by exactly the same arguments. Finally, since $\delta > 0$ was arbitrary, we can replace 4 with $\frac{1}{2}$ in the statement of the result. The final upper bound follows from applying Proposition 5.1(iii). \square

As a consequence of Lemma 6.2, we deduce the following.

Lemma 6.3. *P-almost surely under Assumption 1.3, we have for all sufficiently large t that*

$$(i) \sup_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \xi(v) \leq (\log t)^{-\frac{1}{\alpha}} a(t),$$

$$(ii) \quad \tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}} = \hat{Z}_t^{(2)} \text{ and } \tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}} = \hat{Z}_t^{(1)},$$

$$(iii) \quad \text{For any } \varepsilon > 0 \text{ and each } i = 1, 2, \tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(i)}\}} \geq a(t)(\log t)^{-\varepsilon}.$$

Proof. (i) Choose some $\delta < \frac{2}{\alpha-2}$. We have from Lemma 6.2 that

$$\#\Lambda_t \leq (\log t)^{-(z-\delta)+\frac{2}{\alpha-2}} r(t)^2$$

eventually almost surely. On this event, since the potentials at different vertices are independent of each other, and since conditioning on $u \neq \hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}$ can only (stochastically) reduce $\xi(u)$, we have from a union bound that on the event $\{\#\Lambda_t \leq (\log t)^{-(z-\delta)+\frac{2}{\alpha-2}} r(t)^2\}$, for any $m > 0$,

$$P\left(\sup_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \xi(v) \geq (\log t)^{-m} a(t)\right) \leq (\log t)^{-(z-\delta)+\frac{2}{\alpha-2}} r(t)^2 (\log t)^{m\alpha} a(t)^{-\alpha} \leq (\log t)^{-y},$$

where $y = (z-\delta) - \frac{2}{\alpha-2} - m\alpha > 1$ when $m = \frac{1}{\alpha}$ by our choice of z . Then, by Lemma 5.4 we can choose $\varepsilon > 0$ small enough that

$$\begin{aligned} P\left(\sup_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \xi(v) \geq (\log t)^{-m} a(t) \text{ i.o.}\right) &\leq \limsup_{T \rightarrow \infty} \sum_{t \geq T: \hat{Z}_t^{(2)} \neq \hat{Z}_{t-}^{(2)}} P\left(\sup_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \xi(v) \geq (\log t)^{-m} a(t)\right) \\ &\leq \limsup_{N \rightarrow \infty} \left\{ o(1) + \sum_{n=N}^{\infty} \sum_{\substack{t \in [2^n, 2^{n+1}]: \\ \hat{Z}_t^{(2)} \neq \hat{Z}_{t-}^{(2)}}} (\log t)^{-y} \right\} \\ &\leq \limsup_{N \rightarrow \infty} \left\{ o(1) + \sum_{n=N}^{\infty} c n^\varepsilon n^{-y} \right\} = 0. \end{aligned}$$

This gives the result.

(ii) Take some $0 < \varepsilon < m$. It follows from (i) and Proposition 5.1(ii) that eventually almost surely, for each $i = 1, 2$,

$$\xi(\hat{Z}_t^{(i)}) - \deg(\hat{Z}_t^{(i)}) \geq a(t)(\log t)^{-\varepsilon} > a(t)(\log t)^{-\frac{1}{\alpha}} \geq \sup_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} (\xi(v) - \deg(v)).$$

(iii) For $0 < \varepsilon < m$, the statement follows from (i) since $\xi(\hat{Z}_t^{(i)}) - \deg \hat{Z}_t^{(i)} \geq 2a(t)(\log t)^{-\varepsilon}$ eventually almost surely by Proposition 5.1(ii) (clearly this implies the result for all $\varepsilon \geq m$ as well). \square

We now use Lemma 6.3 to control the paths contributing to (40), first defining some notation.

Given $x \in \Lambda_t$, we define the set of all paths from x to $\hat{Z}_t^{(1)}$ that stay in $\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}$ and end on the first hit to $\hat{Z}_t^{(1)}$ as $\Gamma_{x,t}^{(1)}$. Similarly, let $\Gamma_{x,t}^{(2)}$ denote the set of all paths from x to $\hat{Z}_t^{(2)}$ that stay in $\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}$ and end on the first hit to $\hat{Z}_t^{(2)}$. If $x \in \Gamma_t^{(1)} \cup \Gamma_t^{(2)}$ we denote the direct path from x to $\hat{Z}_t^{(1)}$ (respectively $\hat{Z}_t^{(2)}$) as $\gamma_{x,t}^{(1)}$ (respectively $\gamma_{x,t}^{(2)}$). Given $t > 0$, we also introduce the notation $\pi(X_{[0,t]})$ to denote the path that consists of all the steps taken by the random walk $(X_s)_{s \geq 0}$ between times 0 and t .

For $t > 0$ and $x \in \Lambda_t$, we let

$$\begin{aligned} \Gamma_{x,t}^{(1,\text{bad})} &:= \left\{ \gamma \in \Gamma_{x,t}^{(1)} : \prod_{v \in \gamma \setminus \gamma_{x,t}^{(1)}} \frac{\deg v}{\deg v - [\xi(v) + \deg \hat{Z}_t^{(1)} - \xi(\hat{Z}_t^{(1)})]} \geq 1 \right\}, \\ \Gamma_{x,t}^{(2,\text{bad})} &:= \left\{ \gamma \in \Gamma_{x,t}^{(2)} : \prod_{v \in \gamma \setminus \gamma_{x,t}^{(2)}} \frac{\deg v}{\deg v - [\xi(v) + \deg \hat{Z}_t^{(2)} - \xi(\hat{Z}_t^{(2)})]} \geq 1 \right\}. \end{aligned} \tag{43}$$

Lemma 6.4. *P-almost surely under Assumption 1.3, there exists $t_0 < \infty$ such that for all $t \geq t_0$ and all $x \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$, $\Gamma_{x,t}^{(1,bad)} = \Gamma_{x,t}^{(2,bad)} = \emptyset$.*

Proof. It follows exactly as in the proofs of Proposition 5.1(ii) and (iii) that eventually almost surely, for all $v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$,

$$\xi(v) < \xi(\hat{Z}_t^{(i)}) - \deg \hat{Z}_t^{(i)}$$

for each $i = 1, 2$, which implies that each term in the products in (43) is less than 1. \square

Lemma 6.5. *Fix some constant $C_{\alpha,\beta}$. Under Assumption 1.3, for any $\varepsilon > 0$, $\#B(\hat{Z}_t^{(2)}, C_{\alpha,\beta}) \leq (\log t)^\varepsilon$ eventually P-almost surely.*

Proof. By very similar arguments to Proposition 2.3(ii), it holds that there exists $C < \infty$ such that $\mathbf{P}(\#B(\hat{Z}_t^{(2)}, C_{\alpha,\beta}) \geq \lambda) \leq C\lambda^{-K}$ for any $K < \infty$ and all $\lambda > 1$. The proposition therefore follows by choosing $K > 2\varepsilon^{-1}$, setting $\lambda = (\log t)^\varepsilon$ and applying Borel-Cantelli and Lemma 5.4 similarly to the proofs of Lemmas 6.2 and 6.3. \square

6.3 Bounds on eigenfunctions

In this section we apply the results on Λ_t to bound $\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}$ and $\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}}^{(1)}$. In what follows we will assume that $\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}$ is always normalised so that $\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(\hat{Z}_t^{(2)}) = 1$, and vice versa.

Lemma 6.6. *Eventually P-almost surely under Assumption 1.3, for each $i = 1, 2$ and each $x \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$,*

$$\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(x) \leq \prod_{v \in \gamma_{x,t}^{(2)}} \frac{\deg v}{\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}} \quad \text{and} \quad \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}}^{(1)}(x) \leq \prod_{v \in \gamma_{x,t}^{(1)}} \frac{\deg v}{\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}}}.$$

Proof. The proof follows a similar strategy to [KLMS09, Lemma 2.3]. We only prove the first statement; the proof of the second statement works analogously. Firstly, it follows from the Rayleigh-Ritz formula (e.g. see [dKd20, Equation (2.7)]) for the principal eigenvalue of the Anderson Hamiltonian that

$$\begin{aligned} \lambda_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)} &= \sup\{\langle (\xi + \Delta)f, f \rangle_{\ell^2(T_\infty)} : f \in \ell^2(T_\infty), \text{supp}(f) \subset \Lambda_t \setminus \{\hat{Z}_t^{(1)}\}, \|f\|_2 = 1\} \\ &\geq \sup_{z \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}\}} \{\langle (\xi + \Delta)\delta_z, \delta_z \rangle_{\ell^2(T_\infty)}\} \\ &= \sup_{z \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}\}} \{\xi(z) - \deg(z)\} = \xi(\tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}) - \deg(\tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}). \end{aligned}$$

Then, taking $y = \hat{Z}_t^{(2)}$ in (40) we have for $x \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$,

$$\begin{aligned} \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(x) &= \mathbb{E}_x \left[\exp \left\{ \int_0^{H_{\hat{Z}_t^{(2)}}} (\xi(X_s) - \lambda_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}) \, ds \right\} \mathbb{1}_{\{H_{\hat{Z}_t^{(2)}} < \tau_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}\}} \right] \\ &\leq \mathbb{E}_x \left[\exp \left\{ \int_0^{H_{\hat{Z}_t^{(2)}}} \left(\xi(X_s) - [\xi(\tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}) - \deg \tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}] \right) \, ds \right\} \mathbb{1}_{\{H_{\hat{Z}_t^{(2)}} < \tau_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}\}} \right]. \end{aligned}$$

Now let $(T_v)_{v \in \gamma_{x,t}^{(2)}}$ denote a set of independent exponential random variables with respective parameters $(\deg v)_{v \in \gamma_{x,t}^{(2)}}$. By Lemma 6.3(ii), we can work on the event $\{\tilde{Z}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}} = \hat{Z}_t^{(2)}\}$. We therefore have eventually

P -almost surely that

$$\begin{aligned}
\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(x) &\leq \mathbb{E}_x \left[\exp \left\{ \int_0^{H_{\hat{Z}_t^{(2)}}} (\xi(X_s) - [\xi(\hat{Z}_t^{(2)}) - \deg \hat{Z}_t^{(2)}]) \, ds \right\} \mathbb{1}_{\{H_{\hat{Z}_t^{(2)}} < \tau_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}\}} \right] \\
&\leq \sum_{\gamma \in \Gamma_{x,t}^{(2)}} \mathbb{P} \left(\pi(X_{[0, H_{\hat{Z}_t^{(2)}}]}) = \gamma \right) \prod_{v \in \gamma} \mathbb{E} \left[\exp \left\{ T_v [\xi(v) + \deg \hat{Z}_t^{(2)} - \xi(\hat{Z}_t^{(2)})] \right\} \right] \\
&\leq \sum_{\gamma \in \Gamma_{x,t}^{(2)}} \mathbb{P} \left(\pi(X_{[0, H_{\hat{Z}_t^{(2)}}]}) = \gamma \right) \prod_{v \in \gamma} \frac{\deg v}{\deg v - [\xi(v) + \deg \hat{Z}_t^{(2)} - \xi(\hat{Z}_t^{(2)})]}.
\end{aligned}$$

Note that every path $\gamma \in \Gamma_{x,t}^{(2)}$ contains $\gamma_{x,t}^{(2)}$ as a subset. Applying Lemma 6.4 and decomposing the product according to whether vertices are in $\gamma_{x,t}^{(2)}$ or not (if some vertices in $\gamma_{x,t}^{(2)}$ are visited more than once, we put repeat visits into the second product), we obtain that

$$\begin{aligned}
\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(x) &\leq \prod_{v \in \gamma_{x,t}^{(2)}} \frac{\deg v}{\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}} \sum_{\gamma \in \Gamma_{x,t}^{(2)}} \mathbb{P} \left(\pi(X_{[0, H_{\hat{Z}_t^{(2)}}]}) = \gamma \right) \prod_{v \in \gamma \setminus \gamma_{x,t}^{(2)}} \frac{\deg v}{\deg v - [\xi(v) + \deg \hat{Z}_t^{(2)} - \xi(\hat{Z}_t^{(2)})]} \\
&\leq \prod_{v \in \gamma_{x,t}^{(2)}} \frac{\deg v}{\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}}. \quad \square
\end{aligned}$$

Finally, we will need the following lemma to bound the function appearing in the statement of Lemma 6.6.

Lemma 6.7. *P -almost surely under Assumption 1.3 we have the following.*

(i) *There exist deterministic $D, E < \infty$ such that for all sufficiently large t and each $i = 1, 2$,*

$$\sup_{x \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \left\{ \sum_{v \in \gamma_{x,t}^{(i)}} \log(\deg v) - [D \log t + E |\gamma_{x,t}^{(i)}|] \right\} \leq 0.$$

(ii) *Fix any $C < \infty$. Then for any $\delta > 0$, it holds for all sufficiently large t and each $i = 1, 2$ that*

$$\sup_{x \in (\Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}) \cap V(B(\hat{Z}_t^{(i)}, C))} \left\{ \sum_{v \in \gamma_{x,t}^{(i)}} \log(\deg v) - \delta \log \log t \right\} \leq 0.$$

Proof. (i) The statement follows from Lemma 2.9 and Proposition 5.1(iii).

(ii) Fix $i = 1$ or $i = 2$. Take any $\varepsilon > 0$, and choose $c > 2(\varepsilon + 1)\delta^{-1} + 1$. Working backwards along the path $\gamma_{x,t}^{(i)}$ starting from $\hat{Z}_t^{(i)}$ and ending at x , the degrees of the vertices independently satisfy the size-biased tail bound of (42) (by the same logic), so that $A := \log \left(\sup_{t \geq 0} \sup_{v \in \gamma_{x,t}^{(i)}} \mathbf{E}[(\deg v)^c] \right) < \infty$.

Now let $x \in V(B(\hat{Z}_t^{(i)}, C)) \setminus \{\hat{Z}_t^{(i)}\}$. We compute, using a Chernoff bound, that

$$\begin{aligned}
\mathbf{P} \left(\sum_{v \in \gamma_{x,t}^{(i)}} \log(\deg v) \geq \delta \log \log t \right) &\leq \mathbf{E} \left[\exp \left(c \sum_{v \in \gamma_{x,t}^{(i)}} \log(\deg v) \right) \right] \exp(-c\delta \log \log t) \\
&\leq \exp \left(A |\gamma_{x,t}^{(i)}| - c\delta \log \log t \right) \\
&\leq e^{AC} (\log t)^{-c\delta}.
\end{aligned}$$

Now for $t > 0$ let us define the event in question as

$$A_t := \left\{ \sup_{x \in (\Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}) \cap V(B(\hat{Z}_t^{(i)}, C))} \left\{ \sum_{v \in \gamma_{x,t}^{(i)}} \log(\deg v) - \delta \log \log t \right\} \geq 0 \right\}.$$

Applying a union bound over all vertices in $B(\hat{Z}_t^{(i)}, C)$ with the bound above and the bound in the proof of Lemma 6.5 yields

$$P(A_t) \leq (\log t)^{\delta/2} e^{AC} (\log t)^{-c\delta} + \mathbf{P}(\#B(\hat{Z}_t^{(i)}, C) > (\log t)^{\delta/2}) = e^{AC} (\log t)^{-(2c-1)\delta/2} + c'(\log t)^{-c\delta}.$$

Combining with Lemma 5.4 and a union bound, we deduce that

$$\begin{aligned} P(A_t \text{ i.o.}) &= \limsup_{N \rightarrow \infty} P\left((A_t)_{t \geq 2^N} \text{ i.o.}\right) \leq P\left(\exists n \geq N : \#\left\{\hat{Z}_t^{(i)} : t \in [2^n, 2^{n+1}]\right\} \leq (\log t_n)^\varepsilon\right) + \sum_{n \geq N} n^\varepsilon P(A_{2^n}) \\ &\leq o(1) + C' \sum_{n \geq N} n^\varepsilon n^{-(2c-1)\delta/2} \rightarrow 0, \end{aligned}$$

by our choice of c . This concludes the proof. \square

Lemma 6.8. *Eventually P -almost surely under Assumption 1.3, as $t \rightarrow \infty$*

$$\|\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}\|_2^2 \sum_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(v) + \|\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}}^{(1)}\|_2^2 \sum_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(2)}\}}^{(1)}(v) \rightarrow 0. \quad (44)$$

Proof. We start by bounding the first sum above. Take $\varepsilon \in (0, \beta - 1)$ and let $c := \beta - 1 - \varepsilon > 0$. We set

$$C_{\alpha, \beta} = \frac{3(q+1)}{q}$$

and split the sum into two parts, summing separately over $v \in U_t := \Lambda_t \setminus (\{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\} \cup B(\hat{Z}_t^{(2)}, C_{\alpha, \beta}))$ and $v \in W_t := (\Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}) \cap B(\hat{Z}_t^{(2)}, C_{\alpha, \beta})$.

We estimate the two parts separately in the following.

- **Part 1:** U_t . Take some $\delta > 0, \varepsilon > 0$ and $E < \infty$ as in Lemma 6.7(i). Also let $h(t) = 8(\log t)^{-(z-\delta) + \frac{2}{\alpha-2}} r(t)^2$ (as in Lemma 6.2). Define the event

$$\begin{aligned} E_1 := & \{\#\Lambda_t \leq h(t)\} \cap \left\{ \sup_{v \in U_t} \left\{ \sum_{x \in \gamma_{v,t}^{(2)}} \log(\deg x) - 2E|\gamma_{v,t}^{(2)}| \right\} \leq 0 \right\} \\ & \cap \left\{ \tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}} \geq a(t)(\log t)^{-\varepsilon} \right\} \cap \left\{ \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(v) \leq \prod_{x \in \gamma_{v,t}^{(2)}} \frac{\deg x}{\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}} \text{ for all } v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\} \right\}. \end{aligned}$$

Note that E_1 holds eventually almost surely by Proposition 5.1(i) and Lemmas 6.2, 6.7, 6.3(iii) and 6.6. Almost surely, i.e. on the event E_1 , we can therefore calculate for all sufficiently large t that

$$\begin{aligned} \sum_{v \in U_t} \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(v) &\leq \sum_{v \in U_t} \exp \left(\sum_{x \in \gamma_{v,t}^{(2)}} [\log(\deg(x)) - \log(\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}})] \right) \\ &\leq h(t) \sup_{v \in U_t} \exp \left(-|\gamma_{v,t}^{(2)}| (q \log t - O(\log \log t)) \right) \\ &\leq \exp \left(-(C_{\alpha, \beta} q - 2(q+1)) \log t + O(\log \log t) \right), \end{aligned}$$

which converges to 0 as $t \rightarrow \infty$ by our choice of $C_{\alpha, \beta}$.

- **Part 2:** W_t . We define the event

$$E_2 := E_1 \cap \{\#B(\hat{Z}_t^{(2)}, C_{\alpha,\beta}) \leq \log t\}.$$

By Lemma 6.5 and Part 1 above, it follows that E_2 holds eventually almost surely. We therefore calculate similarly to the first part that almost surely for sufficiently large t , i.e. on the event E_2 ,

$$\begin{aligned} \sum_{v \in W_t} \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(v) &\leq \sum_{v \in W_t} \exp \left(\sum_{x \in \gamma_{v,t}^{(2)}} [\log(\deg(x)) - \log(\tilde{g}_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}})] \right) \\ &\leq \exp(2EC_{\alpha,\beta} - q \log t + O(\log \log t)), \end{aligned}$$

which again goes to 0 as $t \rightarrow \infty$.

Combining these two calculations we deduce that $\sum_{v \in \Lambda_t \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}(v) \rightarrow 0$ almost surely.

This also implies that $\|\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}\|_2^2 \leq 2\|\phi_{\Lambda_t \setminus \{\hat{Z}_t^{(1)}\}}^{(1)}\|_1 \leq (1 + o(1))^2$ almost surely for all sufficiently large t , so we deduce that the first term in (44) goes to 0 almost surely as $t \rightarrow \infty$. The proof for the second sum works analogously. \square

6.4 High probability spectral statements

To prove the single site localisation whp (Theorem 1.6) we will have to prove localisation on a slightly bigger set. The strategy is the same as in the previous subsections, except that the results hold whp rather than almost surely. For this reason we do not give the full details of the proofs.

Define $\bar{R}_t := |\hat{Z}_t^{(1)}| \left(1 + \frac{1}{(\log t)^{\frac{1}{2}(1+\frac{1}{\alpha})}}\right)$. We will prove localisation on $B_{\bar{R}_t}$ whp, for which we will need the following results.

Recall from Section 3.3 that

$$\begin{aligned} Z_{B_{\bar{R}_t}} &= \arg_{z \in B_{\bar{R}_t}} \max\{\xi(z)\}, & \tilde{Z}_{B_{\bar{R}_t}} &= \arg_{z \in B_{\bar{R}_t}} \max\{\xi(z) - \deg z\}, \\ g_{B_{\bar{R}_t}} &= \xi(Z_{B_{\bar{R}_t}}) - \max_{z \in B_{\bar{R}_t}, z \neq Z_{B_{\bar{R}_t}}} \{\xi(z)\}, & \tilde{g}_{B_{\bar{R}_t}} &= \xi(\tilde{Z}_{B_{\bar{R}_t}}) - \deg(\tilde{Z}_{B_{\bar{R}_t}}) - \max_{z \in B_{\bar{R}_t}, z \neq \tilde{Z}_{B_{\bar{R}_t}}} \{\xi(z) - \deg(z)\}. \end{aligned}$$

Furthermore, we define the direct path from $x \in V(B_{\bar{R}_t})$ to $\tilde{Z}_{B_{\bar{R}_t}}$ as $\gamma_{x,t}$ and we define the set of all paths from x to $\tilde{Z}_{B_{\bar{R}_t}}$ that stay in $B_{\bar{R}_t}$ and end on the first hit to $\tilde{Z}_{B_{\bar{R}_t}}$ as $\Gamma_{x,t}$.

Lemma 6.9. [cf Lemma 6.6]. Under Assumption 1.1,

$$P \left(\phi_{B_{\bar{R}_t}}^{(1)}(x) \leq \prod_{v \in \gamma_{x,t}} \frac{\deg v}{\tilde{g}_{B_{\bar{R}_t}}} \text{ for all } x \in V(B_{\bar{R}_t}) \right) \rightarrow 1 \text{ as } t \rightarrow \infty.$$

Proof. Exactly as in Lemma 6.6,

$$\phi_{B_{\bar{R}_t}}^{(1)}(x) \leq \prod_{v \in \gamma_{x,t}} \frac{\deg v}{\tilde{g}_{B_{\bar{R}_t}}} \sum_{\gamma \in \Gamma_{x,t}} \mathbb{P} \left(\pi(X_{[0, H_{\tilde{Z}_{B_{\bar{R}_t}}}]}) = \gamma \right) \prod_{v \in \gamma \setminus \gamma_{x,t}} \frac{\deg v}{\deg v - [\xi(v) + \deg \tilde{Z}_{B_{\bar{R}_t}} - \xi(\tilde{Z}_{B_{\bar{R}_t}})]}.$$

Now recall from Proposition 5.1(ii'), Lemma 3.4(ii) and Lemma 3.7 that $\tilde{Z}_{B_{\bar{R}_t}} = Z_{B_{\bar{R}_t}}$ and $\deg \tilde{Z}_{B_{\bar{R}_t}} \leq (\log t)^B$ whp, and from Lemma 3.6 and Proposition 5.1(i') that $g_{B_{\bar{R}_t}} \geq 2a(t)(\log t)^{\frac{-(1+\varepsilon)}{\alpha}}$ for all sufficiently large t whp. On these events, we have for all $v \neq \tilde{Z}_{B_{\bar{R}_t}}$ that

$$\xi(\tilde{Z}_{B_{\bar{R}_t}}) - \deg \tilde{Z}_{B_{\bar{R}_t}} - \xi(v) > g_{B_{\bar{R}_t}} - \deg \tilde{Z}_{B_{\bar{R}_t}} > 0,$$

which implies that the corresponding term in the product on the right-hand side above is at most 1. \square

We will also need the following lemma concerning the degrees of vertices in the direct path $\gamma_{x,t}$.

Lemma 6.10. *[cf Lemma 6.7]. Under Assumption 1.1,*

(i) *There exist constants $D, E < \infty$ such that*

$$P \left(\sup_{x \in B_{\bar{R}_t}} \left\{ \sum_{v \in \gamma_{x,t}} \log(\deg v) - [D \log \bar{R}_t + E |\gamma_{x,t}|] \right\} \leq 0 \right) \rightarrow 1 \text{ as } t \rightarrow \infty,$$

(ii) *Fix any $C < \infty$. Then for any $\delta > 0$,*

$$P \left(\sup_{x \in V(B(\tilde{Z}_{B_{\bar{R}_t}}, C) \cap B_{\bar{R}_t} \setminus \tilde{Z}_{B_{\bar{R}_t}})} \left\{ \sum_{v \in \gamma_{x,t}} \log(\deg v) - \delta \log \log \bar{R}_t \right\} \leq 0 \right) \rightarrow 1 \text{ as } t \rightarrow \infty.$$

Proof. The proof is exactly the same as that of Lemma 6.7, except that some of the relevant events hold whp but not almost surely. For part (ii) we adapt the proof slightly by taking some $c \in (0, \beta - 1)$ and considering the event $\{\#B(\tilde{Z}_{\bar{R}_t}, C) > (\log t)^{c\delta/2}\}$. \square

We are now able to show that the principal eigenfunction is concentrated in $\tilde{Z}_{B_{\bar{R}_t}}$.

Lemma 6.11. *Under Assumption 1.1,*

$$\|\phi_{B_{\bar{R}_t}}^{(1)}\|_2^2 \sum_{z \in B_{\bar{R}_t} \setminus \tilde{Z}_{B_{\bar{R}_t}}} \phi_{B_{\bar{R}_t}}^{(1)}(z) \rightarrow 0 \text{ in } P\text{-probability as } t \rightarrow \infty.$$

Proof. The proof is exactly the same as that of Lemma 6.8, except that some of the relevant events hold whp but not almost surely, we use Proposition 2.3(i) and Proposition 5.1(i'), (iii') to consider the event $\{\#B_{\bar{R}_t} \leq \bar{R}_t^d (\log \bar{R}_t)^\varepsilon\}$ in place of $\{\#\Lambda_t \leq h(t)\}$, and we instead take $C_{\alpha,\beta} = \frac{(d+1)(q+1)}{q}$. \square

7 Almost sure two site localisation

In this section we prove Theorem 1.5. The strategy is a refined version of that of [KLMS09, Section 5] and proceeds roughly as follows. Although this section is written under Assumption 1.3 in which case $d = 2$, we write some proofs under Assumption 1.1 in order to use them later for the proof of Theorem 1.6.

Recall from Section 3.2 that $R(t) = r(t)(\log t)^p$ where $p = q + 2$. Recall also that $z = \frac{d\alpha}{\alpha-d}$, $C = \frac{pd}{\alpha} + z + 1$, and

$$\begin{aligned} F_t &= \{v \in B_{R(t)} : \xi(v) \geq R(t)^{\frac{d}{\alpha}} (\log t)^{-C}\}, \\ E_t &= \{v \in B_{R(t)} \setminus F_t : \exists u \in F_t \text{ such that } \xi(u) - \deg(u) \leq \xi(v) - \deg(v)\}, \\ \Lambda_t &= \bigcup_{i=1,2} \left\{ v \in V(T_\infty) : |v - \hat{Z}_t^{(i)}| + \min\{|v|, |\hat{Z}_t^{(i)}|\} \leq (1 + (\log t)^{-z}) |\hat{Z}_t^{(i)}| \right\}. \end{aligned}$$

Now, for each $v \in V(T_\infty)$, we define

$$u_1(t, v) := \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{X_t = v\} \mathbb{1}\{\tau_{\Lambda_t} > t, H_{\{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \leq t\} \right]. \quad (45)$$

and set $U_1(t) := \sum_{v \in V(T_\infty)} u_1(t, v)$.

Now take some $\varepsilon > 0$, and also let J_t denote the number of jumps made by $(X_s)_{s \geq 0}$ up until time t , and set $j_t = t^{\frac{d+\varepsilon}{\beta}+1}r(t)$. We also define the five following contributions which upper bound the remaining mass:

$$\begin{aligned}
U_2(t) &:= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \right. \\
&\quad \left. \mathbb{1}\{\tau_{B_{R(t)}} > t, \tau_{\Lambda_t} < t, H\{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\} \leq t, \arg \max_{s \in [0, t]} [\xi(X_s) - \deg(X_s)] \in \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}, J_t \leq j_t\} \right], \\
U_3(t) &:= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{R(t)}} \leq t\} \right], \\
U_4(t) &:= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{R(t)}} > t, \arg \max_{s \in [0, t]} [\xi(X_s) - \deg(X_s)] \in (F_t \cup E_t) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}, J_t \leq j_t\} \right], \\
U_5(t) &:= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{R(t)}} > t, H_{F_t} > t\} \right], \\
U_6(t) &:= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{R(t)}} > t, J_t \geq j_t\} \right],
\end{aligned} \tag{46}$$

We will show in Proposition 7.3 that $u_1(t, v)$ localises on $\{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$, and in Propositions 7.4 to 7.9 that $\frac{U_i(t)}{U(t)} \rightarrow 0$ for all $i \in \{2, 3, 4, 5, 6\}$. This implies the result since $u(t, v)$ is non-negative and $\sum_{v \in V(T_\infty)} u(t, v) \leq \sum_{i \leq 6} U_i(t)$ for all $t > 0, v \in V(T_\infty)$.

We first derive some bounds for the contribution to $U(t)$ from a specific path. This will be particularly useful for proving a lower bound on $U(t)$ in Proposition 7.2, since the strategy employed to prove the analogous result in [KLMS09, Proposition 4.2] does not carry through in the variable degree setting.

Lemma 7.1. *Let $\gamma = v_0, \dots, v_n$ be a path in T_∞ , and let $i^* = \arg \max_{i \leq n} [\xi(v_i) - \deg(v_i)]$ (taking i^* to be the index of the first visit to v_{i^*} if v_{i^*} is visited more than once). Then:*

(i)

$$\mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\pi(X_{[0, t]}) = \gamma\} \right] \leq e^{t(1 + \xi(v_{i^*}) - \deg(v_{i^*}))} \prod_{i \neq i^*} \frac{1}{|1 + [\xi(v_{i^*}) - \deg(v_{i^*})] - [\xi(v_i) - \deg(v_i)]|}.$$

(ii)

$$\mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\pi(X_{[0, t]}) = \gamma\} \right] \geq e^{t(\xi(v_{i^*}) - \deg(v_{i^*}))} \prod_{i \neq i^*} \frac{1 - e^{-\frac{t}{n} |[\xi(v_{i^*}) - \deg(v_{i^*})] - [\xi(v_i) - \deg(v_i)]|}}{|[\xi(v_{i^*}) - \deg(v_{i^*})] - [\xi(v_i) - \deg(v_i)] + 1|}.$$

If γ is a direct path, the “+1” terms in the denominators are not necessary.

Proof. Exactly the same computation as in the proof of Proposition 2.13, but exchanging the roles of s_0 and s_{i^*} (i.e. instead applying the substitution $s_n = t - \sum_{i=0}^{n-1} s_i = t - \sum_{i \neq i^*} s_i - s_{i^*}$ to the integral over s_{i^*}), shows that

$$\begin{aligned}
&\mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\pi(X_{[0, t]}) = \gamma\} \right] \\
&= e^{t(\xi(v_{i^*}) - \deg v_{i^*})} \int_{(0, \infty)^n} \exp \left\{ \sum_{i \neq i^*} s_i ([\xi(v_i) - \deg v_i] - [\xi(v_{i^*}) - \deg v_{i^*}]) \right\} \mathbb{1} \left\{ \sum_{i \neq i^*} s_i < t \right\} \prod_{i \neq i^*} ds_i.
\end{aligned}$$

- (i) To obtain the upper bound, we suppose instead that $\xi(v_{i^*})$ is replaced with $\xi(v_{i^*}) + 1$ only on the first visit to v_{i^*} . This can only increase the expectation in question. We can then remove the indicator function from the above integral and integrate over $(0, \infty)^n$ to obtain the stated upper bound.

- (ii) For the lower bound, we instead replace $\xi(v_i)$ with $\xi(v_i) - 1$ for all i such that $v_i = v_{i^*}$ but $i \neq i^*$. We then integrate each variable over the interval $(0, \frac{t}{n})$. \square

We start by using this to give a lower bound for the total mass $U(t)$.

Proposition 7.2. (i) Set $c = \frac{q}{8}$. Under Assumption 1.3 we have for any $\varepsilon > 0$ that

$$U(t) = \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \right] \geq \exp \{ t \psi_t(\hat{Z}_t^{(1)}) - r(t) t^{-c} \} \geq \exp \{ -ta(t)(\log t)^{-\varepsilon} \}$$

eventually almost surely.

- (ii) Under Assumption 1.1, the same statement holds with high probability as $t \rightarrow \infty$.

Proof. (i) From Lemma 7.1(ii), taking γ to be the direct path from O to $\hat{Z}_t^{(1)}$, in which case $v_{i^*} = \hat{Z}_t^{(1)}$, $n = |\hat{Z}_t^{(1)}|$ and $\psi_t(\hat{Z}_t^{(1)}) > 0$, we see that

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \{ \pi(X_{[0,t]}) = \gamma \} \right] \\ & \geq e^{t(\xi(v_{i^*}) - \deg(v_{i^*}))} \prod_{i \neq i^*} \frac{1 - e^{-\frac{t}{n} |\xi(v_{i^*}) - \deg(v_{i^*})| - [\xi(v_i) - \deg(v_i)]}}{|\xi(v_{i^*}) - \deg(v_{i^*})|} \\ & \geq e^{t(\xi(v_{i^*}) - \deg(v_{i^*})) - |v_{i^*}| \log |\xi(v_{i^*}) - \deg(v_{i^*})|} \prod_{i \neq i^*} \left(1 - e^{-\frac{t}{n} |\xi(v_{i^*}) - \deg(v_{i^*})| - [\xi(v_i) - \deg(v_i)]} \right). \end{aligned} \quad (47)$$

Since $v_{i^*} = \hat{Z}_t^{(1)}$, $n = |\hat{Z}_t^{(1)}|$ and $\psi_t(\hat{Z}_t^{(1)}) > 0$, it follows from Proposition 5.1(ii) that eventually almost surely,

$$\xi(v_{i^*}) - \deg(v_{i^*}) \geq \frac{q}{2} \frac{n \log t}{t}.$$

Therefore, additionally applying Lemma 6.3(i) and Proposition 5.1(ii) we deduce that eventually almost surely for all $i \neq i^*$

$$\frac{t}{n} |\xi(v_{i^*}) - \deg(v_{i^*})| - [\xi(v_i) - \deg(v_i)] \geq \frac{1}{2} \frac{t}{n} [\xi(v_{i^*}) - \deg(v_{i^*})] \geq \frac{q}{4} \log t.$$

Therefore, the final line in (47) above is lower bounded by

$$e^{t(\xi(v_{i^*}) - \deg(v_{i^*})) - |v_{i^*}| \log |\xi(v_{i^*}) - \deg(v_{i^*})|} \prod_{i \neq i^*} \left(1 - e^{-\frac{q}{4} \log t} \right) = e^{t \psi_t(\hat{Z}_t^{(1)})} \left(1 - e^{-q(\log t)/4} \right)^{|\hat{Z}_t^{(1)}|}.$$

Therefore we have for all sufficiently large t that (using that $\log(1-x) \geq -2x$ for $|x| < 1/2$ and then Proposition 5.1(iii)),

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \{ \pi(X_{[0,t]}) = \gamma \} \right] & \geq \exp \left\{ t \psi_t(\hat{Z}_t^{(1)}) + |\hat{Z}_t^{(1)}| \log \left(1 - e^{-q(\log t)/4} \right) \right\} \\ & \geq \exp \{ t \psi_t(\hat{Z}_t^{(1)}) - 2 |\hat{Z}_t^{(1)}| e^{-q(\log t)/4} \} \\ & \geq \exp \{ t \psi_t(\hat{Z}_t^{(1)}) - r(t) e^{-q(\log t)/8} \}. \end{aligned}$$

The final lower bound then follows from Proposition 5.1(ii).

- (ii) We can modify the proof above, noting that for any $\varepsilon > 0$, $|v_{i^*}| = |\hat{Z}_t^{(1)}| \leq r(t)(\log t)^\varepsilon$ whp by Proposition 5.1(iii'). Moreover, the same proof as in Lemma 6.3(i) shows that $\inf_{i \neq i^*} |\xi(v_{i^*}) - \deg(v_{i^*})| - [\xi(v_i) - \deg(v_i)] \geq c \log t$ whp as $t \rightarrow \infty$, as above. The final lower bound then follows from Proposition 5.1(ii'). \square

Proposition 7.3. *Under Assumption 1.3, P -almost surely as $t \rightarrow \infty$*

$$\frac{\sum_{v \in V(T_\infty) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} u_1(t, v)}{U(t)} \rightarrow 0. \quad (48)$$

Proof. By the definition in (45), u_1 is of the form (41) with $\Lambda = \Lambda_t$ and $\Omega = \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$. Moreover $\tilde{g}_{\Omega, \Lambda} > 0$ by Lemma 6.3(ii). The result therefore follows directly from Lemmas 6.1(ii) and 6.8. \square

Proposition 7.4. *Under Assumption 1.3, P -almost surely as $t \rightarrow \infty$*

$$\frac{U_2(t)}{U(t)} \rightarrow 0.$$

Proof. Recall the definition of F_t from Section 3.2.

The proof is similar to [KLMS09, Lemma 4.1]. Wlog we assume that $\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}) \geq \xi(\hat{Z}_t^{(2)}) - \deg(\hat{Z}_t^{(2)})$ (otherwise we can just switch the roles of $\hat{Z}_t^{(1)}$ and $\hat{Z}_t^{(2)}$ in all that follows). We first consider the following sets of paths:

$$\begin{aligned} \hat{\Gamma}_t^{(1)} &:= \left\{ \gamma \subset B_{R(t)} : \hat{Z}_t^{(1)} \in \gamma, \gamma \not\subset \Gamma_t^{(1)}, \arg \max_{v \in \gamma} [\xi(v) - \deg(v)] = \hat{Z}_t^{(1)}, |\gamma| \leq j_t \right\}, \\ \hat{\Gamma}_t^{(2)} &:= \left\{ \gamma \subset B_{R(t)} : \hat{Z}_t^{(2)} \in \gamma, \hat{Z}_t^{(1)} \notin \gamma, \gamma \not\subset \Gamma_t^{(2)}, \arg \max_{v \in \gamma} [\xi(v) - \deg(v)] = \hat{Z}_t^{(2)}, |\gamma| \leq j_t \right\}. \end{aligned}$$

Furthermore, recall that $\gamma_{O,t}^{(1)}$ denotes the direct path from O to $\hat{Z}_t^{(1)}$. Now let $\gamma \in \hat{\Gamma}_t^{(1)}$ with $\gamma = v_0, v_1, \dots, v_n$ and let $v^* = \arg \max_{v \in \gamma} (\xi(v) - \deg(v))$, $i^* = \inf \{j \geq 1 : v_j = v^*\}$. From Lemma 7.1(i) we have that (note that $v_{i^*} = \hat{Z}_t^{(1)}$ for $\gamma \in \hat{\Gamma}_t^{(1)}$)

$$\mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \{ \pi(X_{[0,t]}) = \gamma \} \right] \leq e^{t(1+\xi(v_{i^*}) - \deg(v_{i^*}))} \prod_{i \neq i^*} \frac{1}{1 + [\xi(v_{i^*}) - \deg(v_{i^*})] - [\xi(v_i) - \deg(v_i)]}. \quad (49)$$

Note that since γ escapes $\Gamma_t^{(1)}$ and hits $\hat{Z}_t^{(1)}$ we have that $n \geq (1 + (\log t)^{-z})|\hat{Z}_t^{(1)}|$ by definition of $\Gamma_t^{(1)}$. Since γ hits $\hat{Z}_t^{(1)}$ it must also contain $\gamma_{O,t}^{(1)}$ as a subpath, which contains at least $|\hat{Z}_t^{(1)}| - (\log t)^D$ vertices in F_t^c by Proposition 3.4(i). Furthermore, for the excursions from the direct path we have by Proposition 3.5(ii) and the fact that all excursions must have even length that at least $\frac{n - |\hat{Z}_t^{(1)}|}{2}$ vertices are in F_t^c . The intuition behind the proof is that each of these vertices contributes a penalty term to (49), which reduces the contribution from paths in $\hat{\Gamma}_t^{(1)}$.

Each term in the product above corresponding to a vertex in F_t^c is upper bounded by $\frac{1}{1 + \xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}) - R(t)^{\frac{d}{\alpha}} (\log t)^{-C}}$.

Moreover, for any $\delta > 0$, $\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}) \geq a(t)(\log t)^{-\delta} = R(t)^{d/\alpha} (\log t)^{-(C-1-z+\delta)}$ eventually almost surely by Proposition 5.1(ii). Incorporating this logic into (49), and using that $\log(1-x) \geq -2x$ for all sufficiently small x , we therefore deduce that $\mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1} \{ \pi(X_{[0,t]}) = \gamma \} \right]$ is upper bounded by

$$\begin{aligned} & e^{t(1+\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}))} \left(\frac{1}{1 + [\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})] - R(t)^{\frac{d}{\alpha}} (\log t)^{-C}} \right) \\ & \leq \exp \left\{ t(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})) - |\hat{Z}_t^{(1)}| \log(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})) + t \right. \\ & \quad \left. - \left(\frac{n - |\hat{Z}_t^{(1)}|}{2} - (\log t)^D \right) \log(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})) - \left(\frac{n + |\hat{Z}_t^{(1)}|}{2} - (\log t)^D \right) \log \left(1 - \frac{R(t)^{\frac{d}{\alpha}} (\log t)^{-C} - 1}{\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})} \right) \right\} \\ & \leq \exp \left\{ t\psi_t(\hat{Z}_t^{(1)}) - \frac{n - |\hat{Z}_t^{(1)}|}{2} \left[\log(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})) - 2(\log t)^{-(1+z-\delta)} \right] + 2|\hat{Z}_t^{(1)}|(\log t)^{-(1+z-\delta)} + O(t) \right\} \\ & \leq \exp \left\{ t\psi_t(\hat{Z}_t^{(1)}) - c_q(n - |\hat{Z}_t^{(1)}|) \log t + 3|\hat{Z}_t^{(1)}|(\log t)^{-(1+z-\delta)} \right\}, \end{aligned}$$

for example taking $c_q = \frac{q}{3}$ (in fact we can take any $c_q < \frac{q}{2}$). Here we used that $\log(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})) - (\log t)^{-(1+z-\delta)} \geq \frac{2q}{3} \log t$ eventually almost surely by Proposition 5.1(ii).

Thus, defining $U(\hat{\Gamma}_t^{(1)}) := \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}_{\{\pi(X_{[0,t]}) \in \hat{\Gamma}_t^{(1)}\}} \right]$ and applying Lemma 2.6, we deduce from the final line above that for any $\varepsilon > 0$, eventually almost surely:

$$\begin{aligned} & \log(U(\hat{\Gamma}_t^{(1)})) \\ &= \log \left(\sum_{(1+(\log t)^{-z})|\hat{Z}_t^{(1)}| \leq n \leq j_t} \sum_{\gamma: |\gamma|=n, \gamma \in \hat{\Gamma}_t^{(1)}} \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}_{\{\pi(X_{[0,t]}) = \gamma\}} \right] \right) \\ &\leq \sup_{(1+(\log t)^{-z})|\hat{Z}_t^{(1)}| \leq n} \left\{ \max_{\gamma \in \hat{\Gamma}_t^{(1)}: |\gamma|=n} \log \left(\mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}_{\{\pi(X_{[0,t]}) = \gamma\}} \right] \right) + \eta_{R(t)}(n, \hat{Z}_t^{(1)}) \right\} + \log(j_t) \\ &\leq \sup_{(1+(\log t)^{-z})|\hat{Z}_t^{(1)}| \leq n} \left\{ t\psi_t(\hat{Z}_t^{(1)}) + (n - |\hat{Z}_t^{(1)}|) \left(\varepsilon \log(R(t)) + \log \left(\frac{n}{n - |\hat{Z}_t^{(1)}|} \right) - c_q \log t \right) + 3|\hat{Z}_t^{(1)}|(\log t)^{-(1+z-\delta)} \right\}. \end{aligned}$$

For all $n \geq (1 + (\log t)^{-z})|\hat{Z}_t^{(1)}|$, the coefficient of $n - |\hat{Z}_t^{(1)}|$ in the final line above is eventually almost surely upper bounded by

$$-c_q \log t + \varepsilon \log(R(t)) + \log \left(\frac{n}{n - |\hat{Z}_t^{(1)}|} \right) \leq -c_q \log t + \varepsilon \log(R(t)) + \log \left(\frac{1 + (\log t)^{-z}}{(\log t)^{-z}} \right) < -\frac{c_q}{2} \log t,$$

provided we chose $\varepsilon > 0$ sufficiently small, and moreover the coefficient becomes more negative as n increases. This implies that the supremum above is attained at the minimal value of n . Therefore, we deduce that

$$\log(U(\hat{\Gamma}_t^{(1)})) \leq t\psi_t(\hat{Z}_t^{(1)}) - (n - |\hat{Z}_t^{(1)}|) \frac{c_q}{2} \log t + 3|\hat{Z}_t^{(1)}|(\log t)^{-(1+z-\delta)} \leq t\psi_t(\hat{Z}_t^{(1)}) - \frac{c_q}{3} |\hat{Z}_t^{(1)}|(\log t)^{1-z}.$$

Invoking Proposition 5.1(i) and Proposition 7.2 thus yields that, almost surely,

$$\log \left(\frac{U(\hat{\Gamma}_t^{(1)})}{U(t)} \right) \leq -\frac{c_q}{3} r(t)(\log t)^{-z} + r(t)t^{-c} \rightarrow -\infty.$$

An identical argument shows that same result with $\hat{\Gamma}_t^{(2)}$ in place of $\hat{\Gamma}_t^{(1)}$. This proves the result since $U_2(t) = U(\hat{\Gamma}_t^{(1)}) + U(\hat{\Gamma}_t^{(2)})$. \square

Remark 7.5. In the proof above, if the ε in the upper bound for $\eta_{R(t)}(n, \hat{Z}_t)$ is replaced by a constant larger than c_q (as could be the case under Assumption 1.1, see Remark 2.7) this argument fails. Since we can take any $c_q < \frac{q}{2}$, this in particular fails when $\frac{d(q+1)}{2\beta} > \frac{q}{2}$, which is in fact always the case under the assumption $\alpha > d$.

Proposition 7.6. Under Assumption 1.1, P -almost surely as $t \rightarrow \infty$

$$\frac{U_3(t)}{U(t)} \rightarrow 0.$$

Proof. Recall that $p = q + 2$. Since $\alpha > d$, we can therefore choose $\varepsilon > 0$ small enough that

$$p > \max \left\{ q + 1, (q + 1) \frac{1}{\alpha} (d + \varepsilon) \right\}. \quad (50)$$

By Corollary 2.11 and Corollary 3.2 we have that P -almost surely for t sufficiently large

$$\begin{aligned} U_3(t) &\leq \sum_{r \geq R(t)} \mathbb{E} \left[\exp \left\{ t \sup_{v \in B_r} \xi(v) \right\} \mathbb{1}_{\left\{ \sup_{s \leq t} |X_s| = r \right\}} \right] \leq \sum_{r \geq R(t)} \exp \left\{ t \sup_{v \in B_r} \xi(v) \right\} \mathbb{P}(\tau_{B_{r-1}} \leq t) \\ &\leq \sum_{r \geq r(t)(\log t)^p} \exp \left\{ tr^{\frac{d}{\alpha}} (\log r)^{\frac{1}{\alpha}(d+\varepsilon)} - \frac{r}{5} \log \left(\frac{r}{et} \right) \right\}. \end{aligned}$$

Now note that for $r \geq r(t)(\log t)^p$,

$$tr^{\frac{d}{\alpha}}(\log r)^{\frac{1}{\alpha}(d+\varepsilon)} < \frac{1}{10}r \log\left(\frac{r}{et}\right). \quad (51)$$

provided t is sufficiently large. Thus we have for all sufficiently large t that

$$U_3(t) \leq \sum_{r \geq r(t)(\log t)^p} \exp\left\{-\frac{1}{10}r \log\left(\frac{r}{et}\right)\right\} \rightarrow 0$$

as $t \rightarrow \infty$. Since $U(t) \geq 1$ eventually almost surely by Proposition 7.2, the proof is complete. \square

Proposition 7.7. *Under Assumption 1.3, P -almost surely as $t \rightarrow \infty$*

$$\frac{U_4(t)}{U(t)} \rightarrow 0.$$

Proof. We proceed similarly to the proof of Proposition 7.4. For each $u \in (F_t \cup E_t) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}$, let us define the following set of paths:

$$\hat{\Gamma}_t^{(4)}(u) := \left\{ \gamma \subset B_{R(t)} : \arg \max_{v \in \gamma} [\xi(v) - \deg(v)] = u, |\gamma| \leq j_t \right\}.$$

Now fix some $\gamma = v_0, \dots, v_n \in \bigcup_{u \in (F_t \cup E_t) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \hat{\Gamma}_t^{(4)}(u)$. Let $v^* = \arg_{v \in \gamma} \max(\xi(v) - \deg v)$ and let $i^* = \inf\{0 \leq j \leq n : v_j = v^*\}$. Take some $\delta < \frac{1}{2\alpha(q+1)}$ and consider G_t as defined in (23). Then, using again (49) and the fact that γ contains at least $\frac{n - |v_{i^*}|}{2} + |v_{i^*}| - t^{2\alpha(q+1)\delta}$ vertices in G_t^c by Lemma 3.5(i), (ii), and $v_{i^*} \in B_{R(t)}$ by assumption, we can deduce exactly as in the proof of Proposition 7.4 that

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\pi(X_{[0,t]}) = \gamma\} \right] \\ & \leq \exp \left\{ t\psi_t(v_{i^*}) + t - \left(\frac{n - |v_{i^*}|}{2} - t^{2\alpha(q+1)\delta} \right) \log(\xi(v_{i^*}) - \deg(v_{i^*})) - \left(\frac{n + |v_{i^*}|}{2} - t^{2\alpha(q+1)\delta} \right) \log \left(1 - R(t)^{-\frac{\delta}{2}} \right) \right\}. \end{aligned}$$

Therefore, since we chose $\delta < \frac{1}{2\alpha(q+1)}$ and $\log(1-x) \leq -x$ for $x \in (0, 1)$, we have for any $\varepsilon > 0$ that

$$\begin{aligned} & \log(U_4(t)) \\ & = \log \left(\sum_{v_{i^*} \in (F_t \cup E_t) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \sum_{|v_{i^*}| \leq n \leq j_t} \sum_{\gamma \in \hat{\Gamma}_t^{(4)}(v_{i^*}) : |\gamma| = n} \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\pi(X_{[0,t]}) = \gamma\} \right] \right) \\ & \leq \sup_{v_{i^*} \in (F_t \cup E_t) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} \sup_{|v_{i^*}| \leq n} \left\{ t\psi_t(v_{i^*}) \right. \\ & \quad \left. - \left(\frac{n - |v_{i^*}|}{2} \right) \log(\xi(v_{i^*}) - \deg(v_{i^*})) + (n + |v_{i^*}|)R(t)^{-\frac{\delta}{2}} + \eta_{R(t)}(n, v_{i^*}) \right\} + O(t). \end{aligned}$$

Since $v_{i^*} \in F_t \cup E_t$ the same argument as in the proof of Proposition 7.4 yields that the maximum in the expression above is attained at the smallest possible value of n , provided that we choose $\varepsilon > 0$ small enough when applying the upper bound in Lemma 2.6. Therefore, substituting $n = |v_{i^*}|$ we obtain that

$$\log(U_4(t)) \leq t\psi(v_{i^*}) + O(R(t)^{1-\frac{\delta}{2}}). \quad (52)$$

Invoking Propositions 5.6 and 7.2 therefore yields that almost surely,

$$\log \left(\frac{U_4(t)}{U(t)} \right) \leq \sup_{v_{i^*} \in (F_t \cup E_t) \setminus \{\hat{Z}_t^{(1)}, \hat{Z}_t^{(2)}\}} t(\psi_t(v_{i^*}) - \psi_t(\hat{Z}_t^{(1)})) + r(t)t^{-c} + O(r(t)t^{-\frac{(q+1)\delta}{3}}) \rightarrow -\infty,$$

as required. \square

Proposition 7.8. *Under Assumption 1.3, P -almost surely as $t \rightarrow \infty$*

$$\frac{U_5(t)}{U(t)} \rightarrow 0.$$

Proof. Clearly, $U_5(t) \leq \exp\{t \sup_{u \in F_t^c} \xi(u)\}$. Taking C as in (21) and taking $\varepsilon = \frac{1}{2}(C - \frac{pd}{\alpha})$ in Proposition 7.2(i), we deduce that

$$\log \left(\frac{U_5(t)}{U(t)} \right) \leq ta(t)[(\log t)^{-(C - \frac{pd}{\alpha})} - (\log t)^{-\varepsilon}] \rightarrow -\infty$$

as required. \square

Proposition 7.9. *Under Assumption 1.3, P -almost surely as $t \rightarrow \infty$,*

$$\frac{U_6(t)}{U(t)} \rightarrow 0.$$

Proof. Note that, for any $\varepsilon > 0$, $\sup_{v \in B_{R(t)}} \deg(v) \leq t^{\frac{d+\varepsilon}{\beta}}$ eventually almost surely by Lemma 2.5(i). Therefore,

$$P\left(J_t \geq t^{\frac{d+\varepsilon}{\beta}+1}r(t)\right) \leq P\left(\text{Poi}\left(t^{\frac{d+\varepsilon}{\beta}+1}\right) \geq t^{\frac{d+\varepsilon}{\beta}+1}r(t)\right) \leq e^{-ct^{\frac{\varepsilon}{2}}r(t)}$$

for all sufficiently large t . Therefore, again applying Proposition 3.2 we deduce that

$$\mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{R(t)}} > t, J_t \geq t^{\frac{d+\varepsilon}{\beta}+1}r(t)\} \right] \leq \exp\left\{t \sup_{v \in B_{R(t)}} \xi(v) - ct^{\frac{\varepsilon}{2}}r(t)\right\} \leq \exp\{r(t)t^{\frac{\varepsilon}{3}} - ct^{\frac{\varepsilon}{2}}r(t)\}.$$

The result then follows from Proposition 7.2 similarly to the other cases. \square

Proof of Theorem 1.5. Since $U(t) - U_1(t) \leq \sum_{i=2}^6 U_i(t)$, the theorem is a direct consequence of Propositions 7.3-7.8. \square

8 High probability single site localisation

Unlike in [KLMS09], we cannot directly read off a single site localisation result from the almost sure result because stronger analogues of Propositions 7.4 and 7.7 for U_2 and U_4 do not necessarily hold with high probability under Assumption 1.1. In particular since the number of paths through a given vertex of length n grows fairly fast (cf Lemma 2.6) we cannot write high probability analogues of these results since we cannot justify the restriction to paths that are “as direct as possible”.

Instead we give a somewhat modified version of the proof, which nevertheless follows a similar structure to that used in [KMS06]. For this we keep $p = q + 2$ and define

$$R(t) = r(t)(\log t)^p, \quad \bar{R}_t = |\hat{Z}_t^{(1)}| \left(1 + \frac{1}{(\log t)^{\frac{1}{2}(1+\frac{1}{\alpha})}} \right).$$

Note that $\bar{R}_t \leq R(t)$ with high probability as $t \rightarrow \infty$, by Proposition 5.1 (iii').

For ease of notation we define $h_t = (\log t)^{-\frac{1}{2}(1+\frac{1}{\alpha})}$ for the rest of this section. Similar to Section 7 we define

$$\begin{aligned} \tilde{u}_1(t, v) &= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{X_t = v\} \mathbb{1}\{\tau_{B_{\bar{R}_t}} > t\} \mathbb{1}\{H_{\hat{Z}_t^{(1)}} \leq t\} \right], \\ \tilde{U}_2(t) &= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{\bar{R}_t}} \leq t, \tau_{B_{R(t)}} > t\} \right], \\ \tilde{U}_3(t) &= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{R(t)}} \leq t\} \right], \\ \tilde{U}_4(t) &= \mathbb{E}_0 \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{\tau_{B_{\bar{R}_t}} > t\} \mathbb{1}\{H_{\hat{Z}_t^{(1)}} > t\} \right]. \end{aligned}$$

We show that \tilde{u}_1 localises on $\hat{Z}_t^{(1)}$ whp, and that \tilde{U}_2, \tilde{U}_3 and \tilde{U}_4 are asymptotically negligible whp. For this we define a modified functional

$$\bar{\psi}_t(v) = \xi(v) - \frac{|v|}{t} \log \left(\frac{|v|}{et} \right). \quad (53)$$

Although $\bar{\psi}_t$ is “less precise” than ψ_t , the advantage of working with $\bar{\psi}_t$ is that it is easier to use it to give upper bounds on $u(t, v)$ for different v . It is less of a natural lower bound for $u(t, v)$ since we did not incorporate ρ as in (5), but in order to show that $\exp\{t\bar{\psi}_t(\hat{Z}_t^{(1)})\}$ is almost a lower bound for $u(t, \hat{Z}_t^{(1)})$ one only has to condition on certain high probability events at the single site $\hat{Z}_t^{(1)}$. It would be harder to show that $\exp\{t\psi_t(v)\}$ is an asymptotic upper bound for any $v \neq \hat{Z}_t^{(1)}$, since this involves conditioning on certain events occurring for *all* other $v \neq \hat{Z}_t^{(1)}$, which is much harder (and already we saw that the upper bounds in Propositions 7.4 and 7.7 do not carry through in the high probability setting).

Proposition 8.1. *Take any $\varepsilon > 0$. Under Assumption 1.1, it holds with high P -probability as $t \rightarrow \infty$ that*

$$\log U(t) \geq t\bar{\psi}_t(\hat{Z}_t^{(1)}) + o\left(\frac{ta(t)}{(\log t)^{1-\varepsilon}}\right).$$

Proof. By Proposition 7.2(ii), it is enough to show that

$$\bar{\psi}_t(\hat{Z}_t^{(1)}) - \psi_t(\hat{Z}_t^{(1)}) = \deg(\hat{Z}_t^{(1)}) + \frac{|\hat{Z}_t^{(1)}|}{t} \log \left(\frac{(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}))te}{|\hat{Z}_t^{(1)}|} \right) = o\left(\frac{a(t)}{(\log t)^{1-\varepsilon}}\right)$$

whp as $t \rightarrow \infty$. This is immediate from Proposition 5.1(i'), (iii') and Lemma 3.1 since whp we have:

$$\frac{|\hat{Z}_t^{(1)}|}{t} \log \left(\frac{(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}))te}{|\hat{Z}_t^{(1)}|} \right) \leq \frac{r(t)(\log t)^{\varepsilon/2}}{t} \log \left(\frac{a(t)t \log t}{r(t)} \right) = \frac{a(t)}{(\log t)^{1-\varepsilon/2}} \log((\log t)^2).$$

A similar result holds for $\deg(\hat{Z}_t^{(1)})$ by Proposition 3.4(ii) and since $\hat{Z}_t^{(1)} \in F_t$ whp by Proposition 5.1(ii'). \square

We will need the following result to complete the localisation proof for \tilde{u}_1 .

Lemma 8.2. *Under Assumption 1.1,*

- (i) $\lim_{t \rightarrow \infty} P(\hat{Z}_t^{(1)} = Z_{B_{\bar{R}_t}} = \tilde{Z}_{B_{\bar{R}_t}}) = 1$.
- (ii) $P(\hat{Z}_t^{(1)} \text{ is also the global maximiser of } \bar{\psi}_t) \rightarrow 1$.

Proof. (i) By Lemma 3.7, it is sufficient just to establish the first equality. Since $\psi_t(\hat{Z}_t^{(1)}) \geq \psi_t(Z_{B_{\bar{R}_t}})$ by definition, we can rearrange to deduce that

$$\begin{aligned} & \xi(Z_{B_{\bar{R}_t}}) - \xi(\hat{Z}_t^{(1)}) \\ & \leq \deg(Z_{B_{\bar{R}_t}}) - \deg(\hat{Z}_t^{(1)}) + \frac{|Z_{B_{\bar{R}_t}}|}{t} \log(\xi(Z_{B_{\bar{R}_t}}) - \deg(Z_{B_{\bar{R}_t}})) - \frac{|\hat{Z}_t^{(1)}|}{t} \log(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})) \\ & \leq \deg(Z_{B_{\bar{R}_t}}) + \left(\frac{|Z_{B_{\bar{R}_t}}|}{t} - \frac{|\hat{Z}_t^{(1)}|}{t} \right) \log(\xi(Z_{B_{\bar{R}_t}}) - \deg(Z_{B_{\bar{R}_t}})) + \frac{|\hat{Z}_t^{(1)}|}{t} \log \left(\frac{\xi(Z_{B_{\bar{R}_t}}) - \deg(Z_{B_{\bar{R}_t}})}{\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)})} \right). \end{aligned}$$

Now take any $\varepsilon \in (0, \frac{1}{8\alpha})$ and note that:

- By Proposition 5.1(ii') and Lemma 3.4(ii), $\deg(Z_{B_{\bar{R}_t}}) \leq (\log t)^B$ whp as $t \rightarrow \infty$.
- By Proposition 5.1(iii'), $|\hat{Z}_t^{(1)}| \leq r(t)(\log t)^\varepsilon = ta(t)(\log t)^{\varepsilon-1}$ whp as $t \rightarrow \infty$.
- Since $\bar{R}_t \leq R(t)$ whp as $t \rightarrow \infty$ and by Lemma 3.1 applied to $B_{R(t)}$, it follows that $\xi(Z_{B_{\bar{R}_t}}) - \deg(Z_{B_{\bar{R}_t}}) \leq R(t)^{d/\alpha}(\log t)^\varepsilon \leq a(t)(\log t)^{p+\varepsilon}$ whp as $t \rightarrow \infty$.

- By Proposition 5.1(ii'), $\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}) \geq a(t)(\log t)^{-\varepsilon}$ whp as $t \rightarrow \infty$.

Substituting these bounds above, and using the fact that we chose $\varepsilon < \frac{1}{8\alpha}$, we deduce that, whp as $t \rightarrow \infty$ (also using that $\alpha \geq 2$):

$$\begin{aligned} \xi(Z_{B_{\bar{R}_t}}) - \xi(\hat{Z}_t^{(1)}) &\leq (\log t)^B + a(t)(\log t)^{\varepsilon - \frac{1}{2}(1 + \frac{1}{\alpha})} + (p + 2\varepsilon)a(t)(\log t)^{\varepsilon - 1} \log \log t \\ &\leq (2 + p + 2\varepsilon)a(t)(\log t)^{\varepsilon - \frac{1}{2}(1 + \frac{1}{\alpha})} \\ &< a(t)(\log t)^{2\varepsilon - \frac{3}{2\alpha}}. \end{aligned}$$

By Lemma 3.6 and Proposition 5.1(i'), we also have that $g_{B_{\bar{R}_t}} \geq (\bar{R}_t)^{\frac{d}{\alpha}} (\log(\bar{R}_t))^{-(\frac{1}{\alpha} + \varepsilon)} \geq a(t)(\log t)^{-(\frac{1}{\alpha} + 2\varepsilon)}$ whp as $t \rightarrow \infty$.

Combining these two calculations, we deduce that $\xi(Z_{B_{\bar{R}_t}}) - \xi(\hat{Z}_t^{(1)}) < g_{B_{\bar{R}_t}}$ whp, which in turn implies that $Z_{B_{\bar{R}_t}} = \hat{Z}_t^{(1)}$ whp.

- (ii) By part (i), it follows that $\bar{\psi}_t(\hat{Z}_t^{(1)}) \geq \bar{\psi}_t(v)$ for all $v \in B_{\bar{R}_t} \setminus B_{|\hat{Z}_t^{(1)}|}$ whp. Additionally, by Proposition 5.1(ii') and Lemma 3.1, it follows that $\bar{\psi}_t(\hat{Z}_t^{(1)}) \geq \bar{\psi}_t(v)$ for all $v \in B_{r(t)(\log t)^{-\varepsilon}}$. Finally, by the same proof used to prove the upper bound in Theorem 1.7, it follows that the maximiser of $\bar{\psi}_t$ is in $B_{r(t)(\log t)^{\varepsilon}}$.

Now suppose that both v is in $B_{|\hat{Z}_t^{(1)}|} \setminus B_{r(t)(\log t)^{-\varepsilon}}$ and $v \neq \hat{Z}_t^{(1)}$. If $v \in F_t$, then we have that

$$\begin{aligned} &\bar{\psi}_t(v) - \bar{\psi}_t(\hat{Z}_t^{(1)}) \\ &\leq \psi_t(v) - \psi_t(\hat{Z}_t^{(1)}) + \frac{|v|}{t} \log \left(\frac{et(\xi(v) - \deg v)}{|v|} \right) - \frac{|\hat{Z}_t^{(1)}|}{t} \log \left(\frac{et(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}))}{|\hat{Z}_t^{(1)}|} \right) + \deg(v) - \deg(\hat{Z}_t^{(1)}) \\ &\leq \psi_t(v) - \psi_t(\hat{Z}_t^{(1)}) + \frac{|\hat{Z}_t^{(1)}|}{t} \log \left(\frac{|\hat{Z}_t^{(1)}|(\xi(v) - \deg v)}{|v|(\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}))} \right) + \deg(v). \end{aligned}$$

Take $\varepsilon \in (0, \frac{1}{6})$. By Lemma 3.1 and Proposition 5.1(ii'), we can assume that $\xi(\hat{Z}_t^{(1)}) - \deg(\hat{Z}_t^{(1)}) \geq (\xi(v) - \deg v)(\log t)^{3\varepsilon}$, and since $v \in B_{r(t)(\log t)^{\varepsilon}} \setminus B_{r(t)(\log t)^{-\varepsilon}}$ whp, we can also assume that $\frac{|\hat{Z}_t^{(1)}|}{|v|} \leq (\log t)^{2\varepsilon}$ and $|v| \leq r(t)(\log t)^{\varepsilon}$ by Proposition 5.1(iii'). Moreover, by Lemma 5.5, we can assume that $\psi_t(v) - \psi_t(\hat{Z}_t^{(1)}) \leq -a(t)(\log t)^{-\varepsilon}$. Assuming these and applying Proposition 3.4(ii), we therefore have that

$$\bar{\psi}_t(v) - \bar{\psi}_t(\hat{Z}_t^{(1)}) \leq \psi_t(v) - \psi_t(\hat{Z}_t^{(1)}) + \frac{r(t)(\log t)^{3\varepsilon}}{t} \log \log t + (\log t)^B \leq -a(t)(\log t)^{-\varepsilon} + a(t)(\log t)^{4\varepsilon - 1} < 0,$$

as required.

If $v \notin F_t$, then $\bar{\psi}_t(v) \leq \xi(v) \leq a(t)(\log t)^{-(z+1)}$ and also $\bar{\psi}_t(\hat{Z}_t^{(1)}) \geq \psi_t(\hat{Z}_t^{(1)}) \geq a(t)(\log t)^{-1}$ whp by Proposition 5.1 (ii'). Combining these gives the same result. \square

Additionally, exactly the same proof as in Lemma 5.5 gives the following (note that Lemma 5.5 was written only under Assumption 1.1, and since $\bar{\psi}_t(v) \geq \psi_t(v)$ for all v the same almost sure lower bounds hold for $\bar{\psi}_t(v)$). Note also that by Lemma 8.2 we can assume that $\hat{Z}_t^{(1)}$ is also the maximiser of the functional $\bar{\psi}$.

Lemma 8.3. *Take any $D > 0$. Under Assumption 1.1, for any $0 < K < D$, we have for all t large enough that*

$$P \left(\bar{\psi}_t(\hat{Z}_t^{(1)}) - \sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v) < a(t)(\log t)^{-D} \right) \leq 4\alpha(\log t)^{-K}. \quad (54)$$

Proposition 8.4. *Under Assumption 1.1,*

$$\frac{\sum_{v \in V(T_\infty) \setminus \{\hat{Z}_t^{(1)}\}} \tilde{u}_1(t, v)}{U(t)} \rightarrow 0 \quad (55)$$

in P -probability as $t \rightarrow \infty$.

Proof. This follows from Lemmas 3.7, 6.1, 6.11 and 8.2. \square

Proposition 8.5. *Under Assumption 1.1, $\frac{\tilde{U}_2(t)}{U(t)} \rightarrow 0$ whp.*

Proof. We will show that

$$\log \left(\tilde{U}_2(t) \right) \leq \max \left\{ t \sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v), t\xi(\hat{Z}_t^{(1)}) - \bar{R}_t \log \left(\frac{\bar{R}_t}{et} \right) \right\} + o \left(\frac{ta(t)}{\log t} \right) \quad (56)$$

whp as $t \rightarrow \infty$. To do this, first note that by the definition of \tilde{U}_2 and Lemma 2.12, we have with high probability that

$$\begin{aligned} \log \left(\tilde{U}_2(t) \right) &\leq \log \left(\sum_{r \in [\bar{R}_t, r(t)(\log t)^p]} \exp \left\{ t \sup_{v \in B_r} \xi(v) \right\} \mathbb{P}(\tau_{B_{r-1}} \leq t) \right) \\ &\leq t \max_{r \in [\bar{R}_t, r(t)(\log t)^p]} \left\{ \sup_{v \in B_r} \xi(v) - \frac{r}{t} \log \left(\frac{r}{et} \right) \right\} + o \left(\frac{ta(t)}{\log t} \right). \end{aligned}$$

Let $\hat{r}(t)$ denote the r for which the maximum is attained, and $\hat{v}(t) = \arg \max \{ \sup \xi(v) : v \in B_{\hat{r}(t)} \}$. If $\hat{v}(t) = \hat{Z}_t^{(1)}$, then

$$t \max_{r \in [\bar{R}_t, r(t)(\log t)^p]} \left\{ \sup_{v \in B_r} \xi(v) - \frac{r}{t} \log \left(\frac{r}{et} \right) \right\} \leq t\xi(\hat{Z}_t^{(1)}) - \bar{R}_t \log \left(\frac{\bar{R}_t}{et} \right).$$

If $\hat{v}(t) \neq \hat{Z}_t^{(1)}$, then

$$\begin{aligned} t \max_{r \in [\bar{R}_t, r(t)(\log t)^p]} \left\{ \sup_{v \in B_r} \xi(v) - \frac{r}{t} \log \left(\frac{r}{et} \right) \right\} &= t\xi(\hat{v}(t)) - \hat{r}(t) \log \left(\frac{\hat{r}(t)}{et} \right) \leq t\xi(\hat{v}(t)) - |\hat{v}(t)| \log \left(\frac{|\hat{v}(t)|}{et} \right) \\ &\leq t \sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v). \end{aligned}$$

This establishes (56). To complete the proof, we invoke Proposition 8.1 which implies that, for any $\varepsilon > 0$, we have with high probability as $t \rightarrow \infty$,

$$\frac{\tilde{U}_2(t)}{U(t)} \leq \exp \left\{ \max \left\{ t \left(\sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v) - \bar{\psi}_t(\hat{Z}_t^{(1)}) \right), |\hat{Z}_t^{(1)}| \log \left(\frac{|\hat{Z}_t^{(1)}|}{et} \right) - \bar{R}_t \log \left(\frac{\bar{R}_t}{et} \right) \right\} + o \left(\frac{ta(t)}{(\log t)^{1-\varepsilon}} \right) \right\}.$$

- To deal with the first term in the maximum above, take $\varepsilon = \frac{1}{4}$ and note from Lemma 8.3 that for $\eta_t = (\log t)^{-\frac{1}{2}}$ we have that $P \left(\bar{\psi}_t(\hat{Z}_t^{(1)}) - \sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v) < \eta_t a(t) \right) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$t \left(\sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v) - \bar{\psi}_t(\hat{Z}_t^{(1)}) \right) + o \left(\frac{ta(t)}{(\log t)^{3/4}} \right) \rightarrow -\infty$$

in probability as $t \rightarrow \infty$.

- To deal with the second term in the maximum above, recall that $\bar{R}_t = |\hat{Z}_t^{(1)}|(1 + h_t)$ where $h_t = \frac{1}{(\log t)^{\frac{1}{2}(1+\frac{1}{\alpha})}}$, and choose ε small enough such that $h_t \geq (\log t)^{-(1-2\varepsilon)}$. We then have by Proposition 5.1(i') that, with high probability as $t \rightarrow \infty$

$$\begin{aligned} \bar{R}_t \log \left(\frac{\bar{R}_t}{et} \right) - |\hat{Z}_t^{(1)}| \log \left(\frac{|\hat{Z}_t^{(1)}|}{et} \right) + o \left(\frac{ta(t)}{(\log t)^{1-\varepsilon}} \right) &\geq |\hat{Z}_t^{(1)}| h_t \log \left(\frac{|\hat{Z}_t^{(1)}|}{et} \right) + o \left(\frac{ta(t)}{(\log t)^{1-\varepsilon}} \right) \\ &\geq C |\hat{Z}_t^{(1)}| h_t \log t + o \left(\frac{ta(t)}{(\log t)^{1-\varepsilon}} \right) \\ &\geq C ta(t) \frac{(\log t)^{-\varepsilon}}{(\log t)^{1-2\varepsilon}} + o \left(\frac{ta(t)}{(\log t)^{1-\varepsilon}} \right), \end{aligned}$$

which diverges as $t \rightarrow \infty$.

This completes the proof. □

Proposition 8.6. *Under Assumption 1.1, $\frac{\tilde{U}_3(t)}{U(t)} \rightarrow 0$ whp.*

Proof. This holds by Proposition 7.6 (note that we proved it under the weaker assumption), since we chose the same value of p when defining $R(t)$, and in particular $p > q + 1$. □

Proposition 8.7. *Under Assumption 1.1, $\frac{\tilde{U}_4(t)}{U(t)} \rightarrow 0$ whp.*

Proof. We split the sum into two parts by writing,

$$\begin{aligned} \tilde{U}_4(t) &\leq \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{H_{\hat{Z}_t^{(1)}} > t\} \mathbb{1}\{\tau_{B_{\bar{R}_t}} > t\} \right] \\ &\leq \sum_{r \leq \bar{R}_t} \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{H_{\hat{Z}_t^{(1)}} > t\} \mathbb{1}\{\sup_{s \leq t} |X_s| = r\} \right] \\ &= \sum_{r \leq r(t)(\log t)^{-1}} \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{H_{\hat{Z}_t^{(1)}} > t\} \mathbb{1}\{\sup_{s \leq t} |X_s| = r\} \right] \\ &\quad + \sum_{r(t)(\log t)^{-1} \leq r \leq \bar{R}_t} \mathbb{E} \left[\exp \left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}\{H_{\hat{Z}_t^{(1)}} > t\} \mathbb{1}\{\sup_{s \leq t} |X_s| = r\} \right]. \end{aligned} \quad (57)$$

We now bound the two sums appearing here.

- By Lemma 3.1 it holds with high P -probability as $t \rightarrow \infty$ that the logarithm of the first sum in (57) is upper bounded by

$$t \sup_{v \in B_{r(t)(\log t)^{-1}}} \xi(v) + \log(r(t)(\log \log t)^{-1}) = o \left(\frac{ta(t)}{(\log t)^{d/(2\alpha)}} \right).$$

- By Lemma 2.12, it holds whp that the logarithm of the second sum in (57) is upper bounded by

$$\begin{aligned} &\max_{\substack{r \in [r(t)(\log t)^{-1}, \\ r(t)(\log t)^p]}}, \left\{ t \sup_{v \in B_r \setminus \{\hat{Z}_t^{(1)}\}} \xi(v) + \log(\mathbb{P}(\tau_{B_{r-1}} \leq t)) \right\} + \log(r(t)(\log(t))^p) \\ &\leq \max_{\substack{r \in [r(t)(\log t)^{-1}, \\ r(t)(\log t)^p]}}, \left\{ t \sup_{v \in B_r \setminus \{\hat{Z}_t^{(1)}\}} \xi(v) - r \log \left(\frac{r}{et} \right) + o(r(t)) \right\} + \log(r(t)(\log(t))^p) \leq t \sup_{v \neq \hat{Z}_t^{(1)}} \bar{\psi}_t(v) + o(r(t)). \end{aligned}$$

Combining these two calculations and applying Proposition 8.1 (with $\varepsilon = 1/2$) and then Lemma 8.3 shows that whp as $t \rightarrow \infty$,

$$\log \left(\frac{\tilde{U}_2(t)}{U(t)} \right) \leq t \sup_{v \neq \hat{Z}_t^{(1)}} \{ \bar{\psi}_t(v) - \bar{\psi}_t(\hat{Z}_t^{(1)}) \} + o \left(\frac{ta(t)}{(\log t)^{d/2\alpha}} + r(t) + \frac{ta(t)}{(\log t)^{1/2}} \right) \leq -\frac{ta(t)}{(\log t)^{d/4\alpha}} + o \left(\frac{ta(t)}{(\log t)^{d/2\alpha}} \right) \rightarrow -\infty,$$

as required. \square

Proof of Theorem 1.6. Since $U(t) - \sum_{v \in V(T_\infty)} \tilde{u}_1(t, v) \leq \sum_{i=2}^4 \tilde{U}_i(t)$, the theorem is a direct consequence of Propositions 8.4-8.7. \square

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