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Hom-Alternative, Hom-Lie and Hom-Jordan rings

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Abstract

The purpose of this paper is to introduce Hom-alternative rings, Hom-Lie rings and Hom-Jordan rings. We study some of their properties and provide construction procedures using ordinary alternative rings or Lie rings or Jordan rings. Also, we show that a polarization of Hom-associative ring leads to Hom-Lie ring or Hom-Jordan ring.

Keywords: Alternative rings, Hom-rings, Jordan rings, Lie rings, Rings

Introduction

The notion of Hom-Lie algebra first appeared in q-deformations of algebras of vector fields, such as Witt and Virasoro algebras [1, 3, 5]. The concept of Hom-Lie algebras generalizes the one for Lie algebras where the Jacobi identity is twisted by a homomorphism [7, 11]. Hom-associative algebras were introduced and studied in [14], moreover Hom-alternative algebra and Hom-Jordan algebras were studied in [13, 16]. In the last years, many classical algebraic concepts have been extended to the framework of Hom-structures. For examples the concept of Hom-group [12],[8], [9] and Hom-Lie group [10, 15] were studied.

The definition of Hom-ring first introduced in [6]. In this work, Fregier-Gohr defined a Hom-ring as a set A with two binary operations $+$: $A \times A \rightarrow A$ and \cdot : $A \times A \rightarrow A$, a self-map β : $A \rightarrow A$ and a special element $0 \in A$ such that

- $(A, +, 0)$ is an abelian group.
- The multiplication is distributive on both sides.
- β is an abelian group homomorphism.
- β and \cdot satisfy the Hom-associativity condition; $\beta(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z)$.

Motivated by the concept of Hom-group, the authors of [4] modified the definition of Hom-ring. It is a more general definition of Hom-ring, such that the Hom-rings defined by Fregier-Gohr will be a subclass of these hom-rings.

The aim of this paper is to introduce Hom-alternative rings, Hom-Lie rings and Hom-Jordan rings which are twisted version of the ordinary alternative rings, Lie rings and Jordan rings. We discuss some of their properties and provide construction procedures using ordinary alternative rings or Lie rings or Jordan rings. Also, we show that a polarization of a Hom-associative ring leads to a Hom-Lie ring or a Hom-Jordan ring.

The paper is organized as follows. In Section 1, we recall some basic definitions concerning Hom-group. Also, we introduce the definition of Hom-multilinear maps. This notion will be used in some proofs of our work. The Section 2 is devoted to remind the definition of Hom-ring introduced in [4]. A definition of Hom-algebra over a commutative Hom-ring is also given in this section. In section 3, we introduce Hom-alternative rings, study their properties and provide construction procedures using ordinary alternative rings. The last Section is dedicated to Hom-Lie rings and Hom-Jordan rings. We introduce the notion of Hom-Lie rings and Hom-Jordan rings and show that it fits with the Hom-associative structure, that is a Hom-associative ring leads to Hom-Lie ring or Hom-Jordan ring by polarization. Also, we provide a way to construct a Hom-Lie ring (resp. Hom-Jordan ring) starting from an ordinary Lie ring (resp. Jordan ring).

1 Hom-groups

Here we recall the definition of a Hom-group from [12], [8] and [9], and we will expose some properties of it. Also, a notion of Hom-multilinear maps is introduced.

Definition 1.1. A Hom-group is a quadruplet (G, μ, e, α) consisting of a set G with a distinguished member e of G , an operation $\mu : G \times G \rightarrow G$ and a set map $\alpha : G \rightarrow G$, such that the following axioms are satisfied:

1. The product map $\mu : G \times G \rightarrow G$ and the set map $\alpha : G \rightarrow G$ satisfy the Hom-associativity property

$$\mu(\alpha(g), \mu(h, k)) = \mu(\mu(g, h), \alpha(z)). \quad (1.1)$$

For simplicity when there is no confusion we omit the multiplication sign μ .

2. The map α is multiplicative, i.e., $\alpha(\mu(g, h)) = \mu(\alpha(g), \alpha(h))$.
3. The element e is called unit and it satisfies the Hom-unity condition

$$\mu(g, e) = \mu(e, g) = \alpha(g), \quad \alpha(e) = e. \quad (1.2)$$

4. The map $g \mapsto g^{-1}$ is an antimorphism; $\mu(g, h)^{-1} = \mu(h^{-1}, g^{-1})$.
5. For any $g \in G$, there exists a natural number k satisfying the Hom-invertibility condition

$$\alpha^k(\mu(g, g^{-1})) = \alpha^k(\mu(g^{-1}, g)) = e. \quad (1.3)$$

The smallest such k is called the invertibility index of g .

If α is invertible the relation (1.3) will be simplified to the following condition ;

For every element $g \in G$, there exists an element g^{-1} which $\mu(g, g^{-1}) = \mu(g^{-1}, g) = e$.

A Hom-group, such as α is invertible, is called a regular Hom-group [9, 10]. If moreover the product μ is commutative, we say that G is an abelian (commutative) Hom-group. In this case, the product μ will be denoted by $+$ and the element e by 0. Therefore the multiplicativity axiom will change to $\alpha(g+h) = \alpha(g)+\alpha(h)$, $\forall g, h \in G$.

Remarks 1.2.

1. From the Hom-unity and the Hom-associativity conditions, we can show that the multiplicativity of the map α is intuitive; $\alpha(\mu(g, h)) = \mu(e, \mu(g, h)) = \mu(\alpha(e), \mu(g, h)) = \mu(\mu(e, g), \alpha(h)) = \mu(\alpha(g), \alpha(h))$.
2. Since we have the antimorphism $g \mapsto g^{-1}$, therefore by the definition, inverse of any element $g \in G$ is unique although different elements may have different invertibility index.
3. The inverse of the element e of a Hom-group is itself, because $\alpha(\mu(e, e)) = \alpha(e) = e$.
4. For any Hom-group (G, μ, e, α) we have $\alpha(g^{-1}) = \alpha(g)^{-1}$ and if the invertibility index of g is k , then the index invertibility of $\alpha(g)$ is $k - 1$.

Example 1.3. Let (G, μ, e) be a group and $\alpha : G \rightarrow G$ be a group homomorphism. We define a new product $\mu_\alpha : G \times G \rightarrow G$ by

$$\mu_\alpha(g, h) = \alpha(\mu(g, h)) = \mu(\alpha(g), \alpha(h)).$$

Then $(G, \mu_\alpha, e, \alpha)$ is a Hom-group and denote this by G_α . It is said the twisted group of G .

Example 1.4. [4] Let $(\mathbb{Z}, +)$ be the additive group of integers. We have $\text{End}(\mathbb{Z}) \simeq \mathbb{Z}$, i.e., any endomorphism $\alpha \in \text{End}(\mathbb{Z})$ is completely determined by the element $\alpha(1) \in \mathbb{Z}$ and for every $q \in \mathbb{Z}$ there exists an endomorphism $\alpha \in \text{End}(\mathbb{Z})$ such that $\alpha(1) = q$. In other words, the correspondence $\alpha \mapsto \alpha(1)$ is an isomorphism between $\text{End}(\mathbb{Z})$ and \mathbb{Z} . Then, for every $q \in \mathbb{Z}$, we can define a Hom-group denoted by $(\mathbb{Z}, +_q, 0, \alpha)$ such as $+_q$ is defined by $n +_q m = q(n + m)$, $\forall n, m \in \mathbb{Z}$ and checks the following condition

$$qn +_q (m +_q p) = (n +_q m) +_q qp = q^2(n + m + p), \forall n, m, p \in \mathbb{Z}. \quad (1.4)$$

The Hom-group $(\mathbb{Z}, +_q, 0, \alpha)$ is called the q -additive group of integers.

Definition 1.5. Let $(G, \mu_G, e_G, \alpha_G)$ and $(H, \mu_H, e_H, \alpha_H)$ be two Hom-groups. A homomorphism of Hom-groups is a map $f : G \rightarrow H$ such that

$$f(\mu_G(g, h)) = \mu_H(f(g), f(h)), \forall g, h \in G \quad f \circ \alpha_G = \alpha_H \circ f.$$

It said to be a weak homomorphism if $f(\mu_G(g, h)) = \mu_H(f(g), f(h))$, $\forall g, h \in G$.

Remark 1.6. Let $f : G \rightarrow H$ be a weak homomorphism of Hom-groups such that $f(e_G) = e_H$ and g be an element in G . Then, we have

$$f(\alpha_G(g)) = f(\mu_G(e_G, g)) = \mu_H(f(e_G), f(g)) = \mu_H(e_H, f(g)) = \alpha_H(f(g)).$$

So any weak homomorphism of Hom-groups $f : (G, \mu_G, e_G, \alpha_G) \rightarrow (H, \mu_H, e_H, \alpha_H)$ such that $f(e_G) = e_H$ is a homomorphism.

Definition 1.7. Two Hom-groups $(G, \mu_G, e_G, \alpha_G)$ and $(H, \mu_H, e_H, \alpha_H)$ are called isomorphic if there exists a bijective homomorphism of Hom-groups $f : (G, \mu_G, e_G, \alpha_G) \rightarrow (H, \mu_H, e_H, \alpha_H)$.

Definition 1.8. Let (G, μ, e, α) be a Hom-group. A nonempty subset H of the Hom-group G is a Hom-subgroup of G if (H, μ, e, α) is a Hom-group. We use the notation $H \leq G$ to indicate that H is a Hom-subgroup of G .

If H is a Hom-subgroup, we see that e the identity for G is also the identity for H . Consequently the following theorem is obvious:

Theorem 1.9. A subset H of the Hom-group (G, \cdot, e, α) is a Hom-subgroup of G if and only if

1. $e \in H$,
2. $h \in H \Rightarrow h^{-1} \in H$,
3. $h_1, h_2 \in H \Rightarrow h_1 \cdot h_2 \in H$.

Remarks 1.10. 1. From the third item of the previous theorem, one notes that if H is a Hom-subgroup of G then $\alpha(h) = e \cdot h \in H$ for all $h \in H$. Therefore $\alpha(H) \subset H$.

2. Let (G, μ, e, α) be a Hom-group and H be a subset of G . A Hom-group structure on H is defined by the inclusion map.

Example 1.11. Let G be a group and $\alpha : G \rightarrow G$ be a group homomorphism. If H is a subgroup of G which $\alpha(H) \subset H$, then H_α is a Hom-subgroup of G_α .

Example 1.12. Let $(G, \mu_G, e_G, \alpha_G)$ and $(H, \mu_H, e_H, \alpha_H)$ be Hom-groups. If $f : G \rightarrow H$ is a Hom-group homomorphism such that $f(e_G) = e_H$, then the reciprocal image of each Hom-subgroup of H is a Hom-subgroup of G .

In what follows, we recall a property of in a Hom-group introduced in [4].

Proposition 1.13. Let (G, \cdot, e, α) be a regular abelian Hom-group. For all g, h, k and l in G , we have

$$(g \cdot h) \cdot (k \cdot l) = (g \cdot k) \cdot (h \cdot l). \quad (1.5)$$

Proof. For all g, h, k and l in G , we have

$$\begin{aligned}(g \cdot h) \cdot (k \cdot l) &= \alpha(g) \cdot (h \cdot \alpha^{-1}(k \cdot l)) = \alpha(g) \cdot (\alpha^{-1}(h \cdot k) \cdot l) = \alpha(g) \cdot (\alpha^{-1}(k \cdot h) \cdot l) \\ &= \alpha(g) \cdot (k \cdot \alpha^{-1}(h \cdot l)) = (g \cdot k) \cdot (h \cdot l).\end{aligned}$$

Corollary 1.14. Let (G, \cdot, e, α) be a regular abelian Hom-group. For all $g \in G$, we have

$$((gg)\alpha(g))((g^{-1}g^{-1})\alpha(g^{-1})) = e.$$

Proof. From Proposition 1.13, we have

$$((gg)\alpha(g))((g^{-1}g^{-1})\alpha(g^{-1})) = ((gg)(g^{-1}g^{-1}))(\alpha(g)\alpha(g^{-1})) = e\alpha(gg^{-1}) = e.$$

This completes the proof

So, one can show the following

Corollary 1.15. Let (G, \cdot, e, α) be a regular abelian Hom-group. For all $g \in G$, we have

$$(((gg)\alpha(g))\alpha^2(g))\alpha((g^{-1}g^{-1})\alpha(g^{-1})) = \alpha^4(g).$$

1.1 Hom-multilinear maps

Definition 1.16. Let $(A, +_A, 0, \alpha_A)$ be an abelian Hom-group and let A^n denote the n -fold Cartesian product $A \times A \times \cdots \times A$. In other words, an element of A^n is an n -tuple (a_1, \dots, a_n) , where each $a_i \in A$, $i = 1, 2, \dots, n$. If $(B, +_B, 0, \alpha_B)$ is another abelian Hom-group, then a mapping $f : A^n \rightarrow B$ is said to be Hom-multilinear if

1. For each $i = 1, 2, \dots, n$, we have

$$\begin{aligned}f(\alpha_A(a_1), \dots, \alpha_A(a_{i-1}), a_i + a'_i, \alpha_A(a_{i+1}), \dots, \alpha_A(a_n)) &= f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \\ &\quad +_B f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n),\end{aligned}$$

for all $a_i, a'_i \in A$.

2. $f(\alpha_A(a_1), \dots, \alpha_A(a_n)) = \alpha_B(f(a_1, \dots, a_n))$.

In the case $n = 2$ ($n = 3$), one speaks of a Hom-bilinear mapping (respectively, a Hom-trilinear mapping).

Lemma 1.17. Let $(A, +_A, 0, \alpha_A)$ and $(B, +_B, 0, \alpha_B)$ be two abelian Hom-groups and $f : A^n \rightarrow B$ a Hom-multilinear mapping. Then for each $i, j = 1, 2, \dots, n$, $j < i$, we have

$$\begin{aligned}&f(\alpha_A^2(a_1), \dots, \alpha_A^2(a_{j-1}), \alpha_A(a_j +_A a'_j), \alpha_A^2(a_{j+1}), \dots, \alpha_A^2(a_{i-1}), \alpha(a_i + a'_i), \alpha_A^2(a_{i+1}), \dots, \alpha_A^2(a_n)) \\ &= \left(f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{i-1}, a_i, a_{i+1}, a_n) +_B f(a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_{i-1}, a_i, a_{i+1}, a_n) \right) \\ &\quad +_B \left(f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{i-1}, a'_i, a_{i+1}, a_n) +_B f(a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_{i-1}, a'_i, a_{i+1}, a_n) \right),\end{aligned}$$

for all $a_i, a_j, a'_i, a'_j \in A$.

Proof. The proof is straightforward.

Definition 1.18. A Hom-multilinear mapping f is called skew-symmetric if it changes sign when two of its arguments are interchanged. That is, f is skew-symmetric if for each $i, j = 1, 2, \dots, n$, we have

$$f(a_1, \dots, a_j, \dots, a_i, \dots, a_n) = -f(a_1, \dots, a_i, \dots, a_j, \dots, a_n).$$

Lemma 1.19. Let $(A, +_A, 0, \alpha_A)$ and $(B, +_B, 0, \alpha_B)$ be two abelian Hom-groups and $f : A^n \rightarrow B$ a Hom-multilinear mapping. Then f is skew-symmetric if and only if f takes the value 0 whenever at least two of the x_i 's are equal.

Proof. Suppose that for each $i = 1, 2, \dots, n$ we have

$$f(a_1, \dots, a_i, \dots, a_i, \dots, a_n) = 0,$$

and we will prove that

$$f(a_1, \dots, a_j, \dots, a_i, \dots, a_n) = -f(a_1, \dots, a_i, \dots, a_j, \dots, a_n).$$

From Lemma 1.17, we have

$$\begin{aligned} 0 &= f(\alpha_A^2(a_1), \dots, \alpha_A^2(a_{j-1}), \alpha_A(a_i +_A a_j), \alpha_A^2(a_{j+1}), \dots, \alpha_A^2(a_{i-1}), \alpha(a_i + a_j), \alpha_A^2(a_{i+1}), \dots, \alpha_A^2(a_n)) \\ &= \left(f(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_i, a_{i+1}, a_n) +_B f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{i-1}, a_i, a_{i+1}, a_n) \right) \\ &+_B \left(f(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, a_n) +_B f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, a_n) \right). \end{aligned}$$

But

$$f(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_i, a_{i+1}, a_n) = 0,$$

and

$$f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, a_n) = 0.$$

These give

$$0 = \alpha_B \left(f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{i-1}, a_i, a_{i+1}, a_n) \right) +_B \alpha_B \left(f(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, a_n) \right).$$

Since α_B is bijective, the lemma holds.

2 Hom-rings

Definition 2.1. A Hom-ring of type (1) is a tuple $(A, +, \cdot, 0, \alpha, \beta)$ with a set A together two binary operations $+: A \times A \rightarrow A$ (the addition) and $\cdot: A \times A \rightarrow A$ (the multiplication) and two set maps $\alpha, \beta: A \rightarrow A$, such that:

1. $(A, +, 0, \alpha)$ is an abelain Hom-group.
2. β is an endomorphism of the abelian Hom-group $(A, +, 0, \alpha)$, i.e., $\beta(x + y) = \beta(x) + \beta(y)$, $\forall x, y \in A$, and $\alpha \circ \beta = \beta \circ \alpha$.
3. α and β are multiplicative maps, i.e., $\alpha(xy) = \alpha(x)\alpha(y)$ and $\beta(xy) = \beta(x)\beta(y)$, for all $x, y \in A$.
4. The map β and the product \cdot satisfy the Hom-associativity condition;

$$\beta(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z). \quad (2.1)$$

5. The multiplication is Hom-distributive over the addition on both sides;

$$(a) \quad \alpha(x) \cdot (y + z) = x \cdot y + x \cdot z, \quad (2.2)$$

$$(b) \quad (y + z) \cdot \alpha(x) = y \cdot x + z \cdot x, \quad (2.3)$$

for all $x, y, z \in A$.

If $(A, +, \cdot, 0, \alpha, \beta)$ admits an unit element $1 \in A$ satisfying the properties

$$1. \quad x \cdot 1 = 1 \cdot x = \beta(x), \quad \forall x \in A, \quad (2.4)$$

$$2. \quad \alpha(1) = \beta(1) = 1, \quad (2.5)$$

the Hom-ring A is said to be unitary.

Remark 2.2. Note that if $\alpha = id$, we automatically have the definition of Hom-ring in [6] and when $\alpha = \beta = id$, we have the definition of ring.

Example 2.3. Let $(A, +, \cdot, 0)$ be an unitary ring. Suppose that $r \neq 1$ is a central idempotent element of A and $u \in A$ is an invertible element. Then the maps $\alpha_r : A \rightarrow A$ and $\beta_u : A \rightarrow A$ defined respectively by $\alpha_r(x) = rx$ and $\beta_u(x) = uxu^{-1}$ are two commuting endomorphisms of the ring A . In this case, we have a new addition $\tilde{+} : A \times A \rightarrow A$ and a new multiplication $\tilde{\cdot} : A \times A \rightarrow A$ given by

- $x \tilde{+} y = \alpha_r(x + y) = r(x + y)$,
- $x \tilde{\cdot} y = \beta_u(x \cdot y) = ux y u^{-1}$, for all $x, y \in A$.

According to [4], $(A, \tilde{+}, \tilde{\cdot}, 0, \alpha_r, \beta_u)$ is a Hom-ring of type (1). We denote this by A_{α_r, β_u} .

Definition 2.4. A Hom-ring of type (2) is a tuple $(A, +, \cdot, 0, \alpha, \beta)$ with a set A together two binary operations $+: A \times A \rightarrow A$ (the addition) and $\cdot : A \times A \rightarrow A$ (the multiplication) and two set maps $\alpha, \beta : A \rightarrow A$, such that:

1. $(A, +, 0, \alpha)$ is an abelain Hom-group.
2. β is an endomorphism of the abelian Hom-group $(A, +, 0, \alpha)$, i.e., $\beta(x + y) = \beta(x) + \beta(y)$, $\forall x, y \in A$, and $\alpha \circ \beta = \beta \circ \alpha$.
3. α and β are multiplicative maps, i.e., $\alpha(xy) = \alpha(x)\alpha(y)$ and $\beta(xy) = \beta(x)\beta(y)$, for all $x, y \in A$.
4. The maps α, β and the product \cdot satisfy $\beta(\alpha^2(x)) \cdot (\beta(\alpha(y)) \cdot \beta^2(z)) = (\alpha^2(x) \cdot \beta(\alpha(y))) \cdot \beta^2(\alpha(z))$.
5. The multiplication is Hom-distributive over the addition on both sides;
 - (a) $\alpha^2(x) \cdot \beta(y + z) = \alpha(x) \cdot \beta(y) + \alpha(x) \cdot \beta(z)$,
 - (b) $\alpha(y + z) \cdot \alpha(\beta(x)) = \alpha(y) \cdot \beta(x) + \alpha(z) \cdot \beta(x)$,
 for all $x, y, z \in A$.

If $(A, +, \cdot, 0, \alpha, \beta)$ admits a unit element $1 \in A$ satisfying the following properties

1. $x \cdot 1 = \beta(x)$ and $1 \cdot x = \alpha(x)$, $\forall x \in A$,
2. $\alpha(1) = \beta(1) = 1$,

the Hom-ring A is said to be unitary.

Example 2.5. Let $(A, +, \cdot, 0)$ be a unitary ring and C the set of all central idempotents of A . For any two elements e and f in C , we define respectively the maps $\alpha_e : A \rightarrow A$ and $\beta_f : A \rightarrow A$ by $\alpha_e(x) = ex$ and $\beta_f(x) = fx$. Then α_e and β_f are two commuting endomorphisms of the ring A . Defining a new addition $\tilde{+} : A \times A \rightarrow A$ and a new multiplication $\tilde{\cdot} : A \times A \rightarrow A$ given by

- $x \tilde{+} y = \alpha_e(x + y) = e(x + y)$,
- $x \tilde{\cdot} y = \beta_f(x) \cdot \alpha_e(y) = efxy$, for all $x, y \in A$,

then, from [4], $(A, \tilde{+}, \tilde{\cdot}, 0, \alpha_e, \beta_f)$ is a Hom-ring of type (2). We denote this by A_{α_e, β_f} .

Definition 2.6. • We say that a Hom-ring $(A, +, \cdot, 0, \alpha, \beta)$ is a Hom-nonassociative ring of type (1) (resp. of type (2)) if the 4th property in Definition 2.1 (resp. Definition 2.4) is not satisfied for it.

- A Hom-ring $(A, +, \cdot, 0, \alpha, \beta)$ is called a commutative Hom-ring if for all $x, y \in A$ we have $x \cdot y = y \cdot x$. Also, a Hom-ring is said to be regular if the twisted maps α and β are bijective. The inverse of an element x in a Hom-ring $(A, +, \cdot, 0, \alpha, \beta)$ with respect to the addition will be named the additive inverse and will be denoted by $(-x)$.

In what follows we will provide some calculation properties, the definition of Hom-ring homomorphism and Hom-subring.

Proposition 2.7. [4] Let $(A, +, \cdot, 0, \alpha, \beta)$ be a unitary regular Hom-ring of type (1) or of type (2).

1. For all $a \in A$, we have $a \cdot 0 = 0 \cdot a = 0$.
2. Let $a, b \in A$. Then, the additive inverse of $(a \cdot b)$ is $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$. One can notes that $\alpha(-a) = -\alpha(a)$.

Remark 2.8. In the proof of the above proposition, only the fact that α is bijective has been used.

2.1 Hom-algebra over commutative Hom-ring of type (1)

Definition 2.9. Let $(A, +_A, \cdot, 0, \alpha, \beta)$ be a unitary Hom-ring of type (1) with unit 1_A . A left A -module M consists of an abelian Hom-group $(M, +_M, \gamma)$ and an operation $\lambda_l : A \otimes M \rightarrow M$ such that for all a, b in A and m, m_1, m_2 in M , we have:

$$\lambda_l(\alpha(a) \otimes (m_1 +_M m_2)) = \lambda_l(a \otimes m_1) +_M \lambda_l(a \otimes m_2), \quad (2.6)$$

$$\lambda_l((a +_A b) \otimes \gamma(m_1)) = \lambda_l(a \otimes m_1) +_M \lambda_l(b \otimes m_1), \quad (2.7)$$

$$\lambda_l((a \cdot b) \otimes \gamma(m)) = \lambda_l(\beta(a) \otimes \lambda_l(b \otimes m)), \quad (2.8)$$

$$\lambda_l(1_A \otimes m_1) = \gamma(m_1). \quad (2.9)$$

The operation of the Hom-ring on M is called scalar multiplication, and is usually written by am for a in A and m in M . The notation ${}_A M$ indicates a left A -module M . If γ is invertible, a such left module ${}_A M$ is said regular. A right A -module M or M_A is defined similarly, except that the ring acts on the right; i.e., the operation takes the form $\lambda_r : M \otimes A \rightarrow M$, and the above axioms are written with an scalar multiplication on the right.

Example 2.10. If A is a unitary Hom-ring, then we may consider A as either a left or right A -module, where the action is given by the multiplication.

Proposition 2.11. Let $(A, +_A, \cdot, 0, \alpha, \beta)$ be a unitary Hom-ring. If A is commutative, then these left modules are the same as right modules.

Proof. Let M be a right A -module, i.e., there exists $\lambda_r : M \otimes A \rightarrow M$ by $\lambda_r(m \otimes a) = ma$. Now, we define $\lambda_l : A \otimes M \rightarrow M$ by $\lambda_l(a \otimes m) = ma$. Using the commutativity of the Hom-ring A , we can verify that λ_l defines a left module structure for M over A .

Definition 2.12. Let $(A, +_A, \cdot_A, 0_A, \alpha_A, \beta_A)$ be a commutative Hom-ring of type (1). An A -Hom-algebra is a Hom-ring $(B, +_B, \cdot_B, 0_B, \alpha_B, \beta_B)$ of type (1) such that the multiplication " \cdot_B " satisfies

$$\beta_A(a) * (b_1 \cdot_B b_2) = (a * b_1) \cdot_B \beta_B(b_2) = \beta_B(b_1) \cdot (a * b_2) \quad \forall a \in A, b_1, b_2 \in B.$$

Remark 2.13. An Hom-algebra can be defined as a Hom-ring $(B, +_B, \cdot_B, 0_B, id_B, \beta_B)$ of type (1) and a \mathbb{K} -Hom-algebra.

Remark 2.14. In the rest of this paper, the word Hom-ring refers to a Hom-ring of type (1).

3 Alternative-Hom-rings

Definition 3.1. Let $(A, +, \cdot, 0, \alpha, \beta)$ be a Hom-ring (not necessarily Hom-associative). The Hom-commutator $[,] : A \times A \rightarrow A$ and the β -associator $as_\beta : A \times A \times A \rightarrow A$ are defined by

$$[x, y] = xy - yx \quad as_\beta(x, y, z) = (xy)\beta(z) - \beta(x)(yz),$$

for all $x, y, z \in A$.

Lemma 3.2. Let $(A, +, \cdot, 0, \alpha, \beta)$ be a Hom-ring (not necessarily Hom-associative) such that α is bijective. Then, the Hom-commutator is a skew-symmetric Hom-bilinear mapping, the β -associator is Hom-trilinear and they verify the following identities

$$([xy, \beta(z)] - \alpha \circ \beta(x)[y, z]) - \alpha([x, z]\alpha \circ \beta(y)) = (as_\beta(x, y, z) - as_\beta(x, z, y)) + \alpha(as_\beta(z, x, y)), \quad (3.1)$$

$$\begin{aligned} (as_\beta(wx, \beta(y), \beta(z)) - as_\beta(\beta(w), xy, \beta(z))) + \alpha(as_\beta(\beta(w), \beta(x), yz)) &= \alpha^2 \circ \beta^2(w)\alpha(as_\beta(x, y, z)) \\ &+ \alpha(as_\beta(w, x, y))\alpha^2 \circ \beta^2(z). \end{aligned} \quad (3.2)$$

Proof. The skew-symmetry of the Hom-commutator is straightforward. Let $x, y, z, t \in A$. Then we have,

$$[x + y, \alpha(z)] = (x + y)\alpha(z) - \alpha(z)(x + y).$$

But the multiplication is Hom-distributive over the addition on both sides, i.e., the multiplication and α satisfy the equations (2.2) and (2.3). This gives

$$[x + y, \alpha(z)] = (xz + yz) - (zx + zy).$$

From the Proposition 1.13, we obtain

$$[x + y, \alpha(z)] = (xz - zx) + (yz - zy) = [x, z] + [y, z].$$

According to the skew-symmetry of the Hom-commutator, we can also see that

$$[\alpha(z), x + y] = [z, x] + [z, y].$$

This shows that the Hom-commutator is a skew-symmetric Hom-bilinear mapping. Now we're going to show the Hom-trilinearity of the β -associator:

$$as_\beta(x + y, \alpha(z), \alpha(t)) = \left((x + y)\alpha(z) \right) \beta(\alpha(t) - \beta(x + y)) \left((\alpha(z)\alpha(t)) \right).$$

By the equations (2.2), (2.3), the multiplicativity of α and the fact that $\alpha \circ \beta = \beta \circ \alpha$, we have

$$as_\beta(x + y, \alpha(z), \alpha(t)) = (xz + yz)\alpha(\beta(t) - \beta(x + y)\alpha(zt)).$$

The fact $\beta(x + y) = \beta(x) + \beta(y)$ gives

$$as_\beta(x + y, \alpha(z), \alpha(t)) = (xz + yz)\alpha(\beta(t) - (\beta(x) + \beta(y))\alpha(zt)).$$

Using formerly the equations (2.2), (2.3), we find

$$as_\beta(x + y, \alpha(z), \alpha(t)) = \left((xz)\beta(t) + (yz)\beta(t) \right) - \left(\beta(x)(zt) + \beta(y)(zt) \right).$$

The Proposition 1.13 implies

$$\begin{aligned} as_\beta(x + y, \alpha(z), \alpha(t)) &= \left((xz)\beta(t) - \beta(x)(zt) \right) + \left((yz)\beta(t) - \beta(y)(zt) \right) \\ &= as_\beta(x, z, t) + as_\beta(y, z, t). \end{aligned}$$

Similarly, we can show that

$$as_\beta(\alpha(x), y + z, \alpha(t)) = as_\beta(x, y, t) + as_\beta(x, z, t),$$

and

$$as_\beta(\alpha(x), \alpha(y), z + t) = as_\beta(x, y, z) + as_\beta(x, y, t).$$

This shows the Hom-trilinearity of the β -associator. Now, taking into account Proposition 1.13, the equations (2.2), (2.3), the multiplicativity of β and that $(A, +, 0, \alpha)$ is regular abelian Hom-group, we will prove the identities (3.1) and (3.2).

For identity (3.1):

$$\begin{aligned}
& \left([xy, \beta(z)] - \alpha \circ \beta(x)[y, z] \right) - \alpha \left([x, z] \alpha \circ \beta(y) \right) \\
&= \left(((xy)\beta(z) - \beta(z)(xy)) - \alpha(\beta(x))(yz - zy) \right) - \alpha \left((xz - zx)\alpha(\beta(y)) \right) \\
&= \left(((xy)\beta(z) - \beta(z)(xy)) - (\beta(x)(yz) - \beta(x)(zy)) \right) - \alpha((xz)\beta(y) - (zx)\beta(y)) \\
&= \left(((xy)\beta(z) - \beta(x)(yz)) - (\beta(z)(xy) - \beta(x)(zy)) \right) - \alpha((xz)\beta(y) - (zx)\beta(y)) \\
&= \left(as_\beta(x, y, z) - (\beta(z)(xy) - \beta(x)(zy)) \right) - \alpha \left((xz)\beta(y) - (zx)\beta(y) \right) \\
&= \alpha(as_\beta(x, y, z)) - \left((\beta(z)(xy) - \beta(x)(zy)) + ((xz)\beta(y) - (zx)\beta(y)) \right) \\
&= \alpha(as_\beta(x, y, z)) - \left((\beta(z)(xy) - (zx)\beta(y)) + ((xz)\beta(y) - \beta(x)(zy)) \right) \\
&= \alpha(as_\beta(x, y, z)) - \left(as_\beta(x, z, y) - as_\beta(z, x, y) \right) \\
&= \left(as_\beta(x, y, z) - as_\beta(x, z, y) \right) + \alpha(as_\beta(z, x, y)).
\end{aligned}$$

For identity (3.2):

$$\begin{aligned}
& \left(as_\beta(wx, \beta(y), \beta(z)) - as_\beta(\beta(w), xy, \beta(z)) \right) + \alpha(as_\beta(\beta(w), \beta(x), yz)) \\
&= \left(\left(((wx)\beta(y))\beta^2(z) - \beta(wx)(\beta(y)\beta(z)) \right) - \left((\beta(w)(xy))\beta^2(z) - \beta^2(w)((xy)\beta(z)) \right) \right) \\
&\quad + \alpha \left((\beta(w)\beta(x))\beta(yz) + \beta^2(w)(\beta(x)(yz)) \right) \\
&= \left(\left(((wx)\beta(y))\beta^2(z) - (\beta(w)(xy))\beta^2(z) \right) - \left(\beta(wx)\beta(yz) - \beta^2(w)((xy)\beta(z)) \right) \right) \\
&\quad + \alpha \left((\beta(w)\beta(x))\beta(yz) + \beta^2(w)(\beta(x)(yz)) \right) \\
&= \left(\left(((wx)\beta(y) - \beta(w)(xy))\alpha(\beta^2(z)) \right) - \left(\beta(wx)\beta(yz) - \beta^2(w)((xy)\beta(z)) \right) \right) \\
&\quad + \alpha \left((\beta(w)\beta(x))\beta(yz) + \beta^2(w)(\beta(x)(yz)) \right) \\
&= \left(as_\beta(w, x, y)\alpha(\beta^2(z)) - \left(\beta(wx)\beta(yz) - \beta^2(w)((xy)\beta(z)) \right) \right) \\
&\quad + \alpha \left((\beta(w)\beta(x))\beta(yz) + \beta^2(w)(\beta(x)(yz)) \right) \\
&= \alpha(as_\beta(w, x, y)\alpha(\beta^2(z))) - \left(\left(\beta(wx)\beta(yz) - \beta^2(w)((xy)\beta(z)) \right) \right. \\
&\quad \left. - \left((\beta(wx))\beta(yz) - \beta^2(w)(\beta(x)(yz)) \right) \right) \\
&= \alpha(as_\beta(w, x, y)\alpha(\beta^2(z))) - \left(\left(\beta(wx)\beta(yz) - \beta(wx)\beta(yz) \right) \right. \\
&\quad \left. - \left(\beta^2(w)((xy)\beta(z)) - \beta^2(w)(\beta(x)(yz)) \right) \right) \\
&= \alpha(as_\beta(w, x, y)\alpha(\beta^2(z))) + \alpha \left(\beta^2(w)((xy)\beta(z)) - \beta^2(w)(\beta(x)(yz)) \right) \\
&= \alpha(as_\beta(w, x, y)\alpha(\beta^2(z))) + \alpha \left(\alpha(\beta^2(w)((xy)\beta(z) - \beta(x)(yz))) \right) \\
&= \alpha^2 \circ \beta^2(w)\alpha(as_\beta(x, y, z)) + \alpha(as_\beta(w, x, y))\alpha^2 \circ \beta^2(z).
\end{aligned}$$

Definition 3.3. A Hom-alternative-ring is a Hom-nonassociative ring satisfying the condition

$$as_\beta(x, x, y) = as_\beta(y, x, x) = 0. \quad (3.3)$$

Remark 3.4. If $\alpha = \beta = id$, we automatically have the definition of alternative ring.

Lemma 3.5. *Let $(A, +, \cdot, 0, \alpha, \beta)$ be a Hom-alternative-ring such that α is bijective. Then, the β -associator is skew-symmetric and we have*

$$[xy, \beta(x)] = -\alpha \circ \beta(x)[x, y], \quad (3.4)$$

$$[xy, \beta(y)] = [x, y]\alpha \circ \beta(y), \quad (3.5)$$

for all $x, y \in A$.

Proof. Lemma 1.19 gives the skew-symmetry of the β -associator. Using Definition 3.3 in (3.1), it follows

$$\alpha([xy, \beta(x)] - \alpha \circ \beta(x)[y, x]) = 0. \quad (3.6)$$

Since α is bijective, (3.4) holds. In the similar way, we can obtain (3.5).

Now, we consider the (multilinear) function $f(w, x, y, z)$ defined by

$$f(w, x, y, z) = \alpha(as_\beta(wx, \beta(y), \beta(z)) - (\alpha(\beta^2(x))as_\beta(w, y, z) + as_\beta(x, y, z)\alpha(\beta^2(w))). \quad (3.7)$$

If we write

$$3f(w, x, y, z) = (f(w, x, y, z) + f(w, x, y, z)) + \alpha(f(w, x, y, z)),$$

we have the following lemma.

Lemma 3.6. *In a Hom-alternative ring $(A, +, \cdot, 0, \alpha, \beta)$ such that α is bijective, the function f defined by (3.7) is skew-symmetric and satisfies the identities*

$$3f(w, x, y, z) = \alpha([\alpha(\beta^2(w)), as_\beta(x, y, z)] - [\alpha(\beta^2(x)), as_\beta(w, y, z)]) \quad (3.8)$$

$$+ \alpha([\alpha(\beta^2(y)), as_\beta(z, w, x)] - [\alpha(\beta^2(z)), as_\beta(w, x, y)]), \quad (3.9)$$

$$f(w, x, y, z) = as_\beta([w, x], \alpha(\beta(y)), \alpha(\beta(z))) + as_\beta([y, z], \alpha(\beta(w)), \alpha(\beta(x))). \quad (3.10)$$

Proof. In all this proof we take into account the Proposition 1.13 and the fact that $(A, +, 0, \alpha)$ is a regular abelian Hom-group. By the skew-symmetry of the β -associator and from the equation (3.2), $\alpha(as_\beta(wx, \beta(y), \beta(z)))$ can be written as follows

$$\begin{aligned} \alpha(as_\beta(wx, \beta(y), \beta(z))) &= (\alpha \circ \beta^2(w)\alpha(as_\beta(x, y, z)) + as_\beta(w, x, y)\alpha \circ \beta^2(z)) \\ &+ (as_\beta(\beta(w), xy, \beta(z)) - as_\beta(\beta(w), \beta(x), yz)). \end{aligned}$$

Substitution from (3.7) gives

$$(f(w, x, y, z) - f(x, y, z, w)) + \alpha(f(y, z, w, x)) = F(w, x, y, z), \quad (3.11)$$

where $F(w, x, y, z)$ denotes the right-hand side of (3.8), and hence changes sign when w, x, y, z are permuted cyclically. Thus, from (3.11), we get

$$0 = F(w, x, y, z) + F(x, y, z, w) = \alpha^2(f(w, x, y, z) + f(z, w, x, y)).$$

Since α is bijective, we find

$$f(w, x, y, z) = -f(z, w, x, y). \quad (3.12)$$

Hence f changes sign when its arguments are permuted cyclically (by (3.12)) and when the last two are interchanged (by (3.7)), and therefore when any two are interchanged. So f is skew-symmetric. In particular, (3.11) reduces to (3.8). Since f is skew-symmetric, and if we write $2f(w, x, y, z) = f(w, x, y, z) + f(w, x, y, z)$, we have

$$\begin{aligned} 2f(w, x, y, z) &= f(w, x, y, z) - f(x, w, y, z) \\ &= \alpha(as_\beta([w, x], \alpha(\beta(y)), \alpha(\beta(z))) \\ &\quad + ([\alpha(b^2(w)), as_\beta(x, y, z)] - [\alpha(b^2(x)), as_\beta(w, y, z)]). \end{aligned} \quad (3.13)$$

Therefore, one can find

$$\begin{aligned} \alpha^2(3f(w, x, y, z)) &= \left(2f(w, x, y, z) - \alpha^2(as_\beta([w, x], \alpha(\beta(y)), \alpha(\beta(z)))\right) \\ &\quad + \left(2f(w, x, y, z) - \alpha^2(as_\beta([y, z], \alpha(\beta(w)), \alpha(\beta(x)))\right). \end{aligned}$$

According to the Hom-associativity of α and the addition "+", we obtain the identity (3.10). This completes the proof.

From Lemma 3.6 we deduce a certain number of identities which appear in Hom-alternate rings:

Proposition 3.7. *For all x, y, z of a Hom-alternate ring $(A, +, \cdot, 0, \alpha, \beta)$ such that α is bijective, we have*

$$\alpha(as_\beta(x^2, \beta(y), \beta(z))) = as_\beta(x, y, z)\alpha(\beta^2(x)) + \alpha(\beta^2(x))as_\beta(x, y, z), \text{ where } x^2 = xx, \quad (3.14)$$

$$as_\beta(\beta(x), xy, \beta(z)) = as_\beta(\beta(x), \beta(y), xz) = as_\beta(x, y, z)\alpha(\beta^2(x)), \quad (3.15)$$

$$as_\beta(\beta(x), yx, \beta(z)) = as_\beta(\beta(x), \beta(y), zx) = \alpha(\beta^2(x))as_\beta(x, y, z), \quad (3.16)$$

and the Hom-Moufang identities

$$\beta(xy)\beta(zx) = \beta^2(x)((yz)\beta(x)), \quad (3.17)$$

$$((xy)\beta(x))\beta^2(z) = \beta^2(x)(\beta(y)(xz)), \quad ((zx)\beta(y))\beta^2(x) = \beta^2(z)(\beta(x)(yx)). \quad (3.18)$$

Proof. Considering $w = x$ in (3.7), we obtain

$$0 = f(x, x, y, z) = \alpha(as_\beta(x^2, \beta(y), \beta(z))) - \left(\alpha(\beta^2(x))as_\beta(x, y, z) + as_\beta(x, y, z)\alpha(\beta^2(x))\right).$$

As α is bijective, we deduce (3.14). Again by (3.7), we have

$$0 = f(x, y, z, x) = \alpha(as_\beta(xy, \beta(z), \beta(x))) - \left(\alpha(\beta^2(y))as_\beta(x, z, x) + as_\beta(y, z, x)\alpha(\beta^2(x))\right).$$

Since f is skew-symmetric, we find

$$0 = \alpha(as_\beta(\beta(x), xy, \beta(z))) - \alpha(as_\beta(x, y, z)\alpha(\beta^2(x))).$$

Therefore, the bijectivity of α gives $as_\beta(\beta(x), xy, \beta(z)) = as_\beta(x, y, z)\alpha(\beta^2(x))$. Similarly, by (3.7), we have

$$0 = f(x, z, y, x) = \alpha(as_\beta(xz, \beta(y), \beta(x))) - \left(\alpha(\beta^2(z))as_\beta(x, y, x) + as_\beta(z, y, x)\alpha(\beta^2(x))\right).$$

As α is bijective and f is skew-symmetric, we deduce $as_\beta(\beta(x), \beta(y), xz) = as_\beta(x, y, z)\alpha(\beta^2(x))$. So the identity (3.15) is well shown, similarly for (3.16). Using (3.16), we get (3.17) as follows:

$$\begin{aligned}\alpha^2(\beta(xy)\beta(zx)) &= as_\beta(\beta(x), \beta(y), zx) + \alpha(\beta^2(x)(\beta(y)(zx))) \\ &= \alpha(\beta^2(x))as_\beta(x, y, z) + \alpha(\beta^2(x)(\beta(y)(zx))).\end{aligned}$$

Writing the expression of $as_\beta(x, y, z)$ in the definition of β -associator and using (2.2) and the fact that $(A, +, 0, \alpha)$ is a regular abelian Hom-group, we find

$$\alpha^2(\beta(xy)\beta(zx)) = \alpha^2(\beta^2(x)((yz)\beta(x))).$$

As α is bijective, we deduce (3.17). The first equation of (3.18) comes by

$$\begin{aligned}\alpha^3(((xy)\beta(x))\beta^2(z)) &= \alpha(as_\beta(xy, \beta(x), \beta(z))) + \alpha^2(\beta(xy)\beta(xz)) \\ &= \alpha^2(\beta^2(x)(\beta(y)(xz))) + (as_\beta(\beta(x), \beta(y), xz) - as_\beta(\beta(x), xy, \beta(z))) \\ &= \alpha^3(\beta^2(x)(\beta(y)(xz))).\end{aligned}$$

Similarly, we find the second equation of (3.18).

The following proposition gives new identities for a Hom-alternative ring.

Proposition 3.8. *For all x, y, z of a Hom-alternative ring $(A, +, \cdot, 0, \alpha, \beta)$ such that α is bijective, we have*

$$\begin{aligned}as_\beta(as_\beta(x, y, z), \alpha(\beta^2(x)), \alpha(\beta^2(y))) &= [\alpha(\beta^2(x)), \alpha(\beta^2(y))]\alpha(\beta(as_\beta(x, y, z))) \\ &= -\alpha(\beta(as_\beta(x, y, z)))[\alpha(\beta^2(x)), \alpha(\beta^2(y))],\end{aligned}\tag{3.19}$$

$$as_\beta(as_\beta(x, y, z)^2, \alpha(\beta^3(x)), \alpha(\beta^3(y))) = 0.\tag{3.20}$$

Proof. Proposition 3.7 gives us

$$\alpha(as_\beta(\beta(x^2), \beta^2(y), \beta(zx))) = \left(\alpha(\beta^2(y))as_\beta(x, y, z)\right)\alpha(\beta^3(x)) + \alpha(\beta^3(x))\left(\alpha(\beta^2(y))as_\beta(x, y, z)\right),$$

and

$$\alpha(as_\beta(\beta(x^2), \beta^2(y), \beta(zx))) = \alpha(\beta^3(y))\left(as_\beta(x, y, z)\alpha(\beta^2(x))\right) + \alpha(\beta^3(y))\left(\alpha(\beta^2(x))as_\beta(x, y, z)\right).$$

By subtraction and using (1.13), we have

$$\begin{aligned}0 &= as_\beta(as_\beta(x, y, z), \alpha(\beta^2(x)), \alpha(\beta^2(y))) + \left(\alpha(\beta^3(x))\left(\alpha(\beta^2(y))as_\beta(x, y, z)\right)\right. \\ &\quad \left.- \alpha(\beta^3(y))\left(\alpha(\beta^2(x))as_\beta(x, y, z)\right)\right).\end{aligned}$$

But $as_\beta(\alpha(\beta^2(x)), \alpha(\beta^2(y)), as_\beta(x, y, z)) = -as_\beta(\alpha(\beta^2(y)), \alpha(\beta^2(x)), as_\beta(x, y, z))$. This gives

$$\alpha(\beta^3(x))\left(\alpha(\beta^2(y))as_\beta(x, y, z)\right) - \alpha(\beta^3(y))\left(\alpha(\beta^2(x))as_\beta(x, y, z)\right) = -[\alpha(\beta^2(x)), \alpha(\beta^2(y))]\alpha(\beta(as_\beta(x, y, z))).$$

So, we can see that

$$as_\beta(as_\beta(x, y, z), \alpha(\beta^2(x)), \alpha(\beta^2(y))) = [\alpha(\beta^2(x)), \alpha(\beta^2(y))]\alpha(\beta(as_\beta(x, y, z))).$$

Similarly, by applying the Proposition 3.7 in two ways to $\alpha(as_\beta(\beta(x^2), \beta^2(y), \beta(zx)))$, we can show that

$$as_\beta(as_\beta(x, y, z), \alpha(\beta^2(x)), \alpha(\beta^2(y))) = -\alpha(\beta(as_\beta(x, y, z)))[\alpha(\beta^2(x)), \alpha(\beta^2(y))].$$

This proves (3.19). (3.20) proves in the following way:

$$\begin{aligned}as_\beta(as_\beta(x, y, z)^2, \alpha(\beta^3(x)), \alpha(\beta^3(y))) &= as_\beta(as_\beta(x, y, z), \alpha(\beta^2(x)), \alpha(\beta^2(y)))\alpha(\beta^2(as_\beta(x, y, z))) \\ &\quad + \alpha(\beta^2(as_\beta(x, y, z)))as_\beta(as_\beta(x, y, z), \alpha(\beta^2(x)), \alpha(\beta^2(y))) \\ &= \left(-\alpha(\beta(as_\beta(x, y, z)))[\alpha(\beta^2(x)), \alpha(\beta^2(y))]\right)\alpha(\beta^2(as_\beta(x, y, z))) \\ &\quad + \alpha(\beta^2(as_\beta(x, y, z)))\left([\alpha(\beta^2(x)), \alpha(\beta^2(y))]\alpha(\beta(as_\beta(x, y, z)))\right) \\ &= -as_\beta(\alpha(\beta(as_\beta(x, y, z))), [\alpha(\beta^2(x)), \alpha(\beta^2(y))], \alpha(\beta(as_\beta(x, y, z)))) \\ &= 0.\end{aligned}$$

The following lemma helps to get a result for Hom-alternative ring:

Lemma 3.9. *In a Hom-alternative ring $(A, +, \cdot, 0, \alpha, \beta)$ such that α is bijective, the (multilinear) function $g(u, v, w, x, y)$, defined for all $u, v, w, x, y \in A$ by*

$$\begin{aligned} \alpha^2(f(uv, \beta(w), \beta(x), \beta(y))) &= \left(\left(\alpha^2(\beta^3(u))f(v, w, x, y) + f(u, w, x, y)\alpha^2(\beta^3(v)) \right) \right. \\ &\quad \left. + \alpha^2(\beta(as_\beta(u, x, y)))\alpha^2(\beta^2([v, w])) \right) \\ &\quad + \left(\alpha^2(\beta^2([u, w]))\alpha^2(\beta(as_\beta(v, x, y))) + g(u, v, w, x, y) \right), \end{aligned} \quad (3.21)$$

is skew-symmetric in u, v, w and in x, y . Equivalently,

$$f(uv, \beta(u), \beta(x), \beta(y)) = \alpha^2(\beta^3(u))f(v, u, x, y) + \alpha(\beta(as_\beta(u, x, y)))\alpha(\beta^2([v, u])), \quad (3.22)$$

$$f(uv, \beta(v), \beta(x), \beta(y)) = f(u, v, x, y)\alpha^2(\beta^3(v)) + \alpha(\beta^2([u, v]))\alpha(\beta(as_\beta(v, x, y))), \quad (3.23)$$

for all $u, v, x, y \in A$.

Proof. Since f is multilinear, so is g . Also, g is zero for $y = x$, hence skew-symmetric in x, y . Considering $G(u, v, w) = g(u, v, w, x, y)$ for fixed x, y , from (3.22) and (3.23), we deduce that G is skew-symmetric in u, v, w . To prove (3.22), note that

$$f(vu, \beta(u), \beta(x), \beta(y)) = as_\beta([vu, \beta(u)], \alpha(\beta^2(x)), \alpha(\beta^2(y))) + as_\beta([\beta(x), \beta(y)], \alpha(\beta(vu)), \alpha(\beta^2(u))),$$

by (3.10). From (3.16), we get

$$as_\beta([\beta(x), \beta(y)], \alpha(\beta(vu)), \alpha(\beta^2(u))) = \alpha^2(\beta^3(u))as_\beta([x, y], \alpha(\beta(v)), \alpha(\beta(u))).$$

Also (3.5) implies

$$as_\beta([vu, \beta(u)], \alpha(\beta^2(x)), \alpha(\beta^2(y))) = as_\beta([v, u]\alpha(\beta(u)), \alpha(\beta^2(x)), \alpha(\beta^2(y))).$$

Moreover, (3.7) implies

$$\begin{aligned} f([v, u], \alpha(\beta(u)), \alpha(\beta(x)), \alpha(\beta(y))) &= \alpha(as_\beta([v, u]\alpha(\beta(u)), \alpha(\beta^2(x)), \alpha(\beta^2(y)))) \\ &\quad - \left(as_\beta(\alpha(\beta(u)), \alpha(\beta(x)), \alpha(\beta(y)))\alpha(\beta^2([v, u])) \right. \\ &\quad \left. + \alpha^2(\beta^3(u))as_\beta([v, u], \alpha(\beta(x)), \alpha(\beta(y))) \right). \end{aligned}$$

So, we get

$$\begin{aligned} \alpha^3(as_\beta([vu, \beta(u)], \alpha(\beta^2(x)), \alpha(\beta^2(y)))) &= f([v, u], \alpha(\beta(u)), \alpha(\beta(x)), \alpha(\beta(y))) \\ &\quad + \alpha \left(as_\beta(\alpha(\beta(u)), \alpha(\beta(x)), \alpha(\beta(y)))\alpha(\beta^2([v, u])) \right. \\ &\quad \left. + \alpha^2(\beta^3(u))as_\beta([v, u], \alpha(\beta(x)), \alpha(\beta(y))) \right). \end{aligned} \quad (3.24)$$

Using (2.2) and the fact that $(A, +, 0, \alpha)$ is a regular abelian Hom-group and replacing (3.24) in the expression of $\alpha^3(f(vu, \beta(u), \beta(x), \beta(y)))$, we obtain

$$\begin{aligned} \alpha^3(f(vu, \beta(u), \beta(x), \beta(y))) &= \alpha(f([v, u], \alpha(\beta(u)), \alpha(\beta(x)), \alpha(\beta(y)))) \\ &\quad + \alpha \left(\alpha^3(\beta^3(u))f(v, u, x, y) + \alpha^2(\beta(as_\beta(u, x, y)))\alpha^2(\beta^2([v, u])) \right). \end{aligned}$$

The Hom-multilinearity of f gives

$$\begin{aligned} \alpha^3(f(vu, \beta(u), \beta(x), \beta(y))) &= \alpha^2(f(vu, \beta(u), \beta(x), \beta(y))) - \left(\alpha(f(uv, \beta(u), \beta(x), \beta(y))) \right. \\ &\quad \left. - \left(\alpha^3(\beta^3(u))f(v, u, x, y) + \alpha^2(\beta(as_\beta(u, x, y)))\alpha^2(\beta^2([v, u])) \right) \right). \end{aligned}$$

So, we can see that

$$0 = f(uv, \beta(u), \beta(x), \beta(y)) - \left(\alpha^2(\beta^3(u))f(v, u, x, y) + \alpha(\beta(as_\beta(u, x, y)))\alpha(\beta^2([v, u])) \right).$$

This completes the proof of (3.22). To prove (3.23), we operate similarly on $f(vu, \beta(v), \beta(x), \beta(y))$.

Definition 3.10. Let $(A, +, \cdot, 0, \alpha, \beta)$ be a Hom-ring. A non-zero element a in A is called a left (resp., right) zero-divisor if there exists a non-zero element b such that

$$ab = 0 \quad (ba=0).$$

Proposition 3.11. Let x, y, z be any elements of a Hom-alternative ring $(A, +, \cdot, 0, \alpha, \beta)$ such that α is bijective. If $(as_\beta(x, y, z))$ is not a divisor of zero and β is injective, then

$$\beta^2([as_\beta(x, y, z)^2, \alpha(\beta^3(x))]) = 0. \quad (3.25)$$

Proof. Consider $u = as_\beta(x, y, z)$, $v = uu = u^2$ and $w = f(\alpha(\beta^3(x)), \alpha(\beta^3(y)), \alpha(\beta^3(z)), v)$. (3.21) gives us

$$\begin{aligned} & \alpha^2(f(\alpha(\beta^3(x))v, \alpha(\beta^4(y)), \alpha(\beta^4(x)), \alpha(\beta^4(z)))) \\ &= \left(\left(\alpha^3(\beta^6(x))f(v, \alpha(\beta^3(y)), \alpha(\beta^3(x)), \alpha(\beta^3(z))) \right. \right. \\ & \quad \left. \left. + f(\alpha(\beta^3(x)), \alpha(\beta^3(y)), \alpha(\beta^3(x)), \alpha(\beta^3(z)))\alpha^2(\beta^3(v)) \right) \right. \\ & \quad \left. + \alpha^2(\beta(as_\beta(\alpha(\beta^3(x)), \alpha(\beta^3(x)), \alpha(\beta^3(z))))\alpha^2(\beta^2([v, \alpha(\beta^3(y))])) \right) \\ & \quad + \left(\alpha^3(\beta^5([x, y]))\alpha^2(\beta(as_\beta(v, \alpha(\beta^3(x)), \alpha(\beta^3(z)))) \right) \\ & \quad \left. + g(\alpha(\beta^3(x)), v, \alpha(\beta^3(y)), \alpha(\beta^3(x)), \alpha(\beta^3(z))) \right). \end{aligned}$$

Since $as_\beta(v, \alpha(\beta^3(x)), \alpha(\beta^3(z))) = 0$, we find

$$\begin{aligned} \alpha^2(f(\alpha(\beta^3(x))v, \alpha(\beta^4(y)), \alpha(\beta^4(x)), \alpha(\beta^4(z)))) &= \alpha^2\left(\alpha^3(\beta^6(x))w\right) \\ & \quad + \alpha(g(\alpha(\beta^3(x)), v, \alpha(\beta^3(y)), \alpha(\beta^3(x)), \alpha(\beta^3(z)))). \end{aligned}$$

But

$$f(\alpha(\beta^3(x))v, \alpha(\beta^4(y)), \alpha(\beta^4(x)), \alpha(\beta^4(z))) = -f(\alpha(\beta^3(x))v, \alpha(\beta^4(x)), \alpha(\beta^4(y)), \alpha(\beta^4(z))).$$

Now, from (3.22), we get

$$\begin{aligned} f(\alpha(\beta^3(x))v, \alpha(\beta^4(x)), \alpha(\beta^4(y)), \alpha(\beta^4(z))) &= \alpha^3(\beta^6(x))f(v, \alpha(\beta^3(x)), \alpha(\beta^3(y)), \alpha(\beta^3(z))) \\ & \quad + \alpha^2(\beta^4(as_\beta(x, y, z)))\alpha(\beta^2([v, \alpha(\beta^3(x))])) \\ &= -\alpha^3(\beta^6(x))w + \alpha^2(\beta^4(u))\alpha(\beta^2([v, \alpha(\beta^3(x))])). \end{aligned}$$

Therefore

$$g(\alpha(\beta^3(x)), v, \alpha(\beta^3(y)), \alpha(\beta^3(x)), \alpha(\beta^3(z))) = -\alpha^3(\beta^4(u))\alpha^2(\beta^2([v, \alpha(\beta^3(x))])). \quad (3.26)$$

If we operate similarly on $f(v\alpha(\beta^3(x)), \alpha(\beta^4(y)), \alpha(\beta^4(x)), \alpha(\beta^4(z)))$, we can prove that

$$g(\alpha(\beta^3(x)), v, \alpha(\beta^3(y)), \alpha(\beta^3(x)), \alpha(\beta^3(z))) = \alpha^2(\beta^2([v, \alpha(\beta^3(x))]))\alpha^3(\beta^4(u)). \quad (3.27)$$

So

$$\alpha(\beta^4(u))\beta^2([v, \alpha(\beta^3(x))])) = -\beta^2([v, \alpha(\beta^3(x))]))\alpha(\beta^4(u)). \quad (3.28)$$

(3.7) implies

$$\begin{aligned} f(\alpha(\beta^4(x)), \alpha(\beta^4(y)), \alpha(\beta^3(x))v, \alpha(\beta^4(z))) &= \alpha(as_\beta(\alpha(\beta^4(xy)), \beta(y), \beta(z))) \\ & \quad - \left(\alpha(\beta^2(x))as_\beta(w, y, z) + as_\beta(x, y, z)\alpha(\beta^2(w)) \right). \end{aligned}$$

Using (3.7), (3.15) and (3.20), one can shows that

$$\alpha(\beta^4(u))\beta^2([v, \alpha(\beta^3(x))])) = 0 = \beta^2([v, \alpha(\beta^3(x))]))\alpha(\beta^4(u)). \quad (3.29)$$

Since α is bijective and β is injective, we deduce the proposition.

Remark 3.12. Setting $\alpha = \beta = id$, one recovers all identities in [2].

Now we have introduced the concept of Hom-alternative rings, it's natural to provide a way for constructing them.

Proposition 3.13. Let $(A, +, \cdot, 0)$ be an alternative ring and $\alpha, \beta : A \rightarrow A$ be two ring endomorphisms such that $\alpha \circ \beta = \beta \circ \alpha$. Then $(A, +_\alpha, \cdot_\beta, 0, \alpha, \beta)$ is a Hom-alternative ring, where

$$x +_\alpha y = \alpha(x + y), \quad x \cdot_\beta y = \beta(xy).$$

Proof. The β -associator is defined by

$$as_\beta(x, y, z) = \alpha\left((x \cdot_\beta y) \cdot_\beta \beta(z) - \beta(x) \cdot_\beta (y \cdot_\beta z)\right) = \alpha \circ \beta^2(as(x, y, z)),$$

where $as(x, y, z) = (xy)z - x(yz)$. As $(A, +, \cdot, 0)$ is an alternative ring, i.e., $as(x, x, y) = 0 = as(y, x, x)$, we deduce the proposition.

4 Hom-Lie rings and Hom-Jordan rings

Definition 4.1. A Hom-Lie ring is a sextuplet $(A, +, [,], 0, \alpha, \beta)$ with a set A together two operations $+: A \times A \rightarrow A$ (the addition) and $[], : A \times A \rightarrow A$ (the Hom-Lie bracket) and two set maps $\alpha, \beta : A \rightarrow A$, such that $(A, +, 0, \alpha)$ is a regular abelian Hom-group, the map β is an homomorphism of abelian Hom-group, and the following properties are satisfied:

Hom-Additive in left coordinate: $[x + y, \alpha(z)] = [x, z] + [y, z]$,

Hom-Additive in right coordinate: $[\alpha(z), x + y] = [z, x] + [z, y]$,

Skew symmetry: $[x, y] + [y, x] = 0$,

Alternation: $[x, x] = 0$,

Hom-Jacobi Identity:

$$\begin{aligned} & \left([\alpha \circ \beta(x), [y, z]] + [\alpha \circ \beta(y), [z, x]] \right) + \alpha([\alpha \circ \beta(z), [x, y]]) = \alpha([\alpha \circ \beta(x), [y, z]]) \\ & \quad + \left([\alpha \circ \beta(y), [z, x]] + [\alpha \circ \beta(z), [x, y]] \right) \\ & = 0. \end{aligned} \tag{4.1}$$

A Hom-Lie ring $(A, +, [,], 0, \alpha, \beta)$ is called a regular Hom-Lie ring if β is an invertible map.

Remark 4.2. A Hom-Lie ring $(A, +, [,], 0, \alpha, \beta)$ with $\alpha = \beta = id$ is a Lie-ring.

Proposition 4.3. Let $(A, +, \cdot, 0, \alpha, \beta)$ be a Hom-ring such that the map α is invertible. Then $(A, +, [,], 0, \alpha, \beta)$ is a Hom-Lie ring, where

$$[x, y] = xy - yx, \quad \forall x, y \in A.$$

Proof. Let x, y and z in A . One can shows that

$$[x + y, \alpha(z)] = (x + y)\alpha(z) - \alpha(z)(x + y) = (xz + yz) - (zx + zy) = (xz - zx) + (yz - zy) = [x, z] + [y, z].$$

Also, we get

$$[\alpha(z), x + y] = \alpha(z)(x + y) - (x + y)\alpha(z) = [z, x] + [z, y].$$

The proof of skew symmetry and alternation conditions is obvious. Moreover, we have the Hom-Jacobi Identity

as follows:

$$\begin{aligned}
\alpha([\alpha \circ \beta(z), [x, y]]) &= \alpha([\alpha \circ \beta(z), xy - yx]) \\
&= \alpha([\beta(z), xy] - [\beta(z), yx]) \\
&= \alpha(\beta(z)(xy) - (xy)\beta(z)) - \alpha(\beta(z)(yx) - (yx)\beta(z)) \\
&= [(\beta(z)(xy) - (xy)\beta(z)) + 0] - [(\beta(z)(yx) - (yx)\beta(z)) + 0] \\
&= [(\beta(z)(xy) - (xy)\beta(z)) + ((yz)\beta(x) - (yz)\beta(x))] \\
&\quad - [(\beta(z)(yx) - (yx)\beta(z)) + (\beta(x)(zy) - \beta(x)(zy))] \\
&= [((yz)\beta(x) - \beta(x)(yz)) + ((zx)\beta(y) - \beta(y)(zx))] \\
&\quad - [((zy)\beta(x) - \beta(x)(zy)) + ((xz)\beta(y) - \beta(y)(xz))] \\
&= ([yz, \beta(x)] + [zx, \beta(y)]) - ([zy, \beta(x)] + [xz, \beta(y)]) \\
&= ([yz, \beta(x)] - [zy, \beta(x)]) + ([zx, \beta(y)] - [xz, \beta(y)]) \\
&= [[y, z], \alpha \circ \beta(x)] + [[z, x], \alpha \circ \beta(y)].
\end{aligned}$$

Proposition 4.4. Let $(A, +, [,]_L)$ be a Lie ring and α, β two homomorphism of Lie rings such that α is invertible and $\alpha \circ \beta = \beta \circ \alpha$. Then $(A, \hat{+}, [,]_{HL}, 0, \alpha, \beta)$ is a Hom-Lie ring, where the addition $\hat{+}$ and the bracket $[,]_{HL}$ are given by:

- $x\hat{+}y = \alpha(x + y)$,
- $[x, y]_{HL} = [\alpha \circ \beta(x), \alpha \circ \beta(y)]_L$, for all $x, y \in A$.

Proof. We are interested in showing the Hom-Additive in left coordinate condition and the Hom-Jacobi identity. Let x, y and z in A . Then we have

1. $[x\hat{+}y, \alpha(z)]_{HL} = [\beta \circ \alpha^2(x+y), \beta \circ \alpha^2(z)]_L = \alpha([\alpha \circ \beta(x), \alpha \circ \beta(z)]_L + [\alpha \circ \beta(y), \alpha \circ \beta(z)]_L) = [x, z]_{HL} \hat{+} [y, z]_{HL}$
- 2.

$$\begin{aligned}
&([\alpha \circ \beta(x), [y, z]_{HL}]_{HL} \hat{+} [\alpha \circ \beta(y), [z, x]_{HL}]_{HL}) \hat{+} \alpha([\alpha \circ \beta(z), [x, y]_{HL}]_{HL}) \\
&= \alpha^2([\alpha^2 \circ \beta^2(x), \alpha^2 \circ \beta^2[y, z]_L]_L + [\alpha^2 \circ \beta^2(y), \alpha^2 \circ \beta^2[z, x]_L]_L) \\
&\quad + \alpha^2([\alpha^2 \circ \beta^2(z), \alpha^2 \circ \beta^2[x, y]_L]_L) \\
&= \alpha^4 \circ \beta^2([x, [y, z]_L]_L + [y, [z, x]_L]_L) + \alpha^4 \circ \beta^2([z, [x, y]_L]_L) \\
&= 0.
\end{aligned}$$

In the similar way we can prove the following:

Proposition 4.5. Let $(A, +, \cdot)$ be a ring and $\alpha, \beta : A \rightarrow A$ be two homomorphisms of ring such that α is invertible and $\alpha \circ \beta = \beta \circ \alpha$. Then $(A, \hat{+}, [,], 0, \alpha, \beta)$ is a Hom-Lie ring, where the addition $\hat{+}$ and the bracket $[,]$ are given by:

- $x\hat{+}y = \alpha(x + y)$,
- $[x, y] = \alpha \circ \beta(xy - yx)$, for all $x, y \in A$.

Definition 4.6. A Hom-Jordan ring is a commutative Hom-nonassociative ring $(A, \hat{+}, \cdot, 0, \alpha, \beta)$ that respects the Hom-Jordan identity:

$$\alpha^2(\beta^2(x))(\alpha(y)(xx)) = \alpha((\beta(x)y)\beta(xx)). \quad (4.2)$$

Remark 4.7. Note if $\alpha = \beta = \text{id}$, we have the definition of Jordan ring.

Proposition 4.8. To any Hom-associative ring $(A, +, \cdot, 0, \alpha, \beta)$ such that α is bijective, one may associate a Hom-Jordan ring $(A, +, \circ, 0, \alpha, \beta)$, where the multiplication " \circ " is defined for all $x, y \in A$ by

$$x \circ y = xy + yx.$$

Proof. The commutativity of " \circ " is obvious. Let $x, y \in A$, we will prove $\alpha^2(\beta^2(x)) \circ (\alpha(y) \circ (x \circ x)) = \alpha((\beta(x) \circ y) \circ \beta(x \circ x))$. Using (2.2), (2.3) and (1.13), we find

$$\begin{aligned} \alpha((\beta(x) \circ y) \circ \beta(x \circ x)) &= \left(((\beta(x)y)\beta(xx) + (\beta(x)y)\beta(xx)) + ((y\beta(x))\beta(xx) + (\beta(x)y)\beta(xx))) \right) \\ &\quad + \left((\beta(xx)(\beta(x)y) + \beta(xx)(\beta(x)y)) + (\beta(xx)(y\beta(x)) + \beta(xx)(y\beta(x))) \right). \end{aligned}$$

The Hom-associativity condition (2.1) gives

$$\begin{aligned} (\beta(x)y)\beta(xx) &= \beta^2(x)(y(xx)), \\ (y\beta(x))\beta(xx) &= \beta(y)(\beta(x)(xx)) = \beta(y)((xx)\beta(x)) = (y(xx))\beta^2(x), \\ \beta(xx)(\beta(x)y) &= ((xx)\beta(x))\beta(y) = (\beta(x)(xx))\beta(y) = \beta^2(x)((xx)y), \beta(xx)(y\beta(x)) = ((xx)y)\beta^2(x). \end{aligned}$$

Using these identities and using again (2.2), (2.3) and (1.13), we deduce the proposition.

Proposition 4.9. Let $(A, +, \cdot)$ be a Jordan ring and $\alpha, \beta : A \rightarrow A$ be two ring endomorphisms. Then $(A, +_\alpha, \cdot_\beta, 0, \alpha, \beta)$ is a Hom-Jordan ring, where

$$x +_\alpha y = \alpha(x + y), \quad x \cdot_\beta y = \beta(xy).$$

Proof. The demonstration is left for the reader.

Remark 4.10. From the Remark 2.13, all results of this article will be true for Hom-algebras.

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