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# A RECURSIVE FORMULA FOR THE PRODUCT OF ELEMENT ORDERS OF FINITE ABELIAN GROUPS

SUBHRAJYOTI SAHA

## Abstract

Let  $G$  be a finite group and let  $\psi(G)$  denote the sum of element orders of  $G$ ; later this concept has been used to define  $R(G)$  which is the product of the element orders of  $G$ . Motivated by the recursive formula for  $\psi(G)$ , we consider a finite abelian group  $G$  and obtain a similar formula for  $R(G)$ .

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## 1. Introduction

Let  $G$  be a finite group. For any non-empty subset  $S$  of  $G$ , let  $\psi(S)$  denote the sum of element orders of  $S$ . This has been introduced in [2] and later in [5], the notion  $R(G)$  was introduced which stands for the product of element orders of  $G$ . In the same paper, a formula for computing  $R(G)$  when  $G$  is a finite abelian group was obtained. In [3, 6], an explicit recursive formula for computing  $\psi(G)$  were obtained in case  $G$  is abelian. Motivated by these results, in this paper, we obtain a similar recursive formula for computing  $R(G)$  when  $G$  is a finite abelian group.

Throughout this paper, we let  $\varphi(n)$  denote the Euler totient function of the positive integer  $n$  and let  $p$  denote a prime number. A cyclic group of order  $n$  will be denoted by  $C_n$  whereas  $C_p^{(r)}$  will denote the elementary abelian  $p$ -group of rank  $r$ . We always assume  $G$  to be finite. For a group  $G$  and element  $x \in G$ , the notation  $o(x)$  denotes the order of  $x$ . For any group  $G$ , we take

$$R(G) = \prod_{x \in G} o(x).$$

For a group  $G$ , the notation  $\exp(G)$  denotes the exponent of  $G$  which is the smallest positive integer  $z$  such that  $g^z = 1_G$  for all  $g \in G$  where  $1_G$  is the identity element of  $G$ ; without any ambiguity we will denote this identity element as 1.

## 2. Explicit formulas for finite abelian groups

In this section, we obtain explicit recursive formula for  $R(G)$  where  $G$  is a finite abelian group. We present this in different cases starting from a finite cyclic group.

This format is inspired by [3, Section 2]. We will then consider the direct product of a finite cyclic  $p$ -group and a (not necessarily abelian)  $p$ -group. Finally, we will consider a the most general case of finite abelian groups. The proofs of our results in this section are motivated by the methods used in [3]. We begin with the following important preliminary results.

**THEOREM 2.1.** [5, Proposition 1.1] *Let  $G_1, G_2, \dots, G_k$  be finite groups having co-prime orders and  $G \cong G_1 \times G_2 \times \dots \times G_k$ . Then*

$$R(G) = \prod_{i=1}^k R(G_i)^{n_i}, \quad \text{where } n_i = \prod_{j=1, j \neq i}^k |G_j|, i = 1, 2, \dots, k.$$

**LEMMA 2.2.** [3, Lemma 2.6] *Let  $H \cong C_{p^{r_1}} \times C_{p^{r_2}} \times \dots \times C_{p^{r_n}}$  where  $1 \leq r_1 \leq r_j$  for all  $j$  with  $2 \leq j \leq n$ . Then for any  $i \in \{1, \dots, r_1\}$ , there are  $(p^i)^n - (p^{i-1})^n$  elements of  $H$  of order  $p^i$*

The following lemma, motivated by [3, Lemma 1.1], follows easily from the fact that  $C_n$  has exactly  $\varphi(d)$  elements of order  $d$  for each divisor  $d$  of  $n$ .

**LEMMA 2.3.** *Let  $n$  be any positive integer. Then  $\psi(C_n) = \sum_{d|n} d\varphi(d)$  and  $R(C_n) = \prod_{d|n} d^{\varphi(d)}$ .*

**2.1. Finite Cyclic Groups** Let  $G$  be a cyclic group of order  $n$ . Then we know that  $G \cong C_{m_1} \times \dots \times C_{m_k}$ , where  $m_1, \dots, m_k$  are co-prime to each other and  $n = m_1 \dots m_k$ .

**LEMMA 2.4.** *Let  $G$  be a cyclic group of order  $p^n$  where  $p$  is a prime number and  $n$  is a positive integer, then  $R(G) = p^{\left(\frac{p+n p^{n+2} - (n+1)p^{n+1}}{p(p-1)}\right)}$ .*

**PROOF.** Using Lemma 2.3, we get  $R(G) = \prod_{r=1}^n p^{\varphi(p^r)r} = p^{\sum_{r=1}^n \varphi(p^r)r}$ . So we have  $R(G) = p^z$  where  $z = \sum_{r=1}^n r(p^r - p^{r-1}) = (1 - \frac{1}{p}) \sum_{r=1}^n r p^r$ . Now the sum  $\sum_{r=1}^n r p^r$  is an arithmetic-geometric series with common difference 1 and common ration  $p$ . Thus  $\sum_{r=1}^n r p^r = \frac{p+n p^{n+2} - (n+1)p^{n+1}}{(p-1)^2}$ . This shows that  $z = \frac{p+n p^{n+2} - (n+1)p^{n+1}}{p(p-1)}$ . So we get  $R(G) = p^{\left(\frac{p+n p^{n+2} - (n+1)p^{n+1}}{p(p-1)}\right)}$ .  $\square$

**LEMMA 2.5.** *Let  $G$  be a cyclic group of order  $s = p_1^{r_1} \dots p_k^{r_k}$  where the  $p_i$  are distinct primes with  $r_i \geq 1$  for  $i = 1, \dots, k$ . Then*

$$R(G) = \prod_{i=1}^k p_i^{\left(\frac{p_i+r_i p_i^{r_i+2} - (r_i+1)p_i^{r_i+1}}{p_i(p_i-1)}\right) n_i}, \quad n_i = p_i^{-r_i} \prod_{j=1}^k p_j^{r_j}.$$

**PROOF.** We know that  $G \cong C_{p_1^{r_1}} \times \dots \times C_{p_k^{r_k}}$  and  $p_i$  are all distinct primes. Hence we can apply Theorem 2.1. Thus by Lemma 2.4, we arrive at the required result.  $\square$

**2.2. Direct product of a cyclic  $p$ -Group and a  $p$ -Group** In this section, we obtain a recursive formula for  $R(G)$  where  $G$  is a direct product of a finite cyclic  $p$ -group and any  $p$ -group. We begin with the following lemma from [3, Lemma 2.3].

**LEMMA 2.6.** *Let  $G = H \times K$  where  $H$  and  $K$  are  $p$ -groups. Then for any  $(x_1, x_2) \in G$ ,  $o((x_1, x_2)) = \max \{o(x_1), o(x_2)\}$*

We now prove the following using techniques motivated by [3].

**PROPOSITION 2.7.** *Let  $G = C_{p^r} \times H$  where  $r \geq 1$ , and  $H$  is a  $p$ -group with  $\exp(H) \geq p^r$ . Let  $N_j$  be the number of elements in  $H$  that have order  $p^j$ . Then*

$$R(G) = \begin{cases} p^{p-1} R(H)^{p^r} \prod_{i=2}^r \left( \prod_{j=1}^{i-1} (p^{i-j})^{N_j} \right)^{p^i - p^{i-1}}, & \text{if } r > 1 \\ p^{p-1} R(H)^p, & \text{if } r = 1 \end{cases}$$

**PROOF.** Note that  $G$  is a finite group whose elements are of the form  $(x, y)$  where  $x \in C_{p^r}$  and  $y \in H$ . We now partition  $G$  based on the order of the elements in the first component. In particular we have,  $G = \bigcup_{k=0}^r F_k$  where  $F_k = \{(x_1, x_2) \in G \mid o(x_1) = p^k\}$ . Since the  $F_i \cap F_j = \emptyset$  for  $i \neq j$ , we have  $R(G) = \prod_{k=0}^r R(F_k)$ . Now let  $x_1 \in C_{p^r}$  with  $o(x_1) = p^i$  for some  $i$  with  $0 \leq i \leq r$ . For each such  $x_1$ , define  $F_{i,x_1} = \{(x_1, x_2) \mid x_2 \in H\}$ . Then we have  $F_i = \bigcup_{x_1 \in C_{p^r}, o(x_1)=p^i} F_{i,x_1}$  and  $R(F_i) = \prod_{x_1 \in C_{p^r}, o(x_1)=p^i} R(F_{i,x_1})$  for  $i = 0, 1, \dots, r$ . There are  $(p^i - p^{i-1})$  elements of order  $p^i$  in  $C_{p^r}$ , see Lemma 2.2. As a result,  $R(F_i) = R(F_{i,x_1})^{(p^i - p^{i-1})}$ .

Taking  $i = 0$ , we have  $F_0 = F_{0,1}$  and thus,  $R(F_0) = R(H)$ . For  $i = 1$ , each element  $(x_1, x_2)$  in  $F_{1,x_1}$  has order same as  $o(x_2)$  except for  $(x_1, 1)$  which has order  $p$ . Thus  $R(F_{1,x_1}) = pR(H)$ . So  $R(F_1) = (pR(H))^{p-1}$ .

If  $r = 1$  then  $R(G) = R(F_0)R(F_1) = p^{p-1}R(H)^p$ .

Now consider  $r > 1$  and let  $i \in \{2, \dots, r\}$ . For  $(x_1, x_2) \in F_{i,x_1}$  with  $o(x_2) = p^j$ , we have  $o((x_1, x_2)) = p^j$  if  $j < i$  and  $o((x_1, x_2)) = p^i$  if  $j \geq i$ . If  $r' = \exp(H)$  then

$$\begin{aligned} R(F_{i,x_1}) &= \prod_{j=0}^{i-1} (p^i)^{N_j} \prod_{j=i}^{r'} (p^j)^{N_j} \\ &= p^i \prod_{j=1}^{i-1} (p^i)^{N_j} \prod_{j=i}^{r'} (p^j)^{N_j} \\ &= p^i \prod_{j=1}^{r'} (p^j)^{N_j} \prod_{j=1}^{i-1} (p^{i-j})^{N_j} \\ &= p^i R(H) \prod_{j=1}^{i-1} (p^{i-j})^{N_j} \end{aligned}$$

Thus  $\prod_{i=2}^r R(F_i) = R(H)^{(p^r - p)} \prod_{i=2}^r \left( \prod_{j=1}^{i-1} (p^{i-j})^{N_j} \right)^{p^i - p^{i-1}}$  and finally the result follows from a direct calculation  $R(G) = R(F_0)R_1(F_1) \prod_{i=2}^r R_i(F_i)$ .  $\square$

**2.3. Finite abelian groups** We can now state how to compute  $R(G)$  for any finite abelian group  $G$ . In view of Theorem 2.1, the following is a direct application of Proposition 2.7 and Lemma 2.2.

**THEOREM 2.8.** *Let  $G$  be a finite abelian group with  $G \cong H_1 \times \cdots \times H_k$  where each  $H_i$  is an abelian  $p_i$ -group and  $p_i$  are distinct primes for  $i = 1, \dots, k$ . Then*

$$R(G) = R(H_1) \dots R(H_k)$$

where  $R(H_i)$  for  $i = 1, \dots, k$  are computed as follows:

a) If  $H_i \cong C_{p_i^n}$  then

$$R(H_i) = p_i^{\left( \frac{p_i + n p_i^{n+2} - (n+1) p_i^{n+1}}{p_i(p_i-1)} \right)}.$$

b) If  $H_i \cong C_{p_i^{r_1}} \times C_{p_i^{r_2}} \times \cdots \times C_{p_i^{r_n}}$ , where  $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ , and  $r_1 + \dots + r_n = r$  then  $R(H_i)$  can be determined recursively as follows

i) If  $r_1 > 1$  then

$$R(H_i) = p_i^{p_i-1} R(C_{p_i^{r_2}} \times \cdots \times C_{p_i^{r_n}})^{p_i^{r_1}} \prod_{z=2}^{r_1} \left( \prod_{j=1}^{z-1} (p_i^{z-j})^{N_j} \right)^{p_i^z - p_i^{z-1}}$$

$$\text{where } N_j = \left( (p_i^j)^{n-1} - (p_i^{j-1})^{n-1} \right)$$

ii) If  $r_1 = 1$  then

$$R(H_i) = p_i^{p_i-1} R(C_{p_i^{r_2}} \times \cdots \times C_{p_i^{r_n}})^{p_i}.$$

### 3. Some Examples

In this final section we compute some examples using Theorem 2.8.

**EXAMPLE 3.1.** *We compute  $R(C_p^{(r)} \times C_{p^n})$  where  $n$  and  $r$  are positive integers. By part a) of Theorem 2.8 we have*

$$R(C_{p^n}) = p^{\left( \frac{p + n p^{n+2} - (n+1) p^{n+1}}{p(p-1)} \right)}.$$

Then by part b) of Theorem 2.8 we have

$$R(C_p \times C_{p^n}) = p^{p-1} p^{\left( \frac{p + n p^{n+2} - (n+1) p^{n+1}}{p(p-1)} \right)} = p^{\left( \frac{(p-1)^2 + (p+n) p^{n+2} - (n+1) p^{n+1}}{(p-1)} \right)}.$$

Similarly

$$R(C_p \times C_p \times C_{p^n}) = p^{\left( \frac{(p-1)^2(1+p) + p(p+n) p^{n+2} - (n+1) p^{n+1}}{(p-1)} \right)}.$$

Thus inductively it is easy to show that

$$R(C_p^{(r)} \times C_{p^n}) = p^{\left( \frac{(p-1)^2(1+p+\dots+p^{r-1}) + p^{r-1}(p+n) p^{n+2} - (n+1) p^{n+1}}{(p-1)} \right)}.$$

Thus we have

$$R(C_p^{(r)} \times C_{p^n}) = p^{\left( \frac{(p-1)(p^r-1) + p^{r-1}(p+n) p^{n+2} - (n+1) p^{n+1}}{(p-1)} \right)}.$$

The next example is an application of Theorem 2.8 in the case where  $r_1 > 1$ .

**EXAMPLE 3.2.** *In this example we compute  $R(C_{p^2} \times C_{p^n})$  where  $n$  is a positive integers. By part a) of Theorem 2.8 we have*

$$R(C_{p^n}) = p^{\left(\frac{p+n p^{n+2} - (n+1)p^{n+1}}{p(p-1)}\right)}.$$

Then by part b) of Theorem 2.8 we have

$$R(C_{p^2} \times C_{p^n}) = p^{p-1} p^{p^2 \left(\frac{p+n p^{n+2} - (n+1)p^{n+1}}{p(p-1)}\right)} p^{p(p-1)(p^{n-1}-1)}.$$

This gives

$$R(C_{p^2} \times C_{p^n}) = p^{\left(\frac{(p-1)^2(p^{n+1}-p) + p(p+n p^{n+2} - (n+1)p^{n+1})}{p-1}\right)}.$$

In the following example we compute  $R(C_p^{(r)} \times C_{p^2} \times C_{p^n})$  where  $n$  and  $r$  are positive integers with  $n \geq 2$ .

**EXAMPLE 3.3.** *In example 3.2 we have computed  $R(C_{p^2} \times C_{p^n})$ . Then by part b) of Theorem 2.8 we have*

$$R(C_p \times C_{p^2} \times C_{p^n}) = p^{p-1} p^p \left(\frac{(p-1)^2(p^{n+1}-p) + p(p+n p^{n+2} - (n+1)p^{n+1})}{p-1}\right).$$

This gives

$$R(C_p \times C_{p^2} \times C_{p^n}) = p^p \left(\frac{(p-1)^2(1+p-p^2+p^{n+1}) + p^2(p+n p^{n+2} - (n+1)p^{n+1})}{p-1}\right).$$

Then inductively one can show that

$$R(C_p^{(r)} \times C_{p^2} \times C_{p^n}) = p^p \left(\frac{(p-1)^2(1+p+\dots+p^r - p^{r+1} + p^{n+r}) + p^{r+1}(p+n p^{n+2} - (n+1)p^{n+1})}{p-1}\right)$$

where  $n$  and  $r$  are integers with  $n \geq 2$ .

Note that formulas obtained in Examples 3.1, 3.2 and 3.3 are straightforward to compute even when  $n$  and  $r$  are large. For any finite abelian  $p$ -group, an explicit formula for computing  $R(G)$  was obtained in [5, Theorem 1.1]. By expanding and simplifying the formula obtained in [5, Theorem 1.1] one can compare the computations for  $R(C_p^{(r)} \times C_{p^2} \times C_{p^n})$ ; the recursive formula used in Example 3.3 provides a more efficient method for obtaining  $R(C_p^{(r)} \times C_{p^2} \times C_{p^n})$ .

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