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Florent Barret, Niccolò Torri

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Abstract

In the present article we consider the Schelling model, an agent-based model describing a segregation dynamics when we have a cohabitation of two social groups. As for several social models, the behavior of the Schelling model was analyzed along several directions, notably by exploiting theoretical physics tools and computer simulations. This approach led to conjecture a phase diagram in which either different social groups were segregated in two large clusters or they were mixed. As far as the authors know, a rigorous mathematical analysis of some aspect of the model has been made by Holden and Sheffield in [12]. In this article, we describe and analyze a perturbation of the the Schelling model as a particle system model by adding a Glauber and Kawasaki dynamics to the original Schelling dynamics. We prove the existence of an hydrodynamic limit described by a reaction-diffusion equation with a discontinuous non-linear reaction term. The existence and uniqueness of the solution is non trivial and the analysis of the limit PDE is interesting in its own. Based on our results, we conjecture, as in other variations of this model, the existence of a phase diagram in which we have a mixed, a segregated and a metastable segregation phase.

2010 Mathematics Subject Classification: 60K35, 82C22, 82D99.

Keywords: Schelling model, particle systems, hydrodynamics limit, reaction-diffusion equation, Ising model.

1 Introduction

Schelling's model of segregation was introduced by Thomas Schelling in 1971 [21, 22]. The original model is defined on a square grid of N^2 sites (or, more generally, on a regular graph with N sites) where agents (individuals) belonging to two groups are disposed. Each agent located at a given site of the grid compares its group with the group of its neighbors. More precisely, we fix a tolerance threshold $T \in [0, 1]$. We call r_x the fraction of neighbors belonging to the agent's group at site x and we say that the agent is satisfied if $r_x \geqslant T$. If the agent is unsatisfied, then he *moves* on a site that makes him satisfied.

Several variations of this model exists, and these variations depend on several parameters [21]:

(1) the neighborhood (its size, its geometry),

(2) the initial distribution of the agents,

(3) the choice of the satisfaction condition (e.g. the value of the tolerance parameter, or one could introduce a different tolerance for each group),

(4) the local dynamics between agents (swapping between two unsatisfied agents, or between the exterior and the grid...)

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In the original model, some sites are assumed to be empty. Several variants of Schelling's model have been considered in the recent literature in order to study the behavior of the model when the fundamental parameters are modified. We refer to [18] for a complete overview on the subject. Among the different variations, let us mention that there can be more than two groups of agents [12], or/and that the Schelling dynamics can be perturbed: each site has a positive probability to switch regardless of its satisfaction.

The main concern is the behavior of the model for large times, does the model reach a stationary state ? a stationary distribution ? If so what are the features of this equilibrium ?

A common result of the considered variants is the existence of three stationary states separated by two critical thresholds T_f and T_c (a *frozen* state $T < T_f$, a *segregated* state $T_f < T < T_c$ and a *mixed* state $T > T_c$) towards which the system evolves, suggesting an *universal* behavior of the model. When the system is open (agents can change the group) the behavior of the system is expected to be symmetric with respect to $T = 1/2$, see [19].

In the present paper, we approach the model from a physical point of view, by interpreting the agent dynamics as a particle systems in interaction. This approach was adopted by the physical community to study this model, see for instance [3, 8, 11, 19]. In particular we consider the setting where the points of the grid (a discrete torus $\mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d$, $d \geqslant 1$) are fully occupied and unsatisfied agents flip their state if it makes them satisfied (Glauber dynamics). The size of the neighborhood taken into account to compute the fraction r_x, grows at most logarithmically with N.

Moreover we introduce random perturbations, either by flipping a state of an agent at rate β (spontaneous Glauber dynamics) or exchanging the position of two agents at rate αN^2 (accelerated Kawasaki dynamics).

To summarize, we assume the following features:

(1) the neighborhood size is going to infinity with N^d, the number of sites,

(2) the initial distribution of the agents is fixed (deterministic) and converges as N goes to infinity,

(3) we fix the tolerance parameter $T \in [0, 1]$,

(4) we introduce two random perturbations of the Schelling mechanism: regardless of their satisfaction, a site can change type (spontaneous Glauber dynamics), and a site can swap type with a closest-neighbor (spontaneous accelerated Kawasaki dynamics).

Our main result (Theorem 3.1) is to prove by rescaling the space at $\frac{1}{N}$, an hydrodynamic limit described by a reaction-diffusion equation and to give a complete description of the limit PDE that we get. In the case where the size of the neighborhood stays finite in the limit, we obtain a classical reaction-diffusion equation, this is the case where the size of the interaction term stays finite and thus microscopic. However, when the size of the neighborhood goes to infinity we get a non-linearity (the reaction term) which is discontinuous at two points. In this case, the interaction of the Schelling dynamics takes into account more and more agents but in the limit, the reaction term is still purely local. In this "mesoscopic" limit, the existence and uniqueness of the solution of the reaction-diffusion equation with discontinuities is one of the major technical points of the paper. More precisely, the system does not have a unique solution for some class of initial condition, and some values of β, the parameter tuning the spontaneous Glauber dynamics. Finally, we conjecture the existence of a rich phase diagram in which beyond a mixed and a segregated phase there is a transition in between. In this phase, the system has the potential to show a metastable segregation: the mixing phase should be the most stable one but symmetrical stable segregation phases also exist. The critical points depend on the parameters β and T but not on α, see Figure 1. A rigorous proof of the phase diagram requires a delicate analysis of the local dynamics that goes beyond the techniques used in the present paper. We reserve this for future work.

Let us stress that in [12] the authors prove also a convergence of a discrete model of Schelling dynamics to the solution of a reaction equation (without diffusion) baptized a continuous Schelling dynamics. We point out that the model is quite different since, in their work, the authors consider a macroscopic neighborhood (which gives at the limit, an integro-differential equation), do not assume any spontanous random perturbation (either Glauber or Kawasaki) and consider a model with $M \geqslant 2$ groups. Also, the authors consider a fixed tolerance parameter of $T = \frac{1}{2}$ and assume that the initial configuration is given by random independent uniform variables. The proof of the convergence is based on a coupling between the discrete and the continuous Schelling dynamics.

Our method of proof is based on the technique of the relative entropy method in the framework developed by Jara and Mezenes in [14], [13], and also used by Funaki and Tsunoda in [7] for a finite number of particle in the interaction. However, in our setting, we need to improve their bounds to cover the case where the number of particles in the interaction is going to infinity. More precisely, in order to use the relative entropy method, a central step is the control of $\partial_t \mathcal{H}_N(t)$, the derivative in time of the relative entropy between the law of the process $\mu_N(t)$ and a discrete measure which approximates the density solution of the reaction-diffusion equation, see Proposition 4.3 and (4.13). Since the number of the particles in the interaction grows with N, the size of the system, we need to retrace the bounds obtained in [14], [13] and [7] by taking into account the size of the interactions. This is done in Theorem 4.5. With our bound (4.13), we get that as soon as the diameter of the interaction grows at most as $\delta(\log(N))^{1/d}$ (see Assumption 1), the relative entropy is $O(N^{d-\varepsilon})$ for some $\varepsilon > 0$ (see Equation (4.14)) which entails that the empirical measure is close in probability to a deterministic discrete process defined by Equation (4.10).

To complete the proof of the main result (Theorem 3.1), we also prove that this deterministic discrete process, defined by Equation (4.10), converges to a solution of a limiting reaction-diffusion equation (Equation (3.3)). This is done in two steps: we first prove that the limiting PDE has a solution (in Proposition 6.3), and for some class of initial conditions, this solution is locally unique (in Proposition 6.4). The existence result is done via an approximating sequence of smooth non-linearities which are natural in our framework (defined by Equation (6.13)). Note that we do not use the deterministic process defined by Equation (4.10) which is discrete in space. The second step is therefore to prove that the deterministic process, defined by Equation (4.10), has accumulation points in a uniform norm on compact set which are all solutions of Equation (3.3) (this is Theorem 7.1). If we have uniqueness for solutions of Equation (3.3), we have the main result.

We establish local uniqueness for (3.3) only for a class of initial conditions (in Proposition 6.4), we use and adapt arguments of Gianni [10] and Deguchi [4] (which proves existence and uniqueness with only one discontinuity). Note that for some initial conditions, (3.3) does not have a unique solution, see Remark 6.2 for a concrete example. It would be therefore quite interesting to understand if for such initial conditions, the empirical measure process converges in some sense. We also reserve this for future work.

Note also that, still in [12], the continuous Schelling dynamics does not have a unique solution for all initial conditions. However, starting from a random Gaussian field, the authors prove the solution exists and is a.s unique.

A detailed plan of the method of proof and the article is given at the end of Section 3 containing the main result and assumptions. In the following Section, we define the model.

Acknowledgments

The authors thank *Antoine Lucquiaud* and *Alessandra Faggionato* for the fruitful discussion on the model and on the article. The authors also thank *Oriane Blondel* for pointing out reference [12]. This research has been conducted within the FP2M federation (CNRS FR 2036) and as part of the project Labex MME-DII (ANR11-LBX-0023-01).

2 The model

2.1 Configurations

For $N \in \mathbb{N} = \{1, 2, 3, \dots\}$ we let $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$ be the discrete torus and let $\Omega_N = \{0,1\}^{\mathbb{T}_N^d}$ be the space of all possible configurations. We call $\eta \in \Omega_N$ a configuration and $i \in \mathbb{T}_N^d$ a site. Let $\mathcal{V}_N \subset \mathbb{T}_N^d \backslash \{0\}$ be a bounded set. We say that two sites $i, j \in \mathbb{T}_N^d$ are neighbors if $i - j \in \mathcal{V}_N$. We denote $K_N = |\mathcal{V}_N|$ its cardinality. For a configuration $\eta \in \Omega_N$ and a site $i \in \mathbb{T}_N^d$ we let

$$r_i(\eta) = \frac{1}{K_N} \sum_{j \in \mathcal{V}_N} \mathbb{1}_{\{\eta_i = \eta_{i+j}\}} \qquad \text{and} \qquad \rho_i(\eta) = \frac{1}{K_N} \sum_{j \in \mathcal{V}_N} \eta_{i+j}. \tag{2.1}$$

The quantity $\rho_i(\eta)$ is the mean field of η on the neighborhood $\mathcal{V}_N + i$. Let us observe that $\rho_i(\eta)$ is independent of η_i. We note that

$$r_i(\eta) = \rho_i(\eta)\mathbb{1}_{\{\eta_i=1\}} + (1 - \rho_i(\eta))\mathbb{1}_{\{\eta_i=0\}}. \tag{2.2}$$

For a given configuration, we now introduce the definition of *stable, unstable* and *potentially stable site.*

Definition 2.1. *For a given site i, let us denote η^i the configuration where we change η_i to $1 - \eta_i$.*
Let $T \in [0,1]$. If $r_i(\eta) < T$, the site i is said unstable for η, otherwise if $r_i(\eta) \geqslant T$ the site is said stable for η. An unstable site i for η which is stable for η^i is said potentially stable.

Note that $r_i(\eta^i) = 1 - r_i(\eta)$. Thus

1. a site i is potentially stable if and only if $r_i(\eta) < T$ and $r_i(\eta) \leqslant 1 - T$. In particular if $T \leqslant \frac{1}{2}$ an unstable site for η is automatically potentially stable.
2. if $T > \frac{1}{2}$ and $1 - T < r_i(\eta) < T$, we have $r_i(\eta^i) < T$ and the site i is unstable for η and η^i.

2.2 Infinitesimal generator, construction of the process

Fix $\alpha > 0$ and $\beta > 0$. Let us consider the following dynamics: starting from a configuration η

1. if a site i is potentially stable, we flip the value at i with rate 1,
2. two nearest-neighbors i and j exchange their values with rate αN^2 (accelerated Kawasaki dynamics),
3. a site i can change its value at rate $\beta > 0$ (spontaneous Glauber dynamics).

This dynamics defines an infinitesimal generator $\mathcal{L}_N$ defined for F cylinder function on Ω_N by

$$\mathcal{L}_N F(\eta) = \sum_{i \in \mathbb{T}_N^d} (F(\eta^i) - F(\eta))(\mathbb{1}_{\{r_i(\eta)<T, r_i(\eta) \leqslant 1-T\}} + \beta) + \alpha N^2 \sum_{\substack{i,j \in \mathbb{T}_N^d \\ |i-j|=1,}} (F(\eta^{ij}) - F(\eta)), \tag{2.3}$$

where η^{ij} is the configuration where the values at site i and j have been exchanged.
The following proposition states that the process is well defined, since the state space is finite.

Proposition 2.1. *Given an initial configuration η_0, $\mathcal{L}_N$ is the infinitesimal generator of a Feller process, denoted $(\eta^N(t))_{t \geqslant 0}$.*

We let μ_t^N be the distribution of $\eta^N(t)$.

Remark 2.1. *In this article we focus on the compact setting (torus) because a non compact framework, as $\mathbb{R}^d$, presents technical problems for the convergence of the process, nevertheless the discrete model can be well defined on $\mathbb{Z}^d$ (cf.* (2.3) *and Proposition 2.1).*

3 Main results

We let $u_0^N(i) := \mathbb{E}_{\mu_N}\big[\eta_i^N(0)\big]$, $i \in \mathbb{T}_N^d$ be the initial distribution of our process, that is, $\eta_i^N(0)$ is distributed as a Bernoulli of parameter $u_0^N(i)$.

For a vector v, $|v|$ denotes its euclidean norm and $|v|_\infty$ its uniform norm.

Assumption 1. *Assumptions on $\mathcal{V}_N$.*

1. *Let $\ell_{\mathcal{V}}$ be the diameter of $\mathcal{V}$. Then, $\ell_{\mathcal{V}} \leqslant \delta\big(\log N\big)^{\frac{1}{d}}$ for some $\delta > 0$.*
2. *If $d = 1$, suppose that $\mathcal{V}_N \subset \mathbb{Z}\backslash\mathbb{N}$.*

Assumption 2. *Assumptions on u_0^N.*

1. *There exists $\varepsilon > 0$ such that $\varepsilon \leqslant u_0^N(i) \leqslant 1-\varepsilon$ uniformly on $i \in \mathbb{T}_N^d$ and $N \in \mathbb{N}$.*
2. *There exists $C_0 > 0$ independent of N such that $|\nabla u_0^N(i)|_\infty \leqslant \frac{C_0}{N}$, where $\nabla u_0^N(i) = (u_0^N(i + e_k) - u_0^N(i))_{k=1}^d$, with $e_k \in \mathbb{Z}^d$ the unit vector of direction k.*
3. *Let $\upsilon_{u_0^N}^N = \bigotimes_{i\in\mathbb{T}_N^d} \mathrm{B}\big(u_0^N(i)\big)$ be the law of a sequence of independent Bernoulli of parameter $u_0^N(i)$. Suppose that $\mathcal{H}(\mu_0^N \,|\, \upsilon_{u^N(0)}^N) = O(N^{d-\varepsilon_0})$ for some $\varepsilon_0 > 0$ small, where $\mathcal{H}(\mu\,|\,\upsilon)$ is the relative entropy of $\mu \ll \upsilon$,*
$$\mathcal{H}(\mu\,|\,\upsilon) := \int \frac{\mathrm{d}\mu}{\mathrm{d}\upsilon} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\upsilon}\right)\mathrm{d}\upsilon \tag{3.1}$$
4. *Let $\widetilde{u}_0^N(x)$ be the linear interpolation on $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, the d dimensional torus, of $u_0^N(i)$ such that $\widetilde{u}_0^N(i/N) = u_0^N(i)$. Then, there exists $u_0 \in \mathcal{C}(\mathbb{T}^d)$ such that $\widetilde{u}_0^N$ converges to u_0 in $\mathcal{C}(\mathbb{T}^d)$.*

Remark 3.1. *The assumption 1(2.) is only technical and it could be removed by considering the dimension $d = 1$ separately from the rest of the dimensions, cf. Remark 4.3.*

Define
$$\pi_t^N = \pi_t^N(\eta, \mathrm{d}v) = \frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \eta_i(t)\delta_{\frac{i}{N}}(\mathrm{d}v) \tag{3.2}$$
the empirical measure associated to the Markov process η where the space is rescaled by $\frac{1}{N}$, which is a positive measure on $\mathbb{T}^d$.

We now state our main result, which concerns the convergence in probability of the empirical measure.

Theorem 3.1. *Under Assumptions 1 and 2, if u_0 (the limit of the initial condition, according to Assumption 2-4) is such that*

1. *$u_0 \in \mathcal{C}^1(\mathbb{T}^d)$, with ∇u_0 Lipschitz,*
2. *$\nabla u_0(p) \neq 0$ for $p = \min(T, 1-T)$ and $p = 1 - \min(T, 1-T)$,*

then the reaction-diffusion equation
$$\begin{cases} \partial_t u(t,x) = 2\alpha\Delta u(t,x) + \beta(1-2u(t,x)) + g_\infty(u(t,x)), \\ u(0,x) = u_0(x), \end{cases} \tag{3.3}$$
has an unique density solution $u = u(t,x)$ with $(t,x) \in [0,\tau] \times \mathbb{T}^d$ for some $\tau > 0$, where $\mathbb{T}^d$ is the d-dimension torus and $g_\infty(p) := (1-p)\mathbb{1}_{\{1-p<\min(T,1-T)\}} - p\mathbb{1}_{\{p\leqslant\min(T,1-T)\}}$. Moreover, for every test function $\varphi : \mathbb{T}^d \to \mathbb{R}$ and for every $\varepsilon > 0$,
$$\lim_{N\to+\infty} \mu^N\left(\Big|\langle\pi^N, \varphi\rangle - \langle u, \varphi\rangle\Big| > \varepsilon\right) = 0\,, \qquad \forall\, t \in [0,\tau]\,, \tag{3.4}$$
where $\langle\pi^N, \varphi\rangle$ and $\langle u, \varphi\rangle$ denote the integral of φ with respect to the measure π^N or $u(x)\mathrm{d}x$ respectively.

Remark 3.2. *If the solution of* (3.3) *is not unique, which is not technical difficulty but a real possibility for some initial conditions (see Remark 6.2 for a concrete example), then any accumulation point of the sequence of empirical measure is a solution of* (3.3).

3.1 Organisation of the paper

To prove Theorem 3.1 we first prove that the empirical measure is close to a discrete measure u^N which is a solution of a discrete analogous of (3.3), this is Theorem 4.1. Its proof is based on a entropy method approach in which the relative entropy between μ^N and $v^N_{u^N}$, see Theorem 4.2. Even if this technique is quite standard in the particle system theory, some new technical estimations arising from the geometry of the system are needed, this is Theorem 4.5. In Section 5 we discuss some central technical estimations about u^N, in order to describe the behaviour of the discrete model. Then, in Section 6 we discuss the existence and uniqueness of (3.3) and in Section 7 we show the convergence of the u^N toward the density u by completing the proof of Theorem 3.1. We stress that the proof of the existence and uniqueness is not standard and the analysis of this PDE is interesting in its own.

In Section 3.2 we state our conjecture on the phase diagram of the model.

3.2 Conjecture on the phase diagram

In this Section, we discuss the phase diagram that describes the mixed and segregated phases. We start by setting the Equation (3.3) in a more convenient form. Set $p_0(T) := \min(T, 1-T) \in [0, \frac{1}{2}]$. For $p \in [0,1]$ we define

$$\gamma_{\infty,\beta}(p) = \begin{cases} \beta\left(p-\dfrac{1}{2}\right)^2 + \dfrac{1}{2}\left(p^2 - p_0(T)^2\right) & \text{for } 0 \leqslant p < p_0(T), \\ \beta\left(p-\dfrac{1}{2}\right)^2 & \text{for } p_0(T) \leqslant p < 1-p_0(T), \\ \beta\left(p-\dfrac{1}{2}\right)^2 + \dfrac{1}{2}\left((1-p)^2 - p_0(T)^2\right) & \text{for } 1-p_0(T) \leqslant p \leqslant 1. \end{cases} \tag{3.5}$$

Our conjecture is based on the analysis of $\gamma_{\infty,\beta}$ and it is represented in Figure 1. We observe that $\gamma_{\infty,\beta}$ is continuous and satisfies $\gamma_{\infty,\beta}(p) = \gamma_{\infty,\beta}(1-p)$. For $p \neq p_0(T), 1-p_0(T)$, we have that $\gamma'_{\infty,\beta}(p) = -\beta(1-2p) - g_\infty(p)$ and (3.3) can be written as

$$\partial_t u(t,x) = 2\alpha\Delta u(t,x) - \gamma'_{\infty,\beta}(u(t,x)). \tag{3.6}$$

To discuss the phase transition we can look at the structure of $\gamma_{\infty,\beta}(p)$. The function $p \mapsto \beta\left(p-\frac{1}{2}\right)^2 + \frac{1}{2}\left(p^2 - p_0(T)^2\right)$ has a unique minimum at $p = p^\ell := \frac{\beta}{1+2\beta}$. Therefore, if $0 \leqslant p^\ell < p_0(T) < \frac{1}{2}$, the function $\gamma_{\infty,\beta}$ has three regular minima: $p^c := \frac{1}{2}$, p^ℓ and $p^r := 1-p^\ell$. Note that

$$\gamma_{\infty,\beta}(p^c) = 0 \quad \text{and} \quad \gamma_{\infty,\beta}(p^r) = \gamma_{\infty,\beta}(p^\ell) = \frac{\beta}{4(1+2\beta)} - \frac{p_0(T)^2}{2}.$$

Then we get that if $p_0(T) < p^m := \sqrt{\frac{\beta}{2(1+2\beta)}}$, we have $\gamma_{\infty,\beta}(p^c) < \gamma_{\infty,\beta}(p^\ell)$ and if $p^m < p_0(T)$, we have that $\gamma_{\infty,\beta}(p^c) > \gamma_\infty(p^\ell)$. If $p^\ell > p_0(T)$, p^c is the only minimum.

The two thresholds for $p_0(T)$ are then p^ℓ and p^m, see Figure 1. Since $p^\ell < p^m$, we have the following picture: as T is close to 0 and below p^ℓ, we have a unique minimum of $\gamma_{\infty,\beta}$, so that typical configurations are close to $p = 1/2$ which is of lowest energy $\gamma_{\infty,\beta}$. It means that, at equilibrium, we expect a configuration balanced between 0 and 1 and we do not have segregation. Then, as T goes above the threshold p^ℓ but stays below p^m, other minima at $p = p^\ell$ and $p = p^r$ appear, and these

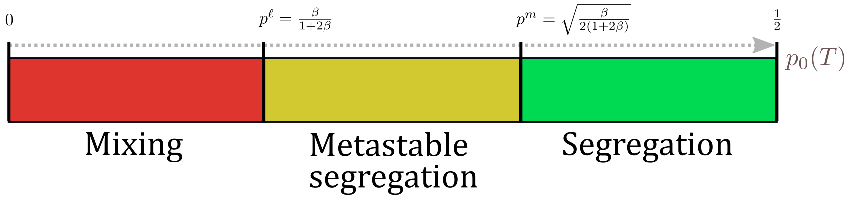

Figure 1: Representation of the different phases of the system as function of the parameter $p_0(T) \in [0, \frac{1}{2}]$. When T is close to 0 or 1 ($p_0(T) \in (0, p^\ell)$) we do not have segregation (red parts) and typical configurations are provided by a mixing of 0 and 1. If T is close to 1/2 ($p_0(T) \in (p^m, \frac{1}{2})$) we have segregation (green parts): a very large parts of the configuration are composed of 0 or 1. We have also intermediate values of T ($p_0(T) \in (p^\ell, p^m)$) for which the segregation is metastable (yellow parts).

two configurations are metastable since their energy is higher, so we can have segregation for a small proportion of the time. The next threshold is p^m, at which the two metastable configurations become stable and $p = 1/2$ is the metastable one so that we expect stable segregation. For T above $\frac{1}{2}$ the picture is symmetric.

Remark 3.3. *The hydrodynamic limit* (3.3) *has, at least, two different formulations as a gradient flow:*

1. *in the classical $L^2(\mathbb{T}^d)$ setting, with the potential $\mathcal{F}$ defined for $u : \mathbb{T}^d \to [0, 1]$*

$$\mathcal{F}(u) = \int \alpha \|\nabla u\|^2 + \gamma_{\infty,\beta}(u), \tag{3.7}$$

Equation (6.2) *can be written as $\partial_t u = -\delta\mathcal{F}(u)$ where $\delta\mathcal{F}$ denotes the Fréchet derivative of $\mathcal{F}$.*

2. *in a Wasserstein-like setting defined in [17] with the entropy potential $\mathcal{H}$, for $u : \mathbb{T}^d \to]0, 1]$, and $\xi : \mathbb{T}^d \to \mathbb{R}$ (seen as an element in the tangent bundle)*

$$\mathcal{H}(u) = \frac{1}{2} \int (2u \log(2u) - 2u + 1) \geqslant 0, \qquad \mathcal{K}(u)\xi = -2\alpha \nabla \cdot (u \nabla \xi) + \frac{\gamma'_{\infty,\beta}(u)}{\log(2u)} \xi, \tag{3.8}$$

Equation (3.3) *can be written as*

$$\partial_t u = -\mathcal{K}(u)(\delta\mathcal{H}).$$

Since $\delta\mathcal{H} = \log(2u)$ and $\mathcal{K}(u)(\delta\mathcal{H}) = -2\alpha\Delta u + \gamma'_{\beta,\infty}(u)$.

Both formulations could be useful to establish a rigorous proof of the phase diagram given in Figure 1. In particular, along a gradient flow the potential is non-increasing, thus for all $t > 0$, along a solution u we have $\mathcal{F}(u(t)) \leqslant \mathcal{F}(u(t = 0))$ and if u converges to a stationary solution v, it must be a stationary point of $\mathcal{F}$ (i.e. $\delta\mathcal{F}(v) = 0$).

For the second formulation, note that

$$\frac{\gamma'_{\infty,\beta}(u)}{\log(2u)} > 0, u \in]0, 1] \Longleftrightarrow p_0(T) < p^\ell = \frac{\beta}{1 + 2\beta} \Longleftrightarrow \beta > \frac{p_0(T)}{1 - 2p_0(T)}. \tag{3.9}$$

We are thus in the mixing phase of the diagram and $\mathcal{K}(u)$ is positive definite in the sense that

$$\int \xi \mathcal{K}(u) \xi = \int 2\alpha u |\nabla \xi|^2 + \frac{\gamma'_{\infty,\beta}(u)}{\log(2u)} \xi^2 \geqslant 0. \tag{3.10}$$

Then, one can prove that along a solution u, we get that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u) = \int \partial_t u \delta \mathcal{H}(u) \leqslant -\int \gamma'_{\infty,\beta}(u)\log(2u) \leqslant -c\mathcal{H}(u) \tag{3.11}$$

where $c = \beta - \frac{p_0(T)}{1-2p_0(T)} > 0$, and we get that $\mathcal{H}(u(t)) \leqslant \mathcal{H}(u(0))e^{-ct}$. Thus $\mathcal{H}(u(t))$ goes to 0 as $t \to \infty$, this entails that u converges to the only stationary point of $\mathcal{H}$ which is the constant $\frac{1}{2}$. Heuristically, it suggests that an exponential relaxation is taking place in the mixing phase of Figure 1.

4 Relative entropy method

Using the *relative entropy* method, in this section we prove that the empirical measure $\pi^N(t)$ is close to $u^N(t) = (u^N(t,i))_{i\in\mathbb{T}_N^d}$ the solution of a suitable discrete PDE.

In order to state the main result of this section, we observe that the generator $\mathcal{L}_N$ in (2.3) can be written as $\mathcal{L}_N = \mathcal{G}_N + 2\alpha N^2 \mathcal{K}_N$ where $\mathcal{G}_N$ is the generator which describes the Glauber-Schelling dynamics and $\mathcal{K}_N$ the generator which describes the Kawasaki dynamics, that is,

$$\mathcal{G}_N F(\eta) = \sum_{i\in\mathbb{T}_N^d} \Big(\mathbb{1}_{\{r_i(\eta)<T, r_i(\eta)\leqslant 1-T\}} + \beta\Big)(F(\eta^i) - F(\eta)), \tag{4.1}$$

$$\mathcal{K}_N F(\eta) = \frac{1}{2} \sum_{\substack{i,j\in\mathbb{T}_N^d \\ |i-j|=1}} (F(\eta^{ij}) - F(\eta)). \tag{4.2}$$

Let us stress that $\mathcal{G}_N$ can be written as follows

$$\mathcal{G}_N F(\eta) = \sum_{i\in\mathbb{T}_N^d} (c_i(\eta) + \beta)\big(F(\eta^i) - F(\eta)\big), \tag{4.3}$$

where $c_i(\eta)$ is a local function which describes the dynamics (Glauber-Schelling dynamics). To be more precise, we write $c_i(\eta) = c_0(\tau_i\eta)$ where c_0 is the flipping rate of a particle at the origin, that is,

$$c_0(\eta) := \mathbb{1}_{\{r_0(\eta)<T, r_0(\eta)\leqslant 1-T\}} \tag{4.4}$$

and $(\tau_i\eta)_j = \eta_{i+j}$, likewise τ_i acts on u. Note that c_0 is a random variable and take the value 0 or 1.

We let

$$\kappa_N = \kappa_N(T) := \min\Big\{\lfloor K_N T\rfloor - 1; \lfloor K_N(1-T)\rfloor\Big\}. \tag{4.5}$$

We observe that $\lim\limits_{n\to+\infty} \frac{\kappa_N}{K_N} = \min(T, 1-T)$.

Since $c_0(\eta) = \mathbb{1}_{\{r_0(\eta)\leqslant \frac{\kappa_N}{K_N}\}} = \mathbb{1}_{\{r_0(\eta)\leqslant \frac{\kappa_N}{K_N}, \eta_0=0\}} + \mathbb{1}_{\{r_0(\eta)\leqslant \frac{\kappa_N}{K_N}, \eta_0=1\}}$, we define

$$c_0^+(\eta) := \mathbb{1}_{\{\rho_0(\eta)\geqslant 1-\frac{\kappa_N}{K_N}\}} \qquad \text{and} \qquad c_0^-(\eta) := \mathbb{1}_{\{\rho_0(\eta)\leqslant \frac{\kappa_N}{K_N}\}}, \tag{4.6}$$

so that $c_0(\eta) = c_0^+(\eta)(1-\eta_0) + c_0^-(\eta)\eta_0$, by (2.2). The functions c^+ and c^- can be viewed as the rate of creation and annihilation of a particle at $i = 0$ respectively.

For any function $u = (u_i)_{i\in\mathbb{T}_N^d}$ we define

$$\upsilon_u(\mathrm{d}\eta) = \upsilon_u^N(\mathrm{d}\eta) := \bigotimes_{i\in\mathbb{T}_N^d} \mathrm{B}\big(u_i\big) \tag{4.7}$$

and we let $c_0^+(u)$ and $c_0^-(u)$ be the expectation of $c_0^+(\eta)$ and $c_0^-(\eta)$ under υ_u, that is,

$$c_0^+(u) := \mathbb{P}_{\upsilon_u}\left(\rho_0(\eta) \geqslant 1 - \frac{\kappa_N}{K_N}\right) \qquad \text{and} \qquad c_0^-(u) := \mathbb{P}_{\upsilon_u}\left(\rho_0(\eta) \leqslant \frac{\kappa_N}{K_N}\right). \tag{4.8}$$

We finally define

$$G(u) := c_0^+(u)(1-u_0) - c_0^-(u)u_0 \quad \text{and} \quad G(i,u) := G(\tau_i u). \tag{4.9}$$

Let $u^N(t) = (u^N(t,i))_{i\in\mathbb{T}_N^d}$ be the solution of

$$\begin{cases} \partial_t u^N(t,i) = 2\alpha N^2 \Delta u^N(t,i) + \beta(1-2u^N(t,i)) + G(i,u^N) \\ u^N(t,i) = u_0^N(i), \end{cases} \tag{4.10}$$

where $\Delta u^N(t,i)$ is the discrete Laplacian on the torus. Note that (4.10) can be interpreted as a discretized version of (3.3).

To prove Theorem 3.1 we first define a discrete approximation of u and we show an equivalent of Theorem 3.1 for u^N defined in (4.10). More precisely, we consider u^N as a measure on $\mathbb{T}^d$, that is,

$$u^N(t,\mathrm{d}v) := \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d} u^N(t,i)\delta_{\frac{i}{N}}(\mathrm{d}v)\,, \tag{4.11}$$

The main result of this section is the following theorem.

Theorem 4.1. *Under Assumptions 1 and 2, for every test function $\varphi : \mathbb{T}^d \to \mathbb{R}$ and for every $\delta > 0$ there exists $\tau > 0$ such that*

$$\lim_{N\to+\infty} \mu^N\left(\left|\langle \pi^N, \varphi\rangle - \langle u^N, \varphi\rangle\right| > \delta\right) = 0\,, \qquad \forall\, t \in [0,\tau]\,.$$

To prove Theorem 4.1, the main ingredient is that the relative entropy of $\upsilon^N_{u^N(t)}$ (cf. (4.7)) with respect to μ_t^N stays small in time, if it is small at $t = 0$.

Theorem 4.2. *Under Assumptions 1 and 2, with $\delta > 0$ sufficiently small, we have that for some $\varepsilon > 0$ small*

$$\mathcal{H}_N(t) := \mathcal{H}(\mu_t^N \,|\, \upsilon^N_{u^N(t)}) = O(N^{d-\varepsilon}), \quad \forall\, t \in [0,\tau] \tag{4.12}$$

with $\tau = \tau(\delta) > 0$.

To prove Theorem 4.2 it is enough to show that

$$\partial_t \mathcal{H}_N(t) \leqslant C\ell_{\mathcal{V}}^d\Big(\mathcal{H}_N(t) + O(N^{d-a})\Big) \tag{4.13}$$

with $a \in (0,1)$ if $d \geqslant 2$ and $a \in (0,\frac{1}{2})$ if $d = 1$, indeed in this case Gronwall's inequality gives

$$\mathcal{H}_N(t) \leqslant \Big(\mathcal{H}_N(0) + tO(N^{d-a})\Big)e^{C\ell_{\mathcal{V}}^d t}. \tag{4.14}$$

Since $e^{C\ell_{\mathcal{V}}^d t} \leqslant N^{C\delta t}$ by Assumption 1, the proof of Theorem 4.2 is complete. We prove (4.13) in Section 4.1

Let us observe that Theorem 4.2 implies Theorem 4.1. Indeed, we recall the entropy inequality stated for a set $\mathcal{A}$ and two measures $\upsilon \ll \mu$, cf. A1.8.2 of [15] or Section 2.2 of [7],

$$\mu(\mathcal{A}) \leqslant \frac{\log 2 + \mathcal{H}(\mu \,|\, \upsilon)}{\log\left(1 + \frac{1}{\upsilon(\mathcal{A})}\right)}. \tag{4.15}$$

For a given test function φ and $\delta > 0$, we let

$$\mathcal{A}^{\delta}_{N,t,\varphi} := \{\eta \in \Omega_N : |\langle \pi^N, \varphi\rangle - \langle u^N, \varphi\rangle| > \delta\}, \tag{4.16}$$

so that the proof follows by Theorem 4.2 and (4.15) if

$$\upsilon^N_{u^N(t)}(\mathcal{A}^{\delta}_{N,t,\varphi}) \leqslant e^{-C_\delta N^d}. \tag{4.17}$$

Since $u^N \in (0,1)$ (cf. Proposition 5.2), the proof of (4.17) is model independent and follows line by line the proof of Proposition 2.2 in [7], we omit the details.

Remark 4.1. *Let us note that c_0 can be expressed as a polynomial on the variables η_j's, as in relation (1.5) of [7]. This remark will be useful in the sequel of the paper. For this purpose, let $A \subset \mathcal{V}_N$, we denote:*

$$c^+_A(\eta) = \prod_{j\in A}(1-\eta_j) \prod_{j\in \bar{A}\cap\mathcal{V}_N} \eta_j \quad \textit{and} \quad c^-_A(\eta) = \prod_{j\in A}\eta_j \prod_{j\in \bar{A}\cap\mathcal{V}_N}(1-\eta_j) = c^+_A(1-\eta) \tag{4.18}$$

where $1-\eta$ is the configuration with $(1-\eta)_i = 1-\eta_i$ since $r_0(\eta) = r_0(1-\eta)$. Note that

$$c^+_A(\eta) = \begin{cases} 1 \text{ if } \eta_j = 0 \text{ for } j \in A \text{ and } \eta_j = 1 \text{ for } j \in \bar{A}\cap\mathcal{V}_N \\ 0 \text{ otherwise.} \end{cases} \tag{4.19}$$

By an abuse of notation, for a function $u = (u_j)_{j\in\mathbb{T}^d_N}$ we let let also $c^+_A(u) = \prod_{j\in A}(1-u_j)\prod_{j\in\bar{A}\cap\mathcal{V}_N} u_j$ and accordingly for $c^-_A(u)$. By (4.6) we have that

$$c^+_0(\eta) = \sum_{k=0}^{\kappa_N} \mathbb{1}_{\{r_0(\eta)=\frac{k}{K_N},\eta_0=0\}} = (1-\eta_0) \sum_{\substack{A\subset\mathcal{V}_N\\ |A|\leqslant\kappa_N}} c^+_A(\eta). \tag{4.20}$$

Accordingly, we have

$$c^-_0(\eta) = \sum_{k=0}^{\kappa_N} \mathbb{1}_{\{r_0(\eta)=\frac{k}{K_N},\eta_0=1\}} = \eta_0 \sum_{\substack{A\subset\mathcal{V}_N\\ |A|\leqslant\kappa_N}} c^-_A(\eta). \tag{4.21}$$

In the rest of this section we prove Theorem 4.2. The strategy that we use follows the one used to prove the analogous result in [7] and [13], but some extra-technicality is required due to the geometry of our problem.

4.1 Proof of Theorem 4.2

To make the notation lighter, for $t \geqslant 0$, $j \in \mathbb{T}^d_N$, $p \in (0,1)$ and u^N the solution of (4.10), we let

$$u_j(t) := u^N(t,j), \qquad \chi(p) := p(1-p), \qquad \omega_j := \frac{\eta_j - u_j}{\chi(u_j)}. \tag{4.22}$$

and, more generally, whenever the context is clear we omit the superscript N, so that $\upsilon^N_{u^N(t)}$ and μ^N_t will be denoted simply by $\upsilon_{u(t)}$ and μ_t respectively.

To compare μ and υ_u we introduce

$$\vartheta_\alpha(\mathrm{d}\eta) := \bigotimes_{i\in\mathbb{T}^d_N} \mathrm{B}(\alpha) \tag{4.23}$$

be a sequence of independent Bernoulli of parameter $\alpha \in (0,1)$ defined on the space of configurations. We define

$$f_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\upsilon_{u(t)}} \qquad \text{and} \qquad \psi_t := \frac{\mathrm{d}\upsilon_{u(t)}}{\mathrm{d}\vartheta_\alpha}. \tag{4.24}$$

We have all the ingredients to state Yau's inequality in our context. The proof is quite standard (cf. proof of Lemma A.1 in [13]), so that it is omitted.

Proposition 4.3. *For any $t \geqslant 0$ we have that*

$$\partial_t \mathcal{H}_N(t) \leqslant -\int \Gamma_N\left(\sqrt{f_t(\eta)}\right) \upsilon_{u(t)}(\mathrm{d}\eta) + \int f_t(\eta)\left[\mathcal{L}_N^{*,\upsilon_{u(t)}}\mathbf{1} - \partial_t \log \psi_t\right] \upsilon_{u(t)}(\mathrm{d}\eta), \tag{4.25}$$

where $\mathcal{L}_N^{*,\upsilon_{u(t)}}$ is the adjoint of $\mathcal{L}_N$ with respect to the measure $\upsilon_{u(t)}$ and $\Gamma_N(h)(\eta) = \mathcal{L}_N h^2(\eta) - 2h(\eta)\mathcal{L}_N h(\eta)$ is the carré du champ operator.

We define the *current* $\mathcal{J}_t = \mathcal{J}_t^N(\eta)$ as

$$\mathcal{J}_t := \mathcal{L}_N^{*,\upsilon_{u(t)}}\mathbf{1} - \partial_t \log \psi_t. \tag{4.26}$$

Our main goal is to estimate the current $\mathcal{J}_t$ to control the right hand side of (4.25) and get (4.13).

4.1.1 The current $\mathcal{J}_t^N$

To control $\mathcal{L}_N^{*,\upsilon_{u(t)}}\mathbf{1}$ we have to compute the adjoint of $\mathcal{K}_N$ and of $\mathcal{G}_N$, cf. (4.1) ans (4.2). We follow the computations done in [7]. By Lemma 2.4 of [7], we get

$$\mathcal{K}_N^{*,\upsilon_{u(t)}}\mathbf{1} = -\frac{1}{2}\sum_{\substack{i,j\in\mathbb{T}_N^d,\\ |i-j|=1}} (u_i - u_j)^2 \omega_j \omega_i + \sum_{i\in\mathbb{T}_N^d} (\Delta u)_i \omega_i \tag{4.27}$$

where $(\Delta u)_i = \sum_{j\in\mathbb{T}_N^d, |i-j|=1}(u_j - u_i)$ is the discrete Laplacian. Since c_0 satisfies the condition (1.5) of [7] (see Remark 4.1), by Lemma 2.5 of [7] we get

$$\begin{aligned}
\mathcal{G}_N^{*,\upsilon_{u(t)}}\mathbf{1} &= \sum_{i\in\mathbb{T}_N^d} \left(c_i^+(\eta)(1-u_i) - c_i^-(\eta)u_i + \beta(1-2u_i)\right)\omega_i \\
&= \sum_{i\in\mathbb{T}_N^d} \left((c_i^+(\eta) - c_i^+(u))(1-u_i) - (c_i^-(\eta) - c_i^-(u))u_i\right)\omega_i \\
&\qquad + \sum_{i\in\mathbb{T}_N^d} \left(c_i^+(u)(1-u_i) - c_i^-(u)u_i + \beta(1-2u_i)\right)\omega_i.
\end{aligned} \tag{4.28}$$

In the second equality we centered the variables $c_i^+(\eta)$ $c_i^-(\eta)$ since under υ_u, $c_i^+(\eta)$ and $c_i^-(\eta)$ are Bernouilli random variables with expectation $c_i^+(u)$ and $c_i^-(u)$ respectively. Finally, by Lemma 2.6 of [7] we get

$$\partial_t \log \psi_t(\eta) = \sum_{i\in\mathbb{T}_N^d} (\partial_t u_i)\omega_i. \tag{4.29}$$

Summarizing, we obtain the following result.

Proposition 4.4. *The current $\mathcal{J}_t(\eta)$ satisfies*

$$\begin{aligned}
\mathcal{J}_t(\eta) = \sum_{i\in\mathbb{T}_N^d} (-\partial_t u_i + 2\alpha N^2(\Delta u)_i + \beta(1-2u_i) + G(i,u))\omega_i \\
- V(u,\eta) + V^+(u,\eta) - V^-(u,\eta),
\end{aligned} \tag{4.30}$$

where G was defined in (4.9) *and*

$$V^+(u,\eta) = \sum_{i\in\mathbb{T}_N^d} \left(c_i^+(\eta) - c_i^+(u)\right)(1-u_i)\omega_i, \tag{4.31}$$

$$V^-(u,\eta) = \sum_{i\in\mathbb{T}_N^d} \left(c_i^-(\eta) - c_i^-(u)\right) u_i\omega_i, \tag{4.32}$$

$$V(u,\eta) = -\alpha N^2 \sum_{\substack{i,j\in\mathbb{T}_N^d,\\ |i-j|=1}} (u_i-u_j)^2\omega_j\omega_i. \tag{4.33}$$

In particular, if u satisfies (4.10) *the current reduces to the second line.*

In the rest of the section we provide estimates of V^+, V^- and V.

4.1.2 Estimates of V^+ and V^-

Let us denote $\{e_1, e_2, \ldots, e_d\}$ the canonical basis of $\mathbb{Z}^d$. For $\varphi : \mathbb{T}_N^d \to \mathbb{R}$ and $i \in \mathbb{T}_N^d$ and $k \in \{1,\ldots,d\}$, let $\nabla_k\varphi(i) = \varphi(i+e_k) - \varphi(i)$. We denote $\|\nabla\varphi\|_\infty = \max_{i,k}|\nabla_k\varphi(i)|$.

We note that

$$V^-(u,\eta) = -V^+(1-u, 1-\eta), \tag{4.34}$$

so that the bound for V^+ can be transferred to V^-, see Remark 4.2. In the following we get an upper-bound for V^+. Denote $\vartheta_i^+ = c_i^+(\eta) - c_i^+(u)$ and $\omega_i^+ = (1-u_i)\omega_i = \frac{\eta_i - u_i}{u_i}$.

Then

$$V^+(u,\eta) = \sum_{i\in\mathbb{T}_N^d} \vartheta_i^+\omega_i^+ \tag{4.35}$$

To bound $V^+(u,\eta)$ we follow the method used by Jara and Menezes [13] and by Funaki and Tsuneda [7]. For this purpose let us observe that the carré du champ referred to the generator $\mathcal{L}_N$, namely, $\Gamma_N(h) = \mathcal{L}_N h^2 - 2h\mathcal{L}_N h$ can be decomposed as

$$\Gamma_N(h) = \Gamma_N^{\mathcal{G}}(h) + 2\alpha N^2\Gamma_N^{\mathcal{K}}(h), \tag{4.36}$$

where $\Gamma_N^{\mathcal{G}}(h)$ and $\Gamma_N^{\mathcal{K}}(h)$ are the carré du champ related to the generator $\mathcal{G}_N$ and $\mathcal{K}_N$ respectively (cf. (4.1) and (4.2)). In particular,

$$\Gamma_N^{\mathcal{K}}(h) = \frac{1}{2}\sum_{\substack{i,j\in\mathbb{T}_N^d:\\ |i-j|=1}} \left(h(\eta^{i,j}) - h(\eta)\right)^2. \tag{4.37}$$

In the next result we provide the control that we need for V^+.

Theorem 4.5. *Under Assumption 1, we have that for any $u : \mathbb{T}_N^d \to [0,1]$ such that*

1. $\varepsilon := 1 - |||u|||_\infty > 0$ *with* $|||u|||_\infty := \min\Big\{\|u\|_\infty, \|1-u\|_\infty\Big\}$,
2. $\|\nabla u(i)\|_\infty \leqslant \frac{C_0}{N}$ *with C_0 independent of N,*

and for any density f with respect to υ_u we have that

$$\int V^+(u,\eta)f(\eta)\upsilon_u(\mathrm{d}\eta) \leqslant \delta N^2 \int \Gamma_N^{\mathcal{K}}\big(\sqrt{f}\big)(\eta)\upsilon_u(\mathrm{d}\eta) + C_1\ell_{\mathcal{V}}^d\mathcal{H}(f\upsilon_u \,|\, \upsilon_u) + C_2N^{d-a}, \tag{4.38}$$

for any $\delta > 0$ and $C_1 = 3^d d(d+1)\frac{6}{\varepsilon^2}\left(\frac{1}{\delta} + \frac{C_0}{\varepsilon^2}\right)$, $C_2 = \frac{6C_0}{\varepsilon^4}$ for $N > \frac{2C_0}{\varepsilon^2}$.

Remark 4.2. *Note that if the estimate* (4.38) *holds, then it holds for V^-. Indeed, using* (4.34), *the fact that $|||u|||_\infty = |||1-u|||_\infty = 1-\varepsilon$ and that, under $\upsilon_{1-u}(\mathrm{d}\eta)$, $1-\eta$ has for law υ_u we get*

$$\begin{aligned}\int V^-(u,\eta)f(\eta)\upsilon_u(\mathrm{d}\eta) &= -\int V^+(1-u,1-\eta)f(\eta)\upsilon_u(\mathrm{d}\eta)\\ &= \int V^+(1-u,\eta)f(1-\eta)\upsilon_{1-u}(\mathrm{d}\eta)\\ &\leqslant \delta N^2\int \Gamma_N^{\mathcal{K}}\big(\sqrt{f}\big)(1-\eta)\upsilon_{1-u}(\mathrm{d}\eta) + C_1\ell_{\mathcal{V}}^d\mathcal{H}(f(1-\cdot)\upsilon_{1-u}\,|\,\upsilon_{1-u}) + C_2N^{d-a}\\ &= \delta N^2\int \Gamma_N^{\mathcal{K}}\big(\sqrt{f}\big)(\eta)\upsilon_u(\mathrm{d}\eta) + C_1\ell_{\mathcal{V}}^d\mathcal{H}(f\upsilon_u\,|\,\upsilon_u) + C_2N^{d-a}.\end{aligned}$$

Proof of Theorem 4.5. The proof will proceed in several steps.

Note that, according to Hoeffeding Inequality (Lemma A.4), under the probability measure υ_u, ϑ_i^+ is sub-Gaussian with variance parameter $\frac{1}{4}$ and ω_i^+ is sub-Gaussian with variance parameter $\frac{1}{4u_i^2}$.

As in Jara and Menezes [13], we proceed to use an averaged version of V^+. Let $\ell>0$ and consider $\Lambda_\ell=\{0\dots,\ell-1\}^d$ the cube of size ℓ starting at 0. Let $p_\ell(i)=\ell^{-d}\mathbb{1}_{\{i\in\Lambda_\ell\}}$ and $\hat p_\ell(i)=\ell^{-d}\mathbb{1}_{\{i\in-\Lambda_\ell\}}=p_\ell(-i)$. Then for φ defined on $\mathbb{T}_N^d$, we set

$$\overleftarrow{\varphi}(i) := p_\ell*\varphi(i) = \sum_{j+k=i}p_\ell(j)\varphi(k) = \sum_j p_\ell(j)\varphi(i-j) = \ell^{-d}\sum_{j\in\Lambda_\ell}\varphi(i-j) \tag{4.39}$$

$$\overrightarrow{\varphi}(i) := \hat p_\ell*\varphi(i) = \sum_{j+k=i}p_\ell(-j)\varphi(k) = \sum_j p_\ell(-j)\varphi(i-j) = \ell^{-d}\sum_{j\in\Lambda_\ell}\varphi(i+j). \tag{4.40}$$

Let us denote $q_\ell=p_\ell*p_\ell$. We have $q_\ell(i)=\ell^{-2d}|\Lambda_\ell\cap(i-\Lambda_\ell)|$. Thus, $0\leqslant q_\ell(i)\leqslant\ell^{-d}$ and $q_\ell(i)=0$ if and only if $i\notin\Lambda_{2\ell-2}$. All sum being finite, we get for φ and ψ defined on $\mathbb{T}_N^d$:

$$\begin{aligned}\sum_{i,j}\varphi(i)\psi(i+j)q_\ell(j) &= \sum_{i,j,k}\varphi(i)\psi(i+j)p_\ell(k)p_\ell(j-k) = \sum_{i,k}\varphi(i)p_\ell(k)\sum_j\psi(i+j)p_\ell(j-k)\\ &= \sum_{i,k}\varphi(i)p_\ell(k)(\hat p_\ell*\psi)(k+i) = \sum_j\sum_i\varphi(i)p_\ell(j-i)(\hat p_\ell*\psi)(j)\\ &= \sum_j(p_\ell*\varphi)(j)(\hat p_\ell*\psi)(j) = \sum_j\overleftarrow{\varphi}(j)\overrightarrow{\psi}(j).\end{aligned}$$

Thus let $V^{+,\ell}:=\sum_{i\in\mathbb{T}_N^d}\vartheta_i^+\omega_i^{+,\ell}$ with $\omega_i^{+,\ell}=\sum_j\omega_{i+j}^+q_\ell(j)$. We have that

$$V^{+,\ell} = \sum_{i,j\in\mathbb{T}_N^d}\vartheta_i^+\omega_{i+j}^+q_\ell(j) = \sum_{i\in\mathbb{T}_N^d}\overleftarrow{\vartheta^+}_i\,\overrightarrow{\omega^+}_i. \tag{4.41}$$

Then Theorem 4.5 is proved with the following two estimates.

Lemma 4.6. *Suppose that $|||u|||_\infty<1$ and let $\varepsilon=1-|||u|||_\infty$. Then for any $\ell>\ell_{\mathcal{V}}$*

$$\int V^{+,\ell}f\upsilon_u(\mathrm{d}\eta) \leqslant C\varepsilon^{-1}\ell_{\mathcal{V}}^{d/2}\left(\mathcal{H}(f\upsilon_u\,|\,\upsilon_u)+\frac{N^d}{\ell^d}\right). \tag{4.42}$$

The constant C depends only on the dimension, namely it can be taken as $C=3^d2^{d/2+2}$.

Lemma 4.7. *Suppose $\varepsilon = 1 - |||u|||_\infty > 0$ and $\|\nabla u(i)\|_\infty \leqslant \frac{C_0}{N}$ for some $C_0 > 0$ independent of N, where $\nabla u(i) = (\nabla_k u(i))_{k=1}^d$. Then, for any $\ell > \ell_{\mathcal{V}}$, with $\ell = N^\kappa$ for some $\kappa > 0$, and $\delta > 0$*

$$\int (V^+ - V^{+,\ell}) f \upsilon_u(\mathrm{d}\eta) \leqslant \delta N^2 \int \Gamma_N^{\mathcal{K}}(\sqrt{f})\, \upsilon_u(\mathrm{d}\eta) + C_1 \ell_{\mathcal{V}}^d \frac{g_d(\ell)\ell^d}{N} \left(\left(2 + \frac{\ell_{\mathcal{V}}}{\ell}\right)^d \mathcal{H}(f\upsilon_u \,|\, \upsilon_u) + \frac{N^d}{\ell^d} \right) + C_2 N^{d-1} \quad (4.43)$$

where $C_2 = \frac{3C_0}{\varepsilon^4}\left(1 + \frac{2C_0}{N\varepsilon^2}\right)$ and $C_1 = 3^d d(d+1)\left(\frac{2}{\delta\varepsilon^2}\left(1 + \frac{C_0}{N\varepsilon^2}\right) + C_2\right)$ depends only on the dimension and $g_d(\ell)$ is defined in (4.47).

Indeed, we choose ℓ such that $\frac{\ell^d g_d(\ell)}{N} \leqslant C_0$: more precisely, for any $\delta_0 \in (0,1)$ we take

$$\ell = N^{\frac{1}{2}(1-\delta_0)} \quad \text{for} \quad d = 1, \qquad \text{and} \qquad \ell = N^{\frac{1}{d}(1-\delta_0)} \quad \text{for} \quad d \geqslant 2. \quad (4.44)$$

Since $\ell_{\mathcal{V}}^d$ has a log-growth (cf. Assumption 1), this choice of ℓ together with Lemmas 4.6 and 4.7 concludes the proof of Theorem 4.5. $\square$

We now prove Lemma 4.6 and Lemma 4.7.

Proof of Lemma 4.6. We start by recalling that $V^{+,\ell} = \sum_{i\in\mathbb{T}_N^d} \overleftarrow{\vartheta^+}_i \overrightarrow{\omega^+}_i$. Note that, under υ_u, using Lemma A.3, $\overrightarrow{\omega^+}_i$ is a sub-Gaussian variable with variance parameter $\sum_{i\in\Lambda_\ell} \frac{1}{4u_i^2\ell^{2d}} \leqslant \frac{1}{4\varepsilon^2\ell^d}$.

Since $c_i^+(\eta)$ is a function of $(\eta_{i+j})_{j\in\mathcal{V}_N}$, then for i and j such that $|i-j|_\infty > \ell_{\mathcal{V}}$, ϑ_i^+ and ϑ_j^+ are independent, and sub-Gaussian with variance parameter $\frac{1}{4}$. Using Lemma A.3, $\overleftarrow{\vartheta^+}_i$ is a sub-Gaussian variable with variance parameter $\frac{\ell_{\mathcal{V}}^d}{4\ell^d}\left(1 + \frac{\ell_{\mathcal{V}}}{\ell}\right)^d \leqslant 2^{d-2}\frac{\ell_{\mathcal{V}}^d}{\ell^d}$.

We note that all the sites involved in the averages $\overleftarrow{\vartheta^+}_i \overrightarrow{\omega^+}_i$ are in $i + Q_{\ell_{\mathcal{V}}/2+\ell}$, where $Q_m = \{-m, \dots, m\}^d$ is the d-dimensional cube centered at 0. In particular, for i and j such that $|i-j|_\infty > \ell_{\mathcal{V}} + 2\ell$, the corresponding averages are independent (under υ_u). Then, we can take a partition of $\mathbb{T}_N^d$ into independent sites by letting $i = j + (\ell_{\mathcal{V}} + 2\ell)k$ where $j \in \Lambda_{\ell_{\mathcal{V}}+2\ell}$ and $k \in \Lambda_{\lceil N/(\ell_{\mathcal{V}}+2\ell)\rceil}$. The entropy inequality (cf. (B.3) in [13]) gives

$$\begin{aligned}
\int V^{+,\ell} f\upsilon_u(\mathrm{d}\eta) &= \sum_{j\in\Lambda_{\ell_{\mathcal{V}}+2\ell}} \int \sum_{k\in\Lambda_{\lceil N/(\ell_{\mathcal{V}}+2\ell)\rceil}} \overleftarrow{\vartheta^+}_{j+(\ell_{\mathcal{V}}+2\ell)k} \overrightarrow{\omega^+}_{j+(\ell_{\mathcal{V}}+2\ell)k} f\upsilon_u(\mathrm{d}\eta) \\
&\leqslant \sum_{j\in\Lambda_{\ell_{\mathcal{V}}+2\ell}} \frac{1}{\gamma}\Bigg(\mathcal{H}(f\upsilon_u \,|\, \upsilon_u) \\
&\qquad + \sum_{k\in\Lambda_{\lceil N/(\ell_{\mathcal{V}}+2\ell)\rceil}} \log \int \exp\left\{\gamma \overleftarrow{\vartheta^+}_{j+(\ell_{\mathcal{V}}+2\ell)k} \overrightarrow{\omega^+}_{j+(\ell_{\mathcal{V}}+2\ell)k}\right\} \upsilon_u(\mathrm{d}\eta)\Bigg) \\
&\leqslant \frac{(\ell_{\mathcal{V}}+2\ell)^d}{\gamma}\mathcal{H}(f\upsilon_u \,|\, \upsilon_u) + \frac{1}{\gamma}\sum_{i\in\mathbb{T}_N^d} \log \int \exp\left\{\gamma \overleftarrow{\vartheta^+}_i \overrightarrow{\omega^+}_i\right\} \upsilon_u(\mathrm{d}\eta)
\end{aligned}$$

According to Lemma A.2, for $\gamma^{-1} = 2^{d/2+2}\ell^{-d}\varepsilon^{-1}\ell_{\mathcal{V}}^{d/2}$, we have that

$$\log \int \exp\left\{\gamma \overleftarrow{\vartheta^+}_i \overrightarrow{\omega^+}_i\right\} \upsilon_u(\mathrm{d}\eta) \leqslant \log 3.$$

Thus we obtain

$$\int V^{+,\ell} f \upsilon_u(\mathrm{d}\eta) \leqslant 2^{d/2+2}\varepsilon^{-1}\ell_{\mathcal{V}}^{d/2} \left(\left(2 + \frac{\ell_{\mathcal{V}}}{\ell}\right)^d \mathcal{H}(f\upsilon_u \,|\, \upsilon_u) + \frac{N^d}{\ell^d}\log 3\right).$$

$\square$

Proof of Lemma 4.7. We first prove the Lemma by also assuming that $\mathcal{V}_N \subset \mathbb{Z}^d \backslash \mathbb{N}^d$. In Remark 4.3 at the end of the proof we show how to remove this assumption in dimension $d > 1$.

We then proceed as in Jara and Menezes [13]. We use the fact that

$$\begin{aligned} V^+ - V^{+,\ell} &= \sum_{i \in \mathbb{T}_N^d} \vartheta_i^+ (\omega_i^+ - \omega_i^{+,\ell}) \\ &= \sum_{i,j \in \mathbb{T}_N^d} \vartheta_i^+ \omega_{i+j}^+ (\mathbb{1}_{\{0\}}(j) - q_\ell(j)). \end{aligned} \tag{4.45}$$

We now use Lemma 3.2 in [13] stating that there exists a function $\Phi_\ell : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ which is a flow connecting the distribution $\mathbb{1}_{\{0\}}$ to q_ℓ, i.e

- $\Phi_\ell(i,j) = -\Phi_\ell(j,i)$
- $\sum_{j:\,|i-j|=1} \Phi_\ell(i,j) = \mathbb{1}_{\{0\}}(i) - q_\ell(i)$
- $\Phi_\ell(i,j) = 0$ for $i, j \notin \Lambda_{2\ell-1}$
- there is a constant $C = C(d)$ independent of ℓ such that

$$\sum_{|i-j|=1} |\Phi_\ell(i,j)|^2 < C g_d(\ell) \qquad \text{and} \qquad \sum_{|i-j|=1} |\Phi_\ell(i,j)| < C\ell \tag{4.46}$$

  where

$$g_d(\ell) = \begin{cases} \ell \text{ for } d = 1 \\ \log(\ell) \text{ for } d = 2 \\ 1 \text{ for } d \geqslant 3 \end{cases} \tag{4.47}$$

Using the flow Φ_ℓ we therefore have

$$\begin{aligned} V^+ - V^{+,\ell} &= \sum_{i,j \in \mathbb{T}_N^d} \vartheta_i^+ \omega_{i+j}^+ \sum_{k:\,|j-k|=1} \Phi_\ell(j,k) \\ &= \sum_{i,j \in \mathbb{T}_N^d} \vartheta_i^+ \omega_{i+j}^+ \sum_{k=1}^d \left\{\Phi_\ell(j, j+e_k) - \Phi_\ell(j-e_k, j)\right\} \\ &= \sum_{k=1}^d \sum_{i,j \in \mathbb{T}_N^d} \vartheta_i^+ \Phi_\ell(j, j+e_k)(\omega_{i+j}^+ - \omega_{i+j+e_k}^+) \\ &= \sum_{k=1}^d \sum_{i,j \in \mathbb{T}_N^d} \vartheta_{i-j}^+ \Phi_\ell(j, j+e_k)(\omega_i^+ - \omega_{i+e_k}^+) \\ &= \sum_{k=1}^d \sum_{i \in \mathbb{T}_N^d} h_i^k (\omega_i^+ - \omega_{i+e_k}^+) \end{aligned} \tag{4.48}$$

where

$$h_i^k = h_i^{\ell,k} := \sum_{j\in\mathbb{T}_N^d} \vartheta_{i-j}^+ \Phi_\ell(j, j+e_k) = \sum_{j\in\Lambda_{2\ell-1}} \vartheta_{i-j}^+ \Phi_\ell(j, j+e_k). \tag{4.49}$$

To complete the result we need to estimate (4.48). We apply Lemma 3.5 of [7]. Note that the hypothesis of Lemma 3.5 are satisfied in our case: $u_- = \varepsilon$ and $u_+ = 1-\varepsilon$ and $h_i^k(\eta^{i,i+e_k}) = h_i^k(\eta)$ for any configuration η inasmuch h_i^k is only a function of the sites η_{i-j+s}, for $s \in \mathcal{V}_N$, $j \in \Lambda_{2\ell-1}$, so that it does not depend on η_i and η_{i+e_k} since $\mathcal{V}_N \subset \mathbb{Z}^d \backslash \mathbb{N}^d$. Moreover, we observe that Lemmas 3.4 and 3.5 apply to our case replacing $\chi(u_i)$ by u_i. Therefore, for any $\alpha' > 0$ we have that

$$\begin{aligned}\int h_i^k(\omega_i^+ - \omega_{i+e_k}^+) f \upsilon_u(\mathrm{d}\eta) \leqslant \frac{\alpha'}{2} \int \left(\sqrt{f(\eta^{i,i+e_k})} - \sqrt{f(\eta)}\right)^2 \upsilon_u(\mathrm{d}\eta) \\ + \frac{C_{1,\varepsilon}}{\alpha'} \int (h_i^k)^2 f \upsilon_u(\mathrm{d}\eta) + R_i^k.\end{aligned} \tag{4.50}$$

with $C_{1,\varepsilon} = \frac{2}{\varepsilon^2}\left(1 + \frac{C_0}{N\varepsilon^2}\right)$ and where the rest term R_i^k is controlled as

$$R_i^k \leqslant C_{2,\varepsilon} \left| \nabla_k u_i \right| \int \left| h_i^k(\eta) \right| f \upsilon_u(\mathrm{d}\eta) , \tag{4.51}$$

with, for $\varepsilon < \frac{1}{2}$, $C_{2,\varepsilon} = \frac{3}{\varepsilon^4}\left(1 + \frac{2C_0}{N\varepsilon^2}\right)$. We now take $\alpha' = \delta N^2$, with $\delta > 0$. We get (cf. (4.37))

$$\begin{aligned}\int (V^+ - V^{+,\ell}) f \upsilon_u(\mathrm{d}\eta) &= \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \int h_i^k(\omega_i^+ - \omega_{i+e_k}^+) f \upsilon_u(\mathrm{d}\eta) \\ &\leqslant \delta N^2 \int \Gamma_N^{\mathcal{K}}(\sqrt{f})\, \upsilon_u(\mathrm{d}\eta) + \frac{C_{1,\varepsilon}}{\delta N^2} \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \int (h_i^k)^2 f \upsilon_u(\mathrm{d}\eta) + \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} R_i^k.\end{aligned} \tag{4.52}$$

The first term is the same of (4.43), then to conclude the proof we have to upper bound the second and the third term of (4.52).

Let us start from the third one. Using that $\left|\nabla_k u_i\right| \leqslant \frac{C_0}{N}$ and $|h_i^k| \leqslant 1 + (h_i^k)^2$ in (4.51) we get

$$R_i^k \leqslant \frac{C_0 C_{2,\varepsilon}}{N} \int \left(1 + (h_i^k)^2\right) f \upsilon_u(\mathrm{d}\eta) .$$

So that, the last term of (4.52) is bounded by

$$C_0 C_{2,\varepsilon} \left(N^{d-1} + \frac{1}{N} \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \int (h_i^k)^2 f \upsilon_u(\mathrm{d}\eta) \right).$$

Let us note that this last term dominates the second term of (4.52). To conclude the proof we need to upper-bound

$$\sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \int (h_i^k)^2 f \upsilon_u(\mathrm{d}\eta).$$

As we noted above, h_i^k depends only on the sites η_{i-j+s}, for $s \in \mathcal{V}_N$ and $j \in \Lambda_{2\ell-1}$, so that the random variables h_i^k and $h_{i'}^k$ are independent if $|i - i'|_\infty > (2\ell + \ell_{\mathcal{V}})$. We then decompose $\mathbb{T}_N^d$ as a disjoint union of cubes of size $2\ell + \ell_{\mathcal{V}}$, that is, we write $i = j + (\ell_{\mathcal{V}} + 2\ell)z$ where $j \in \Lambda_{\ell_{\mathcal{V}}+2\ell}$ and $z \in \Lambda_{\lceil N/(\ell_{\mathcal{V}}+2\ell)\rceil}$,

$$\sum_{i\in\mathbb{T}_N^d} \int (h_i^k)^2 f \upsilon_u(\mathrm{d}\eta) = \sum_{j\in\Lambda_{2\ell+\ell_{\mathcal{V}}}} \sum_{z\in\Lambda_{\lceil N/(\ell_{\mathcal{V}}+2\ell)\rceil}} \int (h_{j+z}^k)^2 f \upsilon_u(\mathrm{d}\eta).$$

We then apply the entropy inequality and we get

$$
\begin{aligned}
\sum_{i\in\mathbb{T}_N^d}\int (h_i^k)^2 f\upsilon_u(\mathrm{d}\eta) &\leqslant \frac{1}{\gamma}\sum_{j\in\Lambda_{2\ell+\ell_{\mathcal{V}}}}\left(\mathcal{H}\big(f\upsilon_u\,|\,\upsilon_u\big)+\log\int\exp\Big\{\gamma\sum_{z\in\Lambda_{\lceil N/(\ell_{\mathcal{V}}+2\ell)\rceil}}(h_{j+z}^k)^2\Big\}\upsilon_u(\mathrm{d}\eta)\right)\\
&=\frac{(2\ell+\ell_{\mathcal{V}})^d}{\gamma}\mathcal{H}\big(f\upsilon_u\,|\,\upsilon_u\big)+\frac{1}{\gamma}\sum_{i\in\mathbb{T}_N^d}\log\int\exp\Big\{\gamma(h_i^k)^2\Big\}\upsilon_u(\mathrm{d}\eta)
\end{aligned}
\tag{4.53}
$$

To conclude we use a concentration inequality. We have that h_i^k is sub-Gaussian random variable, let σ^2 be its variance parameter. By Proposition A.1 we have that for any $\gamma \leqslant \frac{1}{4\sigma^2}$,

$$
\int \exp\Big\{\gamma(h_i^k)^2\Big\}\upsilon_u(\mathrm{d}\eta)\leqslant \log 3\,.
$$

Moreover, to get an upper bound on the variance parameter, we use the same decomposition of the sum (4.49) into subsets which are independent (since ϑ_j^+ is a function of η_{i+j}, for $i\in\mathcal{V}_N$ and is sub-Gaussian with variance parameter $\frac{1}{4}$). This is done in Lemma F.12 in [13]. We then have that $\sigma^2\leqslant C_d\ell_{\mathcal{V}}^d g_d(\ell)$, where C_d is a constant which depends only on the dimension. By getting γ as large as possible, namely $\gamma^{-1}=(d+1)\ell_{\mathcal{V}}^d\,g_d(\ell)$ we obtain that

$$
\sum_{i\in\mathbb{T}_N^d}\int (h_i^k)^2 f\upsilon_u(\mathrm{d}\eta)\leqslant (d+1)\ell_{\mathcal{V}}^d\,g_d(\ell)\ell^d\left(\Big(2+\frac{\ell_{\mathcal{V}}}{\ell}\Big)^d\mathcal{H}\big(f\upsilon_u\,|\,\upsilon_u\big)+\frac{N^d}{\ell^d}\log 3\right).
\tag{4.54}
$$

To conclude the proof we have to remove the assumption $\mathcal{V}_N\subset\mathbb{Z}^d\backslash\mathbb{N}^d$. We show that in Remark 4.3. □

Remark 4.3. *By (4.48) we recall that $V^+-V^{+,\ell}=\sum_{k=1}^d\sum_{i\in\mathbb{T}_N^d}h_i^k(\omega_i^+-\omega_{i+e_k}^+)$ where h_i^k is defined in (4.49). We also observe that, by assumption, $\ell>\ell_{\mathcal{V}}$. Since h_i^k is a function of the sites η_{i-j+s}, for $s\in\mathcal{V}_N$, $j\in\Lambda_{2\ell-1}$, whenever $|j|>\ell_{\mathcal{V}}$, then ϑ_{i-j}^+ does not depend on η_i and η_{i+e_k}. Therefore we split h_i^k into the sum of two functions h_i' and h_i''. The first one is independent of η_i and η_{i+e_k}, while the second one depends on these sites,*

$$
h_i' := \sum_{\substack{j\in\Lambda_{2\ell-1}\\ |j|>2\ell_{\mathcal{V}}}}\vartheta_{i-j}^+\Phi_\ell(j,j+e_k),\qquad \textit{and}\qquad h_i'' := \sum_{\substack{j\in\Lambda_{2\ell-1}\\ |j|\leqslant 2\ell_{\mathcal{V}}}}\vartheta_{i-j}^+\Phi_\ell(j,j+e_k).
\tag{4.55}
$$

In such a way

$$
V^+-V^{+,\ell}=(V^+-V^{+,\ell})'+(V^+-V^{+,\ell})'',
\tag{4.56}
$$

where

$$
(V^+-V^{+,\ell})' := \sum_{k=1}^d\sum_{i\in\mathbb{T}_N^d}h_i'(\omega_i^+-\omega_{i+e_k}^+)\quad \textit{and}\quad (V^+-V^{+,\ell})'' := \sum_{k=1}^d\sum_{i\in\mathbb{T}_N^d}h_i''(\omega_i^+-\omega_{i+e_k}^+)
\tag{4.57}
$$

We can apply the method used above to $(V^+-V^{+,\ell})'$ obtaining that (4.54) holds. We control $(V^+-V^{+,\ell})''$ by showing that $\int(V^+-V^{+,\ell})''f\upsilon_u(\mathrm{d}\eta)=O(N^{d-\varepsilon})$, for some $\varepsilon>0$. For this purpose we apply Cauchy-Swartz inequality to the measure $f\upsilon_u(\mathrm{d}\eta)$, which gives

$$
\int h_i''(\omega_i^+-\omega_{i+e_k}^+)f\upsilon_u(\mathrm{d}\eta)\leqslant\Big(\int(h_i'')^2f\upsilon_u(\mathrm{d}\eta)\Big)^{\frac12}\Big(\int(\omega_i^+-\omega_{i+e_k}^+)^2f\upsilon_u(\mathrm{d}\eta)\Big)^{\frac12}
\tag{4.58}
$$

Therefore

$$\begin{aligned}\int (V^+ - V^{+,\ell})'' f\upsilon_u(\mathrm{d}\eta) &= \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \int h_i''(\omega_i^+ - \omega_{i+e_k}^+) f\upsilon_u(\mathrm{d}\eta) \\ &\leqslant \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \Big(\int (h_i'')^2 f\upsilon_u(\mathrm{d}\eta)\Big)^{\frac12} \Big(\int (\omega_i^+ - \omega_{i+e_k}^+)^2 f\upsilon_u(\mathrm{d}\eta)\Big)^{\frac12} \\ &\leqslant \left(\sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \Big(\int (h_i'')^2 f\upsilon_u(\mathrm{d}\eta)\Big) \sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \Big(\int (\omega_i^+ - \omega_{i+e_k}^+)^2 f\upsilon_u(\mathrm{d}\eta)\Big)\right)^{\frac12}, \end{aligned} \tag{4.59}$$

where in the last inequality we used again Cauchy-Swartz inequality.

We observe that $(\omega_i^+ - \omega_{i+e_k}^+) = (\frac{\eta_i}{u_i} - \frac{\eta_{i+e_k}}{u_{i+e_k}}) = \eta_i(\frac{1}{u_i} - \frac{1}{u_{i+e_k}}) + \frac{1}{u_{i+e_k}}(\eta_i - \eta_{i+e_k})$, *so that*

$$\begin{aligned}\int (\omega_i^+ - \omega_{i+e_k}^+)^2 f\upsilon_u(\mathrm{d}\eta) &\leqslant 2\int \eta_i\Big(\frac{1}{u_i} - \frac{1}{u_{i+e_k}}\Big)^2 f\upsilon_u(\mathrm{d}\eta) + 2\int \frac{1}{u_{i+e_k}^2}(\eta_i - \eta_{i+e_k})^2 f\upsilon_u(\mathrm{d}\eta) \\ &\leqslant C_\varepsilon |\nabla u_k|_\infty^2 + C_\varepsilon \int (\eta_i - \eta_{i+e_k})^2 f\upsilon_u(\mathrm{d}\eta) \leqslant C'_\varepsilon,\end{aligned}$$

uniformly on i *and* N. *So that,*

$$\sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \Big(\int (\omega_i^+ - \omega_{i+e_k}^+)^2 f\upsilon_u(\mathrm{d}\eta)\Big) \leqslant C'_\varepsilon N^d. \tag{4.60}$$

Therefore, by (4.59), *to conclude the proof it is enough to show that*

$$\sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \Big(\int (h_i'')^2 f\upsilon_u(\mathrm{d}\eta)\Big) = O(N^{d-\varepsilon}), \tag{4.61}$$

for some $\varepsilon > 0$ *small. For this purpose we have to look more carefully at the function* $\Phi_\ell(j, j+e_k)$ *which defines the flow. We recall (cf. Appendix G of [13]) that in the construction of the flow connecting* $\mathbb{1}_{\{0\}}$ *and* $p_\ell * p_\ell$, *we first define a flow* Ψ_ℓ *connecting* $\mathbb{1}_{\{0\}}$ *and* p_ℓ *supported in* Λ_ℓ *which satisfies* (4.46) *and then we define*

$$\Phi_\ell(j, j+e_k) := \sum_{i\in\mathbb{T}_N^d} \Psi_\ell(i, i+e_k) p_\ell(j-i),$$

therefore

$$\begin{aligned}\sum_{\substack{j\in\Lambda_{2\ell-1}\\ |j|\leqslant 2\ell_{\mathcal{V}}}} \big|\Phi_\ell(j, j+e_k)\big| &\leqslant \sum_{\substack{j\in\Lambda_{2\ell-1}\\ |j|\leqslant 2\ell_{\mathcal{V}}}} \sum_{i\in\mathbb{T}_N^d} \big|\Psi_\ell(i, i+e_k)\big| p_\ell(j-i) \\ &\leqslant \sum_{i\in\mathbb{T}_N^d} \big|\Psi_\ell(i, i+e_k)\big| \sum_{\substack{j\in\Lambda_{2\ell-1}\\ |j|\leqslant 2\ell_{\mathcal{V}}}} p_\ell(j-i) \leqslant C\ell \frac{\ell_{\mathcal{V}}^d}{\ell^d},\end{aligned}$$

where we used that $\sum_{\substack{j\in\Lambda_{2\ell-1}\\ |j|\leqslant 2\ell_{\mathcal{V}}}} p_\ell(j-i) \leqslant \frac{\ell_{\mathcal{V}}^d}{\ell^d}$ *uniformly on* i *and* (4.46) *applied to* Ψ_ℓ. *We deduce that, since* $|\vartheta_{i-j}^+| \leqslant 2$,

$$\sum_{k=1}^d \sum_{i\in\mathbb{T}_N^d} \Big(\int (h_i'')^2 f\upsilon_u(\mathrm{d}\eta)\Big) \leqslant CN^d \frac{\ell_{\mathcal{V}}^{2d}}{\ell^{2(d-1)}}.$$

Since $\ell = N^\kappa$ *for some* $\kappa > 0$ *and* $\ell_{\mathcal{V}}^d$ *grows logarithmically, the result follows (in dimension* $d \geqslant 2$*).*

4.1.3 Estimate of V

We show that under the hypothesis of Theorem 4.5, $V(u,\eta)$ satisfies (4.38). For $i \in \mathbb{T}_N^d$ and $k \in \{1,\ldots,d\}$ we let

$$\widetilde{\omega}_i^k := -\alpha N^2 (u_i - u_{i+e_k})^2 \omega_i.$$

In such a way we get

$$\begin{aligned} V(u,\eta) &= -\alpha N^2 \sum_{\substack{i,j \in \mathbb{T}_N^d \\ |i-j|=1}} (u_i - u_j)^2 \omega_i \omega_j \\ &= -\alpha N^2 \sum_{k=1}^d \sum_{i \in \mathbb{T}_N^d} \left\{ (u_i - u_{i+e_k})^2 \omega_i \omega_{i+e_k} + (u_i - u_{i-e_k})^2 \omega_i \omega_{i-e_k} \right\} \\ &= \sum_{k=1}^d \sum_{i \in \mathbb{T}_N^d} \left\{ \widetilde{\omega}_i^k \omega_{i+e_k} + \widetilde{\omega}_{i-e_k}^k \omega_i \right\} = 2 \sum_{k=1}^d \sum_{i \in \mathbb{T}_N^d} \widetilde{\omega}_{i-e_k}^k \omega_i. \end{aligned}$$

For $k \in \{1,\ldots,d\}$ we let $\widetilde{V}^k := \sum_{i \in \mathbb{T}_N^d} \widetilde{\omega}_{i-e_k}^k \omega_i$ and $\widetilde{V}^{k,\ell} := \sum_{i \in \mathbb{T}_N^d} \overleftarrow{\widetilde{\omega}_{i-e_k}^k} \, \overrightarrow{\omega_i}$. We have that $\widetilde{V}^{k,\ell}$ and $\widetilde{V}^k - \widetilde{V}^{k,\ell}$ satisfy (4.42) and (4.43) respectively, which implies that $V(u,\eta)$ satisfies (4.38). The proof follows the same ideas of V^+ and V^- and it is actually simpler since we do not have to deal with the diameter of $\mathcal{V}_N$. We omit the details.

4.1.4 Conclusion: Gronwall's inequality (4.13)

We observe that since $\Gamma_N^{\mathcal{K}}(\sqrt{f}) \leqslant \Gamma_N(\sqrt{f})$, cf. (4.36) and that the carré du champ operator is non-negative, (4.13) is a consequence of Proposition 4.3 and Theorem 4.5 with $u(t) = u^N(t)$, the solutions of (4.10), $f = \frac{\mathrm{d}\mu_t}{\mathrm{d}\nu_{u(t)}}$ if we show that u^N satisfies Assumption 2 for any $t \in [0,\tau]$. This is one of the goal of the Section 5, see Propositions 5.2 and 5.3.

5 Estimates on the solutions (u^N) of (4.10)

Note that (4.10) is a first order ordinary differential equation in $\mathbb{R}^{\mathbb{T}_N^d}$. By standard theory, we have a solution, locally in time, starting from every initial condition.

We say that u is supersolution of (4.10) if $\partial_t u_i \geqslant 2\alpha N^2 (\Delta u)_i + \beta(1-2u_i) + G(i,u)$ and that it is a subsolution if $\partial_t u_i \leqslant 2\alpha N^2 (\Delta u)_i + \beta(1-2u_i) + G(i,u)$.

Note that any solution is both a super- and a subsolution. We have a comparaison lemma between super and subsolutions.

Proposition 5.1. *Let u be a supersolution, and v be a subsolution such that $u(0,i) \geqslant v(0,i)$ for all $i \in \mathbb{T}_N^d$. Then, for all $t \geqslant 0$ and all $i \in \mathbb{T}_N^d$, $u(t,i) \geqslant v(t,i)$.*

Proof. Since u and v have derivative in time they are continuous, consider i and t such that $u(t,i) = v(t,i)$ and $u(t,j) \geqslant v(t,j)$. Then, we have $G(i,u(t)) \geqslant G(i,v(t))$ by Proposition B.1, and thus

$$\partial_t (u(i,t) - v(i,t)) = 2\alpha N^2 \sum_{j, |i-j|=1} (u(j,t) - v(j,t)) + G(i,u(t)) - G(i,v(t)) \geqslant 0. \tag{5.1}$$

This proves that u stays above v at all time. $\square$

From the two previous propositions we conclude:

Proposition 5.2. *Let $\delta = \frac{\beta}{1+4\beta}$ and suppose $0 < \delta < \frac{1}{4e}$. Let also $4\delta \leqslant T \leqslant 1 - 4\delta$ and $0 \leqslant \varepsilon < 2\delta$. For all $K_N > \frac{|\log(\varepsilon/2)|}{\varepsilon^2}$, and all N, consider the solution $(u^N(t,i))_{i\in\mathbb{T}_N^d}$ of Equation (4.10) starting from $u_0 \in [\varepsilon, 1-\varepsilon]$, then we have that $u^N(t,i) \in [\varepsilon, 1-\varepsilon]$ for all i and $t \geqslant 0$.*

Note that the proposition holds for $\varepsilon = 0$.

Proof. Let $p \in [0,1]$, and set $u(i) = p$, for all i. Using that $g_{K_N}(p) = G(i,p)$, we have that

- if $g_{K_N}(p) + \beta(1-2p) \geqslant 0$, then u is a subsolution;
- if $g_{K_N}(p) + \beta(1-2p) \leqslant 0$, then u is a supersolution.

Then, by using the result of Proposition B.2 on the analysis of $g_K(p)$ close to $p = 0$ and $p = 1$, it is easy to check that $u(i) = \varepsilon$ is a subsolution and $u(i) = 1 - \varepsilon$ is supersolution, for ε satisfying the hypothesis of the proposition.

In particular, for $p = 1$, $g_{K_N}(1) = 0$, we have a supersolution. For $p = 0$, $g_{K_N}(0) = 0$, we have a subsolution. □

In the next result, we show that if u solves (4.10) and $\|\nabla u(0,i)\|_\infty \leqslant \frac{C}{N}$, then $\|\nabla u(t,i)\|_\infty \leqslant \frac{C\sqrt{t}}{N}$ for any $t > 0$.

Proposition 5.3. *Let u be a solution of (4.10) with $u(0) = u_0$ such that, there exists $C_0 > 0$ for which $|\nabla u_0|_\infty \leqslant \frac{C_0}{N}$. Then, there exists $C > 0$ such that $\|\nabla u(t,i)\|_\infty \leqslant \frac{C_0 + C\sqrt{t}}{N}$ for all $t \geqslant 0$.*

To prove Proposition 5.3 we follow [7] and the reference therein, in particular [5]. Using the notation of [5] we let $p(t,x,z)$ be the heat kernel of discrete Laplacian

$$\Delta u(t,i) = \sum_{j\in\mathbb{Z}^d} a(t,i,i+j)\big[u(t,i+j) - u(t,i)\big], \tag{5.2}$$

with $a(t,i,j) = \mathbb{1}_{\{j-i\in\Gamma\}}$ and $\Gamma = \{\pm e_i,\ i = 1,\dots,d\}$, that is,

$$\begin{cases} p(\cdot, i_0, i) = \delta_{i_0}(\cdot), \\ \partial_t p(t,i_0,i) = \Delta p(t,i_0,i). \end{cases}$$

In this case (comments below (1.2) of [5]) we have that $a^*(t,i,j) = a(t,i,j)$ so that $p^*(t,i,j) = p(t,i,j)$. Let $\nabla_k u(t,i) = u(t,i+e_k) - u(t,i)$, then since p is a uniform transitions function, there exist $c, C > 0$ independent of t, k such that (cf. (1.3) of [5])

$$\big|\nabla_k p(t,0,i)\big| \leqslant C\frac{p(ct,0,i)}{\sqrt{1\vee t}}. \tag{5.3}$$

We refer to [5], Section 4 and the reference therein for a proof. We stress that the delicate point is to extend the classical theory of E. De Giorgi, J. Nash and J. Moser to discrete operators. The authors follow mainly [6], but similar results can be also found in [9] (Appendix B) and [23].

Proof of Proposition 5.3. Using Duhamel's formula (i.e., variation of constant) we get that

$$u(t,i) = \sum_{j\in\mathbb{T}_N^d} u(0,j)p_N(t,i,j) + \int_0^t \mathrm{d}s \sum_{j\in\mathbb{T}_N^d} (\beta(1-2u_j) + G_j(u))p_N(t-s,i,j), \tag{5.4}$$

where $p_N(t,i,j) = \sum_{z\in N\mathbb{Z}^d} p(2\alpha N^2 t, i, j+z)$ is the heat kernel of discrete Laplacian on the torus speeds up by a factor $2\alpha N^2$ and $p(t,i,j)$ is the heat kernel introduced above. We observe that (5.3) gives

$$\left|\nabla_k p_N(t,0,i)\right| \leqslant \frac{C}{N}\frac{p_N(ct,0,i)}{\sqrt{t}}. \tag{5.5}$$

For the first term, using that $p_N(t,i,j) = p_N(t,i+z,j+z)$ for any $i,j,z \in \mathbb{T}_N^d$ and the assumption on ∇u_0 (cf. Assumptions 2), an integration by parts gives

$$\begin{aligned}
\left|\nabla_k\left\{\sum_{j\in\mathbb{T}_N^d} u(0,j)p_N(t,i,j)\right\}\right| &= \left|\sum_{j\in\mathbb{T}_N^d} u(0,j)\big(p_N(t,i+e_k,j) - p_N(t,i,j)\big)\right| \\
&= \left|\sum_{j\in\mathbb{T}_N^d} u(0,j)p_N(t,i,j-e_k) - \sum_{j\in\mathbb{T}_N^d} u(0,j)p_N(t,i,j)\right| \\
&= \left|\sum_{j\in\mathbb{T}_N^d} \Big(u(0,j+e_k) - u(0,j)\Big)p_N(t,i,j)\right| \leqslant \frac{C}{N}.
\end{aligned}$$

By using that $(\beta(1-2u_j) + G_j(u))$ is bounded and (5.5) we get that for any $k = 1,\dots,d$,

$$\left|\nabla_k\left\{\int_0^t \mathrm{d}s \sum_{j\in\mathbb{T}_N^d} (\beta(1-2u_j) + G_j(u))p_N(t-s,i,j)\right\}\right| \leqslant \frac{C}{N}\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{d}s = \frac{C}{N}\sqrt{t}\,.$$

□

6 Existence and uniqueness of reaction-diffusion PDE

In Section 7, we prove that in the limit $N \to \infty$, the solution of the discretized Equation 4.10, with $u^N(t=0) = u_0^N \in [0,1]^{\mathbb{T}_N^d}$ satisfying Assumption 2, converges to a solution of the scalar nonlinear reaction diffusion equation

- when $K_N \to K$ as $N \to \infty$:

$$\partial_t u(t,x) = 2\alpha\Delta u(t,x) + \beta(1-2u(t,x)) + g_K(u(t,x)), \tag{6.1}$$

- when $K_N \to \infty$ as $N \to \infty$:

$$\partial_t u(t,x) = 2\alpha\Delta u(t,x) + \beta(1-2u(t,x)) + g_\infty(u(t,x)). \tag{6.2}$$

The main difference between the two equations is that in the first case ($K < +\infty$), g_K is a C^1 function on $[0,1]$ (thus Lipschitz) so the reaction diffusion equation (6.1) is very classical, whereas for the second case ($K = +\infty$), since g_∞ is not even continuous we need to consider (6.2) as a subdifferential inclusion.

The main results of this section are

(i) Proposition 6.3 which proves existence of a solution, in a suitable sense, for the equation (6.2),

(ii) Proposition 6.4 which proves local uniqueness of the solution, in a suitable sense, for the equation (6.2), starting from a suitable class of initial conditions,

(iii) and Theorem 7.1 which proves that all accumulation points of $(u^N)_N$ is a solution, in a suitable sense, for the equation (6.2).

In the rest of this section we change our notations and define $v = 2u - 1$. We center the solution around the constant steady state $u = \frac{1}{2}$. It simplifies the presentation and proofs of our results. The original form of our equations can be retrieved by letting $u = \frac{1}{2}(v+1)$. In such a way (6.1) takes the form

$$\partial_t v(t,x) - 2\alpha\Delta v(t,x) + 2\beta v - 2g_K\left(\frac{1}{2}(v+1)\right) = 0. \tag{6.3}$$

6.1 Solution of (6.1)

Let us denote $s(t,x,y)$ the semigroup of the operator $\frac{1}{2}\Delta$ on $\mathbb{T}^d$, that is,

$$s(t,x,y) = \frac{1}{(2\pi t)^{d/2}} \sum_{k\in\mathbb{Z}^d} \exp\left(-\frac{\|x-y-k\|^2}{2t}\right). \tag{6.4}$$

Denote also $s_0(t,x,y)$ the semigroup of the operator $\frac{1}{2}\Delta$ on $\mathbb{R}^d$,

$$s_0(t,x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x-y\|^2}{2t}\right). \tag{6.5}$$

Note that $\xi \mapsto s_0(t,0,\xi)$ is the density of d independent normal random variables with variance t.

Let us consider $(S_t^{\lambda,\gamma})$ the semigroup on $L^1(\mathbb{T}^d)$ defined by, for $f \in L^1(\mathbb{T}^d)$, $\lambda \geqslant 0$ and $\gamma > 0$

$$S_t^{\lambda,\gamma} f(x) = \int_{\mathbb{T}^d} e^{-\lambda t} s(\gamma t,x,y) f(y)\mathrm{d}y = \int_{\mathbb{R}^d} e^{-\lambda t} s_0(\gamma t,x,y)\widetilde{f}(y)\mathrm{d}y, \tag{6.6}$$

where for a measurable function f on $\mathbb{T}^d$, we denoted $\widetilde{f}$ its extension on $\mathbb{R}^d$ defined by $\widetilde{f}(x) = f(x - \lfloor x\rfloor)$.

Another way to define $S_t^{\lambda,\gamma}$ is to use the Brownian motion: denote by X a Brownian motion on $\mathbb{R}^d$ starting from x on some probability space $(\Omega,\mathcal{F},\mathbb{P}_x)$, indeed we have $S_t^{\lambda,\gamma}f(x) = e^{-\lambda t}\mathbb{E}_x(\widetilde{f}(X_{\gamma t}))$, and for all $\lambda \geqslant 0$, and $\gamma > 0$, $S^{\lambda,\gamma}$ is a C_0-contraction semigroup on $L^p(\mathbb{T}^d)$ for $p \in [1,+\infty]$.

As we will look at (6.1) in its mild form, the following result is crucial to study the regularity of the solution.

Proposition 6.1. *For $v_0 \in L^\infty(\mathbb{T}^d)$ and $g \in L^\infty([0,\tau]\times\mathbb{T}^d)$, define*

$$v(t,x) := S_t^{\lambda,\gamma} v_0(x) + \int_0^t S_{t-s}^{\lambda,\gamma}(g(s,\cdot))(x)\mathrm{d}s. \tag{6.7}$$

then $v \in C([0,\tau],\mathbb{T}^d)$. We have the following estimates, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^d$,

$$|v(t,x)| \leqslant e^{-\lambda t}\|v_0\|_\infty + \frac{1}{\lambda}(1-e^{-\lambda t})\|g\|_\infty \leqslant \|v_0\|_\infty + \frac{1}{\lambda}\|g\|_\infty \tag{6.8}$$

and for all $\tau > 0$, there exists a constant C depending only on τ,γ,λ and d, such that for all $(t,x),(s,y) \in [1/\tau,\tau]\times\mathbb{T}^d$ with $s < t$

$$|v(t,x) - v(s,y)| \leqslant C((t-s)|\log(t-s)| + \|x-y\|)(\|g\|_\infty + \|v_0\|_\infty). \tag{6.9}$$

Remark 6.1. *v is called a mild solution of the equation $\partial_t v - \frac{\gamma}{2}\Delta v + \lambda v = g$ with initial value v_0. The fact that a mild solution is a classical solution if g is sufficiently regular is a result from Pazy ([20], Corollary 4.2.5).*

Estimates (6.8) and (6.9) are quite standard but we include the proof for the sake of completeness.

Proof of Proposition 6.1. The fact that $v \in C([0,\tau], L^\infty(\mathbb{T}^d))$ is a consequence of the fact that $S^{\lambda,\gamma}$ is a C_0 contraction semigroups on L^∞. For the first estimate (6.8), we have that:

$$\begin{aligned}
|v(t,x)| &\leqslant |S_t^{\lambda,\gamma} v_0(x)| + \left|\int_0^t S_{t-u}^{\lambda,\gamma}(g(u,\cdot))(x)\mathrm{d}u\right| \\
&\leqslant e^{-\lambda t}\|v_0\|_\infty \int_{\mathbb{R}^d} s_0(\gamma t, x, z)\mathrm{d}z + \|g\|_\infty \int_0^s e^{-\lambda(t-u)} \int_{\mathbb{R}^d} s_0(\gamma(t-u), x, z)\mathrm{d}z\mathrm{d}u \\
&= e^{-\lambda t}\|v_0\|_\infty + \frac{1}{\lambda}(1-e^{-\lambda t})\|g\|_\infty.
\end{aligned}$$

For the estimate (6.9), we start by letting $s < t$, we have

$$\begin{aligned}
|v(t,x) - v(s,y)| &\leqslant |S_t^{\lambda,\gamma} v_0(x) - S_s^{\lambda,\gamma} v_0(y)| + \left|\int_0^t S_{t-u}^{\lambda,\gamma}(g(u,\cdot))(x)\mathrm{d}u - \int_0^s S_{s-u}^{\lambda,\gamma}(g(u,\cdot))(y)\mathrm{d}u\right| \\
&\leqslant I_1\|v_0\|_\infty + (I_2 + I_3)\|g\|_\infty
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} \left| e^{-\lambda t} s_0(\gamma t, x, z) - e^{-\lambda s} s_0(\gamma s, y, z) \right| \mathrm{d}z, \\
I_2 &= \int_0^s \int_{\mathbb{R}^d} \left| e^{-\lambda(t-u)} s_0(\gamma(t-u), x, z) - e^{-\lambda(s-u)} s_0(\gamma(s-u), y, z) \right| \mathrm{d}z\mathrm{d}u, \\
I_3 &= \int_s^t \int_{\mathbb{R}^d} e^{-\lambda(t-u)} s_0(\gamma(t-u), x, z)\mathrm{d}z\mathrm{d}u.
\end{aligned}$$

We have that $I_3 \leqslant t - s$. For I_1 and I_2, we use the fact that, for $i = 1 \dots d$

$$\begin{aligned}
\partial_t s_0(t,0,\xi) &= \frac{1}{2}\Delta_\xi s_0(t,0,\xi) = \frac{1}{2}\left(\frac{\|\xi\|^2}{t^2} - \frac{d}{t}\right) s_0(t,0,\xi) \\
\partial_{\xi_i} s_0(t,0,\xi) &= -\frac{\xi_i}{t} s_0(t,0,\xi).
\end{aligned}$$

First for I_1, denote $c_1(r) = rt + (1-r)s$ and $c_2(r) = r(z-x) + (1-r)(z-y)) = z - (rx + (1-r)y)$ for $r \in [0,1]$, We have that

$$\begin{aligned}
|(\partial_t s_0)(\gamma c_1(r), 0, c_2(r))| &\leqslant \frac{1}{2}\left(\frac{\|c_2(r)\|^2}{\gamma^2 c_1(r)^2} + \frac{d}{\gamma c_1(r)}\right) s_0(\gamma c_1(r), 0, c_2(r)) \\
|(\partial_{\xi_i} s_0)(\gamma c_1(r), 0, c_2(r))| &\leqslant \frac{|c_2(r)_i|}{\gamma c_1(r)} s_0(\gamma c_1(r), 0, c_2(r))
\end{aligned}$$

Therefore

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} \left| e^{-\lambda t} s_0(\gamma t, 0, z-x) - e^{-\lambda s} s_0(\gamma s, 0, z-y) \right| \mathrm{d}z \\
&\leqslant \int_{\mathbb{R}^d} \int_0^1 |c_1'(r)\partial_t(e^{-\lambda t} s_0(\gamma t, 0, \xi))_{|t=c_1(r),\xi=c_2(r)}|\mathrm{d}r\mathrm{d}z \\
&\quad + \int_{\mathbb{R}^d} \int_0^1 \sum_{i=1}^d |c_2'(r)_i \partial_{\xi_i}(e^{-\lambda t} s_0(\gamma t, 0, \xi))_{|t=c_1(r),\xi=c_2(r)}|\mathrm{d}r\mathrm{d}z \\
&\leqslant I_{1,1} + I_{1,2},
\end{aligned}$$

where we have, by applying Fubini and a change of variable

$$\begin{aligned}
I_{1,1} &= (t-s)\int_{\mathbb{R}^d}\int_0^1 \left[\lambda e^{-\lambda c_1(r)} + \frac{\gamma}{2}\left(\frac{\|c_2(r)\|^2}{c_1(r)^2} + \frac{d}{c_1(r)}\right)\right] s_0(\gamma c_1(r), 0, c_2(r))\mathrm{d}r\mathrm{d}z \\
&\leqslant (t-s)\int_0^1 \left[\lambda e^{-\lambda c_1(r)} + \frac{\gamma}{2}\left(\frac{d\gamma c_1(r)}{\gamma^2 c_1(r)^2} + \frac{d}{\gamma c_1(r)}\right)\right]\mathrm{d}r \\
&\leqslant (t-s)\int_0^1 \left[\lambda e^{-\lambda c_1(r)} + \frac{d}{c_1(r)}\right]\mathrm{d}r = \int_s^t \left(\lambda e^{-\lambda u} + \frac{d}{u}\right)\mathrm{d}u \\
&= e^{-\lambda s}(1 - e^{-\lambda(t-s)}) + d\log\left(\frac{t}{s}\right) \leqslant \left(\lambda + \frac{d}{s}\right)(t-s).
\end{aligned}$$

For the term $I_{1,2}$, we get, using Cauchy-Schwarz, Fubini and a change of variable

$$\begin{aligned}
I_{1,2} &= \int_{\mathbb{R}^d}\int_0^1 \sum_{i=1}^d |y_i - x_i| e^{-\lambda c_1(r)} \frac{|c_2(r)_i|}{\gamma c_1(r)} s_0(\gamma c_1(r), 0, c_2(r))\mathrm{d}r\mathrm{d}z \\
&\leqslant \|x-y\| \int_0^1 \frac{e^{-\lambda c_1(r)}}{\gamma c_1(r)} \int_{\mathbb{R}^d} \|c_2(r)\| s_0(\gamma c_1(r), 0, c_2(r))\mathrm{d}z\mathrm{d}r = \|x-y\| \int_0^1 \frac{e^{-\lambda c_1(r)}}{\sqrt{\gamma c_1(r)}}\mathrm{d}r C_1 \\
&\leqslant C_1\|x-y\| \int_s^t \frac{e^{-\lambda u}}{\sqrt{\gamma u}} \frac{\mathrm{d}u}{t-s} \leqslant \frac{C_1}{\sqrt{\gamma s}}\|x-y\|
\end{aligned}$$

where C_1 is the expectation of the quadratic norm of $X = (X_1, X_2, \dots X_d)$ of d independent standard normal variables: $C_1 = \mathbb{E}(\|X\|) \leqslant \mathbb{E}(\|X\|^2)^{1/2} = \sqrt{d}$ (we also have $C_1 = \sqrt{2}\frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \sim \sqrt{d}$). We get that, for $s \geqslant \frac{1}{T}$,

$$I_1 \leqslant \left(\lambda + \frac{d}{s}\right)(t-s) + \sqrt{\frac{d}{\gamma s}}\|x-y\| \leqslant (\lambda + d\tau)(t-s) + \sqrt{\frac{d\tau}{\gamma}}\|x-y\|.$$

For I_2, we make the same computations with $c_1(r) = r(t-u) + (1-r)(s-u) = rt + (1-r)s - u$ where $u \in [0,s]$, we have, since $c_1'(r) = t - s$,

$$I_2 \leqslant \int_0^s \int_{s-u}^{t-u} \left(\lambda e^{-\lambda v} + \frac{d}{v}\right)\mathrm{d}v\mathrm{d}u + C_1\|x-y\| \int_0^s \int_{s-u}^{t-u} \frac{e^{-\lambda v}}{\sqrt{\gamma v}} \frac{\mathrm{d}v}{t-s}\mathrm{d}u$$

For the first integral, we have

$$\begin{aligned}
\int_0^s \int_{s-u}^{t-u} \left(\lambda e^{-\lambda v} + \frac{d}{v}\right)\mathrm{d}v\mathrm{d}u &= (1 - e^{-\lambda(t-s)}) \int_0^s e^{-\lambda(s-u)}\mathrm{d}u + d\int_0^s \log\left(\frac{t-u}{s-u}\right)\mathrm{d}u \\
&= \frac{1}{\lambda}(1 - e^{-\lambda(t-s)})(1 - e^{-\lambda s}) + d\left[t\log(t) - s\log(s) - (t-s)\log(t-s)\right] \\
&\leqslant \lambda s(t-s) + d\left[s(\log(t) - \log(s)) + (t-s)(\log(t) - \log(t-s))\right] \\
&\leqslant \lambda s(t-s) + d(t-s) + d(t-s)(|\log(t)| + |\log(t-s))|) \\
&\leqslant (\lambda s + d + |\log(t)|)(t-s) + d(t-s)|\log(t-s)|.
\end{aligned}$$

For the second integral, we get

$$\int_0^s \int_{s-u}^{t-u} \frac{e^{-\lambda v}}{\sqrt{\gamma v}}\mathrm{d}v\mathrm{d}u \leqslant (t-s)\int_0^s \frac{1}{\sqrt{\gamma(s-u)}}\mathrm{d}u = (t-s)\sqrt{\frac{2s}{\gamma}}.$$

Then we obtain, for $1/\tau \leqslant s < t \leqslant \tau$

$$I_2 \leqslant (\lambda s + d + |\log(t)|)(t-s) + d(t-s)|\log(t-s)| + C_1\|x-y\|\sqrt{\frac{2s}{\gamma}}$$
$$\leqslant (\lambda\tau + d + |\log(\tau)|)(t-s) + d(t-s)|\log(t-s)| + \sqrt{\frac{2d\tau}{\gamma}}\|x-y\|.$$

At last, we get the following estimate

$$|v(t,x) - v(s,y)| \leqslant \left[(\lambda + d\tau)(t-s) + \sqrt{\frac{d\tau}{\gamma}}\|x-y\|\right]\|v_0\|_\infty$$
$$+ \left[(\lambda\tau + |\log(\tau)| + d + 1)(t-s) + d(t-s)|\log(t-s)| + \sqrt{\frac{2d\tau}{\gamma}}\|x-y\|\right]\|g\|_\infty.$$

□

We modify a little our equation (6.1), both in order to obtain a sharper estimate on the uniform norm of the solution and to get a coherent notation with the solution of the limit equation when $K \to +\infty$.

We define for $q \in [-1,1]$,

$$r_K(q) := -\int_{\frac{1}{2}}^{\frac{1}{2}(q+1)} 4g_K(s)ds. \tag{6.10}$$

We have that

$$r'_K(q) = -2g_K\left(\frac{1}{2}(q+1)\right) = -(1-q)\mathbb{P}_{\frac{1-q}{2}}\big[X < \kappa(K,T)\big] + (1+q)\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big]. \tag{6.11}$$

So we let h_K be

$$h_K(q) = \begin{cases} -1 \text{ for } q < -1 \\ -r'_K(q) + q \text{ for } q \in [-1,1] \\ 1 \text{ for } q > 1 \end{cases} \tag{6.12}$$

Since $r'_K(1) = 0$ and $r'_K(-1) = 0$, h_K is continuous on $\mathbb{R}$. We have that, for $q \in [-1,1]$,

$$h_K(q) = -r'_K(q) + q = (1-q)\mathbb{P}_{\frac{1-q}{2}}\big[X < \kappa(K,T)\big] - (1+q)\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + q.$$

We now solve the following equation

$$\partial_t v(t,x) - 2\alpha\Delta v(t,x) + (2\beta+1)v = h_K(v(t,x)). \tag{6.13}$$

Note that, this equation and (6.3) are exactly the same with the term v added on both sides if $\|v\|_\infty \leqslant 1$, since for $q \notin [-1,1]$, $h_K(q) \neq q$. Thus, a solution v of (6.3) with $\|v\|_\infty \leqslant 1$ will also be a solution of (6.13) and reciprocally.

Proposition 6.2. *For $v_0 \in L^\infty(\mathbb{T}^d)$ with $\|v_0\|_\infty \leqslant 1$, there exists a unique solution $(v(t,x), t \geqslant 0, x \in \mathbb{T}^d)$ to the problem*

- *v is continuous from $\mathbb{R}_+^*$ to $L^\infty(\mathbb{T}^d)$*
- *v satisfies, for all $t > 0$ and $x \in \mathbb{T}^d$*

$$v(t,x) = S_t^{2\beta+1,4\alpha}v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}[h_K(v(s,\cdot))](x)\mathrm{d}s. \tag{6.14}$$

We say that v is a mild solution to (6.13)*. We have also that $\|v\|_\infty \leqslant 1$ and v satisfies*

$$v(t,x) = S_t^{2\beta,4\alpha} v_0(x) - \int_0^t S_{t-s}^{2\beta,4\alpha}[r'_K(v(s,\cdot))](x)\mathrm{d}s \tag{6.15}$$

and thus is a mild solution of (6.3)*.*

Proof. The first part of the proposition comes from a fixed point argument (see also Pazy [20], Theorem 6.1.2) applied to the following functional let $\tau > 0$ and define the functional $F : C(]0,\tau], L^\infty(\mathbb{T}^d)) \to C(]0,\tau], L^\infty(\mathbb{T}^d))$ defined by

$$F(v)(t,x) := S_t^{2\beta+1,4\alpha} v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}[h_K(v(s,\cdot))](x)\mathrm{d}s. \tag{6.16}$$

We equip $C(]0,\tau], L^\infty(\mathbb{T}^d))$ with the uniform topology on all compact subset. We can apply the Banach fixed point Theorem to F (see the proof of Pazy [20], Theorem 6.1.2. Moreover, the mapping $v_0 \mapsto v$ is Lipschitz continuous from L^∞ to $C(]0,\tau], L^\infty(\mathbb{T}^d))$.

An application of Proposition 6.1 proves that $\|v\|_\infty \leqslant 1$.

Since h_K is differentiable, then, if $v_0 \in C^2(\mathbb{T}^d)$, v_0 is in the domain of Δ and thus v is classical solution of (6.13) (Theorem 6.1.5 [20]). Thus v is a classical solution of (6.3) since $\|v\|_\infty \leqslant 1$ and a mild solution of (6.3). Now consider an approximating sequence $(v_{0,n})$ in $C^2(\mathbb{T}^d)$ of $v_0 \in L^\infty$, and (v_n) the sequence of mild solutions with initial value $v_{0,n}$ and v the mild solution of (6.13) with initial value v. Then, since $v_0 \mapsto v$ is Lipschitz continuous, by the dominated convergence theorem, we get that, uniformly on $[t_0,\tau] \times \mathbb{T}^d$ for all $t_0 > 0$, the right hand side of

$$v_n(t,x) = S_t^{2\beta,4\alpha} v_{0,n}(x) - \int_0^t S_{t-s}^{2\beta,4\alpha}[r'_K(v_n(s,\cdot))](x)\mathrm{d}s \tag{6.17}$$

converges to $S_t^{2\beta,4\alpha} v_0(x) - \int_0^t S_{t-s}^{2\beta,4\alpha}[r'_K(v(s,\cdot))](x)\mathrm{d}s$, whereas the left hand side converges to v. So we obtain that v is a mild solution of (6.3). □

6.2 Solution of (6.2)

6.2.1 Existence of a solution

We use the same transform as before, and let h_∞ be the pointwise limit of h_K. The equation (6.2) is now formally

$$\partial_t v(t,x) - 2\alpha\Delta v(t,x) + (2\beta+1)v = h_\infty(v(t,x)). \tag{6.18}$$

For the limiting equation, we prove first that the family $(v_K)_K$ of solutions associated to h_K with common initial value $v_0 \in L^\infty(\mathbb{T}^d)$ in $C(\mathbb{R}_*^+ \times \mathbb{T})$ is compact (with uniform norm on all compact subset). Then, by taking the limit, any accumulation point v_∞ of the sequence satisfy the mild formulation of the limiting equation relaxed as a subdifferential inclusion.

In order to prove this, we set some notations:

$$r_\infty(q) := -\int_{\frac{1}{2}}^{\frac{1}{2}(q+1)} 4g_\infty(s)\mathrm{d}s \tag{6.19}$$

h_∞, the pointwise limit of h_K, is the function on $\mathbb{R}$

$$h_\infty(q) = 2g_\infty\left(\frac{1+q}{2}\right) + q = -\mathbb{1}_{q\leqslant -2\rho} + q\mathbb{1}_{-2\rho<q\leqslant 2\rho} + \mathbb{1}_{q>2\rho}. \tag{6.20}$$

Then h_∞ is non-decreasing and is the left-derivative of the convex function H_∞

$$\begin{aligned} H_\infty(q) &:= -r_\infty(q) + \frac{q^2}{2} \\ &= [-q + 2\rho + 2\rho^2]\mathbb{1}_{\{q \leqslant -2\rho\}} + \frac{1}{2}q^2\mathbb{1}_{\{-2\rho < q \leqslant 2\rho\}} + [q - 2\rho + 2\rho^2]\mathbb{1}_{\{2\rho < q\}} \end{aligned} \tag{6.21}$$

The subdifferential of H_∞ at q is defined as

$$\partial H_\infty(q) = \{p \in \mathbb{R}, H_\infty(q') - H_\infty(q) \geqslant p(q' - q), \text{ for all } q' \in [-1, 1]\}.$$

In particular we have,

$$\partial H_\infty(q) = \begin{cases} \{-1\} & \text{for } q < -2\rho, \\ [-1, -2\rho] & \text{for } q = -2\rho, \\ \{q\} & \text{for } -2\rho < q < 2\rho, \\ [2\rho, 1] & \text{for } q = 2\rho, \\ \{1\} & \text{for } q > 2\rho. \end{cases} \tag{6.22}$$

We adopt the following definition for a solution of the equation:

$$\partial_t v(t, x) - 2\alpha\Delta v(t, x) + (2\beta + 1)v \in \partial H_\infty(v(t, x)). \tag{6.23}$$

Definition 6.1. *We say that v is a mild solution of Equation* (6.23) *if it satisfies, for some $\tau > 0$ and all $t \leqslant \tau$*

$$v(t, x) = S_t^{2\beta+1,4\alpha} v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}[w(s, \cdot)](x)\mathrm{d}s. \tag{6.24}$$

where $w \in L^2([0, \tau] \times \mathbb{T}^d)$, with $w(t, x) \in \partial H_\infty(v(t, x))$ almost everywhere.

Proposition 6.3. *For $v_0 \in L^\infty(\mathbb{T}^d)$, any accumulation point (in $C(\mathbb{R}_+^* \times \mathbb{T}^d)$ equipped with uniform norm on each compact set), of the sequence (v_K) of solutions given by Proposition 6.2 is a mild solution of* (6.23).

As a consequence of the proposition, there exists $(v(t, x), t \geqslant 0, x \in \mathbb{T}^d)$ a mild solution of (6.23) such that v is continuous from $\mathbb{R}_+^*$ to $L^\infty(\mathbb{T}^d)$, and $\|v\|_\infty \leqslant 1$.

The existence of a solution for a given initial condition v_0 is not difficult and can be prove in different ways. Here we adopt some kind of regularization procedure, since we have a natural family of differentiable functions (namely the (h_K)) approximating h_∞ and we use the convergence of the sequence (v_K) in the next section to prove the convergence of the stochastic process. We also present the proof because we need its arguments in order to prove the Theorem 7.1.

In the Remark 6.3, we present another construction of solution(s) using the monotonicity of h_∞ which is interesting since it also gives an insight on the problem of non-uniqueness.

Proof. For each K, we have a mild solution v_K from Proposition 6.2. From Proposition 6.1, we have that each solution is uniformly bounded, uniformly continuous on $[1/\tau, \tau] \times \mathbb{T}^d$, and the modulus of continuity only depends on $\tau > 1$ (since the others parameters are fixed).

Therefore, by the Arzela-Ascoli Theorem, the sequence $(v_K)_K$ is compact on $C([1/\tau, \tau] \times \mathbb{R}^d)$ and we can extract a subsequence converging uniformly in $C([1/\tau, \tau] \times \mathbb{R}^d)$, and then by a diagonal argument, a sequence converging to a limit v_∞ in $C(]0, \infty[\times \mathbb{R}^d)$, uniformly on each compact. Note that since $\|v_K\| \leqslant 1$ for all K, we also have $\|v_\infty\| \leqslant 1$. We show that v_∞ satisfies (6.18).

Let us assume that $\rho > 0$. Denote for any $s > 0$, $\mathcal{A}_\rho(s) := \{y \in \mathbb{R}^d \colon |v_\infty(s, y) \pm 2\rho| = 0\}$. Note that since h_∞ is uniformly continuous on $[-1, -2\rho[$, $]-2\rho, 2\rho[$ and $]2\rho, 1]$, we have that for any

$y \notin \mathcal{A}_\rho(s)$, $\lim_{K\to\infty} h_K(v_K(s,y)) = h_\infty(v_\infty(s,y))$. Let $y \in \mathcal{A}_\rho(s)$ and assume $v_\infty(s,y) = 2\rho$ without loss of generality, since $v_K(s,y)$ converges to $v_\infty(s,y)$, for all $\varepsilon > 0$ such that $2\rho - \varepsilon > 0$, there exists K_0 such that for all $K \geqslant K_0$, $2\rho - \varepsilon < v_K(s,y) < 2\rho + \varepsilon$. Thus, using Lemma B.5, we have that, for all $K \geqslant K_0$, $2\rho - 2\varepsilon \leqslant h_K(v_K(s,y)) \leqslant 1$. Then taking the limits in K and $\varepsilon \to 0$, we get

$$h_\infty(2\rho) = 2\rho \leqslant \liminf_{K\to\infty} h_K(v_K(s,y)) \leqslant \limsup_{K\to\infty} h_K(v_K(s,y)) \leqslant 1 = h_\infty(2\rho^+). \tag{6.25}$$

For $\rho = 0$, we have the same inequality since then $h_\infty(0) = -1$ and $h_\infty(0+) = 1$.

Let $w_+(s,y) = \limsup_{K\to\infty} h_K(v_K(s,y))$ and $w_-(s,y) = \liminf_{K\to\infty} h_K(v_K(s,y))$, thus we have that for all $(s,y) \in]0,+\infty[\times\mathbb{T}^d$:

$$h_\infty(v_\infty(s,y)) \leqslant w_-(s,y) \leqslant w_+(s,y) \leqslant h_\infty(v_\infty(s,y)^+). \tag{6.26}$$

Since $h_K(v_K)$ is bounded, by the Banach Alaoglu Theorem, the sequence is weakly compact in $L^2(]0,\tau[\times\mathbb{T}^d)$, and we have a subsequence of $(h_K(v_K))_K$ converging weakly to $w \in L^2_{loc}(]0,+\infty[\times\mathbb{T}^d)$. Since the density of the semigroup $S^{2\beta+1,4\alpha}$ is in $L^2(]0,\tau]\times\mathbb{R}^d)$ for all $T > 0$, we have as $K \to +\infty$,

$$v_\infty(t,x) = S_t^{2\beta+1,4\alpha} v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha} w(s,\cdot)(x)\mathrm{d}s. \tag{6.27}$$

Moreover, w_- and w_+ are bounded and therefore in $L^2(]0,\tau]\times\mathbb{T}^d)$. Let $\varphi \in L^2(]0,\tau]\times\mathbb{T}^d)$ and $\varphi \geqslant 0$, by the Fatou Lemma we get

$$\begin{aligned} 0 &= \int_{]0,\tau]\times\mathbb{T}^d} \liminf_{K\to+\infty} (h_K(v_K) - w_-)\varphi \\ &\leqslant \liminf_{K\to+\infty} \int_{]0,\tau]\times\mathbb{T}^d} (h_K(v_K) - w_-)\varphi = \int_{]0,\tau]\times\mathbb{T}^d} (w - w_-)\varphi. \end{aligned} \tag{6.28}$$

We also have

$$\begin{aligned} 0 &= \int_{]0,\tau]\times\mathbb{T}^d} \liminf_{K\to+\infty} (w_+ - h_K(v_K))\varphi \\ &\leqslant \liminf_{K\to+\infty} \int_{]0,\tau]\times\mathbb{T}^d} (w_+ - h_K(v_K))\varphi = \int_{]0,\tau]\times\mathbb{T}^d} (w_+ - w)\varphi. \end{aligned} \tag{6.29}$$

Thus, almost everywhere on $]0,+\infty[\times\mathbb{T}^d$, we have that

$$h_\infty(v_\infty) \leqslant w_- \leqslant w \leqslant w_+ \leqslant h_\infty(v_\infty^+). \tag{6.30}$$

Therefore, $w \in \partial H_\infty(v_\infty)$ a.e. $\square$

6.2.2 Uniqueness of solution of (6.23)

The main problem concerns the uniqueness of a solution. We prove first that, we do not have uniqueness for a constant initial condition $v_0(x) = 2\rho$ when $2\rho < \frac{1}{1+2\beta}$, so we are in the case of segregation or metastable segragation described by Figure 1.

Remark 6.2. *We describe three possible solutions starting from the initial condition $v_0(x) = 2\rho$ when $2\rho < \frac{1}{1+2\beta}$.*

Note that

$$S_t^{2\beta+1,4\alpha} v_0(x) = \int_{\mathbb{R}^d} e^{-(2\beta+1)t} s_0(t,x,y) 2\rho \mathrm{d}y = 2\rho e^{-(2\beta+1)t}. \tag{6.31}$$

Suppose that v does not depend on x, $v(t,x) = c(t)$ for all $x \in \mathbb{T}^d$, we have

$$\int_0^t S_{t-s}^{2\beta+1,4\alpha}[h_\infty(v(s,\cdot))](x)\mathrm{d}s = \int_0^t h_\infty(c(s))e^{-(2\beta+1)(t-s)}\mathrm{d}s. \tag{6.32}$$

Let us consider the functions

$$v^1(t,x) = c^1(t) = 2\rho e^{-2\beta t} \tag{6.33}$$

$$\begin{aligned} v^2(t,x) = c^2(t) &= 2\rho e^{-(2\beta+1)t} + \frac{1}{1+2\beta}\left(1 - e^{-(2\beta+1)t}\right) \\ &= 2\rho + \left(\frac{1}{1+2\beta} - 2\rho\right)\left(1 - e^{-(2\beta+1)t}\right). \end{aligned} \tag{6.34}$$

Since, $c^1(t) \in [0, 2\rho[$, for $t > 0$, we have $h_\infty(c^1(t)) = c^1(t)$ and then

$$\int_0^t h_\infty(c^1(s))e^{-(2\beta+1)(t-s)}\mathrm{d}s = \int_0^t 2\rho e^{-2\beta s}e^{-(2\beta+1)(t-s)}\mathrm{d}s = 2\rho e^{-(2\beta+1)t}(e^t - 1). \tag{6.35}$$

Therefore

$$\begin{aligned} S_t^{2\beta+1,4\alpha}v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}[h_\infty(v^1(s,\cdot))](x)\mathrm{d}s &= 2\rho e^{-(2\beta+1)t} + 2\rho e^{-(2\beta+1)t}(e^t - 1) \\ &= 2\rho e^{-2\beta t} = v^1(t,x). \end{aligned} \tag{6.36}$$

Since $2\rho < \frac{1}{1+2\beta}$, $c^2(t) \in]2\rho, 1]$, for $t > 0$, we have $h_\infty(c^2(t)) = 1$ and then by the same computation,

$$\int_0^t h_\infty(c^2(s))e^{-(2\beta+1)(t-s)}\mathrm{d}s = \frac{1}{1+2\beta}(1 - e^{-(2\beta+1)t}). \tag{6.37}$$

Therefore, we also have

$$S_t^{2\beta+1,4\alpha}v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}[h_\infty(v^2(s,\cdot))](x)\mathrm{d}s = v^2(t,x). \tag{6.38}$$

Thus, both v^1 and v^2 are mild solutions to (6.18) and thus to (6.23) with the same initial conditions. Note that at $t = 0$, we have $v^1(0,x) = v^2(0,x) = 2\rho$ and

$$\partial_t v^1(0,x) = -4\beta\rho = -(2\beta+1)2\rho + 2\rho = -(2\beta+1)2\rho + h_\infty(2\rho^-) \tag{6.39}$$

$$\partial_t v^2(0,x) = -2\rho(2\beta+1) + 1 = -(2\beta+1)2\rho + h_\infty(2\rho^+). \tag{6.40}$$

We see that non uniqueness comes from the fact that at $t = 0$, where $v(t,x) = 2\rho$, we have at least two choices for the derivative due to the fact that h_∞ is not continuous.

Note that if we consider the mild solution to the subdiffrential inclusion (6.23), then we have at least a third solution: $v^3(t,x) = 2\rho$. We consider $w(t,x) = 2\rho(2\beta+1)$, we have

$$\begin{aligned} S_t^{2\beta+1,4\alpha}v_0(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}[w(s,\cdot)](x)\mathrm{d}s &= 2\rho e^{-(2\beta+1)t} + 2\rho(2\beta+1)\int_0^t e^{-(2\beta+1)(t-s)}\mathrm{d}s \\ &= 2\rho. \end{aligned} \tag{6.41}$$

Since, $2\rho < \frac{1}{1+2\beta}$, we have that $2\rho < 2\rho(2\beta+1) < 1$ so $w(t,x) \in \partial H_\infty(2\rho)$.

Therefore, we cannot expect uniqueness for all initial condition, we have to impose some condition on the initial condition if we want a unique solution.

In the literature, we can find different conditions ensuring that the solution of Equation (6.23) is unique. Adapting [10] and [4], we prove that the regularity of the initial condition at the levels where the non-linearity h_∞ is not continuous is sufficient.

Definition 6.2. *A function $v_0 : \mathbb{T}^d \to [-1,1]$ in $C^1(\mathbb{T}^d)$ is* regular at level $q \in]-1,1[$ *if for all $x \in \mathbb{T}^d$, such that $v_0(x) = p$, we have $\nabla v_0(x) \neq 0$.*

Proposition 6.4. *For $v_0 \in C^1(\mathbb{T}^d)$, such that ∇v_0 is Lipschitz on $\mathbb{T}^d$ and regular at levels 2ρ and -2ρ, the solution v to Equation (6.23) is locally unique. Moreover, the Lebesgue measure of the set $\mathcal{A}_\rho(s) := \{y \in \mathbb{R}^d \colon |v(s,y) \pm 2\rho| = 0\}$ is zero.*

We adapt two arguments by [10] and [4].

Lemma 6.5. *If v is a mild solution of (6.23) with $v_0 \in C^1(\mathbb{T}^d)$, and such that ∇v_0 is Lipschitz on $\mathbb{T}^d$, then, for all $\tau > 0$, there exists a constant $C > 0$ such that, for all $t \in [0,\tau]$*

$$\|v(t) - v_0\|_\infty \leqslant Ct^{1/2}, \qquad \|\nabla v(t) - \nabla v_0\|_\infty \leqslant Ct^{1/2}. \tag{6.42}$$

Proof. Since v is a mild solution of (6.23) there exists $w \in L^\infty_{loc}([0+\infty[\times \mathbb{T}^d)$ with $\|w\|_\infty \leqslant 1$ since $w \in \partial H_\infty(v)$. Thus we get

$$v(t) - v_0 = S_t^{2\beta+1,4\alpha} v_0 - v_0 + \int_0^t S_{t-s}^{2\beta+1,4\alpha} w(s)\mathrm{d}s. \tag{6.43}$$

Then, we have for the last integral

$$\left\| \int_0^t S_{t-s}^{2\beta+1,4\alpha} w(s)\mathrm{d}s \right\|_\infty \leqslant \int_0^t e^{-(2\beta+1)(t-s)}\mathrm{d}s \leqslant t.$$

We have also

$$\begin{aligned} |S_t^{2\beta+1,4\alpha} v_0(x) - v_0(x)| &\leqslant e^{-(2\beta+1)t} \int_{\mathbb{T}^d} s(4\alpha t, x, y)|v_0(y) - v_0(x)|\mathrm{d}y \\ &\leqslant Le^{-(2\beta+1)t} \int_{\mathbb{R}^d} s_0(4\alpha t, x, y)\|y - x\|_2 \mathrm{d}y \\ &\leqslant Le^{-(2\beta+1)t}\sqrt{4\alpha t d} \leqslant L\sqrt{4\alpha d}t^{1/2} \end{aligned}$$

where L is the Lipschitz constant of v_0 and the third inequality comes from the computation of the upper bound of the quadratic norm of d independent random variables with common variance $4\alpha t$. Therefore we obtain

$$\|v(t) - v_0\|_\infty \leqslant (L\sqrt{4\alpha d} + \sqrt{\tau})t^{1/2} \tag{6.44}$$

For the second bound, we use the fact that $\frac{\mathrm{d}}{\mathrm{d}x_i} s_0(t,x,y) = -\frac{\mathrm{d}}{\mathrm{d}y_i} s_0(t,x,y)$, thus we have, using an integration by parts

$$\partial_{x_i} S_t^{2\beta+1,4\alpha}(v_0) = S_t^{2\beta+1,4\alpha}(\partial_{x_i} v_0). \tag{6.45}$$

Then, we have

$$\begin{aligned} \partial_{x_i} v(t) - \partial_{x_i} v_0 = S_t^{2\beta+1,4\alpha}&(\partial_{x_i} v_0) - (\partial_{x_i} v_0) \\ &+ \int_0^t \int_{\mathbb{R}^d} e^{-(2\beta+1)(t-s)} \partial_{x_i} s_0(4\alpha(t-s), x, y) w(s,y)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{6.46}$$

We can treat both integrals as before, for the last integral:

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}^d} e^{-(2\beta+1)(t-s)} \partial_{x_i} s_0(4\alpha(t-s), x, y) w(s,y) \mathrm{d}y \mathrm{d}s\right| \\
&\quad \leqslant \int_0^t \frac{e^{-(2\beta+1)(t-s)}}{4\alpha(t-s)} \int_{\mathbb{R}^d} |x_i - y_i| s_0(4\alpha(t-s), x, y) \mathrm{d}y \mathrm{d}s \\
&\quad \leqslant \int_0^t \frac{e^{-(2\beta+1)(t-s)}}{4\alpha(t-s)} \sqrt{2\alpha(t-s)} \mathrm{d}s \\
&\quad \leqslant \frac{1}{\sqrt{2\alpha}} \int_0^{\sqrt{t}} e^{-(2\beta+1)u^2} \mathrm{d}u \leqslant \frac{\sqrt{t}}{\sqrt{2\alpha}}.
\end{aligned}$$

For the first integral, we have the same estimates as before

$$|S_t^{2\beta+1,4\alpha}(\partial_{x_i} v_0)(x) - \partial_{x_i} v_0(x)| \leqslant L' \sqrt{4\alpha d} t^{1/2} \tag{6.47}$$

where L' is the maximum of the Lipschitz constants of $(\partial_{x_i} v_0)_i$. Thus, we get

$$\|\nabla v(t) - \nabla v_0\|_\infty \leqslant \left(L'\sqrt{4\alpha d} + \frac{1}{\sqrt{2\alpha}}\right) t^{1/2} \tag{6.48}$$

□

We now prove Proposition 6.4.

Proof. Let us assume that we have two solutions, v_1 and v_2 and let $e(t) = \|v_1 - v_2\|_{L^\infty([0,t]\times\mathbb{R}^d)}$. Note that the previous Lemma entails that, for all $\tau > 0$, there exists C, such that for $t < \tau$, we have $e(t) \leqslant C\sqrt{t}$.

We define $I_{s,t}^+ = \{(s,y), s \leqslant t, |v_1(s,y) - 2\rho| \leqslant e(t)\}$ and $I_{s,t}^- = \{(s,y), s \leqslant t, |v_1(s,y) + 2\rho| \leqslant e(t)\}$.

Since v_1 and v_2 are solutions of (6.23), there exists w_1 and w_2 such that $w_1 \in \partial H_\infty(v_1)$ a.e. and $w_2 \in \partial H_\infty(v_2)$. We can decompose each w_i as $w_i(t,x) = f_\infty(v_i(t,x)) + g_i(t,x)$ where f_∞ is the continuous part of ∂H_∞:

$$f_\infty(q) = \begin{cases} -2\rho \text{ for } q \in [-1, -2\rho] \\ q \text{ for } q \in [-2\rho, 2\rho] \\ 2\rho \text{ for } q \in [2\rho, 1] \end{cases} \tag{6.49}$$

and $g_i(t,x) = w_i(t,x) - f_\infty(v_i(t,x))$. Note that

$$\begin{aligned}
g_i(t,x) &= -1 + 2\rho \text{ a.e. on } \{(t,x), v_i(t,x) < -2\rho\}, \\
g_i(t,x) &= 0 \text{ a.e. on } \{(t,x), -2\rho < v_i(t,x) < 2\rho\}, \\
g_i(t,x) &= 1 - 2\rho \text{ a.e. on } \{(t,x), v_i(t,x) > 2\rho\},
\end{aligned} \tag{6.50}$$

since $w_i = h_\infty(v_i)$ a.e. on $\{(t,x), |v_i(t,x)| = 2\rho\}$.

As a consequence we have that, up to a negligible set, $\{(s,y), s \leqslant t, g_1(s,y) \neq g_2(s,y)\} \subset I_{s,t}^+ \cup I_{s,t}^-$ since, $g_1(s,y) \neq g_2(s,y)$ entails that one of the following inequalities is true $v_2(s,y) < 2\rho < v_1(s,y)$ or $v_2(s,y) < -2\rho < v_1(s,y)$ or $v_1(s,y) < 2\rho < v_2(s,y)$ or $v_1(s,y) < -2\rho < v_2(s,y)$. For each case, the inclusion is true: for the first one for example, if $s \leqslant t$ and $v_2(s,y) < 2\rho < v_1(s,y)$

$$e(t) \geqslant v_1(s,y) - v_2(s,y) = v_1(s,y) - 2\rho + 2\rho - v_2(s,y) \geqslant v_1(s,y) - 2\rho \geqslant 0 \tag{6.51}$$

thus $(s,y) \in I_{s,t}^+$. The same is true for the other cases.

Therefore we obtain the following expression for the difference $v_1 - v_2$:

$$\begin{aligned} v_1(t,x) - v_2(t,x) &= \int_0^t \int_{\mathbb{T}^d} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)(f_\infty(v_1(s,y)) - f_\infty(v_2(s,y)))\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_{I^+_{s,t}\cup I^-_{s,t}} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)(g_1(s,y) - g_2(s,y))\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{6.52}$$

For the first integral in (6.52) we note that f_∞ is 1-Lipschitz, thus

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{T}^d} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)(f_\infty(v_1(s,y)) - f_\infty(v_2(s,y)))\mathrm{d}y\mathrm{d}s \right| \\ &\leqslant \int_0^t \int_{\mathbb{T}^d} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)e(t)\mathrm{d}y\mathrm{d}s \leqslant te(t). \end{aligned}$$

For the second integral in (6.52) we note that $|g_i| \leqslant 1 - 2\rho$, then we first have

$$\begin{aligned} &\left| \int_0^t \int_{I^+_{s,t}\cup I^-_{s,t}} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)(g_1(s,y) - g_2(s,y))\mathrm{d}y\mathrm{d}s \right| \\ &\leqslant 2(1-2\rho) \int_0^t \int_{I^+_{s,t}\cup I^-_{s,t}} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)\mathrm{d}y\mathrm{d}s. \end{aligned}$$

Let $s \leqslant t$, since v_0 is regular on the level set $\{v_0 = 2\rho\}$ which is compact (since $\mathbb{T}^d$ is) and ∇v_0 is a Lipschitz function, we can find $\delta, \eta > 0$ such that on $\{v_0 = 2\rho\} + B_\delta(0)$, $|\nabla v_0(x)| > \eta$. Using the second part of Lemma 6.5, and since $e(t) \leqslant C\sqrt{t}$, there exists $T > 0$ such that for $s \leqslant t \leqslant T$, $I^+_{s,t} \subset \{v_0 = 2\rho\} + B_\delta(0)$ and on $I^+_{s,t}$, $|\nabla v_1(s)| > \eta/2$.

Since $I^+_{s,t}$ is compact and $\nabla v_1(s) \neq 0$, by the implicit function theorem, we can find a finite cover by open balls $(B_i)_{1\leqslant i\leqslant N}$ centered on points on $I^+_{s,t}$ such that locally on each ball B_i, the level set $\{v_1(s,y) = 2\rho\}$ is the graph of a function, e.g $y_1 = \varphi(y_2, \dots y_d)$. Note that since $\{v_0 = 2\rho\}$ is compact, N is uniform in $s \leqslant T$, since by the lemma, we can make the cover of open balls on $\{v_0 = 2\rho\}$ and take their traces on $\{v_1(s,y) = 2\rho\}$.

By the mean value theorem on the first coordinate y_1 of $v_1(s)$, we have $I^+_{s,t}\cap B_i \subset [-2e(t)/\nu, 2e(t)/\nu] \times \Pi_1(B_i)$, where Π_1 is the projection along the first coordinate. Thus,

$$\begin{aligned} &\int_0^t \int_{I^+_{s,t}\cap B_i} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)\mathrm{d}y\mathrm{d}s \\ &\leqslant \int_0^t e^{-(2\beta+1)(t-s)} \frac{4e(t)}{\nu\sqrt{4\alpha(t-s)}} \int_{\Pi_1(B_i)} s(4\alpha(t-s),0,(0,y_2,\dots y_d))\mathrm{d}y_2 \cdots \mathrm{d}y_d\mathrm{d}s \\ &\leqslant \int_0^t e^{-(2\beta+1)(t-s)} \frac{2e(t)}{\nu\sqrt{\alpha(t-s)}}\mathrm{d}s \leqslant \frac{2e(t)t^{1/2}}{\nu\sqrt{\alpha}}. \end{aligned}$$

Thus

$$\int_0^t \int_{I^+_{s,t}} e^{-(2\beta+1)(t-s)} s(4\alpha(t-s),x,y)\mathrm{d}y\mathrm{d}s \leqslant \frac{2Ne(t)t^{1/2}}{\nu\sqrt{\alpha}}. \tag{6.53}$$

Since the same holds for $I^-_{s,t}$, we obtain, that for some constant $C > 0$, and all $t < \tau$

$$|v_1(t,x) - v_2(t,x)| \leqslant (t + Ct^{1/2})e(t) \leqslant (\tau + C\tau^{1/2})e(\tau). \tag{6.54}$$

Then $e(\tau) \leqslant (\tau + C\tau^{1/2})e(\tau)$, and taking τ small enough, we obtain $e(\tau) = 0$ thus $v_1 = v_2$ on $[0,\tau] \times \mathbb{T}^d$. □

Remark 6.3. ***Maximal and minimal solutions.*** *Another approach to (6.2) is to use a monotone construction of solutions, which arises from a comparison principle close to the one developed in Proposition 5.1. This was done initially in [2] and also in [4] Define $\overline{h_\infty}$ the right continuous version of h_∞ (Equation 6.20) by*

$$\overline{h_\infty}(q) := -\mathbb{1}_{q<-2\rho} + q\mathbb{1}_{-2\rho\leqslant q<2\rho} + \mathbb{1}_{q\geqslant 2\rho}. \tag{6.55}$$

Note that h_∞ and $\overline{h_\infty}$ are non decreasing (recall that $2\rho \in [0,1]$). Recall that (S_t) is the semigroup on $L^1(\mathbb{T}^d)$ associated to $-2\alpha\Delta$, we denote

$$\underline{F}(v)(t,x) := e^{-(2\beta+1)t}S_t v_0(x) + \int_0^t e^{-(2\beta+1)(t-s)}S_{t-s}[h_\infty(v(s,\cdot))](x)\mathrm{d}s \tag{6.56}$$

$$\overline{F}(v)(t,x) := e^{-(2\beta+1)t}S_t v_0(x) + \int_0^t e^{-(2\beta+1)(t-s)}S_{t-s}[\overline{h_\infty}(v(s,\cdot))](x)\mathrm{d}s. \tag{6.57}$$

Then, fixed points of the maps above are mild solutions of these two formulations of our subdifferential inclusion:

$$\partial_t v(t,x) - 2\alpha\Delta v(t,x) + (2\beta+1)v(t,x) = h_\infty(v(t,x)) \tag{6.58}$$

$$\partial_t v(t,x) - 2\alpha\Delta v(t,x) + (2\beta+1)v(t,x) = \overline{h_\infty}(v(t,x)) \tag{6.59}$$

Since h_∞ (resp. $\overline{h_\infty}$) is non decreasing and that $h_\infty(p) \leqslant \overline{h_\infty}(p)$ for all $p \in [-1,1]$, we have that, for u, v two functions such that $-1 \leqslant v \leqslant u \leqslant 1$,

1. $\underline{F}(v)(t,x) \leqslant \underline{F}(u)(t,x)$
2. $\overline{F}(v)(t,x) \leqslant \overline{F}(u)(t,x)$
3. $\underline{F}(u)(t,x) \leqslant \overline{F}(u)(t,x)$

We define the sequences $(V^n)_n$ and $(W^n)_n$ of functions on $\mathbb{R}^+ \times \mathbb{T}^d$: $V^0(t,x) = 1$, $W^0(t,x) = -1$ and for all $n \geqslant 1$

$$V^n(t,x) = e^{-(2\beta+1)t}S_t v_0(x) + \int_0^t e^{-(2\beta+1)(t-s)}S_{t-s}[\overline{f}(V^{n-1}(s,\cdot))](x)\mathrm{d}s \tag{6.60}$$

$$W^n(t,x) = e^{-(2\beta+1)t}S_t v_0(x) + \int_0^t e^{-(2\beta+1)(t-s)}S_{t-s}[\underline{f}(W^{n-1}(s,\cdot))](x)\mathrm{d}s. \tag{6.61}$$

Thus, for $-1 \leqslant v_0(x) \leqslant 1$, we can prove by induction that the sequences (V^n) and (W^n) satisfy, for all n

$$-1 \leqslant W^1 \leqslant W^2 \cdots \leqslant W^n \leqslant V^n \leqslant \cdots \leqslant V^2 \leqslant V^1 \leqslant 1. \tag{6.62}$$

By a compactness and monotony argument, one can prove that (W^n) and (V^n) converge to functions w and w which are mild solutions of the subdifferential inclusion. These are the minimal and maximal solutions of the subdifferential inclusion, in the sense that any other solution (Definition 6.1) must be bounded below by w and above by v. Uniqueness follows if one can prove that $w = v$.

7 Convergence of the discrete PDE

In analogy with the continuous setting, we define $v^N = 2u^N - 1$ where u^N is the solution of the discretized Equation (4.10) and $H(i, v^N) = 2G(i, \frac{v^N+1}{2}) + v^N(i)$.

In such a way (4.10) becomes

$$\begin{cases} \partial_t v^N(t,i) = 2\alpha N^2 \Delta u^N(t,i) - (2\beta+1)v^N(t,i) + H(i,v^N) \\ v^N(0,i) = 2u_0^N(i) - 1, \end{cases} \tag{7.1}$$

We main goal of this section is to prove the following result which states the convergence of v^N.

Theorem 7.1. *Let v^N be the solution of (7.1). Then (v^N) is pre-compact for the uniform convergence on each compact sets of $\mathbb{T}^d\times]0,+\infty[$ and any accumulation points v_∞ is a solution of (6.23). In particular, whenever the solution of (7.1) is a.e. unique, v_∞ is also (the) mild solution of (6.2) and the whole sequence v^N converges to v_∞, uniformly on all compact sets of $\mathbb{T}^d\times]0,+\infty[$.*

To prove Theorem 7.1 we need some technical results. Let consider the semigroup of the discrete Laplacian $\frac{1}{2}N^2\Delta^N$ on $\mathbb{T}_N^d$ and $\mathbb{Z}^d$, denoted by $s^N(t,i,j)$ and $s_0^N(\gamma t,i,j)$ respectively. In particular we have that $s^N(t,i,j) = p_N(t,i,j)$, where $p_N(t,i,j)$ is heat kernel of discrete Laplacian on the discrete torus, cf. (5.2).

For any $\lambda,\gamma \geqslant 0$ and $f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$, we let $(S_t^{N,\lambda,\gamma})$ be the semigroup defined by

$$S_t^{N,\lambda,\gamma} f(x) = \sum_{y\in\frac{1}{N}\mathbb{T}_N^d} e^{-\lambda t} s^N(\gamma t, Nx, Ny) f(y) = \sum_{y\in\frac{1}{N}\mathbb{Z}^d} e^{-\lambda t} s_0^N(\gamma t, Nx, Ny)\widetilde{f}(y) \tag{7.2}$$

where, as in (6.6), $\widetilde{f}$ is the periodic extension of f on $\frac{1}{N}\mathbb{Z}^d$.

In the remaining part of this article we will consider $S_t^{N,\lambda,\gamma}f(x)$ with $f \in \mathcal{C}(\mathbb{T}^d)$. In that case we mean that the function f is restricted on $\frac{1}{N}\mathbb{T}_N^d \subset \mathbb{T}^d$, which is equivalent to consider $f^N(x) := f(\lfloor Nx\rfloor/N)$. We observe that if f is also Lipschitz, then $\|f - f^N\|_{\mathbb{T}^d} \leqslant \frac{c}{N^d}$ for some $c > 0$ and $\|f^N\|_{\mathbb{T}^d} \leqslant \|f\|_{\mathbb{T}^d}$. Then, with the same extension to $\mathbb{T}^d$ for s^N we can write, for any $x \in \mathbb{T}^d$

$$S_t^{N,\lambda,\gamma} f(x) = \int_{\mathbb{T}^d} e^{-\lambda t} N^d s^N(\gamma t, Nx, Ny) f^N(y)\mathrm{d}y. \tag{7.3}$$

By a slight abuse of notation, we still denote by v^N the linear interpolation on $\mathbb{T}^d$ such that $v^N(t,\frac{i}{N}) = v_i^N(t)$. We also redefine the function H on the torus $\mathbb{T}^d$ by the linear interpolation such that $H(\frac{i}{N}, v^N) = H(i, v^N)$ and we define H^N as f^N in (7.3).

Definition 7.1. *Let $N \in \mathbb{N}$ and $v_0^N \in L^\infty(\mathbb{T}^d)$. We say that $(v^N(t,x), t \geqslant 0, x \in \mathbb{T}^d)$ is a mild solution of (7.1) if*

- *for any N, v^N is continuous from $\mathbb{R}_+^*$ to $L^\infty(\mathbb{T}^d)$,*
- *for all $t > 0$ and $x \in \mathbb{T}^d$*

$$v^N(t,x) = S_t^{N,2\beta+1,4N^2\alpha}(v_0^N)(x) + \int_0^t S_{t-s}^{N,2\beta+1,4N^2\alpha}\Big[H^N(\cdot, v^N(s,\cdot))\Big](x)\mathrm{d}s\,. \tag{7.4}$$

Let u^N be the unique solution of (4.10), so that $v^N = 2u^N - 1$ satisfies (7.1). Of course, for any N the solution v^N of (7.4) exists and it is unique.

The proof of Theorem 7.1 is based on the representation of v^N as in (7.4). We define $\widetilde{v}^N$ as a slight modification of (7.4), that is,

$$\widetilde{v}^N(t,x) := S_t^{2\beta+1,4\alpha}(v_0)(x) + \int_0^t S_{t-s}^{2\beta+1,4\alpha}\Big[H(\cdot, v^N(s,\cdot))\Big](x)\mathrm{d}s\,. \tag{7.5}$$

Lemma 7.2. *For any $\tau > 0$*

$$\lim_{N\to+\infty} \|\widetilde{v}^N - v^N\|_{[1/\tau,\tau]\times\mathbb{T}^d} = 0. \tag{7.6}$$

Proof. Let $\tau > 0$, then

$$\begin{aligned}\sup_{t\in[\frac{1}{\tau},\tau],\,x\in\mathbb{T}^d} \Big|\widetilde{v}^N(t,x) - v^N(t,x)\Big| &\leqslant \sup_{t\in[\frac{1}{\tau},\tau],\,x\in\mathbb{T}^d} \Big|S_t^{N,\lambda,N^2\gamma} v_0^N(x) - S_t^{\lambda,\gamma} v_0(x)\Big| \\ &+ \sup_{t\in[\frac{1}{\tau},\tau],\,x\in\mathbb{T}^d} \Big| \int_0^t \Big\{ S_{t-s}^{N,2\beta+1,4N^2\alpha}\Big[H^N(\cdot, v^N(s,\cdot))\Big](x) - S_{t-s}^{2\beta+1,4\alpha}\Big[H(\cdot, v^N(s,\cdot))\Big](x)\Big\}\mathrm{d}s\Big|. \end{aligned} \tag{7.7}$$

We show that the right hand side of (7.7) converges to 0. We detail the convergence of the second term, which is more delicate. The argument can be adapted to the first term by using Assumption 2 which ensures that v_0^N converges to v_0 in $C(\mathbb{T}^d)$.

We fix $\varepsilon \in (0, \frac{1}{\tau})$ and we get that

$$\begin{aligned}&\Big| \int_0^t S_{t-s}^{N,2\beta+1,4N^2\alpha}\Big[H^N(\cdot, v^N(s,\cdot))\Big](x) - S_{t-s}^{2\beta+1,4\alpha}\Big[H(\cdot, v^N(s,\cdot))\Big](x)\mathrm{d}s\Big| \\ &\quad\leqslant \Big| \int_0^{t-\varepsilon} S_{t-s}^{N,2\beta+1,4N^2\alpha}\Big[H^N(\cdot, v^N(s,\cdot))\Big](x) - S_{t-s}^{2\beta+1,4\alpha}\Big[H(\cdot, v^N(s,\cdot))\Big](x)\mathrm{d}s\Big| + C\varepsilon\,, \end{aligned} \tag{7.8}$$

where we used that Lemma B.6 which implies that $H(\cdot, v^N)$ is bounded by 1 uniformly on N. The integral on the right hand side of (7.8) is bounded from above by

$$\begin{aligned}&\int_0^{t-\varepsilon} e^{-(2\beta+1)(t-s)} \Big| \int_{\mathbb{T}^d} \Big(N^d s^N(4N^2\alpha(t-s), Nx, Ny) - s(4\alpha(t-s), x, y)\Big) H^N(y, v^N(s,y))\,\mathrm{d}y\,\Big|\,\mathrm{d}s \\ &\quad + \int_0^{t-\varepsilon} e^{-(2\beta+1)(t-s)} \int_{\mathbb{T}^d} s(4\alpha(t-s), x, y) \Big|H^N(y, v^N(s,y)) - H(y, v^N(s,y))\Big|\,\mathrm{d}y\mathrm{d}s\,. \end{aligned} \tag{7.9}$$

Since $\sup_{s\geqslant 0,\, y\in\mathbb{T}^d} \Big|H^N(y, v^N(s,y)) - H(y, v^N(s,y))\Big| \leqslant \frac{1}{N}$, the second integral is smaller than $\frac{c_{\alpha,\beta}}{N}$. For the first integral, we first use that $H(\cdot, v^N)$ is bounded by 1 uniformly on N and then we operate the change of variable $u = t - s$ which gives that it is bounded from above by

$$\int_\varepsilon^t e^{-(2\beta+1)u} \int_{\mathbb{T}^d} \Big| N^d s^N(4N^2\alpha u, Nx, Ny) - s(4\alpha u, x, y)\Big|\,\mathrm{d}y\,\mathrm{d}u \tag{7.10}$$

We now use the local central limit theorem (cf. Theorem 2.1.1 and (2.5) in [16]): let ρ be the Gaussian Kernel and $\rho(u,x,y) = \frac{1}{u^{d/2}}\rho\Big(\frac{x-y}{u^{1/2}}\Big)$, then

$$\psi_{\varepsilon,\tau,N} := \sup_{u\in[\varepsilon,\tau],\, y\in\frac{1}{N}\mathbb{Z}^d} \Big|N^d p(uN^2, Nx, Ny) - \rho(u,x,y)\Big| \xrightarrow{N\to+\infty} 0. \tag{7.11}$$

We also observe that by symmetry the supremium in (7.11) is independent of x. Moreover, by Proposition 2.4.6 in [16] we have that there exist $c_1, c_2 > 0$ independent of x, y, u such that

$$\Big|N^d p(uN^2, Nx, Ny) - \rho(u,x,y)\Big| \leqslant \frac{c_1}{u^{\frac{d}{2}}} e^{-c_2\frac{\|x-y\|^2}{u}}. \tag{7.12}$$

In such a way, for any $M > 0$ fixed we write $\mathbb{T}^d = B_x(M) \cup B_x(M)^c$ and we get that (7.10) is smaller than

$$c_d \int_\varepsilon^t e^{-(2\beta+1)u}\Big(M^d \psi_{N,\varepsilon,\tau} + \frac{c_1}{M^d u^{\frac{d}{2}}} e^{-c_2\frac{M^{2d}}{u}}\Big)\,\mathrm{d}s \leqslant C_d\Big(M^d\psi_{N,\varepsilon,\tau} + \frac{1}{M^d}\Big), \tag{7.13}$$

where $c_d, C_d > 0$ are two positive constants that depend only on the dimension d.

We conclude that the right hand side of (7.8) is bounded by $\frac{c_{\alpha,\beta}}{N} + cM^d\psi_{N,\varepsilon,\tau} + \frac{c}{M^d} + C\varepsilon$, uniformly on $x \in \mathbb{T}^d$ and $t \in [\frac{1}{\tau}, \tau]$. Therefore, by taking the limit on $N \to +\infty$ and then on $M \to +\infty$ and $\varepsilon \to 0$ we conclude the proof. □

Proof of Theorem 7.1. We control $\widetilde{v}^N$ to get the convergence of v^N. We observe that since $H(\cdot, v^N(s,\cdot))$ is uniformly bounded so that by Proposition 6.1 $\widetilde{v}^N$ is uniformly bounded in N, uniformly continuous on $[1/\tau, \tau] \times \mathbb{T}^d$, and the modulus of continuity only depends on $\tau > 1$. By the Ascoli-Arzela Theorem, the sequence $(\widetilde{v}^N)_N$ is pre-compact on $C([1/\tau, \tau] \times \mathbb{R}^d)$ and therefore, by Lemma (7.2), (v^N) also. By a diagonal argument, we can extract from (v^N) a subsequence converging uniformly to a limit v_∞ in $C(]0, \infty[\times\mathbb{R}^d)$, uniformly on each compact.

Using Corollary B.4, we can adapt the argument used in the proof of Proposition 6.3 (6.25–6.30) to get that each accumulation point v_∞ is a mild solution of (6.23), we omit the details. □

7.1 Proof of Theorem 3.1

Theorem 3.1 is now a consequence of Theorems 4.1 and 7.1.

A Concentration inequalities

We follow the definitions in Jara and Menezes [13] and [14] and Boucheron, Lugosi, Massart [1], Section 2.3. We omit the proofs since there are present in the references.

Definition A.1 ([13] and [1], Section 2.3). *Let X be a real random variable. X is said to be sub-Gaussian with variance parameter σ^2 if, for all $t \in \mathbb{R}$*

$$\psi_X(t) := \log \mathbb{E}(\exp(tX)) \leqslant \sigma^2 \frac{t^2}{2}. \tag{A.1}$$

We denote $\mathcal{G}(\sigma^2)$ the set of real sub-Gaussian random variables with variance parameter σ^2.

Proposition A.1. *[[1] and [14], Proposition F.7] The following statements are equivalent:*

1. $X \in \mathcal{G}(\sigma^2)$
2. $\mathbb{P}(|X| > t) \leqslant 2\exp(-\frac{t^2}{2\sigma^2})$
3. $\mathbb{E}(\exp(\gamma X^2)) \leqslant 3$ *for all* $0 \leqslant \gamma < \frac{1}{4\sigma^2}$.

Let us complete our family of inequality:

Lemma A.2. *[[14], Proposition F.8] Let $X \in \mathcal{G}(\sigma_1^2)$ and $Y \in \mathcal{G}(\sigma_2^2)$, then for all $0 \leqslant \gamma < \frac{1}{4\sigma_1\sigma_2}$,*

$$\mathbb{E}(\exp(\gamma XY)) \leqslant 3$$

Lemma A.3. *[[14], Proposition F.12] Let $X_1, \ldots, X_n$ be random variables with $X_i \in \mathcal{G}(\sigma_i^2)$, such that there is a partition of K subsets $P_1, \ldots, P_K$ each containing L variables that $\Sigma_k = \sigma(X_i, i \in P_k)$ are independent σ-algebra then, for all real $\alpha_1, \ldots, \alpha_n$, the random variable $Y = \sum_i \alpha_i X_i$ is sub-Gaussian with variance parameter $L\sum_i \alpha_i^2\sigma_i^2$.*

Note that if $L = 1$, the variables are independent.

Lemma A.4 (Hoeffeding Inequality, [1], Section 2.3). *Let X be a bounded random variable with $X \in [a, b]$, then $X - \mathbb{E}X \in \mathcal{G}\left(\frac{(b-a)^2}{4}\right)$.*

B Controls for the non-linearities

In this section, we collect some results about the specifics of our model. Recall the notations: for $\eta \in \{0,1\}^{\mathbb{T}_N^d}$

$$c_0^+(\eta) := \mathbb{1}_{\{\rho_0(\eta) \geqslant 1 - \frac{\kappa_N}{K_N}\}}, \qquad c_0^-(\eta) := \mathbb{1}_{\{\rho_0(\eta) \leqslant \frac{\kappa_N}{K_N}\}},$$

and $\kappa_N = \kappa_N(T) := \min\Big\{\lfloor K_N T\rceil - 1; \lfloor K_N(1-T)\rfloor\Big\}$. For any function $u = (u_i)_{i\in\mathbb{T}_N^d}$, we define $\upsilon_u(\mathrm{d}\eta) = \upsilon_u^N(\mathrm{d}\eta) := \bigotimes_{i\in\mathbb{T}_N^d} \mathrm{B}\big(u_i\big)$ where $B(u_i)$ denote a Bernoulli distribution with parameter u_i. and we let $c_0^+(u)$ and $c_0^-(u)$ be the expectations of $c_0^+(\eta)$ and $c_0^-(\eta)$ under υ_u. We set $(\tau_i\eta)_j = \eta_{i+j}$, and likewise τ_i acts on u. Then,

$$G(u) := (1-u_0)c_0^+(u) - u_0 c_0^-(u) = (1-u_0)\mathbb{P}_{\upsilon_u}\left[\rho_0(\eta) \geqslant 1 - \frac{\kappa_N}{K_N}\right] - u_0\mathbb{P}_{\upsilon_u}\left[\rho_0(\eta) \leqslant \frac{\kappa_N}{K_N}\right] \quad \text{(B.1)}$$

and $G(i,u) := G(\tau_i u)$. We start with some results on the non linearity G:

Proposition B.1. *Let $u, v \in [0,1]^{\mathbb{T}_N^d}$ such that $u_i \geqslant v_i$ for all $i \in \mathcal{V}_N$ and $u_0 = v_0$. Then $G(0,u) \geqslant G(0,v)$.*

Proof. We construct a coupling between υ_u and υ_v: let (U_i) be independent and identically distributed random variables uniform on $[0,1]$. Define $\eta_i = \mathbb{1}_{\{U_i \leqslant u_i\}}$ and $\eta'_i = \mathbb{1}_{\{U_i \leqslant v_i\}}$. We have $\eta_i \geqslant \eta'_i$ for all $i \in \mathcal{V}_N$, therefore $\rho_0(\eta) \geqslant \rho_0(\eta')$. This proves that $c_0^+(u) \geqslant c_0^+(v)$ and $c_0^-(u) \leqslant c_0^-(v)$. The results follows since $u_0 = v_0$. ☐

For $p \in [0,1]$, $T \in [0,1]$, $K \in \mathbb{N}^*$, we let $\kappa(K,T) = \min\Big\{\lceil KT\rceil - 1; \lfloor K(1-T)\rfloor\Big\} \leqslant K/2$, and we define $g_K(p)$ as

$$g_K(p) := (1-p)\mathbb{P}_p\big[X > K - \kappa(K,T)\big] - p\mathbb{P}_p\big[X \leqslant \kappa(K,T)\big]$$

where X is a random variable with binomial distribution with parameter (K,p) under $\mathbb{P}_p$. In particular, we have that $g_{K_N}(p) = G(i,u)$ for $u(t,i) = p$ for all $i \in \mathbb{T}_N^d$. Note that g_K is $C^\infty([0,1])$. We also have

$$\begin{aligned} g_K(p) &= (1-p)\mathbb{P}_{1-p}\big[X < \kappa(K,T)\big] - p\mathbb{P}_p\big[X \leqslant \kappa(K,T)\big] \\ &= \sum_{k=0}^{\kappa(K,T)-1} \binom{K}{k} \Big[(1-p)^{k+1}p^{K-k} - p^{k+1}(1-p)^{K-k}\Big] \\ &\quad - \binom{K}{\kappa(K,T)} p^{\kappa(K,T)+1}(1-p)^{K-\kappa(K,T)}. \end{aligned} \quad \text{(B.2)}$$

We recall $g_\infty(p) := (1-p)\mathbb{1}_{\{1-p<p_0(T)\}} - p\mathbb{1}_{\{p\leqslant p_0(T)\}}$, where $p_0(T) = \min(T, 1-T)$.

The following proposition estimates the convergence of g_K to g_∞, in particular we prove that close to $p = 0$ (resp. $p = 1$), g_K is negative (resp. positive).

Proposition B.2. *For all $K \in \mathbb{N}^* \cup \{\infty\}$, we have $g_K(0) = g_K(1) = 0$. For all $K \in \mathbb{N}^*$ and $p \in [0,1]$, we have, for $|p - p_0(T)| > \frac{1}{K}$ and $|(1-p) - p_0(T)| > \frac{1}{K}$*

$$|g_K(p) - g_\infty(p)| \leqslant 2e\exp(-2K(p_0(T)-p)^2) + 2e\exp(-2K(p_0(T)-(1-p))^2) \quad \text{(B.3)}$$

In particular, for any $0 < \delta < \frac{1}{4e}$, $T \in [4\delta, 1-4\delta]$, and $K > \frac{|\log(\delta)|}{4\delta^2}$, we have that $g_K(2\delta) < -\delta$ and $g_K(1-2\delta) > \delta$.

Proof. We consider g_K written as in (B.2). Then, the values at $p=0$ and $p=1$ are obvious.

Note that, under $\mathbb{P}_p$, $\frac{X}{K}-p$ converges to 0 (in $L^2(\Omega_N)$) and is sub-Gaussian with variance parameter $\frac{1}{4K}$, thus, for any $t>0$,

$$\mathbb{P}_p\left(\left|\frac{X}{K}-p\right|>t\right)\leqslant 2\exp(-2Kt^2).$$

We also have that

$$0\leqslant p_0(T)-\frac{\kappa(K,T)}{K}\leqslant\frac{1}{K}.$$

Then, for $p>p_0(T)$,

$$\begin{aligned}\mathbb{P}_p\big[X\leqslant\kappa(K,T)\big]&=\mathbb{P}_p\left[p-\frac{X}{K}\geqslant p-p_0(T)+p_0(T)-\frac{\kappa(K,T)}{K}\right]\leqslant\mathbb{P}_p\left[p-\frac{X}{K}\geqslant p-p_0(T)\right]\\&\leqslant 2\exp(-2K(p-p_0(T))^2),\end{aligned}$$

and for $p<p_0(T)-\frac{1}{K}$,

$$\begin{aligned}\mathbb{P}_p[X>\kappa(K,T)]&=\mathbb{P}_p\left[\frac{X}{K}-p>\frac{\kappa(K,T)}{K}-p_0(T)+p_0(T)-p\right]\\&\leqslant\mathbb{P}_p\left[\frac{X}{K}-p\geqslant p_0(T)-p-\frac{1}{K}\right]\leqslant 2\exp(-2K(p_0(T)-p-1/K)^2)\\&\leqslant 2e^{2(p_0(T)-p)}\exp(-2K(p_0(T)-p)^2)\leqslant 2e\exp(-2K(p_0(T)-p)^2).\end{aligned}$$

Thus, we have the following, for $|p-p_0(T)|>\frac{1}{K}$,

$$\left|\mathbb{P}_p[X\leqslant\kappa(K,T)]-\mathbb{1}_{\{p\leqslant p_0(T)\}}\right|<2e\exp(-2K(p_0(T)-p)^2).$$

The results follows with the same estimates with $(1-p)$ instead of p. This prove (B.3).

Let $\delta>0$ and set $p=2\delta$, and $T\in[4\delta,1-4\delta]$, then $|2\delta-p_0(T)|\leqslant 2\delta$ for $K>\frac{1}{2\delta}$ and we have the same for $1-p=1-2\delta$. Applying the result, we have

$$|g_K(2\delta)-g_\infty(2\delta)|=|g_K(2\delta)+2\delta|\leqslant 4e\exp(-8K\delta^2)$$

Then, if $4e\exp(-8K\delta^2)<\delta$, we get the result. This happens if $\exp(-8K\delta^2)<\delta^2$ and $\delta<\frac{1}{4e}$, which gives the condition on K. $\square$

Proposition B.3. *Let $v:\mathbb{T}^d\to[0,1]$ be a continuous fixed density on the torus. For any $i\in\mathbb{T}_N^d$ we let $u_i:=v_{i/N}$. Then as $i/N\to x$, we have that u_i converges to v_x. Let $\upsilon_u=\bigotimes_{i\in\mathbb{T}_N^d}\mathrm{B}(u_i)$. Then, $\rho_0(\eta)$ converges to v_0 in probability.*

Proof. We use the coupling introduced in Lemma B.1: we let (U_i) be i.i.d uniform random variables on $[0,1]$ so that under υ_u we have that $\rho_0(\eta)$ and $\frac{1}{K_N}\sum_{i\in\mathcal{V}_N}\mathbb{1}_{\{U_i<u_i\}}$ are equal in law. By the Tchebychev inequality, we have that for any $\varepsilon>0$

$$\mathbb{P}\left(\left|\frac{\sum_{i\in\mathcal{V}_N}\mathbb{1}_{\{U_i<u_i\}}}{K_N}-\frac{\sum_{i\in\mathcal{V}_N}u_i}{K_N}\right|>\varepsilon\right)\leqslant\frac{1}{4K_N\varepsilon^2}.$$

Moreover, the sequence $K_N^{-1}\sum_{i\in\mathcal{V}_N}u_i$ converges to v_0 because for any $i\in\mathcal{V}_N$, $i/N\to 0$. $\square$

We have the following corollary.

Corollary B.4. *Let $u = (u_i)_{i\in\mathbb{T}_N^d}$ as in Proposition B.3. Then, $G(u)$ and $g_{K_N}(v_0)$ converge both to $g_\infty(v_0)$ as $N \to +\infty$.*

Recall that, for $q \in [-1,1]$,

$$h_K(q) = 2g_K\left(\frac{1}{2}(q+1)\right) + q = (1-q)\mathbb{P}_{\frac{1-q}{2}}\big[X < \kappa(K,T)\big] - (1+q)\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + q.$$

Recall the critical parameter $\rho := \rho(T) = \left|T - \frac{1}{2}\right| = \frac{1}{2} - p_0(T) \in \left[0, \frac{1}{2}\right]$. Note that h_K converges pointwise to the function $h_\infty(q) = 2g_\infty\left(\frac{1+q}{2}\right) + q = -\mathbb{1}_{q\leqslant -2\rho} + q\mathbb{1}_{-2\rho<q\leqslant 2\rho} + \mathbb{1}_{q>2\rho}$. The points $q = \pm 2\rho$ are the discontinuities of h_∞, and compare the Lemma to the fact that $2\rho = h_\infty(2\rho^-)$ and $1 = h_\infty(2\rho^+)$. A similar estimate holds at $q = -2\rho$.

Lemma B.5. *For all $q \in [-1,1]$, $|h_K(q)| \leqslant 1$. Moreover, for all $\varepsilon > 0$ such that $2\rho - \varepsilon > 0$, there is $K_0 > 0$ such that for $K \geqslant 0$,*

$$\begin{aligned} 2\rho - 2\varepsilon = h_\infty(2\rho^-) - 2\varepsilon \leqslant &h_K(q) \leqslant h_\infty(2\rho^+) = 1 && \text{for } q \in [2\rho-\varepsilon, 2\rho+\varepsilon] \\ -1 = h_\infty(-2\rho^-) \leqslant &h_K(q) \leqslant h_\infty(-2\rho^+) + 2\varepsilon = -2\rho + 2\varepsilon && \text{for } q \in [-2\rho-\varepsilon, -2\rho+\varepsilon] \end{aligned}$$

Proof. We start by observing that for all $q \in [-1,1]$, $\mathbb{P}_{\frac{1-q}{2}}\big[X < \kappa(K,T)\big] = \mathbb{P}_{\frac{1+q}{2}}\big[X > K - \kappa(K,T)\big]$. We also have that

$$1 = \mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + \mathbb{P}_{\frac{1+q}{2}}\big[\kappa(K,T) < X \leqslant K - \kappa(K,T)\big] + \mathbb{P}_{\frac{1+q}{2}}\big[X > K - \kappa(K,T)\big].$$

Thus, using the definiton of h_K, we get

$$\begin{aligned} h_K(q) = &- \mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + q\mathbb{P}_{\frac{1+q}{2}}\big[\kappa(K,T) < X \leqslant K - \kappa(K,T)\big] \\ &+ \mathbb{P}_{\frac{1+q}{2}}\big[X > K - \kappa(K,T)\big]. \end{aligned}$$

This gives us the result. Indeed, for $q \in [0,1]$, we have

$$\begin{aligned} h_K(q) &\leqslant q\mathbb{P}_{\frac{1+q}{2}}\big[\kappa(K,T) < X \leqslant K - \kappa(K,T)\big] + \mathbb{P}_{\frac{1+q}{2}}\big[X > K - \kappa(K,T)\big] \\ &\leqslant \mathbb{P}_{\frac{1+q}{2}}\big[\kappa(K,T) < X\big] \leqslant 1 \end{aligned}$$

and

$$\begin{aligned} h_K(q) &\geqslant -\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + \mathbb{P}_{\frac{1+q}{2}}\big[X > K - \kappa(K,T)\big] \\ &\geqslant \mathbb{P}_{\frac{1-q}{2}}\big[X < \kappa(K,T)\big] - \mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] \geqslant -\mathbb{P}_{\frac{1+q}{2}}\big[X = \kappa(K,T)\big]. \end{aligned}$$

The last inequality comes form the fact that, by a coupling argument, $p \mapsto \mathbb{P}_p\big[X < \kappa(K,T)\big]$ is decreasing on $[0,1]$, and since $q \geqslant 0$, $\frac{1-q}{2} \leqslant \frac{1+q}{2}$.

In particular, for $q \in [2\rho-\varepsilon, 2\rho+\varepsilon]$ such that $2\rho - \varepsilon > 0$, we have the same upper bound as before for $h_K(q)$, and for the lower bound:

$$\begin{aligned} h_K(q) &\geqslant -\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + q\mathbb{P}_{\frac{1+q}{2}}\big[X > \kappa(K,T)\big] \\ &\geqslant -\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] + (2\rho - \varepsilon)\left(1 - \mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big]\right) \\ &\geqslant 2\rho - \varepsilon - 2\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big]. \end{aligned}$$

From the proof of Proposition B.2, we have that, for $K \geqslant K_0$:

$$\mathbb{P}_{\frac{1+q}{2}}\big[X \leqslant \kappa(K,T)\big] \leqslant 2\exp\left(-K(q+2\rho)^2/2\right) \leqslant 2e^{-2K_0\rho^2}$$

Choosing K_0 large enough such that the right hand side is less than $\varepsilon/2$, we get the result.

For $q \in [-1,0]$, the proof is completely similar.

□

Lemma B.6. *Let $u = (u_i)_{i\in\mathbb{T}_N^d}$, with $u_i \in [0,1]$ and let $v = 2u-1$. Then, $|H(i,v)| \leqslant 1$, uniformly on $i \in \mathbb{T}_N^d$.*

Proof. We recall (4.9), in particular that $G(i,u) = G(\tau_i u)$. So that we only prove that $|H(v)| \leqslant 1$, where $H(v) = G((v+1)/2) + v_0$. The proof is similar to Lemma B.5. Indeed, again by (4.9) we have that

$$\begin{aligned}2G\left(\frac{v+1}{2}\right) + v_0 &= c_0^+\left(\frac{v+1}{2}\right)(1-v_0) - c_0^-\left(\frac{v+1}{2}\right)(1+v_0) + v_0\\ &= c_0^+\left(\frac{v+1}{2}\right) - c_0^-\left(\frac{v+1}{2}\right) + v_0\left(1 - c_0^+\left(\frac{v+1}{2}\right) - c_0^-\left(\frac{v+1}{2}\right)\right).\end{aligned}$$

We observe that, by definition, $1 - c_0^+\left(\frac{v+1}{2}\right) - c_0^-\left(\frac{v+1}{2}\right) > 0$. This implies that $-1 \leqslant H(v) \leqslant 1$. $\square$

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