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## The sexagesimal place-value notation and abstract numbers in mathematical cuneiform texts

Christine Proust (Laboratoire SPHERE, CNRS & Université de Paris, France)\*

**Abstract** – The discovery at the end of the 19<sup>th</sup> century of the mathematical cuneiform texts posed to historians the question of the nature of the numbers used in them, i.e. that of the sexagesimal place-value notation. This notation, although familiar to us today since it is the one we use to measure time, has, in the cuneiform texts, specificities which still raise challenges of interpretation. One of these specificities is the fact that the cuneiform writing does not indicate the order of magnitude of the numbers (for example, 1, 60, 1/60 or any other power of 60 are written in the same way). This article outlines the way in which historians of the late 19<sup>th</sup> and early 20<sup>th</sup> centuries interpreted this specificity. The focus here is on the interpretation proposed by the Assyriologist François Thureau-Dangin, who in 1930 considered numbers in sexagesimal place-value notation as “abstract numbers”, as opposed to “concrete numbers”.

**Résumé** – La découverte à la fin du 19<sup>e</sup> siècle des premières tablettes mathématiques cunéiformes a posé aux historiens la question de la nature des nombres qui y étaient utilisés, c’est-à-dire celle de la notation sexagésimale positionnelle. Cette notation, quoique familière aujourd’hui puisque c’est celle que nous utilisons pour la mesure du temps, revêt dans les textes cunéiformes des spécificités qui soulèvent encore aujourd’hui des défis d’interprétation. Une de ces spécificités est le fait que l’écriture cunéiforme n’indique pas l’ordre de grandeur des nombres (par exemple, 1, 60, 1/60 ou toute autre puissance de 60 s’écrivent de la même façon). Cet article retrace la façon dont les historiens de la fin du 19<sup>e</sup> siècle et du début du 20<sup>e</sup> siècle ont interprété cette spécificité. L’accent est mis ici sur l’interprétation de François Thureau-Dangin, qui, en 1930, considérait les nombres en notation sexagésimale positionnelle comme des « nombres abstraits », qu’il opposait aux « nombres concrets ».




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**Mots clés :** Mésopotamie, histoire de l’écriture, textes mathématiques cunéiformes, Thureau-Dangin, notation sexagésimale positionnelle, nombres abstraits, notation flottante, opérations

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### 1- The sexagesimal place-value notation in cuneiform texts


The sexagesimal place-value notation is found, in the cuneiform documents, almost exclusively in astronomical and mathematical texts.<sup>1</sup> As their name indicates, these numbers are written in base sixty on a positional principle. The 59 sexagesimal digits are composed of signs for the ones (vertical wedges ) , and signs for the tens (chevrons ) , repeated as many times as necessary. For example, the digit  (three chevrons and two wedges) corresponds to 32 in modern notation. Digits are thus

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<sup>1</sup> The context, date and proveniences of both categories are different. The Astronomical texts come from South Mesopotamia and are dated to the second part of the first millennium BCE. The mathematical texts originate from a vast area—Mesopotamia, Syria and Elam in modern western Iran—and are dated to different periods over three millennia, from the mid-third millennium to the very end of the first millennium; but the bulk of them is dated to the Old Babylonian period, i.e. the beginning of the 2<sup>nd</sup> millennium BCE. In rare cases, the sexagesimal place-value notation is attested in other categories of texts, for example in administrative texts – See (Middeke-Conlin 2020); (Ouyang and Proust forthcoming) and related bibliography.

noted in a decimal additive system. Numbers are composed of sequences of sexagesimal digits, each of the signs worth sixty times more than the same sign in the previous place (on its right). For example,

the number  (1:21) is composed of a sequence of two sexagesimal digits, 1 and 21. The wedge in the left-hand position is worth sixty times more than the wedge in the right-hand position. This system is similar to our modern sexagesimal system for measuring time, but with an important difference: the cuneiform notation does not indicate the position of the units in the number, thus the order of magnitude is not specified. The notation is not only sexagesimal and place-valued, but also

floating. For example, the notation  (1:21) corresponds, in modern notation, at the same time to  $60+21$ , to  $1+21/60$ , and to one of these numbers multiplied by any power of sixty.

This description of the sexagesimal place-value notation reflects my point of view.<sup>2</sup> In particular, I refer to the absence of indication of the orders of magnitude in the cuneiform writing by the adjective “floating”. However, this description differs slightly from that of other authors, especially with respect to the “floating” character of the place-value numbers. For example, the absence of indication of the orders of magnitude in cuneiform writing may be referred to by the expressions “relative numbers”, or “abstract numbers”, or “absence of zero”, or just ignored.<sup>3</sup> What is the meaning and what are the consequences of these differences?

The discovery of the sexagesimal place-value notation in mathematical cuneiform texts dates back to the end of the 19th century. Since its discovery by Assyriologists, this notation attracted the attention of historians of mathematics. For example, in his *History of Mathematics*, Cajori devotes a brief chapter to “The Babylonians” (pp. 5–9), where he refers to the sexagesimal place-value notation as follows:

Not to be overlooked is the fact that in the sexagesimal notation of integers the “principle of position” was employed. Thus, in 1.4 (=64), the 1 is made to stand for 60, the unit of the second order, by virtue of its position with respect to the 4. The introduction of this principle at so early a date is the more remarkable, because in the decimal notation it was not introduced till about the fifth or sixth century after Christ. (Cajori 1893, 7, 1894 edition).

Cajori notes the problem of the absence of indication of the orders of magnitude in cuneiform writing and points out that a consequence of this is an absence of distinction between integers and sexagesimal fractions:

The sexagesimal system was used also in fractions. Thus, in the Babylonian inscriptions,  $1/2$  and  $1/3$  are designated by 30 and 20, the reader being expected, in his mind, to supply the word “sixtieths”. (Cajori 1893, 7, 1894 edition).

At the time Cajori published his “History of Mathematics”, historians relied on very few sources to discuss the sexagesimal place-value notation. Cajori was aware of the existence of cuneiform astronomical texts that began to be published at that time (Epping et Strassmeier 1889). As for mathematical sources, he knew of only two texts: “We possess two Babylonian tablets which exhibit its [sexagesimal system] use” (Cajori 1893, 6). One of them was the so-called “Senkereh Tablet” which contains numerical tables that circulated at that time, but was not critically edited before 1930 by Thureau-Dangin (see Section 2-1). Relying on the same sources, Moritz Cantor relates the same observations as Cajori on sexagesimal place-value notation (Cantor 1894 (Third Edition 1907), 73-104).

Cajori provides much more detail and explanations on the sexagesimal place-value notation used in the “Senkereh Table” in his 1928 *History of Mathematical Notation* vol. I:

<sup>2</sup> For more details on this point of view, see (Proust 2013).

<sup>3</sup> See, for example, the descriptions of the sexagesimal place-value notation in (Thureau-Dangin 1932b: 49-72, that I deal with in Section 2; Neugebauer and Sachs 1945: 2; Høytrup 2002: 5-11; Friberg 2007: 5-11).

Hincks's explanation [i.e. “the sexagesimal scale”] was confirmed by the decipherment of tablets found at Senkereh, near Babylon, in 1854, and called the Tablets of Senkereh. One tablet was found to contain a table of square numbers, from  $1^2$  to  $60^2$ , a second one a table of cube numbers from  $1^3$  to  $32^3$ . The tablets were probably written between 2300 and 1600 B.C. Various scholars contributed toward their interpretation. Among them were George Smith (1872), J. Oppert, Sir H. Rawlinson, Fr. Lenormant, and finally R. Lepsius. (Cajori 1928, 3).

Between Cajori’s two publications (1893 and 1928), the corpus of mathematical texts had grown dramatically thanks to Hermann Hilprecht’s publications of tablets from Nippur in 1906. Hilprecht clearly exhibits the use of the sexagesimal place-value notation in mathematical texts. To him, the consequence of the “absence of zero” is that the sign  $\Uparrow$  may represent all the powers of sixty.

deciphering. For owing to the absence of a zero in Babylonian, every  $\Uparrow$  or  $\triangleleft$ , as illustrated by the following scale (which may be continued indefinitely),

$\Uparrow$	$\triangleleft$	$\Uparrow$	$\triangleleft$	$\Uparrow$	$\triangleleft$	$\Uparrow$	$\triangleleft$	$\Uparrow$
12,960,000	2,160,000	216,000	36,000	3,600	600	60	10	1

can be read in many different ways, unless determined by the context and a sufficient number of smaller figures following. Each line of the left column of *K 2069* beginning

**Figure 1** Hilprecht 1906, 26

As Neugebauer did later, Hilprecht considered that the power of sixty had to be “determined by the context” (Hilprecht 1906, 26). Hence, he felt it necessary to determine the absolute value of the

number written with one wedge ( $\Uparrow$ ) in different contexts. Hilprecht gave the value  $60^4$  to the number denoted by one wedge in reciprocal tables. Indeed, he interpreted these tables as the list of the divisors

of  $\Uparrow$ , starting from the *a priori* that all the divisors given by the table were integers.<sup>4</sup> The issue of the absolute value of the number written with one wedge in reciprocal tables has been subsequently much discussed, and the value 60 was the most often attributed to this sign. For example Scheil (1915, 195)

wonders “why would  $\Uparrow$  be worth 12960000 and not its documented value of 60?”<sup>5</sup>

As we can see, in the first attempts to understand the mathematical cuneiform texts, the description of the sexagesimal place-value notation texts raised many questions. One of the thorniest, even today, is that of the absence of indication of order of magnitude. François Thureau-Dangin, in his 1930 article “*Nombres concrets et nombres abstraits dans la numération babylonienne*” (*Concrete numbers and abstract numbers in Babylonian numeration*), provided a possible solution to this problem by characterizing the numbers in sexagesimal place-value notation as “abstract numbers”:

This very abstract system, which did not distinguish between integers and fractions, which ignored the order of magnitude of the numbers, was used for arithmetic operations, particularly for *igi-aré*, that is, “divisions and multiplications”, which it greatly facilitated. The so-called Esagil tablet perfectly illustrates

<sup>4</sup> From there he developed a rather fanciful theory that the number 12,960,000 is the “Plato number” found in Plato’s *Republic*, Book III (p. 29 ff). This theory was much mocked, which obscured the important contributions that Hilprecht made on the school context of cuneiform mathematics and on the use of the sexagesimal place-value notation in arithmetic texts.

<sup>5</sup> My translation. Original text: “Pourquoi  $\Uparrow$  vaudrait-il 12960000 et non pas sa valeur documentée de 60 ?”.

the method used by the Babylonians and shows how, in their calculations, they went from the concrete to the abstract, then back from the abstract to the concrete. (Thureau-Dangin 1930b, 117)<sup>6</sup>

The analysis of the sexagesimal place-value notation by Thureau-Dangin as abstract numbers is quite original, and has hardly had any echo in the historiography of cuneiform mathematics. In this article, I present Thureau-Dangin's conception of "abstract numbers", and confront it with that of other contemporary or later historians.

## 2- Thureau-Dangin's conception of abstract numbers

François Thureau-Dangin (1872–1944) was one of the pioneers of Assyriology. From 1895 to 1928, he was successively attaché, then curator, then director, of Oriental Antiquities at the Louvre Museum (Paris). He devoted much of his career to building up the Louvre's collection of tablets and published the cuneiform texts they contain. He made a decisive contribution to the deciphering and understanding of the numerical and metrological systems used in administrative and mathematical texts, and was one of the decipherers of the Sumerian language.<sup>7</sup> The following discussion is based on two brief "Assyriological Notes" that Thureau-Dangin published in 1930 in the *Revue d'Assyriologie: Note LV "La table de Senkereh"*, and *Note LVI "Nombres concrets et nombres abstraits dans la numération babylonienne"*.<sup>8</sup>

### 2-1 The "Senkereh Table"

The note "*La Table de Senkereh*" (Thureau-Dangin 1930a) is of great historiographical importance because it offers a partial transcription of one of the earliest mathematical cuneiform tablets that had been brought to the attention of historians of mathematics. The tablet<sup>9</sup> contains metrological and numerical tables dated to the Old-Babylonian period. The numerical tables, which are tables of square roots and cube roots, played an essential role in the earliest modern attempts to understand the numerical notations used in mathematical cuneiform texts and provided the material for discussions on the sexagesimal place-value notation to many historians of mathematics in the late 19<sup>th</sup> and early 20<sup>th</sup> centuries, as evoked in Section 1.<sup>10</sup>

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<sup>6</sup> My translation. Original text: "Ce système très abstrait, qui ne distinguait pas entre les entiers et les fractions, qui ignorait l'ordre de grandeur des nombres, servait aux opérations arithmétiques, notamment aux *igi-aré*, c'est-à-dire aux "divisions et multiplications", qu'il facilitait grandement. La tablette dite de l'Esagil illustre parfaitement la méthode employée par les Babyloniens et montre comment, dans leurs calculs, ils passaient du concret à l'abstrait, puis revenaient de l'abstrait au concret".

<sup>7</sup> On Thureau-Dangin's biography, his works on numbers and metrology and related bibliography, see Pierre Chaigneau's PhD thesis (Chaigneau 2019: Chapters 1, 3).

<sup>8</sup> *Senkereh Table and Concrete numbers and abstract numbers in Babylonian numeration* (Thureau-Dangin 1930a, 1930b).

<sup>9</sup> The tablet is now kept at the British Museum under the number 92698, its copy was published by Rawlinson (Rawlinson et al. 1861-1884), its first interpretation by Weissbach (1915), and its complete edition by Neugebauer (1935: 69). The photo is available on the British Museum website ([https://www.britishmuseum.org/collection/object/W\\_92698](https://www.britishmuseum.org/collection/object/W_92698), accessed June 2021).

<sup>10</sup> Neugebauer insists on the historiographical importance of this tablet as follows: "The arithmetic tables form the earliest known group of "mathematical" texts. With the famous "Tablets of Senkereh"[...] the sexagesimal and positional character of the cuneiform numbers was verified for the first time (1854/55), and Hilprecht's large publication of the "Mathematical, Metrological and chronological tablets" (1906) treated as "mathematical" only texts such as those compiled in this chapter [Chapter I of *Mathematische Keilschrift-Texte* vol. I]. Thus, the knowledge of "mathematical" cuneiform texts was limited until 1916 ... almost exclusively to such tabular texts." Original German citation: "Die Rechentabellen bilden die am längsten bekannte Gruppe "mathematischer" Texte. An den berühmten "Täfelchen von Senkereh" ... wurde zuerst (1854/55) der sexagesimale und zugleich positionelle Charakter der Keilschrift-Ziffern verifiziert, und Hilprechts grosse Publikation der "Mathematische, Metrologische und chronologische Tafeln" (1906) behandelte an "mathematischen" Texten nur solche, wie sie in diesen Kapitel zusammengestellt sind. So beschränkte sich die Kenntnis von "mathematischen" Keilschrifttexten bis 1916 ... so gut wie ausschliesslich auf derartige Tabellentexte." (Neugebauer 1935: 4). This excerpt also reflects the fact that Neugebauer minimized the interest of the metrological tables, which he did not consider as truly mathematical.

Thureau-Dangin echoes this role as follows:

The “*Table de Senkereh*” has, since the beginning of Assyriology, been a *crux interpretatum*. If one studies this text in the light of the very similar text that I have published above, p. 74 [...], one realizes that it owes its obscurities and contradictions only to errors attributable either to the scribe or to the editor. [...] The restitution that Weissbach proposes is probably excessive. We give below the part of the text which can be restored with certainty [...] (Thureau-Dangin 1930a, 115)<sup>11</sup>

The historical importance of the “Senkereh Table” comes from the fact that it begins with a metrological table, i.e. a correspondence between measurement values and numbers in sexagesimal place-value notation. Unlike the two numerical tables, this metrological table was seldom commented on historiographically. Yet, this metrological table offers a crucial key to the understanding of the relationship between measurement values and numbers as conceived by the ancient scribes, authors or users of the Old Babylonian mathematical texts. Let us therefore examine a translation of this text, or at least of the beginning of the fragment that was known to Thureau-Dangin (my own translation; bold is mine).<sup>12</sup>

Obverse, col. 1

<b>1 šu-si</b>	<b>10</b>
2 šu-si	20
3 šu-si	30
4 šu-si	40
5 šu-si	50
6 šu-si	1
7 šu-si	1:10
8 šu-si	1:20
9 šu-si	1:30
$\frac{1}{3}$ kuš	1:40
$\frac{1}{2}$ kuš	2:30
$\frac{2}{3}$ kuš	3:20
$\frac{2}{3}$ kuš 1 šu-si	3:30
$\frac{2}{3}$ kuš 2 šu-si	3:40
$\frac{2}{3}$ kuš 3 šu-si	3:50
$\frac{2}{3}$ kuš 4 šu-si	4
$\frac{2}{3}$ kuš 5 šu-si	4:10
$\frac{2}{3}$ kuš 6 šu-si	4:20
$\frac{2}{3}$ kuš 7 šu-si	4:30
$\frac{2}{3}$ kuš 8 šu-si	4:40
$\frac{2}{3}$ kuš 9 šu-si	4:50
1 kuš	5
1 $\frac{1}{3}$ kuš	6:40
1 $\frac{1}{2}$ kuš	7:30
1 $\frac{2}{3}$ kuš	8:20
<b>2 kuš</b>	<b>10</b>
3 kuš	15
4 kuš	20
5 kuš	25

<sup>11</sup> My translation. Original text: “La « Table de Senkereh » est, depuis les débuts de l’assyriologie, une *crux interpretatum*. Si on étudie ce texte à la lumière du texte tout à fait semblable que j’ai publié ci-dessus, p. 74, on se rend compte qu’il ne doit ses obscurités et ses contradictions qu’à des erreurs imputables soit au scribe, soit à l’éditeur. [...] Nous donnons ci-dessous la partie du texte qui peut être restaurée avec certitude [...]”

<sup>12</sup> 1 *ninda* (1 pole, ca. 6 m) is worth 12 *kuš*; 1 *kuš* (1 cubit, ca. 50 cm) is worth 30 *šu-si* (*šu-si* means digit, ca. 1,6 cm).

$\frac{1}{2}$ <i>ninda</i>	30
$\frac{1}{2}$ <i>ninda</i> 1 <i>kuš</i>	35
$\frac{1}{2}$ <i>ninda</i> 2 <i>kuš</i>	40
$\frac{1}{2}$ <i>ninda</i> 3 <i>kuš</i>	45
$\frac{1}{2}$ <i>ninda</i> 4 <i>kuš</i>	50
$\frac{1}{2}$ <i>ninda</i> 5 <i>kuš</i>	55
1 <i>ninda</i>	1
1 $\frac{1}{2}$ <i>ninda</i>	1:30
2 <i>ninda</i>	2
2 $\frac{1}{2}$ <i>ninda</i>	2:30
3 <i>ninda</i>	3
Obverse, col. 2	
[...]	
<b>10 <i>ninda</i></b>	<b>10</b>

This metrological table is presented in two sub-columns, the left one containing length measurements in ascending order, the right one containing numbers written in sexagesimal place-value notation. The correspondence between measurement values and numbers in sexagesimal place-value notation is, at first sight, proportional:

- To 1 *šu-si* (1 finger, ca. 1,6 cm) corresponds the number 10, so to 2 *šu-si* (2 fingers) corresponds the number 20, etc.
- To 1 *kuš* (1 cubit = 30 fingers) corresponds the number 5, so to 2 *kuš* (2 cubits) corresponds 10, etc.
- To 1 *ninda* (1 pole = 12 cubits) corresponds 1, so to 2 *ninda* corresponds 2, etc.

However, the proportionality seems only local because it appears to be disturbed by a phenomenon of cyclicity: the same number in the right sub-column corresponds to several length measurements in the left sub-column. For example, in this table, the number 10 corresponds to three length measurements, 1 *šu-si*, 2 *kuš* and 10 *ninda*, which differ by a factor 60 (these items are underlined in bold in the translation above). This phenomenon is due to the fact underlined in Section 1 that the cuneiform notation does not indicate the position of the units in the number. Consequently, the left sub-column appears as a regular and increasing progression, while the right sub-column appears as cyclic. Another difference between the sub-columns is that the numbers are not written in the same numerical system. For example, in the item “1  $\frac{1}{2}$  *ninda* 1:30”, the number of *ninda*, 1  $\frac{1}{2}$ , includes a fraction and its order of magnitude is determined, whereas the number that corresponds to this length measurement, 1:30, is written in floating sexagesimal place-value notation. Another striking difference is that the left sub-column is composed of measurement values, i.e. numbers followed by a measurement unit (“concrete numbers” in Bézout’s sense), while the right-hand sub-column is composed of numbers alone (“abstract numbers” in Bézout’s sense).<sup>13</sup>

<sup>13</sup> Bézout’s definition is the most commonly adopted nowadays: for him, a concrete number is followed by a denomination (measurement units or names of the items counted), while an abstract number is not. “A number which is stated without designating the kind of units, as when one simply says *three* or *three times*, *four* or *four times*, is called an *abstract number*; and when one states at the same time the kind of units, as when one says *four pounds*, *one hundred barrels*, it is called a *concrete number*”. My translation. Original text: “Un nombre qu'on énonce sans désigner l'espèce des unités, comme quand on dit simplement *trois* ou *trois fois*, *quatre* ou *quatre fois*, s'appelle un *nombre abstrait*; et lorsqu'on énonce en même temps l'espèce des unités, comme quand on dit *quatre livres*, *cent tonneaux*, on l'appelle *nombre concret*” (Bézout 1764 (ed. 1781)). For more comments on Bézout’s definition, see other chapters of the present Special Issue (Vandendriessche & Proust, Ferreira & Schubring, Chambris & Visnovska).

## 2-2 “Nombres concrets et nombres abstraits”

Just after the note on the “*Table de Senkereh*”, Thureau-Dangin published a note entitled “Concrete numbers and abstract numbers in the Babylonian numeration” (“*Nombres concrets et nombres abstraits dans la numération babylonienne*”) where he offers a penetrating analysis of the role of sexagesimal place-value notation in calculations.

[In the “Esagila Tablet”] it is a question of calculating three rectangular surfaces: that of the large courtyard, that of the courtyard of Ištar and Zababa and that of the base of the multi-story tower. The concrete data of each problem are translated into abstract numbers, without a determined order of magnitude, and it is on these numbers that the calculator operates. [...] The result is still an abstract number. The last operation consists in going from the abstract to the concrete.<sup>14</sup>

However, surprisingly, Thureau-Dangin describes the process of calculation through the so-called “Esagil Tablet”, an administrative text dating from the Hellenistic period, more than one millennium after the Old-Babylonian period. Thus, he makes no explicit connection between *Note LV* on the “*Table de Senkereh*” and the process from the concrete to the abstract and vice versa, which he describes in the next note.

Today we know abundant sources—unknown in Thureau-Dangin’s time—which are much more relevant than the Esagil tablet to serve as evidence for an explanation of the use of the sexagesimal place-value notation in calculation. These sources are school texts which date to the same period, and come from the same region, South Mesopotamia, and were produced in the same environment, scribal schools, as the “Senkereh Table”. In my view, the process described by Thureau-Dangin is clearly apparent in school tablets devoted to the calculation of the area of a square, such as those found in large numbers in Nippur.<sup>15</sup> For example, on the tablet from Nippur kept at the University of Philadelphia under the number UM 29-15-192 (Figure 2), one sees the “concrete” data, the length of the sides of the square (here 2 šu-si, that is, 2 fingers, ca. 3,2 cm), and the multiplication implemented on “abstract” numbers in sexagesimal place value notation, in this case the square of the number 20, which produces 6:40. As I already stressed elsewhere several times,<sup>16</sup> the key point is that the relationship between the “concrete” length of the sides and the “abstract” numbers in sexagesimal place value notation is precisely that given by the metrological tables. For example, 2 šu-si, the side of the square given in the tablet, corresponds to the number 20 in the metrological table “Senkereh Table” (BM 92698) published by Thureau-Dangin (see translation above); 20 is the number noted on the left top of the tablet, of which the square, 6:40, is given. The number 6:40 corresponds to the area  $1/3 \text{ še}$  ( $1/3$  grain, ca. 10 cm<sup>2</sup>) according to metrological tables for surfaces, abundantly attested in school archives.

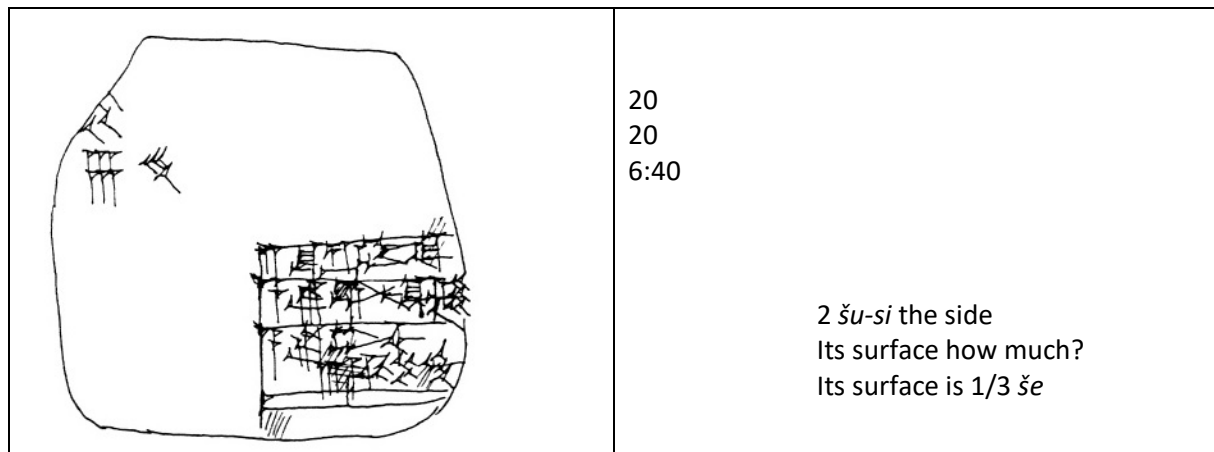
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<sup>14</sup> My translation. Original text: « [Dans la « Tablette de l’Esagil »], il s’agit de calculer trois surfaces rectangulaires : celle de la grande cour, celle de la cours d’Ištar et Zababa et celle de la base de la tour à étages. Les données concrètes de chaque problème sont traduites en nombres abstraits, sans ordre de grandeur déterminé, et c’est sur ces nombres qu’opère le calculateur. ... Le résultat est encore un nombre abstrait. La dernière opération consiste à passer de l’abstrait au concret. » Thureau-Dangin 1930b : 117, from which the quotation in Section 1 is also extracted.

<sup>15</sup> These tablets were not published until the 1940s by Otto Neugebauer and Abraham Sachs.

<sup>16</sup> See in particular Proust 2013.





**Figure 2** Tablet UM 29-15-192, copy Neugebauer and Sachs 1984, 251; my translation

This tablet, which was unknown to Thureau-Dangin, perfectly illustrates the process he described in 1930: “the method used by the Babylonians [which] shows how, in their calculations, they went from the concrete to the abstract, then back from the abstract to the concrete” (Thureau-Dangin 1930b, 117, already quoted in Section 1).

Thureau-Dangin does not explain how precisely, in his view, the scribes, in their calculations, passed from the concrete to the abstract and vice versa. He does not refer to the metrological tables, in particular to the table of length which is precisely included in the “Senkereh Table”. However, if *Notes LV* (“*La table de Senkereh*”) and *LVI* (“*Nombres concrets et nombres abstraits*”) were placed by Thureau-Dangin one after the other in the same issue of the “*Revue d’Assyriologie*”, it was probably intentional, and we can guess that Thureau-Dangin had in mind a link between these two notes. However, this link is not explicitly stated. Thureau-Dangin’s caution can be explained by the lack of comparative material available at the time, and perhaps also by the intellectual context.<sup>17</sup>

### 2–3 Abstract numbers: Thureau-Dangin versus Bézout

The opposition between concrete and abstract numbers in Thureau-Dangin’s sense refer specifically to certain properties of numbers in cuneiform mathematical texts. These two categories are not only modern distinctions imposed by historians, but they were undoubtedly recognized as such by the ancient actors. It is clear in the layout of school texts, particularly the metrological tables and the calculations of the area of squares discussed above, where the two categories are separated. This opposition may have existed in the ancient terminology. In the Sumerian language used in school texts, two terms for “numbers”, or “calculation” are attested: “*šid*” seems to refer to calculations with numbers in sexagesimal place-value notation, and “*nig*” seems to refer to counting with quantities. In any case, it is clear that these two categories correspond to distinct moments of the calculation, carefully identified by the ancient masters of scribal schools.

However, the opposition between concrete and abstract numbers as it was formalized by Thureau-Dangin may be confusing to a modern reader. Indeed, it induces the idea (which Thureau-Dangin did not have, by the way) that an abstract number is a concrete number from which one has removed the measurement units. Yet, Thureau-Dangin’s “abstract” numbers are different in nature from the “concrete numbers” associated with measurement units: the former are written in floating sexagesimal place-value notation, and therefore they are unsuitable to be used to express quantities; the latter are written in notations based on an additive principle (a different sign is used for each order

<sup>17</sup> This paradox is discussed with great finesse by Pierre Chaigneau in Chapter 3 of his PhD thesis (Chaigneau 2019). The battle of wits between Neugebauer and Thureau-Dangin in the 1920s about the sexagesimal place-value notation is particularly revealing in this regard (*Ibid.*: Section 3.5.2). One can suspect that Neugebauer could have intimidated the Assyriologist with his aura as a mathematician. I will pursue this point later.

of magnitude, for example for 1, 10, 60, 600, 3600, etc.), so their order of magnitude is perfectly defined, and they are suitable the expression of quantities. For Thureau-Dangin, it is not the presence or absence of measurement units that fundamentally distinguishes concrete numbers from abstract numbers. As we can see, Thureau-Dangin does not give to the expression “abstract number” the same meaning as Bézout in his arithmetic.

However, Thureau-Dangin’s definition coincides in most cases with Bézout’s. Indeed, as a general rule, the “abstract” numbers (in Thureau-Dangin’s sense) are not followed by measurement units, and the “concrete” numbers are followed by measurements units. This general rule, which Thureau-Dangin had certainly noticed but did not mention, probably justifies the choice he made of the expressions “abstract numbers and concrete numbers”.

There are, however, exceptions to this general rule. Actually, it happens in some instances that “concrete numbers” in Thureau-Dangin’s sense are “abstract numbers” in Bézout’s sense and vice versa. Thureau-Dangin remarked that in some instances, an “abstract number” is followed by a measurement unit.<sup>18</sup> This measurement unit is not a part of a measurement value, like in the notation “1 ½ *ninda*”, where 1 ½ is the number of *ninda* that composes the measurement of a length. The measurement unit is simply apposed to the number. The measurement unit plays the role of a kind of determinative: it is an indication of the order of magnitude of the quantity which corresponds to a number in sexagesimal place-value notation.<sup>19</sup>

Such a notation [sexagesimal place-value notation] could easily lead to ambiguity. Sometimes, in order to prevent a possible misinterpretation, the scribe follows the learned notation by the common notation. Thus, in problems 44 [...] and 231 [...], the number 5, which here designates 5’ of NINDA [i.e. 5/60 NINDA], is followed by the indication “1 cubit”. It was also written in abbreviated form 5 *ammatum*, which in this case does not mean “five cubits”, but “5’ (or 1) cubit”. (Thureau-Dangin 1938, xvi)<sup>20</sup>

Conversely, a “concrete number” may be deprived of measurement units or name of the items counted that would be attached to it. This phenomenon is rare, and was unknown to Thureau-Dangin because it appears in texts that were published after his death. It is attested for example in a lexical list (CBS 11319+) where the signs used to count are enumerated systematically by increasing order, like in the rhyme “1, 2, 3, etc.” (Proust 2008). It also happens that “concrete numbers” (in Thureau-Dangin’s sense) are separated from measurement units in the margins of administrative tablets where they reflect traces of additions and subtractions (Ouyang and Proust forthcoming). Removing the measurement units is interpreted by some historians as a process of decontextualization and abstraction, leading to the concept of abstract number, as we will see in Section 4. However, the examples just evoked show that removing the measurement units may reflect other types of processes than abstraction, for example a lexical approach in the former example, and an accounting calculation in the latter.

For Thureau-Dangin in 1930, an “abstract number” is not a “concrete number” deprived of its measurement unit, but a calculation tool, essentially for multiplication and division. There is thus an essential difference between the way Thureau-Dangin considers the concrete *versus* abstract dichotomy, and the one found in the histories of arithmetic of the 19th century, for example that of

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<sup>18</sup> In his later publications, Thureau-Dangin replaced the expression “abstract numbers” by “learned notation” (*notation savante*). More on this revealing change of terminology in Section 3.

<sup>19</sup> See an inventory of all attestations of apposition in the mathematical texts from the Diyala region, located in Northern Mesopotamia, in (Gonçalves forthcoming).

<sup>20</sup> My translation. Original text: « Un tel mode de notation [la notation sexagésimale positionnelle] pouvait aisément prêter à ambiguïté. Parfois, en vue de prévenir une erreur d’interprétation possible, le scribe fait suivre la notation savante par la notation commune. Ainsi, dans les problèmes 44 ... et 231 ..., le chiffre 5, qui désigne ici 5’ de NINDA [i. e. 5/60 NINDA], est suivi de la mention " 1 coudée". On écrivait aussi en abrégé 5 *ammatum*, qui, dans ce cas, signifie non pas "cinq coudées", mais « 5’ (ou 1) coudée. »

Peacock or Bézout - See other chapters of the present Special Issue, for example (Proust & Vandendriessche, Vandendriessche, Chambris & Visnovska, Ferreira & Schubring). Indeed, for them the operations act in the same way on both kinds of numbers. In Peacock's "Arithmetic", for example, we can read:

We are thus lead to the distinction of numbers into abstract and concrete, though the abstraction exist merely in the word by which any number is designated, or in the equivalent symbol by which it is represented in different arithmetical systems. In Arithmetic we consider both kinds of numbers, though the operations are in all cases the same as if the numbers were perfectly abstract; the association of qualities being merely of use in directing us to the particular operations or reductions to be performed, and in assisting us in the proper interpretation of the result. (Peacock 1826 (ed. 1845), 147, note to section 2)

By contrast, in the case of cuneiform texts, the operations are not the same according to whether they are performed on "concrete" numbers or on "abstract" numbers. In the first case, the numbers are quantities on which additions and subtractions essentially operate; in the second case, numbers are calculation tools written in floating sexagesimal place-value notation on which multiplications and divisions essentially operate.<sup>21</sup> Thureau-Dangin does not clearly formulate the opposition between addition-subtraction on the one hand, and multiplication-division on the other, but he does draw a link between "this very abstract system" (the sexagesimal place-value notation) and "divisions and multiplications" (Thureau-Dangin 1930b , 117, already quoted in Section 1).

Note that some historians of mathematics of the beginning of the 20th century, like David Eugene Smith or Florian Cajori, as well as Otto Neugebauer subsequently, rejected or ignored the dichotomy "abstract numbers / concrete numbers" (in Bézout's sense), considering that only the "abstract numbers" (or "pure numbers") belong to mathematics, the concrete numbers, i.e. the numbers specified by a measurement unit, being excluded from the mathematical field.<sup>22</sup>

### 3- How "abstractness" is reflected in transcriptions of numbers?

For Thureau-Dangin in 1930, the floating character of the numbers in sexagesimal place-value notation is clear when he refers to "This very abstract system, which did not distinguish between integers and fractions, which ignored the order of magnitude of the numbers" (Thureau-Dangin 1930b , 117, already quoted in Section 1). What is abstract in these numbers is first and foremost the absence of an order of magnitude.

Thureau-Dangin's understanding of numbers in cuneiform texts was largely ignored by contemporaries and successors.<sup>23</sup> Thureau-Dangin's description of the place-value notation in 1930, with his emphasis on the absence of indication of the order of magnitude, is subtly different from that of later historians,

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<sup>21</sup> For a more thorough discussion on the fundamental reasons why the additive systems used in metrological notations are suitable for additions and subtractions, and the sexagesimal place-value notation, a floating system, is suitable for multiplications and related operations, see (Proust 2013) To fix ideas without going into details, an example will suffice. One cannot add the numbers 1:30 and 30 if one does not know their orders of magnitude, at least relative; depending on their relative positions, one will obtain 31:30, or 2, or 2:30:30, etc. On the other hand, multiplying these two numbers in floating notation gives 45, no matter how one places them relative to each other. Similarly, dividing 1:30 by 30 gives 3 in floating notation.

<sup>22</sup> (Smith 1925 : 11-12; Cajori 1893). In fact, Cajori does not explicitly mention this dichotomy, but implicitly assumes it insofar as metrology is absent from his history of mathematics.

<sup>23</sup> Jens Høyrup insists on Thureau-Dangin's deep understanding of the numerical notations in cuneiform texts, in particular in his *Esquisse d'une histoire du système sexagésimal* (Thureau-Dangin 1932b), and on the little historiographical impact of Thureau-Dangin's point of view: "Thureau-Dangin's *Esquisse d'une histoire du système sexagésimal* [...] points out very explicitly that the place value system was introduced as an *instrument de calcul* (p. 51). This publication [...] gives much more insight into the overall numerical culture of ancient Mesopotamia than Neugebauer's papers on the topic from 1930 to 1932. However, [...] this study never had much impact on the historiography of mathematics" (Høyrup 2016: 184).

who interpreted this absence as a default of the cuneiform notation. As a consequence, most of the historians of mathematics specify the order of magnitudes of the numbers in sexagesimal place-value notation in their translations and/or commentaries, following Neugebauer: “In my view, indication of order of magnitude only seems justified when it comes to a translation of a text” (Neugebauer 1932-3).<sup>24</sup>

What is more puzzling is that Thureau-Dangin himself watered down his first analysis later and abandoned the expression “abstract numbers”. In 1938, he no longer speaks of “abstract numbers” but of “learned notation” (*notation savante*) (Thureau-Dangin 1938). This change of terminology is associated with a change in the representations of numbers in his publications. In 1930, he does not specify the orders of magnitude, while in 1938, he does so.

For example, in 1938, describing the “Esagil Tablet”, Thureau-Dangin states that the length “10 GAR 6 pas  $\frac{2}{3}$ ” corresponds to the abstract number “10.33.20” and formulates the operations as follows:

$$10.33.20 \times 4.30 = 47.30; \quad 47.30 \times 18 = 14.15 \text{ (Thureau-Dangin 1930b, 117)}$$

These operations are multiplications of numbers corresponding to the length and width of rectangles, producing the areas. These “equalities” show clearly that the orders of magnitude are ignored.

In 1932, Thureau-Dangin does not specify the orders of magnitude in the translations (Figure 3a) but specify them in brackets in his explanations (Figure 3b).<sup>25</sup>

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<sup>24</sup> Neugebauer specified the orders of magnitude of the numbers by introducing a semi-column (“;”) to separate the integer and the fractional part of a number (eg. 1;30 means  $1 + \frac{30}{60}$ ) and by completing the numbers with zeros if necessary (eg. 1.0 means 60, or 0;1 means  $\frac{1}{60}$ ). More on this in (Proust forthcoming).

<sup>25</sup> Thureau-Dangin specified the orders of magnitude of the numbers by using an extended version of the system “degree, minutes, seconds, etc.” that we use today for time, and he fixed the degree as unit. For example,  $3^{\circ}30'15''$  means  $3 \times 60 + 30 + \frac{15}{60}$ .

Toi, en opérant, ajoute 27, la somme du flanc et du front, à 3.3, (cela te donne) 3.30. Ajoute 2 à 27, (cela te donne) 29. Tu fractionneras en deux 29, (cela te donne) 14.30).  $14.30 \times 14.30 = 3.30.15$ . De 3.30.15 tu retrancheras 3.30; 15 est le reste.  $15 = 30$  au carré. Ajoute 30 au premier 14.30, (cela te donne) 15 le flanc. Tu soustrairas 30 du deuxième 14.30, (cela te donne) 14 le front. 2 que tu as ajouté à 27, tu le retrancheras de 14 le front, (cela te donne) 12 le front exact. 15 le flanc et 12 le front, multiplie-les (entre eux) :  $15 \times 12 = 3$  la surface. De combien 15 le flanc excède-t-il 12 le front ? Il excède de 3. Ajoute 3 à 3 la surface, (cela te donne) 3.3 la surface<sup>4</sup>.

Figure 3a (Thureau-Dangin 1932a, 5)

(1 <sup>er</sup> ouvrier)	7 ( $\times 12 =$ )	1.24	(1.24 $\times$ 2.13.20 =)	3.6.40 (3°6'40")
(2 <sup>e</sup> — )	11 ( $\times 12 =$ )	2.12	(2.12 $\times$ 2.13.20 =)	4.53.20 (4°53'20")
(3 <sup>e</sup> — )	13 ( $\times 12 =$ )	2.36	(2.36 $\times$ 2.13.20 =)	5.46.40 (5°46'40")
(4 <sup>e</sup> — )	14 ( $\times 12 =$ )	2.48	(2.48 $\times$ 2.13.20 =)	6.13.20 (6°13'20")

Figure 3b (Thureau-Dangin 1932a, 9)

**Figure 3** Partial translation and explanation of problem 1 in tablet AO 8862 (Thureau-Dangin 1932a, 5, 9).

As we can see in Figure 3a, Thureau-Dangin translates the beginning of the procedure without indication of the orders of magnitude of the numbers:

You, in your procedure, add 27, the sum of the flank [length] and the front [width], to 3.3, (this gives you) 3.30. Add 2 to 27, (this give you) 29. You will break into two 29, (this gives you 14.30).  $14.30 \times 14.30 = 3.30.15$ .

In 1938, Thureau-Dangin specifies the orders of magnitude, as shown in Figure 4 by the translation of the same problem as in Figure 3 (problem 1 in tablet AO 8862):

Toi, en opérant, ajoute 27, la somme du flanc et du front, à [3'3] : 3'30. Ajoute 2 à 27 : 29. Tu fractionneras en deux 29 : (14°30'). 14°30' fois 14°30' : 3'30°15'. De 3'30°15' tu soustrairas 3'30 : 15', le reste. 15' est le carré de 30'. Ajoute 30' au premier 14°30' : 15, le flanc. Tu retrancheras 30' du deuxième 14°30' : 14, le front. 2, que tu as ajouté à 27, tu soustrairas de 14, le front : 12, le front vrai. Croise 15, le flanc, et 12, le front. 12 fois 15 : 3', la surface. De quoi 15, le flanc, excède 12, le front ? Il excède de 3. Ajoute 3 à 3', la surface : 3'3, la surface<sup>1</sup>.)”

**Figure 4** Partial translation of problem 1 in tablet AO 8862 (Thureau-Dangin 1938, 65)

As we can see in Figure 4, Thureau-Dangin translates the beginning of the same procedure by introducing an indication of the orders of magnitude of the numbers (the units are followed by the symbol for degree [°], the sixtieth by the symbol for minute ['], and so on):

You, in your procedure, add 27, the sum of the flank [length] and the front [width], to [3'.3]: 3'.30. Add 2 to 27: 29. You will fraction in two 29: 14°30'.  $14°30' \times 14°30' : 3'30°15'$ .

In Thureau-Dangin’s publications, the expression “abstract numbers” disappears together with the introduction of the orders of magnitudes of numbers in translations. This confirms that, for Thureau-Dangin, the “abstract” character of numbers is linked to their “floating” nature.

The thousands of mathematical school tablets known today allow us to revive and systematize the first intuitions of Thureau-Dangin.<sup>26</sup> We know that metrological tables belong to the corpus of “school texts”, i.e. texts that were used in Old Babylonian scribal schools for the education of future bureaucrats and scholars. The metrological tables served to teach the metrological and numerical systems used in mathematics, trade and administration. In the context of elementary education in the Old-Babylonian scribal schools, the numbers in sexagesimal place-value notation were used exclusively for multiplication and division (and derived operations). There is no addition nor subtraction in elementary mathematical training. It appears that the floating character of the place-value numbers would be inherent to the numerical system, and that the place of the units is not indicated, not because it is implicit, but because it does not exist. Anyway, without any indication of their order of magnitude, numbers in sexagesimal place-value notation are suitable for multiplication and division, but not for quantification. This case can be compared to the case of numbers represented on the abacus in certain contexts in China (Chemla forthcoming).

#### 4- Abstraction seen as a step in a historical process

These considerations lead to an interest in the use of the word “abstract” in historiography. Eric Vandendriessche’s article in this special issue shows how, in 19<sup>th</sup> century ethnographic writings, abstraction is seen as a process of thought development. Ferrara & Valerio’s article in this same issue shows how the history of number in Mesopotamia is told as a process from the concrete to the abstract. To complete these analyses, I propose some comments on the way in which the history of arithmetic is considered in the landmark book *Archaic Bookkeeping. Writing and Techniques of Economic Administration in the Ancient Near East* (Nissen, Damerow, and Englund 1993). I concentrate my comments on this book because it is very influential and popular among Assyriologists and the general public. The book, which has become an indispensable textbook for teaching the early intellectual history of Mesopotamia, is based on a first-hand analysis of numerous sources dating from the invention of writing in the middle of the 4<sup>th</sup> millennium BCE to the development of mathematics in the Old-Babylonian period. The authors, who are among the foremost specialists in archaic documents, detail the history of the writing of numbers and measurement units, essentially in connection with their accounting uses.

Chapter 16, written by Peter Damerow and Robert Englund, deals with “The development of Arithmetic”. The chapter distinguishes three main stages: “proto-arithmetic”, “archaic arithmetic”, and “arithmetic” proper. The authors contrast the modern “concept of number” with the earliest forms of representation of numbers by tokens, or “counting aids”.

There is a fundamental difference between modern numerical signs, on the one hand, for example, the sign “5,” and symbolic counting aids on the other, as in the case of fingers used in counting. The ability to use the sign “5” requires the understanding of its meaning, namely, an understanding of what a number is and the knowledge of the specific number “5” represented by the sign. With finger-aided calculations, the situation is clearly different. In this case, fingers do not represent numbers but rather represent merely the counted objects or higher counting units such as counting groups. This use of symbols, which stand for discrete, concrete objects and so function only as an indirect means of carrying information about their number, assumes no developed forms of arithmetic thought and no concept of number, but rather only the ability to establish correspondences between the number symbols and the counted objects. Since these aids and their related techniques precede the developmental stages of proper arithmetic, such techniques are called “proto-arithmetical.” (Nissen, Damerow, and Englund 1993, 125)

We recognize here an issue often discussed in the field of the history of mathematics, the question of knowing if some ancient actors have or have not acceded to the “concept of number”. For Damerow and Englund, a material system that serves to count concrete objects does not testify to an

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<sup>26</sup> This is the thesis I have defended in numerous publications - See (Proust 2013) and the related bibliography.

understanding of what “a number is”, and thus does not fall within the concept of number. The “proto-arithmetic” stage is described as primitive, and as often in such cases, especially in ethnographic historiography (see Vandendriessche’s article in the present Special Issue), is paralleled with that of “primitive societies” and that of small children:

Proto-arithmetical aids are known from virtually all studied primitive cultures. They also occur in modern society, since they are of great assistance in conveying our notion of abstract numbers to small children. In every culture, the proto-arithmetical aids form the foundation and necessary precondition for the development of an explicit number concept. (Nissen, Damerow, and Englund 1993, 125)

Along the same lines as Schmandt-Besserat (see Ferrara & Valerio’s chapter in this Special Issue), Damerow and Englund analyze the clay tokens, as well as their printed traces on the outside of the bullae that sometimes contained these tokens, as precursors of the numerical signs written on archaic tablets. These first-written signs conveyed quantitative information, but, according to the authors, were not “abstract numbers”.

[...] before the appearance of the first true ideograms, numerical notations were already known. In other words, the first written signs were therefore exclusively “numerical”. During this phase, their shapes most probably did not follow strict conventions. They certainly were not numerals in the modern sense. In fact, they were signs used for counting units with qualitative connotations and by no means signs for abstract numbers. [...] This development still did not seem to lead to a modern kind of numerical notation, but rather to a peculiar symbol system for the codification of quantitative information unparalleled in the later history of arithmetic. (Nissen, Damerow, and Englund 1993, 130)

After the “proto-arithmetic”—according to the authors—comes the “archaic arithmetic”. In the periods of the Early Dynasties (ED I-II, ca. 2900-2700 BCE), the calculation of areas illustrates what the authors call “archaic arithmetic”, that is an arithmetic which does not have the abstract concept of number nor general methods of calculation (Nissen, Damerow, and Englund 1993, 134). For them, only with the texts of the Fara period (ca. 2600-2500 BCE) arithmetic begins (Nissen, Damerow, and Englund 1993, 138). They consider that the decisive following step is the invention of the sexagesimal place-value notation.

The appearance of the sexagesimal place-value notation in these texts is generally dated to the Ur III period (ca. 2100-2000 BCE). Indeed, the earliest evidence of a systematic use of this notation is found in reciprocal tables from this period. However, Damerow and Englund date the probable appearance of the sexagesimal place-value notation later, to the Old-Babylonian period (Nissen, Damerow, and Englund 1993, 142, col. 2).<sup>27</sup> For these authors, the invention of the sexagesimal place-value notation seems to be linked to the need for a “short” notation for large numbers. The sexagesimal place-value notation “offered virtually unlimited possibilities of calculating according to uniform rules with arbitrarily large or small numbers”. (Nissen, Damerow, and Englund 1993, 142). According to the authors, the scribes had to invent a system to convert numbers written in the old notation into writing in the “new” more convenient notation, to express small and large numbers and perform calculations. This conversion tool was, they thought, the role of the metrological tables.

In order to use the new calculation techniques in the scribal profession, the traditional numerical notations had to be translated into the new notations. Scribes really did perform such exercises, as is evidenced by the fragments of the tablets shown in figures 125–126 [these figures represent metrological tables]. (Nissen, Damerow, and Englund 1993, 143).

In the rest of the chapter, the authors oppose the “traditional notation” and the “new notation”. They do not mention—as Thureau-Dangin did—any link between the sexagesimal place-value notation and two particular operations: multiplication and division.

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<sup>27</sup> On the problem of dating these reciprocal tables, see (Ouyang and Proust forthcoming).

In summary, the authors of chapter 16 of “Archaic Bookkeeping”, by recognizing three stages, proto-arithmetic, archaic arithmetic and arithmetic proper, adopt an evolutionary scheme to outline the history of arithmetic. This evolution would be based on changes in the conceptions of numbers, from a “concrete” approach—the numbers being dependent on what is quantified—to an increasingly abstract approach of the number concept, suitable for arithmetic thinking. The “concept of number” functions as the focus point toward which the subsequent generations of scribes evolved. The sexagesimal place-value notation is presented as not only a conceptual progress, but also as a technical one, facilitating the writing of large numbers and calculation. In this evolutionary scheme, the sexagesimal place-value notation was intended to replace the ancient non-positional notations.

However, the cuneiform sources of the 2<sup>nd</sup> and 1<sup>st</sup> millennia show that several systems of numeration, additive and positional, have always coexisted in mathematical cuneiform texts. The sexagesimal place-value notation has never driven out the non-positional notations, as pointed out by Thureau-Dangin as soon as 1932:

This system [the sexagesimal place-value notation] was, as we can see, eminently abstract. It was not created in order to be substituted, in the expression of concrete numbers, for the traditional system, but in order to be an instrument of calculation. (Thureau-Dangin 1932b, 51–52).<sup>28</sup>

The place-value notation is therefore an invention with a precise function, different from that of other notations. By establishing a link between the “abstract numbers”, to wit, the sexagesimal place-value notation without indication of the order of magnitude, with the techniques of multiplication and division, Thureau-Dangin developed a profound analysis which, however, remained largely ignored by his contemporaries and successors.

#### 5- “Abstract numbers”: flash idea or historiographical chimera?

I consider that Thureau-Dangin’s characterization of place-value numbers as “abstract numbers” in 1930 was visionary in that it highlighted their floating character. For this reason, in my view, this characterization was not adopted by other historians of mathematics, who saw the indeterminacy of the order of magnitude as a defect of cuneiform writing to be corrected in translations and commentaries, and not as an intrinsic property of the numbers themselves. Thureau-Dangin became subsequently influenced by this dominant trend. Going against the grain, when I began studying cuneiform mathematics, Thureau-Dangin’s characterization of the numbers in place-value notations as “abstract” seemed insightful to me, and I adopted this expression in my early publications. However, I question this choice today because I realized, particularly in discussions concerning the notions of “abstract number” and “abstraction” that took place in the SPHERE<sup>29</sup> seminars, that the word “abstract” conveys many implicit meanings (I give some examples in Section 4). Moreover, while my earliest works focused on Old Babylonian texts (dated ca. 2000-1600 BCE), later on I explored older documents (4<sup>th</sup> and 3<sup>rd</sup> millennia), and I was confronted with a profusion of forms and syntaxes of quantification that defied the very definition given to the word “abstract”, and even to the word “number”. Therefore, I now, prefer to use the expression “abstract numbers” more cautiously because of the parasitic undertones that the notion of abstraction carries in historiography.

The examples given in this brief overview show that the notion of abstraction is quite relative. The shift of numbers from concrete to abstract is not perceived in the same way by the different historians who studied it. For Denise Schmandt-Besserat, the shift occurred at the time of the invention of writing (end of the 4<sup>th</sup> millennium). For Damerow and Englund, it marks the beginning of arithmetic with the emergence of the concept of number (mid-3<sup>rd</sup> millennium). For Thureau-Dangin, it is embodied by the

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<sup>28</sup> My translation. Original text: “Ce système était, on le voit, éminemment abstrait. Il a d’ailleurs été créé non pas en vue d’être substitué, dans l’expression des nombres concrets, au système traditionnel, mais en vue d’être un instrument de calcul.”

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sexagesimal place-value notation, and this kind of “abstraction” is specific to numbers which are tools of calculation. More generally, Eric Vandendriessche’s work has highlighted the historiographical biases conveyed by the notion of abstraction, particularly in the milieu of 19th and 20th century anthropologists. Abstraction turns out to be an evanescent notion, falling under fluctuating categories, or conveying anachronistic views.

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