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## Chapter 1

# Foundations of mathematics buried in school garbage (Southern Mesopotamia, early second millennium BCE)

Christine Proust  
CNRS & Université Paris Diderot  
christine.proust@orange.fr

### Abstract

In this chapter, I suggest an analysis of mathematical texts based on notions, concepts and tools that were instilled during their early education in scholars who were active in Southern Mesopotamia during the Old Babylonian period (early second millennium BCE). The sources considered are mathematical texts from scribal schools that flourished in the Ancient Land of Sumer. I propose to examine the content of the garbage from these schools to access the "internal meanings" of advanced mathematical texts. First, I outline some characteristics of the mathematics taught in elementary education, drawing mainly from sources found in Nippur. Then, I discuss two mathematical texts in detail and show how they can be interpreted using the tools that were taught to (or invented by) their authors or users. The first example (CBS 1215) deals with the extraction of reciprocals by factorization; the second (CBS 12648) deals with volume problems using coefficients. The conclusion underlines the mathematical work that some ancient erudite scribes accomplished in order to emancipate them from the world of elementary education.

### Keywords

coefficient; cuneiform mathematics; curriculum; factorization; reciprocal; scribal schools; sexagesimal place-value notation; volume.

### 1. Basic knowledge, fundamental knowledge

In this chapter, I suggest a way of addressing some scholarly mathematical texts based on notions, concepts and tools that were instilled in the ancient scribes during their early education. This approach could be termed as "emicist", in the sense of "approach to societies or social groups through 'internal' meanings", given to this term by Olivier de Sardan (2015, p. 65). As far as ancient mathematics are concerned, such a search for "internal meanings" has been brilliantly undertaken by Jens Høyrup, who developed a method of "close reading" of mathematical cuneiform texts and revealed the geometrical dimension of apparently algebraic procedures (Høyrup 2002 and Chap. II, this volume). In the present chapter, I argue that another point of access to "internal meanings" would be to consider the mathematical education of the authors themselves. How did the fundamental notions that scholars acquired in their early education impact their mathematical production? There is generally no hope of answering this kind of question due to the scarcity of sources and information on the ancient contexts of education. However, cuneiform sources offer a unique opportunity to lift the veil: thousands of school exercises, mainly from the city of Nippur, a cultural centre situated at the Northern limit of the Ancient Land of Sumer, have come down to us. In general, school drafts were not

intended to be preserved or archived, and they were usually destroyed, which explains their rarity. This is also true of the scribal schools which were active during the Old Babylonian period (early second millennium BCE) in the Ancient Near East. However, a particular property of the writing support lead to the unexpected conservation of the school garbage. Once dry, tablets are solid, almost indestructible objects and were often recycled as building materials. Part of the scribal schools' huge production was incorporated in the foundations of houses, walls, floors or fillings, and these recycling practices contributed to protecting the modest school writings against destruction. I propose to examine the content of this school garbage to access the "internal meanings" of advanced mathematical texts. In the following, I show how whole facets of the mathematical world in which the young scholars were immersed during their elementary education can be reconstructed through school texts. I argue that a modern historian can benefit from a familiarity with the basics of mathematics taught to scholars in their youth in order to penetrate the meaning of some mathematical texts.

The elementary curriculum, notably that of Nippur, is well known, and in the first part (Section 2), I only summarize the main results of previous studies.<sup>1</sup> My aim in this section is to outline some characteristics of the mathematics taught in elementary education, drawing mainly from sources found in Nippur, and to provide the reader with some basic texts, which allow an understanding of the subsequent sections. Then, I discuss some mathematical texts in detail in order to analyze how they can be interpreted using the tools that were taught to (or invented by) the authors or users of these texts. The first example deals with the extraction of reciprocals as detailed on tablet CBS 1215 (Sect. 3); the second is a third degree problem dealing with volumes as detailed on tablet CBS 12648 (Sect. 4). Both tablets are dated to the Old Babylonian period and come from the same region, the Ancient Land of Sumer in Southern Mesopotamia (the lower valley of the Tigris and Euphrates). Whatever the uncertainties about their provenience, it is quite clear that these texts, and others that are mentioned in this chapter, were produced in the context of scribal schools by authors immersed in a mathematical world similar to that developed in Nippur.

## 2. The mathematical world in Nippur schools

At the beginning of the second millennium BCE, scribal schools were particularly active in Nippur. This city has delivered abundant school archives to modern archaeologists. The thousands of school drafts found by American archaeologists between 1888 and 1990 provide a remarkably clear picture of teaching. Among the texts produced in the school context, Sumerian literary texts composed for educational purposes echo the content of the teaching, that is, the "scribal art" to use their own terms. For example, a literary composition called "Dialogue between two scribes" by its translators (Civil 1985 and Vanstiphout 1997) summarizes the school program as follows:<sup>2</sup>

Having been at school for the required period,  
I am now an expert in Sumerian, in the scribal art, in reading tablets, in calculation and in accounting.  
I can even speak Sumerian.

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<sup>1</sup> For example, Veldhuis 1997, Robson 2001, Proust 2007.

<sup>2</sup> Below is the translation of lines 36-38 of the text "Dialogue between two scribes", reconstructed after several duplicates from Nippur and some other Southern cities (my translation after Civil and Vanstiphout).

This extract indicates what was learnt in schools: cuneiform script, writing, reading and speaking Sumerian<sup>3</sup>, calculation and accounting. Exercises written by the young scribes provide more detail on this content and on the order in which it was learnt. The curriculum in Nippur has recently been reconstructed, and it has been shown that schooling was organized in three levels, called by Veldhuis (1997) elementary, intermediate and advanced. The mathematical curriculum in Nippur was in the same way organized in three levels (Proust 2007; Proust forthcoming-a). The elementary level consisted in learning metrological systems, multiplication and reciprocal tables, the sexagesimal place-value system (see explanation Sect. 2.1.2.), and correspondences between measurement values and numbers in sexagesimal place-value notation. The intermediate level consisted in solving short problems dealing with the surfaces of squares, the calculation of reciprocals, and diverse linear situations. I do not enter into the detail of the curriculum, to which an abundant literature has already been devoted,<sup>4</sup> but limit myself here to presenting a selection of some school texts of elementary, intermediate and advanced level that illuminate the main characteristics of the mathematical world to which the scholars were exposed in their early years of education.

## ***2.1. Elementary level***

The first of the mathematical entities introduced in elementary education were measurement values. All the measurement values commonly used in mathematical and economic practices were enumerated in increasing order, in lists of capacity, weight, surface and length measurements, the so-called metrological lists. Then, a set of numerical tables was introduced (reciprocals, multiplication tables, square and square root) operating on numbers in sexagesimal place-value notation (hereafter SPVN). At the same time, a system of correspondence between measurement values and numbers in sexagesimal place-value notation was established through metrological tables similar to that translated in Sect. 2.1.1.

Metrological and numerical tables provided the basic toolbox used in mathematics in scribal schools, and, for this reason, they are also indispensable for a modern reader to understand mathematical cuneiform texts. I give below (Sect. 2.1.1) the translations of the tables that will be useful for the following discussions.<sup>5</sup> The order adopted below is not necessarily that of the curriculum.<sup>6</sup>

### **2.1.1 Metrological tables**

The metrological tables of lengths establish a correspondence between length measurements and numbers in SPVN. These tables were used for linear dimensions belonging to a horizontal

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<sup>3</sup> Sumerian is a dead language at this time, while the mother tongue of the young scribes attending the schools was Akkadian, a Semitic language.

<sup>4</sup> See Veldhuis 1997, Robson 2001, Proust 2007 and bibliography provided by these publications.

<sup>5</sup> The same metrological and numerical tables are attested in many duplicates from Nippur and other proveniences. The variations from one source to another being insignificant, any copy can be used indifferently. The sources that I present in this section come from Nippur if possible. However, in some cases, I use sources from other proveniences that are in better condition than those from Nippur. All the intermediate level sources used in part 2.2 come from Nippur. The tablet possibly reflecting the advanced level quoted in part 2.3 comes from another Southern city, possibly Larsa. All the sources quoted in this chapter are dated to the Old Babylonian period.

<sup>6</sup> The order adopted in the curriculum, metrological tables of capacity, weight, surface, length and height, numerical tables of reciprocal, multiplication, and so on, is not the most intuitive for a modern reader. For an approach closer to the curriculum, see Proust 2007, Proust forthcoming-a.

plan, for example, for the sides of a field or the base of a canal. The metrological table of lengths translated below (Table 1) is of unknown provenience.<sup>7</sup>

**Table 1.1** Metrological table of lengths (MS 3869/11)<sup>a</sup>

Obverse		Reverse	
1 <i>šu-si</i>	10	5 <i>kuš</i>	25
2 <i>šu-si</i>	20	$\frac{1}{2}$ <i>ninda</i>	30
3 <i>šu-si</i>	30	$\frac{1}{2}$ <i>ninda</i> 1 <i>kuš</i>	35
4 <i>šu-si</i>	40	$\frac{1}{2}$ <i>ninda</i> 2 <i>kuš</i>	40
5 <i>šu-si</i>	50	$\frac{1}{2}$ <i>ninda</i> 3 <i>kuš</i>	45
6 <i>šu-si</i>	1	$\frac{1}{2}$ <i>ninda</i> 4 <i>kuš</i>	50
7 <i>šu-si</i>	1:10	$\frac{1}{2}$ <i>ninda</i> 5 <i>kuš</i>	55
8 <i>šu-si</i>	1:20	1 <i>ninda</i>	1
9 <i>šu-si</i>	1:30	1 $\frac{1}{2}$ <i>ninda</i>	1:30
$\frac{1}{3}$ <i>kuš</i>	1:40	2 <i>ninda</i>	2
$\frac{1}{2}$ <i>kuš</i>	2:30	2 $\frac{1}{2}$ <i>ninda</i>	2:30
$\frac{2}{3}$ <i>kuš</i>	3:20	3 <i>ninda</i>	3
1 <i>kuš</i>	5	3 $\frac{1}{2}$ <i>ninda</i>	3:30
1 $\frac{1}{3}$ <i>kuš</i>	6:40	4 <i>ninda</i>	4
1 $\frac{1}{2}$ <i>kuš</i>	7:30	4 $\frac{1}{2}$ <i>ninda</i>	4:30
1 $\frac{2}{3}$ <i>kuš</i>	8:20	5 <i>ninda</i>	5
2 <i>kuš</i>	10		
3 <i>kuš</i>	15		
4 <i>kuš</i>	20		

<sup>a</sup> The length unit *šu-si* (a finger) represents about 1.6 cm; the units *kuš* (a cubit) is 30 *šu-si*, and the unit *ninda* is 12 *kuš* (see synthesis in Appendix).

The metrological tables of heights establish a correspondence between measurements values of vertical lines and numbers in SPVN. These tables were used, for example, for the depth of a canal or the thickness of a brick in order to calculate their volume. Indeed, a volume unit is defined in the Old Babylonian metrology as a surface unit with a thickness of 1 *kuš* (ca. 50 cm), which corresponds to the number 1 in SPVN – see Appendix. The metrological table of heights translated below (Table 2) comes from Nippur.<sup>8</sup>

**Table 1.2** Metrological table of heights, HS 243, from Nippur

Obverse		Reverse	
1 <i>šu-si</i>	2	2 <i>kuš</i>	2
2 <i>šu-si</i>	4	3 <i>kuš</i>	3
3 <i>šu-si</i>	6	4 <i>kuš</i>	4
4 <i>šu-si</i>	8	5 <i>kuš</i>	5
5 <i>šu-si</i>	10	$\frac{1}{2}$ <i>ninda</i>	6
6 <i>šu-si</i>	12	$\frac{1}{2}$ <i>ninda</i> 1 <i>kuš</i>	7
7 <i>šu-si</i>	14	$\frac{1}{2}$ <i>ninda</i> 2 <i>kuš</i>	8
8 <i>šu-si</i>	16	$\frac{1}{2}$ <i>ninda</i> 3 <i>kuš</i>	9
9 <i>šu-si</i>	18	$\frac{1}{2}$ <i>ninda</i> 4 <i>kuš</i>	10
$\frac{1}{3}$ <i>kuš</i>	20	$\frac{1}{2}$ <i>ninda</i> 5 <i>kuš</i>	11
$\frac{1}{2}$ <i>kuš</i>	30	1 <i>ninda</i>	12
$\frac{2}{3}$ <i>kuš</i>	40		
1 <i>kuš</i>	1		

<sup>7</sup> Tablet MS 3869/11 belongs to the Schøyen Collection and is of unknown provenience, probably from Southern Mesopotamia. This tablet was published in (Friberg 2007, p. 119); photo available at <https://cdli.ucla.edu/P252942>.

<sup>8</sup> Tablet HS 243 is now kept at the University of Jena, and was published in (Hilprecht 1906, pl. 27; Proust 2008, n°31). The photo is available at <https://cdli.ucla.edu/P388162>.

The metrological tables of weights establish a correspondence between weight measurements and numbers in SPVN. The following tablet contains a section for small values, which could be used also for surfaces and volume.<sup>9</sup>

**Table 1.3** Metrological table of weights, surfaces and volumes, MS 2186

<b>Obverse</b>		<b>Reverse</b>	
1/2 še silver	10	27 še	9
1 še	20	28 še	9:20
1 1/2 še	30	29 še	9:40
2 še	40	1/6 gin	10
2 1/2 še	50	1/6 gin 10 še	13:20
3 še	1	1/4 gin	15
4 še	1:20	1/4 gin 5 še	16:40
5 še	1:40	1/3 gin	20
6 še	2	1/3 gin 15 še	25
7 še	2:20	1/2 gin	30
8 še	2:40	1/2 gin 15 še	35
9 še	3	2/3 gin	40
10 še	3:20	2/3 gin 15 še	45
11 še	3:40	5/6 gin	50
12 še	4	5/6 gin 15 še	55
13 še	4:20	1 gin	1
14 še	4:40		
15 še	5		
16 še	5:20		
17 še	5:40		
18 še	6		
19 še	6:20		
20 še	6:40		
21 še	7		
22 še	7:20		
23 še	7:40		
24 še	8		
25 še	8:20		
26 še	8:40		

<sup>9</sup> Tablet MS 2186 of the Schøyen Collection (Friberg, 2007, p. 110), is of unknown provenience, probably from Southern Mesopotamia. The photo is available online (<https://cdli.ucla.edu/P250902>). About the use of the section for small values of tables of weights as tables of surfaces and volume, see Proust forthcoming-b.

### 2.1.2. A multiplication table

A multiplication table (HS 217a)<sup>10</sup> provides an excellent example to explain the sexagesimal place-value notation.

**Table 1.4** Multiplication table HS 217a

Obverse		Reverse	
1	9	15	2:15
2	18	16	2:24
3	27	17	2:33
4	36	18	2:42
5	45	19	2:51
6	54	20	3
7	1:3	30	4:30
8	1:12	40	6
9	1:21	50	7:30
10	1:30	8:20 times 1	8:20
11	1:39		
12	1:48		
13	1:57		
14	2:6		

In the left-hand column of the table (see photo <https://cdli.ucla.edu/P254585>), we see one wedge, two wedges, and so on, nine wedges, one chevron, which represents ten wedges, and so on. The wedge represents the number one, and the chevron represents the number ten. So we recognize the sequence of numbers 1, 2, and so on, 10, 11, and so on, 19, 20, 30, 40 and 50 (converted into the modern decimal system). In the right-hand column, in front of 1 we read nine wedges, that is, the number 9, in front of 2, one chevron and eight wedges, that is, the number 18, in front of 3, the number 27, and so on. We recognize the multiplication table by 9. In front of 7, we expect 63, but we see one wedge, a space, and three wedges. It means that the “sixty” of “sixty-three” is represented by a wedge in the left position. We understand that the notation uses the base 60 and a place-value principle, exactly in the same way as we represent 63 seconds, that is, 1 minute 3 seconds by the number 1:3. This is the so-called “sexagesimal place-value notation”.

Let us continue the reading until the entry 20, which exhibits a feature of considerable importance in the following. In front of 20, we expect 20 times 9, that is, 180, that is, 3 times 60. Indeed, we see three wedges. But the notation on the clay does not indicate that these wedges represent sixties, and not units. The “3” in front of 20 is the same as the “3” in the third line. We see that the cuneiform sexagesimal place-value notation does not indicate the position of the units in the number. 1, 60, 1/60 are noted in the same way. The notation is “floating”. The fact that numbers in SPVN do not convey absolute quantitative information is one of the most remarkable aspects of school mathematics, even if quite puzzling for a modern observer. To sum up, the numerical system adopted in this table, as well as in all the numerical tables produced in a school context is:

- Sexagesimal
- Place-valued
- Floating

The following section show how all these three properties, including the third, “floating”, play an essential role in the arithmetic developed in scribal schools.

<sup>10</sup> Tablet HS 217a from Nippur is now kept at the University of Jena and was published in (Hilprecht 1906; Proust 2008); photo at <https://cdli.ucla.edu/P254585>.

### 2.1.3. A table of reciprocal (ERM 14645)

The first of the numerical tables that was learnt in scribal schools was the table of reciprocals. Tablet ERM 14645 is an example similar to many others found in Nippur and elsewhere.<sup>11</sup>

**Table 1.5** Table of reciprocals ERM 14645

Obverse		Reverse	
The reciprocal of 2 is	30	reciprocal of 27	2:13:20
reciprocal of 3	20	reciprocal of 30	2
reciprocal of 4	15	reciprocal of 32	1:52:30
reciprocal of 5	12	reciprocal of 36	1:40
reciprocal of 6	10	reciprocal of 40	1:30
reciprocal of 8	7:30	reciprocal of 45	1:20
reciprocal of 9	6:40	reciprocal of 48	1:15
reciprocal of 10	6	reciprocal of 50	1:12
reciprocal of 12	5	reciprocal of 1:4	56:15
reciprocal of 15	4	reciprocal of 1:21	44:26:40
reciprocal of 16	3:45		
reciprocal of 18	3:20		
reciprocal of 20	3		
reciprocal of 24	2:30		
reciprocal of 25	2:24		

This table states that:

The reciprocal of 2 is 30

The reciprocal of 3 is 20 etc.

Indeed, 2 times 30 is 60, that is 1 in floating notation; 3 time 20 is 60, that is 1, and so on. Entry 7 does not appear in the table because the reciprocal of 7 cannot be written with a finite number of digits in base 60. In modern terms, numbers, which have finite reciprocals in a given base are said to be “regular” in this base, and the others “irregular”. In cuneiform texts, irregular numbers in base 60 appear as “igi nu”, which means in Sumerian “no reciprocal” (“igi” means reciprocal, and “nu” is the negation).

The reciprocal is a fundamental notion in cuneiform mathematics because of the way in which the divisions are performed. Indeed, division of a number  $a$  by a regular number  $b$  is a sequence of two operations: finding the reciprocal of  $b$ , and multiplying it by  $a$ .

### 2.2. Intermediate level

The written traces of the intermediate level of education are much less uniform and standardized than that of the elementary level. The intermediate level exercises in Nippur are diverse, but deal with few topics, the most attested being the computation of reciprocals, the evaluation of the surface of squares, and small linear problems. The examples detailed below, the first on reciprocal and the second on surface, show how intermediate level exercises are based on tables learnt in the elementary level.

<sup>11</sup> Tablet ERM 14645 is kept at the Hermitage Museum, St Petersburg (Friberg and Al-Rawi 2017: § 3.5). Photo <https://cdli.ucla.edu/P211991>.



### 2.2.1. An extraction of reciprocals

For example, the tablet translated below contains a typical example of extraction of a reciprocal<sup>12</sup>. Many school exercises of similar content, generally written on small round or square tablets, have been found in Nippur and elsewhere.

**Ist Ni 10241**

Obverse

4:26:40

Its reciprocal is 13:30.

Reverse

4:26:40            9

40                    1:30

13:30

On the obverse, we read 4:26:40, its reciprocal 13:30. On the reverse, the algorithm used to extract the reciprocal is displayed. We can observe that the operations that produce this sequence of numbers are only multiplications and extractions of reciprocals.<sup>13</sup>

The reciprocal of 6:40 is 9 (obtained using the reciprocal table, see Sect. 2.1.3)

4:26:40 × 9 gives 40 (obtained using the multiplication table by 9, see Sect. 2.1.2)

The reciprocal of 40 is 1:30 (obtained using the reciprocal table)

1:30 × 9 gives 13:30 (obtained using the multiplication table by 9)

The algorithm can be summarized in modern terms as follows:

4:26:40 is decomposed into the product 6:40 × 40

Thus the reciprocal of 4:26:40 is the product of the reciprocals of these factors, 1:30 × 9, that is, 13:30.

The algorithm is based on the factorization of the number the reciprocal of which is to be sought, and on the rule “the reciprocal of a product of factors is the product of the reciprocals of these factors”. The mathematical world in which the algorithm works is that of numbers in SPVN on which multiplications and extractions of reciprocals are carried out. These operations do not require the determination of the place of units in numbers. Floating numbers are particularly well adapted to this multiplicative world.

### 2.2.2 The calculation of the surface of a square

The young scribes had to navigate between two different types of entities, numbers in SPVN, and measurement values. A dozen exercises of similar content and layout devoted to the calculation of the surface of a square found in Nippur shed lights on this navigation between numbers and quantities. All of these exercises have the same layout: on the top left we see three numbers in SPVN, on the bottom right we see a small problem text. One of these exercises, UM 29-15-192, is based on quite simple data, and shows the process well. The numbers noted on the top left corner are 20, 20, and their product 6:40.

20  
20  
6:40

The statement noted on the bottom right corner is.

<sup>12</sup> Tablet Ist Ni 10241 from Nippur is now kept at the Archaeological Museums of Istanbul (Proust 2007); photo <https://cdli.ucla.edu/P368962>.

<sup>13</sup> Here and elsewhere in the chapter, the reader is invited to check the calculations using MesoCalc, the Mesopotamian calculator implemented by Baptiste Mèlès at <http://baptiste.meles.free.fr/site/mesocalc.html>.

2 šu-si the side of a square  
How much is the surface?  
Its surface is 1/3 še.

The relationship between the numbers written on the top left (20 and 6:40) and the length and surface measurements written on the bottom right (2 šu-si and 1/3 še) is exactly that established by the metrological tables: in a metrological table of lengths such as MS 3869/11 (see Sect. 2.1.1.), the measurement value 2 šu-si corresponds to 20, and in a metrological table of surfaces such as MS 2186, the measurement value 1/3 še corresponds to 6:40 (actually, an extrapolation is necessary to reach this correspondence, since the smallest entry on the table is 1/2 še). As we see, the two different types of entities are displayed on the tablet in two opposite corners, the corner for quantities (bottom right), and the corner for numbers in SPVN (top left), correlated in metrological tables. Tablets of this kind invite us to consider the relationships between numbers in SPVN and quantities as correspondences, and not as equalities.

The interpretation of this text by Neugebauer and Sachs, who published it, is grounded on the implicit assumption that the relationships between the measurement values and the numbers in SPVN are *equalities*. For example the length measurement 2 šu-si is considered to be *equal* to 20, and thus this latter number is understood as 0;0,20 *ninda*.

The meaning of the marginal numbers is clear:  $(2 \text{ šu-si})^2 = (0;0,20 \text{ GAR})^2 = 0;0,0,6,40 \text{ SAR}$ ” (Neugebauer and Sachs 1984, p. 248).<sup>14</sup>

A similar text was recently analyzed in the same manner by Robson (2008, p. 9), who translates the number “1:45” noted on the top left corner as “0;00 01 45”. Indeed, understanding the number “1:45” as equal to a measurement value, here a number of *ninda*, forces one to restore the order of magnitude.

With this simple example, we see how taking in consideration the notions inculcated in the elementary level, namely the nature of the correspondence between numbers in SPVN and quantities, leads to understandings (and translations) of numbers that are quite different. In the following sections, I show how these differences have important consequences on the interpretation of some scholarly mathematical texts.

### 2.3. Advanced level

The mathematical texts reflecting advanced education are more difficult to detect because, unlike those of elementary and intermediate levels, the type of tablets and genre of texts are not specific to this level. A text written in the context of advanced education is often hard to distinguish from a text of pure erudition. Most of the scholarly mathematical texts were probably produced in the context of scribal schools but, as schools were also centers of culture and erudition, mathematical texts were not necessarily designed only for teaching or learning.<sup>15</sup> However, some texts can be interpreted with some reliability as syllabus of advanced mathematics. It is the case of a procedure text from Southern Mesopotamia, perhaps from the city of Larsa, which provides a list of solved problems exploring in a systematic way the methods for solving linear and quadratic problems. This text, YBC 4663, is composed of eight problems dealing with the cost of digging a trench; the parameters are the dimensions of the trench, the daily work assignment and the worker wages.<sup>16</sup> For example, the first problem runs as follows:

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<sup>14</sup> For Neugebauer and Sachs, “marginal numbers” means numbers noted on the top left; they note “GAR” the length unit *ninda*.

<sup>15</sup> A detailed discussion of this issue is developed in Proust 2014.

<sup>16</sup> Tablet YBC 4663 is now kept at the Yale Babylonian Collection, Yale University. It was published in Neugebauer and Sachs 1945. The elements of interpretation proposed here are detailed in Proust forthcoming-a.

YBC 4663, problem 1

Obverse<sup>17</sup>

1. A trench. 5 *ninda* is its length, 1 1/2 *ninda* (is its width), 1/2 *ninda* is its depth, 10 *gin* is the volume of the work assignment, 6 *še* [silver is the wage of the hired man].
2. The base, the volume, the (number) of workers and the silver are how much? You, in your procedure,
3. the length and the width cross, 7:30 it will give you.
4. 7:30 to its depth raise, 45 it will give you.
5. The reciprocal of the work assignment detach, 6 it will give you. To 45 raise, 4:30 it will give you.
6. 4:30 to the wage raise, 9 it will give you. Such is the procedure.

The first six problems are linear. Some of the parameters being known, the other parameters are asked for. The last two problems are quadratic: the length and width are to be found knowing their sum (or difference) and their product. In this text, as well as in others of the same genre (see Proust forthcoming-a), some features are interesting for the present discussions. The data given in the statements are measurement values, but the operations prescribed in the procedures are carried out on numbers in SPVN. The correspondence between the measurement values given in the statements and the numbers in SPVN used in the procedures is that established in metrological tables. Thus, the first step in the procedure (implicit here) is the same as in the exercise on the square explained above (Sect. 2.2.2.): transform the measurement values given by the statement into numbers in SPVN using metrological tables. In the first six problems, the operations are only multiplications and divisions carried out on numbers in SPVN. The arithmetical world is the same as that of the elementary and intermediate levels, a world of numbers in SPVN on which only multiplications and divisions are performed. In the last two problems, additions and subtractions of numbers in SPVN are introduced, which marks a break with the linear world the first six problems evolve in.

#### **2.4. Linear paradigm**

To sum up, the mathematical world of the elementary and intermediate levels of Old Babylonian schools is based on a duality between the domain of quantities, and the domain of numbers in SPVN. In the latter, the fundamental operations are the multiplication and the extraction of reciprocals. In the school mathematical world, quantities are deduced from each other by multiplication or division. One of the basic problems introduced in the early advanced mathematical education was to calculate the area of a rectangle knowing its length and width, and the reverse problems (examples and more details in Proust forthcoming-a). In the same line, the first problem in procedure text YBC 4663 quoted above calculates a cost knowing the task to be performed, the work assignment per day per worker, and the daily wages. Other examples are found in advanced mathematics, such as calculate the value of silver in relation to the value of gold, or calculate a profit knowing the benefit (Middeke-Conlin and Proust 2014). All intermediate level exercises and a majority of the scholarly mathematical texts illustrate this linear paradigm where quantities are deduced from each other by multiplication or division. The term used in later Akkadian texts to designate mathematics, *igi-arê*, which word for word means “reciprocal-multiplication”, could refer to this linear paradigm.

Addition and subtraction belong to the world of accounting documented in the numerous models of contracts found in school archives. In this world, the mathematical entities are quantities, that is, measurement values or numbers of objects, animals or people. The duality between quantities, on which additions and subtractions operate, and numbers in SPVN, on which multiplications and reciprocals operate, could be reflected by the use of different

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<sup>17</sup> My translation. The line numbers were added by the modern editors.

Sumerian terms to designate calculation and accounting, respectively “šid” and “nikkas”. See for example the “Dialogue between two scribes” quoted above: “I am now an expert in [...] calculation and in accounting”.

There is no place in this dual world for addition and subtraction of numbers in SPVN, since the addition and subtraction belong to the world of quantities, and numbers in SPVN do not. The introduction of addition and subtraction on numbers in SPVN is intimately related to the introduction of quadratic problems in the mathematical world, as shown, for example, in the procedure text YBC 4663 (Sect. 2.3). The addition and subtraction of numbers in SPVN appear to be an audacious invention that emancipates the calculator from the rules of the linear paradigm. This invention raises serious problems since addition and subtraction require information on the relative position of numbers, while SPVN do not convey such information.<sup>18</sup> The categories which seem relevant from the point of view of mathematical practices in Old Babylonian Southern Mesopotamia are different from those that were shaped in the historiography of modern or premodern mathematics. The relevant distinctions are not between integers and fractions, or between rational and irrational, for example. The relevant distinctions are between numbers used in quantification, and numbers (in SPVN) used in multiplication, and among the latter, between regular and non-regular (igi nu) numbers. Problems are not classified according to their degree (first degree, second degree, or even higher degree), but rather according to the nature of the operations involved in the procedures: on the one hand linear problems, involving only reciprocals, multiplications and other operations which derive from them (division, square roots), and on the other hand non-linear (mainly quadratic) problems, also involving additions and subtractions.

In sections 3 and 4, I discuss two scholarly texts in which factorization and coefficients, two methods typical of the linear paradigm, are implemented.

### 3. The role of factorization in calculating reciprocals

Extracting a reciprocal was one of the basic techniques taught in the intermediate level of education in Nippur and elsewhere, as explained in Sect. 2.2. This technique is developed in a systematic way in CBS 1215, a tablet dated to the Old Babylonian period and of unknown provenience.<sup>19</sup> The text, which contains only numbers, is divided into six columns, three on the obverse and three on the reverse (right to left). The columns are divided into 21 boxes. The first entry is 2:5, the second 4:10, the double of 2:5, the third 8:20, the double of 4:10, and so on. In each box, the entry is the double of the previous one. Thus, the last entry, in the 21st box, is 2:5 times 2<sup>20</sup> (10:6:48:53:20). In each box, the text is divided into 3 sub-columns (left, center, right), and the input of the calculations is identical to the output. For example, the text in the first box runs as follows:

2:5	12
25	2:24
28:48	1:15
36	1:40
	2:5

The numerical relationships are similar to those observed previously (Sect. 2.2.1) on tablet Ist Ni 10241:

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<sup>18</sup> A discussion on this issue is proposed in Proust 2013.

<sup>19</sup> CBS 1215 is now kept at the Museum of Archaeology and Anthropology, University of Pennsylvania, Philadelphia. It was published by Abraham Sachs (Sachs 1947), who was the first historian to understand this puzzling numerical text. A detailed analysis is proposed in Proust 2012b. Photo: <https://cdli.ucla.edu/P254479>.

- the reciprocal of 5 is 12 (obtained using a reciprocal table, see Sect. 2.1.3.)
- 12 times 2:5 gives 25 (obtained using a multiplication table by 12)
- The reciprocal of 25 is 2:24 (obtained using a reciprocal table)
- 2:24 times 12 gives 28:48 (obtained using a multiplication table by 12)
- 28:48 is the reciprocal of 2:5, the entry.

As in tablet Ist Ni 10241, the operations are reciprocals and multiplications, which are carried out on numbers in SPVN. The algorithm is based on breaking down the entry, here 2:5, into two factors, here 5 and 25. The reciprocal sought, 28:48, is the product of the reciprocal of the factors. Each time a factor is detected, its reciprocal is marked in the right sub-column, which helps to execute and control the algorithm. However, unlike in Ist Ni 10241, the calculation does not stop when the reciprocal of the entry is found. It continues by applying the same process again:

- The reciprocal of 48 is 1:15
- 28:48 is broken down into its factors 48 and 36
- the reciprocals of 48 and 36 are 1:15 and 1:40, marked on the right
- the product of the reciprocals,  $1:15 \times 1:40$ , gives 2:5

As previously, this sequence of operations provides the reciprocal of 28:48, that is 2:5, the entry in the first section. To sum up, the algorithm in the first section is decomposed into two sequences:

- a direct sequence: the reciprocal of the entry 2:5 is calculated, it provides 28:48
- a reverse sequence: the reciprocal of 28:48 is calculated, it provides the entry 2:5.

The algorithm is based on the decomposition into factors of the number the reciprocal of which is sought, and the main issue, for the practitioner, was to identify these factors. The first factor can be detected in the trailing part (the last digits of the entry),<sup>20</sup> and the second factor is obtained by dividing the entry by the first factor.

The entry in the eighth section of CBS 1215 is 4:26:40, the same as in the school exercise Ist Ni 10241, and the direct sequence in this section is similar to the school exercise:

- The trailing part of 4:26:40 is 6:40, the reciprocal of 6:40 is 9
- 4:26:40 is broken down into the factors 6:40 and 40
- the reciprocals of 6:40 and 40 are 9 and 1:30, marked on the right,
- the product of the reciprocals,  $9 \times 1:30$ , gives 13:30, the reciprocal of the entry.

The reverse sequence produces the entry:

- The trailing part of 13:30 is 30, the reciprocal of 30 is 2
- 13:30 is broken down into the factors 30 and 27
- the reciprocals of 30 and 27 are 2 and 2:13:20, marked on the right
- The product of the reciprocals,  $2 \times 2:13:20$ , gives 4:26:40, the entry.

We see the same calculation that appears in the school exercise Ist Ni 10241, but with some differences. Indeed, in the school exercise, there are no clear vertical alignments; here, three sub-columns are formed, which, as noted above, facilitate the execution of the algorithm. In the school exercise, only the reciprocal is asked for, and only the direct sequence is noted; here, we see the direct and the reverse sequence, so that the first and the last number in the section are the same. Section 8 is not the only one to be attested on a school tablet; many other known

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<sup>20</sup> The term "trailing part" was coined by Friberg to designate the last digits of a sexagesimal number.

school tablets from Nippur, Ur, Mari and other places contain calculations of the same genre. All of these school exercises reproduce more or less the direct sequence of sections of CBS 1215. Despite their similarities, the functions of the school exercises and CBS 1215 seem to have been different. I come back to this issue below.

The final sections of CBS 1215 show another feature. For example, section 20 runs as follows:

5:3:24:26:40	[9]
45:30:40	1:30
1:8:16	3:45
4:16	3:45
16	3:45
14:3:45	
5[2:44]:3:45	
1:19:6:5:37:30	
11:51:54:50:37:30	2
23:43:49:41:15	16
25:18:45	16
6:45	1:20
9	6:[40]
8:53:20	
2:22:13: 20	
37:55:33:20	
2:31:42:13:20	
5:3:24:26:40	

The entry is 5:3:24:26:40, that's is 2:5 after 19 doublings. The trailing part is 6:40, reciprocal 9. The entry divided by 6:40, that is, multiplied by 9, is 45:30:40. This second factor does not belong to the table of reciprocals (it is not an elementary regular number), thus its reciprocal is unknown for the practitioner. The process is then iterated: the trailing part of 45:30:40 is 40, reciprocal 1:30. The number 45:30:40 is divided by 40, that is, multiplied by 1:30, which produces 1:8:16. This number is not an elementary regular number, the process is iterated again. The trailing part of 1:8:16 is 16, reciprocal 3:45. The number 1:8:16 is divided by 16, that is, multiplied by its reciprocal 3:45, which produces 4:16. Once more this number is not an elementary regular number, the process is iterated, which produces 16, an elementary regular number. As, at each step, the reciprocal of the trailing part has been marked on the right, the calculator has just to multiply the numbers which form the right sub-column. These multiplications form a central sub-column. The final result is 11:51:54:50:37:30, the reciprocal of the entry. The reverse sequence is similar.

This calculation shows how the algorithm can be iterated. Iterations appear in the text from section 11 onward. The iterations are facilitated by the layout of the algorithm. In the examples we have seen, the factors are numbers which appear in the reciprocal table, that is, elementary regular numbers. The decomposition into elementary regular factors is not unique, as several different regular trailing parts could be chosen as divisor. Hence the question: how was the trailing part selected? We observe that in the direct sequence, the trailing part is the largest elementary regular numbers. There is only one way to implement the algorithm. In modern terms, such decomposition is unique. However in the reverse sequence, there is not apparent rule for selecting the trailing part. It seems that several ways of implementing the algorithm are left open. It is perhaps meaningful to observe that only the direct sequences are

attested in school texts: the algorithm is taught as unique. But the work on the algorithm in CBS 1215 suggests that their author(s) were aware of the fact that the algorithm can be implemented in several ways.

Furthermore, we have seen that the reverse sequence produces the entry, and thus, its result is known in advance. The series of entries appear as a paradigmatic set of values chosen in order to work on the algorithm itself, not to produce any new result. The reverse sequence appears then to play an essential role, that of verification of the algorithm. The purpose of this text seems to be the validation of the algorithm (choice of the entries, reverse algorithm), while the purpose of the school exercises is training in the implementation of the algorithm. The text also underlines the fact that the algorithm can be implemented in several ways, despite the fact that it is taught as unique in scribal schools. It seems to me that this text is not (or not only) a sourcebook which provides school exercises, but a systematic elaboration developed by masters or scholars in order to firmly validate an algorithm, the factorization, routinely used in elementary education and in advanced mathematics.

As in the case of the interpretation of UM 29-15-192 on the surface of a square, the understanding of the nature of numbers has an important impact on the interpretation of the text. For Abraham Sachs, the extraction of the reciprocal is based on a decomposition of the entry into sum, and not into product.<sup>21</sup> He represents the algorithm by the following algebraic formula (the number  $c$ , the reciprocal of which is to be extracted, is decomposed in sum  $a + b$ ; the reciprocal of a number  $n$  is noted  $\bar{n}$ ):

Figure 1: Extract from (Sachs 1947: 222)

$$\bar{c} = \overline{a + b} = \bar{a} \cdot (1 + b\bar{a})$$

Moreover, as Sachs considers the numbers as representing quantities, he restores the orders of magnitude with plenty of zeros, which makes the process quite cumbersome (see figure 2).

Figure 2: Extract from (Sachs 1947: 240)

0;00,00,2,31,42,13,20	(16)	5,3,24,26; 40	(17)
0;0,00,0,37,55,33,20	(15)	0;0,0,0,2,31,42,13,20	(16)
0;0,0,2,22,13,20	(14)	0;0,0,0,37,55,33,20	(15)
0;0,8,53,20	(13)	0;0,2,22,13,20	(14)
0;6,40	(12)	0;0,8,53,20	(13)

#### 4. The role of coefficients in problems of volumes

Nippur has provided archaeologists with hundreds of school mathematical texts and has been by far the most prolific, among the sites of the ancient Near East, for school texts. However, only three procedure texts reflecting scholarly activities have been found in Nippur. This

<sup>21</sup> This interpretation originates in a tablet, probably of later date, VAT 6505, which explains the algorithm for some of the entries found in CBS 1215. The procedure includes an addition. In my view, this addition reflects an incursion into the multiplication algorithm, and is not a step in the procedure (more discussion in Proust 2012a).

contrast may appear as quite surprising, and may have several explanations, such as circulation of tablets during travels by scribes, chance in the excavation sites, or illegal activities of dealers. The three scholarly tablets found in Nippur, CBS 11681, CBS 19761+Ni 5175, and CBS 12648, deal with volumes and the procedures they contain involve only multiplications and divisions.<sup>22</sup> Tablet CBS 11681 offers a calculation of the volume of a rectangular prism from its sides (length, width and height) and a reverse problem, find the unknown side of a prism knowing its volume and two of its sides. Tablet CBS 19761+Ni 5175 contains problems dealing with the volume of a pile of bricks.

In the following, I focus on CBS 12648, which illustrates interestingly one of the key tools used in the linear world, the use of coefficients. While the two other scholarly texts from Nippur (CBS 11681 and CBS 19761+Ni 5175) are written in Akkadian, CBS 12648 is written in Sumerian and bears witness to the very beginning of the Old Babylonian mathematical traditions. Only the lower left part of a large multi-column tablet has come down to us. It contains several problems, only one being well preserved. Friberg (2001, pp. 149-150) was the first historian to decrypt the procedure “vaguely described in the text” (ibid., p. 150).

The preserved problem provides the volume of a rectangular prism and the ratios between its sides, asks for the sides of the prism, and gives the procedure in quite concise terms. It runs as follows:

**Obverse, column *i***

Problem 2

5. 2  $\dot{\text{še}}$  1/12 the volume
6. 2/3 of the length, it is the width;
7. The half of the width, it is
8. the depth.
9. Its length, its width
10. and its depth, how much?
11. The length, the width,
12. and the depth,
13. cross, then
14. its reciprocal detach,
15. to its volume raise.
16. The equal side (cubic root) of
17. 15:37:30
18. Is to be extracted.
19. The equal side of 15:37:30  
<is 2:30>

The procedure may appear quite mysterious at first sight because the results of the different operations, which are prescribed are not specified. For example, in lines 11-13, it is prescribed to multiply the length, the width, and the depth, but the result of these operations does not appear explicitly. However, beyond the unusual style, the method used is characteristic of the mathematics of this period: it consists of transforming a problem by applying a multiplying factor so as to produce a standard problem whose resolution procedure is known.

The problem is the following: the volume (implicitly of a rectangular prism) is 2  $\frac{1}{12}$   $\dot{\text{še}}$  (which represents about 60 dm<sup>3</sup>). The sides of the prism are unknown, but their ratios are given: the width is two-thirds of the length, and the depth is half of the width. The sides of the prism are asked for. The method consists in considering another prism of same shape as the actual prism, calculating the ratio between this reference prism and the actual prism (the volume of which is 2  $\frac{1}{12}$   $\dot{\text{še}}$ ), and deducing the sides of the actual prism.

<sup>22</sup> For a complete study and bibliography, see Proust 2007, Chap. 7.



The sole intermediate result specified in the procedure is 15:37:30, the coefficient expressing the ratio between the reference prism and the actual prism. Finding this coefficient thus appears to be pivotal to the procedure. This number is written in SPVN, which reflects a general feature of Old Babylonian mathematical texts: all the operations prescribed in the procedures are carried out on numbers in SPVN, and produce numbers in SPVN.

With these remarks in mind, the procedure can be reconstructed in detail despite its terse style (see synthesis in Table 6). The dimensions of the reference prism which is introduced in lines 11-14 are 1 *ninda* length,  $\frac{2}{3}$  *ninda* width ( $\frac{2}{3}$  of the length), and  $\frac{1}{3}$  *ninda* depth ( $\frac{1}{2}$  of  $\frac{2}{3}$  of the length). As the procedure applies to numbers in SPVN, these measurement values must be transformed into numbers in SPVN. As in other texts from the same period, these correspondences would have been provided by metrological tables:

The length 1 *ninda* corresponds to 1 according to metrological tables of length (see Table 1)  
 The width  $\frac{2}{3}$  *ninda* corresponds to 40 according to metrological tables of length (see Table 1)  
 The depth  $\frac{1}{3}$  *ninda* corresponds to 4 according to metrological tables of heights (see Table 2).

The operation prescribed in lines 11-13 is the multiplication of the three numbers in SPVN corresponding to the length, the width, and the depth:

$1 \times 40 \times 4$  gives 2:40.

Note that these operations are prescribed indirectly by a geometrical manipulation, to “cross” the length, the width, and the depth. This produces the reference volume.

The coefficient, that is the ratio of the actual volume to the reference volume, is calculated in lines 14-15 by multiplying the former by the reciprocal of the latter. First the reciprocal of 2:40, the reference volume, is calculated, giving 22:30. The actual volume is the number in SPVN which corresponds to 2 *še*  $\frac{1}{12}$  in the metrological table for volumes, 41:40. Indeed, in such tables, for example MS 2186 (see Sect. 2.1.1 Table 3), 2 *še* corresponds to 40, and  $\frac{1}{12}$  *še* corresponds to 1:40, thus the measurement 2 *še* and  $\frac{1}{12}$  *še* corresponds to 41:40.<sup>23</sup> The coefficient of the volumes is  $22:30 \times 41:40$ , which gives 15:37:30. As noted above, this coefficient is the only intermediate result given in the procedure. The ratio of the actual volume to the reference volume being 15:37:30, the ratio of the sides of the actual prism to the sides of the reference prism is the cube root of 15:37:30 that is 2:30.

The sides of the actual prism can be calculated by multiplying the sides of the reference prism by the coefficient 2:30. However, the preserved part of the text ends by indicating that the cube root of 15:37:30 is to be calculated, but provides neither this cube root, nor the dimensions of the sides. The final steps were either omitted, or were written in the first lines of column *ii*, which is broken. I reconstruct these calculations below because the final result gives interesting information. The sides of the actual prism are obtained as follows:

Length:  $1 \times 2:30$  is 2:30  
 Width:  $40 \times 2:30$  is 1:40  
 Depth:  $4 \times 2:30$  is 10

The measurement values these dimensions correspond to are provided by the metrological tables:

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<sup>23</sup> The first entry in the table is  $\frac{1}{2}$  *še*; the entry  $\frac{1}{12}$  *še* is not attested but the corresponding number in SPVN is easily obtained by dividing 20 (which corresponds to 1 *še*) by 12.

Length: 2:30 corresponds to  $1/2$  *kuš* (metrological table of lengths, see Table 1)

Width: 1:40 corresponds to  $1/3$  *kuš* (same table)

Depth: 10 corresponds to 5 *šu-si* (metrological table of heights, see Table 2)

Lines	Text	Operations
5	2 <i>še</i> $1/12$ the volume	<b>2 <i>še</i> <math>1/12</math></b> → 41:40
6-8	$2/3$ of the length, it is the width The half of the width, it is the depth.	1 <i>ninda</i> → 1 $2/3$ <i>ninda</i> → 40 $1/3$ <i>ninda</i> → 4
11-13	The length, the width, and the depth, cross	$1 \times 40 \times 4$ gives 2:40
14-15	its reciprocal detach, to its volume raise.	Recip(2:40) is 22:30 $22:30 \times 41:40$ gives <b>15:37:30</b>
16-19	The equal side (cubic root) of 15:37:30 is to be extracted. The equal side of 15:37:30 <is 2:30>	The cubic root of <b>15:37:30</b> is 2:30.
		$1 \times 2:30$ gives 2:30 $40 \times 2:30$ gives 1:40 $4 \times 2:30$ gives 10
		Length: 2:30 → $1/2$ <i>kuš</i> Width: 1:40 → $1/3$ <i>kuš</i> Depth: 10 → 5 <i>šu-si</i>

Table 6: Operations in problem 2 of CBS 12648 (in bold: data and results which appear explicitly in the text)

The dimensions of the prism sought,  $1/2$  *kuš* length,  $1/3$  *kuš* width, and 5 *šu-si* height are that of a special brick considered as a unit of volume in several Old Babylonian mathematical texts. For example, this brick is described in a mathematical tablet from Southern Mesopotamia, YBC 4607.<sup>24</sup> The first section runs as follows:

A brick.  $1/2$  *kuš* its length,  
 $1/3$  *kuš* its width, 5 *šu-si* its height.  
Its base, its volume and its equivalent capacity how much?  
12 *še* and one-half a *še* its base, 2 *še* and one-12<sup>th</sup> *še* its volume,  
3  $1/3$  *silá* 8  $1/3$  *gin* its equivalent capacity.

The fact that the dimensions chosen for the prism in our problem is that of a standard brick can hardly be considered as being made by chance. It reflects the circulation of mathematical practices and ideas in Southern cities of the time. This shared mathematical culture related to the quantification of space played a central role in mathematics in Old Babylonian Southern Mesopotamia (Proust forthcoming-b). Perhaps more importantly for the present discussion, the data chosen for this problem show that the result of the procedure was known in advance. Constructing problems “backwards from the known solution” (Høyrup 2002, p. 221) is a general phenomenon in Old Babylonian mathematics; we already observed it in CBS 1215. It appears that our text CBS 12648 was not intended to produce unknown or new results, but to implement a method, that of the use of a coefficient.

The two other scholarly texts from Nippur also deal with volumes and rely on a comparison of a prism of unknown dimensions with a reference prism. This method of comparison with a reference solid was thought by Friberg to be a method of “false position” (Friberg 2001, p. 150). However, there is no technical term in these texts such as “false volume” and “true volume”.<sup>25</sup> While from a modern point of view, there is no fundamental difference

<sup>24</sup> This tablet, now kept at the Yale Babylonian Collection, was published by Neugebauer and Sachs (1945). For an updated study and bibliography, see Proust forthcoming-b.

<sup>25</sup> Such terms are attested in other mathematical texts. See for example Str. 368 (Thureau-Dangin 1938a, p. 91-92; Thureau-Dangin 1938b); VAT 8391 #3 (Høyrup 2002, p. 83) the terms « truth » (*gi-na* = *kînum*) length and « false » (*lul* = *sarrum*) length. See a synthesis in Høyrup 2002: 100-101.

between the “false position” method and the use of coefficients, as both consist in the same sequence of operations, it seems to me important to distinguish methods that were considered as different by the ancient actors.

The calculation itself consists only in multiplications and extractions of reciprocals carried out on numbers in floating SPVN. As before (Section 3), the problem is solved in the multiplicative world. No order of magnitude, no measurement unit appears in the procedure, which makes the calculation very simple, terse and efficient. These features are destroyed by the restoration of units and orders of magnitude in interpretations, such as that of Friberg (2001, p. 150):

The ratio between the true and the false volume is  
 $3;28,20 \square n \cdot n \cdot 9/2 = 10;25 \square n \cdot n \cdot 3/2 = 15;37,30 \square n \cdot n$ .  
 [Note of the present author: the notation “ $\square n \cdot n$ ” represents a square of 1 *ninda*-side]

## 5. Conclusion: the linear paradigm and how get out of it

The two examples given in Sect. 3 and 4 are emblematic of the linear paradigm, the world of *igi-arê* in which future scribes who attended scribal schools were educated and in which powerful algorithms were developed. Section 3 shows how the extractions of reciprocals are based on a method of factorization. This method was also used in other multiplicative contexts, for example for extracting square and cube roots (Proust 2012b), or for reducing rectangles in the famous tablet Plimpton 322 (Britton, Proust & Shnider 2011). Section 4 shows the role of coefficients in volume problems. Many other examples of the use of coefficients in solving quadratic or cubic problems can be found in Old Babylonian mathematical texts, as shown by Høystrup, who describes how “scaling and change of variable are procedures that allow the reduction of complex equations to simple standard cases” (Høystrup 2002, p. 100). The use of coefficients was a basic tool in administrative and economic transactions (Robson 1999). Standard coefficients allowed the use of metrological tables for a wide range of situations (Proust forthcoming-b). However, the scholarly mathematical texts show how masters in scribal schools emancipated themselves from the linear paradigm in order to solve new classes of problems, and primarily quadratic problems (Høystrup 2002). A pedagogical list of problems such as YBC 4663 show how the addition and subtraction of numbers in SPVN were introduced in the context of quadratic problems (Section 2.3).

It must be underlined that the scenario described in this chapter, the mathematical work in the linear paradigm and the efforts to break free from it, is based on sources from Southern Mesopotamia and dated to the first part of the Old Babylonian period, that is, before the destruction of Southern cities around the mid-seventeenth century. Different scenarios could be described from other mathematical sources, such as that of central Mesopotamia after the collapse of South and the emigration of scholars from Southern cities (Larsa, Nippur, Ur or Uruk) to Kish or Sippar, or sources from the Old Babylonian city of Susa in Western Iran. To fully perceive the extent of the inventiveness of scholars who gravitated around the scribal schools of the early Old Babylonian Southern Mesopotamia, it seems to me important to be aware of the prevalence of the linear paradigm. This mathematical context makes the introduction of addition and subtraction of numbers in SPVN even more remarkable in a world where these operations made no sense. This mathematical work goes unnoticed in an arithmetic world based on the four operations that we are familiar with. Its innovative and daring character is discernible only if considered from the point of view of a mind imbued with the linear paradigm.

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## Appendix

The diagrams below represent the metrological systems and correspondences with SPVN according to Old Babylonian sources from Nippur. The arrows represent the factors between different units (e.g.  $gin \leftarrow 180 \text{--} \check{s}e$  means 1 *gin* is equal to 180 *še*); the numbers below the units are the numbers in SPVN that correspond to these units in the metrological tables (e.g. 20 below *še* means that 1 *še* corresponds to 20 in metrological tables).

Length and other horizontal dimensions (1 *ninda* represents about 6 m)

<i>ninda</i>	$\leftarrow 12 \text{--}$	<i>kuš</i>	$\leftarrow 30 \text{--}$	<i>šu-si</i>
1		5		10

Heights and other vertical dimensions (1 *ninda* represents about 6 m)

<i>ninda</i>	$\leftarrow 12 \text{--}$	<i>kuš</i>	$\leftarrow 30 \text{--}$	<i>šu-si</i>
5		1		2

Surface and volume (a surface of 1 *sar* is that of a square of 1 *ninda*-side; a volume of 1 *sar* is that of a rectangular cuboid of 1 *sar*-base and 1 *kuš*-high)

<i>gan</i>	$\leftarrow 100 \text{--}$	<i>sar</i>	$\leftarrow 60 \text{--}$	<i>gin</i>	$\leftarrow 180 \text{--}$	<i>še</i>
1:40		1		1		20

Weight (1 *mana* represents about 500 g)

<i>mana</i>	$\leftarrow 60 \text{--}$	<i>gin</i>	$\leftarrow 180 \text{--}$	<i>še</i>
1		1		20

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