



HAL
open science

Both invariant principles implied by Marx's law of value are necessary and sufficient to solve the transformation problem through Morishima's formalism

Norbert Ankri, Païkan Marcaggi

► To cite this version:

Norbert Ankri, Païkan Marcaggi. Both invariant principles implied by Marx's law of value are necessary and sufficient to solve the transformation problem through Morishima's formalism. 2023. hal-03994960v2

HAL Id: hal-03994960

<https://hal.science/hal-03994960v2>

Preprint submitted on 20 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Public Domain

Both invariant principles implied by Marx's law of value are necessary and sufficient to solve the transformation problem through Morishima's formalism

Norbert Ankri & Païkan Marcaggi

UNIS, INSERM, Aix-Marseille Université, France

2023-03-20

Summary

The unit value of a commodity that Michio Morishima's method and its variations enable to determine correctly, is the sum of the value of the commodities it contains (inputs) and the quantity of labor required for its production. However, goods are sold at their market production price only when they meet a solvent social need that involves the entire economy with its interconnections between the different industrial sectors. This condition gives full meaning to Marx's fundamental equalities, which derive from the law of value and constitute invariants that apply to the economy as a whole. These equalities are necessary to determine market production prices. We demonstrate that they also enable to solve the transformation problem for a simple reproduction system without fixed capital by starting from Morishima's formalism and returning to a formalism closer to that used by Marx.

Introduction

In *The Capital* volumes I, Marx explains that prices of a commodity oscillate around an average price, its value (1). Next, in volume III, the notion of market-price of production of a commodity is introduced. It is now around it (no longer around the value of the commodity) that the price of the commodity oscillates (2). At first sight the theory of value may seem to lose coherence. It becomes necessary to show how values are related to market-prices of production, i.e. what operation enables to transform one into the other. To illustrate the difference between values and market-

production prices, Marx considers a model made of five branches, in each of which an identical value of capital of 100 monetary units is invested. In this example, the inputs are not transformed into production prices, Marx uses it to support an aspect of his theory: how surplus value is drawn preferentially towards most capital-intensive sectors and how this draw is masked in the real world where values are unknown. In this example, there is no question of simple reproduction and there is no interconnection between the different industries. These issues are addressed by Marx's law of value, which implies two fundamental equalities: equality between the sum of profits and the sum of surplus-values; and equality between the total invested capital in value and the total invested capital in price (3). These equalities, commonly referred to as invariants (4), only make sense on the global scale, i.e. at the aggregate level. Now, on this global scale, the interconnections between sectors (branches) mean that the amount of capital allocated to them cannot be chosen arbitrarily, it must comply with demand. As previously explained, Marx's fundamental equalities are analogous to laws of conservation in the field of Physics (3)¹, they rule the interconnection between values and prices.

¹ For each industry, socially necessary human labor adds value to the capital invested. Economic crises or wars destroy value, so we have to explain what we mean by conservation of value. In our conception the space of values is interconnected with the space of prices. There is conservation insofar as the profit of an economic sector (in price) cannot result from anything else than the surplus value (in value) of this sector and/or the displacement of surplus value from or to other sectors. In our value tables exhibiting capital distribution across branches accounting for the production meeting solvent social needs, when surplus value is zero, the value produced by one sector corresponds precisely to the needs of the whole economy. When there is surplus value, all of it is distributed to all industries, with no loss. This is the faithful application of Marx's fundamental equality II.

An explanation on how values and production prices are connected is provided in the Capital volume III. But the underlying mathematical solving method to transform values into market-production prices is lacking. Yet, this incompleteness does not appear as a matter of concern for Marx, as if the underlying math was nothing more than a minor solvable problem (2). Since then, however, the so-called “transformation problem” has been the matter of a long-lasting controversy and different formalisms have developed to address it, the input-output Morishima’s formalism having gained influence by providing straight forward linear algebra methods (4-6).

Here, we show how these attempts have carried a systematic error by dealing with production only, and claiming that technical considerations were enough to determine the interdependence between the different industrial sectors. Surely, technological possibilities of an era are essential constraints, but another one is social and economic order. In a capitalist market, for a commodity to be sold with the current average rate of profit, it must meet a certain demand, which we have defined as the solvent social need (3). This condition is in fact implied by Marx’s first fundamental equality. It follows that the condition of simple reproduction, to start with, cannot be validated by considering physical quantities of goods solely. Beforehand values of commodities must be considered. To comply with both Marx’s fundamental equalities, we show how a transition can be made from Morishima’s common input-output formalism, as clearly laid by Basu (4), to a formulation more faithful to Marx’s original

description that we detailed earlier (3). We first express the rational of this transition by using the three-branch example from Basu (4), where two branches produce commodities that can be consumed by industries and humans (bread and meat) while the remaining branch produces a commodity used by industries solely (iron). A generalization to n branches is provided as an appendix.

1. Preliminary remark

We study an idealized steady state of the capitalist economy, i.e. a state where the unit price of a commodity in input and that of this commodity in output are the same and do not vary, where the value produced does not depreciate. This idealization is necessary to convey the coherence of Karl Marx's conceptualization of the functioning of the capitalist economy. Even if in reality, the state of the latter is not steady, but rather unstable, even chaotic, the idealization of its steady state enables a conceptualization analogous to the approach that prevails in the "hard" sciences and the identification of rules governing its state of equilibrium (7).

2. The three-branch example

The economy is defined as follows:

A: Matrix of social-technical coefficients

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

a_{ij} : Amount of commodity i (number of units) required to produce one unit of commodity j .

\mathbf{v} : Wage vector that gives the amount of goods v_1 , v_2 and v_3 (basket) that a worker (regardless of the industry he belongs) gets in one hour of his labor.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

\mathbf{l} : Labor vector that gives the amount of labor required to production. l_i : Labor input requirement for the production of one unit of commodity i . It is use value.

$$\mathbf{l} = (l_1; l_2; l_3)$$

In value unit, we define

Λ_i : Amount of labor embodied in each unit of commodity i , we have:

$$\Lambda_i = \sum_{j=1}^3 \Lambda_j a_{ji} + l_i \quad i, j = 1, 2, 3$$

Let : $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$

We have : $\mathbf{\Lambda} = \mathbf{\Lambda A} + \mathbf{l}$

I.e. $\mathbf{\Lambda(I - A)} = \mathbf{l}$

Where \mathbf{I} is the unit matrix.

If the Hawkins-Simon condition² is fulfilled then the economy is said to be productive, that means it produces at least what it consumes, then the matrix $(\mathbf{I}-\mathbf{A})$ is invertible:

$$\Lambda = \mathbf{I} \cdot (\mathbf{I} - \mathbf{A})^{-1} \quad (1)$$

Λ enables the calculation of the value of the wage basket obtained by the worker in exchange of one hour of work ($\Lambda\mathbf{v}$), from which the exploitation rate e is derived:

$$e = \frac{1 - \Lambda\mathbf{v}}{\Lambda\mathbf{v}}$$

3. Transition to traditional Marxist formalism

In order to obtain an equivalent formulation more faithful to Marx's formalism, we can convert the above formalism to represent value in terms of circulating capital C , variable capital V and surplus value PL .

We base our rational on the three branches model example provided by Basu (4) defined as follows.

Branch 1: Bread **B**, Branch 2: Iron (fer) **F**, Branch3: Meat **M**

In this model, the workers' consumption is only made of bread and meat in a certain value proportion α ($\alpha=1$ means only bread if $v_2 = 0$)³.

² David Hawkins (1948). Some Conditions of Macroeconomic Stability. London: Econometrica, Vol. 16, No. 4 (Oct., 1948), pp. 309-322.

³ Note that this consumption is given in quantity proportion in Basu (2021).

$$\alpha = \frac{\Lambda_1 v_1}{\Lambda_1 v_1 + \Lambda_2 v_2 + \Lambda_3 v_3} = \frac{\Lambda_1 v_1}{\Lambda \mathbf{v}}$$

All the following values in lowercase letters (b,f,m) are given for one unit of commodity.

For j = 1,2,3

b_{jC} : Value of bread input to commodity j, which is part of the circulating capital.

$$b_{jC} = a_{1j} \Lambda_1$$

b_{jV} : Value of bread content of workers' wage basket of the industry j, which is part of the variable capital.

$$b_{jV} = \frac{\alpha \cdot l_j}{1 + e} = \Lambda_1 v_1 l_j$$

(Note that the worker's wage value (v_j) is lower than the value created by work (l_j): $l_j = (1 + e)v_i$. Hence the $(1 + e)$ denominator)

b_j : Total value of bread used for the industry j.

$$b_j = b_{jC} + b_{jV}$$

f_j : Total value of iron (fer) input to commodity j.

$$f_j = a_{2j} \Lambda_2$$

m_{jC} : Value of meat input to commodity j, which is part of the circulating capital.

$$m_{jC} = a_{3j} \Lambda_3$$

m_{jV} : Value of meat content of workers' wage basket of the industry j, which is part of the variable capital.

$$m_{jV} = \frac{(1 - \alpha) \cdot l_j}{1 + e} = \Lambda_3 v_3 l_j$$

m_j : Total meat used for the industry j.

$$m_j = m_{jC} + m_{jV}$$

pl_j : Surplus-value generated in industry j.

$$pl_j = \frac{e \cdot l_j}{1 + e}$$

w_j : Total output value of the industry j for one commodity j.

$$w_j = b_j + f_j + m_j + pl_j$$

Subscript V (used for b_{jV} and m_{jV}) indicates that these values are a part of the variable capital. Subscript C (used for b_{jC} and m_{jC}) indicates that these values are a part of the circulating capital.

k_i (lowercase) defines the amount of capital invested in branch i to produce one unity of commodity i.

$$k_i = b_i + f_i + m_i$$

K_i (uppercase) is the amount of capital invested in branch i

k_{pi} (lowercase): Capital in price invested in branch i to produce one unit of commodity i.

$$k_{pi} = x_1 b_i + x_2 f_i + x_3 m_i$$

x_i being the transformation coefficient from value to price of commodity i

K_{pi} (uppercase) is the amount of capital invested in branch i, in price

s_j : profit in price for the industry j.

$$s_j = w_j x_j - (x_1 b_j + x_2 f_j + x_3 m_j)$$

k_T : Total committed capital for one unity of each commodity

$$k_T = \sum_i k_i$$

Marx's fundamental equality I:

$$\sum_j S_j = \sum_j Pl_j$$

The sum of the surplus values is equal to the sum of the profits.

Marx's fundamental equality II:

$$\sum_j K_{pj} = \sum_j K_j$$

The sum of the capitals invested in values is equal to the sum of the capitals invested in prices.

These two **laws of conservation** (3) or **invariants** (4) only make sense at the level of the global economy. The production of any commodity is determined by solvent social needs, otherwise it would make no contribution to the total value. Therefore, the amounts of capital allocated for each industry

cannot be arbitrary. They must be balanced according solvent social needs. This balance is expressed by Marx's Equality I.

4. Transformation of values into production prices

The total invested capital K_T is defined by:

$$K_T = \sum_{i=1}^3 (b_i + f_i + m_i)$$

The transformation implies finding together the three amounts of capital allocated to branches 1, 2 and 3 and the three coefficients of transformation x_1, x_2, x_3 of values into prices P_i (3, 8). The problem is bilinear in K and x .

$$\forall i, W_i = K_i w_i$$

$$\sum K_i = K_T$$

$$\forall i, P_i = x_i W_i$$

Depending on the size of the economy the total capital can be multiplied by any given factor which will also applies to all capitals K_i .

We call r the rate of profit. The first following set of equations establishes the equality of this rate for the three industries:

$$w_1 x_1 = (1 + r)b_1 x_1 + (1 + r)f_1 x_2 + (1 + r)m_1 x_3$$

$$w_2 x_2 = (1 + r)b_2 x_1 + (1 + r)f_2 x_2 + (1 + r)m_2 x_3$$

$$w_3 x_3 = (1 + r)b_3 x_1 + (1 + r)f_3 x_2 + (1 + r)m_3 x_3$$

Hence the following eigenvalue equation:

$$\begin{bmatrix} b_1/w_1 & f_1/w_1 & m_1/w_1 \\ b_2/w_2 & f_2/w_2 & m_2/w_2 \\ b_3/w_3 & f_3/w_3 & m_3/w_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{(1+r)} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

By calling:

$$\mathbf{M} = \begin{bmatrix} b_1/w_1 & f_1/w_1 & m_1/w_1 \\ b_2/w_2 & f_2/w_2 & m_2/w_2 \\ b_3/w_3 & f_3/w_3 & m_3/w_3 \end{bmatrix}$$

That is :

$$\mathbf{M} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{(1+r)} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2)$$

Note that :

$$(w_1, w_2, w_3) \mathbf{M} = \mathbf{A} + [\mathbf{v} \cdot \mathbf{I}] = \mathbf{A}'$$

Provided $[\mathbf{v} \cdot \mathbf{I}]$ is defined as :

$$[\mathbf{v} \cdot \mathbf{I}] = \begin{bmatrix} v_1 l_1 & v_1 l_2 & v_1 l_3 \\ v_2 l_1 & v_2 l_2 & v_2 l_3 \\ v_3 l_1 & v_3 l_2 & v_3 l_3 \end{bmatrix}$$

\mathbf{A}' is called augmented matrix.

So:

$$r = \frac{1}{\lambda} - 1$$

Where λ is the maximal eigenvalue of the matrix \mathbf{M} .

And:

$$X^* = q \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix}$$

Where X^* is the unit eigenvector of the matrix \mathbf{M} associated with the eigenvalue λ , and q a real number greater than 0.

The quantity l_i of labor necessary for the manufacture of one unit of commodity i is transformed into value only on the condition that this commodity is sold. This commodity must therefore correspond to a social need that has the capacity to buy (solvent social needs).

The rate of profit r determined from the eigenvalue of the matrix is a potential rate of profit that is realized only when commodities are sold, i.e. when solvent social needs are met.

The satisfaction of social needs is the compliance to Marx's fundamental equality I, which can be written (3):

$$\begin{aligned} z = & K_1[x_1(w_1 - b_1) - x_2f_1 - x_3m_1 - pl_1] \\ & + K_2[x_2(w_2 - f_2) - x_1b_2 - x_3m_2 - pl_2] \\ & + K_3[x_3(w_3 - m_3) - x_1b_3 - x_2f_3 - pl_3] = 0 \end{aligned}$$

Rearranging this equation and setting surplus values to zero, we get:

$$\begin{aligned} & [K_1w_1 - (K_1b_1 + K_2b_2 + K_3b_3)]x_1 \\ & + [K_2w_2 - (K_1f_1 + K_2f_2 + K_3f_3)]x_2 \\ & + [K_3w_3 - (K_1m_1 + K_2m_2 + K_3m_3)]x_3 \\ & = 0 \end{aligned} \quad (3)$$

Each of the factor of x_i having to be greater than or equal to zero (The economy is productive), the nullity of the complete equation is ensured only and only if each of these terms is null. This determines in this case a single solution for the triplet of K_i . Obviously, the equation (3) stipulates the condition of respect for social needs. A commodity can only be sold if Marx's equality is complied with, which implies a set of allowed values for capital employed.

When surplus value is non-zero there exists an infinity of solutions of the triplets which form a portion of a segment in the three-dimensional space K_1, K_2, K_3 .

The second Marx's fundamental equality postulates that the sum of capitals committed in price (subscribe p) is equal to the sum of capitals committed in value. That is:

$$\sum K_{pi} = \sum K_i$$

Taking into account this equality, Marx's second equality can be written:

$$K_1 w_1 (1 - x_1) + K_2 w_2 (1 - x_2) + K_3 w_3 (1 - x_3) = 0 \quad (4)$$

5. Simple Reproduction

In the chosen example, the production of branch 2 produces exactly the inputs for goods 2 in the three branches (in quantity, in price and in value).

$$K_2 w_2 x_2 = (K_1 f_1 + K_2 f_2 + K_3 f_3) x_2$$

Or

$$K_1 f_1 + K_2 (f_2 - w_2) + K_3 f_3 = 0$$

5.1 Resolution with iterations

This iterative method is required when the fixed capital is non-zero and the rate of profit cannot be found by solving the eigenvalue equation (3). In addition, as shown below, it can be used to illustrate the similarity of the cases with or without fixed capital.

The resolution proceeds in two stages which alternate to lead to the solution found by linear interpolation. The first step, called **System in K**, determines a candidate solution triplet (K_1 , K_2 , K_3) (calculated with a certain value of q) for the capital invested. The second, called **System in q** , calculates the quantity z obtained with the solution (K_1 , K_2 , K_3) found in step 1, these successive two steps constitute one iteration. The value of q is zero for the first iteration. After each iteration it is incremented by Δq , until the first negative value of z occurs.

1) Step K, (system in K):

From the above we can now write the following system of three equations with three unknowns (the amounts of capital invested in each industry):

$$\begin{aligned} K_1 + K_2 + K_3 &= K_T \\ K_1 w_1 (1 - qx_1^*) + K_2 w_2 (1 - qx_2^*) + K_3 w_3 (1 - qx_3^*) &= 0 \\ K_1 f_1 + K_2 (f_2 - w_2) + K_3 f_3 &= 0 \end{aligned} \quad (5)$$

If the determinant is non-zero this system accepts a unique solution. For this step we accept negative values for K_i .

2) Step q, (system in q):

$$z = K_1\{q[x_1^*(w_1 - b_1) - x_2^*f_1 - x_3^*m_1] - pl_1\} \\ + K_2\{q[x_2^*(w_2 - f_2) - x_1^*b_2 - x_3^*m_2] - pl_2\} \quad (6) \\ + K_3\{q[x_3^*(w_3 - m_3) - x_1^*b_3 - x_2^*f_3] - pl_3\}$$

We still have to look for the particular value of $q=q^*$ so that the quantity z is zero, thus guaranteeing compliance with Marx's equality II. The value of q^* makes it possible to pass from “relative prices” to “absolute prices”.

3) Iterations:

The first iteration starts with $q=0$. At first glance, it could seem that the first value of z should be negative but this is not the case because the result of the coefficients K_i in step K can be negative. The function z is in fact positive for the values $q < q^*$ as we demonstrate in the chapter “variation of the z-function”.

As soon as the first negative value of $z=z_1$ occurs, we keep the last positive value z_0 of z and the solution q^* that cancels z is somewhere between $q(z_0)$ and $q(z_1)$ so it can be specified by linear interpolation. The precision can be increased by restarting the iterations around the solution

with a Δq step as small as required depending on the intended precision. This “zoom” effect around the value of q^* can be repeated several times.

5.2 Direct Resolution

When fixed capital is zero, the rate of profit can be determined independently of the distribution of capital across branches. The latter can then be obtained by the direct resolution of a linear system:

$$\begin{aligned} K_1 + K_2 + K_3 &= K_T \\ K_1 r_1 + K_2 r_2 + K_3 r_3 &= K_T \cdot r \\ K_1 f_1 + K_2 (f_2 - w_2) + K_3 f_3 &= 0 \end{aligned}$$

where:

$$r_i = pl_i = w_i - 1 \quad (i = 1,2,3)$$

The internal profit rate (or surplus value by unit of capital in value) for branch i is r_i and r is the general rate of profit determined from the eigenvalue of the matrix M.

Note that the second equation of the system is equivalent to the cancellation of z (i.e., the satisfaction of social needs):

$$\begin{aligned} &K_1(x_1(w_1 - e_1) - x_2 c_1 - x_3 v_1) \\ &+ K_2(x_2(w_2 - c_2) - x_1 e_2 - x_3 v_2) \\ &+ K_3(x_3(w_3 - v_3) - x_1 e_1 - x_2 c_2) \\ &= K_1 r_1 + K_2 r_2 + K_3 r_3 = K_T \cdot r \end{aligned}$$

This system with three independent equations makes it possible to determine the three unknowns K_i (non-zero determinant).

While, from the following redundant equation (4):

$$K_1 w_1 (1 - q^* x_1^*) + K_2 w_2 (1 - q^* x_2^*) + K_3 w_3 (1 - q^* x_3^*) = 0$$

We get:

$$q^* = \frac{\sum K_i w_i}{\sum x_i^* K_i w_i}$$

So, both the amounts of capital K_i of the various industries and the production prices are determined.

5.3 Variation of the z function

We have defined the z-function as:

$$z = \sum_i S_i - \sum_i PL_i = \sum_i (S_i - PL_i)$$

Where the profit (in price) of the industry i is:

$$S_i = K_i s_i = K_i (w_i x_i - x_1 b_i - x_2 f_i - x_3 m_i)$$

In its first step our algorithm calculates the capitals in such a way as to comply with fundamental equality II, i.e.:

$$\sum_i W_i = \sum_i W_i q x_1^*$$

We consider a value of q close enough⁴ to q^* so that the provisional solutions for the K_i are all positive. We consider the three-branch case for clarity and without impairing generalization. By detailing each side of fundamental equality II we have:

$$\sum_i (B_i + F_i + M_i + PL_i) = \sum_i (B_i x_1 + F_i x_2 + M_i x_3 + S_i)$$

where:

$$B_i = K_i b_i, \quad F_i = K_i f_i, \quad M_i = K_i m_i, \quad PL_i = K_i pl_i, \quad S_i = K_i s_i$$

When $q < q^*$ prices are lower than when $q = q^*$, so:

$$\sum_i (B_i + F_i + M_i) > \sum_i (B_i x_1 + F_i x_2 + M_i x_3)$$

Therefore, when $q < q^*$, fundamental equality II can be complied with only for $S_i > PL_i$, i.e. for $z > 0$.

We conclude that in the vicinity⁵ of the solution q^* and for $q < q^*$,

$$z > 0$$

Therefore, as illustrated in Figure 1, we define q^* as the first value that both cancels z and is such that:

$$\left[\frac{dz}{dq} \right]_{q^*} < 0$$

⁴ Without fixed capital, the function z is monotonically decreasing and it is not necessary to restrict itself to the neighborhood as it is the case with fixed capital

⁵.idem

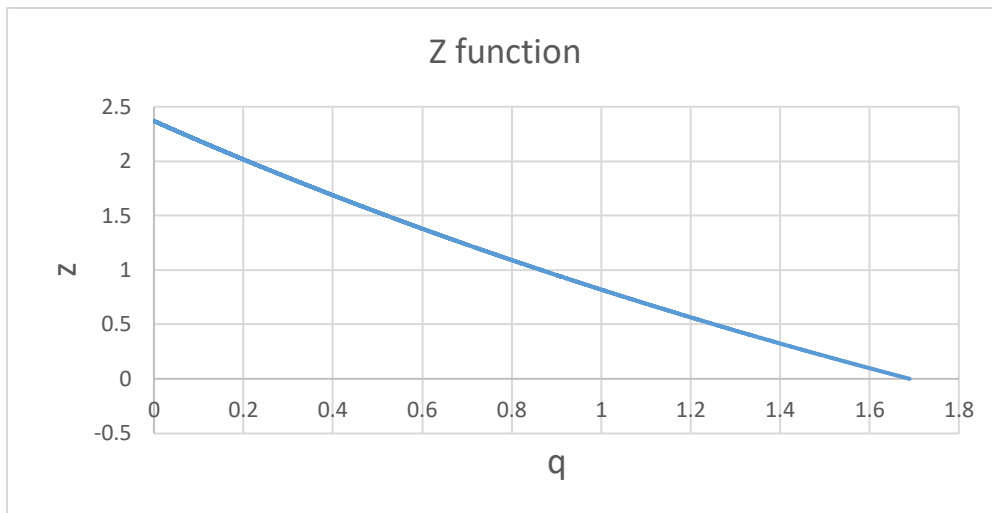


Fig 1:

5.4 Physical quantities of commodities

In production, one must take into account the quantities of goods that the wages of workers allow them to purchase. Therefore, for quantities, the corresponding terms of the initial matrix must be increased accordingly:

$$\mathbf{A}' = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix}$$

If the wage consumption is composed only of goods of industries 1 (b) and 3 (m), only the prime terms of the matrix \mathbf{A}' are increased.

$$a'_{1j} = (a_{1j}\Lambda_1 + b_{jV}) / \Lambda_1$$

$$a'_{3j} = (a_{3j}\Lambda_3 + m_{jV}) / \Lambda_3$$

Which leads to a new vector Λ' :

$$\Lambda' = \mathbf{I} \cdot (\mathbf{I} - \mathbf{A}')^{-1}$$

In the case where the wage is zero (maximum rate of profit) we have:

$$\mathbf{A}' = \mathbf{A} \text{ and } \mathbf{\Lambda}' = \mathbf{\Lambda}$$

In terms of physical quantities, if we call g_j (gross output) the quantity of commodity j produced and y_j (net output) the quantity remaining after its consumption by the three industries, we have:

$$y_j = g_j - (a'_{j1}g_1 + a'_{j2}g_2 + a'_{j3}g_3)$$

$$\mathbf{y} = \mathbf{g} \cdot (\mathbf{I} - \mathbf{A}')$$

$$\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

The quantity y_j of commodity j is given by:

$$(W_1 - \sum_{i=1}^3 b_i)/w_1 = y_1 ; (W_2 - \sum_{i=1}^3 f_i)/w_2 = y_2 ; (W_3 - \sum_{i=1}^3 m_i)/w_3 = y_3$$

The quantities calculated here are those that can be sold on the market (what any good market study can predict). If additional quantities are produced their values can only be zero.

If the economy is in state of simple reproduction in which all iron (fer, $j=2$) is used in production, we have:

$$y_2 = 0$$

This condition has already been imposed at the level of capital in value. In contrast to what has been assumed by previous authors (4), the approach detailed here clearly shows that one cannot decide freely on all the components of the vector \mathbf{y} . Its components are linked together, on the one hand by the need for technical production and on the other hand by the balance between the capitals allocated to each industry according to solvent social needs.

The vector \mathbf{v} components which determine the structure of workers' consumption, do not impact on the rate of exploitation if the amount of socially necessary labor contained in these commodities is the same ($\Lambda \cdot \mathbf{v}$ constant). However, even in the latter case, the modification of its components changes the rate of profit by modifying the M matrix and the organic composition of the branch traditionally denoted V. This is in line with the Marxist conception, and contrary to Duménil⁶ or Lipietz⁷ conclusions that prices should

⁶ « Si cette théorie était juste, elle conférerait à la structure de la consommation ouvrière une importance décisive. L'orientation de cette consommation, par la publicité notamment, devrait permettre au système capitaliste de maximiser son taux de profit sur la base d'un taux de la plus-value déterminé. Cette analyse ne se trouve pas dans *Le Capital*, où seule intervient la valeur de la force de travail qui fixe le taux de la plus-value. Dans l'analyse de K. Marx, la structure de la consommation ouvrière possède une influence sur la valeur de la force de travail, mais tous les « paniers de biens » de même valeur aboutissent au même taux de profit — alors que dans le système de Morishima, pour une même valeur de la force de travail, la structure de la consommation ouvrière détermine le taux de profit. »

⁷ « Ces prix devraient alors être déterminés indépendamment de la consommation ouvrière, mais, comme chez Marx, en fonction de e , de la composition et de la répartition y du capital dans les branches. Il en résulterait

not be based on workers consumption (9, 10). Actually, in the real world, the wage is monetary and the worker spends it as he wants. The structure of the vector v must be understood as the result of a working standard of consumption which is continuously established according to the class struggle.

If this standard vector is oriented towards a consumption of goods produced by industries with a low organic composition, it tends to increase the rate of profit r . However, capitalists are in competition with each other, so there is no possible consensus on r . Besides, some goods obligatorily require processes of production which are capitalistic (regardless of the competition). Furthermore, when such goods fulfill an actual demand and can be produced by capital-intensive industries, the competition between branches would disadvantage those producing the same goods with lower organic composition. There is no consensus to influence consumption in such a way as to increase the rate of profit. Nevertheless, the importation of goods from industries with a lower organic composition of the third countries is certainly a factor in raising the rate of profit.

5.5 Numerical Results

To illustrate the method, we are using the exact same numerical example as the one chosen by D. Basu (4), except

alors, selon le choix des salariés, un ou des paniers de consommation d... qui pourraient alors servir de base de renégociation de w . »

for the vector y which cannot be chosen arbitrarily as demonstrated above.

$$\mathbf{A} = \begin{pmatrix} 186/450 & 54/21 & 30/60 \\ 12/450 & 6/21 & 3/60 \\ 9/450 & 6/21 & 15/60 \end{pmatrix}$$

$$\mathbf{l} = (18/450; 12/21; 30/60)$$

$$\Lambda_i = \sum_{j=1}^3 \Lambda_j a_{j,i} + l_i \quad i, j = 1, 2, 3$$

$$\Lambda = \mathbf{l} \cdot (\mathbf{I} - \mathbf{A})^{-1}$$

$$\Lambda = (0.181818, 1.81818, 0.909091)$$

First case: reference example salary

$$\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1/6 \end{pmatrix}$$

$$\Lambda \mathbf{v} = 0.515152$$

$$e = 0.941176$$

Value table for the production of one unit of each commodity:

	Wheat	Iron	Meat	pl.	w
Wheat	0.08969697	0.048484848	0.024254545	0.019381818	0.181818182
Iron	0.675324675	0.519480519	0.346493506	0.276883117	1.818181818
Meat	0.272727273	0.090909091	0.303181818	0.242272727	0.909090909
Total	1.037748918	0.658874459	0.67392987	0.538537662	2.909090909

Distribution consistent with social need (same total capital;
simple reproduction):

	Wheat	Iron	Meat	pl	W	Wages
Wheat	0.727430628	0.393205745	0.196602872	0.157282298	1.474521543	0.167112
Iron	0.240255014	0.184811549	0.1232077	0.09856616	0.646840423	0.104727
Meat	0.206469385	0.068823128	0.229410428	0.183528342	0.688231284	0.194999
Total	1.174155027	0.646840423	0.549221	0.4393768	2.80959325	

Prices:

	Wheat	Iron	Meat	S	W	Wages
Wheat	0.784547437	0.393805927	0.163247423	0.248698072	1.590298859	0.168036
Iron	0.259119493	0.185093642	0.102304403	0.10131021	0.647827748	0.105305
Meat	0.222681065	0.068928179	0.190488881	0.089368518	0.571466643	0.196076
Total	1.266347995	0.647827748	0.456040706	0.4393768	2.80959325	

r: rate of profit	xi	K	Kp	Unit price
0.185374125	1.078518565	1.317239246	1.341600787	0.196094
0.185374125	1.001526382	0.548274263	0.546517538	1.820957
0.185374125	0.830340986	0.504702942	0.482098125	0.754855
TOTAL		2.37022	2.37022	

The sum of invested capital expressed in value or price is
2.37022 (um)

The sum of surplus value is equal to the sum of profit and is
0.43938 (um)

The system is of simple reproduction; i.e., both Marx's fundamental equalities are complied with: the sum of the profits equals the sum of the surplus values (invariant 1) and the sum of the values equals the sum of the prices (invariant 2).

None of the previous approaches (4) managed to combine invariants 1 and 2, which led to the awkward situation where there was no relevant reason to choose one invariant over the other.

Quantities can be determined using the augmented matrix:

$$\mathbf{A}' = \begin{pmatrix} 0.49333 & 3.71429 & 1.5 \\ 0.02666 & 0.285714 & 0.05 \\ 0.02666 & 0.380952 & 0.333333 \end{pmatrix}$$

$$\mathbf{\Lambda}' = (0.3745, 3.750, 1.875)$$

We check on this example that:

$$(W_1 - \sum_{i=1}^3 b_i)/w_1 = 1.65202$$

$$W_2 - \sum_{i=1}^3 c_i = 0$$

$$(W_3 - \sum_{i=1}^3 m_i)/w_3 = 0.152911$$

Which give:

$$\mathbf{y} = \begin{pmatrix} 1.65202 \\ 0 \\ 0.152911 \end{pmatrix} \quad \mathbf{g} = \begin{pmatrix} 8.1098 \\ 0.35576 \\ 0.757054 \end{pmatrix}$$

These values are valid for a total capital of: 2.37022 monetary units and all greater economies are homothetic to this one.

It can be verified that for any commodity i:

$$w_i g_i = W_i$$

For example:

$$w_1 g_1 = 0.1818 \times 8.1098 = W_1 = 1.474521543$$

Second case: Zero wages, maximum rate of profit

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Start	Wheat	Iron	Meat	pl.	W
Wheat	0.075151515	0.048484848	0.018181818	0.04	0.181818182
Iron	0.467532468	0.519480519	0.25974026	0.571428571	1.818181818
Meat	0.090909091	0.090909091	0.227272727	0.5	0.909090909
Total	0.633593074	0.658874459	0.505194805	1.111428571	2.909090909

Values	Wheat	Iron	Meat	pl	W	Wages
Wheat	0.569569699	0.367464322	0.137799121	0.303158066	1.377991208	0
Iron	0.156686944	0.174096604	0.087048302	0.191506264	0.609338114	0
Meat	0.067777188	0.067777188	0.16944297	0.372774534	0.67777188	0
Total	0.794033831	0.609338114	0.394290393	0.867438864	2.665101202	

Prices	Wheat	Iron	Meat	S	W	Wages
Wheat	0.700396325	0.369983164	0.072598436	0.551529313	1.694507238	0
Iron	0.192676962	0.175289977	0.045860747	0.199687233	0.613514919	0
Meat	0.083345188	0.068241777	0.089269761	0.116222319	0.357079045	0
Total	0.976418475	0.613514919	0.207728944	0.867438864	2.665101202	

r: rate of profit	xi	K	Kp	Unit price
-------------------	----	---	----	------------

0.482537152	1.229693795	1.074833142	1.142977926	0.223580690
0.482537152	1.006854658	0.41783185	0.413827686	1.830644832
0.482537152	0.526842519	0.304997346	0.240856726	0.478947744
TOTAL		1.79766	1.79766	

The sum of invested capital expressed in value or price is:
1.79766

Third case: no surplus value (maximal wage)

a) Maximum Meat

$$\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0.7 \end{pmatrix}$$

Start	Wheat	Iron	Meat	pl.	W
Wheat	0.08969697	0.048484848	0.043636364	0	0.181818182
Iron	0.675324675	0.519480519	0.623376623	0	1.818181818
Meat	0.272727273	0.090909091	0.545454545	0	0.909090909
Total	1.037748918	0.658874459	1.212467532	0	2.909090909

Val=Pr	Wheat	Iron	Meat	pl	W	Wages
Wheat	0.559480213	0.302421737	0.272179563	0	1.134081512	0.249498
Iron	0.218912098	0.168393921	0.202072706	0	0.589378725	0.185233
Meat	0.355689202	0.118563067	0.711378403	0	1.185630672	0.652097
Total	1.134081512	0.589378725	1.185630672	0	2.909090909	

r: rate of profit	xi	K=Kp
0	1	1.134081512
0	1	0.589378725
0	1	1.185630672

The sum of invested capital expressed in value or price is:
2.90909

b) Maximum Wheat

$$\mathbf{v} = \begin{pmatrix} 4.665 \\ 0 \\ 0.167 \end{pmatrix}$$

Start	Wheat	Iron	Meat	pl.	W
Wheat	0.109078788	0.048484848	0.024254545	0	0.181818182
Iron	0.952207792	0.519480519	0.346493506	0	1.818181818
Meat	0.515	0.090909091	0.303181818	0	0.909090909
Total	1.57628658	0.658874459	0.67392987	0	2.909090909

Val=Pr	Wheat	Iron	Meat	pl	W	Wages
Wheat	1.004302041	0.446406063	0.223314633	0	1.674022738	0.368285
Iron	0.366543968	0.199969432	0.133379611	0	0.699893011	0.219966
Meat	0.303176728	0.053517516	0.178480916	0	0.53517516	0.294346
Total	1.674022738	0.699893011	0.53517516	0	2.909090909	

With $pl=0$, Prices equal values and Unit prices are equal to unit values.

r: rate of profit	xi	K=Kp
0	1	1.674296926
0	1	0.699949132
0	1	0.534844851

The sum of invested capital expressed in value or price is:
2.90909

Conclusion

We solved the problem of transformation simply by following the logic impelled by Marx's conceptualization. The path chosen by previous authors, applying an astray materialism that only takes into account the technological aspect of production, starts by an arbitrary choice of net quantities of goods (vector y) produced in each branch. However, in a capitalist market, the quantities of goods depend on their values, which determine their selling in accordance to solvent social needs. Therefore, vector y depends on the economy at the aggregate level. In contrast, decreeing a net product for an economy would rather be in line with a planned economy.

Research in economy most often assesses the quantity of goods likely to be sold. In a troublesome way, the conceptual error of decreeing a net product in order to solve the transformation problem is likely to have led some to consider the notion of value as superfluous. A consistent materialism must start from the overall structure of capitalism and the possibilities it imposes on the sphere of production and not from the idea that physical conditions are the sole determinant of the market.

Since the profit rates are obtained from the maximum eigenvalue of the matrix M , they are identical to those calculated by previous approaches (4). However, D. Basu displays several possibilities of market production prices. This lack of unicity of the solution results from the various possible choice of invariants which is lengthily discussed (4). As we show, as soon as the capital allocation is part of the solution (an implicit consequence of considering the economy as a

whole), the solution is unique (in the case of simple reproduction) and differs from the previously suggested ones.

References

1. K. Marx, *Capital - A Critique of Political Economy - Volume I* (Progress Publishers, 1867).
2. K. Marx, *Capital - A Critique of Political Economy - Volume III* (International Publishers, 1894).
3. N. Ankri, P. Marcaggi (2022) From Marx's fundamental equalities to the solving of the transformation problem - Coherence of the model. (Aix-Marseille Université, Arxiv).
4. D. Basu, *The Logic of Capital - An Introduction to Marxist Economic Theory -Chapter 7* (Cambridge University Press, 2021).
5. W. W. Leontief, *Input-Output Economics, 2nd ed* (New York, ed. Oxford University Press, 1986).
6. M. Morishima, Marx in the light of modern economic theory. *Econometrica* **42**, 611-632 (1974).
7. I. Rubin, *Essays on Marx's Theory of Value* (ed. Black and Red, 1928).
8. J.-M. Harribey, La théorie de la valeur-travail à nouveau reformulée par Vincent Laure van Bambeke. *Revue d'histoire de la pensée économique* **2**, 285-299 (2022).
9. G. Duménil, Une approche fonctionnelle du théorème marxien fondamental d'Okishio-Morishima. *Cahiers d'économie politique* **7**, 129-139 (1982).
10. A. Lipietz, Retour au problème de la "transformation des valeurs en prix de production". *Cahiers d'économie politique* **7** (1982).

APPENDIX: generalization to n branches:

The economy has n sectors, each producing a single commodity using labor and all commodities. The non-negative matrix A is productive which means that each good is produced in a quantity at least equal to its consumption by the whole economy.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

The line vector of direct labor inputs is:

$$\mathbf{l} = (l_1; l_2 \dots l_n)$$

The line vector of values is:

$$\mathbf{\Lambda} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$$

$c_{ij} = a_{ij}\Lambda_i$ is the value of commodity i used in industry j.

$$(w_1, w_2, \dots w_n) \mathbf{M} = \mathbf{A} + [\mathbf{v} \cdot \mathbf{l}]$$

Where M is the augmented matrix

$$\mathbf{v} = (v_1 \dots v_n)^T$$

The exponent T indicates the transposed vector

$$[\mathbf{v} \cdot \mathbf{l}] = \begin{bmatrix} v_1 l_1 & \cdots & v_1 l_n \\ \vdots & \ddots & \vdots \\ v_n l_1 & \cdots & v_n l_n \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} c_{11}/w_1 & \cdots & c_{1n}/w_1 \\ \vdots & \ddots & \vdots \\ c_{n1}/w_n & \cdots & c_{nn}/w_n \end{bmatrix}$$

$$\mathbf{M} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \frac{1}{(1+r)} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

$$r = \frac{1}{\lambda} - 1$$

Where λ is the maximal eigenvalue of the matrix \mathbf{M} .

And:

$$X^* = q \begin{pmatrix} x_1^* \\ \cdot \\ \cdot \\ x_n^* \end{pmatrix}$$

Where X^* is the unit eigenvector of the matrix \mathbf{M} associated with the eigenvalue λ and q a real number greater than 0.

There are $n-2$ equations translating the simple reproduction constraint. It will be assumed that all goods produced by sectors number 2 to $n-1$ are entirely consumed.

$$K_1 c_{11} - K_2 (c_{12} - w_2) + K_3 c_{13} + \dots + K_n c_{1n} = 0$$

$$K_1 c_{11} + K_2 c_{12} - K_3 (c_{13} - w_3) + \dots + K_n c_{1n} = 0$$

.....

$$K_1 c_{11} + K_2 c_{12} + \dots - K_{n-1} (c_{1,n-1} - w_{n-1}) + K_n c_{1n} = 0$$

And the two additional equations:

$$K_1 + K_2 + K_3 + \dots + K_n = K_T$$

$$K_1 w_1 (1 - q x_1^*) + \dots + K_n w_n (1 - q x_n^*) = 0$$

Let n equations for n unknowns K_i ($i = 1, 2 \dots n$). The nonzero determinant, because the system is productive, implies one and only one solution ($K_1 \dots K_n$).

The correct solution that satisfies the social needs is found exactly according to the procedure described for $n=3$ with the following z function.

$$\begin{aligned}
 z = & K_1 \{ q [x_1^* (w_1 - c_{11}) - x_2^* c_{21} - \dots - x_n^* c_{n1}] - pl_1 \} \\
 & + K_2 \{ q [x_2^* (w_2 - c_{12}) - x_1^* c_{22} - \dots - x_n^* c_{n2}] - pl_2 \} \\
 & + \dots \\
 & + K_n \{ q [x_n^* (w_n - c_{1n}) - x_1^* c_{2n} - \dots - x_n^* c_{nn}] \\
 & - pl_n \}
 \end{aligned}$$

Or Direct resolution:

$$K_1 c_{11} - K_2 (c_{12} - w_2) + K_3 c_{13} + \dots + K_n c_{1n} = 0$$

$$K_1 c_{11} + K_2 c_{12} - K_3 (c_{13} - w_3) + \dots + K_n c_{1n} = 0$$

.....

$$K_1 c_{11} + K_2 c_{12} + \dots - K_{n-1} (c_{1,n-1} - w_{n-1}) + K_n c_{1n} = 0$$

And the two additional equations:

$$K_1 + K_2 + K_3 + \dots + K_n = K_T$$

$$K_1 r_1 + K_2 r_2 + \dots + K_n r_n = r K_T$$

$$r_i = pl_i = w_i - 1 \quad (i = 1, 2, \dots, n)$$

$$q^* = \frac{\sum K_i w_i}{\sum x_i^* K_i w_i}$$