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An improved estimator of distortion risk premiums under dependence insured risks with heavy-tailed marginals

Sidya Ahmedou ⁽¹⁾, El Hadji Deme ^(1,*) and Souleymane Fofana ^(1,2)

(1) LERSTAD, UFR SAT, Université Gaston Berger, BP 234, Saint-Louis, Sénégal

(2) Ecole Nationale de la Statistique et de l'Analyse Economique (ENSAE), Dakar, Sénégal

Abstract. In this paper, we introduce a semi parametric estimator of risk premiums for dependent insurance losses with heavy-tailed marginals. Our estimator is based on reduced bias estimators of the tail index and extreme quantiles for stationary insured risks with heavy-tailed distribution under dependence serials. Moreover, we illustrate the behaviour of our proposed estimator and give a comparison between in this estimator and the classical one in terms of the absolute bias and the root median squared error.

Keywords: Risk Premiums, Insurance , Estimation, Heavy-Tailed, Bias reduction, Extreme value, Dependence serials .

(1) Corresponding author: elhadji.deme@ugb.edu.sn

1 Introduction

In recent years, risk measures or premium calculation principles have become important tools in finance and actuarial science. They are used to quantify insurance losses and financial assessments. For the management of these risks, several risk measures have been proposed, we refer to [Gooverts *et al.* (1984)], [Wang (1995)], [Wand (1996)], [Denuit *et al.* (2005)] and references therein. One of the most commonly used one is *the net premium* (the mean) of a non-negative loss random variable X over the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with tail distribution function $\bar{F} := 1 - F$ and defined as

$$\pi = \mathbb{E}(X) = \int_0^{\infty} \bar{F}(x) dx.$$

In general, premiums are required to be greater than or equal to the mean $\mathbb{E}(X)$ in order to avoid that the insurer loses money on average. One way to achieve this goal consists in introducing an increasing, concave function g that maps $[0, 1]$ onto $[0, 1]$, such that $g(0) = 0$ and $g(1) = 1$ and to define the following distortion risk premium introduced by [Wand (1996)]:

$$\pi_g = \int_0^{\infty} g(\bar{F}(x)) dx.$$

Note that the distortion Risk premiums can be seen as the expectation with respect to *distorted probabilities*. The function g is called *distortion function* and is in general parameterized by a one-dimensional parameter $\beta \geq 1$, called the *distortion parameter* and represents the risk aversion. This parameter controls the amount of the risk loading included in the premium for a given riskiness of the loss variable X . Important properties of the distortion risk measure, such as coherence and second order stochastic dominance have been well studied see, for example [Artzner *et al.* (1999)], [Wirch and Hardy(1999)] and [Jones and Zitikis(2003)].

A standard reinsurance product is an excess of loss reinsurance, which means that the reinsurer only compensates the cedant's loss above a certain retention amount $R > 0$. Considering an excess-of-loss reinsurance policy in excess of a high retention level R , the tail distribution function of the total claim amount $(X - R)_+$, with $s_+ = \max(0, s)$, is easily seen to be equal to $\bar{F}_R(y) = \bar{F}(y + R)$, when $y > 0$. Applying Wang's premium principle to this reinsurance setting results in the following expression for the risk premium $\pi_g(R)$ for a reinsurance policy in excess of a high retention level R :

$$\pi_g(R) = \int_0^{\infty} g(\bar{F}_R(y)) dy = \int_0^{\infty} g(\bar{F}(y + R)) dy.$$

By putting $x = y + R$, the risk premium $\pi_g(R)$ can then easily be rewritten as

$$\pi_g(R) = \int_R^{\infty} g(\bar{F}(x)) dx. \tag{1.1}$$

It is assumed throughout the present paper that F is a continuous loss distribution. Let Q be the quantile function corresponding to F and defined by $Q(s) = \inf\{x : F(x) \geq s\}$, for every $s \in [0, 1]$. The quantile function Q plays a pivotal role in defining numerous risk measures, and is a well known risk measure itself, called the Value-at-Risk (VaR). By a change of variables, the distortion risk premium $\pi_g(R)$ can be rewritten in terms of the quantile function Q as follows:

$$\pi_g(R) = - \int_0^{\bar{F}(R)} g(s) dQ(1 - s). \tag{1.2}$$

Several distortion functions $t \rightarrow g(t)$ are equal or can be approximated to a power function $g_\beta(y) = t^{1/\beta}$, $\beta \geq 1$, since they are regularly varying at zero with index $1/\beta$, that is: $g(t) = t^{1/\beta} \ell_g(t)$, where $\ell_g(\cdot)$ is a slowly varying function at zero satisfying $\ell_g(\lambda t)/\ell_g(t) \rightarrow 1$ as $t \rightarrow 0$, for $\lambda > 0$. This condition was used in [Vandewalle and Beirlant (2006)] and [Deme *et al.* (2021)] to estimate the reinsurance risk premiums in the context of independent extreme risks.

We then focus in the rest of this paper on the special case of power distortion function $g(t) := g_\beta(t) = t^{1/\beta}$, $\beta \geq 1$. The corresponding distortion risk measure (see, eg. [Wang (1995)], [Wand (1996)]) is defined in the case of excess of loss reinsurance policy by:

$$\pi_\beta(R) := \pi_{g_\beta}(R) = - \int_0^{\bar{F}(R)} g_\beta(s) dQ(1-s). \quad (1.3)$$

When this parameter is at its minimum value $\beta = 1$, then π_β is the net premium π , and thus there is no loading.

In reinsurance, the purpose of estimating the pure premium is to estimate, for each insured, the expected amount of claims for the studied insurance period. This evaluation is often done using statistical methods. For more details, see [Denuit and Charpentier (2004)]. However, large loss costs require rare event modeling, that is events with a low probability of occurrence, but with a large claim amounts and their disastrous effects. The analysis of these extreme events can be carried out by using the extreme values methodology, where their distribution function F are heavy tailed and mainly characterized by their index which make the possibility to indicate the size and the frequency of some extreme phenomena within the framework of a given probability distribution (See eg. [Vandewalle and Beirlant (2006)]).

The heavy-tailed nature of insurance claims requires that special attention be paid to analyzing the tail distributions of a claims amounts. Thus the extreme value theory (EVT) offers adequate statistical tools to model these tail of the distributions. Also, reinsurance companies must calculate the premiums to cover these excess claims, which are usually very high. The EVT has become one of the leading theories in the development of statistical models for high insurance losses.

Many authors studied the estimation of the premium of these high excess losses by using classical EVT models, mainly based on the independent and identically distributed (i.i.d) assumption of the insured risks with large tails. One can mention among others, [Vandewalle and Beirlant (2006)] [Reiss and Thomas (2007)], [Necir *et al.* (2010)], [Rassoul (2013)], [Deme *et al.* (2013a)], , [Deme *et al.* (2015)], [Deme *et al.* (2021)], etc...

However, this assumption is often violated in practice since, some financial and actuarial losses expose serial dependence. This motivated [de Haan *et al.* (2016)], [Chavez-Demoulin and Guillou (2018)] and [Barry *et al.* (2023)] to investigate reduced bias estimators of the Value-at-Risk (quantile) of financial extreme losses under financial dependence series with heavy-tailed marginals. Also, [Ouadjed (2018)] looked at the high excess reinsurance losses to introduce a semiparametric estimator of the distorted premiums under heavy tailed dependent insured risks. This semiparametric estimator suffers from a bias problem due to the fact it depends on a classical estimator of the Value at Risk estimator (the Weissman's estimator (see, [Weissman (1978)]), which have the same problem.

The aim of this paper is to introduce a reduced bias estimator of the distorted risk premiums under dependent insured risks with heavy tailed marginals. Our consideration is based on the bias reduction approach

proposed by [de Haan *et al.* (2016)] in the estimation of the Value-at Risk under dependence serials.

The rest of the paper is organised as follows. In Section 2, we propose a statistical estimation of the distortion risk premiums under dependent serials. In Section 3, we establish the asymptotic properties of proposed estimator. Then in Section 5&6, we match our theoretical results with a simulation assessment and real reinsurance data analysis in order to highlight the efficiency of our methods. Finally, Section 4 is devoted to the proofs of our main results.

2 EVT under dependence serials

The extreme value theory (EVT) establishes the asymptotic behavior of the largest observations in a sample. It provides methods for extending the empirical distribution functions beyond the observed data. It is thus possible to estimate quantities related to the tail of a distribution such as small exceeding probabilities. or extreme quantiles.

The main result of the extreme value theory is based on the limit distribution of the standardized maximum of $n > 1$, i.i.d random variables sample (X_1, \dots, X_n) :

$$\mathbb{P}\left(a_n^{-1}\left(\max_{1 \leq i \leq n} X_i - b_n\right) \leq x\right) \rightarrow G(x), \quad \text{as } n \rightarrow \infty, \quad (2.4)$$

for all continuity points of G , where $a_n > 0$, $b_n \in \mathbb{R}$ are standardized sequences and G is a non-degenerate limiting distribution function. Necessary, G is the same type of the following generalized extreme value distribution:

$$G_\gamma(x) = \exp\left(-\left(1 + \gamma x\right)_+^{-1/\gamma}\right),$$

where $y_+ = \max(y, 0)$ and $G_\gamma(x) = \exp(e^{-x})$, for $\gamma = 0$. Here, the real-valued parameter is referred to as the extreme value index γ of F , which in turn is said to belong to the maximum domain of attraction of G_γ , denoted by $F \in \mathcal{DM}(G_\gamma)$. We refer to [de Haan and Ferreira (2006)], for general accounts on extreme-value theory.

Now, let X_i , $i \in \mathbb{N}$, be stationary copies from a non negative stationary insured risk X defined over some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common marginal distribution function (df) $F(x) = \mathbb{P}(X \leq x)$. If the X_i , $i \in \mathbb{N}$ are weakly dependent then (2.4) is equivalent to the weak convergence of the distribution function from the standardized maximum of n observations to G . In general, the maximum of a stationary time series is stochastically smaller than the maximum of an independent and identically distributed (i.i.d.) sequence with the same marginal distribution function. Indeed, under mild conditions on the dependence structure, $F \in \mathcal{DM}(G_\gamma)$ implies,

$$\mathcal{L}\left(a_n^{-1}\left(\max_{1 \leq i \leq n} X_i - b_n\right)\right) \rightarrow G_\gamma^\theta \quad \text{weakly}, \quad (2.5)$$

for some $\theta \in [0, 1]$ (see [Leadbetter *et al.* (1983)], Section 3.7) for more details).

Throughout this paper, we assume that the non negative stationary insured risks X_i , $i \in \mathbb{N}$ satisfies the following β -mixing dependence structure condition:

$$\beta(m) := \sup_{p \geq 1} \mathbb{E} \left\{ \sup_{C \in \mathcal{B}_{p+m}^{\infty}} |\mathbb{P}(C | \mathcal{B}_1^p) - \mathbb{P}(C)| \right\} \rightarrow 0,$$

as $m \rightarrow \infty$, where \mathcal{B}_i^j denotes the σ -algebra generated by (X_i, \dots, X_j) . Without loss of generality, $\beta(m)$ measures the total variation distance between the unconditional distribution of the future of the time series and the conditional distribution of the future given the past of the time series when both are detached by m time points.

In addition, we assume that the common marginal distribution function F of the β -mixing insured risks $X_i, i \in \mathbb{N}$, is heavy-tailed (belonging to the the Fréchet domain of attraction). This is equivalent to the fact that its associated tail distribution function \bar{F} is regularly varying at infinity with index $-1/\gamma < 0$, denoted $F \in \mathcal{RV}_{-1/\gamma}$. More precisely,

$$\bar{F}(x) = x^{-1/\gamma} \ell_F(x), \quad x > 0, \quad (2.6)$$

where ℓ_F is a slowly varying function at infinity, *i.e.* for all $x > 0$, $\ell_F(tx)/\ell_F(t) \rightarrow 1$, as $t \rightarrow \infty$. The relation (2.6) is also equivalent to $U(z) = Q(1 - z^{-1}) = z^\gamma \ell_U(z)$, $z > 1$, where $\ell_U(tz)/\ell_U(t) \rightarrow 1$, as $t \rightarrow \infty$, for all $z > 1$. The class of heavy-tailed distributions includes distributions such as Pareto, Burr, Student, Lévy-stable, and log-gamma which are known to be appropriate models in *Extreme value theory* for fitting large insurance claims, large fluctuations of prices, log-returns, etc. (see, e.g., [Beirlant *et al.* (1998)]; [Beirlant *et al.* (2004)]; [Vandewalle and Beirlant (2006)], [Peng and Qi (2004)], [Reiss and Thomas (2007)], [Necir and Meraghni (2009)], [Necir *et al.* (2010)], [Deme *et al.* (2013a)], [Deme *et al.* (2021)]).

From (2.6), one can easily see that for all $x > 0$ and $z > 1$:

$$\lim_{t \rightarrow \infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-1/\gamma} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{U(tz)}{U(t)} = z^\gamma. \quad (2.7)$$

The relation in (2.7) is namely called the first order regularly varying condition. The parameter γ is the tail index (or the extreme value index) and governs the tail behavior, with larger values indicating heavier tails. Its estimation has received a great attention in the extreme value literature, especially in the case of i.i.d. random variables (cf. [de Haan and Ferreira (2006)]). Although only few papers consider that for the case of time series with serial dependence features. We can mention among others, [Drees (2000)], [Drees (2003)], [?]. And very recently [de Haan *et al.* (2016)], [Chavez-Demoulin and Guillou (2018)] and [Barry *et al.* (2023)].

The most popular positive tail index estimators in the framework of extreme value theory is the original Hill's estimator [Hill (1975)], defined as:

$$\hat{\gamma}_k^{(H)} := \frac{1}{k} \sum_{i=1}^{k_n} \log X_{n-i+1,n} - \log X_{n-k,n},$$

where $X_{1,n}, \leq \dots \leq X_{n,n}$ stands for the order statistics and $k = k(n)$ represents an intermediate sequence, that is, a sequence such that $k \rightarrow \infty$ and $k/n \rightarrow 0$, as $n \rightarrow \infty$. To prove the asymptotic normality of the tail index estimators such as the Hill's one, we need a second order condition which specifies the rate of convergence for the left-hand side of the equations in (2.7) to their limits (See, eg. [Bingham *et al.* (1987)], [Geluk and de Haan (1987)] and [de Haan and Ferreira (2006)]). This condition can be formulated in different ways as shown below. We will use the formulation later-on.

Second order regularly varying condition (C_{SO}). Suppose that there exists a positive or negative

function A with $\lim_{t \rightarrow \infty} A(t) = 0$ and a real number $\rho \leq 0$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{A(t)} \left(\frac{U(tx)}{U(t)} - x^\gamma \right) = x^\gamma \frac{x^\rho - 1}{\rho}, \quad \forall x > 0. \quad (2.8)$$

The rate of the convergence for the function A to 0 is essential since it helps to exhibit the bias term of the tail index estimators.

The asymptotic normality of the original Hill's estimator has been established for β -mixing sequences in [Drees (2000)] and [Drees (2003)]. From the assumption that the intermediate sequence k is such that $k^{1/2}A(n/k) \rightarrow \lambda \in \mathbb{R}$, as $n \rightarrow \infty$ and assuming the following regularity conditions on the β -mixing coefficients:

Regularity conditions (C_R). There exist $\epsilon > 0$, a bivariate function r and a sequence ℓ_n such that, as $n \rightarrow \infty$,

(a) $\frac{\beta(\ell)}{\ell} n + \ell \frac{\log^2 k}{\sqrt{k}} \rightarrow 0;$

(b) $\frac{n}{\ell k} Cov \left(\sum_{i=1}^{\ell} \mathbb{I}_{\{X_i > F^{\leftarrow}(1-kx/n)\}}, \sum_{i=1}^{\ell} \mathbb{I}_{\{X_i > F^{\leftarrow}(1-ky/n)\}} \right) \rightarrow r(x, y), \quad \forall 0 \leq x, y \leq 1 + \epsilon;$

(c) For some constant C :

$$\frac{n}{\ell k} \mathbb{E} \left[\left(\sum_{i=1}^{\ell} \mathbb{I}_{\{F^{\leftarrow}(1-ky/n) < X_i \leq F^{\leftarrow}(1-kx/n)\}} \right)^4 \right] \leq C(y - x), \quad \forall 0 \leq x < y \leq 1 + \epsilon \text{ and } n \in \mathbb{N},$$

[Drees (2000)] showed that

$$\sqrt{k}(\hat{\gamma}_k^{(H)} - \gamma) \stackrel{d}{=} \gamma P^{(1)} + \frac{\lambda}{1 - \rho}(1 + o_{\mathbb{P}}(1)), \quad (2.9)$$

where $P^{(1)} = \int_0^1 (t^{-1}W(t) - W(1)) dt$ and $(W(t))_{0 \leq t \leq 1}$ is a centered Gaussian process with covariance structure r , and defined under a Skorohod construction. which leads to

$$\sqrt{k}(\hat{\gamma}_k^{(H)} - \gamma) \stackrel{d}{\rightarrow} \mathcal{N} \left(\frac{\lambda}{1 - \rho}, \gamma^2 r(1, 1) \right). \quad (2.10)$$

In practice, the bias term of $\hat{\gamma}_{k_n}^{(H)}$ depends on whether ρ is close to zero or not, since under the second order condition (C_{SO}), the function $|A|$ is regularly varying at infinity with index ρ . This explains all the literature spread on bias correction in the i.i.d. context, see, e.g., [Beirlant *et al.* (1999)], [Feuerverger and Hall (1999)], [Gomes and Martins (2002)], [Gomes *et al.* (2005)], etc. However, in the case of β -mixing time series, only the very recent papers by [de Haan *et al.* (2016)], [Chavez-Demoulin and Guillou (2016)] and [Barry *et al.* (2023)] deal with this problem and proposed a bias corrected estimator for γ . By the way, they established their asymptotic properties under the regularity conditions (C_R) and the second order assumptions (C_{SO}).

3 Estimating the distortion risk premiums

Under the regularly varying condition (2.7), the first order moments of insured risks and its corresponding distortion risk premiums $\pi_\beta(R)$ defined in (1.3) are not hold when $\gamma \geq 1/\beta$, However, they are both well defined in the case where $0 < \gamma < 1/\beta$. This last situation motivate the need of a specific estimator of the distortion risk premium $\pi_\beta(R)$ for heavy-tailed distributions with tail index $\gamma \in (0, 1/\beta]$.

To better understand the heavier of tail distribution of insured risks, which is governed by the unknown tail index, many authors used the extreme value methodology and investigated semiparametric estimators of the risk adjusted premiums in the case of i.i.d random variables, We can mention among other [?], [Reiss and Thomas (2007)], [Necir and Meraghni (2009)], [Necir *et al.* (2010)], [Deme *et al.* (2013a)], [Deme *et al.* (2015)], et. Although only [Ouadjed (2018)] considered the case of time series with serial dependence features to asses reinsurance risk premiums with an optimal retention level.

The optimal retention level corresponds to the Value-at-Risk $Q(1-k/n)$, which is the minimum amount that a company must have to cover the risk X with an uncertain ruin probability. Since $\bar{F}(Q(1-k/n)) = k/n$, from (1.3), the optimal risk adjusted reinsurance premiums is defined as:

$$\pi_{\beta,n} := \pi_{\beta}(Q(1-k/n)) = - \int_0^{k/n} g_{\beta}(s) dQ(1-s). \quad (3.11)$$

The distorted reinsurance premium $\pi_{\beta,n}$ is unknown since it depends on the unknown high quantile $Q(1-s)$, $s \rightarrow 0$. The most popular of high quantile estimators for heavy-tailed distributions is the one introduced by Weissman [Weissman (1978)] and defined as:

$$\widehat{Q}_k^{(W)}(1-s) = (ns/k)^{-\widehat{\gamma}_k^{(H)}} X_{n-k,n}, \quad s \rightarrow 0, \quad (3.12)$$

where $\widehat{\gamma}_k^{(H)}$ is the above Hill's estimator and the quantity $X_{n-k,n}$ is a moderate quantile and assigned to be the empirical estimator of the optimal retention level $Q(1-k/n)$.

Substituting in (3.11) the extreme quantile $Q(1-s)$, $s \rightarrow 0$ with its estimator $\widehat{Q}_k^{(W)}(1-s)$, [?] introduced the following semiparametric estimator for $\pi_{\beta,n}$:

$$\widehat{\pi}_{\beta,k,n} := - \int_0^{k/n} g_{\beta}(s) d\widehat{Q}_k^{(W)}(1-s), \quad (3.13)$$

which leads to:

$$\widehat{\pi}_{\beta,k,n} = \frac{\widehat{\gamma}_k^{(H)}}{1/\beta - \widehat{\gamma}_k^{(H)}} g_{\beta}(k/n) X_{n-k,n}, \quad (3.14)$$

provided that $\mathbb{P}(\widehat{\gamma}_k^{(H)} > 1/\beta) = o(1)$, for large values of n . Under the second order regularly varying condition (C_{SO}) and regularity conditions on the β -mixing coefficients (C_R), [?] established, the asymptotic normality of the estimator $\widehat{\pi}_{\beta,k,n}$. Clearly, the estimator $\widehat{\pi}_{\beta,k,n}$ suffer from a high bias due to the fact that its depends to Hill's estimator, which from (2.10) as the same problem and the bias heavily depends on the intermediate sequence, making the choice of k difficult in practice.

To overcome to this problem, we propose in next section an asymptotically unbiased estimator of the distortion premium $\pi_{\beta,n}$ for heavy tailed distributions and we establish its asymptotic normality under the β -mixing condition and the regulatory conditions.

4 Main Results

In this section, we first introduce a reduced bias estimator of the distortion risk premiums $\pi_{\beta,n}$ and after, we state our main results on its asymptotic properties.

4.1 Reduced bias estimator for $\pi_{\beta,n}$

Our bias reduction approach is based to the one adopted in [de Haan *et al.* (2016)] to estimate the tail index under the β mixing times series. To this spirit, let $\hat{\rho} := \hat{\rho}_{k_\rho}$ be a consistent estimator of ρ , with $k_\rho := k_{\rho,n}$ an intermediate sequence of integers greater than k , satisfying $k_\rho \rightarrow \infty$ and $k_\rho/n \rightarrow 0$, $k_\rho^{1/2} A(n/k_\rho) \rightarrow \infty$, as $n \rightarrow \infty$. With the second order condition C_{SO} , the β -mixing serials assumptions and the fact that $k \rightarrow \infty$, $k/n \rightarrow 0$ and $k^{1/2} A(n/k) = O(1)$, as $n \rightarrow \infty$, [de Haan *et al.* (2016)] investigated the asymptotic properties of the following unbiased tail index estimator:

$$\hat{\gamma}_{k,\hat{\rho}} := \hat{\gamma}_k^{(H)} - \frac{M_k^{(2)} - 2 \left(\hat{\gamma}_k^{(H)} \right)^2}{2 \hat{\gamma}_k^{(H)} \hat{\rho} (1 - \hat{\rho})^{-1}}, \quad (4.15)$$

where

$$M_k^{(\alpha)} := \frac{1}{k} \sum_{j=1}^k (\log X_{n-j+1,n} - \log X_{n-k,n})^\alpha. \quad (4.16)$$

Clearly, this estimator is a corrected bias version of the Hill's estimator $\hat{\gamma}_k^{(H)}$. It has been initially studied in the i.i.d. context by [?]. According to [de Haan *et al.* (2016)], under the second order regularly varying condition (C_{SO}) and regularity conditions on the β -mixing coefficients,

$$\sqrt{k} (\hat{\gamma}_{k,\hat{\rho}} - \gamma) \xrightarrow{d} \frac{\gamma}{\rho} \left((2 - \rho)P^{(1)} - (1 - \rho)P^{(2)} \right), \text{ as } n \rightarrow \infty, \quad (4.17)$$

where $P^{(\alpha)} = \int_0^1 (-\log t)^{\alpha-1} (t^{-1}W(t) - W(1)) dt$ with $(W(t))_{0 \leq t \leq 1}$, the same centered Gaussian process with covariance structure r , defined in (2.9). Therefore, the process term $P^{(\alpha)}$ is a Gaussian Process with mean zero and covariance $Cov(P^{(\alpha)}, P^{(\beta)}) = c_{\alpha,\beta}$ defined as,

$$c_{\alpha,\beta} = \int_0^1 \int_0^1 (-\log s)^{\alpha-1} (-\log t)^{\beta-1} \left(\frac{r(s,t)}{st} - \frac{r(s,1)}{s} - \frac{r(1,t)}{t} + r(1,1) \right) ds dt. \quad (4.18)$$

Computing the the variance term of $\frac{\gamma}{\rho} \left((2 - \rho)P^{(1)} - \gamma(1 - \rho)P^{(2)} \right)$. with respect to the covariance structure r , [de Haan *et al.* (2016)] deduced the following asymptotic normality, as $n \rightarrow \infty$:

$$\sqrt{k} (\hat{\gamma}_{k,\hat{\rho}} - \gamma) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\gamma,\rho}^2\right),$$

where $\sigma_{\gamma,\rho}^2 = \frac{\gamma^2}{\rho^2} \left((2 - \rho)^2 c_{1,1} + (1 - \rho)^2 c_{2,2} + 2(1 - \rho)(2 - \rho)c_{1,2} \right)$.

A possible choice for $\hat{\rho}_{k_\rho}$ is the most performed estimator among those studied in the i.i.d. case (see, e.g, In order to introduce an asymptotically unbiased estimator of distortion premiums, we , we propose to estimate the extreme quantile $Q(1 - s)$, $s \rightarrow 0$ by using second order condition (S_{OC}), for which the following approximation holds:

$$Q(1 - s) \approx \left(\frac{ns}{k} \right)^{-\gamma} Q(1 - k/n) \left\{ 1 - \frac{A(n/k)}{\rho} \left[1 - \left(\frac{ns}{k} \right)^{-\rho} \right] \right\}, \quad s \rightarrow 0, \quad (4.19)$$

where γ , $A(\cdot)$ and ρ are unknown. The first part $\left(\frac{ns}{k} \right)^{-\gamma} Q(1 - k/n)$ in the right side of (4.19) is exactly estimated by the Weissman's estimator $\hat{Q}_k^{(W)}(1 - s)$. The expression $1 - \rho^{-1} A(n/k_n) \left[1 - \left(\frac{ns}{k} \right)^{-\rho} \right]$ can be viewed as a correcting term since $A(n/k_n)$ tends to zero when n goes to infinity.

A possible choice for $\hat{\rho}_{k_\rho}$ is the most performed estimator among those studied in the i.i.d. case (see, e.g, [Gomes *et.* (2002)], [Deme *et al.* (2013b)]) and used in the β -mixing case by [de Haan *et al.* (2016)] and

[?]:

$$\widehat{\rho}_{k_\rho} = \frac{6S_{k_\rho}^{(2)} - 4 + \left(3S_{k_\rho}^{(2)} - 2\right)^{1/2}}{4S_{k_\rho}^{(2)} - 3}, \quad \text{provided } S_{k_\rho}^{(2)} \in \left(\frac{2}{3}, \frac{3}{4}\right), \quad (4.20)$$

where

$$S_{k_\rho}^{(2)} = \frac{3 \left[M_{k_\rho}^{(4)} - 24 \left(M_{k_\rho}^{(1)} \right)^4 \right] \left[M_{k_\rho}^{(2)} - 2 \left(M_{k_\rho}^{(1)} \right)^2 \right]}{4 \left[M_{k_\rho}^{(3)} - 6 \left(M_{k_\rho}^{(1)} \right)^3 \right]^2}.$$

The consistency and the asymptotic normality of $\widehat{\rho}_{k_\rho}$ have been established in [de Haan *et al.* (2016)] in the case of β -mixing serials.

Now, Using (4.19), [de Haan *et al.* (2016)] proposed the following reduced bias estimator of the high quantile $Q(1-s)$, $s \rightarrow 0$:

$$\widehat{Q}_{k,\widehat{\rho}}(1-s) = \left(\frac{ns}{k}\right)^{-\widehat{\gamma}_{k,\widehat{\rho}}} X_{n-k,n} \left\{ 1 - T_{k,\widehat{\rho}} \left[1 - \left(\frac{ns}{k}\right)^{-\widehat{\rho}} \right] \right\}, \quad (4.21)$$

where $T_{k,\widehat{\rho}}$ is the estimator of $\rho^{-1}A(n/h)$ defined as:

$$T_{k,\widehat{\rho}} = \frac{\left(M_k^{(2)} - 2 \left(\widehat{\gamma}_k^{(H)} \right)^2 \right) (1 - \widehat{\rho})^2}{2 \widehat{\gamma}_k^{(H)} \widehat{\rho}^2}. \quad (4.22)$$

For more details, one can refer to [Barry *et al.* (2023)]. Finally, substituting $Q(1-\cdot)$ with $\widehat{Q}_{k,\widehat{\rho}}(1-\cdot)$ in (3.11), we introduce the following unbiased estimator for $\pi_{\beta,n}$:

$$\widehat{\pi}_{\beta,k,n,\widehat{\rho}} := \left(\frac{\widehat{\gamma}_{k,\widehat{\rho}}}{\frac{1}{\beta} - \widehat{\gamma}_{k,\widehat{\rho}}} + \frac{\widehat{\rho} T_{k,\widehat{\rho}}}{\left(\frac{1}{\beta} - \widehat{\gamma}_{k,\widehat{\rho}}\right)(1 - \beta \widehat{\gamma}_{k,\widehat{\rho}} - \beta \widehat{\rho})} \right) g_\beta(k/n) X_{n-k,n}. \quad (4.23)$$

Before we establish the asymptotic properties of our unbiased estimator $\widehat{\pi}_{\beta,k,n,\widehat{\rho}}$, we start to give in Proposition 4.1 below a Gaussian representation of the empirical process $(X_{n-[kn]t},n)_{t \in [0,1]}$, where $[x]$ is the integer part of x . To his end, let Y_1, \dots, Y_n be stationary β -mixed series and identically distributed from a unit Pareto distribution G , with $G(y) = 1 - y^{-1}$, $y \geq 1$. Clearly, $X_{j,n} \stackrel{d}{=} U(Y_{j,n})$ where $Y_{1,n} \leq \dots \leq Y_{n,n}$ denote the order statistics pertaining to (Y_1, \dots, Y_n) .

Proposition 4.1 *Suppose that (X_1, X_2, \dots) is a stationary β -mixing time series with continuous common marginal distribution function F and assume that (C_{SO}) and (C_R) hold. Let $k = k(n)$ be an intermediate sequence satisfying $k \rightarrow \infty$, $k_n/n \rightarrow 0$ and $k_n^{1/2} A(n/k) = O(1)$, as $n \rightarrow \infty$. Then for a given $\varepsilon > 0$, under a Skorohod construction, there exists a function $\tilde{A} \sim A$, and a centered Gaussian process $(W(t))_{0 \leq t \leq 1}$, with covariance function r , such that, when $n \rightarrow \infty$*

$$\sup_{t \in (0,1)} t^{1/2+\varepsilon} \left| \sqrt{k} \left(\frac{X_{n-[kn]t},n}{U(n/k)} - t^{-\gamma} \right) - \gamma t^{-\gamma-1} W(t) - \sqrt{k} \tilde{A}(n/k) \left(\frac{k}{n} Y_{n-[kt],n} \right)^\gamma \frac{t^{-\rho} - 1}{\rho} \right| \rightarrow 0 \quad (4.24)$$

The following theorem establishes the asymptotic expansion of our distorted risk premiums estimator $\widehat{\pi}_{\beta,k_n,k_\rho}$ in terms of Gaussian process.

Theorem 4.1 *Let (X_1, X_2, \dots) be a stationary β -mixing time series with a continuous common marginal distribution function F and assume that (C_{SO}) and (C_R) hold. Let $\widehat{\rho}_{k_\rho}$ be the estimator for ρ defined in (4.20), where the intermediate sequence $k_\rho := k_\rho(n)$ satisfies $k_\rho \rightarrow \infty$, $k_\rho/n \rightarrow 0$ and $\sqrt{k_\rho} A(n/k_\rho) \rightarrow \infty$, as*

$n \rightarrow \infty$. If $k = k(n)$ is another intermediate sequence such that $k \rightarrow \infty$, $k/n \rightarrow 0$ and $\sqrt{k}A(n/k) = O(1)$, as $n \rightarrow \infty$, then for $1/2 < \gamma < 1$ and $\beta < 1/\gamma$, we have :

$$\frac{\sqrt{k}(\widehat{\pi}_{\beta,k,n,\widehat{\rho}} - \pi_{\beta,n})}{g_{\beta}(k/n)Q(1-k/n)} \xrightarrow{d} a_0W(1) + a_1P^{(1)} + a_2P^{(2)},$$

as $n \rightarrow \infty$, where $P^{(\alpha)} = \int_0^1 (-\log t)^{\alpha-1} (t^{-1}W(t) - W(1)) dt$ is a centered Gaussian process with covariance defined in (4.18) and

$$\begin{aligned} a_0 &= \frac{\beta\gamma^2}{1-\beta\gamma}, \\ a_1 &= \frac{\beta\gamma}{\rho(1-\beta\gamma)^2}(2-\rho) - 2\frac{\beta\gamma(1-\rho)^2}{\rho(1-\beta\gamma-\beta\rho)(1-\beta\gamma)}, \\ a_2 &= -\frac{\beta\gamma}{\rho(1-\beta\gamma)^2}(1-\rho) + \frac{\beta\gamma(1-\rho)^2}{\rho(1-\beta\gamma-\beta\rho)(1-\beta\gamma)}. \end{aligned}$$

Corollary 4.1 Under the assumptions of Theorem 4.1, we have:

$$\frac{\sqrt{k}(\widehat{\pi}_{\beta,k,n,\widehat{\rho}} - \pi_{\beta,n})}{g_{\beta}(k/n)Q(1-k/n)} \xrightarrow{d} \mathcal{N}\left(0, \sigma^2(\gamma, \rho, \beta)\right),$$

where

$$\begin{aligned} \sigma^2(\gamma, \rho, \beta) &= a_0^2 r(1, 1) + a_1^2 c_{1,1} + a_2^2 r(2, 2) + 2a_1 a_2 c_{1,2} \\ &\quad + 2a_0 a_1 \int_0^1 (t^{-1} r(1, t) - r(1, 1)) dt \\ &\quad - 2a_0 a_2 \int_0^1 \log t (t^{-1} r(1, t) - r(1, 1)) dt, \end{aligned}$$

with $c_{i,j}$ is the function in (4.18) and $r(\cdot)$ defined in the regularity conditions (C_R).

5 Simulation Study

The simulation study is based on the following classical stationary models, which satisfy the regularity (C_R) assumptions.:

- **(Autoregressive (AR) model)**: Consider first the stationary solution of the AR(1) equation:

$$X_i = \theta X_{i-1} + Z_i, \quad i = 1, \dots, n, \quad (5.25)$$

for some $\theta \in (0, 1)$ and i.i.d. random variables Z_i . The distribution function of the innovations Z_i is denoted by F_Z . Assume that F_Z admits a positive Lebesgue density which is L_1 Lipschitz-continuous; see [?] eq. (42). Suppose that as $x \rightarrow \infty$, $1 - F_Z(x) \sim px^{-1/\gamma}\ell(x)$ and $F_Z(-x) \sim qx^{-1/\gamma}\ell(x)$, for some slowly varying function ℓ and $p = 1 - q \in (0, 1)$. Then from Sect. 3.2 of [?], we get that $1 - F(x) \sim d_{\theta}(1 - F_Z(x))$, as $x \rightarrow \infty$, where $d_{\theta} = (1 - \theta^{1/\gamma})^{-1}$. Furthermore, the regularity conditions hold with/ $r(x, y) = x \wedge y + \sum_{m=1}^{\infty} (c_m(x, y) + c_m(y, x))$, where $c_m(x, y) = x \wedge y \theta^{m/\gamma}$.

- **(Moving average (MA) model)**: Consider the stationary solution of MA(1) equation:

$$X_i = \theta Z_{i-1} + Z_i, \quad i = 1, \dots, n; \quad (5.26)$$

where the innovation Z_i satisfies the same conditions as in the above AR(1) model. And from Sect. 3.2 of [?], we obtain $1 - F(x) \sim d_{\theta}(1 - F_Z(x))$ as $x \rightarrow \infty$, where $d_{\theta} = 1 + \theta^{1/\gamma}$. One can also compute the

covariance structure as : $r(x, y) = x \wedge y + (1 + \theta^{1/\gamma})^{-1}(x \wedge y\theta^{1/\gamma} + y \wedge x\theta^{1/\gamma})$.

Now, we proceed by generating the data for the three (03) models. This involves an independent model and the two models mentioned above. We first generate the i.i.d innovations (Z_1, \dots, Z_n) , such that:

$$F_Z(z) = \begin{cases} (1-q)(1-\tilde{F}(-z)) & \text{if } z < 0, \\ 1-q+q\tilde{F}(z) & \text{if } z > 0, \end{cases}$$

where \tilde{F} stands for the Fréchet distribution function $\tilde{F}(z) = \exp((-z)^{-1/\gamma})$ for $z > 0$, and $p = 0.75$. Then F_Z belongs to the domain of attraction with extreme value index $\gamma > 0$. In the following table, we generate the three (03) time series models under simulation with their tail distribution, which are needed to compute the true distortion risk premiums:

Description of Models	Independence $X_i = Z_i$	AR(1) $X_i = \theta X_{i-1} + Z_i$	MA(1) $X_i = \theta Z_{i-1} + Z_i$
Tail-distribution	$\bar{F}(x) = \bar{F}_Z(x)$	$\bar{F}(x) \sim (1 - \theta^{1/\gamma})^{-1} \bar{F}_Z(x)$	$\bar{F}(x) \sim (1 + \theta^{1/\gamma}) \bar{F}_Z(x)$
coefficients	$\theta = 0$ and $\gamma = 0.6$	$\theta = 0.3$ and $\gamma = 0.6$	$\theta = 0.3$ and $\gamma = 0.6$

Table 5.1: Description of models under simulation with their tail distribution functions.

For each generating model, we simulate $N = 1000$ samples with size $n = 1000$. To evaluation of the true value of distortion risk premiums $\pi_{\beta,n}$, we use the approximation of the tail distribution \bar{F} given in Table 5.1. Now, we apply to each sample both the estimator $\hat{\pi}_{\beta,k,n}$ in (3.14) and our reduced bias estimator $\hat{\pi}_{\beta,k,n,\hat{\rho}}$ introduced in (4.23), for different integers of top order statistic $k = 1, \dots, m$, where m is the number of positive values of the simulated samples.

The estimator $\hat{\pi}_{\beta,k,n,\hat{\rho}}$ is computed for $\hat{\rho} := \hat{\rho}_{k_\rho}$ defined in (4.20), where k_ρ is selected as follows:

$$k_\rho := \sup \left\{ k : k \leq \min \left(m-1, \frac{2m}{\log \log m} \right) \text{ and } \hat{\rho}_k \text{ exists} \right\}.$$

Next, we compare on the one hand the performance of the mentioned reinsurance premium estimators by computing the absolute value of the median together with the root median squared errors (RMSE) based on the N samples, and defined as the following form:

$$\text{ABias}(\pi, k) := \left| \text{median} \left\{ \frac{\hat{\pi}^{(1)}}{\pi}, \dots, \frac{\hat{\pi}^{(N)}}{\pi} \right\} - 1 \right|$$

and

$$\text{RMSE}(\pi, k) := \sqrt{\text{median} \left\{ \left(\frac{\hat{\pi}^{(1)}}{\pi} - 1 \right)^2, \dots, \left(\frac{\hat{\pi}^{(N)}}{\pi} - 1 \right)^2 \right\}},$$

where $\pi := \pi_{\beta,n}$ is the true value of the distortion risk premium with optimal retention level $Q(1 - k/n)$, and $\hat{\pi}^{(i)}$ is the i -th value ($i = 1, \dots, N$) of an estimator of $\hat{\pi}^{(i)}$ evaluated as mentioned above different number of top order statistics $k = 1, \dots, m$ with different aversion parameter $\beta = 1, 1.1$.

The results are displayed on the graphs in Figure 5.1 and Figure 5.2. Regarding the estimation of the distortion risk risk premiums, we observe from the graphs in the left-side of Figure 5.1 and Figure 5.2 that our goal in reducing the bias is well illustrated on finite sample behavior, when using large values of top order statistics k . In addition, on the graphs in right-side of Figure 5.1 and Figure 5.2, the RMSE of our reduced bias estimator $\hat{\pi}_{\beta,k,n,\hat{\rho}}$, stays at

a lower level than that of the classical estimator $\widehat{\pi}_{\beta,k,n}$ proposed by [Oudjed (2018)] for large values of k , whatever the value of the aversion index β . We also observe that the reduction in RMSE is higher for dependent series than for independent series. We conclude that the simulation studies show that under bias reduction procedure, the estimators for the distortion risk premiums remain stable for a wider range of k values even if the observations of insured risks exhibit serial dependence. Thus, the bias reduction method under dependence serials helps to tackle the two major critiques for applying extreme value statistics to time series in insurance data.

As mentioned above, it is also crucial to compare the estimators at their optimal number k of top extreme risks. To this end, we use the algorithm of [?], p.137, which gives an automatic choice of k for any estimator $\widehat{\gamma}_k^\bullet$ of the tail index γ . According to these authors, an automatic choice of top extremes as the value k^* that minimizes

$$\frac{1}{k} \sum_{j=1}^k j^\delta \left| \widehat{\gamma}_j^\bullet - \text{median}(\widehat{\gamma}_1^\bullet, \dots, \widehat{\gamma}_k^\bullet) \right|, \quad (5.27)$$

where $1 \leq k \leq m$ and $0 \leq \delta < 1/2$. By the way, choosing $\delta = 1/4$, we compute the optimal values k^* as in (5.27) for each tail index estimator used in the computation of their associated distortion risk premium estimators $\widehat{\pi}_{\beta,k,n}$ and $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}$.

In Table 5.2, we present the results of the estimated values of the above mentioned distortions risk premium estimators. Since their asymptotic variances depend on some unknown parameters, we opt to use a block bootstrapping method to construct a 95% confidence interval for the reinsurance premiums.

The block bootstrapping follows the routine `boot` of the package `boot` in R software. By repeating such bootstrapping procedure $T = 10,000$ times, we obtain T bootstrapped estimates for each distortion risk premium estimator. The sample standard deviation across the T estimates gives an estimate of the standard deviation of the underlying estimators for a given $k \in \{1, \dots, m\}$. We construct the 95% confidence interval using the point estimate and the estimated standard deviation. This procedure is applied to all values of k of each estimator. The point estimates of the distortion risk premium at its optimal value k^* as well as the lower bound (Lb), upper bound (Ub) and the cover of the confidence intervals are given in Table 5.2.

The independence model (defined in Table 5.1)							
$\beta = 1, \pi_{\beta,n} = 0.8133$				$\beta = 1.1, \pi_{\beta,n} = 1.1989$			
$\hat{\pi}_{\beta,k,n}$	0.9680	$\tilde{\pi}_{\beta,k,n,\hat{\rho}}$	0.7916	$\hat{\pi}_{\beta,k,n}$	1.5945	$\tilde{\pi}_{\beta,k,n,\hat{\rho}}$	1.1284
ABais	0.1547	ABais	0.0217	ABais	0.3956	ABais	0.0705
RMSE	0.1917	RMSE	0.1323	RMSE	0.3299	RMSE	0.1741
Lb	0.0137	Lb	0.1898	Lb	0.2329	Lb	0.3133
Ub	6.1595	Ub	1.0116	Ub	7.9907	Ub	1.2749
Cover	6.1458	Cover	0.8218	Cover	7.7578	Cover	0.9616
The AR(1) model (defined in Table 5.1)							
$\beta = 1, \pi_{\beta,n} = 0.9938$				$\beta = 1.1, \pi_{\beta,n} = 1.3090$			
$\hat{\pi}_{\beta,k,n}$	1.2535	$\tilde{\pi}_{\beta,k,n,\hat{\rho}}$	0.9678	$\hat{\pi}_{\beta,k,n}$	1.5049	$\tilde{\pi}_{\beta,k,n,\hat{\rho}}$	1.2229
ABais	0.2597	ABais	0.0260	ABais	0.1959	ABais	0.0861
RMSE	0.2612	RMSE	0.1220	RMSE	0.1666	RMSE	0.1732
Lb	0.1848	Lb	0.2024	Lb	0.2725	Lb	0.3194
Ub	6.8500	Ub	1.1446	Ub	8.9188	Ub	1.4600
Cover	6.6652	Cover	0.9422	Cover	8.6463	Cover	1.1406
The MA(1) model (defined in Table 5.1)							
$\beta = 1, \pi_{\beta,n} = 0.9640$				$\beta = 1.1, \pi_{\beta,n} = 1.3345$			
$\hat{\pi}_{\beta,k,n}$	1.2038	$\tilde{\pi}_{\beta,k,n,\hat{\rho}}$	0.9585	$\hat{\pi}_{\beta,k,n}$	1.7171	$\tilde{\pi}_{\beta,k,n,\hat{\rho}}$	1.2827
ABais	0.2398	ABais	0.0055	ABais	0.3826	ABais	0.0518
RMSE	0.2487	RMSE	0.1164	RMSE	0.2866	RMSE	0.1612
Lb	0.1375	Lb	0.2014	Lb	0.2165	Lb	0.3100
Ub	6.9503	Ub	1.1584	Ub	8.9853	Ub	1.4669
Cover	6.8128	Cover	0.9570	Cover	8.7688	Cover	1.1569

Table 5.2: Estimation results of $\hat{\pi}_{\beta,k^*,n}$ and $\tilde{\pi}_{\beta,k^*,n,\hat{\rho}}$ estimators of the true distortion premium $\pi_{\beta,n} := \pi_{\beta}(Q(1 - k^*/n))$ for $\beta = 1; 1.1$ and with their 95% confidence intervals, computed with their associated optimal numbers of top statistics k^* , based on $N = 1000$ samples of size $n = 1000$, from the different three (03) models listed in Table 5.1

After the inspection of the table, two conclusions can be drawn regardless of the situation. First, we notice that the absolute bias of our reduced bias estimator $\tilde{\pi}_{\beta,k^*,n,\hat{\rho}}$ is lower than the classical estimator $\hat{\pi}_{\beta,k^*,n}$. Second, our reduction bias estimator $\tilde{\pi}_{\beta,k^*,n,\hat{\rho}}$ is more efficient than $\hat{\pi}_{\beta,k^*,n}$ regardless to the the root median squared errors and the cover values. That illustrates well our conclusions drawn from the graphical analysis.

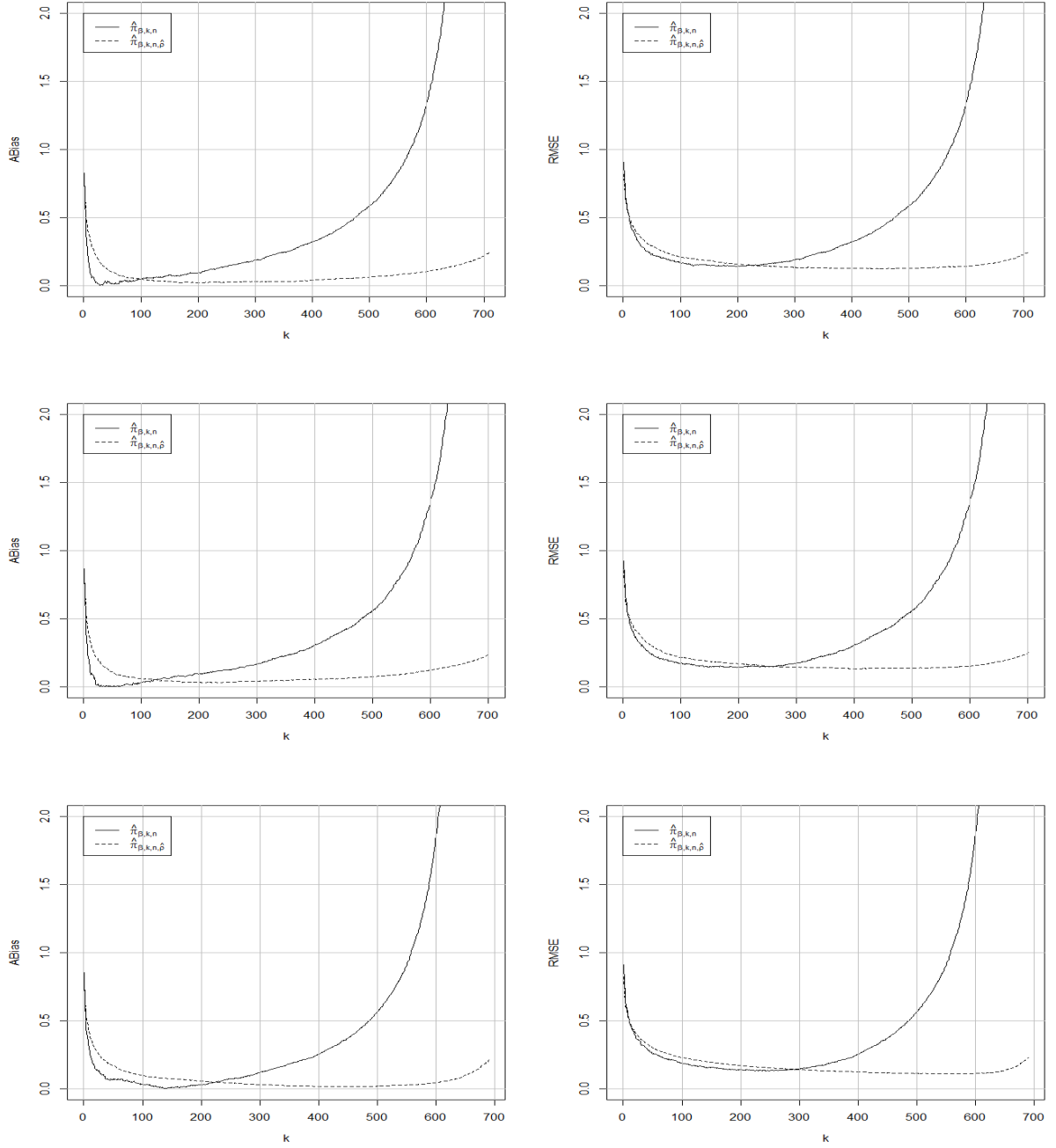


Figure 5.1: Absolute Bias of the median (left column) and Root median squared error (right column) of $\hat{\pi}_{\beta,k,n}$ (full line) and $\tilde{\pi}_{\beta,k,n,\hat{\rho}}$ (dotted line) as a function of k based on $N = 1000$ samples of size 1000 of the models in Table 5.1: independence model (top), AR(1) model (middle) and MA(1) model (down) for the distortion risk premiums with aversion index $\beta = 1$.

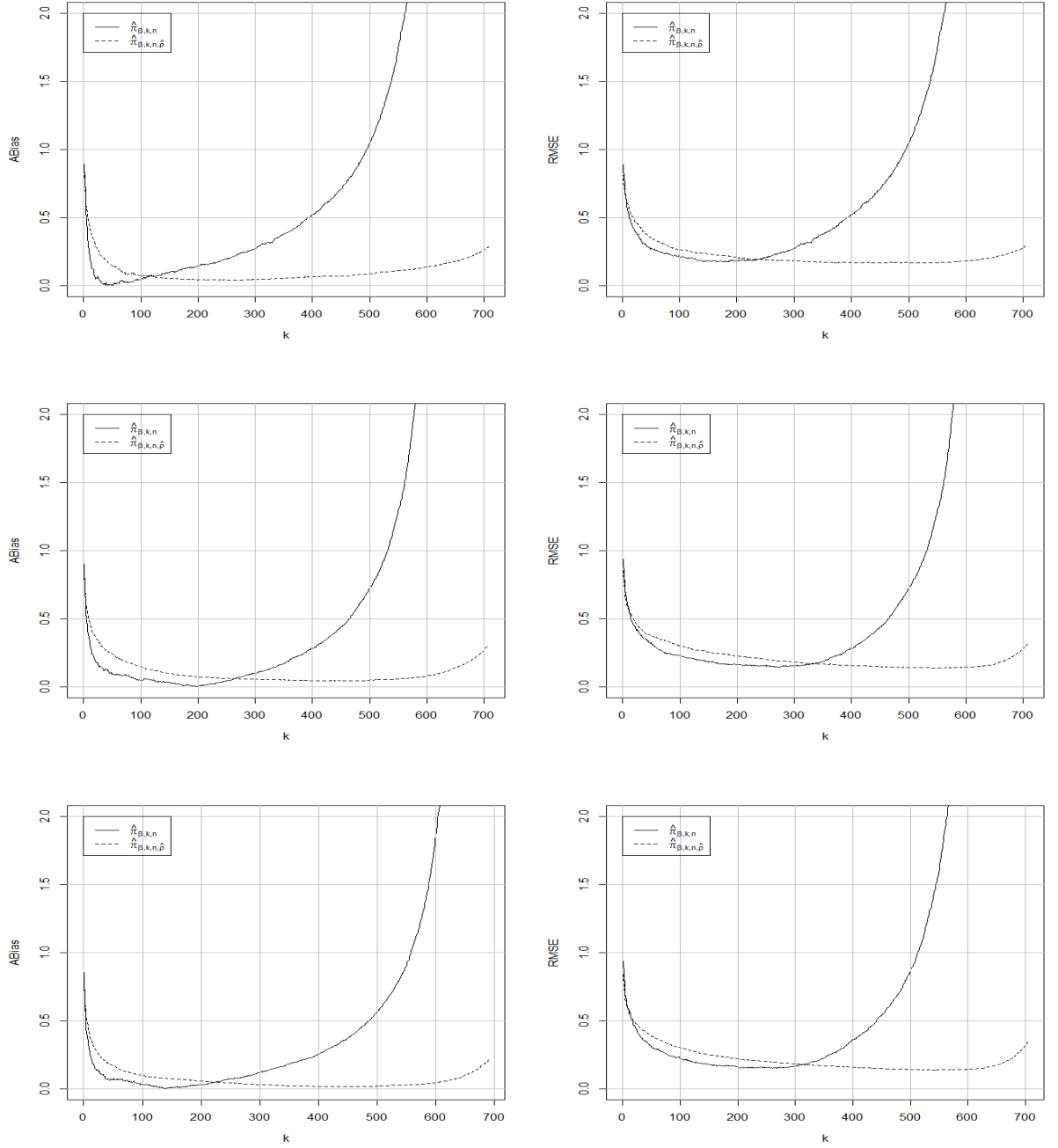


Figure 5.2: Absolute Bias of the median (left column) and Root median squared error (right column) of $\hat{\pi}_{\beta,k,n}$ (full line) and $\tilde{\pi}_{\beta,k,n,\hat{\rho}}$ (dotted line) as a function of k based on $N = 1000$ samples of size 1000 of the models in Table 5.1: independence model (top), AR(1) model (middle) and MA(1) model (down) for the distortion risk premiums with aversion index $\beta = 1.1$.

6 Proofs of the results

Proof of Proposition 4.1. Assume that the second order condition (C_{SO}) holds. From Theorem B.2.18 in [?], we have for all $\varepsilon, \delta > 0$, there exists some positive number $t_0 = t_0(\varepsilon, \delta)$ such that for $tx \geq t_0$:

$$\left| \frac{\left(\frac{U(tx)}{U(t)} - x^\gamma\right)}{\tilde{A}(t)} - x^\gamma \frac{x^\rho - 1}{\rho} \right| \leq \varepsilon x^{\rho+\gamma} \max(x^\delta, x^{-\delta}). \quad (6.28)$$

Now, using the representation $X_i \stackrel{d}{=} U(Y_i)$ where Y_i follows a standard Pareto distribution and (Y_1, Y_2, \dots, Y_n) is a stationary β -mixed series satisfying the (C_R) regularity conditions. Then, according to [?], under a Skorohod construction, there exists a centered Gaussian process $W(t)$, $0 \leq t \leq 1$, with covariance r such that for $\varepsilon > 0$, when $n \rightarrow \infty$

$$\sup_{t \in (0,1]} t^{1/2+\varepsilon} \left| \sqrt{k} \left(t \frac{Y_{n-[kt],n}}{n/k} - 1 \right) - t^{-1} W(t) \right| \rightarrow 0. \quad (6.29)$$

We combine the asymptotic property of $Y_{n-[kt],n}$ in (6.29) with the inequality (6.28) as follows. Taking $t = n/k$ and $tx = Y_{n-[kt],n}$ in (6.28), we get that,

$$\left| \frac{\frac{U(Y_{n-[kt],n})}{U(n/k)} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma}{\tilde{A}(n/k)} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \frac{\left(\frac{k}{n} Y_{n-[kt],n}\right)^\rho - 1}{\rho} \right| \leq \varepsilon \left(\frac{k}{n} Y_{n-[kt],n}\right)^{\rho \pm \delta + \gamma}.$$

This implies that,

$$\begin{aligned} & \left| \left(\frac{U(Y_{n-[kt],n})}{U(n/k)} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \right) \right. \\ & \quad \left. - \tilde{A}(n/k) \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \frac{\left(\frac{k}{n} Y_{n-[kt],n}\right)^\rho - 1}{\rho} \right| \\ & \leq \varepsilon |\tilde{A}(n/k)| \left(\frac{k}{n} Y_{n-[kt],n}\right)^{\rho \pm \delta + \gamma}. \end{aligned}$$

Since $\sqrt{k}A(n/k) = O(1)$, this leads to

$$\begin{aligned} & t^{1/2+\varepsilon} \left| \sqrt{k} \left(\frac{U(Y_{n-[kt],n})}{U(n/k)} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \right) \right. \\ & \quad \left. - \sqrt{k} \tilde{A}(n/k) \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \frac{\left(\frac{k}{n} Y_{n-[kt],n}\right)^\rho - 1}{\rho} \right| \\ & \leq \varepsilon \sqrt{k} |\tilde{A}(n/k)| t^{1/2+\varepsilon} \left(\frac{k}{n} Y_{n-[kt],n}\right)^{\rho \pm \delta + \gamma}. \end{aligned}$$

Furthermore

$$\begin{aligned} & t^{1/2+\varepsilon} \left| \sqrt{k} \left(\frac{U(Y_{n-[kt],n})}{U(n/k)} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \right) \right. \\ & \quad - \sqrt{k} \tilde{A}(n/k) \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \left(\frac{t^{-\rho} - 1}{\rho} - \frac{t^{-\rho} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\rho}{\rho} \right) \\ & \quad \left. + \sqrt{k} t^{-\gamma} - \sqrt{k} t^{-\gamma} + \gamma t^{-\gamma-1} W(t) - \gamma t^{-\gamma-1} W(t) \right| \\ & \leq \varepsilon \sqrt{k} |\tilde{A}(n/k)| t^{1/2+\varepsilon} \left(\frac{k}{n} Y_{n-[kt],n}\right)^{\rho \pm \delta + \gamma}. \end{aligned}$$

Therefore

$$\begin{aligned} & t^{1/2+\varepsilon} \left| \sqrt{k} \left(\frac{X_{n-[kn],n}}{U(n/k)} - t^{-\gamma} \right) - \gamma t^{-\gamma-1} W(t) - \sqrt{k} \tilde{A}(n/k) \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \left(\frac{t^{-\rho} - 1}{\rho} \right) \right. \\ & \quad \left. + \sqrt{k} \tilde{A}(n/k) \left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma \left(\frac{t^{-\rho} - \left(\frac{k}{n} Y_{n-[kt],n}\right)^\rho}{\rho} \right) \right. \\ & \quad \left. - \left(\sqrt{k} \left(\left(\frac{k}{n} Y_{n-[kt],n}\right)^\gamma - t^{-\gamma} \right) - \gamma t^{-\gamma-1} W(t) \right) \right| \\ & \leq \varepsilon \sqrt{k} |\tilde{A}(n/k)| t^{1/2+\varepsilon} \left(\frac{k}{n} Y_{n-[kt],n}\right)^{\rho \pm \delta + \gamma}. \quad (6.30) \end{aligned}$$

Since ε can be arbitrarily close to 0, hence by applying (6.29) to (6.30) and by choosing $\delta \in (0, -\rho)$, the Proposition 4.1 follows. \blacksquare

Proof of Theorem 4.1. Recall that

$$\pi_{\beta,n} := \pi_\beta(Q(1 - k/n)) = \int_{Q(1-k/n)}^{\infty} g_\beta(\bar{F}(x)) dx.$$

Next, we have

$$\begin{aligned} \widehat{\pi}_{\beta,k,n,\widehat{\rho}} - \pi_{\beta,n} &= \left(\frac{\widehat{\gamma}_{k,\widehat{\rho}}}{\frac{1}{\beta} - \widehat{\gamma}_{k,\widehat{\rho}}} + \frac{\widehat{\rho} T_{k,\widehat{\rho}}}{\left(\frac{1}{\beta} - \widehat{\gamma}_{k,\widehat{\rho}}\right)(1 - \beta \widehat{\gamma}_{k,\widehat{\rho}} - \beta \widehat{\rho})} \right) g_{\beta}(k/n) X_{n-k_n,n} \\ &\quad - \int_{Q(1-k_n/n)}^{\infty} g_{\beta}(\overline{F}(x)) dx. \end{aligned}$$

Let consider

$$\begin{aligned} H_1 &= g_{\beta}(k/n) X_{n-k_n,n} \left(\frac{\widehat{\gamma}_{k,\widehat{\rho}}}{\frac{1}{\beta} - \widehat{\gamma}_{k,\widehat{\rho}}} - \frac{\gamma}{\frac{1}{\beta} - \gamma} \right) \\ H_2 &= \frac{\gamma}{\frac{1}{\beta} - \gamma} g_{\beta}(k/n) Q(1 - k/n) \left(\frac{X_{n-k,n}}{Q(1 - k/n)} - 1 \right) \\ H_3 &= \frac{\gamma}{\frac{1}{\beta} - \gamma} g_{\beta}(k/n) Q(1 - k/n) - \int_{Q(1-k/n)}^{\infty} g_{\beta}(\overline{F}(x)) dx \\ H_4 &= T_{k,\widehat{\rho}} X_{n-k,n} g_{\beta}(k/n) \left(\frac{\widehat{\rho}}{\left(\frac{1}{\beta} - \widehat{\gamma}_{k,\widehat{\rho}}\right)(1 - \beta \widehat{\gamma}_{k,\widehat{\rho}} - \beta \widehat{\rho})} \right) \end{aligned}$$

It easy to verify that:

$$\widehat{\pi}_{\beta,k,n,\widehat{\rho}} - \pi_{\beta,n} = H_1 + H_2 + H_3 + H_4.$$

Under assumptions and for all n large enough, we have from (4.17),

$$\sqrt{k} \left(\widehat{\gamma}_{k,\widehat{\rho}} - \gamma \right) \xrightarrow{d} \frac{\gamma}{\rho} \left((2 - \rho) P^{(1)} - (1 - \rho) P^{(2)} \right).$$

where the $P_1^{(\alpha)} = \int_0^1 (-\log t)^{\alpha-1} (t^{-1} W(t) - W(1)) dt$ are normally distributed random variables with mean zero. This Leads to the consistence in probability of $\widehat{\gamma}_{k,\widehat{\rho}}$ to γ . Therefore, using the Delta-method procedure, we get for all n large enough:

$$H_1 \stackrel{d}{=} (1 + o_{\mathbb{P}}(1)) \frac{\beta}{(1 - \beta \gamma)^2} k^{-1/2} g_{\beta}(k/n) X_{n-k,n} \left(\frac{\gamma}{\rho} \left((2 - \rho) P^{(1)} - (1 - \rho) P^{(2)} \right) \right),$$

where $P^{(\alpha)} = \int_0^1 (-\log t)^{\alpha-1} (t^{-1} W(t) - W(1)) dt$ is a centered Gaussian Process with covariance defined in (4.18).

Next, from the Proposition 4.1, we have for all n large enough:

$$\sqrt{k} \left(\frac{X_{n-k,n}}{Q(1 - k/n)} - 1 \right) \stackrel{d}{=} \gamma W(1) + o_{\mathbb{P}}(1). \quad (6.31)$$

This implies that $X_{n-k,n}/Q(1 - k/n) = 1 + o_{\mathbb{P}}(1)$. And then,

$$\frac{\sqrt{k} H_1}{g_{\beta}(k/n) Q(1 - k/n)} \stackrel{d}{=} (1 + o_{\mathbb{P}}(1)) \frac{\gamma \beta}{\rho(1 - \beta \gamma)^2} \left((2 - \rho) P^{(1)} - (1 - \rho) P^{(2)} \right). \quad (6.32)$$

Similarly, we have for all n large enough:

$$\frac{\sqrt{k} H_2}{g_{\beta}(k/n) Q(1 - k/n)} \stackrel{d}{=} (1 + o_{\mathbb{P}}(1)) \frac{\gamma \beta}{1 - \beta \gamma} \sqrt{k} \left(\frac{X_{n-k,n}}{Q(1 - k/n)} - 1 \right).$$

Using again (6.31), we get:

$$\frac{\sqrt{k} H_2}{g_{\beta}(k/n) Q(1 - k/n)} \stackrel{d}{=} (1 + o_{\mathbb{P}}(1)) \frac{\gamma^2 \beta}{1 - \beta \gamma} W(1). \quad (6.33)$$

For the term H_3 , we have:

$$\frac{\sqrt{k} H_3}{g_{\beta}(k/n) Q(1 - k/n)} = \sqrt{k} \left(\frac{\gamma \beta}{1 - \beta \gamma} - \frac{\pi_{\beta,n}}{Q(1 - k/n) g_{\beta}(k/n)} \right)$$

Where $\pi_{\beta,n} = \int_{U(n/k)}^{\infty} g_{\beta}(\overline{F}(x)) dx$ and $U(t) = Q(1 - 1/t)$, $t \geq 1$. A change of variables with $x = U(nt/k)$ yields to:

$$\pi_{\beta,n} = \int_1^{\infty} g_{\beta}(k/nt) dU(nt/k).$$

Since $g_{\beta}(x) = x^{1/\beta}$ and $U(\cdot)$ is a regularly varying function with index $\gamma > 0$, then $g_{\beta}(k/nt) U(k/nt) \rightarrow 0$, as $t \rightarrow \infty$.

Thus, an integration by parts yields to:

$$\pi_{\beta,n} = g_{\beta}(k/n) \frac{1}{\beta} \int_1^{\infty} t^{-1/\beta-1} (U(nx/k) - U(n/k)) dt.$$

Therefore, by using again $U(n/k) = Q(1 - k/n)$, we get:

$$\begin{aligned} \frac{\sqrt{k_n} H_3}{g_\beta(k/n)Q(1 - k/n)} &= \sqrt{k} \left[\frac{\beta\gamma}{1 - \beta\gamma} - \frac{1}{\beta} \int_1^\infty t^{-1/\beta-1} \left(\frac{U(nt/k)}{U(n/k)} - 1 \right) dt \right] \\ &= \frac{-1}{\beta} \sqrt{k} \int_1^\infty t^{-1-1/\beta} \left(\frac{U(nt/k)}{U(n/k)} - t^\gamma \right) dt. \end{aligned}$$

Since $\tilde{A} \sim A$, from (6.28) with $1/2 < \gamma < 1/\beta$, we obtain the following expansion as $n \rightarrow \infty$,

$$\begin{aligned} \frac{\sqrt{k} H_3}{g_\beta(k/n)Q(1 - k/n)} &= -\frac{1}{\beta} \sqrt{k} A(n/k) \int_1^\infty x^{\gamma-1/\beta-1} \frac{x^\rho - 1}{\rho} dx (1 + o(1)) \\ &= -\sqrt{k} A(n/k) \frac{1}{(\frac{1}{\beta} - \gamma)(1 - \beta\gamma - \beta\rho)} (1 + o(1)). \end{aligned}$$

For the term H_4 , using the fact that $X_{n-k,n}/Q(1 - k/n) = 1 + o_{\mathbb{P}}(1)$, we have

$$\frac{\sqrt{k} H_4}{g_\beta(k/n)Q(1 - k/n)} = \sqrt{k} T_{n,\hat{\rho}} \frac{\hat{\rho}}{(\frac{1}{\beta} - \hat{\gamma}_{k,\hat{\rho}})(1 - \beta\hat{\gamma}_{k,\hat{\rho}} - \beta\hat{\rho})} (1 + o_{\mathbb{P}}(1)).$$

Since $\hat{\gamma}_{k,\hat{\rho}}$ and $\hat{\rho} := \hat{\rho}_{k,\rho}$ are respectively consistent to γ and ρ , we obtain:

$$\frac{\sqrt{k} H_4}{g_\beta(k/n)Q(1 - k/n)} \stackrel{d}{=} \sqrt{k} T_{n,\hat{\rho}} \frac{\rho}{(\frac{1}{\beta} - \gamma)(1 - \beta\gamma - \beta\rho)} (1 + o_{\mathbb{P}}(1)).$$

Hence, for all large n , we have

$$\frac{\sqrt{k}(H_3 + H_4)}{g_\beta(k/n)Q(1 - k/n)} = \sqrt{k} \left(T_{n,\hat{\rho}} - \frac{A(n/k)}{\rho} \right) \frac{\rho}{(\frac{1}{\beta} - \gamma)(1 - \beta\gamma - \beta\rho)} (1 + o_{\mathbb{P}}(1)).$$

Recall from (4.22) that

$$T_{k,\hat{\rho}} = \frac{\left(M_k^{(2)} - 2(\hat{\gamma}_k^{(H)})^2 \right) (1 - \hat{\rho})^2}{2\hat{\gamma}_k^{(H)} \hat{\rho}^2},$$

where $M_k^{(2)}$ is as in (4.16). One can rewrite $T_{k,\hat{\rho}}$ as:

$$T_{k,\hat{\rho}} = \left(\frac{M_k^{(2)}}{2(\hat{\gamma}_k^{(H)})} - \hat{\gamma}_k^{(H)} \right) \frac{(1 - \rho)^2}{\rho^2}$$

We have from (2.9) and for all n large enough:

$$\hat{\gamma}_k^{(H)} \stackrel{d}{=} \gamma + \gamma \frac{P^{(1)}}{\sqrt{k}} + \frac{A(n/k)}{1 - \rho} (1 + o_{\mathbb{P}}(1)). \quad (6.34)$$

Next, The Theorem 2.1 in [Barry et al. (2023)] shows that for all n large enough:

$$\frac{M_k^{(2)}}{2(\hat{\gamma}_k^{(H)})} \stackrel{d}{=} \gamma + \gamma \frac{P^{(2)}}{\sqrt{k}} - \gamma \frac{P^{(1)}}{\sqrt{k}} + A(n/k) \left(\frac{1 - (1 - \rho)^2}{\rho(1 - \rho)^2} - \frac{1}{1 - \rho} \right) (1 + o_{\mathbb{P}}(1)). \quad (6.35)$$

Combining (6.34) and (6.35) with the consistency of $\hat{\rho}$ to ρ , we get for all n large enough:

$$\sqrt{k} \left(T_{k,\hat{\rho}} - \frac{\tilde{A}(n/k)}{\rho} \right) \stackrel{d}{=} \frac{\gamma(1 - \rho)^2}{\rho^2} (P^{(2)} - 2P^{(1)}) + o_{\mathbb{P}}(1).$$

Hence,

$$\frac{\sqrt{k}(H_3 + H_4)}{g_\beta(k/n)Q(1 - k/n)} \stackrel{d}{=} \frac{\gamma(1 - \rho)^2}{\rho(\frac{1}{\beta} - \gamma)(1 - \beta\gamma - \beta\rho)} (P^{(2)} - 2P^{(1)}) (1 + o_{\mathbb{P}}(1)). \quad (6.36)$$

Finally,

$$\frac{\sqrt{k}(\hat{\pi}_{\beta,k,n,\hat{\rho}} - \pi_{\beta,n})}{g_\beta(k/n)Q(1 - k/n)} \stackrel{d}{=} \frac{\sqrt{k}(H_1 + H_2 + H_3 + H_4)}{g_\beta(k/n)Q(1 - k/n)},$$

where

Which leads to:

$$\frac{\sqrt{k}(\hat{\pi}_{\beta,k,n,\hat{\rho}} - \pi_{\beta,n})}{g_\beta(k/n)Q(1 - k/n)} \stackrel{d}{\rightarrow} a_0 W(1) + a_1 P^{(1)} + a_2 P^{(2)}, \quad (6.37)$$

where

$$a_0 = \frac{\beta\gamma^2}{1 - \beta\gamma} W(1).$$

$$a_1 = \frac{\gamma\beta}{\rho(1-\beta\gamma)^2}(2-\rho) - 2\frac{\gamma(1-\rho)^2}{\rho(\frac{1}{\beta}-\gamma)(1-\beta\gamma-\beta\rho)}.$$

and

$$a_2 = -\frac{\gamma\beta}{\rho(1-\beta\gamma)^2}(1-\rho)(\rho-1) + \frac{\gamma(1-\rho)^2}{\rho(\frac{1}{\beta}-\gamma)(1-\beta\gamma-\beta\rho)}$$

This completes the proof of Theorem 4.1. ■

Proof of Corollary 4.1. Computing the variance of the centered Gaussian process $a_0P_1^{(1)} + a_1P_1^{(2)} + a_2W(1)$, the Corollary 4.1 holds. ■

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