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# POLYSTABILITY OF STOKES REPRESENTATIONS AND DIFFERENTIAL GALOIS GROUPS

PHILIP BOALCH AND DAISUKE YAMAKAWA

ABSTRACT. Polystability of (twisted) Stokes representations (i.e. wild monodromy representations) will be characterised, in terms of the corresponding differential Galois group (generalising the Zariski closure of the monodromy group in the tame case). This extends some results of Richardson. Further, the intrinsic approach to such results will be established, in terms of reductions of Stokes local systems.

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## 1. INTRODUCTION

We continue our investigations of the nonabelian moduli spaces in 2d gauge theory (the theory of connections on curves). This article is essentially an appendix to the paper [14] that completed the construction of the wild character varieties of smooth curves as affine algebraic Poisson varieties (completing the sequence [5, 6, 9]). This is a purely algebro-geometric approach, complementary to the earlier analytic approaches [3, 1] that constructed wild nonabelian Hodge moduli spaces as (complete) hyperkahler manifolds. Here we will give an intrinsic characterisation of the points of the wild character varieties, generalising existing results in the tame case, and characterise the stable points (generalising a result of [9] in the untwisted wild case). For more background and applications see the reviews in [8, 10, 12] (the first large class of examples of wild character varieties is due to Birkhoff [2] and the simplest case underlies  $U_q(\mathfrak{g})$ , [8] §4).

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First we will recall the basic statements in the tame case. Let  $G$  be a connected complex reductive group, such as  $\mathrm{GL}_n(\mathbb{C})$ , and let  $\Sigma^\circ$  be a smooth complex algebraic curve. Thus  $\Sigma^\circ = \Sigma \setminus \alpha$  for some smooth compact complex algebraic curve  $\Sigma$  (i.e. a compact Riemann surface), and a finite subset  $\alpha \subset \Sigma$ . Given a basepoint  $b \in \Sigma^\circ$  one can consider the representation variety

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma^\circ, b), G)$$

which is a complex affine variety equipped with an action of  $G$ , conjugating representations. A point  $\rho \in \mathcal{R}$  is a  $G$ -representation, i.e. a group homomorphism  $\rho : \pi_1(\Sigma^\circ, b) \rightarrow G$ . In turn the  $G$ -character variety, or Betti moduli space

$$\mathcal{M}_B = \mathcal{M}_B(\Sigma^\circ, G) = \mathcal{R}/G = \mathrm{Hom}(\pi_1(\Sigma^\circ, b), G)/G$$

is the (affine) geometric invariant theory quotient of  $\mathcal{R}$  by  $G$ . By definition this means that the points of the variety  $\mathcal{M}_B$  are the *closed*  $G$  orbits in  $\mathcal{R}$ . The representations  $\rho$  whose  $G$  orbits are closed are called the *polystable* representations. A basic question is thus to characterise the polystable representations intrinsically. The answer (due to Richardson, building on earlier work in the general linear case), is as follows.

Let  $\mathrm{Gal}(\rho) = A(\rho) \subset G$  be the Zariski closure (*adhérence*) of the image of  $\rho$ . A theorem of Schlesinger implies that if  $\rho$  is the monodromy representation of an algebraic connection  $(\nabla, E)$  on an algebraic principal  $G$ -bundle  $E \rightarrow \Sigma^\circ$ , with regular singularities at  $\alpha$ , then  $\mathrm{Gal}(\rho)$  is the differential Galois group of  $(\nabla, E)$ . Recall that a complex affine algebraic group is a *linearly reductive group* if its identity component is reductive (i.e. has trivial unipotent radical). The basic characterisation is then:

**Theorem 1** ([27]). *A point  $\rho \in \mathcal{R}$  is polystable if and only if  $\mathrm{Gal}(\rho)$  is a linearly reductive group. Further,  $\rho$  is stable if and only if  $\mathrm{Gal}(\rho)$  is not contained in a proper parabolic subgroup of  $G$ .*

A representation  $\rho \in \mathcal{R}$  whose Galois group  $\mathrm{Gal}(\rho)$  is linearly reductive is often called a *semisimple* representation<sup>1</sup>. Thus the theorem says that  $\rho \in \mathcal{R}$  is polystable if and only if  $\rho$  is semisimple, and so the points of  $\mathcal{M}_B$  are the isomorphism classes of semisimple representations. Note that for  $G = \mathrm{GL}_n(\mathbb{C})$  this statement is older (Artin/Procesi) and a full account appears in the book of Lubotzky–Magid [24].

This paper is concerned with the extension of this result to the case of Stokes representations, generalising the fundamental group representations, and the characterisation of the points of the wild character varieties, generalising the (tame) character varieties appearing above.

In brief any algebraic connection  $(\nabla, E)$  on an algebraic principal  $G$ -bundle  $E \rightarrow \Sigma^\circ$ , has an invariant, its irregular class, at each point  $a \in \alpha$ . A connection is regular singular if and only if its irregular class is trivial. Our aim is to give the generalisation of Richardson’s results when the irregular classes are arbitrary. Due to work of many people it is known how to describe algebraic connections completely topologically, so

<sup>1</sup>Indeed if  $V \cong \mathbb{C}^n$  and  $G = \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$  then  $\rho : \pi_1(\Sigma^\circ, b) \rightarrow G$  is a semisimple representation if and only if  $V$  is a semisimple  $\pi_1(\Sigma^\circ, b)$ -module.

we can work algebraically on the Betti/Stokes side (see the review of this story in [12]). The basic notions from the tame case are generalised as follows:

$$\begin{aligned} \text{local system} &\rightsquigarrow \text{Stokes local system} \\ \text{fundamental group (or groupoid)} &\rightsquigarrow \text{wild surface group (or groupoid)} \\ \text{fundamental group representation} &\rightsquigarrow \text{Stokes representation.} \end{aligned}$$

Once these generalisations are understood then the story proceeds similarly to the tame case (defining a wild representation variety  $\mathcal{R}$  parameterising framed Stokes local systems, and then acting by a reductive group to forget the framings). A key novelty in the wild setting is that there is a breaking of structure group (“fission”) near the marked points so the group acting involves a (reductive) subgroup of  $G$ . This is intimately related to extra generators in Ramis’ description of the differential Galois group [26], generalising Schlesinger’s density theorem. However, as we will recall, the basic feature of the tame case remains, that  $\mathcal{R}$  can be written explicitly in terms of a product of simpler pieces (doubles  $\mathbb{D}$  or fission spaces  $\mathcal{A}$ ), one for each marked point or handle on  $\Sigma$ :

$$\mathcal{R} \cong (\mathbb{D}^{\otimes g} \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_m) // G$$

if  $\Sigma$  has genus  $g$  and  $m$  marked points.

**1.1. Summary of main results.** In this section we will summarise the main results in a short uncluttered form, with links to the references to the full definitions (mainly in [14]). (An overview of the simpler  $\mathrm{GL}_n(\mathbb{C})$  set-up is in [12] §13.)

Let  $\Sigma$  be a smooth compact complex algebraic curve (i.e. a compact Riemann surface) and let  $\alpha \subset \Sigma$  be a finite non-empty subset. Let  $G$  be a connected complex reductive group.

Choose a  $G$ -irregular class  $\Theta_a$  at each point  $a \in \alpha$  (possibly twisted) as in [14] §3.5. Let  $\Sigma = (\Sigma, \alpha, \Theta)$  be the resulting wild Riemann surface with structure group  $G$  as in [14] §4 (and [9] §8.1, Rmk 10.6).

The topological notion of Stokes local system  $\mathbb{L}$  on  $\Sigma$  is then well-defined, as in [14] Defn. 13 (and [9] Rmk. 8.4, [10] §4.3, [12] §8).

Although we will not need it here, note that the Stokes local systems on  $\Sigma$  encode the algebraic connections on algebraic principal  $G$ -bundles on the open curve  $\Sigma^\circ := \Sigma \setminus \alpha$ , with irregular class  $\Theta_a$  at each  $a \in \alpha$ .

As in [14] §5, [9] §8, it follows that if we choose suitable basepoints  $\beta$  then the wild surface groupoid  $\Pi$  is well defined, as is the wild representation variety (the space of Stokes representations of  $\Pi$ ):

$$\mathcal{R}(\Sigma, \beta) = \mathrm{Hom}_{\mathbb{S}}(\Pi, G) \subset \mathrm{Hom}(\Pi, G).$$

It is an affine variety equipped with an action of a complex reductive group  $\mathbf{H}$ .

Any framed Stokes  $G$ -local system  $\mathbb{L}$  determines a Stokes representation  $\rho = \rho_{\mathbb{L}} \in \mathcal{R}(\Sigma, \beta)$ . Two Stokes local systems are isomorphic if and only if their Stokes representations are in the same  $\mathbf{H}$ -orbit in  $\mathcal{R}(\Sigma, \beta)$ .

The notion of  $\rho$  being polystable or stable for the action of  $\mathbf{H}$  on  $\mathcal{R}$  is well-defined, as for any action of a reductive group on an affine variety (as in [27]). The points of the Poisson wild character variety  $\mathcal{M}_{\mathbf{B}}(\Sigma)$  are the polystable (i.e. closed)  $\mathbf{H}$ -orbits.

On the other hand  $\rho$  determines the Galois group  $\text{Gal}(\rho) \subset G$ , as in Ramis' density theorem [26] (see §§4.1, 5.2 below). It involves not just the usual monodromy, but also the formal monodromy, Stokes automorphisms and the Ramis tori.

Finally we can define  $\mathbb{L}$  to be irreducible if it has no proper parabolic reductions, and to be reductive (or “semisimple”) if it has an irreducible Levi reduction.

**Theorem 2.** *Let  $\mathbb{L}$  be a Stokes local system on  $\Sigma$ , and let  $\rho$  be its Stokes representation. The following are equivalent:*

- 1)  $\rho$  is polystable,
- 2)  $\text{Gal}(\rho)$  is linearly reductive,
- 3)  $\mathbb{L}$  is a semisimple Stokes local system.

Thus the points of the wild character variety  $\mathcal{M}_{\mathbf{B}}(\Sigma)$  are the isomorphism classes of semisimple Stokes local systems on  $\Sigma$ . In the tame case (with each irregular class trivial) the groupoid  $\Pi$  becomes Poincaré's fundamental groupoid (with a finite number of basepoints), and the theorem is already known [27].

Further we will consider stability (not just polystability). This requires possibly adding a few extra punctures to control the kernel of the action, but no generality is lost (see the discussion after (5)).

**Theorem 3.** *Let  $\mathbb{L}$  be a Stokes local system on  $\Sigma$ , and let  $\rho$  be its Stokes representation. The following are equivalent:*

- 4)  $\rho$  is a stable point of  $\mathcal{R}(\Sigma, \beta)$ ,
- 5)  $\text{Gal}(\rho)$  is not contained in a proper parabolic subgroup of  $G$ ,
- 6)  $\mathbb{L}$  is irreducible.

This result was already established in [9] in the case where each irregular class was not twisted.

Recall from [14] that two types of twist are possible: the formal twists (twisted irregular classes, as above), and also interior twists, over the interior of the curve, where we start with a local system of groups  $\mathcal{G} \rightarrow \Sigma^\circ$ , with each fibre isomorphic to  $G$ . Similarly we will establish the analogous results in this fully twisted setting. We will suppose that  $\mathcal{G}$  is “out-finite” in the sense of §5.1 below.

The first main difference (in the presence of interior twists) is that the wild representation variety is replaced by a space of twisted Stokes representations:

$$\mathcal{R} = \mathrm{THom}_{\mathbb{S}}(\Pi, G) \subset \mathrm{Hom}(\Pi, G \rtimes \mathrm{Aut}(G))$$

which is an affine variety equipped with an action of a complex reductive group  $\mathbf{H}$ . Secondly  $\mathrm{Gal}(\rho)$  is not a subgroup of  $G$ , but rather it comes with a homomorphism  $\mathrm{Gal}(\rho) \rightarrow \mathrm{Aut}(G)$ , so naturally acts on  $G$  by automorphisms. The image of  $\mathrm{Gal}(\rho)$  in  $\mathrm{Aut}(G)$  will be denoted  $\overline{\mathrm{Gal}}(\rho)$ . Then we can define irreducibility (§5.3) and semisimplicity (§5.4) for Stokes  $\mathcal{G}$ -local systems, and will prove analogues of the above results:

**Theorem 4.** *Let  $\Sigma = (\Sigma, \alpha, \Theta)$  be a wild Riemann surface with group  $\mathcal{G} \rightarrow \Sigma^\circ$ . Let  $\mathbb{L}$  be a Stokes  $\mathcal{G}$ -local system on  $\Sigma$ , and let  $\rho \in \mathrm{THom}_{\mathbb{S}}(\Pi, G)$  be its twisted Stokes representation. The following three conditions are equivalent:*

- 1')  $\rho$  is a polystable point of  $\mathcal{R}(\Sigma, \beta) = \mathrm{THom}_{\mathbb{S}}(\Pi, G)$ ,
- 2')  $\overline{\mathrm{Gal}}(\rho)$  is linearly reductive,
- 3')  $\mathbb{L}$  is a semisimple Stokes  $\mathcal{G}$ -local system.

Moreover the following three conditions are also equivalent:

- 4')  $\rho$  is a stable point of  $\mathcal{R}(\Sigma, \beta) = \mathrm{THom}_{\mathbb{S}}(\Pi, G)$ ,
- 5')  $\overline{\mathrm{Gal}}(\rho) \subset \mathrm{Aut}(G)$  does not preserve a proper parabolic subgroup of  $G$ ,
- 6')  $\mathbb{L}$  is an irreducible Stokes  $\mathcal{G}$ -local system.

In the set-up of Theorems 2,3 with  $\mathcal{G}$  constant it is true that  $a') \Leftrightarrow a)$  for all  $a = 1, 2, \dots, 6$ . Thus Thm. 4 implies both Theorems 2,3.

## 1.2. Layout of the article.

Sections 2, 3 generalise some of Richardson's results, in two steps. These are extrinsic results, and may well have further applications, beyond the wild character varieties of curves.

Section 4 then applies these results to the spaces of Stokes representations leading to the wild character varieties  $\mathcal{M}_{\mathbb{B}}(\Sigma) = \mathcal{R}/\mathbf{H}$ . The main results are the equivalences 1')  $\Leftrightarrow$  2') in Cor. 25, and 4')  $\Leftrightarrow$  5') in Cor. 28.

Section 5 then discusses the intrinsic objects, Stokes local systems, and how stability/polystability can be read off in terms of (twisted) reductions of structure group. The main results are the equivalences 1')  $\Leftrightarrow$  3') and 4')  $\Leftrightarrow$  6') in parts 2) and 1) of Thm. 29 respectively.

**1.3. Some other directions.** Note that for constant general linear groups the irreducible Stokes local systems are equivalent to the input data in the construction of wild harmonic metrics in [28]. Further note that in the fully untwisted case (but any  $G$ , as in [9]) the irreducible Stokes local systems fit into the ‘‘Betti weight zero’’ case of the recent extension of the wild nonabelian Hodge correspondence due to

Huang–Sun [21] (which looks to be in line with our general conjecture in [11] Rmk. 6, that the “good” meromorphic connections/Higgs fields on parahoric torsors are the right objects to look at). For nontrivial Betti weights one can also give an algebraic construction as in [9] Rmk A.5, [12] §1.5 1), [30]. (Beware that [21] mis-states the main results of [1] (“a moduli correspondence for these objects is still missing” p.3): the paper [1] constructs moduli spaces of wild harmonic bundles, proves they form (complete) hyperkahler manifolds ([1] Thm. 5.4) and identifies them with two purely holomorphic moduli spaces of stable objects (meromorphic connections or Higgs bundles on parabolic vector bundles), establishing the wild nonabelian Hodge correspondence on curves at the level of moduli spaces ([1] Thm. 6.1). Summaries appear in [7, 8, 11].)

In another direction one of the key motivations for this work is the fact that any admissible deformation of a wild Riemann surface leads to a local system of wild character varieties, and its monodromy generalises the usual mapping class group actions on character varieties in the tame case. As explained in [3, 4] the motivating examples for this whole line of thought were the Dubrovin–Ugaglia Poisson varieties whose braid group actions come from the braiding of counts of BPS states ([20] Rmk 3.10, related to earlier work of Cecotti–Vafa); these are examples of *twisted* wild character varieties, involving the non-trivial outer automorphism of  $\mathrm{GL}_n(\mathbb{C})$  (so we now have an intrinsic general framework encompassing such examples). Indeed it was by forgetting this twist that the Poisson variety  $G^*$  underlying  $U_q(\mathfrak{g})$  was recognised and identified as a wild character variety [4]. See e.g. [18, 19, 13] for some recent developments concerning the generalised braid groups that act on wild character varieties, from admissible deformations of more general wild Riemann surfaces.

**Acknowledgements.** These results were completed in 2019 before the first named author moved departments and then learnt of the thesis work leading to [29], that has some overlap with this paper in the tame setting with interior twists, although expressed in a slightly different language. In the intervening years we have not yet managed to incorporate possible simplifications suggested by [29] but thought it reasonable to release our original approach anyway since the scope is larger.

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## 2. TWISTED VERSION OF RICHARDSON’S RESULTS

As in [27] we use the terminology that an affine algebraic group over  $\mathbb{C}$  is *reductive* if it is connected and has trivial unipotent radical. It is *linearly reductive* if its identity component is reductive.

Let  $G$  be a linearly reductive group over  $\mathbb{C}$ . Recall that a point  $x$  of an affine  $G$ -variety is said to be *polystable* if the orbit  $G \cdot x$  is closed. It is said to be *stable* if it is polystable and the kernel of the action has finite index in its stabiliser  $G_x$ .

Let  $n$  be a positive integer. For  $\mathbf{x} = (x_i)_{i=1}^n \in G^n$ , let  $A(\mathbf{x}) \subset G$  be the Zariski closure of the subgroup generated by  $x_1, x_2, \dots, x_n$ . In [27], Richardson examined the simultaneous conjugation action of  $G$  on the product  $G^n$  and obtained the following results:

**Theorem 5** ([27, Thm. 3.6]). *If  $G$  is linearly reductive, a point  $\mathbf{x} \in G^n$  is polystable if and only if  $A(\mathbf{x})$  is linearly reductive.*

**Theorem 6** ([27, Thm. 4.1]). *If  $G$  is reductive, a point  $\mathbf{x} \in G^n$  is stable if and only if  $A(\mathbf{x})$  is not contained in any proper parabolic subgroup of  $G$ .*

In this section we will establish twisted versions of these results, in Thm. 7 and Thm. 12 (2) respectively. Two other characterisations of polystability will also be established, in Thm. 12 (1) and Cor. 18. The subsequent section (§3) will give a further generalisation.

Assume that  $G$  is reductive and choose  $\phi_1, \phi_2, \dots, \phi_n \in \text{Aut}(G)$ . Let  $\Gamma$  be the subgroup of  $\text{Aut}(G)$  generated by the inner automorphism group  $\text{Inn}(G)$  and  $\phi_1, \phi_2, \dots, \phi_n$ . We assume that the quotient  $\bar{\Gamma} := \Gamma / \text{Inn}(G)$  is finite and regard  $\Gamma$  as an algebraic group with identity component  $\text{Inn}(G) \cong G/Z(G)$ . For any  $\phi \in \Gamma$  let  $G(\phi)$  denote the  $G$ -bitorsor/“twisted group”  $G \times \{\phi\} \subset G \rtimes \Gamma$ , as in [14] §2 (as explained there, the monodromy of a  $\mathcal{G}$ -local system lies in such a space, and  $G(\phi)$  embeds in the group of set-theoretic automorphisms of a fibre). Put

$$X = \prod_{i=1}^n G(\phi_i) \subset (G \rtimes \Gamma)^n,$$

on which  $G = G(\text{Id})$  acts by the simultaneous conjugation. For  $\mathbf{x} = (x_i)_{i=1}^n \in X$ , let  $A(\mathbf{x})$  be the Zariski closure (in the algebraic group  $G \rtimes \Gamma$ ) of the subgroup generated by  $x_1, x_2, \dots, x_n$ . Let  $\bar{A}(\mathbf{x}) \subset \Gamma$  be the image of  $A(\mathbf{x})$  under the homomorphism

$$\text{Ad}: G \rtimes \Gamma \rightarrow \Gamma, \quad x = (g, \phi) \mapsto \text{Ad}(x) = \text{Ad}(g) \circ \phi,$$

where  $\text{Ad}(g) = g(\cdot)g^{-1}$ . Note that  $\bar{A}(\mathbf{x})$  is contained in  $\text{Aut}(G)$  and hence naturally acts on  $G$ . Thus in turn, via  $\text{Ad}$ , the group  $A(\mathbf{x})$  naturally acts on  $G$ .

**Theorem 7.** *A point  $\mathbf{x} \in X$  is polystable if and only if  $\bar{A}(\mathbf{x})$  is linearly reductive.*

*Remark 8.* The notation  $A(\mathbf{x})$  for the Zariski closure presumably stems from [15] §3, where  $A$  stands for *adhérence*. This may help to avoid possible confusion (since  $\bar{A}(\mathbf{x})$  denotes the image in  $\text{Aut}(G)$  here). Note that [15] §3.3 shows that if  $H \subset G$  is any subgroup then the Zariski closure of the set  $H$  is a Zariski closed subgroup of  $G$ .

To see that Thm. 7 generalises Thm. 5, first note that:

**Proposition 9.** *Let  $\tilde{G}$  be a linearly reductive group with identity component  $G$ , and let  $A$  be a closed subgroup of  $\tilde{G}$ . Let  $\bar{A}$  be the image of  $A$  under the map  $\text{Ad}|_G: \tilde{G} \rightarrow \text{Aut}(G)$ , and regard it as a quotient algebraic group of  $A$ . Then  $A$  is linearly reductive if and only if  $\bar{A}$  is linearly reductive.*

**Proof.** Note that we may view  $\text{Ad}|_G$  as a map of algebraic groups:  $\text{Aut}(G) \cong \text{Inn}(G) \rtimes \text{Out}(G)$  may have an infinite number of components, but  $\text{Ad}|_G(\tilde{G})$  will only

encounter a finite number of them. Let  $K$  be the kernel of the homomorphism  $A \rightarrow \bar{A}$ . Note  $K \cap G \subset Z(G)$ , and the identity component  $K^0$  of  $K$  is contained in  $(\tilde{G})^0 = G$ . Thus  $K^0 \subset Z(G)$ , which implies  $K^0$  is a torus and so  $K$  is linearly reductive. Now [27] 1.2.2 implies  $A$  is linearly reductive if and only if  $\bar{A}$  is linearly reductive.  $\square$

**Lemma 10.** *Let  $\tilde{G}$  be a linearly reductive group with identity component  $G$  and let  $\mathbf{x}$  be a point in an affine  $\tilde{G}$ -variety  $Y$ . Then the following hold.*

- (1)  $\mathbf{x}$  is polystable for the  $\tilde{G}$ -action if and only if  $\mathbf{x}$  is polystable for the  $G$ -action.
- (2)  $\mathbf{x}$  is stable for the  $\tilde{G}$ -action if and only if  $\mathbf{x}$  is stable for the  $G$ -action.

**Proof.** The  $\tilde{G}$ -orbit  $\tilde{G} \cdot \mathbf{x}$  is a disjoint union of a finite number of  $G$ -orbits of the same dimension, one of which is  $G \cdot \mathbf{x}$ . Hence  $\tilde{G} \cdot \mathbf{x}$  is closed if and only if  $G \cdot \mathbf{x}$  is closed. The second assertion follows from the equality  $G_{\mathbf{x}} = \tilde{G}_{\mathbf{x}} \cap G$  for the stabilisers and a similar one for the kernels.  $\square$

Thus in particular it is now clear that Thm. 7 generalises Thm. 5.

**Proof** (of Theorem 7). Since  $\bar{\Gamma}$  is finite, we can find a finite subgroup  $\Gamma'$  of  $\Gamma$  such that  $\Gamma = \text{Inn}(G) \cdot \Gamma'$  thanks to a result of Borel–Serre and Brion (see [16, 17]). For each  $i = 1, 2, \dots, n$  take  $g_i \in G$  so that  $\phi'_i := \text{Inn}(g_i) \circ \phi_i \in \Gamma'$ . Then we have isomorphisms of bitorsors

$$f_i: G(\phi'_i) \rightarrow G(\phi_i), \quad (g, \phi'_i) \mapsto (gg_i, \phi_i) \quad (i = 1, 2, \dots, n),$$

which induce a  $G$ -equivariant isomorphism

$$f: X' := \prod_{i=1}^n G(\phi'_i) \rightarrow X, \quad (x'_i)_{i=1}^n \mapsto (f_i(x'_i))_{i=1}^n.$$

Since  $f$  is equivariant, a point  $\mathbf{x} \in X$  is stable (resp. polystable) if and only if  $f^{-1}(\mathbf{x}) \in X'$  is stable (resp. polystable). Also, observe that  $\text{Ad} \circ f_i = \text{Ad}$  ( $i = 1, 2, \dots, n$ ) and hence  $\bar{A}(\mathbf{x}) = \bar{A}(f^{-1}(\mathbf{x}))$  for all  $\mathbf{x} \in X$ . Therefore without loss of generality we may assume that  $\phi_i \in \Gamma'$  for all  $i = 1, 2, \dots, n$ .

Put  $\tilde{G} = G \rtimes \Gamma'$ . It is a linearly reductive group with identity component  $G$  and  $X$  is a closed subvariety of  $\tilde{G}^n$ . By Theorem 5, a point  $\mathbf{x} \in \tilde{G}^n$  is polystable with respect to the simultaneous  $\tilde{G}$ -conjugation if and only if  $A(\mathbf{x}) \subset \tilde{G}$  is linearly reductive. By Prop. 9 this happens if and only if  $\bar{A}(\mathbf{x})$  is linearly reductive. Together with Lemma 10 this implies the assertion.  $\square$

Note that in general it really is necessary to work with  $\bar{A}(\mathbf{x})$  rather than  $A(\mathbf{x})$ :

**Lemma 11.** *There are examples of  $\mathbf{x} \in X$  which are polystable but  $A(\mathbf{x})$  is not linearly reductive.*

**Proof.** Write  $x_i = (g_i, \phi_i) \in G(\phi_i)$ . Choose  $g_1, \dots, g_n \in G$  generating a Zariski dense subgroup of an abelian unipotent subgroup  $U \subset G$ , such as a root group. Then define  $\phi_i$  to be the inner automorphism  $\phi_i(g) = g_i^{-1}gg_i$ . It follows that  $A(\mathbf{x}) \cong U$  is not reductive, but  $\overline{A}(\mathbf{x}) = \{1\}$  is trivial (and so  $\mathbf{x}$  is polystable by Thm. 7).  $\square$

**Theorem 12.** *For any point  $\mathbf{x} \in X$  the following hold:*

- (1)  $\mathbf{x}$  is polystable if and only if any  $A(\mathbf{x})$ -invariant parabolic subgroup  $P \subset G$  has an  $A(\mathbf{x})$ -invariant Levi subgroup  $L \subset P$ .
- (2)  $\mathbf{x}$  is stable if and only if there are no proper  $A(\mathbf{x})$ -invariant parabolic subgroups of  $G$ .

We prepare three lemmas.

**Lemma 13.** *Let  $P$  be a parabolic subgroup of  $G$  and  $L$  be a Levi subgroup of  $P$ . Then  $N_P(L) = L$ .*

**Proof.** Let  $\pi: P \rightarrow P/R_u(P)$  be the quotient of  $P$  by the unipotent radical. Suppose that  $g \in P$  normalises  $L$  and decompose it as  $g = hu$ ,  $h \in L$ ,  $u \in R_u(P)$ . Then for any  $x \in L$  we have  $gxg^{-1} \in L$  and hence  $uxu^{-1} \in L$ . Since  $u \in \text{Ker } \pi$  we have  $\pi(uxu^{-1}) = \pi(x)$ , which implies  $uxu^{-1} = x$  since the restriction of  $\pi$  to  $L$  is injective. Thus  $u$  commutes with  $L$ . On the other hand,  $L$  coincides with its centraliser since  $L = C_G(S)$  for some torus  $S \subset L$ . Thus  $u \in L$  and hence  $u = 1$ , i.e.  $g \in L$ .  $\square$

**Lemma 14.** *The kernel of the  $G$ -action on  $X$  is equal to  $Z(G)^\Gamma$ .*

**Proof.** If  $k \in G$  lies in the kernel, then for any  $g \in G$  and  $i = 1, 2, \dots, n$  we have  $k(g, \phi_i) = (g, \phi_i)k$ , i.e.

$$kg = g\phi_i(k).$$

Taking  $g$  to be 1 we obtain  $\phi_i(k) = k$  ( $i = 1, 2, \dots, n$ ). Thus  $kg = gk$  for all  $g \in G$ , i.e.,  $k \in Z(G)$ . Since  $\Gamma$  is generated by  $\text{Inn}(G), \phi_1, \phi_2, \dots, \phi_n$  and  $\text{Inn}(G)$  trivially acts on  $Z(G)$  we obtain  $k \in Z(G)^\Gamma$ . The converse is clear.  $\square$

**Lemma 15.**  *$Z(G) \cap G_{\mathbf{x}} = Z(G)^\Gamma$  for any  $\mathbf{x} \in X$ .*

**Proof.** Take any  $\mathbf{x} = (x_i)_{i=1}^n \in X$ . For each  $i = 1, 2, \dots, n$  we have  $\text{Ad}(x_i)\phi_i^{-1} \in \text{Inn}(G)$ , which trivially acts on  $Z(G)$ . Thus for any  $g \in Z(G) \cap G_{\mathbf{x}}$  we have

$$\phi_i(g) = \text{Ad}(x_i)(g) = g \quad (i = 1, 2, \dots, n),$$

which implies  $g \in Z(G)^\Gamma$ . The inclusion  $Z(G)^\Gamma \subset Z(G) \cap G_{\mathbf{x}}$  is clear.  $\square$

**Proof** (of Theorem 12). As in the proof of Theorem 7, we may assume that  $\phi_i$ ,  $i = 1, 2, \dots, n$  are all contained in a common finite subgroup  $\Gamma' \subset \Gamma$  such that

$\Gamma = \text{Inn}(G) \cdot \Gamma'$  and put  $\tilde{G} = G \rtimes \Gamma'$ . Under this assumption  $\overline{A}(\mathbf{x})$  is linearly reductive if and only if  $A(\mathbf{x}) \subset \tilde{G}$  is linearly reductive, by Prop. 9.

(1) We should show that a subgroup  $H \subset \tilde{G}$  is linearly reductive if and only if any  $H$ -invariant parabolic in  $G$  has an  $H$ -invariant Levi subgroup. First suppose that  $H \subset \tilde{G}$  is linearly reductive and  $P \subset G$  is an  $H$ -invariant parabolic subgroup. Since  $H$  is linearly reductive and contained in  $N_{\tilde{G}}(P)$ , it is contained in some Levi subgroup  $\tilde{L}$  of  $N_{\tilde{G}}(P)$ . Then  $L := \tilde{L} \cap G$  is a Levi subgroup of  $N_{\tilde{G}}(P) \cap G = N_G(P) = P$  and normalised by  $H$  (so it is  $H$ -invariant). Conversely, suppose any  $H$ -invariant parabolic in  $G$  has an  $H$ -invariant Levi subgroup. By [27, Prop. 2.6], there exists a one-parameter subgroup  $\lambda$  of  $G$  such that  $H \subset P_{\tilde{G}}(\lambda)$  and  $R_u(H) \subset U_{\tilde{G}}(\lambda)$ , where

$$(1) \quad P_{\tilde{G}}(\lambda) = \{x \in \tilde{G} \mid \lim_{t \rightarrow 0} \lambda(t)x\lambda(t)^{-1} \text{ exists}\},$$

$$(2) \quad U_{\tilde{G}}(\lambda) = \{x \in \tilde{G} \mid \lim_{t \rightarrow 0} \lambda(t)x\lambda(t)^{-1} = 1\},$$

and  $R_u(H)$  is the unipotent radical of  $H$ . Put  $P = P_G(\lambda) = P_{\tilde{G}}(\lambda) \cap G$ , which is a parabolic subgroup of  $G$ . Since  $P$  is normalised by  $P_{\tilde{G}}(\lambda)$ , it is normalised by  $H$ . Hence  $P$  has an  $H$ -invariant Levi subgroup  $L$  by assumption. We have  $H \subset N_{\tilde{G}}(L)$  and hence  $R_u(H) \subset N_G(L)$  (recall that the unipotent radical is connected). Note that  $R_u(H)$  is also contained in  $U_{\tilde{G}}(\lambda) \cap G = R_u(P)$ , while any non-trivial element of  $R_u(P)$  does not normalise the Levi subgroup  $L$  by Lemma 13. Thus  $R_u(H)$  is trivial, i.e.  $H$  is linearly reductive.

(2) Suppose that  $\mathbf{x} = (x_i)_{i=1}^n \in X$  is stable and let  $P$  be a  $A(\mathbf{x})$ -invariant parabolic subgroup of  $G$ . By Theorem 7,  $\overline{A}(\mathbf{x})$  (and hence  $A(\mathbf{x})$ ) is linearly reductive. Since  $A(\mathbf{x})$  normalises  $P$ , there exists a Levi subgroup  $\tilde{L}$  of  $N_{\tilde{G}}(P)$  containing  $A(\mathbf{x})$ . By [27, Prop. 2.4], there exists a one-parameter subgroup  $\lambda$  of  $G$  such that  $N_{\tilde{G}}(P) = P_{\tilde{G}}(\lambda)$  and  $\tilde{L} = C_{\tilde{G}}(\text{Im } \lambda)$ . Since  $A(\mathbf{x}) \subset C_{\tilde{G}}(\text{Im } \lambda)$  each  $x_i$  commutes with  $\text{Im } \lambda$  and hence  $\text{Im } \lambda \subset G_{\mathbf{x}}$ . The stability now implies that  $\text{Im } \lambda$  is contained in the kernel  $Z(G)^\Gamma$  and hence  $P = G$ . Conversely, suppose that  $\mathbf{x} \in X$  is not stable. Then if the orbit  $G \cdot \mathbf{x}$  is not closed we argue as follows: By the Hilbert–Mumford criterion, there exists a one-parameter subgroup  $\lambda$  of  $G$  and an element  $\mathbf{y} = (y_i)_{i=1}^n \in X$  such that  $\lim_{t \rightarrow 0} \lambda(t) \cdot \mathbf{x} = \mathbf{y}$  and  $G \cdot \mathbf{y}$  is closed. We show that the parabolic subgroup  $P := P_G(\lambda) = P_{\tilde{G}}(\lambda) \cap G$  is  $A(\mathbf{x})$ -invariant and proper. For any  $g \in P$  and  $i = 1, 2, \dots, n$ , the limit of

$$\lambda(t)\text{Ad}(x_i)(g)\lambda(t)^{-1} = \lambda(t)x_i\lambda(t)^{-1} \cdot \lambda(t)g\lambda(t)^{-1} \cdot \lambda(t)x_i^{-1}\lambda(t)^{-1}$$

as  $t \rightarrow 0$  exists. Hence  $P$  is  $A(\mathbf{x})$ -invariant. If  $P$  is not proper,  $\text{Im } \lambda$  is contained in  $Z(G) \cap G_{\mathbf{y}} = Z(G)^\Gamma$ . Thus  $\lambda(t) \cdot \mathbf{x} = \mathbf{x}$  ( $t \in \mathbb{C}^*$ ) and hence  $\mathbf{x} = \mathbf{y}$ , which contradicts the assumption that  $G \cdot \mathbf{x}$  is not closed. Hence  $P$  is proper. Finally suppose  $G \cdot \mathbf{x}$  is closed, but of the wrong dimension. Then the stabiliser  $G_{\mathbf{x}}$  is linearly reductive ([27] 1.3.3) and the quotient  $G_{\mathbf{x}}/Z(G)^\Gamma$  has non-trivial identity component. Hence there exists a one-parameter subgroup  $\lambda$  of  $G_{\mathbf{x}}$  such that  $\text{Im } \lambda \not\subset Z(G)^\Gamma$ . Since each  $x_i$

commutes with  $\lambda$ , the parabolic subgroup  $P := P_G(\lambda)$  is  $A(\mathbf{x})$ -invariant. It is proper since Lemma 15 implies  $\text{Im } \lambda \not\subset Z(G)$ .  $\square$

Let us rephrase/abstract the first part of this in a way that will be useful later. Let  $G$  be a reductive group and  $\Lambda$  an algebraic group acting on  $G$  by (algebraic) group automorphisms. Suppose that the action of  $\Lambda$  is effective and the identity component  $\Lambda^0$  acts by inner automorphisms. Then we may regard  $\Lambda$  as a subgroup of  $\text{Aut}(G)$  and its image in  $\text{Out}(G)$  is finite.

**Proposition 16.**  *$\Lambda$  is linearly reductive if and only if any  $\Lambda$ -invariant parabolic subgroup of  $G$  has a  $\Lambda$ -invariant Levi subgroup.*

**Proof.** By the result of Borel–Serre and Brion, there exists a finite subgroup  $\Lambda' \subset \Lambda$  such that  $\Lambda^0 \Lambda' = \Lambda$ . Put  $\tilde{G} = G \rtimes \Lambda'$  and let  $K \subset \tilde{G}$  be the preimage of  $\Lambda$  under  $\text{Ad}$ , so that Prop. 9 implies  $\Lambda$  is linearly reductive if and only if  $K$  is linearly reductive. Thus the equivalence follows from the proof of Thm. 12 (1).  $\square$

In the same setting there is a further characterisation:

**Proposition 17.**  *$\Lambda$  is linearly reductive if and only if there exists a torus  $S \subset G$  such that  $C_G(S)$  is  $\Lambda$ -invariant and has no proper  $\Lambda$ -invariant parabolic subgroups.*

**Proof.** Suppose that  $\Lambda$  is linearly reductive. We show that there exists a decreasing sequence of  $\Lambda$ -invariant closed subgroups

$$G = L_0 \supset P_1 \supset L_1 \supset P_2 \supset L_2 \supset \cdots \supset P_r \supset L_r$$

such that each  $P_i$  is a proper parabolic subgroup of  $L_{i-1}$ , each  $L_i$  ( $i > 0$ ) is a Levi subgroup of  $P_i$  and  $L_r$  has no proper  $\Lambda$ -invariant parabolic subgroups. If  $L_0 = G$  has no proper  $\Lambda$ -invariant parabolic subgroups we have nothing to do (just put  $r = 0$ ). Otherwise we take any proper  $\Lambda$ -invariant parabolic subgroup  $P_1 \subsetneq L_0$ . Then by Prop. 16 there exists a  $\Lambda$ -invariant Levi subgroup  $L_1 \subset P_1$ . Let  $\Lambda_1$  be the quotient of  $\Lambda$  by the kernel of the induced  $\Lambda$ -action on  $L_1$ , so that  $\Lambda_1$  effectively acts on  $L_1$ . Note that its identity component  $\Lambda_1^0$  acts by inner automorphisms of  $L_1$ ; indeed, the action of any element of  $\Lambda_1^0$  is induced from some inner automorphism of  $G$  preserving  $P_1, L_1$  and hence is inner by Lemma 13. Since  $\Lambda$  is linearly reductive  $\Lambda_1$  is also linearly reductive, and a subgroup of  $L_1$  is  $\Lambda_1$ -invariant if and only if it is  $\Lambda$ -invariant as a subgroup of  $L_0$ . If  $L_1$  has no proper  $\Lambda_1$ -invariant parabolic subgroup, the sequence  $L_0 \supset P_1 \supset L_1$  is as desired. Otherwise we take any proper  $\Lambda_1$ -invariant parabolic subgroup  $P_2 \subsetneq L_1$ . Then by Prop. 16 there exists a  $\Lambda_1$ -invariant Levi subgroup  $L_2 \subset P_2$ . Iterating this procedure, we obtain a desired decreasing sequence. Since each  $L_i$  ( $i > 0$ ) is a Levi subgroup of  $P_i$  there exists a torus  $S_i \subset L_i$  such that  $L_i = C_{L_{i-1}}(S_i)$ . Then  $L_r$  is the common centraliser of  $S_1, S_2, \dots, S_r$  in  $G$ . Hence the torus  $S \subset G$  generated by  $S_1, S_2, \dots, S_r$  (note that they commute with each other) is as desired.

Conversely, suppose that there exists a torus  $S \subset G$  such that  $L := C_G(S)$  is  $\Lambda$ -invariant and has no proper  $\Lambda$ -invariant parabolic subgroups. Let  $P \subset G$  be a  $\Lambda$ -invariant parabolic subgroup. Then the intersection  $P \cap L$  is a  $\Lambda$ -invariant parabolic subgroup of  $L$  and hence  $P \cap L = L$ , i.e.  $L \subset P$ . Since  $L$  is reductive, there exists a Levi subgroup  $M \subset P$  containing  $L$ . For any  $\psi \in \Lambda$  the image  $\psi(M)$  of  $M$  is also a Levi subgroup of  $P$  containing  $L$ . Since  $L$  contains a maximal torus of  $P$  and any maximal torus of  $P$  is contained in a unique Levi subgroup, we have  $\psi(M) = M$ . Hence  $M$  is  $\Lambda$ -invariant, which together with Prop. 16 shows that  $\Lambda$  is linearly reductive.  $\square$

Applying this to the set-up of the present section (with  $\Lambda = \overline{A}(\mathbf{x})$ ) yields:

**Corollary 18.** *The following are equivalent:*

- 0) A point  $\mathbf{x} \in X$  is polystable,
- 1) The group  $\overline{A}(\mathbf{x})$  is linearly reductive,
- 2) Any  $A(\mathbf{x})$ -invariant parabolic in  $G$  has an  $A(\mathbf{x})$ -invariant Levi subgroup,
- 3) There exists a subtorus  $S \subset G$  such that  $C_G(S)$  is  $A(\mathbf{x})$ -invariant and has no proper  $A(\mathbf{x})$ -invariant parabolic subgroups,
- 4) There exists an  $A(\mathbf{x})$ -invariant Levi subgroup  $L$  of a parabolic of  $G$ , such that  $L$  has no proper  $A(\mathbf{x})$ -invariant parabolic subgroups.

Note that 3) and 4) are trivially equivalent since centralisers of tori in  $G$  are exactly the Levi subgroups of parabolics.

### 3. MORE GENERAL SET-UP

The results of the previous section will now be generalised, in a form more directly useful in the context of Stokes local systems. Return to the set-up of Thm. 7 (with  $n \geq 1$ ), but now further choose an integer  $m \geq 1$  and tori  $\mathbb{T}_1, \dots, \mathbb{T}_m \subset G$ . Write  $\mathbf{T} = \mathbb{T}_1 \times \dots \times \mathbb{T}_m \subset G^m$ , let  $H_i = C_G(\mathbb{T}_i)$  and  $\mathbf{H} = H_1 \times \dots \times H_m = C_{G^m}(\mathbf{T}) \subset G^m$ . We allow some of the  $\mathbb{T}_i$  to be a point, in which case  $H_i = G$ . In this section we will study the stability and polystability for the action of  $\mathbf{H}$  on

$$(3) \quad X := G^{m-1} \times \prod_{i=1}^n G(\phi_i)$$

given by

$$\mathbf{h} \cdot (\mathbf{C}, \mathbf{M}) = (h_2 C_2 h_1^{-1}, \dots, h_m C_m h_1^{-1}, h_1 M_1 h_1^{-1}, \dots, h_1 M_n h_1^{-1})$$

where  $\mathbf{h} = (h_1, \dots, h_m)$ ,  $\mathbf{C} = (C_2, \dots, C_m)$ ,  $\mathbf{M} = (M_1, \dots, M_n)$ ,  $C_i \in G$ ,  $M_i \in G(\phi_i)$ .

Thus if  $m = 1$  and  $\mathbb{T}_1 = 1$  we recover the situation of Thm. 7. The case  $m = 1$  and  $\mathbb{T}_1$  arbitrary but each  $\phi_i = 1$  was studied by Richardson in [27] Thm. 13.2, 14.1 (taking  $S = \mathbb{T}_1$  acting on  $G$  by conjugation). More generally, in effect, [9] Cor. 9.6

studied the notion of stability in the case with  $m$  arbitrary and each  $\phi_i = 1$ , making the link to the differential Galois group of irregular connections (whence the  $\mathbb{T}_i$  are the Ramis/exponential tori).

For  $\mathbf{x} = (\mathbf{C}, \mathbf{M}) \in X$  let  $A(\mathbf{x}) \subset G \times \Gamma$  be the Zariski closure of the subgroup generated by

$$M_1, M_2, \dots, M_n, \mathbb{T}_1, C_2^{-1}\mathbb{T}_2C_2, \dots, C_m^{-1}\mathbb{T}_mC_m.$$

Let  $\bar{A}(\mathbf{x}) \subset \Gamma$  be the image of  $A(\mathbf{x})$  in  $\Gamma \subset \text{Aut}(G)$ , as before. Recall that a subset of  $G$  is  $A(\mathbf{x})$ -invariant if it is preserved by  $\bar{A}(\mathbf{x}) \subset \text{Aut}(G)$ .

**Theorem 19.** (1) *A point  $\mathbf{x} \in X$  is polystable for the  $\mathbf{H}$  action if and only if  $\bar{A}(\mathbf{x})$  is linearly reductive.*

(2) *A point  $\mathbf{x} \in X$  is stable for the  $\mathbf{H}$  action if and only if there are no proper  $A(\mathbf{x})$ -invariant parabolic subgroups in  $G$ .*

As in the last section, one can rephrase polystability in several different ways:

**Corollary 20.** *A point  $\mathbf{x} \in X$  is polystable if and only if*

- 1) *The group  $\bar{A}(\mathbf{x})$  is linearly reductive, or*
- 2) *Any  $A(\mathbf{x})$ -invariant parabolic in  $G$  has an  $A(\mathbf{x})$ -invariant Levi subgroup, or*
- 3) *There exists a subtorus  $S \subset G$  such that  $C_G(S)$  is  $A(\mathbf{x})$ -invariant and has no proper  $A(\mathbf{x})$ -invariant parabolic subgroups.*

**Proof** (of Thm. 19). Let  $G \times \mathbf{H}$  act on  $\tilde{X} := G^m \times \prod_1^N G(\phi_i)$  via

$$(g, \mathbf{h}) \cdot (\mathbf{C}, \mathbf{M}) = (h_1C_1g^{-1}, h_2C_2g^{-1}, \dots, h_mC_mg^{-1}, gM_1g^{-1}, \dots, gM_n g^{-1})$$

where  $\mathbf{h} = (h_1, \dots, h_m)$ ,  $\mathbf{C} = (C_1, \dots, C_m)$ ,  $\mathbf{M} = (M_1, \dots, M_n)$ ,  $C_i \in G$ ,  $M_i \in G(\phi_i)$ . (Up to relabelling and incrementing  $m$  this is the special case where  $H_1 = G$ .) Consider the  $\mathbf{H}$ -equivariant map  $\tilde{X} \rightarrow X$  taking  $(\mathbf{C}; \mathbf{M})$  to

$$(C_2C_1^{-1}, \dots, C_mC_1^{-1}; C_1M_1C_1^{-1}, \dots, C_1M_nC_1^{-1}) \in X.$$

It expresses  $\tilde{X}$  as a principal  $G$ -bundle over  $X$  (the fibres are exactly the  $G$ -orbits). Any point  $\mathbf{x} \in X$  has a unique lift  $\tilde{\mathbf{x}} \in \tilde{X}$  with  $C_1 = 1$ . It follows that the  $\mathbf{H}$  orbit of  $\mathbf{x} \in X$  is closed if and only if the  $G \times \mathbf{H}$  orbit of  $\tilde{\mathbf{x}}$  is closed in  $\tilde{X}$ .

Now choose  $\mathbf{t} = (t_1, \dots, t_m) \in \mathbf{T}$  so that  $t_i$  generates a Zariski dense subgroup of  $\mathbb{T}_i$  for each  $i$ . In particular  $H_i = C_G(t_i)$ . Consider the simultaneous conjugation action of  $G$  on  $Y := G^m \times \prod_1^N G(\phi_i)$ , and the  $G$ -equivariant embedding

$$\pi : \tilde{X}/\mathbf{H} \hookrightarrow Y; \quad [(\mathbf{C}; \mathbf{M})] \mapsto (\mathbf{C}^{-1}\mathbf{t}\mathbf{C}, \mathbf{M}).$$

Thus  $\tilde{X}/\mathbf{H}$  is identified with a closed subvariety of  $Y$  and  $\tilde{X}$  is a  $G$ -equivariant principal  $\mathbf{H}$ -bundle over  $\tilde{X}/\mathbf{H}$ . It follows that the  $G \times \mathbf{H}$  orbit of  $\tilde{\mathbf{x}}$  is closed in  $\tilde{X}$  if and only if the  $G$  orbit of  $\pi(\tilde{\mathbf{x}})$  is closed in  $Y$ .

Hence part (1) of the theorem follows from Thm. 7 (applied to  $Y$ ). To deal with stability we need to consider the stabilisers and the kernels of the actions.

**Lemma 21.** *The stabiliser of any point  $\mathbf{x} \in X$  for the  $\mathbf{H}$ -action is canonically isomorphic to the stabiliser of the point  $\pi(\tilde{\mathbf{x}}) \in Y$  for the  $G$ -action.*

**Proof.** Suppose  $g \in G$  fixes  $(\mathbf{C}^{-1}\mathbf{t}\mathbf{C}, \mathbf{M})$  and let  $h_i = C_i g C_i^{-1}$  ( $i = 1, 2, \dots, m$ ) and  $\mathbf{h} = (h_1, \dots, h_m)$ . Note  $h_1 = g$  because  $C_1 = 1$ . Since  $g$  commutes with each  $C_i^{-1}t_i C_i$ , each  $h_i$  commutes with  $t_i$ , which implies  $\mathbf{h} \in \mathbf{H}$ . We have  $h_i C_i g^{-1} = C_i$  ( $i = 1, 2, \dots, m$ ) by the definition of  $h_i$  and  $g M_j g^{-1} = M_j$  ( $j = 1, 2, \dots, N$ ) as  $g$  centralises  $\mathbf{M}$ . Hence the pair  $(g, \mathbf{h}) \in G \times \mathbf{H}$  stabilises  $\tilde{\mathbf{x}} \in \tilde{X}$ , and hence  $\mathbf{h}$  stabilises  $\mathbf{x} \in X$ . Conversely if  $\mathbf{h} = (h_1, \dots, h_m) \in \mathbf{H}$  fixes  $\mathbf{x}$ , then let  $g = h_1$ . It follows immediately that  $(g, \mathbf{h}) \in G \times \mathbf{H}$  fixes  $\tilde{\mathbf{x}}$  and thus that  $g$  fixes  $\pi(\tilde{\mathbf{x}})$ . Clearly the two correspondences are inverses of each other.  $\square$

As in Lemma 14 one has:

**Lemma 22.** (1) *The kernel of the  $G$ -action on  $Y$  (or  $\tilde{X}/\mathbf{H}$ ) is the  $\Gamma$ -invariant subgroup  $Z(G)^\Gamma$  of the center of  $G$ , and (2) *The kernel of the  $\mathbf{H}$ -action on  $X$  is the subgroup of elements  $(h_1, \dots, h_m) \in \mathbf{H}$  satisfying  $h_1 = h_2 = \dots = h_m \in Z(G)^\Gamma$ .**

Note that these two groups correspond to each other under the correspondence of Lem. 21 (for any  $\mathbf{x} \in X$ ). It follows that  $\mathbf{x} \in X$  is stable if and only if  $\pi(\tilde{\mathbf{x}}) \in Y$  is stable. By Thm. 12 (2) applied to  $Y$ , this happens if and only if there are no proper  $A(\mathbf{x})$  invariant parabolic subgroups of  $G$ .  $\square$

#### 4. APPLICATION TO WILD CHARACTER VARIETIES

Recall from [9, 14] that the wild character variety  $\mathcal{M}_B(\Sigma) = \text{THom}_S(\Pi, G)/\mathbf{H}$  is determined by an irregular curve/wild Riemann surface  $\Sigma = (\Sigma, \alpha, \Theta)$  with group  $\mathcal{G}$ , where  $\Sigma$  is a compact smooth complex algebraic curve,  $\alpha \subset \Sigma$  is a non-empty finite subset,  $\mathcal{G} \rightarrow \Sigma^\circ := \Sigma \setminus \alpha$  is a local system of groups over the punctured curve (with each fibre isomorphic to some fixed connected complex reductive group  $G$ ), and  $\Theta$  consists of the data of an irregular class  $\Theta_a$  at each point  $a \in \alpha$  (in the sense of [14] §3.5—it is the class of a graded  $\mathcal{G}$  local system).

As in [14] §4,  $\Sigma$  then determines an auxiliary surface  $\tilde{\Sigma}$ , equipped with boundary circles  $\partial$ , halos  $\mathbb{H} \subset \tilde{\Sigma}$ , and tangential punctures  $e(\mathbb{A})$ . Further  $\Sigma$  determines a local system  $\mathbb{T} \rightarrow \partial$  of finite dimensional complex tori, the Ramis tori ([14] p.9). Choosing a finite set  $\beta = \{b_1, \dots, b_m\} \subset \partial$  of basepoints (with one point in each component circle, as in [14] §4.1) then determines the wild surface groupoid  $\Pi = \Pi_1(\tilde{\Sigma}, \beta)$ , the fundamental groupoid of the auxiliary surface with these basepoints, as in [14] §5. The local system  $\mathcal{G}$  is determined by a map  $f : \Pi \rightarrow \text{Aut}(G)$  and this determines the

space of  $f$ -twisted representations

$$\mathrm{THom}(\Pi, G) = \{\rho \in \mathrm{Hom}(\Pi, G \rtimes \mathrm{Aut}(G)) \mid \rho(\gamma) \in G(f(\gamma)) \text{ for all } \gamma \in \Pi\}$$

of  $\Pi$ , as in [14] §5. Since  $\tilde{\Sigma}$  is just a punctured real surface with boundary, choosing generating paths in  $\Pi$  yields an isomorphism  $\mathrm{THom}(\Pi, G) \cong G^N$  of spaces, for some integer  $N$ , so it is a smooth affine variety. In turn the wild representation variety  $\mathcal{R} = \mathrm{THom}_{\mathbb{S}}(\Pi, G)$  is the closed subvariety of  $\mathrm{THom}(\Pi, G)$  cut out by the two Stokes conditions, in [14] Defn. 18. Intrinsically,  $\mathcal{R}$  is the moduli space of framed Stokes local systems, as in [14] Prop. 19, framed via a graded isomorphism to a standard fibre  $\mathcal{F}_i$  at each basepoint  $b_i \in \beta$  ([14] §4.1). The group  $H_i = \mathrm{GrAut}(\mathcal{F}_i) = C_G(\mathbb{T}_i) \subset G = \mathrm{Aut}(\mathcal{F}_i)$  acts transitively on the set of framings at  $b_i$ , where  $\mathbb{T}_i = \mathbb{T}_{b_i}$ . Thus the group  $\mathbf{H} = \prod_{b_i \in \beta} H_i$  acts naturally on  $\mathcal{R}$ . The wild character variety  $\mathcal{M}_{\mathbb{B}}(\Sigma)$  is the affine geometric invariant theory quotient  $\mathcal{R}/\mathbf{H}$ , and so its points are the closed  $\mathbf{H}$  orbits in  $\mathcal{R}$ . This leads directly to the key statement:

**Proposition 23.** *The wild representation variety  $\mathcal{R} = \mathrm{THom}_{\mathbb{S}}(\Pi, G)$  may be embedded in an  $\mathbf{H}$ -equivariant way, as a closed subvariety of the space  $X = G^{m-1} \times \prod_1^n G(\phi_i)$  of (3), for suitable  $n$  and automorphisms  $\{\phi_i\} \subset \mathrm{Aut}(G)$ , with  $m = \#\alpha$ .*

**Proof.** This comes down to considering the inclusion  $\mathrm{THom}_{\mathbb{S}}(\Pi, G) \subset \mathrm{THom}(\Pi, G)$  as a closed subvariety (forgetting the Stokes conditions, as in [14] Defn. 18), and then identifying  $\mathrm{THom}(\Pi, G) \cong X$  in an  $\mathbf{H}$ -equivariant way. Both of these are straightforward. In particular  $M_j = \rho(\gamma_j)$  for generators  $\gamma_j$  of  $\pi_1(\tilde{\Sigma}, b_1)$ , and  $C_i = \rho(\chi_i)$  for paths  $\chi_i$  from  $b_1$  to  $b_i$ , for  $i = 2, \dots, m$ .  $\square$

**4.1. Galois groups from Stokes representations.** In order to define the Galois group  $\mathrm{Gal}(\rho)$  of  $\rho \in \mathrm{THom}_{\mathbb{S}}(\Pi, G)$  we will assume that  $\mathcal{G}$  is “Out-finite”, in the sense that the monodromy group  $f(\pi_1(\tilde{\Sigma}, b_1)) \subset \mathrm{Aut}(G)$  of  $\mathcal{G}$  has finite image in  $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ . Then the group  $\Gamma$  generated by  $f(\pi_1(\tilde{\Sigma}, b_1))$  and  $\mathrm{Inn}(G)$  is an algebraic group, as in §2 above. Thus any Stokes representation takes values in the algebraic group  $G \rtimes \Gamma \subset G \rtimes \mathrm{Aut}(G)$ , and so we can consider the Zariski closure of its monodromy. The Galois group is defined by adding the Ramis tori as well: If  $t \in \mathbb{T}_i \subset G$  and  $\chi$  is any path in  $\tilde{\Sigma}$  from  $b_1$  to  $b_i$ , consider the element

$$(4) \quad C^{-1}tC \in G = G(\mathrm{Id}) \subset G \rtimes \Gamma,$$

where  $C = \rho(\chi)$ .

**Definition 24.** The differential Galois group  $\mathrm{Gal}(\rho)$  of  $\rho$  is the Zariski closure of the subgroup of  $G \rtimes \Gamma$  generated by  $\rho(\pi_1(\tilde{\Sigma}, b_1))$  and all of the tori (4) (as  $i, t, \chi$  vary).

It follows that  $\mathrm{Gal}(\rho)$  acts on  $G$  by group automorphisms, via the adjoint action of  $G \rtimes \Gamma$  on  $G = G(\mathrm{Id})$ . Let  $\overline{\mathrm{Gal}}(\rho) \subset \Gamma \subset \mathrm{Aut}(G)$  be the resulting image of  $\mathrm{Gal}(\rho)$ . This definition is (of course) motivated by Ramis’ description ([26], [25] Thm. 21,

[23] Thm. III.3.11) of the differential Galois group of an algebraic connection on a vector bundle.

**Corollary 25.** *A Stokes representation  $\rho \in \mathrm{THom}_{\mathbb{S}}(\Pi, G)$  is polystable for the action of  $\mathbf{H}$  if and only if  $\overline{\mathrm{Gal}}(\rho)$  is a linearly reductive group.*

**Proof.** This now follows from part (1) of Thm. 19, via Prop. 23, since  $\overline{\mathrm{Gal}}(\rho)$  matches up with  $\overline{A}(\mathbf{x})$ .  $\square$

Special cases include:

- If  $\mathcal{G}$  has finite monodromy then  $\rho \in \mathrm{THom}_{\mathbb{S}}(\Pi, G)$  is polystable if and only if  $\mathrm{Gal}(\rho)$  is a linearly reductive group.
- If  $\mathcal{G}$  is a constant general linear group then  $\rho \in \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$  is polystable if and only if  $\rho$  is the direct sum of irreducible Stokes representations.

Recall that  $Z(G)^{\Gamma}$  is the  $\Gamma$  invariant subgroup of the centre of  $G$ , and it embeds diagonally in  $\mathbf{H}$ . To deal with stability we will avoid degenerate cases by assuming:

$$(5) \quad \text{The kernel of the action of } \mathbf{H} \text{ on } \mathrm{THom}_{\mathbb{S}}(\Pi, G) \text{ is } Z(G)^{\Gamma}.$$

The lemma below shows one can always add one or two punctures to ensure this condition holds. Note that no generality is lost: any symplectic leaf  $\mathcal{M}_{\mathbb{B}}(\Sigma, \mathcal{C}) \subset \mathcal{M}_{\mathbb{B}}(\Sigma)$  will also be a symplectic leaf of the larger Poisson variety obtained by first making such additional punctures (namely that with trivial monodromy around the new punctures). For example the usual character variety of a genus  $g > 0$  compact Riemann surface  $\Sigma$  is a (very special) symplectic leaf of the character variety of  $\Sigma \setminus a$ , for any point  $a \in \Sigma$ .

**Lemma 26.** *Suppose  $m \geq 1$  and  $a_1 \in \alpha$  has trivial irregular class, and if  $g = 0$  then  $m \geq 2$ . Then  $\mathrm{THom}_{\mathbb{S}}(\Pi, G)$  is a smooth non-empty affine variety and the kernel  $\mathbf{K}$  of the  $\mathbf{H}$  action is  $Z(G)^{\Gamma}$  embedded diagonally in  $\mathbf{H}$ .*

**Proof.** It is nonempty as it is the fusion of some fission spaces and some internally fused doubles: Recall that  $\mathrm{THom}_{\mathbb{S}}(\Pi, G)$  can be described as the quasi-Hamiltonian  $G$ -reduction

$$\mathrm{THom}_{\mathbb{S}}(\Pi, G) \cong (\mathbb{D}_1 \underset{G}{\otimes} \cdots \underset{G}{\otimes} \mathbb{D}_g \underset{G}{\otimes} \mathcal{A}(Q_1) \underset{G}{\otimes} \cdots \underset{G}{\otimes} \mathcal{A}(Q_m)) // G$$

where each  $\mathbb{D}_i$  is a (twisted) internally fused double ([14] p.23, [9] Thm 8.2). Since  $\mathcal{A}(Q_1) \cong \mathbf{D}(G)$  is the double of  $G$ , it follows that

$$\mathrm{THom}_{\mathbb{S}}(\Pi, G) \cong \mathbb{D}_1 \otimes \cdots \otimes \mathbb{D}_g \otimes \mathcal{A}(Q_2) \otimes \cdots \otimes \mathcal{A}(Q_m)$$

so that  $\mathrm{THom}_{\mathbb{S}}(\Pi, G)$  is the product of some smooth nonempty affine varieties. Note that  $\mathbf{H}$  still acts on  $\mathrm{THom}_{\mathbb{S}}(\Pi, G)$  and this includes  $H_1 = G$ .

Suppose  $\mathbf{k} \in \mathbf{K}$  and suppose (as usual) that the framings of  $\mathcal{G}$  are such that the monodromy in  $\mathrm{Aut}(G)$  of  $\mathcal{G}$  is trivial along the  $m - 1$  chosen paths  $\chi_i : b_1 \rightarrow b_i$ .

Then  $\mathbf{k}$  acts on  $C_i = M_{\chi_i}$  as  $k_i C_i k_1^{-1}$ . Taking  $C_2 = C_3 = \dots C_m = 1$  implies  $k_1 = k_2 = \dots = k_m$ . If  $m \geq 2$  then the fact that all  $C_2 \in G$  are fixed implies  $k_1 \in Z(G)$ . On the other hand if  $m = 1$  then  $g > 0$  so looking at  $\mathbb{D}_1$  there is  $\phi \in \text{Aut}(G)$  so that  $k_1 A \phi(k_1^{-1}) = A$  for all  $A \in G$ . This implies  $k_1 = \phi(k_1)$  and  $k_1 \in Z(G)$ . Thus in all case  $k_1$  is central. Then looking at any loop  $\gamma$  based at  $b_1$  leads to a relation of the form  $k_1 M_\gamma \phi_\gamma(k_1^{-1}) = M_\gamma$ . Thus since  $k_1$  is central this implies  $k_1 = \phi_\gamma(k_1)$ , so  $k_1 \in Z(G)^\Gamma$ .  $\square$

*Remark 27.* Note that it follows in general (as in [9] Thm. 8.2) that if  $\text{THom}_{\mathbb{S}}(\Pi, G)$  is nonempty (and  $m > 0$ ) then it is a smooth affine variety.

Part (2) of Theorem 12 then implies:

**Corollary 28.** *A Stokes representation  $\rho \in \text{THom}_{\mathbb{S}}(\Pi, G)$  is stable for the action of  $\mathbf{H}$  if and only if there is no proper parabolic subgroup  $P \subset G$  stabilised by the action of  $\text{Gal}(\rho)$ .*

**Proof.** This follows from Thm. 12 since  $\mathcal{R}$  is closed in  $X$ ,  $\text{Gal}(\rho)$  matches up with  $A(\mathbf{x})$ , and the kernel of the  $\mathbf{H}$  action on  $\mathcal{R}$  and on  $X$  is the same.  $\square$

## 5. STABILITY AND POLYSTABILITY OF STOKES LOCAL SYSTEMS

This section will consider the intrinsic objects (Stokes local systems) underlying Stokes representations, and define the notions of “irreducible” and “reductive” for Stokes local systems, in terms of reductions of structure group. Then we will deduce:

**Theorem 29.** *Suppose  $\mathbb{L}$  is a Stokes local system and  $\rho \in \text{THom}_{\mathbb{S}}(\Pi, G)$  is the monodromy of  $\mathbb{L}$ .*

- 1)  $\rho$  is stable for the action of  $\mathbf{H}$  if and only if  $\mathbb{L}$  is irreducible,
- 2)  $\rho$  is polystable for the action of  $\mathbf{H}$  if and only if  $\mathbb{L}$  is reductive.

**5.1. Graded local systems.** A Stokes local system is a special type of  $\mathbb{T}$ -graded local system (in the sense of Defn. 30 below), so to clarify the ideas we will focus on them here—the results for Stokes local systems follow almost immediately.

Let  $S$  be a connected real oriented surface of finite topological type. Let  $\mathbb{H} \subset S$  be an open subset, and let  $\mathbb{T} \rightarrow \mathbb{H}$  be a local system of complex tori over  $\mathbb{H}$ . We allow the fibres of  $\mathbb{T}$  to have different dimensions in different components of  $\mathbb{H}$ . Fix a connected complex reductive group  $G$ , and a local system  $\mathcal{G} \rightarrow S$  of groups, such that each fibre of  $\mathcal{G}$  is isomorphic to  $G$ . We will assume throughout that  $\mathcal{G}$  is “Out-finite”. This means that the monodromy of  $\mathcal{G}$  has finite image in  $\text{Out}(G)$ . In more detail, given a basepoint  $b \in S$  and a framing  $G \cong \mathcal{G}_b$  of  $\mathcal{G}$  at  $b$ , then the monodromy representation  $f : \pi_1(S, b) \rightarrow \text{Aut}(G)$  of  $\mathcal{G}$  is such that the monodromy group  $f(\pi_1(S, b)) \subset \text{Aut}(G)$

has finite image in  $\text{Out}(G)$ . Of course if  $G$  is semisimple then  $\text{Out}(G)$  is finite and so this is no restriction.

Recall that a  $\mathcal{G}$ -local system over  $S$  is a local system  $\mathbb{L} \rightarrow S$  which is a  $\mathcal{G}$ -torsor (cf. e.g. [14] §2.1), and it determines a local system  $\text{Aut}(\mathbb{L}) \rightarrow S$  of groups (each fibre of which is also isomorphic to  $G$ ).

**Definition 30.** A  $\mathbb{T}$ -graded  $\mathcal{G}$ -local system over  $S$  is a  $\mathcal{G}$ -local system  $\mathbb{L} \rightarrow S$  together with an embedding

$$\mathbb{T} \hookrightarrow \text{Aut}(\mathbb{L})|_{\mathbb{H}}$$

of local systems of groups over  $\mathbb{H}$ .

For brevity we will simply call this a “graded local system on  $S$ ”, and write  $\mathbb{T} \hookrightarrow \text{Aut}(\mathbb{L})$  for the grading. A Stokes local system  $\mathbb{L}$  (in the sense of [9, 14]) is a special type of  $\mathbb{T}$ -graded local system, taking  $S = \tilde{\Sigma}$  to be the auxiliary surface,  $\mathbb{H}$  to be the union of the halos, and  $\mathbb{T}$  to be the image in  $\text{Aut}(\mathbb{L})|_{\mathbb{H}}$  of the exponential torus  $\mathcal{T}$ . (Note that  $\mathbb{T}$  is determined just by the irregular class of  $\mathbb{L}$ , in the sense of [14] §3.5, since as explained there the class determines the finite rank local system  $I \subset \mathcal{I}$  of lattices, and  $\mathbb{T}$  is the local system of tori with character lattice  $I$ .)

**5.2. Galois group.** If  $\mathbb{L} \rightarrow S$  is a  $\mathbb{T}$ -graded local system, and  $b \in S$  is a basepoint, define  $\text{Gal}(\mathbb{L})$  to be the Zariski closure of the group generated by the monodromy of  $\mathbb{L}$  and all the tori  $\mathbb{T}$  (after transporting them to  $b$ ). In more detail, first identify  $\mathcal{G}_b \cong G, \mathbb{L}_b \cong \mathcal{F}$  (the trivial  $G$ -torsor, as in [14] §2) and define  $\Gamma \subset \text{Aut}(G)$  to be the group generated by the monodromy of  $\mathcal{G}$  and  $\text{Inn}(G)$  (as in §2 above). If  $\gamma \in \pi_1(S, b)$  let  $\rho(\gamma) \in G(f(\gamma)) \subset G \rtimes \Gamma \hookrightarrow \text{Perm}(\mathcal{F})$  be the monodromy of  $\mathbb{L}$  around  $\gamma$  (where  $\text{Perm}(\mathcal{F})$  is the group of all permutations of the fibre  $\mathcal{F}$ ). Similarly if  $p \in \mathbb{H}, t \in \mathbb{T}_p \subset \text{Aut}(\mathbb{L})_p \cong G$  and  $\chi$  is any path in  $S$  from  $b$  to  $p$ , consider the element

$$(6) \quad C^{-1}tC \in \text{Aut}(\mathcal{F}) = G = G(\text{Id}) \subset G \rtimes \Gamma,$$

where  $C$  is the transport of  $\mathbb{L}$  along  $\chi$ .

**Definition 31.** The differential Galois group  $\text{Gal}(\mathbb{L})$  of  $\mathbb{L}$  is the Zariski closure of the subgroup of  $G \rtimes \Gamma$  generated by  $\rho(\pi_1(S, b))$  and all of the tori (6) (as  $p, t, \chi$  vary).

It follows that  $\text{Gal}(\mathbb{L})$  acts on  $G$  by group automorphisms, via the adjoint action of  $G \rtimes \Gamma$  on  $G = G(\text{Id})$ . Let  $\overline{\text{Gal}}(\mathbb{L}) \subset \Gamma \subset \text{Aut}(G)$  be the resulting image of  $\text{Gal}(\mathbb{L})$ . Up to isomorphism the affine algebraic group  $\text{Gal}(\mathbb{L})$  and its action on  $G$  do not depend on the choice of basepoint  $b$  or framings.

**5.3. Irreducible graded local systems.** Define a graded local system  $\mathbb{L} \rightarrow S$  to be *reducible* if  $\text{Aut}(\mathbb{L})$  has a sublocal system of proper parabolic subgroups, containing  $\mathbb{T}$ . In other words there is a sublocal system  $\mathcal{P} \subset \text{Aut}(\mathbb{L})$  such that 1) each fibre  $\mathcal{P}_b$  is a proper parabolic subgroup of  $\text{Aut}(\mathbb{L})_b$ , and 2) the grading  $\mathbb{T} \hookrightarrow \text{Aut}(\mathbb{L})$  factors through  $\mathcal{P}$ . Such  $\mathbb{L}$  is *irreducible* if it is not reducible.

**Lemma 32.**  $\mathbb{L}$  is reducible if and only if  $\text{Gal}(\mathbb{L})$  preserves a proper parabolic subgroup of  $G$  (recalling that  $\text{Gal}(\mathbb{L})$  naturally acts on  $G$  by group automorphisms).

**Proof.** Suppose  $P \subset G \cong \text{Aut}(\mathbb{L}_b)$  is preserved by  $\text{Gal}(\mathbb{L})$ . Then  $P$  is the fibre at  $b$  of a local system of parabolic subgroups  $\mathcal{P} \subset \text{Aut}(\mathbb{L})$ , since the monodromy of  $\text{Aut}(\mathbb{L})$  is given by the adjoint action of the monodromy of  $\mathbb{L}$ , which is in  $\text{Gal}(\mathbb{L})$ . Moreover  $\mathbb{T} \hookrightarrow \mathcal{P}$ , since the transport to  $b$  of each fibre of  $\mathbb{T}$  is in  $G$  and preserves  $P$  (so is in  $P$ , since  $N_G(P) = P$ ). The converse is similar, taking the fibre at  $b$  of  $\mathcal{P} \subset \text{Aut}(\mathbb{L})$ .  $\square$

This can be related to (twisted) reductions of structure group as follows (compare [14] Defn. 11).

**Definition 33.** Suppose  $\mathbb{L}$  is a  $\mathbb{T}$ -graded  $\mathcal{G}$ -local system.

- A *reduction* of  $\mathbb{L}$  is a  $\mathbb{T}$ -graded  $\mathcal{P}$ -local system  $\mathbb{P} \rightarrow S$  (for some local system of groups  $\mathcal{P} \rightarrow S$ ), such that  $\mathbb{L}$  is a twisted pushout of  $\mathbb{P}$ . This means that there is a  $\mathcal{G}$  local system  $\mathbb{M}$  with an embedding  $\mathcal{P} \hookrightarrow \text{Aut}(\mathbb{M})$ , together with an isomorphism  $\mathbb{L} \cong \mathbb{P} \times_{\mathcal{P}} \mathbb{M}$  (of graded  $\mathcal{G}$  local systems),

- A reduction  $\mathbb{P}$  of  $\mathbb{L}$  is a *parabolic reduction* if the fibres of  $\mathcal{P}$  embed as parabolic subgroups of the fibres of  $\text{Aut}(\mathbb{M})$  (each of which is isomorphic to  $G$ ),

- Similarly it is a *Levi reduction* if the fibres of  $\mathcal{P}$  embed as Levi subgroups of parabolic subgroups of the fibres of  $\text{Aut}(\mathbb{M})$ ,

- The reduction is *proper* if the fibres of  $\mathcal{P}$  embed as proper subgroups.

**Lemma 34.**  $\mathbb{L}$  is reducible if and only if it has a proper parabolic reduction of structure group.

**Proof.** Given  $\mathcal{P} \subset \text{Aut}(\mathbb{L})$  with  $\mathbb{T} \hookrightarrow \mathcal{P}|_{\mathbb{H}}$ , then taking  $\mathbb{P} = \mathcal{P}$  and  $\mathbb{M} = \mathbb{L}$  gives the desired reduction. Conversely given  $\mathbb{M}, \mathcal{P}, \mathbb{P}$  then  $\text{Aut}(\mathbb{P})$  gives the desired parabolic sublocal system in  $\text{Aut}(\mathbb{L})$ .  $\square$

Note that part 1) of Thm. 29 now follows immediately from Cor. 28, noting that  $\text{Gal}(\mathbb{L}) = \text{Gal}(\rho)$ .

This irreducibility condition can also be spelt out in terms of Stokes representations and compatible systems of parabolics (as in [9] §9).

If  $\mathcal{G}$  is constant then we can use the usual (simpler) notion of reduction of structure group (then  $\text{Gal} \subset G$  and we don't need twisted reductions, i.e. we can take  $\mathbb{M}$  to be the trivial  $G$ -torsor).

**5.4. Reductive/semisimple graded local systems.** If  $\mathbb{L} \rightarrow S$  is a graded local system and  $\mathcal{L} \rightarrow S$  is a Levi reduction of  $\mathbb{L}$  (in the sense of Defn. 33) then  $\mathcal{L}$  is itself a graded local system, and so we can ask if  $\mathcal{L}$  is reducible or not. Define  $\mathbb{L}$  to

be *reductive* (or “semisimple”) if it has an irreducible Levi reduction. Similarly to Lemma 34 one can rephrase this in terms of  $\text{Aut}(\mathbb{L})$ :

**Lemma 35.**  $\mathbb{L}$  is reductive if and only if there is a sublocal system  $\mathcal{E} \subset \text{Aut}(\mathbb{L})$  such that 1) each fibre  $\mathcal{E}_p$  is a Levi subgroup of a parabolic of  $\text{Aut}(\mathbb{L})_p$ , 2)  $\mathbb{T} \subset \mathcal{E}$ , and 3)  $\mathcal{E}$  is irreducible in the sense that it has no proper sublocal systems of parabolic subgroups, containing  $\mathbb{T}$ .

Recall  $\overline{\text{Gal}}(\mathbb{L}) \subset \Gamma \subset \text{Aut}(G)$  is the image of  $\text{Gal}(\mathbb{L})$  in  $\Gamma$ .

**Proposition 36.**  $\mathbb{L}$  is reductive if and only if  $\overline{\text{Gal}}(\mathbb{L})$  is a linearly reductive group.

**Proof.** By Prop. 17  $\overline{\text{Gal}}(\mathbb{L})$  is a linearly reductive group if and only if there is a subgroup  $L \subset G$  such that 1)  $L$  is a Levi subgroup of a parabolic subgroup of  $G$ , 2)  $L$  is preserved by the action of  $\text{Gal}(\mathbb{L})$ , and 3)  $L$  has no proper parabolic subgroups that are  $\text{Gal}(\mathbb{L})$  invariant. (This uses the fact that the centralisers of tori in  $G$  are exactly the Levi subgroups of parabolics.) As in Lem. 32 the existence of such  $L$  is the same as  $\mathbb{L}$  having an irreducible Levi reduction.  $\square$

Part 2) of Thm. 29 now follows immediately from Cor. 25.

Note that if  $\mathcal{G}$  is constant (or has finite monodromy) then this is the same as  $\text{Gal}(\mathbb{L})$  being linearly reductive.

*Remark 37.* Note that if  $\mathbb{L}$  is a Stokes  $\mathcal{G}$ -local system then it makes no difference if we insist on only looking at reductions that are Stokes local systems: i.e.  $\mathbb{L}$  is semisimple if it has a Levi reduction to an irreducible Stokes  $\mathcal{L}$ -local system, and  $\mathbb{L}$  is reducible if it has a parabolic reduction to a Stokes  $\mathcal{P}$ -local system. To see this we need to check 1) that the Stokes conditions on the monodromies (of the reductions) around the tangential punctures are automatic, and 2) that there is no loss of generality in assuming that the local systems of parabolic/Levi subgroups are untwisted around the tangential punctures. This is now an easy exercise: 1) follows since the Stokes groups are controlled by  $\mathbb{T}$ , and 2) follows by considering the proof of Lem. 34, and the analogous proof of Lem. 35.

*Remark 38.* In the case where  $\mathcal{G}$  is a constant general linear group with fibre  $G = \text{GL}_n(\mathbb{C})$  for some  $n$ , then a Stokes  $\mathcal{G}$ -local system  $\mathbb{L}$  is equivalent to a Stokes local system  $\mathbb{V}$  of rank  $n$  vector spaces, as in [12]. Then  $\mathbb{L}$  is irreducible if and only if  $\mathbb{V}$  has no nontrivial proper Stokes sublocal systems, and it is reductive if and only if  $\mathbb{V} = \bigoplus \mathbb{V}_i$  is the direct sum of some irreducible Stokes local systems  $\mathbb{V}_i$ .

Note that there are thus many simple criteria to ensure points of  $\mathcal{R}$  are stable, and they will be studied systematically elsewhere. For example in the constant  $\text{GL}_n(\mathbb{C})$  setting, all the Stokes local systems on  $\Sigma$  are irreducible if at one of the punctures the irregular class just has one Stokes circle  $I \subset \mathcal{I}$  with  $\text{Ram}(I) = n$  (e.g. if  $\text{slope}(I) = k/n$  with  $(k, n) = 1$  this is Katz’s irreducibility criterion [22] (2.2.8)).

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