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ON ENDOMORPHISM ALGEBRAS OF GELFAND–GRAEV REPRESENTATIONS II

Tzu-Jan Li and Jack Shotton

Abstract. Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of characteristic p, with Deligne–Lusztig dual G^*. We show that, over $\overline{\mathbb{Z}}[1/pM]$ where M is the product of all bad primes for G, the endomorphism ring of a Gelfand–Graev representation of $G(\mathbb{F}_q)$ is isomorphic to the Grothendieck ring of the category of finite-dimensional $\overline{\mathbb{F}}_q$-representations of $G^*(\mathbb{F}_q)$.

0. Introduction. — Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of characteristic p, let F be the associated Frobenius endomorphism of G, and let Λ be a subring of $\overline{\mathbb{Q}}$ containing $\overline{\mathbb{Z}}[\frac{1}{p}]$. Let B_0 be an F-stable Borel subgroup of G with (necessarily F-stable) unipotent radical U_0, and let $\psi : U_0^F \longrightarrow \Lambda^\times$ be a regular (also called nondegenerate) character. The Gelfand–Graev representation

$$\Gamma_{G,\psi} := \mathrm{Ind}_{U_0^F}^{G^F} \psi$$

is an important representation of G^F (already studied in [DeLu, Sec. 10] and [DLM]). Its endomorphism ring

$$\Lambda\mathsf{E}_G := \mathrm{End}_{\Lambda G^F}(\Gamma_{G,\psi})$$

is commutative, independent of the choice of ψ up to isomorphism and, over $\overline{\mathbb{Q}}$, may be identified with the ring of $\overline{\mathbb{Q}}$-valued class functions on $G^{*F^*}_{\mathrm{ss}}$, where (G^*, F^*) is a chosen Deligne–Lusztig $\mathbb{F}_q$-dual of (G, F) (see [Cu]). Such an identification only depends on choices of group homomorphisms $(\mathbb{Q}/\mathbb{Z})_{p'} \simeq \overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$, which we fix from now on.

There are then (at least) two natural Λ-lattices in $\overline{\mathbb{Q}}\mathsf{E}_G$: $\Lambda\mathsf{E}_G$ and the lattice $\Lambda\mathsf{K}_{G^*}$ spanned by Brauer characters of irreducible representations of G^{*F^*}; here, K_{G^*} is the Grothendieck group of the category of finite-dimensional $\overline{\mathbb{F}}_q G^{*F^*}$-modules, and it is a ring with tensor product as multiplication. Denoting by $G^{*F^*}_{\mathrm{ss}}/\sim$ the set of semisimple conjugacy classes in G^{*F^*}, we may then identify

$$\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} = \overline{\mathbb{Q}}\mathsf{K}_{G^*}. \tag{0.1}$$

The main result of this paper may now be stated as follows:

Main theorem. *If all bad primes for G are invertible in Λ, then the two Λ-lattices $\Lambda\mathsf{E}_G$ and $\Lambda\mathsf{K}_{G^*}$ of $\overline{\mathbb{Q}}\mathsf{E}_G$ are equal.*

Here, we use the notion of "bad primes for G" from [Sp]. Denoting by R the root system of G, a prime number ℓ is called *bad* for G if one of the following three conditions holds: (i) $\ell = 2$, and R has an irreducible factor other than type A; (ii) $\ell = 3$ or 5, and R has an irreducible factor of exceptional type (G_2, F_4, E_6, E_7, or E_8); (iii) $\ell = 7$, and R has an irreducible factor of type E_8.

In this theorem, the assumption on the bad primes for G is due to the usage of *almost characters* in Lusztig's work on unipotent characters, where bad primes appear in the denominators of the "Fourier transform matrix." We expect that the theorem remains true without this assumption, though our present method cannot prove it.

The above theorem improves the equality $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ (where W is the Weyl group of G) in [Li, Thm. 2.9] whenever the adjoint group of G is simple of type other than F_4 or G_2 (in these two excluded types, the bad primes and the primes dividing the order of the Weyl group coincide). Moreover, via the $\mathbb{Z}$-model E_G of $\Lambda\mathsf{E}_G$ from [Li, Sec. 1.7], if we denote by M is the product of all bad primes for G, then the above theorem implies that $\mathbb{Z}[\frac{1}{pM}]\mathsf{E}_G = \mathbb{Z}[\frac{1}{pM}]\mathsf{K}_{G^*}$.

Relation with invariant theory. Let $\mathsf{B}_{G^\vee}$ be the ring of functions of the $\mathbb{Z}$-scheme $(T^\vee /\!\!/ W)^{F^\vee}$, where $(G^\vee, T^\vee)$ is the split $\mathbb{Z}$-dual of (G, T) with T an F-stable maximal torus of G, $W = N_{G^\vee}(T^\vee)/T^\vee$ is the Weyl group of $(G^\vee, T^\vee)$, and $F^\vee : T^\vee \longrightarrow T^\vee$ is induced by the action of F on $Y(T^\vee) = X(T)$. If G^* has simply-connected derived subgroup, then $\Lambda\mathsf{B}_{G^\vee}$ is also a Λ-lattice of $\overline{\mathbb{Q}}\mathsf{E}_G$ and appears to be significant for the *local Langlands correspondence in families*. Indeed, for GL_n, in the course of constructing this correspondence in joint work with Moss [HeMo], Helm proved in [Hel, Thm. 10.1] the equality $\Lambda\mathsf{E}_{\mathrm{GL}_n} = \Lambda\mathsf{B}_{\mathrm{GL}_n^\vee}$ for Λ being the ring of Witt vectors of $\overline{\mathbb{F}}_\ell$ with $\ell \neq p$. In our current context (G a connected reductive group over $\mathbb{F}_q$), when G^* has simply-connected derived subgroup, it is known that $\mathsf{B}_{G^\vee} = \mathsf{K}_{G^*}$ (see [Li, Thm. 3.22]), so that our main theorem yields the equalities

$$\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*} = \Lambda\mathsf{B}_{G^\vee}$$

for $\Lambda = \overline{\mathbb{Z}}[\frac{1}{pM}]$. In particular, for GL_n, $M = 1$ and so we provide an alternative proof of Helm–Moss's equality.

On the proof of the main theorem. Identify $\Lambda\mathsf{E}_G = e_\psi \Lambda G^F e_\psi \subset \Lambda G^F$ where $e_\psi := \frac{1}{|U_0^F|}\sum_{u \in U_0^F} \psi(u^{-1})u$ is the primitive central idempotent of ΛU_0^F associated with ψ. We may then consider the *symmetrizing form*

$$\tau = \tau_G := |U_0^F| \mathrm{ev}_{1_{G^F}} : \Lambda\mathsf{E}_G \longrightarrow \Lambda$$

($\mathrm{ev}_{1_{G^F}}$ denotes the evaluation map at 1_{G^F}; recall that a symmetrizing form on a finite projective Λ-algebra A is a map $\tau : A \to \Lambda$ such that the map $(a, b) \mapsto \tau(ab)$ is a perfect symmetric bilinear form), and denote its $\overline{\mathbb{Q}}$-linear extension again by τ. It has been shown in [Li, Sec. 2.8] that $\tau(\mathsf{K}_{G^*}) \subset \mathbb{Z}$ and that $\tau|_{\Lambda\mathsf{K}_{G^*}} : \Lambda\mathsf{K}_{G^*} \longrightarrow \Lambda$ is a symmetrizing form. Therefore, the equality $\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*}$ will hold if

$$\tau(h\pi) \in \Lambda \text{ for all } h \in \Lambda\mathsf{E}_G \text{ and } \pi \in \Lambda\mathsf{K}_{G^*}. \tag{0.2}$$

Indeed, (0.2) shows that each of $\Lambda\mathsf{E}_G$ and $\Lambda\mathsf{K}_{G^*}$ is contained in the dual of the other; as each is self-dual, they are equal.

To study (0.2), we will need a family of special elements $\{\pi_\lambda\}$ of K_{G^*} constructed as follows. Let $\mathsf{K}(G^*\text{-mod})$ be the Grothendieck group of the category of finite-$\overline{\mathbb{F}}_q$-dimensional algebraic G^*-modules; it is a ring with multiplication given by tensor product. As shown in [Ja, Ch. II.2], for every maximal torus T^* of G^*, the associated formal character map gives a ring isomorphism

$$\mathrm{ch} : \mathsf{K}(G^*\text{-mod}) \xrightarrow{\sim} \mathbb{Z}[X(T^*)]^W$$

where $X(T^*) = \mathrm{Hom}_{\mathrm{alg}}(T^*, \mathbb{G}_m)$ is the character group of T^* and W is the Weyl group of (G^*, T^*). For $\lambda \in X(T^*)$, set

$$r_\lambda = r_{G,\lambda} := \sum_{\mu \in W\lambda} \mu \in \mathbb{Z}[X(T^*)]^W \quad \text{and} \quad \pi_\lambda = \pi_{G,\lambda} := \mathrm{ch}^{-1}(r_\lambda)|_{G^{*F^*}} \in \mathsf{K}_{G^*},$$

where for $\lambda \in X(T^*)$, $W\lambda$ denotes the W-orbit of λ. Note that the $\mathbb{Z}$-module $\mathbb{Z}[X(T^*)]^W$ is generated by $\{r_\lambda : \lambda \in X(T^*)\}$.

After preparations on Deligne-Lusztig characters and Curtis homomorphisms (Section 1), we will reduce (0.2) to the study of the condition "$\tau(h\pi_\lambda) \in \Lambda$" by fitting G^* into a central extension (Section 2) and studying related compatibility questions (Sections 3 and 4). To study the condition "$\tau(h\pi_\lambda) \in \Lambda$," we will extend the definition of $\tau(h\pi_\lambda)$ to $h \in G^F$ (Section 5), reduce the discussion to the case where the semisimple part s of h is central in G (Section 6), and finally deal with the case of central s (Section 7).

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1. Preliminaries. — In this section, we recall some properties of Deligne–Lusztig characters and Curtis homomorphisms that we will need later on.

Deligne–Lusztig characters. Let S be an F-stable maximal torus of G, let P be a Borel subgroup containing S, and let V be the unipotent radical of P. Then we have the Deligne–Lusztig variety (see [DiMi2, Def. 9.1.1])

$$DL^G_{S\subset P} = \{gV \in G/V : g^{-1}F(g) \in V \cdot F(V)\},$$

which admits a (left) $G^F \times (S^F)^{\mathrm{op}}$-action. When there is no need to specify the chosen Borel subgroup P, we will write $DL^G_{S\subset P}$ simply as DL^G_S.

We consider the virtual ℓ-adic cohomology $H^*_c(\cdot) = \sum_{j\geq 0}(-1)^j H^j_c(\cdot)$, for ℓ a prime distinct from p. For every character $\chi : S^F \longrightarrow \overline{\mathbb{Q}}^\times$, upon choosing a field embedding

$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have the corresponding Deligne–Lusztig character

$$R_S^G(\chi)(-) := \mathrm{Tr}(-|H_c^*(DL_{S\subset P}^G) \otimes_{\overline{\mathbb{Q}}_\ell S^F} \chi) = \frac{1}{|S^F|} \sum_{s\in S^F} \mathrm{Tr}((-,s)|H_c^*(DL_{S\subset P}^G))\chi(s^{-1}),$$

which is independent of the choice of P and which takes values in $\overline{\mathbb{Q}}_\ell$ *à priori*; but by [DeLu, Prop. 3.3], for any $(g,s) \in G^F \times S^F$, the trace $\mathrm{Tr}((g,s)|H_c^*(DL_{S\subset P}^G))$ is an integer independent of ℓ, so in fact $R_S^G(\chi)$ takes values in $\overline{\mathbb{Q}}$, and it can be verified that $R_S^G(\chi)$ is independent of the choices of ℓ and of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

Curtis homomorphisms. For an F-stable maximal torus S of G, we consider the Curtis homomorphism

$$\mathrm{Cur}_S^G : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}S^F$$

defined as in [Li, Sec. 1.11] (see also [Cu, Thm. 4.2]). In terms of the Deligne-Lusztig dual, the map Cur_S^G is simply a "restriction map to a dual torus": indeed, upon fixing an F^*-stable maximal torus S^* of G^* dual to S (whence a duality $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq S^{*F^*}$ and thus a ring isomorphism $\overline{\mathbb{Q}}S^F \simeq \overline{\mathbb{Q}}^{S^{*F^*}}$), the map Cur_S^G is the unique ring homomorphism making the following diagram commutative (see [Li, Lem. 1.13]):

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, "\sim", "(0.1)"'] \arrow[d, "\mathrm{Cur}_S^G"'] & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[d, "\mathrm{Res}"] \\ \overline{\mathbb{Q}}S^F \arrow[r, "\sim"] & \overline{\mathbb{Q}}^{S^{*F^*}} \end{tikzcd} \tag{1.1}$$

We will later need the following formula of Bonnafé–Kessar ([BoKe, Prop. 2.5], with the missing sign factor corrected):

$$\mathrm{Cur}_S^G(h) = \frac{\epsilon_G\epsilon_S}{|S^F|} \sum_{s\in S^F} \mathrm{Tr}((h,s)|H_c^*(DL_{S\subset P}^G))s^{-1} \in \overline{\mathbb{Q}}S^F \quad (h \in \overline{\mathbb{Q}}\mathsf{E}_G \subset \overline{\mathbb{Q}}G^F). \tag{1.2}$$

Here, as usual, $\epsilon_G = (-1)^{\mathrm{rk}_{\mathbb{F}_q}(G)}$ for G any reductive group over $\mathbb{F}_q$. Observe that (1.2) shows that Cur_S^G is independent of the choice of S^*.

2. On central extensions. — For our group G, we can fit its Deligne–Lusztig dual G^* into an F^*-equivariant exact sequence of reductive groups

$$1 \longrightarrow Z^* \longrightarrow H^* \longrightarrow G^* \longrightarrow 1 \tag{2.1}$$

where the derived subgroup of H^* is simply-connected and Z^* is a torus central in H^*.

We fix a choice of F-equivariant exact sequence of reductive groups

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\kappa} Z \longrightarrow 1 \tag{2.2}$$

which is dual to (2.1). Let T_H be an F-stable maximal torus of H, let B_H be a Borel subgroup of H containing T_H, and let V be the unipotent radical of B_H. Then

$$DL^H_{T_H\subset B_H} = \bigsqcup_{z\in Z^F} (DL^H_{T_H\subset B_H})(z)$$

where for each $z \in Z^F$ we have set

$$(DL^H_{T_H\subset B_H})(z) := \{hV \in DL^H_{T_H\subset B_H} : \kappa(h) = z\}.$$

Let $T_G = \ker(\kappa|_{T_H} : T_H \twoheadrightarrow Z)$ (resp. $B_G = \ker(\kappa|_{B_H} : B_H \twoheadrightarrow Z)$), which is an F-stable maximal torus of G (resp. a Borel subgroup of G). Then $T_G \subset B_G$, and the unipotent radical of B_G is also V. As T_G is connected, we have $\kappa(T_H^F) = Z^F$, so for each $z \in Z^F$ we may choose a $\dot{z} \in T_H^F$ such that $\kappa(\dot{z}) = z$. Under the inclusion $G \subset H$, for each $z \in Z^F$ we have

$$DL^H_{T_H\subset B_H}(z) = DL^G_{T_G\subset B_G} \cdot \dot{z} \subset H/V,$$

so that

$$DL^H_{T_H\subset B_H}(z) \simeq DL^G_{T_G\subset B_G} \text{ as } (G^F \times (T_G^F)^{\mathrm{op}})\text{-varieties.}$$

In terms of virtual ℓ-adic cohomology we therefore have

$$H_c^*(DL^H_{T_H\subset B_H}) = \sum_{z\in Z^F} H_c^*(DL^H_{T_H\subset B_H}(z)),$$

and the $H^F \times (T_H^F)^{\mathrm{op}}$-action on $H_c^*(DL^H_{T_H\subset B_H})$ satisfies:

$$\begin{cases} \text{for every } (h,t) \in H^F \times (T_H^F)^{\mathrm{op}}, (h,t) \cdot H_c^*(DL^H_{T_H\subset B_H}(z)) \subset H_c^*(DL^H_{T_H\subset B_H}(\kappa(ht)z)); \\ \text{for every } z \in Z^F,\ H_c^*(DL^H_{T_H\subset B_H}(z)) \simeq H_c^*(DL^G_{T_G\subset B_G}) \text{ as } G^F \times (T_G^F)^{\mathrm{op}}\text{-modules.} \end{cases}$$

In particular, we obtain the following trace formulae: for $(h,t) \in H^F \times (T_H^F)^{\mathrm{op}}$,

$$\begin{cases} \kappa(ht) \neq 1 \Longrightarrow \mathrm{Tr}((h,t)|H_c^*(DL^H_{T_H\subset B_H})) = 0; \\ (h,t) \in G^F \times (T_G^F)^{\mathrm{op}} \Longrightarrow \mathrm{Tr}((h,t)|H_c^*(DL^H_{T_H\subset B_H})) = |Z^F| \cdot \mathrm{Tr}((h,t)|H_c^*(DL^G_{T_G\subset B_G})). \end{cases} \tag{2.3}$$

We will later need the compatibility (for $\chi : T_H^F \longrightarrow \overline{\mathbb{Q}}^\times$):

$$R^H_{T_H}(\chi)|_{G^F} = R^G_{T_G}(\chi|_{T_G^F}). \tag{2.4}$$

This follows immediately from the defining formula of $R^H_{T_H}(\chi)$ and (2.3). (See also [DiMi2, Prop. 11.3.10]).

3. A compatibility lemma. — Notation as in Section 2. We extend the F-stable Borel subgroup B_0 of G in Section 0 (used to determine the Gelfand–Graev module $\Gamma_{G,\psi}$) to the F-stable Borel subgroup B_0' of H, so that $B_0'/B_0 = Z$ under (2.2); the

unipotent radical of B_0' is then equal to U_0 (the unipotent radical of B_0), and the inclusion $G^F \subset H^F$ induced by (2.2) gives rise to a Λ-algebra inclusion

$$\Lambda \mathsf{E}_G = e_\psi \Lambda G^F e_\psi \hookrightarrow e_\psi \Lambda H^F e_\psi = \Lambda \mathsf{E}_H. \tag{3.1}$$

On the other hand, (2.1) yields the identification

$$(G_{\mathrm{ss}}^{*F^*}/\sim) = (H_{\mathrm{ss}}^{*F^*}/\sim)/Z^{*F^*}, \tag{3.2}$$

which enables us to regard functions on $G_{\mathrm{ss}}^{*F^*}/\sim$ as functions on $H_{\mathrm{ss}}^{*F^*}/\sim$ which are constant on each Z^{*F^*}-orbit.

Let us prove the following "compatibility lemma":

Lemma. *The following diagram of rings is commutative:*

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, "\sim", "(0.1)"'] \arrow[d, hook, "(3.1)"'] & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[d, hook, "(3.2)"] \\
\overline{\mathbb{Q}}\mathsf{E}_H \arrow[r, "\sim", "(0.1)"'] & \overline{\mathbb{Q}}^{H_{\mathrm{ss}}^{*F^*}/\sim}
\end{tikzcd} \tag{3.3}$$

Proof. Let T_G and T_H be as in Section 2, and choose a F^*-stable maximal torus T_G^* of G^* dual to T_G (resp. T_H^* of H dual to T_H) such that $T_H^*/Z^* = T_G^*$. Then the Weyl groups of (G, T_G), (G^*, T_G^*), (H, T_H) and (H^*, T_H^*) are all the same, and we denote this common Weyl group by W. For each $w \in W$, choose an F-stable maximal torus $T_{G,w}$ of G whose G^F-conjugacy class corresponds to the F-conjugacy class of w in W (with respect to T_G, so that we may choose $T_{G,1} = T_G$); choose $T_{G,w}^* \subset G^*$, $T_{H,w} \subset H$ and $T_{H,w}^* \subset H^*$ in a similar way.

In the toric case where $(G, H) = (T_G, T_H)$, the commutativity of (3.3) follows from toric dualities.

For the general case of (G, H), we use the Curtis embeddings $\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_{G,w}})_{w \in W}$ and $\mathrm{Cur}^H = (\mathrm{Cur}^H_{T_{H,w}})_{w \in W}$ (see Section 1) to embed (3.3) into the following cubic diagram of rings:

$$\begin{tikzcd}
 & \prod_{w \in W} \overline{\mathbb{Q}} T_{G,w}^F \arrow[rr, "\sim"] \arrow[dd, hook] & & \prod_{w \in W} \overline{\mathbb{Q}}^{T_{G,w}^{*F^*}} \arrow[dd, hook] \\
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[ur, hook, "\mathrm{Cur}^G"] \arrow[rr, "\sim"] \arrow[dd, hook] & & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[ur, hook, "\mathrm{Res}"] \arrow[dd, hook] & \\
 & \prod_{w \in W} \overline{\mathbb{Q}} T_{H,w}^F \arrow[rr, "\sim"] & & \prod_{w \in W} \overline{\mathbb{Q}}^{T_{H,w}^{*F^*}} \\
\overline{\mathbb{Q}}\mathsf{E}_H \arrow[ur, hook, "\mathrm{Cur}^H"] \arrow[rr, "\sim"] & & \overline{\mathbb{Q}}^{H_{\mathrm{ss}}^{*F^*}/\sim} \arrow[ur, hook, "\mathrm{Res}"] &
\end{tikzcd} \tag{3.4}$$

In (3.4), the right face is clearly commutative; the top and the bottom faces are commutative by (1.1); the back face is the toric case of (3.3) and is hence commutative. So to prove the commutativity of (3.3), it remains to show that the left face in (3.4) is commutative.

Using (1.2) and the relation $\epsilon_H \epsilon_{T_{H,w}} = \epsilon_G \epsilon_{T_{G,w}}$, the commutativity of the left face in (3.4) is equivalent to the property that, for all $h \in \overline{\mathbb{Q}}\mathsf{E}_G \subset \overline{\mathbb{Q}}G^F$ and all $w \in W$, we have

$$\frac{1}{|T^F_{H,w}|} \sum_{t \in T^F_{H,w}} \mathrm{Tr}((h,t)|H^*_c(DL^H_{T_{H,w}}))t^{-1} = \frac{1}{|T^F_{G,w}|} \sum_{t \in T^F_{G,w}} \mathrm{Tr}((h,t)|H^*_c(DL^G_{T_{G,w}}))t^{-1}. \quad (3.5)$$

By (2.3) and the fact that $T^F_{H,w}/T^F_{G,w} = Z^F$, we see that (3.5) is true for all $h \in G^F$, so the left face in (3.4) commutes. This completes the proof of the lemma. □

4. Reduction to the study of $\tau(h\pi_\lambda)$. — We keep notations in Section 2. As Z^{*F^*} is central in H^{*F^*}, the association of each irreducible $\overline{\mathbb{F}}_q G^{*F^*}$-module to the restriction to Z^{*F^*} of its central character induces a $\widehat{Z^{*F^*}}$-graded decomposition

$$\mathsf{K}_{H^*} = \bigoplus_{\lambda \in \widehat{Z^{*F^*}}} (\mathsf{K}_{H^*})_\lambda \ \text{ with } \ \mathsf{K}_{G^*} = (\mathsf{K}_{H^*})_0. \quad (4.1)$$

In particular, we have a ring inclusion $\mathsf{K}_{G^*} \subset \mathsf{K}_{H^*}$, and it is evident that the following diagram of rings is commutative (where br denotes the Brauer character map):

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, "\mathrm{br}", "\sim"'] \arrow[d, hook, "(4.1)"'] & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} \arrow[d, hook, "(3.2)"] \\ \overline{\mathbb{Q}}\mathsf{K}_{H^*} \arrow[r, "\mathrm{br}", "\sim"'] & \overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim} \end{tikzcd} \quad (4.2)$$

Recall that we have identified $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ in (0.1). We now establish another "compatibility lemma":

Lemma. *For $h \in \Lambda\mathsf{E}_G \subset \Lambda\mathsf{E}_H$ and $\pi \in \mathsf{K}_{G^*} \subset \mathsf{K}_{H^*}$, we have $\tau_G(h\pi) = \tau_H(h\pi)$.*

Proof. For T_G and T_H as in Section 2, (G, T_G) and (H, T_H) have the same Weyl group W. Let $T_{G,w}$ and $T_{H,w}$ be associated with $w \in W$ as in the proof of (3.3). Let $h \in \Lambda\mathsf{E}_G$ and $\pi \in \mathsf{K}_{G^*}$. By [BoKe, Eq. 3.5], we have:

$$\tau_G(h\pi) = \frac{1}{|W|} \sum_{w \in W} \mathrm{ev}_{1_{T^F_{G,w}}}(\mathrm{Cur}^G_{T_{G,w}}(h\pi));$$
$$\tau_H(h\pi) = \frac{1}{|W|} \sum_{w \in W} \mathrm{ev}_{1_{T^F_{H,w}}}(\mathrm{Cur}^H_{T_{H,w}}(h\pi)).$$

Via the commutative diagrams (3.3) and (4.2), we identify $h\pi$ consistently as an element of $\overline{\mathbb{Q}}\mathsf{E}_G$, $\overline{\mathbb{Q}}\mathsf{E}_H$, $\overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}$, $\overline{\mathbb{Q}}^{H_{\mathrm{ss}}^{*F^*}/\sim}$, $\overline{\mathbb{Q}}\mathsf{K}_{G^*}$ or $\overline{\mathbb{Q}}\mathsf{K}_{H^*}$; we then have $\mathrm{Cur}^G_{T_{G,w}}(h\pi) = (h\pi)|_{T^{*F^*}_{G,w}}$ (see (1.1)). In addition, for an F-stable maximal torus S of G with a chosen dual torus S^* (an F^*-stable maximal torus of G^*), upon denoting by $\widehat{\cdot} : S^{*F^*} \xrightarrow{\sim} \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$ the associated duality and hence identifying $\overline{\mathbb{Q}}^{S^{*F^*}} = \overline{\mathbb{Q}}S^F$:

$$\text{for every } f \in \overline{\mathbb{Q}}^{S^{*F^*}} \text{ and every } t \in S^F,\ \mathrm{ev}_t(f) = \frac{1}{|S^F|}\sum_{s\in S^{*F^*}} f(s)\widehat{s}(t^{-1}). \tag{4.3}$$

Then the following calculation will complete the proof of the lemma:

$$\begin{aligned}
\mathrm{ev}_{1_{T^F_{G,w}}}(\mathrm{Cur}^G_{T_{G,w}}(h\pi)) &= \frac{1}{|T^F_{G,w}|}\sum_{z\in T^{*F^*}_{G,w}} (h\pi)(z) \quad (\text{by } (4.3)) \\
&= \frac{|Z^{*F^*}|}{|T^F_{H,w}|}\sum_{z\in T^{*F^*}_{G,w}} (h\pi)(z) \quad (|T^F_{G,w}| = |T^{*F^*}_{G,w}| = \frac{|T^{*F^*}_{H,w}|}{|Z^{*F^*}|} = \frac{|T^F_{H,w}|}{|Z^{*F^*}|}) \\
&= \frac{1}{|T^F_{H,w}|}\sum_{z\in T^{*F^*}_{H,w}} (h\pi)(z) \quad (h\pi \text{ is constant on every } Z^{*F^*}\text{-orbit}) \\
&= \mathrm{ev}_{1_{T^F_{H,w}}}(\mathrm{Cur}^H_{T_{H,w}}(h\pi)) \qquad (\text{again by } (4.3)).
\end{aligned}$$

□

A reduction. As H^* has simply-connected derived subgroup, the restriction map

$$\mathrm{Res} : \mathsf{K}(H^*\text{-mod}) \longrightarrow \mathsf{K}_{H^*}$$

is surjective ([St, Thm. 7.4] and [Her, Thm. 3.10]), so that the $\mathbb{Z}$-module K_{H^*} is generated by $\{\pi_{H,\lambda} : \lambda \in X(T^*_H)\}$; each $\pi \in \mathsf{K}_{G^*} \subset \mathsf{K}_{H^*}$ is then of the form $\pi = \sum_{j=1}^m n_j\pi_{H,\lambda_j}$ for some $m \in \mathbb{N}^*$, $n_j \in \mathbb{Z}$ and $\lambda_j \in X(T^*_H)$, and then the previous lemma gives

$$\tau_G(h\pi) = \tau_H(h\pi) = \sum_{j=1}^m n_j\tau_H(h\pi_{H,\lambda_j}). \tag{4.4}$$

Formula (4.4) reduces the study of the condition (0.2) to the study of $\tau_H(h\pi_{H,\lambda})$; that is, if we can prove that $\tau_H(h\pi_{H,\lambda}) \in \Lambda$ for every $h \in \Lambda\mathsf{E}_H$ and every $\lambda \in X(T^*_H)$, then we have $\tau_G(h\pi) \in \Lambda$ for every $h \in \Lambda\mathsf{E}_G$ and $\pi \in \Lambda\mathsf{K}_{G^*}$.

Note however that in the study of the condition "$\tau_H(h\pi_{H,\lambda}) \in \Lambda$," we won't need the property that H^* has simply-connected derived subgroup; in the following, we shall thus return to the group G and study the condition "$\tau_G(h\pi_{G,\lambda}) \in \Lambda$."

5. An extension $\widetilde{\tau}$ for $\tau(h\pi)$. — We return to the group G (the derived subgroup of G^* may not be simply-connected) and write $\tau_G = \tau$. Let T be an F-stable maximal

torus of G, let $W = N_G(T)/T$ be the Weyl group of (G,T), and let T_w be an F-stable maximal torus of G associated with $w \in W$ (with respect to T) as in the proof of (3.3). Recall the identification $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ from (0.1). Then, for $h \in \overline{\mathbb{Q}}\mathsf{E}_G$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$:

$$\begin{aligned}
\tau(h\pi) &= \frac{1}{|W|}\sum_{w\in W} \mathrm{ev}_{1_{T_w^F}}(\mathrm{Cur}^G_{T_w}(h\pi)) \quad (\text{by [BoKe, Eq. 3.5]})\\
&= \frac{1}{|W|}\sum_{w\in W} \mathrm{ev}_{1_{T_w^F}}(\mathrm{Cur}^G_{T_w}(h)\cdot \mathrm{Cur}^G_{T_w}(\pi)) \quad (\mathrm{Cur}^G_{T_w} \text{is a ring homomorphism})\\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{t\in T_w^F} \mathrm{Cur}^G_{T_w}(h)(t^{-1})\cdot \mathrm{Cur}^G_{T_w}(\pi)(t)\\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{t\in T_w^F} \frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\cdot \mathrm{Tr}((h,t)|H^*_c(DL^G_{T_w}))\cdot \mathrm{Cur}^G_{T_w}(\pi)(t) \quad (\text{by (1.2)})\\
&= \frac{1}{|W|}\sum_{w\in W} \frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{t\in T_w^F}\sum_{\chi\in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(t)\cdot \mathrm{Cur}^G_{T_w}(\pi)(t) \quad (\text{trace formula})\\
&= \frac{1}{|W|}\sum_{w\in W} \frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{t\in T_w^F}\sum_{\chi\in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(\mathrm{Cur}^G_{T_w}(\pi)). \qquad (5.1)
\end{aligned}$$

Using the formula (5.1), we can extend the $\overline{\mathbb{Q}}$-bilinear map

$$\overline{\mathbb{Q}}\mathsf{E}_G \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}, \quad (h,\pi) \longmapsto \tau(h\pi),$$

to a $\overline{\mathbb{Q}}$-bilinear map $\widetilde{\tau}(\cdot,\cdot) : \overline{\mathbb{Q}}G^F \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ by setting

$$\widetilde{\tau}(h,\pi) := \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{\chi\in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(\mathrm{Cur}^G_{T_w}(\pi)) \quad (h\in\overline{\mathbb{Q}}G^F,\ \pi\in\overline{\mathbb{Q}}\mathsf{K}_{G^*}). \tag{5.2}$$

We then have

$$\tau(h\pi) = \widetilde{\tau}(h,\pi) \text{ for all } h\in\overline{\mathbb{Q}}\mathsf{E}_G \text{ and all } \pi\in\overline{\mathbb{Q}}\mathsf{K}_{G^*}. \tag{5.3}$$

The formula (5.2) for $\widetilde{\tau}$ involves choices of T and T_w; we now derive an intrinsic formula for $\widetilde{\tau}$ as follows.

Let $\mathcal{T}_G$ be the set of F-stable maximal tori of G, and let $\mathcal{T}_G/G^F$ be the set of G^F-conjugacy classes in $\mathcal{T}_G$. For each $S \in \mathcal{T}_G$, let $W_G(S) = N_G(S)/S$. Since the isomorphism class of T_w depends only on the F-twisted conjugacy class of $w \in W$, and the stabiliser of $w \in W$ under F-twisted conjugacy may be identified with $W_G(T_w)^F$, we have that there are $\frac{|W|}{|W_G(S)^F|}$ elements $w \in W$ such that T_w is G^F-conjugate to S. By (5.2), for $h \in \overline{\mathbb{Q}}\mathsf{E}_G$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$, we have:

$$\widetilde{\tau}(h,\pi) = \sum_{S\in\mathcal{T}_G/G^F}\frac{\epsilon_G\epsilon_S}{|W_G(S)^F|}\frac{1}{|S^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R^G_S(\chi)(h)\cdot\chi(\mathrm{Cur}^G_S(\pi))$$

$$= \frac{1}{|G^F|} \sum_{S \in \mathcal{T}_G} \epsilon_G \epsilon_S \sum_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R_S^G(\chi)(h) \cdot \chi(\mathrm{Cur}_S^G(\pi)). \tag{5.4}$$

6. Reduction to the case of central s. — From now on, let $h = su \in G^F$ with $s \in G^F$ (resp. $u \in G^F$) the semisimple (resp. unipotent) part in the Jordan decomposition of h. Recall Deligne–Lusztig's character formula [DeLu, Thm. 4.2] for each F-stable maximal torus S of G: (notation: $\mathrm{ad}(g)x = {}^g x = gxg^{-1}$)

$$R_S^G(\chi)(h) = \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot ({}^g\chi)(s) \tag{6.1}$$

where $Q_S^G = R_S^G(\mathrm{id}) \cdot \mathbf{1}_{G^F_{\mathrm{unip}}}$ denotes the Green function and $C_G(s)^\circ$ is the identity component of the centralizer of s in G.

We shall write $\widetilde{\tau} = \widetilde{\tau}_G$ to specify the group G. Substituting (6.1) into (5.4), we obtain: (below, $\pi \in \mathsf{K}_{G^*}$)

$$\begin{aligned}
\widetilde{\tau}_G(h, \pi) &= \frac{1}{|G^F|} \sum_{S \in \mathcal{T}_G} \epsilon_G \epsilon_S \sum_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot \chi({}^{g^{-1}}s \cdot \mathrm{Cur}_S^G(\pi)) \\
&= \frac{1}{|G^F|} \frac{1}{|C_G(s)^{\circ F}|} \sum_{S \in \mathcal{T}_G} \epsilon_G \epsilon_S |S^F| \sum_{\substack{g \in G^F \\ g^{-1}sg \in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)({}^{g^{-1}}(s^{-1}))
\end{aligned}$$

(where we have applied the orthogonality of characters)

$$= \frac{1}{|G^F|} \frac{1}{|C_G(s)^{\circ F}|} \sum_{g \in G^F} \sum_{\substack{S \in \mathcal{T}_G \\ (\mathrm{ad}(g)S)^F \ni s}} \epsilon_G \epsilon_S |S^F| \cdot Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_{\mathrm{ad}(g)S}^G(\pi)(s^{-1})$$

(where we have used $\mathrm{Cur}_{\mathrm{ad}(g)S}^G(\pi)({}^g x) = \mathrm{Cur}_S^G(\pi)(x)$ for $g \in G^F$)

$$\begin{aligned}
&= \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{S \in \mathcal{T}_G \\ S^F \ni s}} \epsilon_G \epsilon_S |S^F| \cdot Q_S^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)(s^{-1}) \quad (S \longmapsto \mathrm{ad}(g^{-1})S) \\
&= \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{S \in \mathcal{T}_{C_G(s)^\circ} \\ S^F \ni s}} \epsilon_G \epsilon_S |S^F| \cdot Q_S^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)(s^{-1}),
\end{aligned} \tag{6.2}$$

where the last equality holds because for $S \in \mathcal{T}_G$, if S^F contains s then $S \subset C_G(s)^\circ$.

Consider

$$\mathsf{K}_{G^*}^\circ := \{\widetilde{\pi}|_{G^{*F^*}} : \widetilde{\pi} \in \mathsf{K}(G^*\text{-mod})\} \subset \mathsf{K}_{G^*};$$

observe that $\mathsf{K}^\circ_{G^*}$ is generated by $\{\pi_{G,\lambda} : \lambda \in X(T^*)\}$ as a $\mathbb{Z}$-module (see Section 0). Let Λ_0 be a subring of $\overline{\mathbb{Q}}$, and consider the following statement:

$$\widetilde{\tau}_G(h,\pi) \in \Lambda_0 \text{ for all } h = su \in G^F \text{ and all } \pi \in \mathsf{K}^\circ_{G^*}. \tag{6.3}$$

Now we establish the following "reduction lemma" which will reduce the study of (6.3) for general G to the case of central s in G:

Lemma. *Fix a choice of G. Suppose that (6.3) is true whenever G therein is replaced by $C_G(s)^\circ$ with s a semisimple element in G^F. Then (6.3) is true for G.*

Proof. Fix any $h = su \in G^F$, and choose an F-stable maximal torus T of G containing s, so that T is also an F-stable maximal torus of $C_G(s)^\circ$. As $u \in C_G(s)^\circ$ [DiMi, Prop. 1.27], we have $h \in C_G(s)^{\circ F}$. To verify (6.3) for the chosen h, it suffices to show that $\widetilde{\tau}_G(h,\pi_{G,\lambda}) \in \Lambda_0$ for all $\lambda \in X(T^*)$, since $\{\pi_{G,\lambda} : \lambda \in X(T^*)\}$ generates the $\mathbb{Z}$-module $\mathsf{K}^\circ_{G^*}$.

Let $\lambda \in X(T^*)$, and let S be an F-stable maximal torus of G. Choose an F^*-stable maximal torus S^* of G^* dual to S and with a duality $\widehat{\cdot} : S^F \xrightarrow{\sim} \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$, and fix a choice of $g \in G^*$ such that $S^* = {}^gT^*$. This duality and the fixed embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ allow us to identify $\overline{\mathbb{Q}}S^F$ with $\overline{\mathbb{Q}}^{S^{*F^*}}$. For each $\mu \in X(T^*)$, set $\mu_{S^*} = {}^g\mu \in X(S^*)$, and define $\phi_S(\mu) \in S^F$ by the relation $\mu_{S^*}|_{S^{*F^*}} = \widehat{\phi_S(\mu)} \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$. We then have a map $\phi_S : X(T^*) \longrightarrow S^F$ which extends to a ring homomorphism

$$\phi_S : \overline{\mathbb{Q}}[X(T^*)] \longrightarrow \overline{\mathbb{Q}}S^F = \overline{\mathbb{Q}}^{S^{*F^*}}.$$

The following diagram then commutes (where $W = W_{G^*}(T^*) = N_{G^*}(T^*)/T^*$):

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}(G^*\text{-mod}) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \overline{\mathbb{Q}}[X(T^*)]^W \arrow[d, "\phi_S"] \\
\overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, "\mathrm{br}", "\sim"'] & \overline{\mathbb{Q}}^{G^{*F^*}} \arrow[r, "\mathrm{Res}^{G^{*F^*}}_{S^{*F^*}}"] & \overline{\mathbb{Q}}^{S^{*F^*}}
\end{tikzcd}$$

Combining this with (1.1) we see that the following diagram of rings also commutes:

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}(G^*\text{-mod}) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \overline{\mathbb{Q}}[X(T^*)]^W \arrow[ddl, "\phi_S"] \\
\overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[d, equal, "(0.1)"'] & \\
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[d, "\mathrm{Cur}^G_S"'] & \\
\overline{\mathbb{Q}}S^F &
\end{tikzcd} \tag{6.4}$$

The commutative diagram (6.4) gives the relation

$$\mathrm{Cur}_S^G(\pi_{G,\lambda}) = \phi_S(r_{G,\lambda}), \tag{6.5}$$

where we recall that $r_{G,\lambda} = \sum_{\mu \in W_{G^*}(T^*)\lambda} \mu \in \mathbb{Z}[X(T^*)]^{W_{G^*}(T^*)}$. Via the identifications

$$W_{C_G(s)^{\circ *}}(T^*) = W_{C_G(s)^\circ}(T) \leq W_G(T) = W_{G^*}(T^*),$$

we may write $W_{G^*}(T^*)\lambda = \bigsqcup_{\lambda' \in \Omega} W_{C_G(s)^{\circ *}}(T^*)\lambda'$ for some finite subset Ω of $W_{G^*}(T^*)\lambda$, so that $r_{G,\lambda} = \sum_{\lambda' \in \Omega} r_{C_G(s)^\circ,\lambda'}$ and then $\mathrm{Cur}_S^G(\pi_{G,\lambda}) = \sum_{\lambda' \in \Omega} \mathrm{Cur}_S^{C_G(s)^\circ}(\pi_{C_G(s)^\circ,\lambda'})$ by (6.5). Applying (6.2) to $\pi = \pi_{G,\lambda}$, we thus deduce that

$$\widetilde{\tau}_G(h, \pi_{G,\lambda}) = \epsilon_G \epsilon_{C_G(s)^\circ} \sum_{\lambda' \in \Omega} \widetilde{\tau}_{C_G(s)^\circ}(h, \pi_{C_G(s)^\circ,\lambda'}). \tag{6.6}$$

Thanks to (6.6) and the assumption in the lemma, for our choice of $h = su \in G^F$ and T (with $s \in T$), we have $\widetilde{\tau}_G(h, \pi_\lambda) \in \Lambda_0$ for all $\lambda \in X(T^*)$, whence $\widetilde{\tau}_G(h, \pi) \in \Lambda_0$ for all $\pi \in \mathsf{K}^\circ_{G^*}$. As $h \in G^F$ was arbitrary, (6.3) is true for G. □

7. The case of central s. — Keep the notation T, W and T_w as in Section 5. Let $h = su \in G^F$ be as in Section 6, and suppose furthermore that s lies in the centre of G. Then $C_G(s)^\circ = G$, and (6.1) becomes $R_S^G(\chi)(h) = Q_S^G(u)\chi(s)$, so that (5.2) yields

$$\begin{aligned}
\widetilde{\tau}_G(h, \pi_\lambda) &= \frac{1}{|W|} \sum_{w \in W} \frac{\epsilon_G \epsilon_{T_w}}{|T_w^F|} \sum_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} Q_{T_w}^G(u) \cdot \chi(s) \cdot \chi(\mathrm{Cur}_{T_w}^G(\pi_\lambda)) \\
&= \frac{1}{|W|} \sum_{w \in W} \epsilon_G \epsilon_{T_w} Q_{T_w}^G(u) \langle \pi_\lambda|_{T_w^{*F^*}}, \widehat{s^{-1}} \rangle_{T_w^{*F^*}} \quad \text{(orthogonality of characters)} \\
&= \langle \pi_\lambda, \gamma \rangle_{G^{*F^*}}
\end{aligned} \tag{7.1}$$

where (using [DeLu, Prop. 7.3])

$$\gamma := \frac{1}{|W|} \sum_{w \in W} \epsilon_G \epsilon_{T_w} Q_{T_w}^G(u) \mathrm{Ind}_{T_w^{*F^*}}^{G^{*F^*}} \widehat{s^{-1}} = \frac{1}{|W|} \sum_{w \in W} Q_{T_w}^G(u) R_{T_w^*}^{G^*} \widehat{s^{-1}} \otimes \mathrm{St}_{G^*}.$$

As s lies in the centre of G, $\widehat{s^{-1}}$ is in fact a multiplicative character of G^{*F^*}, so

$$\gamma = \gamma' \otimes \widehat{s^{-1}} \otimes \mathrm{St}_{G^*}$$

with

$$\gamma' := \frac{1}{|W|} \sum_{w \in W} Q_{T_w}^G(u) R_{T_w^*}^{G^*}(\mathrm{id}). \tag{7.2}$$

We need some facts from the theory of almost characters, following [Lu, Ch. 3-4]; in the notation of that book, we are considering the case $n = 1$ and L trivial. See also [Ca, Sec. 7.3] for a concise exposition, but with some extraneous hypotheses. Let c be the order of the automorphism F on W (when G is split, we have $c = 1$); denote by $\widehat{W}_{\mathrm{ex}}$ the set of all $\phi \in \mathrm{Irr}_{\mathbb{Q}}(W)$ which can be extended to a $\mathbb{Q}$-valued irreducible character of $\widehat{W} := W \rtimes \langle F \rangle$ (by [Sp2, Cor. 1.15], every irreducible representation of W over a characteristic 0 field is defined over $\mathbb{Q}$); for each $\phi \in \widehat{W}_{\mathrm{ex}}$, there exists such an extension (in fact, exactly two) $\widetilde{\phi} \in \mathrm{Irr}_{\mathbb{Q}}(\widehat{W})$. Fixing a choice of such $\tilde{\phi}$, we then call

$$R_{\widetilde{\phi}}^{G^*} := \frac{1}{|W|} \sum_{w \in W} \widetilde{\phi}(wF) R_{T_w^*}^{G^*}(\mathrm{id})$$

an *almost character* of G^{*F^*}.

Recall from Section 0 the definition of bad primes for G. Note that a prime is bad for G if and only if it is bad for G^*. Define

$$M_G = \text{product of all bad primes for } G. \tag{7.3}$$

Using Lusztig's work on unipotent characters,

$$\begin{aligned} &\text{each almost character } R_{\widetilde{\phi}}^{G^*} \text{ is a } \overline{\mathbb{Z}}[\tfrac{1}{M_G}]\text{-linear combination} \\ &\text{of irreducible } \overline{\mathbb{Q}}\text{-valued unipotent characters of } G^{*F^*}. \end{aligned} \tag{7.4}$$

Indeed, if G^* has connected centre, then [Lu, Thm. 4.23] expresses $R_{\widetilde{\phi}}$ as a linear combination of unipotent characters of G^{*F^*}. By [Lu, (4.21.7)], the denominators divide the orders of certain groups $\mathcal{G}_{\mathcal{F}}$ of the form $\prod \mathcal{G}_{\mathcal{F}_i}$ where the product is over the irreducible factors of the root system of G^*. Each $\mathcal{G}_{\mathcal{F}_i}$ is defined in a case-by-case fashion, in a way depending only on the corresponding irreducible factor of the root system, in [Lu, 4.4–4.13], and has order divisible only by bad primes for that factor. If G^* does not have connected centre then we choose a short exact sequence

$$1 \to G^* \to H^* \to Z^* \to 1$$

as in (2.2) (with the roles of G^* and G reversed). Extending the chosen maximal F^*-stable torus and Borel from G^* to H^* as in Section 2, we may identify the Weyl groups of G^* and H^*. Using (2.4) (with $\chi = \mathbf{1}$ therein), we then have

$$R_{\widetilde{\phi}}^{H^*}|_{G^{*F^*}} = R_{\widetilde{\phi}}^{G^*},$$

whence $R_{\widetilde{\phi}}^{G^*}$ is $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of restrictions to G^{*F^*} of unipotent characters of H^{*F^*}. However, the restriction to G^{*F^*} of a unipotent character of H^{*F^*} is a unipotent character by [DiMi2, Prop. 11.3.8], so (7.4) follows.

Now we prove the following lemma:

Lemma. *The character γ' in (7.2) is a finite $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of almost characters of G^{*F^*}.*

Proof. We have

$$\begin{aligned}\gamma' &= \frac{1}{|W|}\sum_{w\in W} R^G_{T_w}(\mathrm{id})(u)R^{G^*}_{T^*_w}(\mathrm{id})\\ &= \frac{1}{|W|}\sum_{w\in W} R^G_{T_w}(\mathrm{id})(u)\sum_{\phi\in\widehat{W}_{\mathrm{ex}}}\widetilde{\phi}(wF)R^{G^*}_{\widetilde{\phi}} \quad \text{(see [Ca, p. 76])}\\ &= \sum_{\phi\in\widehat{W}_{\mathrm{ex}}} R^G_{\widetilde{\phi}}(u)R^{G^*}_{\widetilde{\phi}};\end{aligned}$$

by (7.4) and the fact that character values of representations of finite groups are algebraic integers, all $R^G_{\widetilde{\phi}}$ must take values in $\overline{\mathbb{Z}}[\frac{1}{M_G}]$. □

Remark. In the above lemma, the character γ' in (7.2) can indeed be written as a finite $\overline{\mathbb{Z}}$-linear combination of almost characters of G^{*F^*}; to see this, one will need to use a theorem of Shoji ([Sh, Thm. 5.5]; see also [DiMi2, Thm. 13.2.3]) which says that $Q^G_{T_w}(u) = \mathrm{Tr}(wF|H^*_c(\mathcal{B}_u))$ for all $w \in W$, where $\mathcal{B}_u$ is the variety of Borel subgroups of G containing u. We won't need this stronger property of γ' later and so we omit its proof here.

Using the previous lemma, (7.1), (7.2) and (7.4), we get the following proposition:

Proposition. *We have $\widetilde{\tau}_G(h,\pi_\lambda) \in \overline{\mathbb{Z}}[\frac{1}{M_G}]$ for all $\lambda \in X(T^*)$ and all $h \in G^F$ whose semisimple part s is central in G. (M_G is as in (7.3).)*

End of proof of the main theorem in Section 0. From now on, we remove the assumption that s is central in G.

Observe that a prime number that is bad for $C_G(x)^\circ$ with x a semisimple element of G^F is also bad for G; indeed, this follows from the definition of bad primes in Section 0 and from the following two facts: (i) if G is simple of type A (resp. of classical type), then the centralizer of every semisimple element of G has only factors of type A (resp. of classical type); (ii) if G is simple of type G_2, F_4, E_6 or E_7, then the centralizer of every semisimple element of G cannot have factors of type E_8 (for dimensional reasons).

Therefore, the previous proposition and the lemma in Section 6 together imply that $\widetilde{\tau}_G(h,\pi_\lambda) \in \overline{\mathbb{Z}}[\frac{1}{M_G}]$ for all $h \in G^F$ and all $\lambda \in X(T^*)$. We then deduce from (5.3) that

$$\tau_G(h\pi_\lambda) = \widetilde{\tau}_G(h,\pi_\lambda) \in \Lambda[\tfrac{1}{M_G}] \text{ for all } h \in \Lambda\mathsf{E}_G \text{ and all } \lambda \in X(T^*). \tag{7.5}$$

Now fit G into the exact sequence (2.2); as H therein has the same type of root datum as G, we have $M_H = M_G$, so (7.5) applied to H gives $\tau_H(h\pi_{H,\lambda}) \in \Lambda[\frac{1}{M_G}]$ for all $h \in \Lambda\mathsf{E}_H$ and all $\lambda \in X(T^*_H)$; for our G, (4.4) then tells us that (0.2) is true when Λ therein is replaced by $\Lambda[\frac{1}{M_G}]$. Consequently, when all bad prime numbers for G are invertible in Λ, we have $\Lambda[\frac{1}{M_G}] = \Lambda$ and $\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*}$. □

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Tzu-Jan Li: Institut de Mathématiques de Jussieu-Paris Rive Gauche (IMJ-PRG), 4 place Jussieu, 75252 Paris cedex 05, FRANCE

Email address: `tzu-jan.li@imj-prg.fr`

Jack Shotton: Department of Mathematical Sciences, Mathematical Sciences & Computer Science Building, Durham University Upper Mountjoy Campus, Stockton Road, Durham DH1 3LE UNITED KINGDOM

Email address: `jack.g.shotton@durham.ac.uk`