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# A FULLY WELL-BALANCED LAGRANGE-PROJECTION TYPE SCHEME FOR THE SHALLOW-WATER EQUATIONS\*

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**Abstract.** This work focuses on the numerical approximation of the Shallow Water Equations (SWE) using a Lagrange-Projection type approach. We propose a fully well-balanced explicit and positive scheme using relevant reconstruction operators. By fully well-balanced, it is meant that the scheme is able to preserve stationary smooth solutions of the model with non zero velocity, including of course also the well-known lake at rest equilibrium. Numerous numerical experiments illustrate the good behaviour of the scheme.

**Key words.** Shallow water equations, Finite Volume Method, Lagrange-Projection scheme

**AMS subject classifications.** 74S10, 35L60, 35L65, 74G15

**1. Introduction.** We are interested in the numerical approximation of the well-known Shallow Water Equations (SWE), given by

$$(1) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\frac{h^2}{2}\right) = -gh\partial_x z, \end{cases}$$

where  $z(x)$  denotes a given smooth topography and  $g > 0$  is the gravity constant. The primitive variables are the water depth  $h \geq 0$  and its velocity  $u$ , which both depend on the space and time variables, respectively  $x \in \mathbb{R}$  and  $t \in [0, \infty)$ . At time  $t = 0$ , we assume that the initial water depth  $h(x, t = 0) = h_0(x)$  and velocity  $u(x, t = 0) = u_0(x)$  are given. In order to shorten the notations, we will use the following condensed form of (1), namely

$$(2) \quad \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}, z),$$

where  $\mathbf{U} = (h, hu)^T$ ,  $\mathbf{F}(\mathbf{U}) = (hu, hu^2 + gh^2/2)^T$  and  $\mathbf{S}(\mathbf{U}, z) = (0, -gh\partial_x z)^T$ . In case of necessity and since the topography is assumed to be constant in time in this work, we will also make use of the following equivalent form of (1), namely

$$(3) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\frac{h^2}{2}\right) + gh\partial_x z = 0, \\ \partial_t z = 0, \end{cases}$$

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27 and  $\mathbf{V} = (h, hu, z)^T$ . At last, recall that (1) is strictly hyperbolic over the phase space  
 28  $\Omega = \{(h, hu)^T \in \mathbb{R}^2 \mid h > 0\}$  and that the eigenstructure is composed by two genuinely  
 29 non linear characteristic fields associated with the eigenvalues  $\{u - c, u + c\}$  where the  
 30 sound speed is defined by  $c = \sqrt{gh}$ . We recall also that the regions where  $u^2 < c^2$  (resp.  
 31  $u^2 > c^2$ ) are called subcritical (resp. supercritical).

32 As already stated, we are especially interested in the design of a numerical scheme  
 33 satisfying the so-called fully well-balanced property, meaning that it should be able to  
 34 preserve discrete approximations of the smooth stationary solutions of (1). These steady  
 35 states are governed by the ordinary differential system  $\partial_x F(\mathbf{U}) = \mathbf{S}(\mathbf{U}, z)$ , namely

$$36 \quad (4) \quad hu = \text{constant}, \quad \frac{u^2}{2} + g(h + z) = \text{constant}.$$

37 Recall that the well-known "lake at rest" solution corresponds to

$$38 \quad (5) \quad h + z = \text{constant}, \quad u = 0,$$

39 and that numerical schemes preserving these particular stationary solutions are said  
 40 to be well-balanced (see for instance the recent book [26] for a review). Here we are  
 41 also interested in preserving equilibriums (4) with a non-zero velocity, leading to the  
 42 so-called fully well-balanced property.

43 In order to reach this objective, we propose to use a strategy based on a Lagrange-  
 44 Projection (or equivalently Lagrange-Remap) decomposition, which allows to naturally  
 45 decouple the acoustic and transport terms of the model. Such a decomposition proved  
 46 to be useful and very efficient to deal with subsonic or near low-Froude number flows.  
 47 In this case, the usual CFL time step limitation of Godunov-type schemes is indeed  
 48 driven by the acoustic waves and can thus be very restrictive. The Lagrange-Projection  
 49 strategy allows to design a very natural implicit-explicit and large time-step schemes  
 50 with a CFL restriction based on the (slow) transport waves and not on the (fast) acoustic  
 51 waves. In this paper, we will stay focused on explicit schemes but the reader is referred  
 52 to the pioneering work [20] and the more recent ones [14], [15], [16], [17] for more details.  
 53 Note that the large time step Lagrange-Projection scheme proposed in [17] for the SWE  
 54 is well-balanced, but not fully well-balanced.

55 There is a huge amount of works about the design of numerical schemes for the  
 56 SWE, but most of them intend to satisfy the well-balanced property, but not the full  
 57 well-balanced property. To mention only a few of them, we refer for instance the reader  
 58 to the following well-known contributions [3], [27], [30], [22], [23], [1], [19], [6], [7], [2].  
 59 We also refer to the books [9] and [26] as well as [29] and [13] which provide additional  
 60 references and overviews. As far as the full well-balanced property is concerned, let us  
 61 quote for instance, [25], [34], [36], [11], [10], [35], [33], [4], [5], [12], [28], [31].

62 The outline of the paper is as follows. In Section 2 we give a general introduction  
 63 of the Lagrange-Projection approach in the case of flat topography. This technique will  
 64 be generalized in Section 3 for the case of arbitrary topography in order to develop a  
 65 fully well-balanced scheme. Finally, in Section 4, some numerical results will be shown  
 66 in order to test this new scheme.

67 **2. The Lagrange-Projection decomposition.** The aim of this section is to  
 68 briefly present the Lagrange-Projection splitting strategy considered in this paper and  
 69 to introduce the basics of the numerical approximation.

70 Let us begin with the Lagrange-Projection decomposition. By using the chain rule  
 71 for the space derivatives, one first writes system (1) under the following equivalent form

72 for smooth solutions, namely

$$73 \quad \begin{cases} \partial_t h + u \partial_x h + h \partial_x u = 0 \\ \partial_t(hu) + u \partial_x(hu) + hu \partial_x u + \partial_x(g \frac{h^2}{2}) = -gh \partial_x z. \end{cases}$$

74 Then, the splitting simply consists in first solving the Lagrangian form of this system,  
75 namely

$$76 \quad (6) \quad \begin{cases} \partial_t h + h \partial_x u = 0, \\ \partial_t(hu) + hu \partial_x u + \partial_x(g \frac{h^2}{2}) = -gh \partial_x z, \end{cases}$$

which gives in Lagrangian coordinates, with  $\tau = 1/h$  and  $\tau \partial_x = \partial_m$ ,

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m(g \frac{h^2}{2}) = -gh \partial_m z, \end{cases}$$

77 and then the transport system

$$78 \quad (7) \quad \begin{cases} \partial_t h + u \partial_x h = 0, \\ \partial_t(hu) + u \partial_x(hu) = 0. \end{cases}$$

79 *Remark 2.1.* This splitting procedure can be interpreted as a two-step algorithm  
80 consisting in solving first the system in Lagrangian coordinates and then projecting the  
81 results in the Eulerian coordinates. See [24] for further details.

82 **2.1. The Lagrange-Projection numerical algorithm.** Space and time will be  
83 discretized using a space step  $\Delta x$  and a time step  $\Delta t$  into a set of cells  $[x_{j-1/2}, x_{j+1/2})$   
84 and instants  $t^{n+1} = n\Delta t$ , where  $x_{j+1/2} = j\Delta x$  and  $x_j = (x_{j-1/2} + x_{j+1/2})/2$  are  
85 respectively the cell interfaces and cell centers, for  $j \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . For a given initial  
86 condition  $x \mapsto \mathbf{U}^0(x)$ , we will consider a discrete initial data  $\mathbf{U}_j^0$  which approximates  
87  $\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}^0(x) dx$ , for  $j \in \mathbb{Z}$ . Therefore, the proposed algorithm aims at computing  
88 an approximation  $\mathbf{U}_j^n$  of  $\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^n) dx$  where  $x \rightarrow \mathbf{U}(x, t^n)$  is the exact solution  
89 of (1) at all time  $t^n$ ,  $n \in \mathbb{N}$ . Given the sequence  $\{\mathbf{U}_j^n\}_j$ , it is a matter of defining the  
90 sequence  $\{\mathbf{U}_j^{n+1}\}_j$ ,  $n \in \mathbb{N}$ , since  $\{\mathbf{U}_j^0\}_j$  is assumed to be known.

91 Using these notations, the overall Lagrange-Projection algorithm can be described  
92 as follows: for a given discrete state  $\mathbf{U}_j^n = (h, hu)_j^n$ ,  $j \in \mathbb{Z}$  that describes the system at  
93 instant  $t^n$ , the computation of the approximation  $\mathbf{U}_j^{n+1} = (h, hu)_j^{n+1}$  at the next time  
94 level is a two-step process defined by

- 95 1. Update  $\mathbf{U}_j^n$  to  $\mathbf{U}_j^{n+1-}$  by approximating the solution of (6),
- 96 2. Update  $\mathbf{U}_j^{n+1-}$  to  $\mathbf{U}_j^{n+1}$  by approximating the solution of (7).

97 **2.2. The case of a flat topography.** In order to prepare the forthcoming devel-  
98 opments, we first consider the simple situation of a flat topography, which corresponds  
99 to the case  $\partial_x z = 0$ . In this case and using standard notations, the Lagrangian step  
100 writes (see [24])

$$\begin{cases} \tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{h_j^n \Delta x} (u_{j+1/2}^* - u_{j-1/2}^*), \\ u_j^{n+1-} = u_j^n - \frac{\Delta t}{h_j^n \Delta x} (p_{j+1/2}^* - p_{j-1/2}^*), \end{cases}$$

with

$$u_{j+1/2}^* = u^*(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) := \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a_{j+1/2}}(p_{j+1}^n - p_j^n),$$

$$p_{j+1/2}^* = p^*(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) := \frac{1}{2}(p_j^n + p_{j+1}^n) - \frac{a_{j+1/2}}{2}(u_{j+1}^n - u_j^n),$$

97 and  $p_j^n = g(h_j^n)^2/2$  for all  $j$ , or equivalently

$$98 \quad (8) \quad \begin{cases} L_j^n h_j^{n+1-} = h_j^n, \\ L_j^n (hu)_j^{n+1-} = (hu)_j^n - \frac{\Delta t}{\Delta x}(p_{j+1/2}^* - p_{j-1/2}^*), \end{cases}$$

with

$$L_j^n = 1 + \frac{\Delta t}{\Delta x}(u_{j+1/2}^* - u_{j-1/2}^*).$$

Here, the constant  $a_{j+1/2}$  has to be chosen sufficiently large for the sake of stability, and more precisely larger than the Lagrangian sound speed  $hc$  according to the well-known subcharacteristic condition. In practice it is required that  $a_{i+1/2}$  is greater than the values  $h\sqrt{gh}$  at the interface. The definition of  $a_{j+1/2}$  will be given later on and we also refer for instance the reader to [9, 24, 14, 16] for more details. Moreover, the Lagrangian step is stable under the Courant-Friedrichs-Lewy (CFL) condition

$$\Delta t \max_j \{ \max(\tau_j^n, \tau_{j+1}^n) a_{j+1/2} \} \leq \frac{1}{2} \Delta x.$$

As far as the the transport step is concerned, an upwind scheme is applied where the value  $u_{j+1/2}^*$  is the velocity at the interface  $x_{j+1/2}$ .

$$\begin{cases} h_j^{n+1} = h_j^{n+1-} - \frac{\Delta t}{\Delta x}(u_{j+1/2}^{*,-}(h_{j+1}^{n+1-} - h_j^{n+1-}) + u_{j-1/2}^{*,+}(h_j^{n+1-} - h_{j-1}^{n+1-})), \\ (hu)_j^{n+1} = (hu)_j^{n+1-} - \frac{\Delta t}{\Delta x}(u_{j+1/2}^{*,-}((hu)_{j+1}^{n+1-} - (hu)_j^{n+1-}) + u_{j-1/2}^{*,+}((hu)_j^{n+1-} - (hu)_{j-1}^{n+1-})), \end{cases}$$

99 where  $u_{j\pm 1/2}^{*,+} = \max(u_{j\pm 1/2}^*, 0)$  and  $u_{j\pm 1/2}^{*,-} = \min(u_{j\pm 1/2}^*, 0)$  for all  $j$ .

The CFL condition associated with the transport step reads

$$\Delta t \max_j \{ (u_{j-1/2})^+ - (u_{j+1/2})^- \} \leq \Delta x.$$

Easy calculations show that the system can be written in the equivalent form

$$\begin{cases} h_j^{n+1} = L_j^n h_j^{n+1-} - \frac{\Delta t}{\Delta x}(h_{j+1/2}^{n+1-} u_{j+1/2}^* - h_{j-1/2}^{n+1-} u_{j-1/2}^*), \\ (hu)_j^{n+1} = L_j^n (hu)_j^{n+1-} - \frac{\Delta t}{\Delta x}((hu)_{j+1/2}^{n+1-} u_{j+1/2}^* - (hu)_{j-1/2}^{n+1-} u_{j-1/2}^*), \end{cases}$$

where

$$h_{j+1/2}^{n+1-} = \begin{cases} h_j^{n+1-} & \text{if } u_{j+1/2}^* \geq 0, \\ h_{j+1}^{n+1-} & \text{if } u_{j+1/2}^* \leq 0, \end{cases} \quad (hu)_{j+1/2}^{n+1-} = \begin{cases} (hu)_j^{n+1-} & \text{if } u_{j+1/2}^* \geq 0, \\ (hu)_{j+1}^{n+1-} & \text{if } u_{j+1/2}^* \leq 0, \end{cases}$$

which gives that the whole scheme is conservative in the usual sense of finite volume methods and writes

$$\begin{cases} h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}(h_{j+1/2}^{n+1-} u_{j+1/2}^* - h_{j-1/2}^{n+1-} u_{j-1/2}^*), \\ (hu)_j^{n+1} = (hu)_j^n - \frac{\Delta t}{\Delta x}((hu)_{j+1/2}^{n+1-} u_{j+1/2}^* + p_{j+1/2}^* - (hu)_{j-1/2}^{n+1-} u_{j-1/2}^* - p_{j-1/2}^*). \end{cases}$$

100 These formulas will be useful in the next sections to deal with a generalization of  
 101 this scheme to the case of a non flat topography and impose the fully well-balanced  
 102 property.

**2.3. An equivalent formulation.** In this section, we propose an equivalent formulation of the previous scheme based on the Reynolds' transport theorem which can be expressed for any scalar value  $X$  as

$$\frac{d}{dt} \int_{\Omega(t)} X(x, t) dV = \int_{\Omega(t)} \left( \frac{\partial}{\partial t} X(x, t) + \nabla \cdot (X(x, t)v(x, t)) \right) dV,$$

where  $dV$  is a volume element at point  $x$  (the integration variable), and  $v(x, t)$  is the velocity at which the domain  $\Omega(t)$  is moving. In the one dimension case considered in this paper, and considering that  $\Omega(t)$  is an interval with limits  $a(t)$  and  $b(t)$ , the Reynolds's theorem reduces to

$$\frac{d}{dt} \int_{a(t)}^{b(t)} X(x, t) dx = \int_{a(t)}^{b(t)} \left( \frac{\partial}{\partial t} X(x, t) + \partial_x (X(x, t)v(x, t)) \right) dx.$$

103 This theorem expresses the time variation of the amount of  $X$  contained inside the  
 104 volume  $\Omega(t)$  which moves at velocity  $v(x, t)$ .

105

106 *The Lagrangian coordinates.* Considering that the velocity  $v(x, t)$  equals the fluid velocity  
 107  $u(x, t)$  in the SWE, the Reynolds' theorem allows to describe the fluid motion  
 108 in the Lagrangian coordinates, assuming that the observer moves with the fluid flow.  
 109 Considering now successively that  $X = 1, h, hu$  and denoting by  $V(t)$  the infinitesimal  
 110 control volume, we get

111 (9) 
$$\frac{d}{dt} V(t) = V(t) \partial_x u(x(t; \xi), t),$$

112

113 (10) 
$$\frac{d}{dt} \left( V(t) h(x(t; \xi), t) \right) = 0,$$

$$\frac{d}{dt} \left( V(t) (hu)(x(t; \xi), t) \right) = -V(t) \partial_x p(x(t; \xi), t),$$

where each point  $x(t; \xi)$  of the control volume is given by the solution of the following ordinary differential equation, with initial condition  $\xi$ ,

$$\begin{cases} \partial_t x(t; \xi) = u(x(t; \xi), t) \\ x(0; \xi) = \xi. \end{cases}$$

Note that with the above notations, and  $\xi_1 = a(0)$  and  $\xi_2 = b(0)$ ,

$$V(t) = \int_{x(t; \xi_1)}^{x(t; \xi_2)} 1 dx = \int_{\xi_1}^{\xi_2} \partial_\xi x(t; \xi) d\xi$$

so that for an infinitesimal control volume one can write

$$V(t) = V(0) \partial_\xi x(t; \xi).$$

If we now denote  $\bar{\alpha}(\xi, t) = \alpha(x(t; \xi), t)$  for any scalar quantity  $\alpha$ , the previous relations also write

$$\frac{d}{dt} V(t) = V(0) \partial_\xi \bar{u}(\xi, t),$$

$$\frac{d}{dt} \left( V(t) \bar{h}(\xi, t) \right) = 0,$$

114

$$(11) \quad \frac{d}{dt} \left( V(t) \bar{hu}(\xi, t) \right) = -V(0) \partial_\xi \bar{p}(\xi, t).$$

116 *The Lagrangian step.* Let us now turn back to the numerical setting and consider for  
 117 a given  $j$  that the control volume is defined at time  $t = 0$  by the cell  $(x_{j-1/2}, x_{j+1/2})$ .  
 118 The endpoints  $x_\pm(t; x_{j\pm 1/2})$  of the control volume at a given time  $t$  are thus defined by  
 119 the solutions of the following ordinary differential equations

$$(12) \quad \begin{cases} \partial_t x_\pm(t; x_{j\pm 1/2}) = u(x_\pm(t; x_{j\pm 1/2}), t) \\ x_\pm(0; x_{j\pm 1/2}) = x_{j\pm 1/2}. \end{cases}$$

121 Denoting  $h_j^{n+1-}$  and  $(hu)_j^{n+1-}$  constant approximate values of  $h$  and  $(hu)$  at time  $t^{n+1}$   
 122 on the moving domain whose volume is  $V_j(\Delta t)$ , a simple way to discretize (9) would be

$$(13) \quad \begin{cases} V_j(\Delta t) h_j^{n+1-} = V(0) h_j^n, \\ V_j(\Delta t) (hu)_j^{n+1-} = V(0) (hu)_j^n - V(0) \frac{\Delta t}{\Delta x} (p_{j+1/2}^* - p_{j-1/2}^*), \end{cases}$$

124 with  $V(0) = \Delta x$  and according to  $dV(t)/dt = V(t) \partial_x u(x(t; \xi), t)$ ,

$$(14) \quad V_j(\Delta t) = V(0) + \int_0^{\Delta t} V(t) \partial_x u(x(t; \xi), t) dt \approx V(0) + \Delta t (u_{j+1/2}^* - u_{j-1/2}^*) = \Delta x L_j^n.$$

Therefore, one clearly sees that the discretizations (13) and (14) of (9) (or equivalently (11)) allow to recover (8). At last, note now that from (12),

$$\int_0^{\Delta t} \partial_t x_\pm(t; x_{j\pm 1/2}) dt = \int_0^{\Delta t} u(x_\pm(t; x_{j\pm 1/2}), t) dt$$

so that

$$x_\pm(\Delta t; x_{j\pm 1/2}) - x_\pm(0; x_{j\pm 1/2}) \approx \Delta t u_{j\pm 1/2}^*,$$

and by subtraction we also have

$$V_j(\Delta t) := x_+(\Delta t; x_{j+1/2}) - x_-(\Delta t; x_{j-1/2}) = V(0) + \Delta t (u_{j+1/2}^* - u_{j-1/2}^*) = \Delta x L_j^n,$$

and therefore

$$L_j^n = 1 + \frac{\Delta t}{\Delta x} (u_{j+1/2}^* - u_{j-1/2}^*) = \frac{x_+(\Delta t; x_{j+1/2}) - x_-(\Delta t; x_{j-1/2})}{\Delta x}.$$

*The transport step.* Let us now turn to the transport or remap step. In the framework of the present equivalent reformulation of our algorithm, we have now at hand some constant approximations denoted  $h_j^{n+1-}$  and  $(hu)_j^{n+1-}$  of  $h$  and  $(hu)$  at time  $t^{n+1}$  on the new cells  $(x_-(\Delta t; x_{j-1/2}), x_+(\Delta t; x_{j+1/2}))$  and whose volume is  $V_j(\Delta t)$ . In this context, the aim of the remap step is just to project these values on the original mesh, which simply writes with clear notations and  $X = h, hu$ ,

$$X_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} X_j^{n+1-}(x) dx,$$

or equivalently

$$X_j^{n+1} = \frac{x_-(\Delta t; x_{j-1/2}) - x_{j-1/2}}{\Delta x} X_{j-1/2}^{n+1-} + \frac{x_+(\Delta t; x_{j+1/2}) - x_-(\Delta t; x_{j-1/2})}{\Delta x} X_j^{n+1-} + \frac{x_{j+1/2} - x_+(\Delta t; x_{j+1/2})}{\Delta x} X_{j+1/2}^{n+1-}$$

where we have set for all  $j$

$$X_{j+1/2}^{n+1-} = \begin{cases} X_j^{n+1-} & \text{if } u_{j+1/2}^* \geq 0, \\ X_{j+1}^{n+1-} & \text{if } u_{j+1/2}^* < 0. \end{cases}$$

126 It is easy to check that these final values coincide with the ones of the previous formu-  
127 lation of the scheme.

**3. The Lagrange-Projection scheme with a non flat topography.** Now we would like to extend this approach to the case of a non flat bottom paying a particular attention to the well-balanced property. Here the Lagrange-Projection approach aims at solving successively the Lagrangian form of our system, namely

$$\begin{cases} \partial_t h + h \partial_x u = 0, \\ \partial_t (hu) + hu \partial_x u + \partial_x \left( g \frac{h^2}{2} \right) + gh \partial_x z = 0, \end{cases}$$

which gives in Lagrangian coordinates, with  $\tau = 1/h$ ,

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \left( g \frac{h^2}{2} \right) + gh \partial_m z = 0, \end{cases}$$

and the transport system

$$\begin{cases} \partial_t h + u \partial_x h = 0, \\ \partial_t (hu) + u \partial_x (hu) = 0. \end{cases}$$

128

129 At this stage, it is important to have in mind that in order to get the fully well-balanced  
130 property, we are going to consider in-cell reconstructions in both Lagrangian and Trans-  
131 port steps. Moreover, these reconstructions will make both steps of the strategy to be  
132 fully well-balanced for  $(hu)$ , in the sense that if the solution at time  $t^n$  is an equilibrium  
133 state, we will have  $(hu)_j^{n+1} = (hu)_j^{n+1-} = (hu)_j^n$  for all  $j$ . This is clearly sufficient but  
134 not necessary as the fully well-balanced property simply asks for  $(hu)_j^{n+1} = (hu)_j^n$  for  
135 all  $j$ . As far as  $h$  is concerned, we will have  $h_j^{n+1-} \neq h_j^n$  in general, but the overall  
136 algorithm will be fully well-balanced, namely  $h_j^{n+1} = h_j^n$  for all  $j$ . Let us now describe  
137 the two steps in details.

138 **3.1. The Lagrangian step..** Considering first the Lagrangian step, following the  
139 ideas used in the flat topography case, we propose the following generalization

$$140 \quad (15) \quad \begin{cases} L_j^n h_j^{n+1-} = h_j^n, \\ L_j^n (hu)_j^{n+1-} = (hu)_j^n - \frac{\Delta t}{\Delta x} (p_{j+1/2}^* - p_{j-1/2}^*) - \Delta t \{gh \partial_x z\}_j, \end{cases}$$

with

$$L_j^n = 1 + \frac{\Delta t}{\Delta x} (u_{j+1/2}^* - u_{j-1/2}^*),$$

and where

$$\{gh\partial_x z\}_j \approx \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} gh(x, t^n) \partial_x z(x, t^n) dx.$$

At this stage, it remains to define  $u_{j+1/2}^*$ ,  $p_{j+1/2}^*$  and  $\{gh\partial_x z\}_j$  for all  $j$  in such a way that the fully well-balanced property holds true for this step.

*Reconstruction procedure.* In order to get this property, we propose to perform in-cell reconstructions of the variables at time  $t^n$ , that is to say just before starting the Lagrangian step. With this in mind, we suggest to understand the cell values  $h_j^n$  and  $(hu)_j^n$  as the averages of stationary solutions defined in the corresponding cell  $C_j$ . Thus, we aim at defining the functions  $x \rightarrow h(x; K_{1,j}, K_{2,j})$  and  $x \rightarrow u(x; K_{1,j}, K_{2,j})$  such that

$$\begin{cases} (hu)(x; K_{1,j}, K_{2,j}) = K_{1,j}, \\ \frac{u(x; K_{1,j}, K_{2,j})^2}{2} + g(h(x; K_{1,j}, K_{2,j}) + z(x)) = K_{2,j}, \end{cases}$$

with two constants  $K_{1,j}$  and  $K_{2,j}$  such that

$$h_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} h(x; K_{1,j}, K_{2,j}) dx$$

and

$$(hu)_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} (hu)(x; K_{1,j}, K_{2,j}) dx = K_{1,j}.$$

141 In practice, and since we are only interested in a first order scheme, it is more conve-  
 142 nient to approximate the above integrals by the mid-point quadrature formula, which  
 143 allows to identify the cell averages  $h_j^n$  (respectively  $(hu)_j^n$ ) and the mid-point values  
 144  $h(x_j; K_{1,j}, K_{2,j})$  (resp.  $x \rightarrow u(x_j; K_{1,j}, K_{2,j})$ ). We are thus led to define the functions  
 145  $x \rightarrow h(x; K_{1,j}, K_{2,j})$  and  $x \rightarrow u(x; K_{1,j}, K_{2,j})$  by

$$146 \quad (16) \quad \begin{cases} (hu)(x; K_{1,j}, K_{2,j}) = (hu)_j^n, \\ \frac{u(x; K_{1,j}, K_{2,j})^2}{2} + g(h(x; K_{1,j}, K_{2,j}) + z(x)) = \frac{(u_j^n)^2}{2} + g(h_j^n + z_j), \end{cases}$$

$$z_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} z(x) dx$$

for all  $j$ . Using the  $\mathbf{U}$  notation, we have thus been able to reconstruct smooth stationary solutions  $x \rightarrow \mathbf{U}(x; K_{1,j}, K_{2,j})$  of the SWE within each cell and the average of which coincides with the solution at time  $t^n$  for the conservative variables and up to second-order accuracy. These stationary solutions are then used to define left and right extrapolated values at the mesh interfaces by setting for all  $j$

$$\mathbf{U}_{j+1/2-} = \mathbf{U}(x_{j+1/2}; K_{1,j}, K_{2,j}), \quad \mathbf{U}_{j-1/2+} = \mathbf{U}(x_{j-1/2}; K_{1,j}, K_{2,j}).$$

147 Remark that, to close the definition of the reconstructed values, the bottom topography  
 148 should be defined at the interface. Following the ideas in [1] and [11], let us consider  
 149 a reconstructed bottom at  $z_{j+1/2}$ , consistent with  $z(x_{j+1/2})$ , whose precise definition  
 150 will be discussed later. Thanks to (16) and using the notation  $q = hu$ , we consider the  
 151 reconstructed states  $h_{j+1/2\pm}$ ,  $q_{j+1/2\pm}$  given by

$$152 \quad (17) \quad \begin{cases} q_{j+1/2+}^n = q_{j+1}^n, \\ \frac{(q_{j+1}^n)^2}{2(h_{j+1/2+}^n)^2} + g(h_{j+1/2+}^n + z_{j+1/2}) = \frac{(q_{j+1}^n)^2}{2(h_{j+1}^n)^2} + g(h_{j+1}^n + z_{j+1}), \end{cases}$$

153 and

$$154 \quad (18) \quad \begin{cases} q_{j+1/2-}^n = q_j^n, \\ \frac{(q_j^n)^2}{2(h_{j+1/2-}^n)^2} + g(h_{j+1/2-}^n + z_{j+1/2}) = \frac{(q_j^n)^2}{2(h_j^n)^2} + g(h_j^n + z_j). \end{cases}$$

155 It is important to notice that the values  $h_{i+1/2\pm}$  may not be uniquely defined or not  
 156 defined at all. Let us recall that this idea of using in-cell reconstruction using equilibria  
 157 has been used, for instance, in [28, 10, 11]. In particular, let us consider a similar  
 158 notation and the following proposition in [10]:

159 **PROPOSITION 3.1.** *Let  $u_0 \in \mathbb{R}, h_0 \geq 0, \delta \in \mathbb{R}$  and define  $q_0 = h_0 u_0$ . Consider the*  
 160 *system*

$$161 \quad (19) \quad \begin{cases} \frac{q_0^2}{2(h^*)^2} + gh^* = \frac{q_0^2}{2h_0^2} + g(h_0 + \delta), \\ h^* u^* = q_0, \\ h^* \geq 0, u^* \in \mathbb{R}, \end{cases}$$

162 and denote  $h_s \equiv h_s(q_0) = \left(\frac{q_0^2}{g}\right)^{\frac{1}{3}}$ . There exists a solution  $(h^*, u^*)$  to (19) if and only  
 163 if

$$164 \quad (20) \quad \delta + \delta_0(h_0, q_0) \geq 0,$$

165 where

$$166 \quad \delta_0(h_0, q_0) = g^{-1} \left( \frac{q_0^2}{2h_0^2} + gh_0 - \frac{3}{2}(g|q_0|)^{\frac{2}{3}} \right) \in \mathbb{R}^+.$$

167 Moreover,

168 i) If we have equality in (20), there is only one solution  $(h^*, u^*)$  to (19), it is given by

$$169 \quad (21) \quad h^* = h_s, \quad u^* = \begin{cases} \frac{q_0}{h_s} & \text{if } q_0 \neq 0, \\ 0 & \text{if } q_0 = 0. \end{cases}$$

170 ii) If we have a strict inequality in (20), then there are exactly two different solutions  
 171  $(h_{sup}^*, u_{sup}^*)$  and  $(h_{sub}^*, u_{sub}^*)$  to (19), with  $h_{sup}^* \leq h_s < h_{sub}^*$ , and  $h_{sup}^* < h_s$  for  
 172  $q_0 \neq 0$ .

173 iii) A solution  $(h^*, u^*)$  to (19), with  $h^* u^* \neq 0$  satisfies:

- 174 • It is a critical point  $((u^*)^2 = gh^*)$  if, and only if,  $h^* = h_s$ ,
- 175 • It is a subcritical point  $((u^*)^2 < gh^*)$  if, and only if,  $h^* > h_s$ ,
- 176 • It is a supercritical point  $((u^*)^2 > gh^*)$  if, and only if,  $h^* < h_s$ .

177 iv) Assume that  $\delta$  is such that (20) is satisfied strictly so that there are exactly two  
 178 solutions  $h_{sub}$  and  $h_{sup}$  for (19).

- 179 a) When  $(h_0, u_0)$  is subcritical, we have  
 180 a.1) if  $\delta > 0$ , then  $h_{sup}^* < h_0 < h_{sub}^*$ ,  
 181 a.2) if  $\delta \leq 0$ , then  $h_{sup}^* < h_{sub}^* \leq h_0$ ,  
 182 b) When  $(h_0, u_0)$  is supercritical, we have  
 183 b.1) if  $\delta \geq 0$ , then  $h_{sup}^* \leq h_0 < h_{sub}^*$ ,  
 184 b.2) if  $\delta < 0$ , then  $h_0 \leq h_{sup}^* < h_{sub}^*$ ,  
 185 c) When  $(h_0, u_0)$  is critical, we have  $h_{sup}^* \leq h_0 < h_{sub}^*$ .

186

□

187 *Remark 3.2.* Based on this result, solving equations (17) and (18) means:

- 188 a) When  $\Delta z = z_{j+1} - z_j \rightarrow 0$ , we can assure the existence of the reconstructed states.  
 189 This means that when  $U_j$  and  $U_{j+1}$  are not critical points ( $\delta_0(h_j, q_j) > 0$  and  
 190  $\delta_0(h_{j+1}, q_{j+1}) > 0$ ) and the bottom is continuous, we can define the reconstructed  
 191 values provided that the spatial resolution is high enough. Nevertheless, some prob-  
 192 lems may arise when we are near a critical point at the discrete level.  
 193 b) When the reconstructed states exist, we have in general two possible choices: a sub-  
 194 critical and supercritical value. We will select in general the solution that preserves  
 195 the same character, that is, for  $U_j$  subcritical (resp. supercritical) we choose  $U_{j+1/2-}$   
 196 and  $U_{j-1/2+}$  subcritical (resp. supercritical). This is true far from sonic points and  
 197 for  $\Delta z$  sufficiently small.  
 198 c) In the general case, the choice of  $z_{j+1/2}$  should be taken with care. We require that

$$199 \quad z_j - z_{j+1/2} \geq -\delta_0(h_j, q_j) \text{ and } z_{j+1} - z_{j+1/2} \geq -\delta_0(h_{j+1}, q_{j+1})$$

200 so that the reconstructed states exist. Moreover, we shall impose that the recon-  
 201 structed states verify

$$202 \quad h_{j+1/2-} \leq h_j \text{ or } h_{j+1/2-} \leq h_s(q_j),$$

203 and

$$204 \quad h_{j+1/2+} \leq h_{j+1} \text{ or } h_{j+1/2+} \leq h_s(q_{j+1}),$$

205 which will play an essential role for the positivity of the scheme.

206 Based on those conditions we propose the following definition:

- 207 i) Case  $z_{j+1} \geq z_j$ :  
 208 • Case  $z_{j+1} \leq z_j + \delta_0(h_j, q_j)$ :  
 209 We set  $z_{j+1/2} = z_{j+1}$ ,  $h_{j+1/2+} = h_{j+1}$  and  $h_{j+1/2-}$  the solution of (18) with  
 210 the same character (subcritical or supercritical) as  $h_j$ .  
 211 • Case  $z_{j+1} > z_j + \delta_0(h_j, q_j)$ :  
 212 We set  $z_{j+1/2} = z_j + \delta_0(h_j, q_j)$ ,  $h_{j+1/2-} = h_s(h_j, q_j)$  and  $h_{j+1/2+}$  the su-  
 213 percritical solution of (17).  
 214 ii) Case  $z_{j+1} < z_j$ :  
 215 • Case  $z_j \leq z_{j+1} + \delta_0(h_{j+1}, q_{j+1})$ :  
 216 We set  $z_{j+1/2} = z_j$ ,  $h_{j+1/2-} = h_j$  and  $h_{j+1/2+}$  the solution of (17) with the  
 217 same character (subcritical or supercritical) as  $h_{j+1}$ .  
 218 • Case  $z_j > z_{j+1} + \delta_0(h_{j+1}, q_{j+1})$ :  
 219 We set  $z_{j+1/2} = z_{j+1} + \delta_0(h_{j+1}, q_{j+1})$ ,  $h_{j+1/2+} = h_s(h_{j+1}, q_{j+1})$  and  $h_{j+1/2-}$   
 220 the supercritical solution of (18).  
 d) Although the previous definitions have been written for a general case, remark that  
 in most situations it becomes simpler: consider continuous bottom and assume a

spatial discretization sufficiently small, such that

$$z_{j+1} - z_j < \delta_0(h_j, q_j), \text{ for } z_{j+1} \geq z_j$$

and

$$z_j - z_{j+1} < \delta_0(h_{j+1}, q_{j+1}), \text{ for } z_j > z_{j+1}.$$

221 Then, we get  $z_{j+1/2} = \max(z_j, z_{j+1})$  and the values  $h_{j+1/2+}$  and  $h_{j+1/2-}$  correspond  
 222 to the solutions of (17) and (18) with the same character (subcritical or supercritical)  
 223 as  $h_{j+1}$  and  $h_j$  respectively.

224 To conclude this paragraph, notice that the proposed reconstruction procedure can  
 225 be understood as a generalization of the well-known hydrostatic reconstruction associ-  
 226 ated with the lake at rest equilibrium and defined by

$$227 \quad (22) \quad \begin{cases} u(x; K_{1,j}, K_{2,j}) = u_j^n, \\ h(x; K_{1,j}, K_{2,j}) + z(x) = h_j^n + z_j, \end{cases}$$

see [1] for more details.

Now that the reconstruction is defined, we use the left and right traces  $\mathbf{U}_{j+1/2\pm}$  to calculate the numerical fluxes  $u_{j+1/2}^*$  and  $p_{j+1/2}^*$  by simply setting

$$u_{j+1/2}^* = u^*(\mathbf{U}_{j+1/2-}, \mathbf{U}_{j+1/2+}) \quad \text{and} \quad p_{j+1/2}^* = p^*(\mathbf{U}_{j+1/2-}, \mathbf{U}_{j+1/2+}),$$

228 and with

$$229 \quad (23) \quad a_{i+1/2} = \max \left\{ h_{i+1/2+}^n \sqrt{gh_{i+1/2+}^n}, h_{i+1/2-}^n \sqrt{h_{i+1/2-}^n} \right\}.$$

Let us now define  $\{gh\partial_x z\}_j$  such that

$$\{gh\partial_x z\}_j \approx \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} gh(x, t^n) \partial_x z(x) dx.$$

Since we have reconstructed stationary solutions in each cell, we clearly have for all  $j$

$$\partial_x (hu^2 + g \frac{h^2}{2})(x; K_{1,j}, K_{2,j}) = -(gh\partial_x z)(x; K_{1,j}, K_{2,j}),$$

which, by integrating in the interior of the cell, gives

$$\begin{aligned} -\{gh\partial_x z\}_j &= \frac{1}{\Delta x} (p(x_{j+1/2}; K_{1,j}, K_{2,j}) - p(x_{j-1/2}; K_{1,j}, K_{2,j})) + \\ &\frac{1}{\Delta x} (hu^2(x_{j+1/2}; K_{1,j}, K_{2,j}) - hu^2(x_{j-1/2}; K_{1,j}, K_{2,j})), \end{aligned}$$

that is to say

$$-\{gh\partial_x z\}_j = \frac{p_{j+1/2-}^n - p_{j-1/2+}^n}{\Delta x} + \frac{(hu^2)_{j+1/2-}^n - (hu^2)_{j-1/2+}^n}{\Delta x},$$

where of course  $p_{j+1/2-}^n = g(h_{j+1/2-}^n)^2/2$  and  $p_{j-1/2+}^n = g(h_{j-1/2+}^n)^2/2$ . Since,  $(hu)_{j+1/2-}^n =$  (hu)\_{j-1/2+}^n, we finally get

$$-\{gh\partial_x z\}_j = \frac{p_{j+1/2-}^n - p_{j-1/2+}^n}{\Delta x} + (hu)_j^n \frac{u_{j+1/2-}^n - u_{j-1/2+}^n}{\Delta x}.$$

230 Remark that the definition of  $\{gh\partial_x z\}_j$  is related to the integral of the source term  
 231 along the path that corresponds to the integral curves defining stationary states. We  
 232 refer the reader to [29] and [13] for further details.  
 233

**3.2. The Transport step.** Considering now the Transport step, we propose first to rewrite equation

$$\partial_t X + u\partial_x X = 0$$

in following way

$$\partial_t X + \partial_x(Xu) - X\partial_x u = 0,$$

with  $X = h, (hu)$ . Now we suggest the following update formulas

$$X_j^{n+1} = X_j^{n+1-} - \frac{\Delta t}{\Delta x} (X_{j+1/2}^{n+1-} u_{j+1/2}^* - X_{j-1/2}^{n+1-} u_{j-1/2}^*) + X_j^{n+1-} \frac{\Delta t}{\Delta x} (u_{j+1/2}^* - u_{j-1/2}^*),$$

234 or equivalently

$$235 \quad (24) \quad X_j^{n+1} = L_j^n X_j^{n+1-} - \frac{\Delta t}{\Delta x} (X_{j+1/2}^{n+1-} u_{j+1/2}^* - X_{j-1/2}^{n+1-} u_{j-1/2}^*),$$

236 where the  $u_{j+1/2}^*$  have the same definition as in the Lagrangian step, and where the  
 237 values  $X_{j+1/2}^{n+1-}$  denote relevant approximations of  $X = h, (hu)$  at the interface  $x_{j+1/2}$  at  
 238 time  $t^{n+1-}$ . Let us notice first that the whole scheme is actually conservative since easy  
 239 calculations show that it writes

$$240 \quad (25) \quad \begin{cases} h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} (h_{j+1/2}^{n+1-} u_{j+1/2}^* - h_{j-1/2}^{n+1-} u_{j-1/2}^*), \\ (hu)_j^{n+1} = (hu)_j^n - \frac{\Delta t}{\Delta x} ((hu)_{j+1/2}^{n+1-} u_{j+1/2}^* + p_{j+1/2}^* - (hu)_{j-1/2}^{n+1-} u_{j-1/2}^* - p_{j-1/2}^*) \\ \quad - \Delta t \{gh\partial_x z\}_j. \end{cases}$$

241 Finally, the values  $X_{j+1/2}^{n+1-}$  have to be defined for  $X = h, (hu)$  in order to get the ex-  
 242 pected well-balanced property. Let us now address this issue.

243

If one first considers the definition of  $h_{j+1/2}^{n+1-}$ , it is a matter of defining an approxi-  
 mation of  $h$  at the interface  $x_{j+1/2}$  and at time  $t^{n+1-}$ . Mimicking the definition used  
 in the case of a flat topography, namely

$$h_{j+1/2}^{n+1-} = \begin{cases} h_j^{n+1-} & \text{if } u_{j+1/2}^* \geq 0, \\ h_{j+1}^{n+1-} & \text{if } u_{j+1/2}^* \leq 0, \end{cases}$$

and the formula  $L_j^n h_j^{n+1-} = h_j^n$  which was used to define an approximation of  $h$  on the  
 $j$ -th cell and at time  $t^{n+1-}$ , we are tempted to set

$$h_{j+1/2}^{n+1-} = \begin{cases} h_{j+1/2-}^{n+1-} & \text{if } u_{j+1/2}^* \geq 0, \\ h_{j+1/2+}^{n+1-} & \text{if } u_{j+1/2}^* \leq 0, \end{cases}$$

244 where the values  $h_{j+1/2\pm}^{n+1-}$  have to be defined.

245

246 *Definition of  $h_{j+1/2\pm}^{n+1-}$ .* Here we propose to define

$$247 \quad (26) \quad h_{j+1/2\pm}^{n+1-} = h_{j+1/2\pm}^n$$

248 which is a simple way to account for the reconstruction procedure introduced in the  
 249 Lagrangian step. It will also guarantee the fully well-balanced property.  
 250 Note that following the same idea introduced in Section 2.3, we may interpret again  
 251 the transport step as the projection of the values in Lagrangian coordinates into the  
 252 Eulerian coordinates. The reconstructed states  $h_{j+1/2\pm}^{n+1-}$  can thus be seen as the values  
 253 at time  $t^{n+1}$  to the left and to the right of  $x_{j+1/2}^* = x(\Delta t; x_{j+1/2})$ , the position at time  
 254  $t^{n+1}$  of the moving Lagrangian intercell. Here we propose  $h_{j+1/2\pm}^n$  as an approximation  
 255 of those values.

256  
 257

Regarding now the definition of  $(hu)_{j+1/2}^{n+1-}$ , it is a matter of defining again a suitable approximation of  $(hu)$  at time  $t^{n+1-}$ . Here, we propose to keep the same definition used in the case of a flat topography, namely

$$(hu)_{j+1/2}^{n+1-} = \begin{cases} (hu)_j^{n+1-} & \text{if } u_{j+1/2}^* \geq 0, \\ (hu)_{j+1}^{n+1-} & \text{if } u_{j+1/2}^* \leq 0, \end{cases}$$

258 which will be sufficient to get the fully well-balanced property, as we show now.

**3.3. Positivity and fully well-balanced property.** In this section, we prove that the Lagrangian-Projection scheme defined by (15)-(24) is fully well-balanced and that it preserves positivity of water height.

*Well-balanced property.* We shall check that the scheme defined previously preserves all smooth stationary states. Let us assume that the solution at time  $t^n$  corresponds to a general steady state of the SWE, that is to say

$$(hu)_j^n = K_1 \quad \text{and} \quad \frac{(u_j^n)^2}{2} + g(h_j^n + z_j) = K_2, \quad \forall j \in \mathbb{Z},$$

where  $K_1$  and  $K_2$  are two given constants. The reconstruction procedure gives  $\mathbf{U}_{j+1/2-} = \mathbf{U}_{j+1/2+}$  and in particular

$$u_{j+1/2}^* = u_{j+1/2\pm}^n =: u_{j+1/2}^n \quad \text{and} \quad p_{j+1/2}^* = p_{j+1/2\pm}^n =: p_{j+1/2}^n.$$

As a consequence, the Lagrangian step (15) gives

$$\begin{cases} L_j^n h_j^{n+1-} = h_j^n, \\ L_j^n (hu)_j^{n+1-} = (hu)_j^n - \frac{\Delta t}{\Delta x} (p_{j+1/2}^n - p_{j-1/2}^n) - \Delta t \{gh\partial_x z\}_j, \end{cases}$$

with

$$L_j^n = 1 + \frac{\Delta t}{\Delta x} (u_{j+1/2}^n - u_{j-1/2}^n),$$

$$-\{gh\partial_x z\}_j = \frac{p_{j+1/2}^n - p_{j-1/2}^n}{\Delta x} + (hu)_j^n \frac{u_{j+1/2}^n - u_{j-1/2}^n}{\Delta x},$$

that is to say

$$\begin{cases} L_j^n h_j^{n+1-} = h_j^n, \\ L_j^n (hu)_j^{n+1-} = L_j^n (hu)_j^n. \end{cases}$$

Under the usual CFL restriction, we thus have  $(hu)_j^{n+1-} = (hu)_j^n$  so that the Lagrangian step is fully well-balanced for  $(hu)$ . Note that in general,  $h_j^{n+1-} \neq h_j^n$ , except if  $u_{j+1/2} =$

$u_{j-1/2}$  gets true, which happens in the particular case of the lake at rest steady state ( $u_j^n = 0$  for all  $j$ ).

We also have

$$h_{j+1/2-}^{n+1-} = h_{j+1/2-}^n = h_{j+1/2+}^n = h_{j+1/2+}^{n+1-}.$$

As a consequence, (25) gives

$$\begin{cases} h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} ((hu)_j^n - (hu)_j^n), \\ (hu)_j^{n+1} = (hu)_j^n - \frac{\Delta t}{\Delta x} ((hu)_j^n u_{j+1/2}^n + p_{j+1/2}^n - (hu)_j^n u_{j-1/2}^n - p_{j-1/2}^n) \\ \quad - \Delta t \{gh \partial_x z\}_j, \end{cases}$$

259 Therefore, the scheme is thus fully well-balanced since  $h_j^{n+1} = h_j^n$  and  $(hu)_j^{n+1} = (hu)_j^n$   
260 for all  $j$ .

261

262 *Positivity property.* It is clear that the Lagrangian step (15) preserves positivity of water  
263 thickness provided that  $L_j^n > 0$ , which will be true under a suitable CFL condition. The  
264 question is then whether the transport step preserves positivity or not. Following the  
265 ideas presented in [1], given a numerical scheme for water height in the form

$$266 \quad h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_{j+1/2}^h - \mathcal{F}_{j-1/2}^h),$$

the scheme will preserve the positivity of  $h$  if whenever  $h_j^n = 0$ , one has

$$\mathcal{F}_{j+1/2}^h - \mathcal{F}_{j-1/2}^h \leq 0.$$

267 The fully well-balanced Lagrange-Projection for variable  $h$  writes

$$268 \quad h_j^{n+1} = h_j^{n+1-} - \frac{\Delta t}{\Delta x} (u_{j+1/2}^* (h_{j+1/2}^{n+1-} - h_j^{n+1-}) + u_{j-1/2}^* (h_j^{n+1-} - h_{j-1/2}^{n+1-})).$$

Now, definitions given in Remark 3.2. c) grants that when  $U_j^n$  is a subcritical state, then

$$\max(h_{j+1/2-}^{n+1-}, h_{j-1/2+}^{n+1-}) \leq h_j^n.$$

Conversely, when  $U_j^n$  is supercritical, we have

$$\max(h_{j+1/2-}^{n+1-}, h_{j-1/2+}^{n+1-}) \leq h_s(q_j^n).$$

It follows then that whenever  $h_j^n = 0$  we get

$$h_j^{n+1-} = 0, \quad h_{j+1/2-}^{n+1-} = 0, \quad \text{and} \quad h_{j-1/2-}^{n+1-} = 0.$$

269 In that case, we get

$$270 \quad h_j^{n+1} = -\frac{\Delta t}{\Delta x} (u_{j+1/2}^* h_{j+1/2}^{n+1-} - u_{j-1/2}^* h_{j-1/2}^{n+1-}).$$

271 Now, it is easy to check that in this case

$$272 \quad u_{j+1/2}^* h_{j+1/2}^{n+1-} = (u_{j+1/2}^*)^- h_{j+1/2+}^{n+1-} \leq 0,$$

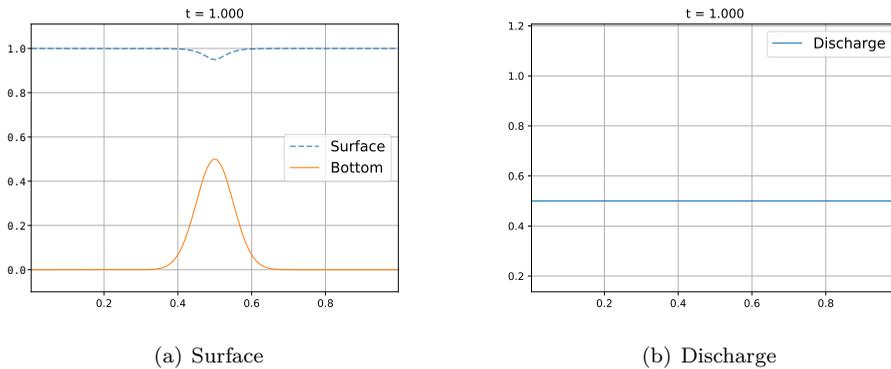


FIGURE 1. *Subcritical steady state*

273 and

274 
$$u_{j-1/2}^* h_{j-1/2}^{n+1-} = (u_{j-1/2}^*)^+ h_{j-1/2+}^{n+1-} \geq 0,$$

275 and the result follows.

276

□

277 **4. Numerical results.**

278 **4.1. Subcritical steady state.** In this first test we will verify the fully well bal-  
 279 anced property of the scheme. To do so, let us consider a bottom topography

280 (27) 
$$z(x) = 0.5 \exp\left(-\frac{(x - 0.5)^2}{0.005}\right),$$

281 and the subcritical steady state solution verifying

282 
$$hu = 0.5, \quad \frac{u^2}{2} + g(h + z) = 0.125.$$

283 The simulation is carried out with 200 cells and open boundary conditions. As  
 284 expected, this steady state is preserved by the scheme with an error  $10^{-12}$ . Figure 1  
 285 shows the surface elevation as well as the discharge at time  $t = 1$  which corresponds  
 286 just to the given initial condition.

287 **4.2. Generation of subcritical steady state.** The objective of this test is to  
 288 study the convergence of the scheme to steady states. To do so let us consider the  
 289 bottom topography (27) and we consider as initial condition a lake at rest situation:  
 290  $h + z = 1$ ,  $u = 0$ . We impose at the left boundary the condition  $hu(x = 0, t) = 0.5$  and  
 291 at the right boundary the condition  $h(x = 1, t) = 1$ . It is expected that, for long time,  
 292 the solution will converge to the steady state presented previously. The simulation  
 293 is carried out with 200 cells and the time evolution of the surface and discharge are  
 294 shown in Figure 2. Here a comparison with the solutions computed with 800 cells are  
 295 also shown. Figure 3 shows the steady state reached for  $t = 50$  and comparison with  
 296 the exact reference solution. Moreover,  $10^{-12}$ . Figure 4 shows the deviation from the  
 297 expected constant values for  $q$  and  $\frac{u^2}{2} + g(h + z)$  at the steady state. We see that the

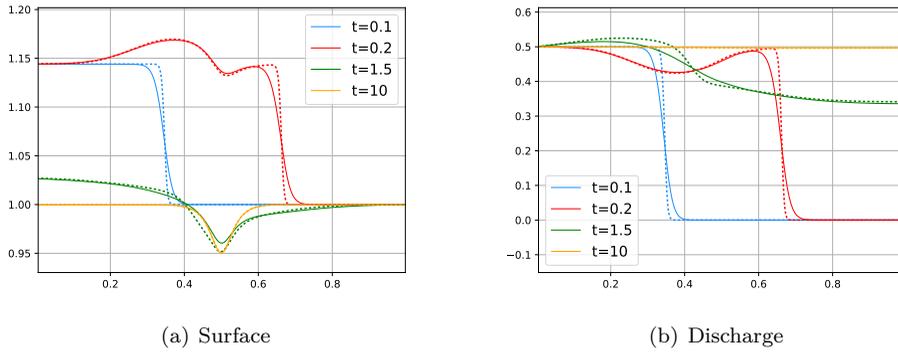


FIGURE 2. Generation of subcritical steady state from stationary initial condition: surface and discharge evolution. Continuous lines correspond to the solution computed with 200 cells and discontinuous to the one computed with 800 cells

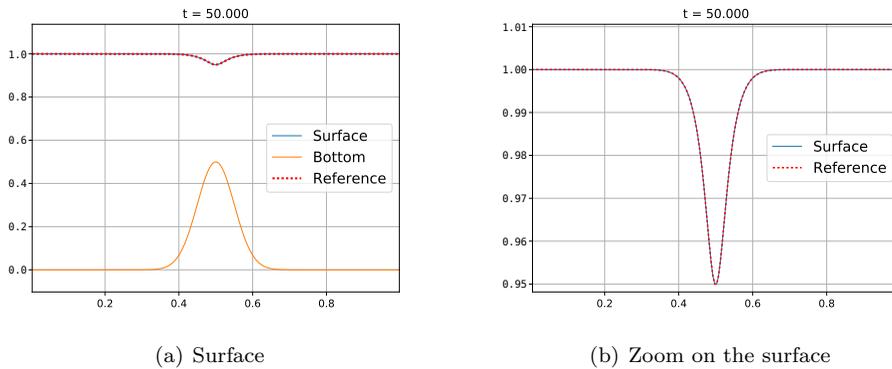


FIGURE 3. Generation of subcritical steady state from stationary initial condition: surface elevation compared to reference solution

298 achieved error is similar to the previous case, order  $10^{-12}$  for the first one and  $10^{-12}$   
 299 for the second.

300 **4.3. Transcritical continuous steady state.** As it is known, the generalized  
 301 hydrostatic reconstruction procedure may present some problems at critical points where  
 302 the reconstructed states may not be well defined. We propose now to take as initial  
 303 condition a transcritical steady state to check how this situation is handled by the  
 304 scheme. We consider again the bottom topography (27) and we solve numerically the  
 305 relations

306 
$$hu = 1.5, \quad \frac{u^2}{2} + g(h + z) = E,$$

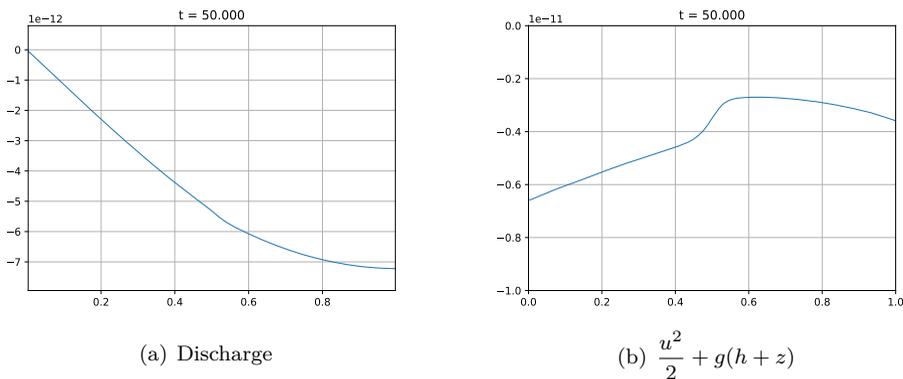


FIGURE 4. *Generation of subcritical steady state from stationary initial condition: deviation from the expected constant state*

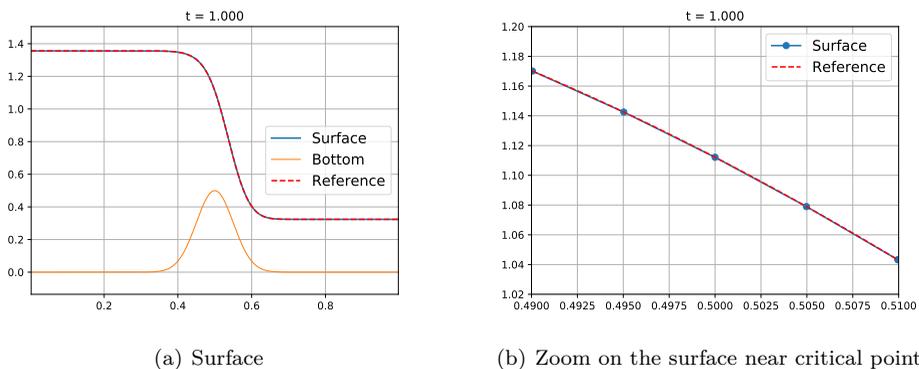


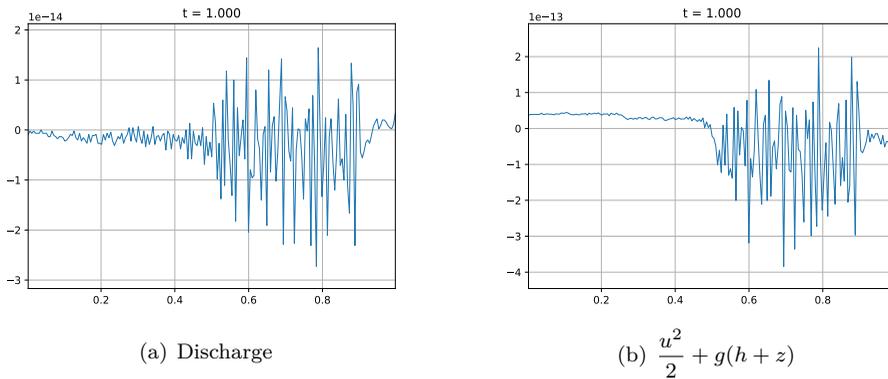
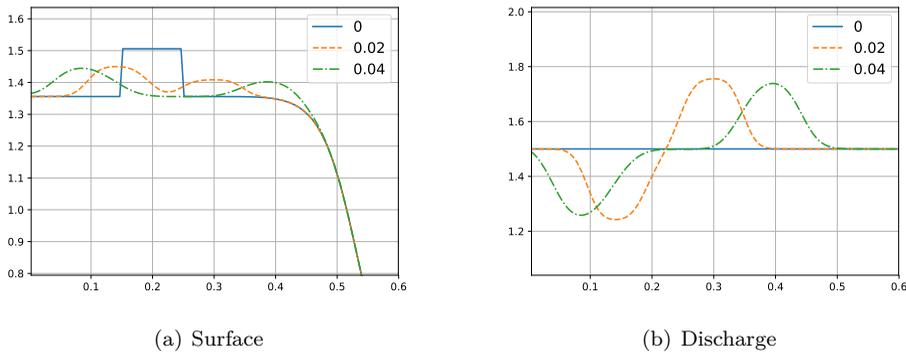
FIGURE 5. *Transcritical steady state: surface elevation*

307 with

308 
$$E = \frac{1.5}{2h_s^2} + g(h_s + 0.5), \quad \text{and} \quad h_s = \left(\frac{1.5^2}{g}\right)^{1/3}.$$

309 We consider the subcritical solution for  $x < 0.5$  and the supercritical solution for  $x > 0.5$ .  
 310 Previous values guarantee that these solutions exist and there is a critical point at  
 311  $x = 0.5$ .

312 Figure 5 shows the surface elevation for time  $t = 1$  using 201 cells. We remark that  
 313 the scheme behaves well at  $x = 0.5$  where the critical point is located. Let us recall that  
 314 critical points may be problematic for the reconstruction using equilibria. This is due  
 315 to the fact that at discrete level we represent the exact smooth functions by a piece-wise  
 316 constant function consisting on the cell averages which gives in practice a jump on the  
 317 bottom at the interfaces. Nevertheless we check in Figure 5(b) that the scheme captures  
 318 correctly the solution. Figure 6 shows the deviation from the expected constant values  
 319 for  $q$  and  $\frac{u^2}{2} + g(h+z)$  at the steady state. We see that the error is of order  $10^{-14}$  for  
 320 the first one and  $10^{-13}$  for the second which shows that the steady state is preserved.

FIGURE 6. *Transcritical steady state: deviation from the expected constant state*FIGURE 7. *Perturbation of a steady state at times  $t = 0, 0.02$  and  $0.04$* 

321 **4.4. Small perturbation of a steady state.** We consider now a small pertur-  
 322 bation of previous transcritical steady state. To do so, let us consider the same bottom  
 323 and initial condition and we add a perturbation in the water thickness by increasing its  
 324 value in 0.15 for  $0.15 < x < 0.25$ .

325 Figures 7 and 8 show the evolution of the surface and the discharge at the beginning  
 326 of the simulation, where the perturbation propagates along the domain. Figure 9 shows  
 327 the surface elevation and discharge at time  $t = 10$  when the steady state is reached.  
 328 Compared to the prescribed initial steady state, we remark that the perturbation has  
 329 not affected the reached steady state. Nevertheless, a small artifact is seen at the critical  
 330 point in Figure 9(b) although it is very small. This small artifact reduces as the number  
 331 of cells is increased.

332 Consider now a smaller perturbation of 0.0001 for  $0.15 < x < 0.2$  and let us compare  
 333 the scheme provided here with a similar based on just an hydrostatic reconstruction [1].  
 334 While the former is fully well-balanced, the latter is only well-balanced for lake at rest  
 335 steady states. The simulations are done using 3201 volume cells. Figure 10 shows at  
 336 time  $t = 0.05$  the difference between the computed values  $h$  and  $q$  with respect to the  
 337 unperturbed steady-state  $h_0$  and  $q_0$  described in Subsection 4.3. As we see, just from  
 338 the beginning, the non fully well-balanced scheme is producing a perturbation in the

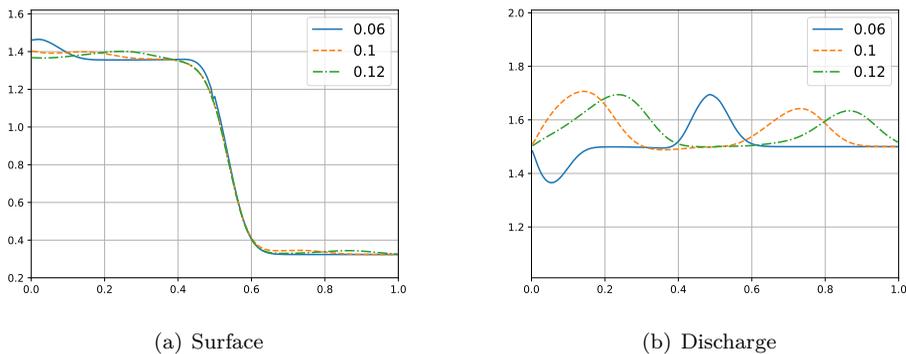


FIGURE 8. *Perturbation of a steady state at times  $t = 0.06, 0.1$  and  $0.12$*

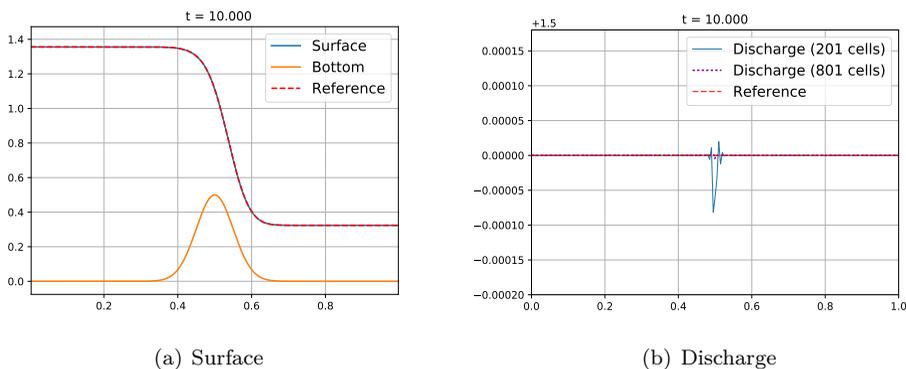


FIGURE 9. *Perturbation of a steady state at times  $t = 10$*

339 middle of the domain which is even bigger than the perturbation introduced on the  
 340 initial condition. Nevertheless, the fully well-balanced scheme introduced here does not  
 341 present this problem.

342 **4.5. Transcritical steady state with shock.** We consider now a numerical test  
 343 taken from [9] consisting of a transcritical flow with a shock over a bump. The interval  
 344 is  $[0, 25]$ , the initial condition and depth bottom is given by

$$\begin{aligned}
 & h(x, t = 0) = 0.33, \quad hu(x, t = 0) = 0.18, \\
 345 \quad (28) \quad & z(x) = \begin{cases} 0.2 + 0.05(x - 10)^2, & \text{if } 8 < x < 12, \\ 0, & \text{otherwise,} \end{cases}
 \end{aligned}$$

346 and the boundary conditions are given by  $hu(x = 0, t) = 0.18, h(x = 25, t) = 0.33$ .

347 We consider a uniform mesh composed by 201 cells.

348 The initial condition will evolve until a transcritical solution with a stationary shock  
 349 is developed. Figure 11 shows the surface and discharge evolution obtained with the  
 350 scheme proposed here. In Figure 12 we see the surface profile obtained and a comparison  
 351 for a reference solution with increasing number of cells. We remark there is a small  
 352 problem for the scheme at the critical point near  $x = 10$  and the position of the shock

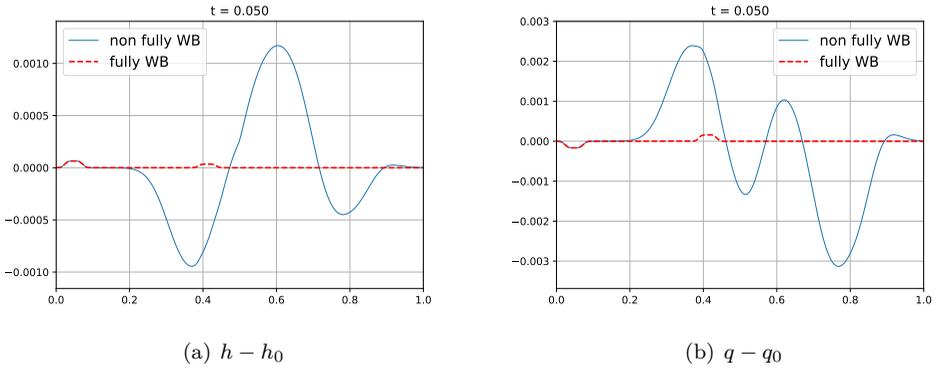


FIGURE 10. *Perturbation of a steady state. Comparison with the unperturbed steady state at time  $t = 0.05$  for fully and non fully well-balanced schemes*

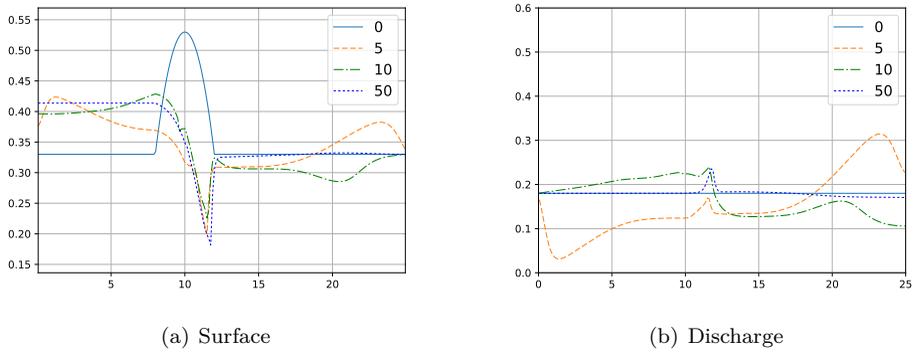


FIGURE 11. *Transcritical steady state with shock: surface elevation and discharge for times  $t=0,5,10,50$*

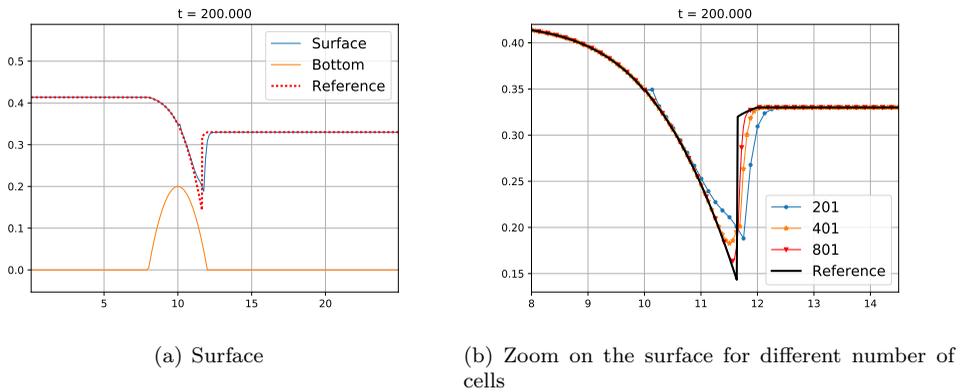


FIGURE 12. *Transcritical steady state with shock: surface elevation*

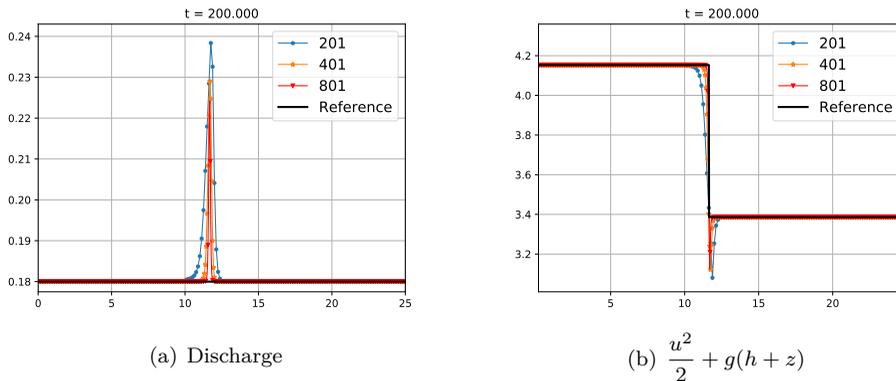


FIGURE 13. Transcritical steady state with shock: steady state condition for different number of cells

No. of cells	$h$		$hu$	
	$L^1$ error	order	$L^1$ error	order
25	2.31e-02	0.00	1.53e-01	0.00
50	1.77e-02	0.38	1.09e-01	0.48
100	1.18e-02	0.59	6.67e-02	0.71
200	7.00e-03	0.75	3.62e-02	0.88
400	3.74e-03	0.91	1.79e-02	1.01

TABLE 1

$L^1$  errors and numerical orders of accuracy.

353 does not exactly coincide with reference solution. Nevertheless those problems disappear  
 354 when the mesh is refined and the numerical solutions converge to the reference solution.  
 355 Figure 13 shows the steady state conditions and comparison with a reference solution  
 356 for different number of cells.

357 **4.6. Formal convergence of the scheme.** We intend now to study the formal  
 358 convergence of the scheme proposed here. To do so, let us consider the initial condition  
 359 given by

$$360 \quad z(x) = 0.1 \cos(2\pi x), \quad h(x, 0) = 1.1 + 0.1 \sin(4\pi x) - z(x), \quad hu(x, 0) = 0.0.$$

361 We compute a reference solution using 3200 points in the interval  $[0, 1]$  and using  
 362 periodic boundary conditions up to time  $t = 0.2$ . Then we make a comparison with the  
 363 solutions given by different number of cells. The  $L^1$  errors are shown in Table 1. As  
 364 expected, the scheme presented here is a first order scheme.

365 **4.7. Traveling wave over an obstacle.** We consider now bottom topography  
 366 and initial condition given by

$$367 \quad (29) \quad z(x) = \begin{cases} 0.5, & \text{if } 0.6 < x < 0.7, \\ 0, & \text{otherwise,} \end{cases} \quad h(x, t = 0) = \begin{cases} 0.5, & \text{if } 0.6 < x < 0.7, \\ 1, & \text{otherwise,} \end{cases}$$

368 and  $hu(x, t = 0) = 0$ . This will produce a rarefaction followed by a shock that travels  
 369 towards the obstacle at the bottom. We compute the solution using 201 cells in the

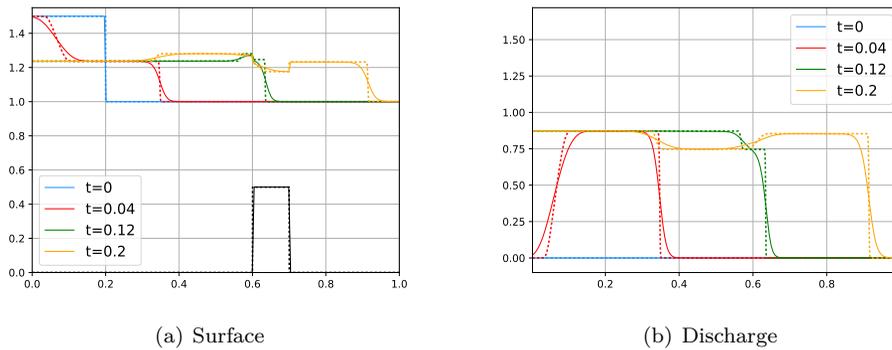


FIGURE 14. *Travelling wave over an obstacle: surface elevation and discharge for times  $t = 0, 0.04, 0.12, 0.2$*

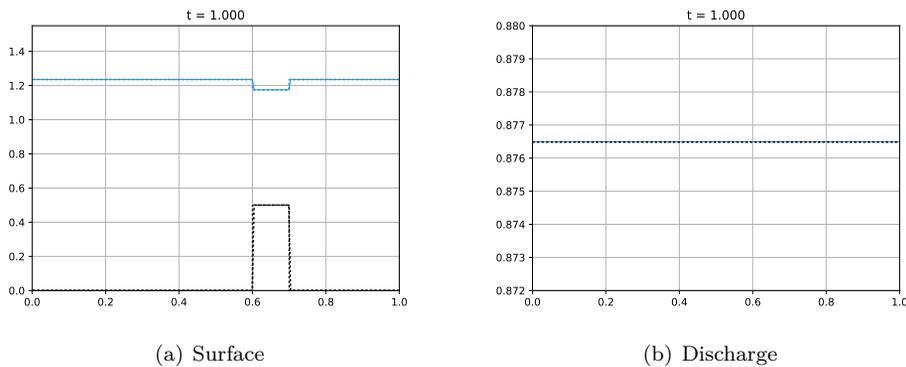


FIGURE 15. *Travelling wave over an obstacle: surface elevation and discharge for time  $t = 1$*

370 interval  $[0, 1]$  with open boundary conditions and compare the results with a reference  
 371 solution computed on a fine grid with 3201 points. The time evolution of the surface  
 372 and discharge are shown in Figure 14 where dashed lines correspond to the results using  
 373 the finer grid and continuous line to the coarser grid. The final time is shown in 15 once  
 374 the waves have left the domain and the steady state is reached. We see that scheme is  
 375 able to reproduce the evolution of the waves in the flat region as well as when the shock  
 376 encounters the obstacle.

377 **5. Conclusions.** We have introduced a positive and fully well-balanced Lagrange-  
 378 Projection scheme for the Shallow Water Equations (SWE). This means that the scheme  
 379 is able to preserve stationary solutions of the model with non zero velocity. Moreover,  
 380 the scheme is positive under suitable CFL condition. Numerical results have shown that  
 381 the scheme behaves well even in transcritical regimes or near critical points, provided  
 382 that a sufficient number of cells is used. In a next work we propose to define semi-  
 383 implicit fully well-balanced numerical schemes for SWE. Fully well-balanced high order  
 384 Lagrange-Projection schemes will be also considered.

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 388 novative Training Networks (ITN) grant 642768 (ModCompShock). The second author  
 389 was partially supported by the ANR project ACHYLLES (ANR-14-CE25-0001-03).

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