A proof of approximate controllability of the 3D Navier–Stokes system via a linear test

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Abstract

We consider the 3D Navier–Stokes system driven by an additive finite-dimensional control force. The purpose of this paper is to show how the approximate controllability of this system can be derived from the approximate controllability of the Euler system linearised around some suitable trajectory. The proof presented here is shorter than the previous ones obtained by Lie algebraic methods and gives some new information about the structure of the control. The dimension of the control space provided by this approach is larger, but it is still uniform with respect to the viscosity.

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0 Introduction

In this paper, we consider the 3D Navier–Stokes (NS) system for incompressible viscous fluids on the torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$:

$$
\partial_t u - \nu \Delta u + \langle u, \nabla \rangle u + \nabla p = f(t, x), \quad \text{div} u = 0, \quad (0.1)
$$

where $\nu > 0$ is the viscosity of the fluid, $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ and $p = p(t, x)$ are the unknown velocity field and pressure, and $f$ is an external force. We fix any $T > 0$ and assume that the force is of the form

$$
f(t, x) = h(t, x) + \eta(t, x), \quad t \in J_T = [0, T], \quad x \in \mathbb{T}^3,
$$

where $h : J_T \times \mathbb{T}^3 \to \mathbb{R}^3$ is a given smooth function and $\eta$ is a control taking values in some subspace $\mathcal{H} \subset H^k(\mathbb{T}^3, \mathbb{R}^3)$, $k \geq 3$. The subspace $\mathcal{H}$ incorporates different constraints that might be imposed on the control; in the examples considered in this paper, it gives the Fourier modes that are directly perturbed by the control force. We are mostly interested here by the situation when $\mathcal{H}$ is a finite-dimensional subspace not depending on the viscosity.

Projecting Eq. (0.1) to the space $H$ of divergence-free vector fields with zero mean value (see (0.9)) and assuming that $h(t)$ and $\eta(t)$ belong to $H$ for any $t \in J_T$, we eliminate the pressure term from the problem and obtain an evolution equation for the velocity field:

$$
\dot{u} + \nu Lu + B(u) = h + \eta. \quad (0.2)
$$

This equation is supplemented with the initial condition

$$
u(0) = u_0. \quad (0.3)
$$

Recall that, for any $u_0 \in H^k = H^k(\mathbb{T}^3, \mathbb{R}^3) \cap H$, problem (0.2), (0.3) has a unique local-in-time strong solution (see Section 1.1).

In this introduction, we formulate a simplified version of our main result, assuming that the subspace $\mathcal{H}$ is given by

$$
\mathcal{H} = \text{span}\{l(\ell) \sin(\ell, x), l(\ell) \cos(\ell, x) : |\ell| \leq 2, \ell \in \mathbb{Z}_3^2\}, \quad (0.4)
$$

where $\{l(\ell), l(-\ell)\}$ is an arbitrary orthonormal basis in $\{x \in \mathbb{R}^3 : \langle \ell, x \rangle = 0\}$.

**Main Theorem.** Eq. (0.2) is approximately controllable in small time by $\mathcal{H}$-valued controls, i.e., for any initial condition $u_0 \in H^{k+1}$, any target $u_1 \in H^{k+1}$, and sufficiently small $\delta > 0$, there is a control $\eta_\delta \in L^2(J_{T\delta}, \mathcal{H})$ and a strong solution $u$ of problem (0.2), (0.3) defined on $J_{T\delta}$ such that

$$
u(T\delta) \to u_1 \quad \text{in } H^k \text{ as } \delta \to 0^+. \quad (0.5)
$$

Moreover, the control $\eta_\delta$ can be chosen in the form

$$
\eta_\delta = R_\delta(u_0, u_1) + \zeta_\delta, \quad (0.6)
$$

where $R_\delta : H^k \times H^k \to L^2(J_{T\delta}, \mathcal{H})$ is a linear bounded operator with a finite-dimensional range and $\zeta_\delta \in L^2(J_{T\delta}, \mathcal{H})$, both $R_\delta$ and $\zeta_\delta$ do not depend on $(u_0, u_1)$. Limit (0.5) is uniform with respect to $u_0$ and $u_1$ in a bounded set of $H^{k+1}$. 

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A more general version of this result is given in Section 2. In particular, we define there a saturation property that implies small time approximate controllability for different subspaces $\mathcal{H}$ spanned by eigenfunctions of the Stokes operator. As a consequence of the Main Theorem, we obtain the following approximate controllability property in fixed time.

**Corollary.** Eq. (0.2) is approximately controllable in time $T > 0$ by $\mathcal{H}$-valued controls, i.e., for any $\varepsilon > 0$ and any $u_0, u_1 \in H^k$, there is a control $\eta \in L^2(J_T, \mathcal{H})$ and a strong solution $u$ of Eq. (0.2) defined on $J_T$ such that

$$\|u(T) - u_1\|_{H^k} < \varepsilon.$$  

Roughly speaking, this result is obtained by applying the Main Theorem on a small time interval, then by forcing the trajectory to remain near $u_1$ for sufficiently long time.

The problem of controllability of PDEs with an additive finite-dimensional force has been studied by many authors in the recent years. Agrachev and Sarychev [AS05, AS06, AS08] were the first who considered this problem; they established the approximate controllability of the NS and Euler systems on the 2D torus. Shirikyan generalised their approach to study the NS system on the 3D torus [Shi06, Shi07] and the Burgers equation on the real line [Shi14] and on a bounded interval with Dirichlet boundary conditions [Shi18]. Rodrigues and Phan [Rod06, PR19] considered the 2D and 3D NS systems on rectangles with Lions boundary conditions. Compressible and incompressible 3D Euler systems were studied by Nersisyan [Ner10, Ner11], and the 2D cubic Schrödinger equation by Sarychev [Sar12]. More recently, the author considered the approximate controllability of Lagrangian trajectories of the 3D NS system [Ner15] and parabolic PDEs with polynomially growing nonlinearities [Ner20]. Boulard et al. [BGN20] considered the 3D system of primitive equations of meteorology and oceanology with control acting directly only on the temperature equation. The proofs of these papers are based on infinite-dimensional extensions of Lie algebraic methods. Most of them provide sharp results, in the sense that they give necessary and sufficient conditions on the Fourier modes that should be perturbed by the control in order to ensure approximate controllability.

In this paper, we take a different route. We proceed by developing an approach by Coron [Cor96b], who considered the approximate controllability of the 2D NS system with Navier slip boundary conditions and used control forces that are localised in the physical space or on the boundary. That approach, called return method, has been later used by Coron and Fursikov [CF96] to study the global exact controllability to trajectories of the 2D NS system on manifolds without boundary, by Coron and Glass [Cor96a, Gla00] to consider the global exact boundary controllability of the 2D and 3D Euler systems, by Fursikov and Imanuilov [FI99] to study the global exact controllability to trajectories of the 3D Boussinesq system, and by many other authors. We refer the reader to the Chapter 6 of the book [Cor07] for a detailed discussion of the return method, for applications to different control problems, and for more references.
The present paper is the first to extend this method to the case of forces that are localised in the Fourier space. The configuration we use here does not provide sharp results\(^1\) in terms of the number of Fourier modes directly perturbed by the control, but gives new and simpler proof with new information about the structure of the control. Roughly speaking, the idea of the proof consists in developing \(u(t)\) as follows:

\[
u(t) = \delta^{-1} w(\delta^{-1} t) + v(\delta^{-1} t) + r_\delta(t) \quad \text{for small } \delta > 0,
\]

where \(w(t)\) is a suitable solution of the Euler system (cf. (1.5)) and \(v(t)\) is a solution of the Euler system linearised around \(w(t)\) (cf. (1.6)); both correspond to some controls taking values in the subspace \(H\) defined by (0.4). We take \(w(t)\) in the form

\[
\psi_\ell^c(t) \cos\langle \ell, x \rangle + \psi_\ell^s(t) \sin\langle \ell, x \rangle,
\]

(0.7)

where the functions \(\{\psi_\ell^c, \psi_\ell^s\} \subset W^{1,2}(J_T, \mathbb{R})\) are chosen such that the boundary conditions

\[
\psi_\ell^c(0) = \psi_\ell^c(T) = \psi_\ell^s(0) = \psi_\ell^s(T) = 0
\]

(0.8)

are satisfied and the derivatives \(\{\dot{\psi}_\ell^c, \dot{\psi}_\ell^s\}\) form an observable family. Replacing the expression (0.7) of the function \(w(t)\) into the left hand side of the Euler system, we infer that \(w(t)\) is indeed a solution corresponding to some \(H\)-valued control. The observability property implies that the linearised Euler system is approximately controllable by \(H\)-valued controls. Furthermore, choosing \(\eta\) in the form (0.6), we show that \(u(t)\) behaves like \(v(t)\) at the endpoints 0 and \(T\delta\) as \(\delta \to 0\). Then the approximate controllability of the linearised Euler equation allows to conclude (0.5). The operator \(R_\delta\) in (0.6) is an approximate right inverse of the resolving operator of the linearised Euler system and \(\zeta_\delta\) is explicitly given in terms of the solution \(w\) and the corresponding control.

The proof of the Main Theorem is general enough and can be applied to many other equations, such as the complex Ginzburg–Landau equation, the Euler system, and parabolic PDEs with polynomial nonlinearities.

This paper is organised as follows. In Section 1, we formulate a perturbative result on solvability of the 3D NS system and explain under what conditions its approximate controllability can be derived from that of the linearised Euler system. In Section 2, we show that the conditions in Section 1 are satisfied when a saturation property holds for the set of controlled Fourier modes. Finally, in Section 3, we discuss the validity of the saturation property.

\(^1\)A sharp version of Corollary is obtained in the papers [Shi06, Shi07, Ner15]. The results of these papers imply, in particular, the approximate controllability in fixed time \(T > 0\) by controls taking values in the smaller subspace \(\text{span}\{l(\ell) \sin(\ell, x), l(\ell) \cos(\ell, x) : |\ell| \leq 1, \ell \in \mathbb{Z}_3^*\}\); see Remark 2.4 for more details.
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Notation

Here we collect some notation used in this paper.

$\mathbb{Z}^3$ is the integer lattice in $\mathbb{R}^3$, $\mathbb{Z}_3^3 = \mathbb{Z}^3\setminus\{0\}$, and $\mathbb{T}^3$ is the torus $\mathbb{R}^3/2\pi\mathbb{Z}^3$.

$L^p(\mathbb{T}^3, \mathbb{R}^3)$, $1 \leq p < \infty$ is the Lebesgue space endowed with the norm $\| \cdot \|_{L^p}$.

$H^k(\mathbb{T}^3, \mathbb{R}^3)$ is the Sobolev space of order $k \geq 1$ endowed with the scalar product $\langle \cdot, \cdot \rangle_k$ and the corresponding norm $\| \cdot \|_k$.

$H^k = H^k(\mathbb{T}^3, \mathbb{R}^3) \cap H = \{ u \in L^2(\mathbb{T}^3, \mathbb{R}^3) : \text{div } u = 0 \text{ in } \mathbb{T}^3, \int_{\mathbb{T}^3} u(x) \, dx = 0 \}$.

$L^p(\mathbb{J}_T, \mathbb{R}^3)$, $1 \leq p < \infty$ is the space of measurable functions $u : \mathbb{J}_T = [0, T] \to \mathbb{R}^3$ with the norm $\| u \|_{L^p(\mathbb{J}_T, \mathbb{R}^3)} = \left( \frac{1}{T} \int_0^T \| u(t) \|_{\mathbb{R}^3}^p \, dt \right)^{1/p}$.

$L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ is the space of measurable functions $u : \mathbb{R}_+ \to \mathbb{R}$ whose restriction to $\mathbb{J}_T$ belongs to $L^p(\mathbb{J}_T, \mathbb{R})$ for any $T > 0$.

$W^{m,p}(\mathbb{J}_T, \mathbb{R})$, $m \geq 1$ is the space of functions $u : \mathbb{J}_T \to \mathbb{R}$ such that $\frac{d^i}{dt^i} u \in L^p(\mathbb{J}_T, \mathbb{R})$ for $0 \leq i \leq m$.

Throughout this paper, the same letter $C$ is used to denote unessential positive constants that may change from line to line.

1 Linear test for approximate controllability

1.1 Perturbative result

Projecting the NS system to the space $H$, we rewrite it in the following equivalent form without pressure term

\[
\begin{align*}
\dot{u} + \nu Lu + B(u) &= f, \\
\quad u(0) &= u_0,
\end{align*}
\]
where $L = -\Delta$ is the Stokes operator, $B(u) = \Pi(\langle u, \nabla \rangle u)$, and $\Pi$ is the Leray orthogonal projection onto $H$ in $L^2$. In this section, we recall a perturbative result for problem (1.1), (1.2). Let us take any integer $k \geq 3$ and define the space

$$X_{T,k} = C(J_T, H^k) \cap L^2(J_T, H^{k+1})$$

endowed with the norm

$$\|u\|_{X_{T,k}} = \|u\|_{C(J_T, H^k)} + \|u\|_{L^2(J_T, H^{k+1})}.$$ 

**Proposition 1.1.** Let $\hat{u}_0 \in H^k$ and $\hat{f} \in L^2(\mathbb{R}^+, H^{k-1})$. There is a maximal time $T_* = T_*(u_0, \hat{f}) > 0$ and a unique solution $\hat{u}$ of problem (1.1), (1.2) with $u_0 = \hat{u}_0$ and $f = \hat{f}$ whose restriction to the interval $J_T$ belongs to $X_{T,k}$ for any $T < T_*$. If $T_* < \infty$, then $\|\hat{u}(t)\|_k \to +\infty$ as $t \to T_*^-$. For any $T < T_*$, there are numbers $\kappa = \kappa(T, \Lambda) > 0$ and $C = C(T, \Lambda) > 0$, where

$$\Lambda = \|\hat{u}\|_{X_{T,k}} + \|\hat{f}\|_{L^2(J_T, H^{k-1})},$$

such that

(i) for any $u_0 \in H^k$ and $f \in L^2(J_T, H^{k-1})$ satisfying

$$\|u_0 - \hat{u}_0\|_k + \|f - \hat{f}\|_{L^2(J_T, H^{k-1})} < \kappa,$$

there is a unique solution $u \in X_{T,k}$ of problem (1.1), (1.2);

(ii) let $S$ be the resolving operator of problem (1.1), (1.2), i.e., the mapping taking $(u_0, f)$ satisfying (1.3) to the solution $u$. Then

$$\|S(u_0, f) - S(\hat{u}_0, \hat{f})\|_{X_{T,k}} \leq C \left( \|u_0 - \hat{u}_0\|_k + \|f - \hat{f}\|_{L^2(J_T, H^{k-1})} \right).$$

Moreover, the problem is regularising in the sense that, when $\hat{f}$ is smooth, the restriction of $S(\hat{u}_0, \hat{f})$ to the interval $(0, T_*)$ belongs to $C^\infty((0, T_*) \times \mathbb{T}^d)$.

See, for example, Chapter 17 in [Tay97] for the proof of local well-posedness and regularising property. The properties (i) and (ii), under these regularity assumptions, are proved in Theorem 1.3 in [Ner15], using standard arguments.

In what follows, we fix any time $T > 0$, any integer $k \geq 3$, and any function $h \in L^2(J_T, H^{k-1})$, and assume that $f = h + \eta$. Let us denote by $\Theta(u_0, h, T)$ the set of functions $\eta \in L^2(J_T, H^{k-1})$ such that problem (1.1), (1.2) has a solution $u \in X_{T,k}$. In view of Proposition 1.1, the set $\Theta(u_0, h, T)$ is open in $L^2(J_T, H^{k-1})$. We denote by $S_t(u_0, h + \eta)$ the restriction of the solution at time $t < T_*(u_0, h + \eta)$. 


1.2 Formulation and proof

By developing the arguments of [Cor96b], we show in this section how the approximate controllability of the NS system

\[
\dot{u} + \nu Lu + B(u) = h + \eta
\]  

(1.4)
can be derived from the approximate controllability of the linearised Euler system. More precisely, we consider the Euler system

\[
\dot{w} + B(w) = \zeta,
\]  

(1.5)
and its linearisation

\[
\dot{v} + Q(v, w) = g,
\]  

(1.6)
where

\[
Q(v, w) = B(v, w) + B(w, v), \quad B(v, w) = \Pi(\langle v, \nabla \rangle w).
\]  

(1.7)
The functions \(\eta, \zeta,\) and \(g\) are considered as controls taking values in the same (finite or infinite-dimensional) subspace \(H\) of \(H^{k+1}\). We will use the following two conditions.

(C1) There is a function \(\zeta \in L^2(J_T, H)\) and a solution \(w \in C(J_T, H^{k+2}) \cap W^{1,2}(J_T, H^{k+1})\) of Eq. (1.5) such that

\[
w(0) = w(T) = 0, \quad Lw(t) \in H \quad \text{for} \ t \in J_T.
\]  

(1.8)  

(1.9)

(C2) The linear Eq. (1.6), with a reference trajectory \(w\) as in Condition (C1), is approximately controllable in time \(T > 0\), i.e., for any \(\varepsilon > 0\) and any \(v_1 \in H^{k+1}\), there is a control \(g \in L^2(J_T, H)\) such that the solution \(v \in C(J_T, H^{k+1}) \cap W^{1,2}(J_T, H^{k})\) of Eq. (1.6) with initial condition \(v(0) = 0\) satisfies

\[
\|v(T) - v_1\|_{k+1} < \varepsilon.
\]

Proposition 1.2. Let \(H\) be a subspace of \(H^{k+1}\) such that Condition (C1) is satisfied. Then for any \(u_0 \in H^{k+1}\), any \(g \in L^2(J_T, H)\), and sufficiently small \(\delta > 0\), there is a control \(\eta_\delta \in \Theta(u_0, h, T\delta) \cap L^2(J_{T\delta}, H)\) such that

\[
S_{T\delta}(u_0, h + \eta_\delta) \to v(T) \quad \text{in} \ H^k \quad \text{as} \ \delta \to 0^+,
\]  

(1.10)
where \(v \in C(J_T, H^{k+1}) \cap W^{1,2}(J_T, H^{k})\) is the solution of Eq. (1.6) with initial condition \(v(0) = u_0\). Moreover, \(\eta_\delta\) is given explicitly by

\[
\eta_\delta = \delta^{-1} g(\delta^{-1} t) + \delta^{-2} \zeta(\delta^{-1} t) + \nu \delta^{-1} Lw(\delta^{-1} t), \quad t \in J_{T\delta},
\]  

(1.11)
and limit (1.10) is uniform with respect to \(u_0\) in a bounded set of \(H^{k+1}\).
Proof. Step 1. Preliminaries. Let us take any \( M > 0 \), any \( u_0 \in B_{H^{k+1}}(0, M) \), and any \( \eta \in L^2_{\loc}(\mathbb{R}_+, \mathcal{H}) \) and denote by \( u(t) = S_t(u_0, h + \eta), t < T_* = T_*(u_0, h + \eta) \) the solution of problem (1.4), (1.2). Following [Cor96b], we make a time substitution and consider the functions
\[
\begin{align*}
\varphi(t) &= v(\delta^{-1}t),
\gamma(t) = \delta^{-1}g(\delta^{-1}t),
\omega(t) = \delta^{-1}w(\delta^{-1}t),
\zeta(t) = \delta^{-2}\zeta(\delta^{-1}t),
\end{align*}
\]
Thus, we need to choose \( \eta \). Assume that we have found a control \( \eta \). Assume that we have found a control \( \eta \), \( \eta \in L^2(J_T, \mathcal{H}) \) such that
\[
T\delta < T_*^\delta = T_*(u_0, h + \eta) \quad \text{for small} \ \delta > 0. \quad (1.13)
\]
Then, in view of the equalities (1.8), (1.12), and \( v(0) = u_0 \), we have
\[
\begin{align*}
r(0) &= 0, \\
r(T\delta) &= u(T\delta) - v(T).
\end{align*}
\]
Hence, we need to choose \( \eta \). Assume that we have found a control \( \eta \), \( \eta \in L^2(J_T, \mathcal{H}) \) such that, in addition to (1.13), also the following limit holds
\[
\|r(T\delta)\|_k \to 0 \quad \text{as} \ \delta \to 0^+, \quad (1.15)
\]
uniformly with respect to \( u_0 \in B_{H^{k+1}}(0, M) \). This will imply limit (1.10).

Step 2. Proof of (1.13) and (1.15). The functions \( \varphi(t) \) and \( \omega(t) \), \( t \in J_T \) satisfy the equations
\[
\begin{align*}
\dot{\varphi} + Q(\varphi, \omega) &= \gamma, \\
\dot{\omega} + Q(\omega) &= \zeta.
\end{align*}
\]
This implies that \( r \) is a solution of the equation
\[
\dot{r} + \nu Lr + B(r, r + \varphi + \omega) + B(\varphi + \omega, r) = \xi, \quad t < \tilde{T}_\delta, \quad (1.16)
\]
where
\[
\xi = h + \eta - \nu Lv + \nu Lw - B(\varphi) - \gamma - \zeta.
\]
Choosing \( \eta = \gamma + \zeta + \nu Lw \in L^2(J_T, \mathcal{H}) \) (cf. (1.11)), we get
\[
\xi = h - \nu Lw - B(\varphi). \quad (1.17)
\]
Taking the scalar product in \( H \) of Eq. (1.16) with \( L^k r \), integrating by parts, then integrating in time, and using the first equality in (1.14), we obtain
\[
\frac{1}{2}\|r\|_k^2 + \nu \int_0^t \|r\|_{k+1}^2 ds = \int_0^t \langle \xi, L^k r \rangle ds - \int_0^t \langle B(r, r + \varphi + \omega), L^k r \rangle ds \\
- \int_0^t \langle B(\varphi + \omega, r), L^k r \rangle ds = I_1 + I_2 + I_3. \quad (1.18)
\]
To estimate $I_1$, we integrate by parts and use (1.17) and the inequalities of Cauchy–Schwarz and Young:

$$|I_1| \leq \int_0^t \|\xi\|_{k-1} \|r\|_{k+1} \, ds$$

$$\leq C \int_0^t (\|h\|_{k-1}^2 + \|\nu L\delta\|_{k-1}^2 + \|B(v_\delta)\|_{k-1}^2) \, ds + \frac{\nu}{4} \int_0^t \|r\|_{k+1}^2 \, ds.$$ 

By a change of variable, we have

$$\int_0^t \|h\|_{k-1}^2 \, ds \leq \int_0^{T\delta} \|h\|_{k-1}^2 \, ds,$$

$$\int_0^t \|L\delta\|_{k-1}^2 \, ds \leq \delta \int_0^T \|v\|_{k+1}^2 \, ds,$$

$$\int_0^t \|B(v_\delta)\|_{k-1}^2 \, ds \leq \delta \int_0^T \|B(v)\|_{k-1}^2 \, ds \leq C\delta \int_0^T \|v\|_{k}^2 \, ds, \quad t \in J_{T\delta}.$$

Thus, there is $\varepsilon_\delta = \varepsilon_\delta(M) > 0$ not depending on $t \in J_{T\delta}$ and $u_0 \in B_{H^{k+1}}(0, M)$ such that $\varepsilon_\delta \to 0$ as $\delta \to 0^+$ and

$$|I_1| \leq \varepsilon_\delta + \frac{\nu}{4} \int_0^t \|r\|_{k+1}^2 \, ds, \quad t < T\delta. \quad (1.19)$$

We estimate $I_2$ and $I_3$ as follows:

$$|I_2| \leq \int_0^t \|\langle B(r, r + v_\delta + w_\delta), L^k r \rangle \| \, ds$$

$$\leq C \int_0^t (\|B(r)\|_{k-1}^2 + \|\langle B(r, v_\delta + w_\delta), L^k r \rangle \|) \, ds + \frac{\nu}{4} \int_0^t \|r\|_{k+1}^2 \, ds,$$

$$|I_3| \leq \int_0^t \|\langle B(v_\delta + w_\delta, r), L^k r \rangle \| \, ds.$$

Note that

$$\int_0^t \|B(r)\|_{k-1}^2 \, ds \leq C \int_0^t \|r\|_{k}^2 \, ds,$$

$$\int_0^t \|\langle B(r, v_\delta + w_\delta), L^k r \rangle \| \, ds \leq C \int_0^t (\|v_\delta\|_{k+1} + \|w_\delta\|_{k+1}) \|r\|_{k}^2 \, ds,$$

$$\int_0^t \|\langle B(v_\delta + w_\delta, r), L^k r \rangle \| \, ds \leq C \int_0^t (\|v_\delta\|_{k} + \|w_\delta\|_{k}) \|r\|_{k}^2 \, ds,$$

where we used the inequalities (see [CF88])

$$\|\langle a, b \rangle, L^k b \| \leq C\|a\|_{k} \|b\|_{k}^2,$$

$$\|\langle a, b \rangle, L^k c \| \leq C\|a\|_{k} \|b\|_{k+1} \|c\|_{k}, \quad a, b \in H^k, \ c \in H^{k+1}.$$
Thus
\[
|I_2 + I_3| \leq C \int_0^t (\|v_\delta\|_{k+1} + \|w_\delta\|_{k+1}) \|r\|_k^2 \, ds + C \int_0^t \|r\|_k^2 \, ds + \frac{\nu}{4} \int_0^t \|r\|_{k+1}^2 \, ds.
\]
Combining this with (1.18) and (1.19), we obtain
\[
\|r\|_k^2 \leq \varepsilon_\delta + C \int_0^t (\|v_\delta\|_{k+1} + \|w_\delta\|_{k+1}) \|r\|_k^2 \, ds + C \int_0^t \|r\|_k^2 \, ds, \quad t < \tilde{T}^\delta.
\]
By the Gronwall inequality,
\[
\|r\|_k^2 \leq (\varepsilon_\delta + C \int_0^t \|r\|_k^4 \, ds) \exp \left( C \int_0^t (\|v_\delta\|_{k+1} + \|w_\delta\|_{k+1}) \, ds \right).
\]
For \(\delta \leq \delta_0(M)\) and \(t \in J_{T^\delta}\), we have
\[
\int_0^t (\|v_\delta\|_{k+1} + \|w_\delta\|_{k+1}) \, ds = \int_0^{t^\delta} (\delta \|v\|_{k+1} + \|w\|_{k+1}) \, ds \leq 1.
\]
Thus
\[
\|r\|_k^2 \leq \varepsilon_\delta + C \int_0^t \|r\|_k^4 \, ds, \quad t < \tilde{T}^\delta, \quad (1.20)
\]
where \(C = C(M) > 0\) does not depend on \(t, \delta,\) and \(u_0\), and by the same letter \(\varepsilon_\delta\) we denote \(e^{C \varepsilon_\delta}\). Let us set
\[
\Phi(t) = \varepsilon_\delta + C \int_0^t \|r\|_k^4 \, ds.
\]
Inequality (1.20) implies that \((\Phi)^{1/2} \leq C \Phi\), which is equivalent to \(\dot{\Phi}/\Phi^2 \leq C\).
Integrating the latter, we obtain
\[
\Phi(t) \leq \varepsilon_\delta (1 - C \varepsilon_\delta t)^{-1}, \quad t < \tilde{T}^\delta.
\]
Choosing \(\delta\) sufficiently small, we see that
\[
\Phi(t) \leq 2 \varepsilon_\delta < 1, \quad t < \tilde{T}^\delta.
\]
This implies both assertions (1.13) and (1.15) and completes the proof of the proposition.

The following is the main result of this section.

**Theorem 1.3.** Let \(\mathcal{H}\) be a subspace of \(H^{k+1}\) such that Conditions (C1) and (C2) are satisfied. Then Eq. (1.4) is approximately controllable in small time by \(\mathcal{H}\)-valued controls, i.e., for any \(u_0, u_1 \in H^{k+1}\) and sufficiently small \(\delta > 0\), there is a control \(\eta_\delta \in \Theta(u_0, h, T^\delta) \cap L^2(J_{T^\delta}, \mathcal{H})\) such that
\[
S_{T^\delta}(u_0, h + \eta_\delta) \to u_1 \quad \text{in} \ H^k \quad \text{as} \ \delta \to 0^+.
\]
Moreover, the control $\eta_\delta$ can be chosen in the form
\[
\eta_\delta = R_\delta(u_0, u_1) + \zeta_\delta,
\]
where $R_\delta : H^k \times H^k \to L^2(J_{T_\delta}, \mathcal{H})$ is a linear bounded operator with a finite-dimensional range and $\zeta_\delta \in L^2(J_{T_\delta}, \mathcal{H})$, both $R_\delta$ and $\zeta_\delta$ do not depend on $(u_0, u_1)$. Limit (1.21) is uniform with respect to $u_0$ and $u_1$ in a bounded set of $H^{k+1}$.

Proof. Let us denote by
\[
A : H^{k+1} \times L^2(J_T, \mathcal{H}) \to C(J_T, H^{k+1}) \cap W^{1,2}(J_T, H^k), \quad (v_0, g) \mapsto v
\]
the resolving operator of Eq. (1.6) with the initial condition $v(0) = v_0$, and let $A_t$ be its restriction at time $t$. By Condition (C₂), the image of the mapping
\[
A_T(0, \cdot) : L^2(J_T, \mathcal{H}) \to H^{k+1}
\]
is dense in $H^{k+1}$. Hence, we can construct an approximate right inverse for $A_T(0, \cdot)$. More precisely, by Proposition 2.6 in [KNS20], for any $\varepsilon > 0$, there is a linear bounded operator $R_\varepsilon : H^k \to L^2(J_T, \mathcal{H})$ such that
\[
\|A_T(0, R_\varepsilon f) - f\|_{k+1} \leq \varepsilon \|f\|_{k+1} \quad \text{for } f \in H^{k+1}.
\]
Now let us take any $M > 0$ and any $u_0, u_1 \in B_{H^{k+1}}(0, M)$. Applying the previous inequality with $f = u_1 - A_T(u_0, 0)$, we get
\[
\|A_T(u_0, g_\varepsilon) - u_1\|_k \leq \varepsilon \|u_1 - A_T(u_0, 0)\|_{k+1} \leq \varepsilon C,
\]
where $g_\varepsilon = R_\varepsilon(u_1 - A_T(u_0, 0))$ and $C = C(M) > 0$ is a constant. Combining this with Propositions 1.1 and 1.2, we complete the proof of the theorem. □

We close this section with the following result.

**Corollary 1.4.** Assume that the conditions of Theorem 1.3 are satisfied. Then Eq. (1.4) is approximately controllable in time $T > 0$ by $\mathcal{H}$-valued controls, i.e., for any $\varepsilon > 0$ and any $u_0, u_1 \in H^k$, there is a control $\eta \in \Theta(u_1, h, T) \cap L^2(J_T, \mathcal{H})$ such that
\[
\|S_T(u_0, h + \eta) - u_1\|_k < \varepsilon.
\]

**Proof.** By the regularising property of the NS system (see Proposition 1.1), the solution corresponding to initial point $u_0 \in H^k$ and control $\eta = 0$ becomes instantaneously smooth. Combining this with Theorem 1.3, we see that Eq. (1.4) is approximately controllable in small time in the sense that, for any $u_0, u_1 \in H^k$, there is a control $\tilde{\eta}_\delta \in \Theta(u_0, h, T\delta) \cap L^2(J_{T\delta}, \mathcal{H})$ such that
\[
S_{T\delta}(u_0, h + \tilde{\eta}_\delta) \to u_1 \quad \text{in } H^k \text{ as } \delta \to 0^+.
\]
Thus, to prove approximate controllability in fixed time, it suffices to show that, for any $T, \varepsilon > 0$ and any $u_1 \in H^k$, there is a control $\eta_1 \in \Theta(u_1, h, T) \cap L^2(J_T, \mathcal{H})$ such that
\[
\|S_T(u_1, h + \eta_1) - u_1\|_k < \varepsilon,
\]
where the initial condition and the target coincide with \( u_1 \). By Proposition 1.1, there is a time \( \tau > 0 \) such that the control \( \eta = 0 \) is in the set \( \Theta(u_1, h, \tau) \) and the function \( S_t(u_1, h) : J_\tau \to H^k \) is continuous. Taking \( \tau \) sufficiently small and using the property (ii) in Proposition 1.1, we find a number \( r \in (0, \varepsilon) \) such that \( \eta = 0 \) belongs to \( \Theta(v, h, \tau) \) for any \( v \in B_{H^k}(u_1, r) \) and

\[
\|S_t(v, h) - u_1\|_k < \varepsilon, \quad t \in J_\tau.
\]

Thus starting from any initial point \( v \in B_{H^k}(u_1, r) \), the solution corresponding to \( \eta = 0 \) remains in the ball \( B_{H^k}(u_1, \varepsilon) \) on the time interval \( J_\tau \). If \( \tau > T \), then the proof is complete. Otherwise, applying (1.23) with initial point \( u'_0 = S_\tau(v, h) \) and target \( u_1 \), we find a small time \( T' < T - \tau \) and a control \( \eta_2 \in \Theta(u'_0, h, T') \cap L^2(J_{T'}, \mathcal{H}) \) such that

\[
\|S_{T'}(u'_0, h + \eta_2) - u_1\|_k < r.
\]

By the choice of \( r \) and \( \tau \), if \( 2\tau + T' > T \), then again the proof is complete. Otherwise, we complete the proof by iterating the above argument finitely many times.

Remark 1.5. In this corollary, the control \( \eta \) is not of the form (1.22). The affine dependence on \((u_0, u_1)\) is lost after the first application of zero control in the \( r \)-neighborhood of \( u_1 \). Indeed, this comes from the fact that \( S_t(u_1, 0) \) is non-linear in \( u_1 \). Analysing the above proof, we easily see that for given \( \varepsilon, M > 0 \) and any \( u_0, u_1 \in B_{H^{k+1}}(0, M) \), the restriction \( \eta|_{[0,T]} \) of the control is of the form (1.22), while the restriction \( \eta|_{[T_\delta,T]} \) does not depend on \( u_0 \).

2 Proof of the Main Theorem

The goal of this section is to show that Conditions (C_1) and (C_2) are verified for different subspaces \( \mathcal{H} \) spanned by a finite number of eigenfunctions of the Stokes operator. Also we prove the Main Theorem formulated in the Introduction.

2.1 More general formulation

Let us fix any numbers \( \alpha_i > 0, \ i = 1, 2, 3 \) and endow the space \( \mathbb{R}^3 \) with the scalar product

\[
\langle x, y \rangle = \sum_{i=1}^{3} \alpha_i^{-1} x_i y_i.
\]

For any \( \ell \in \mathbb{Z}^3_+ \), let us denote

\[
c_\ell(x) = l(\ell) \cos(\langle \ell, x \rangle), \quad s_\ell(x) = l(\ell) \sin(\langle \ell, x \rangle),
\]

where \( \{l(\ell), l(-\ell)\} \) is any orthonormal basis in the hyperplane

\[
\ell^{+\alpha} = \{x \in \mathbb{R}^3 : \langle x, \ell \rangle = 0\}.
\]
The family \( \{c_\ell, s_\ell\}_{\ell \in \mathbb{Z}^3_*} \) is a complete orthogonal system in \( H^k \) composed of eigenfunctions of the Stokes operator. Let \( K \subset \mathbb{Z}^3_* \) be a finite symmetric set (i.e., \( K = -K \)). We associate with \( K \) a non-decreasing sequence of finite-dimensional subspaces by

\[
\mathcal{H}_0(K) = \text{span}\{c_\ell, s_\ell : \ell \in K\}, \quad (2.1)
\]

\[
\mathcal{H}_i(K) = \text{span}\{\eta_1 + Q(\eta_2, \xi) : \eta_1, \eta_2 \in \mathcal{H}_{i-1}(K), \xi \in \mathcal{H}_0(K)\}, \quad i \geq 1, \quad (2.2)
\]

where \( Q \) is the bilinear form defined by (1.7).

**Definition 2.1.** We say that \( K \subset \mathbb{Z}^3_* \) is saturating if the subspace \( \bigcup_{i=1}^{\infty} \mathcal{H}_i(K) \) is dense in \( H^k \).

The following theorem is proved in the next two subsections.

**Theorem 2.2.** Assume that \( K \subset \mathbb{Z}^3_* \) is a saturating set. Then Conditions \((C_1)\) and \((C_2)\) are satisfied for the subspace \( \mathcal{H} = \mathcal{H}_1(K) \), and therefore the conclusions of Theorem 1.3 and Corollary 1.4 hold.

The following theorem provides a practical way for constructing saturating sets. Recall that \( K \subset \mathbb{Z}^3_* \) is a generator if any vector of \( \mathbb{Z}^3 \) is a finite linear combination of vectors of \( K \) with integer coefficients.

**Theorem 2.3.** If a finite symmetric set \( K \subset \mathbb{Z}^3_* \) is a generator, then it is saturating.

See Section 3 for a proof of this result. Now we turn to the proof of the results formulated in the Introduction.

**Proof of the Main Theorem and the Corollary.** For any \( \ell \in \mathbb{R}^3 \), we denote by \( P_\ell \) the orthogonal projection in \( \mathbb{R}^3 \) onto the hyperplane \( \ell^\perp \). Then, for any \( a \in \mathbb{R}^3 \), we have the equalities

\[
\Pi(a \cos(\ell, x)_\alpha) = (P_\ell a) \cos(\ell, x)_\alpha, \quad \Pi(a \sin(\ell, x)_\alpha) = (P_\ell a) \sin(\ell, x)_\alpha.
\]

These equalities and some simple trigonometric computations show that

\[
2Q(a \cos(\ell_1, x)_\alpha, b \sin(\ell_2, x)_\alpha) = \cos(\ell_1 - \ell_2, x)_\alpha P_{\ell_1 - \ell_2} ((a, \ell_2)_\alpha b - (b, \ell_1)_\alpha a) + \cos(\ell_1 + \ell_2, x)_\alpha P_{\ell_1 + \ell_2} ((a, \ell_2)_\alpha b + (b, \ell_1)_\alpha a), \quad (2.3)
\]

\[
2Q(a \cos(\ell_1, x)_\alpha, b \cos(\ell_2, x)_\alpha) = \sin(\ell_1 - \ell_2, x)_\alpha P_{\ell_1 - \ell_2} ((a, \ell_2)_\alpha b - (b, \ell_1)_\alpha a) - \sin(\ell_1 + \ell_2, x)_\alpha P_{\ell_1 + \ell_2} ((a, \ell_2)_\alpha b + (b, \ell_1)_\alpha a), \quad (2.4)
\]

\[
2Q(a \sin(\ell_1, x)_\alpha, b \sin(\ell_2, x)_\alpha) = \sin(\ell_1 - \ell_2, x)_\alpha P_{\ell_1 - \ell_2} ((a, \ell_2)_\alpha b - (b, \ell_1)_\alpha a) + \sin(\ell_1 + \ell_2, x)_\alpha P_{\ell_1 + \ell_2} ((a, \ell_2)_\alpha b + (b, \ell_1)_\alpha a) \quad (2.5)
\]

for any \( \ell_1, \ell_2 \in \mathbb{Z}^3_* \), \( a \in \ell_1^\perp \alpha \), and \( b \in \ell_2^\perp \alpha \). Let us consider the set

\[
\tilde{K} = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}
\]

which is, clearly, a generator. Due to identities \((2.3)-(2.5)\), the subspace \( \mathcal{H}_1(\tilde{K}) \) is contained in the subspace defined by \((0.4)\). Applying Theorems 2.2 and 2.3 with the set \( \tilde{K} \) and taking \( \alpha = (1, 1, 1) \), we obtain the Main Theorem and the Corollary.
Remark 2.4. The papers [Shi06, Shi07, Ner15] provide a sharp version of the Corollary regarding the dimension of the control space. In these papers, a non-linear saturation property is defined for the 3D NS system (1.4), and in the case \( h \equiv 0 \) and \( \alpha = (1, 1, 1) \), the system is proved to be approximately controllable in time \( T > 0 \) by \( \mathcal{H}_0(K) \)-valued controls if and only if \( K \) is a generator (see Theorem 4.5 in [Ner15]). The subspace \( \mathcal{H}_1(K) \) is strictly larger than \( \mathcal{H}_0(K) \).

2.2 Checking Condition (C1)

Let us denote \( \mathcal{H} = \mathcal{H}_1(K) \) and consider the function
\[
w(t) = \sum_{\ell \in K} (\psi^c_\ell(t) c_\ell + \psi^s_\ell(t) s_\ell),
\]
where \( \{\psi^c_\ell, \psi^s_\ell\}_{\ell \in K} \) are any functions in \( W^{1,2}(J_T, \mathbb{R}) \) verifying the boundary conditions
\[
\psi^c_\ell(0) = \psi^c_\ell(T) = \psi^s_\ell(0) = \psi^s_\ell(T) = 0.
\]
As \( c_\ell \) and \( s_\ell \) are eigenfunctions of the Stokes operator, we have \( Lw(t) \in \mathcal{H} \) for \( t \in J_T \). Let us denote \( \zeta = \dot{w} + B(w) \) and show that \( \zeta \in L^2(J_T, \mathcal{H}) \). Indeed, we have \( \dot{w} \in L^2(J_T, \mathcal{H}) \) by the construction. Moreover, the equality
\[
B(w) = \sum_{\ell_1, \ell_2 \in K} Q(\psi^c_{\ell_1}(t)c_{\ell_1} + \psi^c_{\ell_2}(t)s_{\ell_1} + \psi^s_{\ell_1}(t)c_{\ell_2} + \psi^s_{\ell_2}(t)s_{\ell_2})
\]
implies that \( B(w) \in C(J_T, \mathcal{H}) \). Thus, Condition (C1) is satisfied.

2.3 Checking Condition (C2)

Condition (C2) is more subtle and is satisfied under additional hypotheses on the functions \( \{\psi^c_\ell, \psi^s_\ell\}_{\ell \in K} \) entering (2.6). We shall use the notion of observable family of functions from [KNS20].

To prove the approximate controllability of Eq. (1.6), we shall use some arguments close to the ones in Section 4 in [KNS20], where the controllability of the linearised 2D NS system is established. An important difference is that there is no diffusion term in Eq. (1.6), so we do not have a parabolic regularisation property. As a consequence, we cannot use the \( L^2 \)-adjoint problem and the \( H^k \)-adjoint is of rather complicated form and seems to be ill-suited for our purposes. Instead of using the adjoint problem, we use the equation satisfied by the derivative of the resolving operator with respect to the initial condition (see (2.13)).

**Step 1. Observable family.** A family \( \{\phi_i\}_{i=1}^n \subset L^2(J_T, \mathbb{R}) \) is said to be observable\(^2\) if for any subinterval \( J \subset J_T \), any continuous function \( b : J \to \mathbb{R} \), and any \( C^1 \)-functions \( a_i : J \to \mathbb{R} \) the equality
\[
b(t) + \sum_{i=1}^n a_i(t)\phi_i(t) = 0 \quad \text{in } L^2(J, \mathbb{R})
\]
\(^2\)Note that, the observability property we use here is stronger than the one introduced in Definition 4.1 in [KNS20].
implies that \( a_i \equiv b \equiv 0, 1 \leq i \leq n \) on \( J \). An example of observable family can be constructed as follows. Let \( \phi_i : J_T \to \mathbb{R} \) be bounded measurable functions having left and right limits at any point of \( J_T \). Moreover, let there be disjoint countable dense sets \( \{ \mathbb{D}_i \}_{i=1}^n \) in \( J_T \) such that \( \phi_i \) is discontinuous on \( \mathbb{D}_i \) and continuous on \( J_T \setminus \mathbb{D}_i \). Then the family \( \{ \phi_i \}_{i=1}^n \) is observable. Indeed, take any \( 1 \leq i \leq n \) and any \( s \in \mathbb{D}_i \). All the functions \( \phi_j, j \neq i \) are continuous at \( s \), so the jump at \( s \) of the function on the left-hand side of (2.7) is equal to \( a_i(s)(\phi_i(s^+) - \phi_i(s^-)) = 0 \). It follows that \( a_i(s) = 0 \) for any \( s \in \mathbb{D}_i \), hence \( a_i = 0 \) on \( J \), by density and continuity. By (2.7), we have also \( b \equiv 0 \) on \( J \).

Let us now fix an observable family of functions \( \{ \phi^\ell_i, \phi^s_i \}_{\ell \in K} \subset L^2(J_T, \mathbb{R}) \) and denote

\[
\psi^\ell(t) = \phi(t) \int_0^t \phi^\ell(\tau) \, d\tau, \quad \psi^s(t) = \phi(t) \int_0^t \phi^s(\tau) \, d\tau, \quad t \in J_T,
\]

where \( \phi : J_T \to \mathbb{R} \) is a \( C^1 \)-function such that \( \phi(t) = 0 \) if and only if \( t = T \). Of course, Condition (C1) remains true in this case.

**Step 2. Reduction.** Let us fix any \( k \geq 3 \) and denote by \( R(t, \tau) : H^k \to H^k \), \( 0 \leq \tau \leq t \leq 1 \) the two-parameter resolving operator of the linearised problem

\[
\dot{v} + Q(w, v) = 0, \quad v(\tau) = v_0.
\]

Then

\[
A : L^2(J_T, H^k) \to H^k, \quad g \mapsto \int_0^T R(T, \tau)g(\tau) \, d\tau,
\]

is the resolving operator of Eq. (1.6) with initial condition \( v(0) = 0 \). Denote by \( P_H : H^k \to H^k \) the orthogonal projection onto \( H \) in \( H^k \). Our goal is to show that the image of the linear operator

\[
A_1 : L^2(J_T, H^k) \to H^k, \quad A_1 = AP_H
\]

is dense in \( H^k \). It is equivalent to show that the kernel of the adjoint operator

\[
A_1^* : H^k \to L^2(J_T, H^k), \quad z \mapsto P_H R(T, \tau)^* z
\]

is trivial, where \( R(T, \tau)^* : H^k \to H^k \) is the \( H^k \)-adjoint of \( R(T, \tau) \).

**Step 3. Triviality of \( A_1^* \).** Let us take any \( z \in \ker A_1^* \) and show that \( z = 0 \). Indeed, for any \( g \in H \), we have

\[
(g, R(T, \tau)^* z)_k = 0 \quad \text{for a.e. } \tau \in J_T.
\]

This implies that

\[
(R(T, \tau)g, z)_k = 0 \quad \text{for any } \tau \in J_T,
\]

by continuity in \( \tau \) of \( R(T, \tau)g \). Let fix any \( T_1 \in (0, T) \) and rewrite this equality as follows:

\[
(R(T_1, \tau)g, z_1)_k = 0 \quad \text{for any } \tau \in J_{T_1},
\]

for any \( 0 < \tau < T_1 \).
where \( z_1 = R(T, T_1)^*z \). Taking \( \tau = T_1 \), we obtain
\[
(g, z_1)_k = 0,
\]
(2.11)
i.e., \( z_1 \) is orthogonal to \( \mathcal{H} = \mathcal{H}_1(\mathcal{K}) \) in \( H^K \). Let us show that \( z_1 \) is orthogonal also to \( \mathcal{H}_2(\mathcal{K}) \). To this end, let us denote
\[
y(t, \tau) = R(\tau + t, \tau)g,
\]
and note that \( y(t, \tau) \) is the solution of the problem
\[
\dot{y}(t, \tau) + Q(w(\tau + t), y(t, \tau)) = 0, \quad t \in (0, T - \tau),
\]
y(0, \tau) = g.
It follows that \( Y(t, \tau) = \frac{\partial}{\partial \tau}y(t, \tau) \) is the solution of
\[
\dot{Y}(t, \tau) + Q(w(\tau + t), Y(t, \tau)) + Q(\dot{w}(\tau + t), y(t, \tau)) = 0, \quad t \in (0, T - \tau),
\]
\[
Y(0, \tau) = 0.
\]
(2.13)
(2.14)
On the other hand, taking the derivative in \( \tau \) of (2.12) and choosing \( t = T_1 - \tau \), we obtain
\[
\frac{\partial}{\partial \tau}R(T_1, \tau)g = Y(T_1 - \tau, \tau) - \dot{R}(T_1, \tau)g
\]
\[
= Y(T_1 - \tau, \tau) + Q(w(T_1), R(T_1, \tau)g).
\]
(2.15)
In the last equality, we used Eq. (2.8). Taking the derivative of (2.10) in \( \tau \) and using equalities (2.13)-(2.15), we arrive at
\[
0 = \int_0^{T_1-\tau} (Q(w(\tau + t), Y(t, \tau)) + Q(\dot{w}(\tau + t), y(t, \tau)), z_1)_k \, dt
\]
\[
- (Q(w(T_1), R(T_1, \tau)g), z_1)_k
\]
\[
= \int_0^{T_1-\tau} (Q(w(\tau + t), Y(t, \tau)), z_1)_k \, dt + \int_\tau^{T_1} (Q(\dot{w}(t), R(t, \tau)g), z_1)_k \, dt
\]
\[
- (Q(w(T_1), R(T_1, \tau)g), z_1)_k.
\]
Differentiating this in \( \tau \), we get
\[
b(\tau) + \sum_{k \in \mathcal{K}} (a^x_\xi(\tau)\phi^x_\xi(\tau) + a^x_\tau(\tau)\phi^x_\tau(\tau)) = 0 \quad \text{for } \tau \in J_{T_1},
\]
where
\[
b(\tau) = \frac{\partial}{\partial \tau} \int_0^{T_1-\tau} (Q(w(\tau + t), Y(t, \tau)), z_1)_k \, dt - \left( Q(\dot{\phi}(\tau)\dot{w}(\tau), g), z_1 \right)_k
\]
\[
+ \int_\tau^{T_1} \left( Q(\dot{w}(t), \frac{\partial}{\partial \tau} R(t, \tau)g), z_1 \right)_k \, dt - \frac{\partial}{\partial \tau} (Q(w(T_1), R(T_1, \tau)g), z_1)_k.
\]
\[
a^x_\tau(\tau) = -\phi(\tau) \left( Q(c_\ell, g), z_1 \right)_k, \quad a^x_\xi(\tau) = -\phi(\tau) \left( Q(s_\ell, g), z_1 \right)_k;
\]
\[
\dot{w}(\tau) = \sum_{k \in \mathcal{K}} \int_0^{\tau} (\phi^x_\xi(t) c_\ell + \phi^x_\tau(t) s_\ell) \, dt.
\]
The functions \( \{a^\ell_1, a^\ell_2\}_{\ell \in \mathcal{K}} \) are continuously differentiable and \( b \) is continuous on \( J_{T_1} \). By observability of \( \{\phi_1, \phi_2\}_{\ell \in \mathcal{K}} \), we have thus \( a^\ell_1 \equiv a^\ell_2 \equiv 0 \) on \( J_{T_1} \) for any \( \ell \in \mathcal{K} \). As a consequence,

\[
(Q(\ell, g), z)_k = (Q(s, g), z)_k = 0,
\]

which, combined with (2.11), implies that \( z_1 \) is orthogonal to \( \mathcal{H}_2(\mathcal{K}) \). Recalling the definition of \( z_1 \), we conclude that

\[
(R(T, g, z))_k = 0 \quad \text{for any } T_1 \in (0, T) \text{ and } g \in \mathcal{H}_2(\mathcal{K}).
\]

Denoting \( T_1 \) by \( \tau \), we obtain (2.9), but now for any \( g \) in \( \mathcal{H}_2(\mathcal{K}) \). Iterating this argument, we prove (2.9) for any \( g \in \bigcup_{n=1}^{\infty} \mathcal{H}_n(\mathcal{K}) \). Taking \( \tau = T \) and using the saturation hypothesis, we get that \( z = 0 \). This completes the proof of Condition (C2) and that of Theorem 2.2.

3 Saturation property

In this section, we prove Theorem 2.3. In what follows, we write \( \ell_1 \parallel \ell_2 \) to indicate that the vectors \( \ell_1, \ell_2 \in \mathbb{R}^3 \) are non-parallel.

Step 1. Reduction. Let us define a sequence of finite symmetric sets in \( \mathbb{Z}^3 \) as follows:

\[
\mathcal{K}_0 = \mathcal{K}, \quad \mathcal{K}_j = \mathcal{K}_{j-1} \cup \{\ell_1 + \ell_2 : \ell_1 \in \mathcal{K}_{j-1}, \ell_2 \in \mathcal{K}, \ell_1 \parallel \ell_2\}, \quad j \geq 1.
\]

As \( \mathcal{K} \) is a generator, this sequence is strictly increasing and

\[
\bigcup_{j=1}^{\infty} \mathcal{K}_j = \mathbb{Z}^3. \tag{3.1}
\]

Let us assume that we have shown the inclusion

\[
\mathcal{H}_i(\mathcal{K}_j) \subset \mathcal{H}_{i+3}(\mathcal{K}_{j-1}) \quad \text{for any } i \geq 0, j \geq 1, \tag{3.2}
\]

where \( \mathcal{H}_i(\mathcal{K}_j) \) are the subspaces defined by (2.1) and (2.2) with \( \mathcal{K} = \mathcal{K}_j \) and \( c_0 = s_0 = 0 \). Then (3.2) implies that

\[
\mathcal{H}_0(\mathcal{K}_j) \subset \mathcal{H}_3(\mathcal{K}_{j-1}) \subset \mathcal{H}_6(\mathcal{K}_{j-2}) \subset \ldots \subset \mathcal{H}_3(\mathcal{K}).
\]

Combining this with (3.1), we see that the subspace \( \bigcup_{j=1}^{\infty} \mathcal{H}_j(\mathcal{K}) \) is dense in \( H^k \), i.e., \( \mathcal{K} \) is saturating. Thus we need to prove (3.2).

Step 2. Proof of (3.2). We first consider a particular case.

Step 2.1. Let us take any \( \ell_1 \in \mathcal{K}_{j-1} \) and \( \ell_2 \in \mathcal{K} \) such that \( \ell_1 \parallel \ell_2 \) and denote by \( \delta = \delta(\ell_1, \ell_2) \) one of two unit vectors in \( \ell_1^* \cap \ell_2^* \). In this step, we show that

\[
\delta \cos(\ell_1 + \ell_2, x)_\alpha, \delta \sin(\ell_1 + \ell_2, x)_\alpha \in \mathcal{H}_{i+1}(\mathcal{K}_{j-1}). \tag{3.3}
\]

Indeed, by identity (2.3), we have

\[
2Q(b \cos(\ell_2, x)_\alpha, a \sin(\ell_1, x)_\alpha) = -\cos(\ell_1 - \ell_2, x)_\alpha P_{\ell_1 - \ell_2} ((a, \ell_1)_a b - (b, \ell_1)_a a) + \cos(\ell_1 + \ell_2, x)_\alpha P_{\ell_1 + \ell_2} ((a, \ell_2)_a b + (b, \ell_1)_a a) \tag{3.4}
\]
for any \( a \in \ell_{1} \), and \( b \in \ell_{2} \). Summing (2.3) and (3.4), we obtain
\[
\cos(\ell_{1} + \ell_{2}, x)_{a}P_{\ell_{1} + \ell_{2}}(a, \ell_{2}) = Q(a \cos(\ell_{1}, x)_{a}, b \sin(\ell_{2}, x)_{a}) + Q(b \cos(\ell_{2}, x)_{a}, a \sin(\ell_{1}, x)_{a}).
\] (3.5)

We take \( a = \delta \) and \( b \) such that \( \langle b, \ell_{1} \rangle_{a} = 1 \). This choice is possible since \( \ell_{1} \parallel \ell_{2} \).
Then (3.5) becomes
\[
\delta \cos(\ell_{1} + \ell_{2}, x)_{a} = Q(\delta \cos(\ell_{1}, x)_{a}, b \sin(\ell_{2}, x)_{a}) + Q(b \cos(\ell_{2}, x)_{a}, \delta \sin(\ell_{1}, x)_{a}).
\]
This implies that \( \delta \cos(\ell_{1} + \ell_{2}, x)_{a} \in \mathcal{H}_{i+1}(K_{j-1}) \). The second inclusion in (3.3) is proved in a similar way.

**Step 2.2.** Now we prove (3.2). Let us take any \( r \in K \) such that the family \( \mathcal{E} = \{ \ell_{1}, \ell_{2}, r \} \) is a linearly independent. This is possible since \( K \) is a generator. To simplify notation, let us denote
\[
(x, y, z)_{\mathcal{E}} = x\ell_{1} + y\ell_{2} + zr
\]
for any \( x, y, z \in \mathbb{R} \). Then the family \( \{(1, -1, 1)_{\mathcal{E}}, (-1, 1, 1)_{\mathcal{E}}, (1, 1, -1)_{\mathcal{E}}\} \) is linearly independent, so the intersection of the hyperplanes \((1, -1, 1)_{\mathcal{E}}^{1}, (-1, 1, 1)_{\mathcal{E}}^{1}, (1, 1, -1)_{\mathcal{E}}^{1}\), and \((1, 1, -1)_{\mathcal{E}}^{1}\) is \([0]\). To fix the ideas, let us assume that
\[
(1, 1, 1)_{\mathcal{E}} \notin (1, 1, -1)_{\mathcal{E}}^{1},
\] (3.6)
the cases \((1, 1, 1)_{\mathcal{E}} \notin (1, 1, 1)_{\mathcal{E}}^{1}\) and \((1, 1, 1)_{\mathcal{E}} \notin (1, 1, 1)_{\mathcal{E}}^{1}\) are treated in a similar way. By (3.3), we have
\[
\delta(\ell_{1}, \ell_{2}) \cos((1, 1, 0)_{\mathcal{E}}, x)_{a}, \delta(\ell_{1}, \ell_{2}) \sin((1, 1, 0)_{\mathcal{E}}, x)_{a} \in \mathcal{H}_{i+1}(K_{j-1}).
\] (3.7)

Now writing
\[
(1, 1, 1)_{\mathcal{E}} = (1, 1, 0)_{\mathcal{E}} + (0, 0, 1)_{\mathcal{E}},
\]
recalling that \((0, 0, 1)_{\mathcal{E}} = r \in K\), and using (3.5) and (3.7), we obtain
\[
\cos((1, 1, 1)_{\mathcal{E}}, x)_{a}P_{(1, 1, 1)_{\mathcal{E}}}((\delta(\ell_{1}, \ell_{2}), (0, 0, 1)_{\mathcal{E}})_{a}b + \langle b, (1, 1, 0)_{\mathcal{E}} \rangle_{a}\delta(\ell_{1}, \ell_{2})) = Q(\delta(\ell_{1}, \ell_{2}) \cos((1, 1, 0)_{\mathcal{E}}, x)_{a}, b \sin((0, 0, 1)_{\mathcal{E}}, x)_{a}) + Q(b \cos((0, 0, 1)_{\mathcal{E}}, x)_{a}, \delta(\ell_{1}, \ell_{2}) \sin((1, 1, 0)_{\mathcal{E}}, x)_{a}) \in \mathcal{H}_{i+2}(K_{j-1})
\] (3.8)
for any \( b \in (0, 0, 1)_{\mathcal{E}}^{1}\). As the family \( \mathcal{E} \) is a linearly independent, we have that \( \langle \delta(\ell_{1}, \ell_{2}), (0, 0, 1)_{\mathcal{E}} \rangle_{a} \neq 0 \). Hence,
\[
\mathcal{G} = \{ \langle \delta(\ell_{1}, \ell_{2}), (0, 0, 1)_{\mathcal{E}} \rangle_{a}b + \langle b, (1, 1, 0)_{\mathcal{E}} \rangle_{a}\delta(\ell_{1}, \ell_{2}) : b \in (0, 0, 1)_{\mathcal{E}}^{1}\}
\]
is a two-dimensional subspace of \( \mathbb{R}^{3} \) contained in \((1, 1, 1)_{\mathcal{E}}^{1}\), i.e., \( \mathcal{G} = (1, 1, -1)_{\mathcal{E}}^{1}\). Then (3.6) implies that \( P_{(1, 1, 1)_{\mathcal{E}}} \mathcal{G} = (1, 1, 1)_{\mathcal{E}}^{1}\). Combining this with (3.8), we derive that
\[
c_{1(1,1)_{\mathcal{E}}} \subset \mathcal{H}_{i+2}(K_{j-1}).
\] (3.9)
In a similar way, one proves that
\[ s_{\pm(1,1,1)} \subset \mathcal{H}_{i+2}(\mathcal{K}_{j-1}). \]  
(3.10)

Applying the result of Step 2.1 to the difference
\[ (1,1,0) = (1,1,1) - (0,0,1) \]
and using (3.9) and (3.10), we obtain
\[ \delta((1,1,1), (0,0,1)) \cos\langle (1,1,0), x \rangle \in \mathcal{H}_{i+3}(\mathcal{K}_{j-1}). \]

Combining this with the fact that \( \delta((1,1,1), (0,0,1)) \parallel \delta(\ell_1, \ell_2) \) and (3.3), we obtain that \( c_{\pm(\ell_1+\ell_2)} \in \mathcal{H}_{i+3}(\mathcal{K}_{j-1}) \). The proof of \( s_{\pm(\ell_1+\ell_2)} \in \mathcal{H}_{i+3}(\mathcal{K}_{j-1}) \) is similar. This completes the proof of (3.2).

References


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