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Results in Mathematics

Summability of inhomogeneous PDEs with a polynomial semilinearity Pascal Remy Laboratoire de Mathématiques de Versailles, Université de Versailles Saint-Quentin, 45 avenue des Etats-Unis, 78035 Versailles cedex, France  
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# Summability of the formal power series solutions of a certain class of inhomogeneous partial differential equations with a polynomial semilinearity and variable coefficients

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## Abstract

In this article, we investigate the summability of the formal power series solutions in time of a certain class of inhomogeneous partial differential equations with a polynomial semilinearity, and with variable coefficients. In particular, we give necessary and sufficient conditions for the  $k$ -summability of the solutions in a given direction, where  $k$  is a positive rational number entirely determined by the linear part of the equation. These conditions generalize the ones given by the author for the linear case [?, ?] and for the semilinear heat equation [?]. In addition, we present some technical results on the generalized binomial and multinomial coefficients, which are needed for the proof our main theorem.

Summability, Inhomogeneous partial differential equation, Nonlinear partial differential equation, Formal power series, Divergent power series  
35C10, 35C20, 40B05

## 1 Declarations

Not applicable

## 2 Introduction

### 2.1 Setting the problem

The summation theory is a very powerful tool initially developed within the framework of the analytic ordinary differential equations with an irregular singular point (see for instance [?, ?]). In particular, it allows the construction of explicit solutions from formal solutions.

For several years, various works have been done on the divergent solutions of some classes of linear partial differential equations or integro-differential equations in two variables or more, allowing thus to formulate many results on Gevrey

properties, summability or multisummability (e.g. [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] and references inside).

In the case of the nonlinear partial differential equations, the situation is much more complicated. The existing results concern mainly Gevrey properties, especially the convergence (e.g. [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] and references inside), and there are very few results about the summation (see [?, ?, ?, ?, ?, ?]).

In this article, we are interested in the summability of the formal power series solutions in time of the inhomogeneous semilinear partial differential equation

$$(2.1) \quad \begin{cases} \partial_t^\kappa u - a(t, x) \partial_x^p u - P(u) = \tilde{f}(t, x) \\ \partial_t^j u(t, x)|_{t=0} = \varphi_j(x), j = 0, \dots, \kappa - 1 \end{cases}$$

in two variables  $(t, x) \in \mathbb{C}^2$ , where

- $\kappa, p \geq 1$  are two positive integers;
- the coefficient  $a(t, x)$  is analytic on a polydisc  $D_{\rho_0} \times D_{\rho_1}$  centered at the origin  $(0, 0)$  of  $\mathbb{C}^2$  ( $D_\rho$  denotes the disc with center  $0 \in \mathbb{C}$  and radius  $\rho > 0$ ) and satisfies the condition  $a(0, x) \neq 0$ ;
- $P(X) = \sum_{m=2}^d b_m(t, x) X^m \in \mathcal{O}(D_{\rho_0} \times D_{\rho_1})[[X]]$  is a polynomial in  $X$  with analytic coefficients on  $D_{\rho_0} \times D_{\rho_1}$ ;
- the inhomogeneity  $\tilde{f}(t, x)$  is a formal power series in  $t$  with analytic coefficients in  $D_{\rho_1}$  (we denote by  $\tilde{f}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$ ) which may be smooth, or not<sup>1</sup>;
- the initial conditions  $\varphi_j(x)$  are analytic on  $D_{\rho_1}$  for all  $j = 0, \dots, \kappa - 1$ .

Equation (2.1) is fundamental in many physical, chemical, biological, and ecological problems. For example: for  $(\kappa, p) = (1, 2)$ , Eq. (2.1) arises in problems involving diffusion and nonlinear growth such as heat and mass transfer, combustion theory, and spread theory of animal or plant populations (nonlinear heat equation); for  $(\kappa, p) = (2, 2)$ , Eq. (2.1) describes the propagation of nonlinear waves in an inhomogeneous medium (nonlinear Klein-Gordon equation); and, for  $(\kappa, p) = (2, 4)$ , Eq. (2.1) describes the relationship between the beam's deflection and an applied lateral nonlinear force (nonlinear Bernoulli-Euler equation).

The work presented in this article is a natural extension of the work [?] in which the nonlinearity  $P(u)$  is reduced to a term of the form  $b(x)u^2$  and  $(\kappa, p) = (1, 2)$  (see Eq. (2.2) just below). Before stating our main result (see Theorem 3) making explicit a characterization of the summability of the formal series solutions in time of Eq. (2.1), let us first start by recalling some known results about these ones.

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<sup>1</sup>We denote  $\tilde{f}$  with a tilde to emphasize the possible divergence of the series  $\tilde{f}$ .

## 2.2 Formal solutions and known results

First of all, we have the following.

Equation (2.1) admits a unique solution  $\tilde{u}(t, x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$ . Let us write the coefficients  $a(t, x)$  and  $b_m(t, x)$  for  $m = 2, \dots, d$ , and the inhomogeneity  $\tilde{f}(t, x)$  in the form

$$a(t, x) = \sum_{j \geq 0} a_{j,*}(x) \frac{t^j}{j!}, \quad b_m(t, x) = \sum_{j \geq 0} b_{m;j,*}(x) \frac{t^j}{j!}, \quad \tilde{f}(t, x) = \sum_{j \geq 0} f_{j,*}(x) \frac{t^j}{j!}$$

with  $a_{j,*}(x), b_{m;j,*}(x), f_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1})$  for all  $j \geq 0$  and all  $m = 2, \dots, d$ . Looking for  $\tilde{u}(t, x)$  on the same type:

$$\tilde{u}(t, x) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!} \quad \text{with } u_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1}) \text{ for all } j \geq 0,$$

one easily checks that its coefficients  $u_{j,*}(x)$  are uniquely determined for all  $j \geq 0$  by the recurrence relations

$$u_{j+\kappa,*}(x) = f_{j,*}(x) + \sum_{j_0=0}^j \binom{j}{j_0} a_{j_0,*}(x) \partial_x^p u_{j-j_0,*}(x) + \sum_{m=2}^d \sum_{j_0+j_1+\dots+j_m=j} \binom{j}{j_0, j_1, \dots, j_m} b_{m;j_0,*}(x) u_{j_1,*}(x) \dots u_{j_m,*}(x),$$

together with the initial conditions  $u_{j,*}(x) = \varphi_j(x)$  for  $j = 0, \dots, \kappa - 1$ . The notations  $\binom{j}{j_0}$  and  $\binom{j}{j_0, j_1, \dots, j_m}$  stand respectively for the binomial coefficients and for the multinomial coefficients.

In a 1999 article [?], M. Miyake proved in the particular case of the equation

$$\partial_t^\kappa u - \partial_x^p u = 0$$

that the formal solution  $\tilde{u}(t, x)$  is convergent when  $\kappa \geq p$  and  $s$ -Gevrey with  $s = p/\kappa - 1$  otherwise. This result was then generalized by the author, first of all, to the inhomogeneous linear case  $P \equiv 0$  [?] and, afterwards, to the general Eq. (2.1) [?, ?]. In particular, he showed that the Gevrey regularity of  $\tilde{u}(t, x)$  does not depend on the nonlinear term  $P(u)$ , but only on  $\kappa, p$  and  $\tilde{f}(t, x)$ .

[[?, ?]] Let  $s$  be the nonnegative rational number defined by  $s = \max(0, p/\kappa - 1)$ . Then, the formal solution  $\tilde{u}(t, x)$  and the inhomogeneity  $\tilde{f}(t, x)$  are together  $s$ -Gevrey.

Thereby, in the case  $\kappa \geq p$ , Proposition 2.2 provides us a necessary and sufficient condition for  $\tilde{u}(t, x)$  to be convergent, and, in the opposite case  $p > \kappa$ , it naturally leads us to the question of the  $k$ -summability ( $k = 1/s$ ) of  $\tilde{u}(t, x)$ .

In the linear case, M. Miyake [?] and the author [?] gave necessary and sufficient conditions for  $\tilde{u}(t, x)$  to be  $k$ -summable in a given direction  $\arg(t) =$

$\theta$ . To do that, they used two different approaches: the first one based on the definition of the  $k$ -summability in terms of the Borel transformation, and the second one based on the definition of the  $k$ -summability in terms of the successive derivatives.

More recently [?], the author considered Eq. (2.1) with  $(\kappa, p, d) = (1, 2, 2)$  and with constant coefficients in  $t$ , that is the semilinear heat equation

$$(2.2) \quad \begin{cases} \partial_t u - a(x)\partial_x^2 u - b(x)u^2 = \tilde{f}(t, x) \\ u(0, x) = \varphi(x) \end{cases}$$

Using the same approach as the one developed in [?], he gave a necessary and sufficient condition for  $\tilde{u}(t, x)$  to be 1-summable, generalizing thus the condition already proved in the linear case by W. Balser and M. Loday-Richaud [?]. In particular, he showed, as for the Gevrey regularity (see Proposition 2.2), that this condition is not affected by the nonlinear term  $u^2$ .

In this article, we propose to extend all these results to the general Eq. (2.1). In Section 3, we recall some basic definitions and properties about the  $k$ -summable formal series. Then, we state our main result (Theorem 3) which gives, under some various assumptions on the coefficient  $a(t, x)$ , a necessary and sufficient condition for  $\tilde{u}(t, x)$  to be  $k$ -summable in a given direction  $\arg(t) = \theta$  with  $k = 1/s = \kappa/(p - \kappa)$ . The proof of this result is developed in the next two sections. In Section 4, it is detailed in the case of the first assumption, namely  $a(0, 0) \neq 0$ . Our approach is similar to the one presented in [?] for the linear case. However, because of the variable coefficients and the nonlinear terms  $u^m$ , the calculations are much more complicated and require some technical results on the generalized binomial and multinomial coefficients, that is on the binomial and multinomial coefficients with nonnegative real terms. These technical results are all proved in Section 6, which can also be read independently of the rest of the article, so as not to burden the main proof. In Section 5, we show how to adapt the calculations of Section 4 within the framework of the other assumptions on  $a(t, x)$ .

### 3 $k$ -summability of $\tilde{u}(t, x)$

All along the article, we consider  $t$  as the variable and  $x$  as a parameter. Thereby, to define the notion of summability of formal power series in  $\mathcal{O}(D_{\rho_1})[[t]]$ , one extends the classical notion of summability of elements in  $\mathbb{C}[[t]]$  to families parametrized by  $x$  in requiring similar conditions, the estimates being however uniform with respect to  $x$ . Doing that, any formal power series  $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$  can be seen as a formal power series in  $t$  with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a disc  $D_{r_1}$  ( $0 < r_1 < \rho_1$ ) and continuous up to its boundary, equipped with the usual supremum norm. For a general study of the series with coefficients in a Banach space, we refer for instance to [?].

Among the many equivalent definitions of the  $k$ -summability in a given direction  $\arg(t) = \theta$  at  $t = 0$ , we choose in this article a generalization of Ramis' definition which states that a formal series  $\tilde{g}(t) \in \mathbb{C}[[t]]$  is  $k$ -summable in direction  $\theta$  if there exists a holomorphic function  $g$  which is  $1/k$ -Gevrey asymptotic to  $\tilde{g}$  in an open sector  $\Sigma_{\theta, > \pi s}$  bisected by  $\theta$  and with opening larger than  $\pi s$  with  $s = 1/k$  [?, Def. 3.1]. To express the  $1/k$ -Gevrey asymptotic, there also exist various equivalent ways. We choose here the one which sets conditions on the successive derivatives of  $g$  (see [?, p. 171] or [?, Thm. 2.4] for instance).

[ $k$ -summability] Let  $k > 0$  and  $s = 1/k$ . A formal series  $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$  is said to be  $k$ -summable in the direction  $\arg(t) = \theta$  if there exist a sector  $\Sigma_{\theta, > \pi s}$ , a radius  $0 < r_1 \leq \rho_1$  and a function  $u(t, x)$  called  $k$ -sum of  $\tilde{u}(t, x)$  in direction  $\theta$  such that

1.  $u$  is defined and holomorphic on  $\Sigma_{\theta, > \pi s} \times D_{r_1}$ ;
2. For any  $x \in D_{r_1}$ , the map  $t \mapsto u(t, x)$  has  $\tilde{u}(t, x) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!}$  as Taylor series at 0 on  $\Sigma_{\theta, > \pi s}$ ;
3. For any proper<sup>2</sup> subsector  $\Sigma \Subset \Sigma_{\theta, > \pi s}$ , there exist two positive constants  $C > 0$  and  $K > 0$  such that, for all  $\ell \geq 0$ , all  $t \in \Sigma$  and all  $x \in D_{r_1}$ ,

$$|\partial_t^\ell u(t, x)| \leq CK^\ell \Gamma(1 + (s + 1)\ell).$$

We denote by  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  the subset of  $\mathcal{O}(D_{\rho_1})[[t]]$  made of all the  $k$ -summable formal series in the direction  $\arg(t) = \theta$ .

Note that, for any fixed  $x \in D_{r_1}$ , the  $k$ -summability of  $\tilde{u}(t, x)$  coincides with the classical  $k$ -summability. Consequently, Watson's lemma [?, Theorem 5.1.3] implies the unicity of its  $k$ -sum, if any exists.

Note also that the  $k$ -sum of a  $k$ -summable formal series  $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  may be analytic with respect to  $x$  on a disc  $D_{r_1}$  smaller than the common disc  $D_{\rho_1}$  of analyticity of the coefficients  $u_{j,*}(x)$  of  $\tilde{u}(t, x)$ .

Denote by  $\partial_t^{-1} \tilde{u}$  (resp.  $\partial_x^{-1} \tilde{u}$ ) the anti-derivative of  $\tilde{u}$  with respect to  $t$  (resp.  $x$ ) which vanishes at  $t = 0$  (resp.  $x = 0$ ). Proposition 3 below specifies the algebraic structure of  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$ .

Let  $k > 0$  and  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ . Then,  $(\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}, \partial_t, \partial_x)$  is a  $\mathbb{C}$ -differential algebra stable under the anti-derivations  $\partial_t^{-1}$  and  $\partial_x^{-1}$ .

We refer for instance to [?, Prop. 2] for a proof of this result.

With respect to  $t$ , the  $k$ -sum  $u(t, x)$  of a  $k$ -summable series  $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  is analytic on an open sector for which there is no control on the angular opening except that it must be larger than  $\pi/k$  (hence, it contains a closed sector  $\overline{\Sigma}_{\theta, \pi/k}$  bisected by  $\theta$  and with opening  $\pi/k$ ) and no control on the radius except that it must be positive. Thereby, the  $k$ -sum  $u(t, x)$  is well-defined as a section

<sup>2</sup>A subsector  $\Sigma$  of a sector  $\Sigma'$  is said to be a *proper subsector* and one denotes  $\Sigma \Subset \Sigma'$  if its closure in  $\mathbb{C}$  is contained in  $\Sigma' \cup \{0\}$ .

of the sheaf of analytic functions in  $(t, x)$  on a germ of closed sector of opening  $\pi/k$  (that is, a closed interval  $\bar{I}_{\theta, \pi/k}$  of length  $\pi/k$  on the circle  $S^1$  of directions issuing from 0; see [?, 1.1] or [?, I.2]) times  $\{0\}$  (in the plane  $\mathbb{C}$  of the variable  $x$ ). We denote by  $\mathcal{O}_{\bar{I}_{\theta, \pi/k} \times \{0\}}$  the space of such sections.

The operator of  $k$ -summation

$$\begin{aligned} \mathcal{S}_{k; \theta} : \quad \mathcal{O}(D_{\rho_1})\{t\}_{k; \theta} &\longrightarrow \mathcal{O}_{\bar{I}_{\theta, \pi/k} \times \{0\}} \\ \tilde{u}(t, x) &\longmapsto u(t, x) \end{aligned}$$

is a homomorphism of differential  $\mathbb{C}$ -algebras for the derivations  $\partial_t$  and  $\partial_x$ . Moreover, it commutes with the anti-derivations  $\partial_t^{-1}$  and  $\partial_x^{-1}$ .

We are now able to state the main result in this article.

Let  $\arg(t) = \theta \in \mathbb{R}/2\pi\mathbb{Z}$  be a direction issuing from 0.

Let us assume  $p > \kappa$  and let us set  $k = \kappa/(p - \kappa)$ .

Let us also assume that either  $a(0, 0) \neq 0$ , or there exists  $q \in \{1, \dots, p - 1\}$  such that  $\partial_x^n a(t, x)|_{x=0} \equiv 0$  for all  $n = 0, \dots, q - 1$ , and  $\partial_x^q a(0, 0) \neq 0$ . Then,

1. The unique formal series solution  $\tilde{u}(t, x) \in \mathcal{O}(D_{\rho_1})[[t]]$  of Eq. (2.1) is  $k$ -summable in the direction  $\theta$  if and only if the inhomogeneity  $\tilde{f}(t, x)$  and the formal series  $\partial_x^n \tilde{u}(t, x)|_{x=0} \in \mathbb{C}[[t]]$  for  $n = 0, \dots, p - 1$  are  $k$ -summable in the direction  $\theta$ .
2. Moreover, the  $k$ -sum  $u(t, x)$ , if any exists, satisfies Eq. (2.1) in which  $\tilde{f}(t, x)$  is replaced by its  $k$ -sum  $f(t, x)$  in the direction  $\theta$ .

Observe that the necessary condition of the first point is straightforward from Proposition 3 and that the second point stems obvious from Corollary 3. Consequently, we are left to prove the sufficient condition of the first point. This is the subject of the next two sections below. In the first one (Section 4), we focus on the case  $a(0, 0) \neq 0$ . In the second one (Section 5), we show how the calculations made in Section 4 can be adapted within the framework of the other assumptions on  $a(t, x)$ .

From now on, we fix a direction  $\theta$  and we suppose that the inhomogeneity  $\tilde{f}(t, x)$  and the formal power series  $\partial_x^n \tilde{u}(t, x)|_{x=0} \in \mathbb{C}[[t]]$  for  $n = 0, \dots, p - 1$  are all  $k$ -summable in the direction  $\theta$ . To prove that the formal solution  $\tilde{u}(t, x)$  is also  $k$ -summable in this direction, we shall proceed through a fixed point method similar to the ones already used by W. Balser and M. Loday-Richaud in [?] and by the author in [?, ?, ?, ?]. However, as we shall see below, the calculations are much more complicated because of the nonlinear terms  $u^m$ .

## 4 Proof of the sufficient condition: the case $a(0, 0) \neq 0$

All along this section, we assume that the coefficient  $a(t, x)$  satisfies  $a(0, 0) \neq 0$ . Before starting the calculations, let us first begin this proof with a preliminary remark on the coefficients  $\tilde{u}_{*, n}(t) = \partial_x^n \tilde{u}(t, x)|_{x=0}$  of  $\tilde{u}(t, x)$ .

#### 4.1 First step: a preliminary remark

Let us write the coefficients  $a(t, x)$  and  $b_m(t, x)$  for  $m = 2, \dots, d$  in the form

$$a(t, x) = \sum_{n \geq 0} a_{*,n}(t) \frac{x^n}{n!}, \quad b_m(t, x) = \sum_{n \geq 0} b_{m;*,n}(t) \frac{x^n}{n!}$$

with  $a_{*,n}(t), b_{m;*,n}(t) \in \mathcal{O}(D_{\rho_0})$  for all  $n \geq 0$  and all  $m = 2, \dots, d$ . Let us also write the formal solution  $\tilde{u}(t, x)$  and the inhomogeneity  $\tilde{f}(t, x)$  in the same way:

$$\tilde{u}(t, x) = \sum_{n \geq 0} \tilde{u}_{*,n}(t) \frac{x^n}{n!}, \quad \tilde{f}(t, x) = \sum_{n \geq 0} \tilde{f}_{*,n}(t) \frac{x^n}{n!}.$$

Observe that the coefficients  $\tilde{u}_{*,n}(t)$  and  $\tilde{f}_{*,n}(t)$  are divergent in general (hence, the notation with the tilde). By identifying the terms in  $x^n$  in Eq. (2.1), we get the identities

$$(4.1) \quad a_{*,0}(t) \tilde{u}_{*,n+p}(t) + \sum_{n_0=1}^n \binom{n}{n_0} a_{*,n_0}(t) \tilde{u}_{*,n+p-n_0}(t) = \partial_t^\kappa \tilde{u}_{*,n}(t) - \tilde{f}_{*,n}(t) \\ - \sum_{m=2}^d \sum_{n_0+n_1+\dots+n_m=n} \binom{n}{n_0, n_1, \dots, n_m} b_{m;*,n_0}(t) \tilde{u}_{*,n_1}(t) \dots \tilde{u}_{*,n_m}(t)$$

for all  $n \geq 0$ . By assumption,  $a_{*,0}(0) \neq 0$ ; hence,  $1/a_{*,0}(t)$  is well-defined in  $\mathbb{C}[[t]]$  and, consequently, each coefficient  $\tilde{u}_{*,n}(t)$  is uniquely determined from the inhomogeneity  $\tilde{f}(t, x)$  and from the formal series  $\tilde{u}_{*,n'}(t)$  with  $n' = 0, \dots, p-1$ . In particular, the same applies to  $\tilde{u}(t, x)$ .

#### 4.2 Second step: the associated equation

Let us set

$$\tilde{v}(t, x) = \sum_{n=0}^{p-1} \tilde{u}_{*,n}(t) \frac{x^n}{n!}$$

and  $\tilde{u}(t, x) = \tilde{v}(t, x) + \partial_x^{-p} \tilde{w}(t, x)$ . With these notations, Eq. (2.1) becomes

$$(4.2) \quad \tilde{w} - A(t, x) \partial_t^\kappa \partial_x^{-p} \tilde{w} + \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} B_m(t, x) \tilde{v}^{m-j}(t, x) (\partial_x^{-p} \tilde{w})^j = \tilde{g}(t, x)$$

with

$$\tilde{g}(t, x) = A(t, x) \left( \partial_t^\kappa \tilde{v}(t, x) - \sum_{m=2}^d b_m(t, x) \tilde{v}^m(t, x) - \tilde{f}(t, x) \right),$$

where  $A(t, x)$  and  $B_m(t, x)$  stand respectively for the functions

$$A(t, x) = \frac{1}{a(t, x)} \quad \text{and} \quad B_m(t, x) = \frac{b_m(t, x)}{a(t, x)}.$$

Observe that, thanks to the assumption  $a(0,0) \neq 0$ , these functions are all well-defined and holomorphic on a common convenient polydisc  $D_{\rho'_0} \times D_{\rho'_1}$  with  $0 < \rho'_0 \leq \rho_0$  and  $0 < \rho'_1 \leq \rho_1$ .

According to our assumption on the  $k$ -summability of the inhomogeneity  $\tilde{f}(t,x)$  and of the formal power series  $\tilde{u}_{*,n}(t)$  for  $n = 0, \dots, p-1$ , the formal series  $\tilde{v}(t,x)$  and  $\tilde{g}(t,x)$  are both  $k$ -summable in the direction  $\theta$  (see Proposition 3). Thereby, the identity (4.2) above tells us that it is sufficient to prove that it is the same for the formal series  $\tilde{w}(t,x) \in \mathcal{O}(D_{\rho_1})[[t]]$ . To do that, we shall proceed as in [?, ?, ?, ?] by using a fixed point method. Of course, as we shall see below, our calculations will be much more complicated due to the presence of the nonlinear terms  $B_m(t,x)\tilde{v}^{m-j}(t,x)(\partial_x^{-p}\tilde{w})^j$ .

### 4.3 Third step: the fixed point procedure

Let us set  $\tilde{w}(t,x) = \sum_{\mu \geq 0} \tilde{w}_\mu(t,x)$  and let us choose the solution of Eq. (4.2) recursively determined for all  $\mu \geq 0$  by the relations

$$(4.3) \quad \tilde{w}_{\mu+1}(t,x) = A(t,x)\partial_t^k \partial_x^{-p} \tilde{w}_\mu(t,x) - \sum_{m=2}^d \sum_{j=1}^m \sum_{\mu_1+\dots+\mu_j=\mu} \left[ \binom{m}{j} B_m(t,x) \tilde{v}^{m-j}(t,x) \left( \prod_{i'=1}^j \partial_x^{-p} \tilde{w}_{\mu_{i'}}(t,x) \right) \right]$$

together with the initial condition  $\tilde{w}_0 = \tilde{g}$ . Observe that  $\tilde{w}_\mu(t,x) \in \mathcal{O}(D_{\rho'_1})[[t]]$  for all  $\mu \geq 0$ . Observe also that the  $\tilde{w}_\mu(t,x)$ 's are of order  $O(x^{p\mu})$  in  $x$  for all  $\mu \geq 0$ , and, consequently, the series  $\tilde{w}(t,x)$  itself makes sense as a formal series in  $t$  and  $x$ .

Let us now respectively denote by  $w_0(t,x)$  and  $v(t,x)$  the  $k$ -sums of  $\tilde{w}_0$  and  $\tilde{v}$  in the direction  $\theta$  and, for all  $\mu > 0$ , let  $w_\mu(t,x)$  be determined by the relations (4.3) in which  $\tilde{v}$  is replaced by  $v$  and all the  $\tilde{w}_\mu$  are replaced by  $w_\mu$ . By construction, all the functions  $w_\mu(t,x)$  are defined and holomorphic on a common domain  $\Sigma_{\theta, > \pi s} \times D_{\rho'_1}$ , where  $s = 1/k = p/\kappa - 1$ , and where the radius  $\rho''_0$  of  $\Sigma_{\theta, > \pi s}$  and the radius  $\rho''_1$  of  $D_{\rho'_1}$  can always be chosen so that  $0 < \rho''_0 < \rho'_0$  and  $0 < \rho''_1 < \rho'_1$ .

To end the proof, it remains to prove that the series  $\sum_{\mu \geq 0} w_\mu(t,x)$  is convergent and that its sum  $w(t,x)$  is the  $k$ -sum of  $\tilde{w}(t,x)$  in the direction  $\theta$ .

### 4.4 Fourth step: some estimates on the $w_\mu(t,x)$ 's

According to Definition 3, the  $k$ -summability of  $\tilde{w}_0$  and  $\tilde{v}$  implies that there exists  $0 < r'_1 < \min(1, \rho''_1)$  such that, for any proper subsector  $\Sigma \Subset \Sigma_{\theta, > \pi s}$ , there exist two positive constants  $C, K > 0$  such that, for all  $\ell \geq 0$  and all  $(t,x) \in \Sigma \times D_{r'_1}$ , the functions  $w_0$  and  $v$  satisfy the inequalities

$$(4.4) \quad |\partial_t^\ell w_0(t,x)| \leq CK^\ell \Gamma(1+(s+1)\ell) \quad \text{and} \quad |\partial_t^\ell v(t,x)| \leq CK^\ell \Gamma(1+(s+1)\ell).$$

Let us now fix a proper subsector  $\Sigma \Subset \Sigma_{\theta, > \pi s}$ . Let  $r'_0$  denote the radius of  $\Sigma$  and let us choose for the constant  $K$  of the property above a constant  $\geq \max\left(1, \frac{1}{\rho''_0 - r'_0}\right)$ . Observe that such a choice is already possible since the inequalities (4.4) still hold for any constant  $K' \geq K$ . Observe also that the quotient  $1/(\rho''_0 - r'_0)$  makes sense since the definition of a proper subsector (see Footnote 2) implies  $0 < r'_0 < \rho''_0$ .

Proposition 4.4 below provides us some estimates on the derivatives  $\partial_t^\ell w_\mu$ . Let us denote by

- $\alpha$  (resp.  $\beta_m$  for  $m = 2, \dots, d$ ) the maximum of  $|A(t, x)|$  (resp.  $|B_m(t, x)|$ ) on the closed polydisc  $\overline{D}_{\rho''_0} \times \overline{D}_{\rho'_1}$  ( $\overline{D}_\rho$  denotes the closed disc with center  $0 \in \mathbb{C}$  and radius  $\rho > 0$ );
- $C_s = s'(2 + \Gamma(ss'))$ , where  $s'$  is the positive integer  $\geq 1$  defined by

$$s' = \begin{cases} 1 & \text{if } s \geq 1 \\ \left\lfloor \frac{1}{s} \right\rfloor + 1 & \text{if } s < 1 \end{cases} \quad ([x] \text{ stands for the floor of } x \in \mathbb{R});$$

- $C' = \left( \alpha C_s + \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} \beta_m C^{m-1} C_s^m \zeta(p)^{j-1} \right)$ , where  $\zeta$  is the Riemann Zeta function:

$$\zeta(z) = \sum_{n \geq 1} \frac{1}{n^z} \quad \text{for all } z \in \mathbb{C}, \operatorname{Re}(z) > 1.$$

Then, the following inequalities

$$(4.5) \quad |\partial_t^\ell w_\mu(t, x)| \leq CC'^\mu K^{\kappa\mu + \ell} \Gamma(1 + (s+1)(\kappa\mu + \ell)) \frac{|x|^{p\mu}}{(p\mu)!}$$

hold for all  $\ell, \mu \geq 0$  and all  $(t, x) \in \Sigma \times D_{r'_1}$ .

Observe that the constant  $\zeta(p)$  is well-defined since  $p \geq 2$ .

The proof proceeds by recursion on  $\mu$ . The case  $\mu = 0$  is straightforward from the first inequality of (4.4). Let us now suppose that the inequalities (4.5) hold for all the functions  $w_j(t, x)$  with  $j = 0, \dots, \mu$  for a certain  $\mu \geq 0$ .

According to the relations (4.3), we first derive from the generalized Leibniz Formula the identities

$$\begin{aligned} \partial_t^\ell w_{\mu+1}(t, x) &= \sum_{\ell_0=0}^{\ell} \binom{\ell}{\ell_0} \partial_t^{\ell-\ell_0} A(t, x) \partial_t^{\kappa+\ell_0} \partial_x^{-p} w_\mu(t, x) \\ &- \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} \left[ \sum_{\mu_1+\dots+\mu_j=\mu} \sum_{\ell_0+\ell_1+\dots+\ell_m=\ell} \binom{\ell}{\ell_0, \ell_1, \dots, \ell_m} \partial_t^{\ell_0} B_m(t, x) \times \right. \\ &\quad \left. \left( \prod_{i=1}^{m-j} \partial_t^{\ell_i} v(t, x) \right) \left( \prod_{i'=1}^j \partial_t^{\ell_{m-j+i'}} \partial_x^{-p} w_{\mu_{i'}}(t, x) \right) \right] \end{aligned}$$

for all  $\ell \geq 0$  and all  $(t, x) \in \Sigma \times D_{r'_1}$ , with the classical convention that the first product is 1 when  $j = m$ .

Let us now apply the Cauchy Integral Formula to the function  $\partial_t^j A$  with  $j \geq 0$ . Thanks to the definition of the radii  $r'_0$  and  $r'_1$ , we have

$$\partial_t^j A(t, x) = \frac{j!}{(2i\pi)^2} \int_{\substack{|t'-t|=\rho''_0-r'_0 \\ |x'-x|=\rho''_1-r'_1}} \frac{A(t', x')}{(t'-t)^{j+1}(x'-x)} dt' dx',$$

for all  $(t, x) \in \Sigma \times D_{r'_1}$ , and so the estimates

$$\left| \partial_t^j A(t, x) \right| \leq j! \alpha \left( \frac{1}{\rho''_0 - r'_0} \right)^j \leq j! \alpha K^j$$

by definition of the constant  $K$ . In the same way, we have

$$\left| \partial_t^j B_m(t, x) \right| \leq j! \beta_m K^j$$

for all  $j \geq 0$ , all  $m = 2, \dots, d$  and all  $(t, x) \in \Sigma \times D_{r'_1}$ .

Thereby, using the second inequality of (4.4), the inequalities (4.5) applied to the functions  $w_j$  for  $j = 0, \dots, \mu$ , and the fact that  $r'_1 < 1$  and  $K \geq 1$ , we finally get the inequalities

(4.6)

$$\begin{aligned} \left| \partial_t^\ell w_{\mu+1}(t, x) \right| &\leq CC'^\mu K^{\kappa(\mu+1)+\ell} \Gamma(1 + (s+1)(\kappa(\mu+1) + \ell)) \frac{|x|^{p(\mu+1)}}{(p(\mu+1))!} \times \\ &\quad \left( \alpha S_{\mu, \ell} + \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} \beta_m C^{m-1} S'_{\mu, \ell, m, j} \right) \end{aligned}$$

for all  $\ell \geq 0$  and all  $(t, x) \in \Sigma \times D_{r'_1}$ , with, as the constants  $S_{\mu, \ell}$  and  $S'_{\mu, \ell, m, j}$ , the constants respectively defined by

$$S_{\mu, \ell} = \sum_{\ell_0=0}^{\ell} \binom{\ell}{\ell_0} \frac{(\ell - \ell_0)! \Gamma(1 + (s+1)(\kappa\mu + \kappa + \ell_0))}{\Gamma(1 + (s+1)(\kappa\mu + \kappa + \ell))}$$

and

$$\begin{aligned} S'_{\mu, \ell, m, j} = & \sum_{\mu_1 + \dots + \mu_j = \mu} \sum_{\ell_0 + \ell_1 + \dots + \ell_m = \ell} \left( \binom{\ell}{\ell_0, \ell_1, \dots, \ell_m} \frac{(p\mu + p)!}{\prod_{i'=1}^j (p\mu_{i'} + p)!} \times \right. \\ & \left. \frac{\ell_0! \left( \prod_{i=1}^{m-j} \Gamma(1 + (s+1)\ell_i) \right) \left( \prod_{i'=1}^j \Gamma(1 + (s+1)(\kappa\mu_{i'} + \ell_{m-j+i'}) \right)}{\Gamma(1 + (s+1)(\kappa\mu + \kappa + \ell))} \right), \end{aligned}$$

where, as previously, the product on  $i$  is 1 when  $j = m$ .

The inequality (4.5) for  $w_{\mu+1}$  stems then from Lemmas 4.4 and 4.4 below which allow to bound  $S_{\mu,\ell}$  and  $S'_{\mu,\ell,m,j}$ . This completes the proof of Proposition 4.4.

Let  $\mu, \ell \geq 0$  be. Then,  $S_{\mu,\ell} \leq C_s$ .

First of all, let us observe that  $(\ell - \ell_0)! \leq \Gamma(1 + (s + 1)(\ell - \ell_0))$  for all  $\ell_0 \in \{0, \dots, \ell\}$ . This is obvious when  $\ell_0 = \ell$  and stems from the increasing of the Gamma function on  $[2, +\infty[$  otherwise. Thereby, using the notation of the generalized binomial coefficients (see Section 6) and the fact that  $(s + 1)\kappa = p$ , we get

$$S_{\mu,\ell} \leq \sum_{\ell_0=0}^{\ell} \frac{\binom{\ell}{\ell_0}}{\binom{(s+1)\ell + p(\mu+1)}{(s+1)\ell_0 + p(\mu+1)}}.$$

Applying then the Vandermonde's inequality (see Proposition 6, 1)

$$\binom{(s+1)\ell + p(\mu+1)}{(s+1)\ell_0 + p(\mu+1)} \geq \binom{s\ell}{s\ell_0} \binom{\ell}{\ell_0} \binom{p(\mu+1)}{p(\mu+1)} = \binom{s\ell}{s\ell_0} \binom{\ell}{\ell_0},$$

we derive the following

$$S_{\mu,\ell} \leq \sum_{\ell_0=0}^{\ell} \frac{1}{\binom{s\ell}{s\ell_0}}$$

and we conclude by Proposition 6, 1.

Let  $\mu, \ell \geq 0$  be,  $m \in \{2, \dots, d\}$  and  $j \in \{1, \dots, m\}$ .

Then,  $S'_{\mu,\ell,m,j} \leq C_s^m \zeta(p)^{j-1}$ .

★ Let us start with the case  $j = 1$ . We must prove the inequality  $S'_{\mu,\ell,m,1} \leq C_s^m$ , where

$$S'_{\mu,\ell,m,1} = \sum_{\ell_0 + \ell_1 + \dots + \ell_m = \ell} \binom{\ell}{\ell_0, \ell_1, \dots, \ell_m} \times \frac{\ell_0! \left( \prod_{i=1}^{m-1} \Gamma(1 + (s+1)\ell_i) \right) \Gamma(1 + (s+1)(\kappa\mu + \ell_m))}{\Gamma(1 + (s+1)(\kappa\mu + \kappa + \ell))}.$$

Using the inequalities  $\ell_0! \leq \Gamma(1 + (s+1)\ell_0)$  and  $\Gamma(1 + (s+1)(\kappa\mu + \ell_m)) \leq \Gamma(1 + (s+1)(\kappa(\mu+1) + \ell_m))$ , and the fact that  $(s+1)\kappa = p$ , the sum  $S'_{\mu,\ell,m,1}$  can be first majorized in a similar way as the proof of Lemma 4.4 by means of generalized multinomial coefficients (see Section 6):

$$S'_{\mu,\ell,m,1} \leq \sum_{\ell_0 + \ell_1 + \dots + \ell_m = \ell} \frac{\binom{\ell}{\ell_0, \ell_1, \dots, \ell_m}}{\mathcal{M}_{\mu,\ell,m,1}},$$

with

$$\mathcal{M}_{\mu,\ell,m,1} = \binom{(s+1)\ell + p(\mu+1)}{(s+1)\ell_0, (s+1)\ell_1, \dots, (s+1)\ell_{m-1}, (s+1)\ell_m + p(\mu+1)}.$$

Thereby, applying the Vandermonde's inequality (see Proposition 6, 2)

$$\begin{aligned} \mathcal{M}_{\mu,\ell,m,1} &\geq \binom{s\ell}{s\ell_0, s\ell_1, \dots, s\ell_{m-1}, s\ell_m} \binom{\ell}{\ell_0, \ell_1, \dots, \ell_{m-1}, \ell_m} \binom{p(\mu+1)}{0, 0, \dots, 0, p(\mu+1)} \\ &= \binom{s\ell}{s\ell_0, s\ell_1, \dots, s\ell_{m-1}, s\ell_m} \binom{\ell}{\ell_0, \ell_1, \dots, \ell_{m-1}, \ell_m} \end{aligned}$$

we get

$$S'_{\mu,\ell,m,1} \leq \sum_{\ell_0 + \ell_1 + \dots + \ell_m = \ell} \frac{1}{\binom{s\ell}{s\ell_0, s\ell_1, \dots, s\ell_{m-1}, s\ell_m}}$$

and we conclude by Proposition 6, 2.

★ Let us now assume  $j \in \{2, \dots, m\}$  and let us first observe that

$$\begin{aligned} \frac{(p\mu + p)!}{\prod_{i'=1}^j (p\mu_{i'} + p)!} &= \frac{1}{\prod_{i'=1}^{j-1} (p\mu_{i'} + 1) \dots (p\mu_{i'} + p)} \times \frac{(p(\mu+1))!}{(p\mu_1)! \dots (p\mu_{j-1})! (p(\mu_j + 1))!} \\ &\leq \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'} + 1)^p} \binom{p(\mu+1)}{p\mu_1, \dots, p\mu_{j-1}, p(\mu_j + 1)} \\ &= \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'} + 1)^p} \binom{p(\mu+1)}{0, \dots, 0, p\mu_1, \dots, p\mu_{j-1}, p(\mu_j + 1)}, \end{aligned}$$

where 0 occurs  $m - j$  times. Thereby, reasoning as in the case  $j = 1$ , we get

$$\begin{aligned} S'_{\mu,\ell,m,j} &\leq \sum_{\mu_1 + \dots + \mu_j = \mu} \left( \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'} + 1)^p} \times \right. \\ &\quad \left. \sum_{\ell_0 + \ell_1 + \dots + \ell_m = \ell} \frac{\binom{\ell}{\ell_0, \ell_1, \dots, \ell_m} \binom{p(\mu+1)}{0, \dots, 0, p\mu_1, \dots, p\mu_{j-1}, p(\mu_j + 1)}}{\mathcal{M}_{\mu,\ell,m,j}} \right), \end{aligned}$$

where  $\mathcal{M}_{\mu,\ell,m,j}$  stands for the generalized multinomial coefficient

$$\binom{(s+1)\ell + p(\mu+1)}{(s+1)\ell_0, (s+1)\ell_1, \dots, (s+1)\ell_{m-j}, (s+1)\ell_{m-j+1} + p\mu_1, \dots, (s+1)\ell_{m-1} + p\mu_{j-1}, (s+1)\ell_m + p(\mu_j + 1)}.$$

and so

$$\begin{aligned}
S'_{\mu,\ell,m,j} &\leq \sum_{\mu_1+\dots+\mu_j=\mu} \left( \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^p} \sum_{\ell_0+\ell_1+\dots+\ell_m=\ell} \frac{1}{\binom{s\ell}{s\ell_0, s\ell_1, \dots, s\ell_m}} \right) \\
&\leq C_s^m \sum_{\mu_1+\dots+\mu_j=\mu} \left( \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^p} \right).
\end{aligned}$$

Lemma 4.4 follows then from the inequalities

$$\begin{aligned}
\sum_{\mu_1+\dots+\mu_j=\mu} \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^p} &\leq \sum_{\mu_1=0}^{\mu} \dots \sum_{\mu_{j-1}=0}^{\mu} \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^p} \\
&= \left( \sum_{\mu'=0}^{\mu} \frac{1}{(\mu'+1)^p} \right)^{j-1} \\
&\leq \left( \sum_{\mu'=1}^{+\infty} \frac{1}{\mu'^p} \right)^{j-1} \\
&= \zeta(p)^{j-1}.
\end{aligned}$$

This completes the proof.

The following result, which provides the estimates on the  $w_\mu$ 's in view in this section, is a direct consequence of Proposition 4.4.

Let us set  $K_1 = 2^p K$  and  $c = 2^p C' K^\kappa$ . Then, the inequalities

$$|\partial_i^\ell w_\mu(t, x)| \leq CK_1^\ell \Gamma(1 + (s+1)\ell) (c|x|^p)^\mu$$

hold for all  $\ell, \mu \geq 0$  and all  $(t, x) \in \Sigma \times D_{r'_1}$ .

Using the relation  $(s+1)\kappa = p$ , we first derive from the recurrence relation  $\Gamma(1+z) = z\Gamma(z)$  applied  $p\mu$  times the identity

$$\Gamma(1 + (s+1)(\kappa\mu + \ell)) = \Gamma(1 + (s+1)\ell) \prod_{i=1}^{p\mu} ((s+1)\ell + i).$$

Next, applying the inequality  $s+1 \leq p$ , we get the following relations:

$$\begin{aligned}
\Gamma(1 + (s+1)(\kappa\mu + \ell)) &\leq \Gamma(1 + (s+1)\ell) \prod_{i=1}^{p\mu} (p\ell + i) \\
&= \Gamma(1 + (s+1)\ell) \binom{p\ell + p\mu}{p\mu} (p\mu)! \\
&\leq 2^{p\ell + p\mu} (p\mu)! \Gamma(1 + (s+1)\ell).
\end{aligned}$$

Proposition 4.4 stems then from the inequality (4.5).

We are now able to complete the proof of Theorem 3.

## 4.5 Fifth step: conclusion

Let us now choose for  $\Sigma$  a sector containing a proper subsector  $\Sigma'$  bisected by the direction  $\theta$  and opening larger than  $\pi s = \pi/k$  (such a choice is already possible by definition of a proper subsector, see Footnote 2). Let us also choose a radius  $0 < r_1 < \min(r'_1, c^{-1/p})$  and let us set  $C_1 := C \sum_{\mu \geq 0} (cr_1^p)^\mu \in \mathbb{R}_+^*$ .

Thanks to Proposition 4.4, the series  $\sum_{\mu \geq 0} \partial_t^\ell w_\mu(t, x)$  are normally convergent on  $\Sigma \times D_{r_1}$  for all  $\ell \geq 0$  and satisfy the inequalities

$$\sum_{\mu \geq 0} |\partial_t^\ell w_\mu(t, x)| \leq C_1 K_1^\ell \Gamma(1 + (s+1)\ell)$$

for all  $(t, x) \in \Sigma \times D_{r_1}$ . In particular, the sum  $w(t, x)$  of the series  $\sum_{\mu \geq 0} w_\mu(t, x)$  is well-defined, holomorphic on  $\Sigma \times D_{r_1}$  and satisfies the inequalities

$$|\partial_t^\ell w(t, x)| \leq C_1 K_1^\ell \Gamma(1 + (s+1)\ell)$$

for all  $\ell \geq 0$  and all  $(t, x) \in \Sigma \times D_{r_1}$ . Hence, Conditions 1 and 3 of Definition 3 hold, since  $\Sigma' \Subset \Sigma$ .

To prove the second condition of Definition 3, we proceed as follows. The removable singularities theorem implies the existence of  $\lim_{\substack{t \rightarrow 0 \\ t \in \Sigma'}} \partial_t^\ell w(t, x)$  for all  $x \in D_{r_1}$  and, thereby, the existence of the Taylor series of  $w$  at 0 on  $\Sigma'$  for all  $x \in D_{r_1}$  (see for instance [?, Cor. 1.1.3.3]; see also [?, Prop. 1.1.11]). On the other hand, considering recurrence relations (4.3) with  $w_\mu(t, x)$  and the  $k$ -sums  $v(t, x)$  and  $g(t, x)$  instead of  $\tilde{w}_\mu(t, x)$ ,  $\tilde{v}(t, x)$  and  $\tilde{g}(t, x)$ , it is clear that  $w(t, x)$  satisfies equation (4.2) with  $v(t, x)$  in place of  $\tilde{v}(t, x)$  and right-hand side  $g(t, x)$  in place of  $\tilde{g}(t, x)$  and, consequently, so does its Taylor series. Then, since equation (4.2) has a unique formal series solution  $\tilde{w}(t, x)$ , we then conclude that the Taylor expansion of  $w(t, x)$  is  $\tilde{w}(t, x)$ . Hence, Condition 2 of Definition 3 holds.

This achieves the proof of the  $k$ -summability of  $\tilde{w}(t, x)$  and, thereby, the fact that the condition is sufficient when  $a(0, 0) \neq 0$ .

## 5 Proof of the sufficient condition: the other cases

In this section, we assume that there exists  $q \in \{1, \dots, p-1\}$  such that  $\partial_x^n a(t, x)|_{x=0} \equiv 0$  for all  $n = 0, \dots, q-1$ , and  $\partial_x^q a(0, 0) \neq 0$ .

Let us first observe that, under this assumption, the identities (4.1) become

$$(5.1) \quad \partial_t^\kappa \tilde{u}_{*,n}(t) - \tilde{f}_{*,n}(t) - \sum_{m=2}^d \sum_{n_0+n_1+\dots+n_m=n} \binom{n}{n_0, n_1, \dots, n_m} b_{m;*,n_0}(t) \tilde{u}_{*,n_1}(t) \dots \tilde{u}_{*,n_m}(t) = 0$$

for all  $n \leq q-1$ , and

$$(5.2) \quad \binom{n}{q} a_{*,q}(t) \tilde{u}_{*,n+p-q}(t) + \sum_{n_0=q+1}^n \binom{n}{n_0} a_{*,n_0}(t) \tilde{u}_{*,n+p-n_0}(t) = \partial_t^\kappa \tilde{u}_{*,n}(t) - \tilde{f}_{*,n}(t) - \sum_{m=2}^d \sum_{n_0+n_1+\dots+n_m=n} \binom{n}{n_0, n_1, \dots, n_m} b_{m;*,n_0}(t) \tilde{u}_{*,n_1}(t) \dots \tilde{u}_{*,n_m}(t)$$

for all  $n \geq q$ , with  $a_{*,q}(0) \neq 0$ . In particular, the identities (5.2) tell us, as in the case  $a(0,0) \neq 0$ , that each coefficient  $\tilde{u}_{*,n}(t)$  (hence, the formal solution  $\tilde{u}(t,x)$  too) is uniquely determined from the inhomogeneity  $\tilde{f}(t,x)$  and from the formal series  $\tilde{u}_{*,n'}(t)$  with  $n' = 0, \dots, p-1$ .

Observe also that our assumption allows us to write the function  $a(t,x)$  in the form  $a(t,x) = x^q a_q(t,x)$  with  $a_q(0,0) \neq 0$ . Thereby, the functions

$$A_q(t,x) = \frac{1}{a_q(t,x)} \quad \text{and} \quad B_{m;q}(t,x) = \frac{b_m(t,x)}{a_q(t,x)}$$

with  $m = 2, \dots, d$ , are all well-defined and holomorphic on a convenient common polydisc centered at the origin  $(0,0) \in \mathbb{C}^2$ , say  $D_{\rho'_0} \times D_{\rho'_1}$  with  $0 < \rho'_0 \leq \rho_0$  and  $0 < \rho'_1 \leq \rho_1$  to use the same notations as the case  $a(0,0) \neq 0$ .

Setting as before  $\tilde{u}(t,x) = \tilde{v}(t,x) + \partial_x^{-p} \tilde{w}(t,x)$  with

$$\tilde{v}(t,x) = \sum_{n=0}^{p-1} \tilde{u}_{*,n}(t) \frac{x^n}{n!},$$

Eq. (2.1) becomes now

$$(5.3) \quad \tilde{w} - \frac{A_q(t,x)}{x^q} \partial_t^\kappa \partial_x^{-p} \tilde{w} + \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} \frac{B_{m;q}(t,x)}{x^q} \tilde{v}^{m-j}(t,x) (\partial_x^{-p} \tilde{w})^j = \tilde{g}(t,x)$$

where

$$\tilde{g}(t,x) = A_q(t,x) \frac{\partial_t^\kappa \tilde{v}(t,x) - \sum_{m=2}^d b_m(t,x) \tilde{v}^m(t,x) - \tilde{f}(t,x)}{x^q}$$

is again a formal power series in  $t$  and  $x$ . Indeed, due to the identities (5.1), the term

$$\partial_t^\kappa \tilde{v}(t,x) - \sum_{m=2}^d b_m(t,x) \tilde{v}^m(t,x) - \tilde{f}(t,x)$$

is of order  $O(x^q)$  in  $x$ . Assuming then  $\tilde{v}(t, x)$  and  $\tilde{g}(t, x)$  to be  $k$ -summable in the direction  $\theta$ , we can prove as in the case  $a(0, 0) \neq 0$  that  $\tilde{w}(t, x)$  is also  $k$ -summable in the direction  $\theta$ . The proof being similar to the one developed in Section 4, we give below only the key points to modify. In particular, we keep all the notations on the choices of the sectors  $\Sigma$  and  $\Sigma'$ , and on the choices of the various radii.

First of all, let us start by observing that the  $\tilde{w}_\mu(t, x)$ 's are now recursively determined for all  $\mu \geq 0$  by the relations

$$(5.4) \quad \tilde{w}_{\mu+1}(t, x) = \frac{A_q(t, x)}{x^q} \partial_t^\kappa \partial_x^{-p} \tilde{w}_\mu(t, x) \\ - \sum_{m=2}^d \sum_{j=1}^m \sum_{\mu_1 + \dots + \mu_j = \mu} \left[ \binom{m}{j} \frac{B_{m;q}(t, x)}{x^q} \tilde{v}^{m-j}(t, x) \left( \prod_{i'=1}^j \partial_x^{-p} \tilde{w}_{\mu_{i'}}(t, x) \right) \right]$$

together with the initial condition  $\tilde{w}_0 = \tilde{g}$ . In particular, the operator  $\frac{1}{x^q} \partial_x^{-p}$  in place of  $\partial_x^{-p}$  implies that the  $\tilde{w}_\mu(t, x)$ 's are of order  $O(x^{(p-q)\mu})$  in  $x$  for all  $\mu \geq 0$ , instead of  $O(x^{p\mu})$  as in the case  $a(0, 0) \neq 0$ . Still,  $\tilde{w}(t, x)$  is again a formal power series in  $t$  and in  $x$  since  $p - q \geq 1$ .

In doing so, the estimates on the derivatives  $\partial_t^\ell w_\mu$  given in Proposition 4.4 are modified as follows.

For all  $\ell, \mu \geq 0$  and all  $(t, x) \in \Sigma \times D_{r'_1}$ ,

$$|\partial_t^\ell w_\mu(t, x)| \leq CC'^\mu K^{\kappa\mu + \ell} \Gamma(1 + (s+1)(\kappa\mu + \ell)) \frac{|x|^{(p-q)\mu}}{(\mu!)^q ((p-q)\mu)!}$$

with

$$C' = \left( \alpha_q C_s + \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} \beta_{m;q} C^{m-1} C_s^m \zeta(p)^{j-1} \right),$$

where  $\alpha_q$  (resp.  $\beta_{m;q}$  for  $m = 2, \dots, d$ ) stands for the maximum of  $|A_q(t, x)|$  (resp.  $|B_{m;q}(t, x)|$ ) on the closed polydisc  $\overline{D}_{\rho_0''} \times \overline{D}_{\rho_1''}$ .

We proceed as in the proof of Proposition 4.4 by replacing the inequality (4.6) by the inequality

$$(5.5) \quad |\partial_t^\ell w_{\mu+1}(t, x)| \leq \frac{CC'^\mu K^{\kappa(\mu+1) + \ell} \Gamma(1 + (s+1)(\kappa(\mu+1) + \ell)) |x|^{(p-q)(\mu+1)}}{((\mu+1)!)^q ((p-q)(\mu+1)!)^q} \times \\ \left( \alpha_q S_{\mu, \ell, q} + \sum_{m=2}^d \sum_{j=1}^m \binom{m}{j} \beta_{m;q} C^{m-1} S'_{\mu, \ell, m, j, q} \right),$$

where the constants  $S_{\mu, \ell, q}$  and  $S'_{\mu, \ell, m, j, q}$  are respectively defined by

$$S_{\mu, \ell, q} = \sum_{\ell_0=0}^{\ell} \binom{\ell}{\ell_0} \frac{(\ell - \ell_0)! \Gamma(1 + (s+1)(\kappa\mu + \kappa + \ell_0))}{\Gamma(1 + (s+1)(\kappa\mu + \kappa + \ell))} F_{\mu, \ell, q}$$

and

$$S'_{\mu,\ell,m,j,q} = \sum_{\mu_1+\dots+\mu_j=\mu} \sum_{\ell_0+\ell_1+\dots+\ell_m=\ell} \left( \binom{\ell}{\ell_0, \ell_1, \dots, \ell_m} F'_{\mu,\ell,j,q} \times \frac{\ell_0! \left( \prod_{i=1}^{m-j} \Gamma(1+(s+1)\ell_i) \right) \left( \prod_{i'=1}^j \Gamma(1+(s+1)(\kappa\mu_{i'} + \ell_{m-j+i'}) \right)}{\Gamma(1+(s+1)(\kappa\mu + \kappa + \ell))} \right),$$

with

$$F_{\mu,\ell,q} = \frac{((\mu+1)!)^q ((p-q)(\mu+1)!)}{(\mu!)^q ((p-q)\mu)! ((p-q)\mu+1) \dots ((p-q)\mu+p)} \quad \text{and}$$

$$F'_{\mu,\ell,j,q} = \frac{((\mu+1)!)^q ((p-q)(\mu+1)!)}{\prod_{i'=1}^j (\mu_{i'}!)^q ((p-q)\mu_{i'})! ((p-q)\mu_{i'}+1) \dots ((p-q)\mu_{i'}+p)}.$$

As previously, the product on  $i$  in  $S'_{\mu,\ell,m,j,q}$  is 1 when  $j = m$ .

Observe that  $F'_{\mu,\ell,1,q} = F_{\mu,\ell,q}$ . Observe also that the term  $F_{\mu,\ell,q}$  is obvious  $\leq 1$ . Indeed, the inequalities  $p > p - q \geq 1$  imply

$$F_{\mu,\ell,q} = \frac{((\mu+1)!)^q}{(\mu!)^q ((p-q)\mu + p - q + 1) \dots ((p-q)\mu + p)} \leq \frac{((\mu+1)!)^q}{(\mu!)^q (\mu+1)^q} = 1.$$

Thereby, Lemmas 4.4 and 4.4 provide us the inequalities

$$S_{\mu,\ell,q} \leq S_{\mu,\ell} \leq C_s \quad \text{and} \quad S'_{\mu,\ell,m,1,q} \leq S'_{\mu,\ell,m,1} \leq C_s^m = C_s^m \zeta(p)^0.$$

By reasoning in the same way for  $F'_{\mu,\ell,j,q}$  when  $j \geq 2$ , we first have

$$\begin{aligned} F'_{\mu,\ell,j,q} &\leq \frac{((\mu+1)!)^q ((p-q)(\mu+1)!)}{\prod_{i'=1}^j ((\mu_{i'}+1)!)^q ((p-q)(\mu_{i'}+1))!} \\ &= \frac{\binom{\mu+1}{\mu_1, \dots, \mu_{j-1}, \mu_j+1} \binom{(p-q)(\mu+1)}{(p-q)\mu_1, \dots, (p-q)\mu_{j-1}, (p-q)(\mu_j+1)}}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^q ((p-q)\mu_{i'}+1) \dots ((p-q)\mu_{i'}+p-q)} \\ &\leq \frac{\binom{\mu+1}{\mu_1, \dots, \mu_{j-1}, \mu_j+1} \binom{(p-q)(\mu+1)}{(p-q)\mu_1, \dots, (p-q)\mu_{j-1}, (p-q)(\mu_j+1)}}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^p}. \end{aligned}$$

Applying then the calculations made in the proof of Lemma 4.4, we finally get

$$S'_{\mu,\ell,m,j,q} \leq \sum_{\mu_1+\dots+\mu_j=\mu} \left( \frac{1}{\prod_{i'=1}^{j-1} (\mu_{i'}+1)^p} \sum_{\ell_0+\ell_1+\dots+\ell_m=\ell} \frac{1}{\binom{s\ell}{s\ell_0, s\ell_1, \dots, s\ell_m}} \right) \\ \leq C_s^m \zeta(p)^{j-1},$$

which ends the proof of Proposition 5.

Using now the inequality

$$\Gamma(1+(s+1)(\kappa\mu+\ell)) \leq 2^{p\ell+p\mu} (p\mu)! \Gamma(1+(s+1)\ell)$$

proved in the proof of Proposition 4.4, Proposition 5 leads us to the inequality

$$|\partial_t^\ell w_\mu(t, x)| \leq C(2^p K)^\ell (2^p C' K^\kappa |x|^{p-q})^\mu \Gamma(1+(s+1)\ell) \times \binom{p\mu}{\mu, \dots, \mu, (p-q)\mu},$$

where  $\mu$  occurs  $q$  times in the multinomial coefficient. Hence, the following

Let us set  $K_1 = 2^p K$  and  $c = (2q+2)^p C' K^\kappa$ .

Then, the inequalities

$$|\partial_t^\ell w_\mu(t, x)| \leq C K_1^\ell \Gamma(1+(s+1)\ell) (c|x|^{p-q})^\mu$$

hold for all  $\ell, \mu \geq 0$  and all  $(t, x) \in \Sigma \times D_{r_1}$ .

The end of the proof is similar to the one of the case  $a(0,0) \neq 0$  and is left to the reader. This completes the proof of Theorem 3.

## 6 Some technical results on the generalized binomial and multinomial coefficients

In combinatorial analysis, the binomial coefficients  $\binom{n}{m}$  and the multinomial coefficients  $\binom{n}{n_1, \dots, n_q}$  are defined for any nonnegative integers  $0 \leq m \leq n$  and any tuples  $(n, n_1, \dots, n_q)$  of nonnegative integers satisfying  $q \geq 2$  and  $n_1 + \dots + n_q = n$  by the relations

$$\binom{n}{m} = \frac{n!}{k!(n-k)!} \quad \text{and} \quad \binom{n}{n_1, \dots, n_q} = \frac{n!}{n_1! \dots n_q!}.$$

They respectively denote the number of ways of choosing  $m$  objects from a collection of  $n$  distinct objects without regard to order, and the number of ways of putting  $n = n_1 + \dots + n_q$  different objects into  $q$  different boxes with  $n_i$  in the  $i$ -th box for all  $i = 1, \dots, q$ .

Using the fact that  $n! = \Gamma(1+n)$  for any integer  $n \geq 0$ , one can easily extend the definitions of these coefficients to the case where their terms are no longer necessarily integers by setting

$$(6.1) \quad \binom{a}{b} = \frac{\Gamma(1+a)}{\Gamma(1+b)\Gamma(1+a-b)}$$

for any nonnegative real numbers  $0 \leq b \leq a$  and

$$(6.2) \quad \binom{a}{a_1, \dots, a_q} = \frac{\Gamma(1+a)}{\Gamma(1+a_1)\dots\Gamma(1+a_q)} = \frac{\Gamma(1+a)}{\prod_{i=1}^q \Gamma(1+a_i)}$$

for any tuples  $(a, a_1, \dots, a_q)$  of nonnegative real numbers satisfying  $q \geq 2$  and  $a_1 + \dots + a_q = a$ . Observe that all these coefficients are positive. Observe also that one has the following decomposition

$$(6.3) \quad \binom{a}{a_1, \dots, a_q} = \prod_{i=2}^q \binom{a_1 + \dots + a_i}{a_1 + \dots + a_{i-1}}.$$

The four propositions below extend to the generalized binomial coefficients (6.1) and the generalized multinomial coefficients (6.2) some well-known results in combinatorial analysis.

In the proof of Theorem 3 (see Sections 4 and 5), we essentially used the inequalities stated in Propositions 6 and 6. The result of Proposition 6 is useful for the proof of Proposition 6 and the one of Proposition 6 is used in the proof of Proposition 6.

[Pascal's formula] Let  $0 \leq b \leq a$  be two nonnegative real numbers and  $1 \leq m \leq n$  two nonnegative integers. Then,

$$(6.4) \quad \binom{a+n+1}{b+m} = \binom{a+n}{b+m} + \binom{a+n}{b+m-1}.$$

We compute:

$$\begin{aligned}
\binom{a+n}{b+m} + \binom{a+n}{b+m-1} &= \frac{\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m)} \\
&\quad + \frac{\Gamma(1+a+n)}{\Gamma(1+b+m-1)\Gamma(1+a-b+n-m+1)} \\
&= \frac{(a-b+n-m+1)\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&\quad + \frac{(b+m)\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&= \frac{(a+n+1)\Gamma(1+a+n)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&= \frac{\Gamma(1+a+n+1)}{\Gamma(1+b+m)\Gamma(1+a-b+n-m+1)} \\
&= \binom{a+n+1}{b+m};
\end{aligned}$$

hence, the identity (6.4).

[Vandermonde's inequality]

1. (Binomial case) Let  $0 \leq b \leq a$  be two nonnegative real numbers and  $0 \leq m \leq n$  two nonnegative integers. Then,

$$(6.5) \quad \binom{a+n}{b+m} \geq \binom{a}{b} \binom{n}{m}.$$

2. (Multinomial case) Let  $q \geq 2$  be an integer,  $(a, a_1, \dots, a_q)$  a tuple of nonnegative real numbers and  $(n, n_1, \dots, n_q)$  a tuple of nonnegative integers such that  $a_1 + \dots + a_q = a$  and  $n_1 + \dots + n_q = n$ . Then,

$$(6.6) \quad \binom{a+n}{a_1+n_1, \dots, a_q+n_q} \geq \binom{a}{a_1, \dots, a_q} \binom{n}{n_1, \dots, n_q}.$$

★ *First point.* The inequality (6.5) is clear for  $n = m = 0$ . Let us now fix  $0 \leq b \leq a$  and let us prove by induction on  $n \geq 1$  the property

$$(\mathcal{P}_n) : \forall m \in \{0, \dots, n\}, \binom{a+n}{b+m} \geq \binom{a}{b} \binom{n}{m}.$$

A direct calculation gives us the property  $(\mathcal{P}_1)$ :

$$\frac{\binom{a+1}{b}}{\binom{a}{b}} = \frac{\Gamma(1+a+1)\Gamma(1+b)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+a+1-b)} = \frac{a+1}{a+1-b} \geq 1 = \binom{1}{0},$$

$$\frac{\binom{a+1}{b+1}}{\binom{a}{b}} = \frac{\Gamma(1+a+1)\Gamma(1+b)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+b+1)\Gamma(1+a-b)} = \frac{a+1}{b+1} \geq 1 = \binom{1}{1}.$$

Assuming now the property  $(\mathcal{P}_n)$  for a certain  $n \geq 1$ , let us prove the property  $(\mathcal{P}_{n+1})$ . As for the property  $(\mathcal{P}_1)$ , the sought inequality stems from a direct calculation when  $m = 0$  and  $m = n + 1$ :

$$\frac{\binom{a+n+1}{b}}{\binom{a}{b}} = \frac{\Gamma(1+a+n+1)\Gamma(1+b)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+a+n+1-b)} = \frac{\prod_{k=1}^{n+1} (a+k)}{\prod_{k=1}^{n+1} (a+k-b)} \geq 1 = \binom{n+1}{0},$$

$$\frac{\binom{a+n+1}{b+n+1}}{\binom{a}{b}} = \frac{\Gamma(1+a+n+1)\Gamma(1+b)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+b+n+1)\Gamma(1+a-b)} = \frac{\prod_{k=1}^{n+1} (a+k)}{\prod_{k=1}^{n+1} (b+k)} \geq 1 = \binom{n+1}{n+1}.$$

When  $m \in \{1, \dots, n\}$ , it stems from Proposition 6 and the property  $(\mathcal{P}_n)$  as follows:

$$\begin{aligned} \binom{a+n+1}{b+m} &= \binom{a+n}{b+m} + \binom{a+n}{b+m-1} \geq \binom{a}{b} \binom{n}{m} + \binom{a}{b} \binom{n}{m-1} \\ &= \binom{a}{b} \left( \binom{n}{m} + \binom{n}{m-1} \right) \\ &= \binom{a}{b} \binom{n+1}{m}. \end{aligned}$$

This ends the induction and proves the first point of Proposition 6.

★ *Second point.* Let us apply the relation (6.3) and the inequality (6.5) to each factor of the product. We get

$$\begin{aligned} \binom{a+n}{a_1+n_1, \dots, a_q+n_q} &= \prod_{i=2}^q \binom{a_1+\dots+a_i+n_1+\dots+n_i}{a_1+\dots+a_{i-1}+n_1+\dots+n_{i-1}} \\ &\geq \prod_{i=2}^q \binom{a_1+\dots+a_i}{a_1+\dots+a_{i-1}} \binom{n_1+\dots+n_i}{n_1+\dots+n_{i-1}} \\ &= \left( \prod_{i=2}^q \binom{a_1+\dots+a_i}{a_1+\dots+a_{i-1}} \right) \left( \prod_{i=2}^q \binom{n_1+\dots+n_i}{n_1+\dots+n_{i-1}} \right). \end{aligned}$$

The inequality (6.6) follows then by applying again the relation (6.3), which ends the proof of the second point of Proposition 6.

Let  $s, a > 0$  be. Then, the function

$$B_a : b \in [0, a] \mapsto \binom{sa}{sb}$$

is increasing on  $\left[0, \frac{a}{2}\right]$  and decreasing on  $\left[\frac{a}{2}, a\right]$ .

In particular,  $B_a(b) \geq 1$  for all  $b \in [0, a]$ .

The derivative of  $B_a$  is defined for all  $b \in [0, a]$  by

$$B'_a(b) = sB_a(b)(\Psi(1+s(a-b)) - \Psi(1+sb)),$$

where  $\Psi = \Gamma'/\Gamma$  is the Psi (or Digamma) function. The latter being increasing on  $[0, a]$  (the function  $\ln \Gamma$  is indeed convex on  $]0, +\infty[$ ), Proposition 6 follows from Lagrange Theorem.

[Sum of the inverses of binomial and multinomial coefficients] Let  $s > 0$  be and let us set  $C_s = s'(2 + \Gamma(ss'))$ , where  $s'$  is the positive integer  $\geq 1$  defined by

$$s' = \begin{cases} 1 & \text{if } s \geq 1 \\ \left\lfloor \frac{1}{s} \right\rfloor + 1 & \text{if } s < 1 \end{cases},$$

where  $\lfloor x \rfloor$  stands for the floor of  $x \in \mathbb{R}$ .

1. (Binomial case) The following inequality holds for all integers  $n \geq 0$ :

$$(6.7) \quad \sum_{m=0}^n \frac{1}{\binom{sn}{sm}} \leq C_s.$$

2. (Multinomial case) The following inequality holds for all integers  $q \geq 2$  and  $n \geq 0$ :

$$(6.8) \quad \sum_{n_1+\dots+n_q=n} \frac{1}{\binom{sn}{sn_1, \dots, sn_q}} \leq C_s^{q-1}.$$

★ *First point.* The inequality (6.7) is straightforward from Proposition 6 when  $n < 2s'$  since we have the relations

$$\sum_{m=0}^n \frac{1}{\binom{sn}{sm}} \leq n+1 \leq 2s' \leq C_s.$$

Let us now assume  $n \geq 2s'$  and let us write the left hand-side of (6.7) in the form

$$\sum_{m=0}^n \frac{1}{\binom{sn}{sm}} = \sum_{m=0}^{s'-1} \frac{1}{\binom{sn}{sm}} + \sum_{m=s'}^{n-s'} \frac{1}{\binom{sn}{sm}} + \sum_{m=n-s'+1}^n \frac{1}{\binom{sn}{sm}}.$$

Applying Proposition 6, the first and the third sums of the right hand-side are both  $\leq s'$ . Moreover, all the terms of the second sum are  $\geq \binom{sn}{ss'}$ . This brings us to the following relations:

$$\begin{aligned} \sum_{m=0}^n \frac{1}{\binom{sn}{sm}} &\leq 2s' + \frac{n-2s'+1}{\binom{sn}{ss'}} = 2s' + \frac{(n-2s'+1)\Gamma(1+ss')\Gamma(1+sn-ss')}{\Gamma(1+sn)} \\ &= 2s' + s'\Gamma(ss') \frac{n-2s'+1}{n} \frac{\Gamma(1+sn-ss')}{\Gamma(1+sn-1)} \\ &\leq 2s' + s'\Gamma(ss') \frac{\Gamma(1+sn-ss')}{\Gamma(1+sn-1)}. \end{aligned}$$

The inequality (6.7) follows then from the increasing of the Gamma function on  $[2, +\infty[$ . Indeed, the inequality  $ss' \geq 1$  implies  $2 \leq 1+sn-ss' \leq 1+sn-1$ , and thereby  $\frac{\Gamma(1+sn-ss')}{\Gamma(1+sn-1)} \leq 1$ .

★ *Second point.* Applying the relation (6.3) and setting  $n'_k = n_1 + \dots + n_k$  for all  $k = 1, \dots, q-1$ , we first get the identities

$$\begin{aligned} \sum_{n_1+\dots+n_q=n} \frac{1}{\binom{sn}{sn_1, \dots, sn_q}} &= \sum_{n_1+\dots+n_{q-1} \leq n} \frac{1}{\binom{sn}{s(n_1+\dots+n_{q-1})} \dots \binom{s(n_1+n_2)}{sn_1}} \\ &= \sum_{n'_{q-1}=0}^n \sum_{n'_{q-2}=0}^{n'_{q-1}} \dots \sum_{n'_1=0}^{n'_2} \frac{1}{\binom{sn}{sn'_{q-1}} \binom{sn'_{q-1}}{sn'_{q-2}} \dots \binom{sn'_2}{sn'_1}}. \end{aligned}$$

The inequality (6.8) stems then from the inequality (6.7) which we apply  $q-1$  times. This completes the proof.