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## To cite this version:

Cédric Arhancet, Yves Raynaud. 2-positive contractive projections on noncommutative Lp-spaces. 2019. hal-03992515

## HAL Id: hal-03992515
## https://hal.science/hal-03992515v1

Preprint submitted on 16 Feb 2023

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arXiv:1912.03128v1 [math.OA] 3 Dec 2019

# 2-positive contractive projections on noncommutative $\mathrm{L}^p$-spaces

**Cédric Arhancet - Yves Raynaud**

**Abstract**

We prove the first theorem on projections on general noncommutative $\mathrm{L}^p$-spaces associated with non-type I von Neumann algebras where $1 \leqslant p < \infty$. This is the first progress on this topic since the seminal work of Arazy and Friedman [Memoirs AMS 1992] where the problem of the description of contractively complemented subspaces of noncommutative $\mathrm{L}^p$-spaces is explicitly raised. We show that the range of a 2-positive contractive projection on an arbitrary noncommutative $\mathrm{L}^p$-space is completely order and completely isometrically isomorphic to some noncommutative $\mathrm{L}^p$-space. This result is sharp and is even new for Schatten spaces $S^p$. Our approach relies on non tracial Haagerup's noncommutative $\mathrm{L}^p$-spaces in an essential way, even in the case of a projection acting on a Schatten space and is unrelated to the methods of Arazy and Friedman.

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December 9, 2019
*Mathematics subject classification:* Primary 46L51, 46L07.
*Key words and phrases*: noncommutative $\mathrm{L}^p$-spaces, projections, complemented subspaces, conditional expectations.

# 1 Introduction

The study of projections and complemented subspaces has been at the heart of the study of Banach spaces since the inception of the field, see [Rand] and [Mos1] for surveys. Recall that a projection $P$ on a Banach space $X$ is a bounded operator $P\colon X \to X$ such that $P^2 = P$ and that a complemented subspace $Y$ of $X$ is the range of a bounded linear projection $P$. If the projection is contractive, we say that $Y$ is contractively complemented.

Suppose $1 \leqslant p < \infty$. A classical result from seventies essentialy due to Ando [And], and completed by [Tza, BeL] tells that a subspace $Y$ of a classical (=commutative) $\mathrm{L}^p$-space $\mathrm{L}^p(\Omega)$ is contractively complemented if and only if $Y$ is isometrically isomorphic to an $\mathrm{L}^p$-space $\mathrm{L}^p(\Omega')$ (see also [Dou], [HoT], [Ray2], [See]). Moreover, $Y$ is the range of a positive contractive projection if and only if there exists a isometrical order isomorphism from $Y$ onto some $\mathrm{L}^p$-space $\mathrm{L}^p(\Omega')$, see [Rand, Theorem 4.10] and [AbA, Problem 5.4.1]. A positive contractive projection on $\mathrm{L}^p(\Omega)$, $1 \leqslant p < \infty$, $p \neq 2$, acts in the band generated by its range as a weighted conditional expectation [BeL]. This last result was extended later to a larger class of Banach function spaces [DHdP].

It is natural to examine the case of noncommutative $\mathrm{L}^p$-spaces associated to von Neumann algebras. Schatten spaces are the most basic examples of noncommutative $\mathrm{L}^p$-spaces, these are the spaces $S^p$ of all operators $x\colon \ell^2 \to \ell^2$ such that $\|x\|_p = \big(\mathrm{Tr}(|x|^p)\big)^{\frac{1}{p}}$ is finite. It is known from a long time that the range of a contractive projection $P\colon S^p \to S^p$ on a Schatten space $S^p$ is not necessarily isometric to a Schatten space. It is a striking difference with the world of commutative $\mathrm{L}^p$-spaces of mesure spaces. Indeed, in their remarkable memoirs [ArF1] and [ArF2], Arazy and Friedman have succeeded in establishing a complete classification of contractively complemented subspaces of $S^p$. Building blocks of contractively complemented subspaces of $S^p$ are the so called Cartan factors.

The description of general contractively complemented subspaces of noncommutative $\mathrm{L}^p$-spaces is an open problem raised explicitly in [ArF2, page 99]. If $p = 1$, Friedman and Russo [FrB] have given a description of the ranges of contractive projections on preduals (=noncommutative $\mathrm{L}^1$-spaces) of von Neumann algebras. Such a subspace is isometric to the predual of a JW$^*$-triple, that is a weak* closed subspace of the space $\mathrm{B}(H, K)$ of bounded operators between Hilbert spaces $H$ and $K$ which is closed under the triple product $xy^*z + zy^*x$. Actually, the Friedman-Russo result is valid for projections acting on the predual of a JW$^*$-triple, not just on the predual of a von Neumann algebra.

Since Pisier's work [Pis1] [Pis2], we can consider noncommutative $\mathrm{L}^p$-spaces and their complemented subspaces in the framework of operator spaces and completely bounded maps. Using Arazy-Friedman Theorem, Le Merdy, Ricard and Roydor [LRR, Th. 1.1] characterized the completely 1-complemented subspaces of $S^p$. They turn out to be the direct sums of spaces of the form $S^p(H, K)$, where $H$ and $K$ are Hilbert spaces. The strategy of their proof is to examine individually each case provided by Arazy-Friedman Theorem. See also [NO], [NeR] for related results.

Recall that a map $T\colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ is $n$-positive for some integer $n$ if the linear map $\mathrm{Id}_{S^p_n} \otimes T\colon S^p_n(\mathrm{L}^p(M)) \to S^p_n(\mathrm{L}^p(M))$ is a positive map. In particular such a map is positive. Our main result is the following theorem which implies that the range of a 2-positive contractive projection $P\colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ on some noncommutative $\mathrm{L}^p$-space $\mathrm{L}^p(M)$ is completely order and completely isometrically isomorphic to some noncommutative $\mathrm{L}^p$-space $\mathrm{L}^p(N)$.

**Theorem 1.1 (Main Theorem)** *Suppose $1 \leqslant p < \infty$. Let $P\colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ be a 2-positive contractive projection.*

1. *There is a projection $s(P) \in M$, a von Neumann subalgebra $N$ of $s(P)Ms(P)$, a normal*

*semifinite faithful weight $\psi$ on $s(P)Ms(P)$ and a normal faithful conditional expectation $\mathbb{E}\colon s(P)Ms(P) \to s(P)Ms(P)$ with range $N$ leaving $\psi$ invariant such that*

*(a)* $P(\mathrm{L}^p(M)) = s(P)P(\mathrm{L}^p(M))s(P) = P(s(P)\mathrm{L}^p(M)s(P))$,

*(b) the restriction of the projection $P$ on $\mathrm{L}^p(s(P)Ms(P))$ identifies with the conditional expectation $\mathbb{E}_{\psi,p}\colon \mathrm{L}^p(s(P)Ms(P)) \to \mathrm{L}^p(s(P)Ms(P))$ naturally associated with $\mathbb{E}$ and the weight $\psi$,*

*(c) the projection $P$ is $N$-bimodular on $s(P)\mathrm{L}^p(M)s(P)$, that is*

$$P(xhy) = xP(h)y, \quad x, y \in N, h \in s(P)\mathrm{L}^p(M)s(P),$$

*In particular, the range $P(\mathrm{L}^p(M))$ is a $N$-module.*

*(d) the range $P(\mathrm{L}^p(M))$ is completely isometric and completely order $N$-bimodular isomorphic to $\mathrm{L}^p(N)$.*

2. *Moreover, the $\sigma$-finite projections in $N$ are exactly the (left or right) supports of the elements of $P\big(\mathrm{L}^p(M)\big)$.*

3. *Finally, if $1 < p < \infty$ then for any $x \in \mathrm{L}^p(M)$ we have $P(x) = P(s(P)xs(P))$.*

The maps $\mathbb{E}_{\psi,p}$ that we call "conditional expectations" (on noncommutative $\mathrm{L}^p$ spaces) appear in the literature of noncommutative probability, in association with a normal *state* $\psi$, see e.g. the works [Gol, HT, JX, HJX, AcC]. The most achieved one is probably that of [JX]. We cannot limit ourselves to this 'probabilistic' frame, since the von Neumann algebras we are dealing with are not necessarily countably decomposable, and thus may not support a faithful normal state. For this reason we give a definition of theses maps which differs from those of these authors, but follows a very classical scheme in commutative probability, and then verify the coherence of the new definition with that of [JX] in the 'probabilistic case'.

The result stated in the Main Theorem is even new for Schatten spaces $S^p$. The assumption of 2-positivity cannot be dropped. Indeed, if $\sigma\colon S^p \to S^p$, $[x_{ij}] \mapsto [x_{ji}]$ denotes the transpose map then the map $P \overset{\text{def}}{=} \frac{1}{2}(\mathrm{Id}_{S^p} + \sigma)\colon S^p \to S^p$ is a positive contractive projection on the space $\{x \in S^p \ : \ \sigma(x) = x\}$ of symmetric matrices. The more general case of *positive* contractive projections will be investigated in a companion paper [Arh2]. It should also be noted that the use of *non-tracial* Haagerup's noncommutative $\mathrm{L}^p$-spaces is necessary for the case of *tracial* noncommutative $\mathrm{L}^p$-spaces. The paper [Arh1, Th. 5.4] contains an application of this result. Finally, we refer to [PiX] and [HRS] for more information on the structure of noncommutative $\mathrm{L}^p$-spaces.

The paper is organized as follows. Section 2 gives a brief presentation of Haagerup noncommutative $\mathrm{L}^p$-spaces, followed by a presentation of conditional expectation on $\mathrm{L}^p$-space in a general formulation which makes sense for non $\sigma$-finite von Neumann algebras, then some preliminary well-known facts on positive projections in von Neumann algebras and positive linear maps between $\mathrm{L}^p$-spaces are presented or recalled, that are at the root of our results. Finally, Section 3 contains a proof of Theorem 1.1. It is divided in four subsections, the first one gives a local description of the action of a contractive, 2-positive projection. The second subsection presents a short handling of the passage from local to global description in the case where the von-Neumann algebra is $\sigma$-finite, while the third subsection presents the more elaborated treatment of the general case. At this stage the two first points in Main Theorem are proven. The last point of the Main Theorem is a consequence of the general result presented in the last subsection of section 3.

## 2 Preliminaries

The readers are referred to [ER], [Pau] and [Pis1] for details on operator spaces and completely bounded maps and to the surveys [Kos1], [PiX], [Terp] and [Ray1] for noncommutative $\mathrm{L}^p$-spaces and references therein.

### 2.1 Haagerup's noncommutative $\mathrm{L}^p$-spaces.

It is well-known by now that there are several equivalent constructions of noncommutative $\mathrm{L}^p$-spaces associated with a von Neumann algebra. In this paper, we will use Haagerup's noncommutative $\mathrm{L}^p$-spaces introduced in [Haa5] and presented in a more detailed way in [Terp]. $M$ will denote a general von Neumann algebra acting on a Hilbert space $H$ and we denote by $s_l(x)$ and $s_r(x)$ the left support and the right support of an operator $x$. If $x$ is a positive operator then $s_l(x) = s_r(x)$ is called the support of $x$ and denoted by $s(x)$.

If $M$ is equipped with a normal semifinite faithful trace, then the topological $*$-algebra of all (unbounded) $\tau$-measurable operators $x$ affiliated with $M$ is denoted by $\mathrm{L}^0(M,\tau)$. If $a, b \in \mathrm{L}^0(M,\tau)_+$, we have

$$a \leqslant b \iff \operatorname{dom} b^{\frac{1}{2}} \subset \operatorname{dom} a^{\frac{1}{2}} \text{ and } \|a^{\frac{1}{2}}\xi\|_H \leqslant \|b^{\frac{1}{2}}\xi\|_H, \text{ for any } \xi \in \operatorname{dom} b^{\frac{1}{2}}. \tag{2.1}$$

If $a, b \in \mathrm{L}^0(M,\tau)$, we have

$$ab = 0 \Rightarrow s_r(a)b = 0 \text{ and } as_l(b) = 0. \tag{2.2}$$

In the sequel, we fix a normal semifinite faithful weight $\varphi$ on $M$ and $\sigma^\varphi = (\sigma_t^\varphi)_{t\in\mathbb{R}}$ denote the one-parameter modular automorphisms group associated with $\varphi$ [Tak2, page 92].

For $1 \leqslant p < \infty$, the spaces $\mathrm{L}^p(M)$ are constructed as spaces of measurable operators relative not to $M$ but to some semifinite bigger von Neumann algebra, namely, the crossed product $\mathcal{M} \overset{\text{def}}{=} M \rtimes_{\sigma^\varphi} \mathbb{R}$ of $M$ by one of its modular automorphisms groups, that is, the von Neumann subalgebra of $\mathrm{B}(\mathrm{L}^2(\mathbb{R}, H))$ generated by the operators $\pi(x)$ and $\lambda_s$, where $x \in M$ and $s \in \mathbb{R}$, defined by

$$\big(\pi(x)\xi\big)(t) \overset{\text{def}}{=} \sigma_{-t}^\varphi(x)(\xi(t)) \quad \text{and} \quad \lambda_s(\xi(t)) \overset{\text{def}}{=} \xi(t-s), \quad t \in \mathbb{R},\ \xi \in \mathrm{L}^2(\mathbb{R}, H).$$

For any $s \in \mathbb{R}$, let $W(s)$ be the unitary operator on $\mathrm{L}^2(\mathbb{R}, H)$ defined by

$$\big(W(s)\xi\big)(t) \overset{\text{def}}{=} \mathrm{e}^{-\mathrm{i}st}\xi(t), \quad \xi \in \mathrm{L}^2(\mathbb{R}, H).$$

The dual action $\hat{\sigma}\colon \mathbb{R} \to \mathrm{B}(\mathcal{M})$ on $\mathcal{M}$ [Tak2, page 260] is given by

$$\widehat{\sigma}_s(x) \overset{\text{def}}{=} W(s)xW(s)^*, \quad x \in \mathcal{M},\ s \in \mathbb{R}. \tag{2.3}$$

Then, by [Haa4, Lemma 3.6] or [Tak2, page 259], $\pi(M)$ is the fixed subalgebra of $\mathcal{M}$ under the family of automorphisms $\widehat{\sigma}_s$:

$$\pi(M) = \big\{x \in \mathcal{M} \ :\ \widehat{\sigma}_s(x) = x \quad \text{for all } s \in \mathbb{R}\big\}. \tag{2.4}$$

We identify $M$ with the subalgebra $\pi(M)$ in $\mathcal{M}$. If $\psi$ is a normal semifinite weight on $M$, we denote by $\widehat{\psi}$ its Takesaki's dual weight on the crossed product $\mathcal{M}$, see the introduction of [Haa1] for a simple definition using the theory of operator valued weights. This dual weight satisfies the $\widehat{\sigma}$-invariance relation $\widehat{\psi} \circ \widehat{\sigma} = \widehat{\psi}$, see [Terp, (10) page 26]. In fact, Haagerup introduces an

operator valued weight $T\colon \mathcal{M}^+ \to \bar{M}^+$ with values in the extended positive part[1] $\bar{M}^+$ of $M$ and formally defined by

$$T(x) = \int_{\mathbb{R}} \hat{\sigma}_s(x)\,\mathrm{d}s \tag{2.5}$$

and shows that for a normal semifinite weight $\psi$ on $M$, its dual weight is

$$\hat{\psi} \stackrel{\text{def}}{=} \bar{\psi} \circ T \tag{2.6}$$

where $\bar{\psi}$ denotes the natural extension of the normal weight $\psi$ to the whole of $\bar{M}^+$.

By [Str, page 301] [Haa4, Th. 3.7] [Terp, Chap. II, Lemma 1], the map $\psi \to \widehat{\psi}$ is a bijection from the set of normal semifinite weights on $M$ onto the set of normal semifinite $\widehat{\sigma}$-invariant weights on $\mathcal{M}$.

Recall that by [Haa2, Lemma 5.2 and Remark page 343] and [Haa1, Th. 1.1 (c)] the crossed product $\mathcal{M}$ is semifinite and there is a unique normal semifinite faithful trace $\tau = \tau_\varphi$ on $\mathcal{M}$ satisfying $(\mathrm{D}\widehat{\varphi} : \mathrm{D}\tau)_t = \lambda_t$ for any $t \in \mathbb{R}$ where $(\mathrm{D}\widehat{\varphi} : \mathrm{D}\tau)_t$ denotes the Radon-Nikodym cocycle [Str, page 48] [Tak2, page 111] of the dual weight $\widehat{\varphi}$ with respect to $\tau$. Moreover, $\tau$ satisfies the relative invariance $\tau \circ \widehat{\sigma}_s = \mathrm{e}^{-s}\tau$ for any $s \in \mathbb{R}$ by [Haa2, Lemma 5.2].

If $\psi$ is a normal semifinite weight on $M$, we denote by $h_\psi$ the Radon-Nikodym derivative of the dual weight $\widehat{\psi}$ with respect to $\tau$ given by [Str, Theorem 4.10]. By [Str, Corollary 4.8], note that the relation of $h_\psi$ with the Radon-Nikodym cocycle of $\widehat{\psi}$ is

$$(\mathrm{D}\widehat{\psi} : \mathrm{D}\tau)_t = h_\psi^{\mathrm{i}t}, \quad t \in \mathbb{R}. \tag{2.7}$$

By [Terp, Chap. II, Prop. 4], the mapping $\psi \to h_\psi$ gives a bijective correspondence between the set of all normal semifinite weights on $M$ and the set of positive self-adjoint operators $h$ affiliated with $\mathcal{M}$ satisfying

$$\widehat{\sigma}_s(h) = \mathrm{e}^{-s}h, \quad s \in \mathbb{R}. \tag{2.8}$$

Moreover, by [Terp, Chap. II, Cor. 6], $\omega$ belongs to $M_*^+$ if and only if $h_\omega$ belongs to $\mathrm{L}^0(\mathcal{M},\tau)_+$. One may extend by linearity the map $\omega \mapsto h_\omega$ to the whole of $M_*$. The Haagerup space $\mathrm{L}^1(M,\varphi)$ is defined as the set $\{h_\omega : \omega \in M_*\}$, i.e. the range of the previous map. This is a closed linear subspace of $\mathrm{L}^0(\mathcal{M},\tau)$, characterized by the conditions (2.8).

By [Terp, Chap. II, Th. 7], the mapping $\omega \mapsto h_\omega$, $M_* \to \mathrm{L}^1(M,\varphi)$ is a linear order isomorphism which preserves the conjugation, the module, and the left and right actions of $M$. Then $\mathrm{L}^1(M,\varphi)$ may be equipped with a continuous linear functional $\mathrm{Tr}\colon \mathrm{L}^1(M) \to \mathbb{C}$ defined by

$$\mathrm{Tr}\, h_\omega \stackrel{\text{def}}{=} \omega(1), \quad \omega \in M_* \tag{2.9}$$

[Terp, Chap. II, Def. 13]. A norm on $\mathrm{L}^1(M,\varphi)$ may be defined by $\|h\|_1 \stackrel{\text{def}}{=} \mathrm{Tr}(|h|)$ for every $h \in \mathrm{L}^1(M,\varphi)$. By [Terp, Chap. II, Prop. 15], the map $M_* \to \mathrm{L}^1(M,\varphi)$, $\omega \mapsto h_\omega$ is a surjective isometry.

More generally for $1 \leqslant p \leqslant \infty$, the Haagerup $\mathrm{L}^p$-space $\mathrm{L}^p(M,\varphi)$ associated with the normal faithful semifinite weight $\varphi$ is defined [Terp, Chap. II, Def. 9] as the subset of the topological $*$-algebra $\mathrm{L}^0(\mathcal{M},\tau)$ of all (unbounded) $\tau$-measurable operators $x$ affiliated with $\mathcal{M}$ satisfying for any $s \in \mathbb{R}$ the condition

$$\widehat{\sigma}_s(x) = \mathrm{e}^{-\frac{s}{p}}x \quad \text{if } p < \infty \quad \text{and} \quad \widehat{\sigma}_s(x) = x \quad \text{if } p = \infty \tag{2.10}$$

1. If $M = \mathrm{L}^\infty(\Omega)$, $\bar{M}^+$ identifies to the set of equivalence classes of measurable functions $\Omega \to [0,\infty]$.

where $\widehat{\sigma}_s\colon \mathrm{L}^0(\mathcal{M},\tau)\to \mathrm{L}^0(\mathcal{M},\tau)$ is here the continuous $*$-automorphism obtained by a natural extension of the dual action (2.3) on $\mathcal{M}$. By (2.4), the space $\mathrm{L}^\infty(M,\varphi)$ coincides with $\pi(M)$ that we identify with $M$. The spaces $\mathrm{L}^p(M,\varphi)$ are closed self-adjoint linear subspaces of $\mathrm{L}^0(\mathcal{M},\tau)$. They are closed under left and right multiplications by elements of $M$. If $h=u|h|$ is the polar decomposition of $h\in \mathrm{L}^0(\mathcal{M},\tau)$ then by [Terp, Chap. II, Prop. 12] we have

$$h\in \mathrm{L}^p(M,\varphi) \iff u\in M \text{ and } |h|\in \mathrm{L}^p(M,\varphi).$$

Suppose $1\leqslant p<\infty$. By [Terp, Chap. II, Prop. 12] and its proof, for any $h\in \mathrm{L}^0(\mathrm{M},\tau)_+$, we have $h^p\in \mathrm{L}^0(\mathrm{M},\tau)_+$. Moreover, an element $h\in \mathrm{L}^0(\mathrm{M},\tau)$ belongs to $\mathrm{L}^p(M,\varphi)$ if and only if $|h|^p$ belongs to $\mathrm{L}^1(M,\varphi)$. A norm on $\mathrm{L}^p(M,\varphi)$ is then defined by the formula

$$\|h\|_p \overset{\text{def}}{=} (\mathrm{Tr}\,|h|^p)^{\frac{1}{p}} \tag{2.11}$$

if $1\leqslant p<\infty$ and by $\|h\|_\infty \overset{\text{def}}{=} \|h\|_M$, see [Terp, Chap. II, Def. 14].

Let $p,p^*\in[1,\infty]$ with $\frac{1}{p}+\frac{1}{p^*}=1$. By [Terp, Chap. II, Prop. 21], for any $h\in \mathrm{L}^p(M,\varphi)$ and any $k\in \mathrm{L}^{p^*}(M,\varphi)$ we have $hk,kh\in \mathrm{L}^1(M,\varphi)$ and the tracial property $\mathrm{Tr}(hk)=\mathrm{Tr}(kh)$.

If $1\leqslant p<\infty$, by [Terp, Ch. II, Th. 32] the bilinear form $\mathrm{L}^p(M,\varphi)\times \mathrm{L}^{p^*}(M,\varphi)\to\mathbb{C}$, $(h,k)\mapsto \mathrm{Tr}(hk)$ defines a duality bracket between $\mathrm{L}^p(M,\varphi)$ and $\mathrm{L}^{p^*}(M,\varphi)$, for which $\mathrm{L}^{p^*}(M,\varphi)$ is (isometrically) the dual of $\mathrm{L}^p(M,\varphi)$.

On the other hand, if the weight $\varphi$ is tracial, i.e. $\varphi(x^*x)=\varphi(xx^*)$ for all $x\in M$, then the Haagerup space $\mathrm{L}^p(M,\varphi)$ isometrically coincides with Dixmier's classical tracial noncommutative $\mathrm{L}^p$-space, see [Terp, page 62].

It is essentially proved in [Terp, page 59] that $\mathrm{L}^p(M,\varphi)$ is independent of $\varphi$ up to an isometric isomorphism preserving the order and modular structure of $\mathrm{L}^p(M,\varphi)$, as well as the external products and Mazur maps. In fact given two normal semifinite faithful weights $\varphi_1,\varphi_2$ on $M$ there is a $*$-isomorphism $\kappa\colon \mathcal{M}_1\to\mathcal{M}_2$ between the crossed products $\mathcal{M}_i \overset{\text{def}}{=} M\rtimes_{\sigma_i^\varphi}\mathbb{R}$ preserving $M$, as well as the dual actions and pushing the trace on $\mathcal{M}_1$ onto the trace on $\mathcal{M}_2$, that is

$$\pi_2=\kappa\circ\pi_1,\quad \hat{\sigma}_2\circ\kappa=\kappa\circ\hat{\sigma}_1 \quad\text{and}\quad \tau_2=\tau_1\circ\kappa^{-1}. \tag{2.12}$$

Furthermore, $\kappa$ extends naturally to a topological $*$-isomorphism $\hat{\kappa}\colon \mathrm{L}^0(\mathcal{M}_1,\tau_1)\to \mathrm{L}^0(\mathcal{M}_2,\tau_2)$ between the algebras of measurable operators, which restricts to isometric $*$-isomorphisms between the respective $\mathrm{L}^p(M_i,\varphi_i)$, preserving the $M$-bimodule structures.

Moreover it turns out also that for every normal semifinite faithful weight $\psi$ on $M$, the dual weights $\hat{\psi}_i$ corresponds through $\kappa$, that is $\hat{\psi}_2\circ\kappa=\hat{\psi}_1$. It follows that if $\omega\in M_*$ the corresponding Radon-Nikodym derivatives must verify $h_{\omega,2}=\hat{\kappa}(h_{\omega,1})$. In particular if $\omega\in M_*^+$, we have

$$\mathrm{Tr}_1\, h_{\omega,1} \overset{(2.9)}{=} \omega(1) \overset{(2.9)}{=} \mathrm{Tr}_2\, h_{\omega,2}=\mathrm{Tr}_2\,\hat{\kappa}(h_{\omega,1}). \tag{2.13}$$

Hence $\hat{\kappa}\colon \mathrm{L}^1(M,\varphi_1)\to \mathrm{L}^1(M,\varphi_2)$ preserves the functionals Tr:

$$\mathrm{Tr}_1=\mathrm{Tr}_2\circ\hat{\kappa}. \tag{2.14}$$

Since $\hat{\kappa}$ preserves the $p$-powers operations, i.e. $\hat{\kappa}(h^p)=\big(\hat{\kappa}(h)\big)^p$ for any $h\in \mathrm{L}^0(\mathcal{M}_1)$, it induces an isometry from $\mathrm{L}^p(M,\varphi_1)$ onto $\mathrm{L}^p(M,\varphi_2)$. It is not hard to see that this isometry is completely positive and completely isometric, a fact which is of first importance for our study.

This independence allows us to consider $\mathrm{L}^p(M,\varphi)$ as a particular realization of an abstract space $\mathrm{L}^p(M)$. The $M$-bimodule structure and the norm of $\mathrm{L}^p(M)$ are defined unambiguously

by those of any of its particular realization, as well as the trace functional of $\mathrm{L}^1(M)$ and the bilinear products $\mathrm{L}^p(M)\times\mathrm{L}^q(M)\to\mathrm{L}^r(M)$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, and the Mazur maps $\mathrm{L}^p_+(M)\to\mathrm{L}^1_+(M)$, $h\mapsto h^p$ (and their inverses). An element $h\in\mathrm{L}^1(M)$ identifies with the linear form $\psi\in M_*$ defined by the conditions $\psi(x)=\mathrm{Tr}(xh)$, $x\in M$, and the positive part $\mathrm{L}^p_+(M)$ may be seen as the cone of $p$-roots $\psi^{\frac{1}{p}}$ of positive elements of $M_*$.

A key fact for our analysis of positive contractions on noncommutative $\mathrm{L}^p$-spaces is that for every projection $e$ in $M$ the linear subspace $e\mathrm{L}^p(M)e:=\{ehe:h\in\mathrm{L}^p(M)\}$ is completely positively isometric to $\mathrm{L}^p(eMe)$, the $\mathrm{L}^p$-space of the reduced von Neumann algebra $eMe$. This fact is not evident from a given realization of $\mathrm{L}^p(M,\varphi)$, since the restriction $\varphi_e$ of $\varphi$ to $eMe$ may not be semifinite, or the crossed product $\mathbb{R}\rtimes_{\sigma^{\varphi_e}}eMe$ not be a reduct of $\mathbb{R}\rtimes_{\sigma^\varphi}M$.

Recall that the centralizer [Str, page 38] of a normal semifinite faithful weight is the sub-von Neumann algebra $M^\varphi=\{x\in M:\sigma^\varphi_t(x)=x\text{ for all }t\in\mathbb{R}\}$. If $x\in M$, we have by [Str, (2) page 39]

$$x\in M^\varphi\iff x\mathfrak{m}_\varphi\subset\mathfrak{m}_\varphi,\ \mathfrak{m}_\varphi x\subset\mathfrak{m}_\varphi\text{ and }\varphi(xy)=\varphi(yx)\text{ for any }y\in\mathfrak{m}_\varphi. \tag{2.15}$$

If $e$ belongs to the centralizer of $\varphi$, it is well-known that we can identify $\mathrm{L}^p(eMe)$ with the subspace $e\mathrm{L}^p(M)e$ of $\mathrm{L}^p(M)$ ([GoLa, Lemma 4.3], [Wat1, page 508]). Let us give some details.

Let $\varphi$ be a faithful normal semifinite weight on a von Neumann algebra $M$ on a Hilbert space $H$. For each projection $e$ in $M^\varphi$, let $\varphi_e$ be the restriction of $\varphi$ on $eMe$. It results from (2.15) that the weight $\varphi_e$ is still semi-finite. From the KMS-condition, it is easy to see that $\sigma^{\varphi_e}=\sigma^\varphi|eMe$, and it follows that $eMe\rtimes_{\sigma^{\varphi_e}}\mathbb{R}$ coincides with $\bar e(M\rtimes_{\sigma^\varphi}\mathbb{R})\bar e$, where $\bar e$ is the canonical image of $e$ in $\mathcal{M}$, in fact $\bar e=e\otimes\mathrm{Id}_{\mathrm{B}(\mathrm{L}^2(\mathbb{R}))}$. From (2.5) and (2.6) it is clear that the dual weights to $\varphi$ and $\varphi_e$ are linked by the equation

$$\widehat{\varphi_e}=(\widehat{\varphi})_{\bar e}$$

Let $\tau=\tau_\varphi$ be the canonical trace on $\mathcal{M}=M\rtimes_{\sigma^\varphi}\mathbb{R}$, and $h_\varphi=\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\varphi}$. Note that $h_\varphi^{\mathrm{i}t}=\lambda_t$ commutes with $\bar e$, and so does $h_\varphi^{\frac12}$; morover $eh_\varphi$ is a positive self-adjoint operator affiliated with $\bar e\mathcal{M}\bar e$. Then for $x\in\bar e\mathcal{M}\bar e$:

$$\widehat{\varphi}(x)=\tau\big(h_\varphi^{\frac12}xh_\varphi^{\frac12}\big)=\tau\big(h_\varphi^{1/2}\bar exe\bar eh_\varphi^{\frac12}\big)=\tau_{\bar e}\big((\bar eh_\varphi)^{\frac12}x(\bar eh_\varphi)^{\frac12}\big)$$

where $\tau_{\bar e}$ is the reduced (normal, semifinite) trace on $\bar e\mathcal{M}\bar e$. Hence by [Str, Theorem 4.10]

$$(\mathrm{D}\widehat{\varphi_e}:\mathrm{D}\tau_{\bar e})_t=(\bar eh_\varphi)^{\mathrm{i}t}=\bar eh_\varphi^{\mathrm{i}t}=\bar e\lambda_t.$$

Since $\bar e\lambda_t$ is the translation operator on $\bar e\mathcal{M}\bar e$ it follows by uniqueness that $\tau_{\bar e}$ coincides with the canonical trace on $eMe\rtimes_{\sigma^{\varphi_e}}\mathbb{R}$. Then it becomes clear that the *-algebra of $\tau_{\bar e}$-measurable operators on $\bar e\mathcal{M}\bar e$ is realized as a $\mathrm{L}^0$-closed *-subalgebra of $\mathrm{L}^0(\mathcal{M},\tau)$, namely $e\mathrm{L}^0(\mathcal{M},\tau)e$, either abstractly by completing the respective von Neumann algebras in their $\mathrm{L}^0$-topologies, or concretely by representing them as *-subalgebras of (unbounded) closed operators on $\mathrm{L}^2(\mathbb{R},eH)$ and $\mathrm{L}^2(\mathbb{R},H)$ respectively if $M$ is given as a von Neumann subalgebra of some $\mathrm{B}(H)$. Indeed, a (closed, unbounded) operator with dense domain $a$ on $\mathrm{L}^2(\mathbb{R},eH)$ may be trivially extended to an operator $\tilde a$ on $\mathrm{L}^2(\mathbb{R},H)$ by setting

$$\tilde a\xi=ae\xi,\quad\forall\xi\in\operatorname{dom}\tilde a=\operatorname{dom}a\oplus\mathrm{L}^2(\mathbb{R},(1-e)H).$$

Since the dual action of $\sigma^{\varphi_e}$ is the restriction of that of $\sigma^\varphi$ to $e(M\rtimes_{\sigma^\varphi}\mathbb{R})e$, it is not difficult to show that the mapping $a\to ebe$ gives a topological *-isomorphism between $\mathrm{L}^0(\bar e\mathcal{M}\bar e,\tau_e)$ and $\bar e\mathrm{L}^0(\mathcal{M},\tau)\bar e$, sending $\mathrm{L}^p(eMe)$ onto $e\mathrm{L}^p(M)e$ for every $1\leqslant p\leqslant\infty$.

If $\psi \in (eMe)_*^+$ is a normal positive bounded linear form on $eMe$, we may consider its natural extension $i_e\psi_e$ to $M$ ($i_e\psi(x) = \psi(exe)$, $x \in M$). Using (2.5) and (2.6) it is clear that

$$\forall x \in \mathrm{M}\ \widehat{i_e\psi}\,(x) = \widehat{\psi}(\bar{e}x\bar{e})$$

We have $\hat{\psi}(\bar{e}x\bar{e}) = \tau_{\bar{e}}(h_\psi^{1/2}\bar{e}x\bar{e}h_\psi^{1/2}) = \tau(\bar{e}(h_\psi^{1/2}\bar{e}x\bar{e}h_\psi^{1/2}\bar{e})) = \tau(h_\psi^{1/2}xh_\psi^{1/2})$ (since $h_\psi \in \bar{e}\mathrm{L}^0(\mathrm{M},\tau)\bar{e}$). It is then clear by unicity of the Radon-Nikodym derivative that

$$h_{i_e\psi} = h_\psi.$$

It follows that the Haagerup trace $\mathrm{Tr}_\varphi$ restricts to $\mathrm{Tr}_{\varphi_e}$ on $\mathrm{L}^1(eMe)$. Indeed

$$\mathrm{Tr}_\varphi\, h_{i_e\psi} = i_e\psi(1) = \psi(e) = \mathrm{Tr}_{\varphi_e}\, h_\psi$$

It follows at once that the inclusions $\mathrm{L}^p(eMe) \subset \mathrm{L}^p(M)$ are isometric and preserve duality.

Let $e \in M$ be a projection. Let us construct a n.s.f. weight on $M$ with centralizer containing $e$. Choosing with [Str, 10.10] [KaR2, Exercise 7.6.46] two normal semifinite faithful weights $\varphi_1$ and $\varphi_2$ on $eMe$ and $e^\perp Me^\perp$. We can define a normal semifinite faithful weight $\varphi$ on $M$ by

$$\varphi(x) \overset{\text{def}}{=} \varphi_1(exe) + \varphi_2(e^\perp xe^\perp), \quad x \in M_+ \tag{2.16}$$

With (2.15), it is easy to check that $e$ belongs to the centralizer of $\varphi$. In fact it is not hard to see that this example is generic.

We will use the following classical lemma. We add the uniqueness part in part 1.

**Lemma 2.1** *Let $M$ be a (semifinite) von Neumann algebra equipped with a faithful normal semifinite trace $\tau$.*

1. *Let $a, b \in \mathrm{L}^0(\mathcal{M},\tau)^+$ satisfy $a \leqslant b$. Then there exists a unique $x \in \mathcal{M}$ such that $a^{\frac12} = xb^{\frac12}$ and $xs(b) = x$. Moreover, we have necessarily $\|x\|_\infty \leqslant 1$.*
2. *If $a \leqslant b$ in $\mathrm{L}^0(\mathcal{M},\tau)_{\mathrm{sa}}$ and if $x \in \mathrm{L}^0(\mathcal{M},\tau)$ then $x^*ax \leqslant x^*bx$.*

*Proof* : 1. Suppose that $\mathcal{M}$ acts on a Hilbert $H$. If $x$ satisfies the properties then $x$ is obviously null on the subspace $(1-s(b))H$. Moreover, the restriction of $x$ on the subspace $s(b)H$ is given by $x|s(b)H = a^{\frac12}b^{-\frac12}$ where $b^{-\frac12}\colon s(b)H \to s(b)H$. We conclude that $x$ is uniquely determined. The proof of the existence is given in [DeJ1, Remark 2.3] and shows the relation $xs(b) = x$. ■

The second is an easy observation.

**Lemma 2.2** *Let $M$ be a von Neumann algebra and $1 \leqslant p < \infty$. Let $h$ be a positive element of $\mathrm{L}^p(M)$.*

1. *The map $s(h)Ms(h) \to \mathrm{L}^p(M)$, $x \mapsto h^{\frac12}xh^{\frac12}$ is injective.*
2. *Suppose $1 \leqslant p < \infty$. The subspace $h^{\frac12}Mh^{\frac12}$ is dense in $s(h)\mathrm{L}^p(M)s(h)$ for the topology of $\mathrm{L}^p(M)$.*

*Proof* : 1) If $h^{\frac12}xh^{\frac12} = 0$ then $s(h)xs(h) = 0$ by using (2.2) twice. Since $x \in s(h)Ms(h)$, we conclude that $x = s(h)xs(h) = 0$.

2) Suppose that $y \in \mathrm{L}^{p^*}(M)$ belongs to the annihilator $\left(h^{\frac12}Mh^{\frac12}\right)^\perp$ of the subspace $h^{\frac12}Mh^{\frac12}$ of $\mathrm{L}^p(M)$. For any $x \in M$, we have $\mathrm{Tr}\left(h^{\frac12}xh^{\frac12}y\right) = 0$, i.e. $\mathrm{Tr}\left(xh^{\frac12}yh^{\frac12}\right) = 0$. Since $h^{\frac12}yh^{\frac12} \in \mathrm{L}^1(M)$, we deduce that $h^{\frac12}yh^{\frac12} = 0$. We infer that $s(h)ys(h) = 0$. It follows immediately that that $y$ belongs to the annihilator $\left(s(h)\mathrm{L}^p(M)s(h)\right)^\perp$. The proof is complete. ■

The following is a variant of [Sch, Lemma 2.2 (d)]. Our approach is probably more transparent. See [PeT1] and [Kos3, Prop. 3.1] for related facts.

**Lemma 2.3** *Let $M$ be a von Neumann algebra. Suppose $1 \leqslant p < \infty$. If $a, b \in \mathrm{L}^p(M)_+$ verify $a \leqslant b$ then there exists a unique $z \in s(b)Ms(b)$ such that $a = b^{\frac{1}{2}} z b^{\frac{1}{2}}$. Moreover, we have $0 \leqslant z \leqslant s(b)$.*

*Proof* : Let $\mathrm{L}^p(M, \varphi)$ be a realization of $\mathrm{L}^p(M)$ in $\mathcal{M} = M \rtimes_{\sigma^\varphi} \mathbb{R}$. By Lemma 2.1 there exists a unique $x \in \mathcal{M}$ such that $a^{\frac{1}{2}} = x b^{\frac{1}{2}}$ and $xs(b) = x$. Note that $a^{\frac{1}{2}}, b^{\frac{1}{2}} \in \mathrm{L}^{2p}(M)$. Then for any $s \in \mathbb{R}$

$$\mathrm{e}^{-\frac{s}{2p}} a^{\frac{1}{2}} \overset{(2.10)}{=} \hat{\sigma}_s\big(a^{\frac{1}{2}}\big) = \hat{\sigma}_s\big(x b^{\frac{1}{2}}\big) = \hat{\sigma}_s(x)\hat{\sigma}_s\big(b^{\frac{1}{2}}\big) \overset{(2.10)}{=} \hat{\sigma}_s(x)\mathrm{e}^{-\frac{s}{2p}} b^{\frac{1}{2}}.$$

Hence $a^{\frac{1}{2}} = \hat{\sigma}_s(x) b^{\frac{1}{2}}$. On the other hand, since $x = xs(b)$ and $s(b) \in M$, we have

$$\hat{\sigma}_s(x) = \hat{\sigma}_s(xs(b)) = \hat{\sigma}_s(x)\hat{\sigma}_s(s(b)) \overset{(2.10)}{=} \hat{\sigma}_s(x)s(b).$$

Thus by uniqueness of $x$ we get $\hat{\sigma}_s(x) = x$ for every $s \in \mathbb{R}$. That is $x \in M$ by (2.4). By Lemma one has necessarily $\|x\| \leqslant 1$.

From $a^{\frac{1}{2}} = x b^{\frac{1}{2}}$ and $\|x\|_\infty \leqslant 1$ it follows readily that $a = b^{\frac{1}{2}} x^* x b^{\frac{1}{2}}$ and that $0 \leqslant x^*x \leqslant \|x^*x\|_\infty = \|x\|_\infty^2 \leqslant 1$ where we use [Dix, 1.6.9] in the second inequality. If we set $z \overset{\text{def}}{=} s(b)x^*xs(b)$ then we have $z \in s(b)Ms(b)$. Moreover, since $s(b^{\frac{1}{2}}) = s(b)$, we have $b^{\frac{1}{2}} z b^{\frac{1}{2}} = b^{\frac{1}{2}} s(b) x^* x s(b) b^{\frac{1}{2}} = b^{\frac{1}{2}} x^* x b^{\frac{1}{2}} = a$ and $0 \leqslant z \leqslant s(b)$.

The uniqueness of $x$ follows from Lemma 2.2. ■

## 2.2 Conditional expectations on $\mathrm{L}^p$-spaces.

Recall that a positive map $T\colon A \to A$ on a C*-algebra $A$ is said faithful if $T(x) = 0$ for some $x \in A_+$ implies $x = 0$.

Let $B$ be a C*-subalgebra of a C*-algebra $A$. A map $\mathbb{E}\colon A \to A$ is called a conditional expectation on $B$ if it is a positive projection [Str, §9.1] of range $B$ which is $B$-bimodular, that is

$$\mathbb{E}(xyz) = x\mathbb{E}(y)z, \quad y \in A, x, z \in B,$$

Such a map is completely positive [Str, Prop. page 118].

Now, suppose that $A = M$ is a von Neumann algebra, that $B = N$ is a von Neumann subalgebra and that there exists a faithful normal conditional expectation $\mathbb{E}\colon M \to M$ from $M$ onto $N$, and that $\psi$ is a normal semifinite faithful weight on $N$. Then by [Con, Lemme 1.4.3], $\varphi \overset{\text{def}}{=} \psi \circ \mathbb{E}$ is a normal semifinite faithful weight on $M$ and the automorphisms groups of the two weights are linked by the relation

$$\mathbb{E} \circ \sigma_t^\varphi = \sigma_t^\psi \circ \mathbb{E}, \quad t \in \mathbb{R}. \tag{2.17}$$

This implies that $N$ is invariant under $\sigma^\varphi$, i.e. $\sigma_t^\varphi(N) \subset N$ for all $t \in \mathbb{R}$, and that $\sigma_t^\psi\colon N \to N$ is the restriction of $\sigma_t^\varphi\colon M \to M$ to $N$.

Then the crossed product $\mathcal{N} \overset{\text{def}}{=} N \rtimes_{\sigma^\varphi} \mathbb{R}$ is a $*$-subalgebra of $\mathcal{M} \overset{\text{def}}{=} M \rtimes_{\sigma^\varphi} \mathbb{R}$, and the natural representation of $N$ in $\mathcal{N}$ is the restriction of that of $M$ in $\mathcal{M}$. It is clear that the dual action $\hat{\sigma}^\varphi$ of $\mathbb{R}$ on $\mathcal{M}$ restricts to $\hat{\sigma}^\phi$ on $\mathcal{N}$.

Let now $\hat{\mathbb{E}}$ be the restriction to $\mathcal{M}$ of $\mathrm{Id}_{\mathrm{B}(\mathrm{L}^2(\mathbb{R}))} \otimes \mathbb{E}$, when $\mathcal{M}$ is considered as a von Neumann subalgebra of $\mathrm{B}(\mathrm{L}^2(\mathbb{R}))\overline{\otimes}M$. Note that $\hat{\mathbb{E}}$ is normal and faithful since $\mathrm{Id} \otimes \mathbb{E}$ is ([Tom1, Th. 2]). Since $\hat{\mathbb{E}}$ preserves the operators $\lambda_s$, $s \in \mathbb{R}$ and sends $\pi(M)$ onto $\pi(N)$ (due to relation (2.17)) it sends $\mathcal{M}$ onto $\mathcal{N}$. The $\mathcal{M}$-bimodularity of $\hat{\mathbb{E}}$ results from the fact that $\mathrm{Id}_{\mathrm{B}(\mathrm{L}^2(\mathbb{R}))} \otimes \mathbb{E}$ is $\mathrm{B}(\mathrm{L}^2(\mathbb{R}))\overline{\otimes}N$-bimodular. We obtain therefore a normal faithful conditional expectation $\hat{\mathbb{E}}$ from $\mathcal{M}$ onto $\mathcal{N}$.

**Lemma 2.4** *The dual weights $\hat{\varphi}$, $\hat{\phi}$ on $\mathcal{M}$, resp. $\mathcal{N}$ to the weights $\varphi$, $\phi$ satisfy the relation*

$$\hat{\varphi} = \hat{\phi} \circ \hat{\mathbb{E}}. \tag{2.18}$$

*Proof* : It is easy to see that each automorphism $\hat{\sigma}_s$ commutes with $\hat{\mathbb{E}}$. It follows from (2.5) that the operator valued weights $T_{\mathcal{M}}$ and $T_{\mathcal{N}}$ verify the relation

$$T_{\mathcal{N}} \circ \hat{\mathbb{E}} = \bar{\mathbb{E}} \circ T_{\mathcal{M}} \tag{2.19}$$

where $\bar{\mathbb{E}}\colon \bar{M}^+ \to \bar{N}^+$ is the natural extension of the operator $\mathbb{E}\colon M^+ \to N^+$. Let us also denote by $\bar{\omega}$ the natural extension to $\bar{\mathcal{M}}_+$ of any normal weight $\omega$ on $\mathcal{M}$ [Haa2, Prop 1.10]. Then

$$\hat{\varphi} \overset{(2.6)}{=} \bar{\varphi} \circ T_{\mathcal{M}} = \overline{\phi \circ \mathbb{E}} \circ T_{\mathcal{M}} = \bar{\phi} \circ \bar{\mathbb{E}} \circ T_{\mathcal{M}} \overset{(2.19)}{=} \bar{\phi} \circ T_{\mathcal{N}} \circ \hat{\mathbb{E}} \overset{(2.6)}{=} \hat{\phi} \circ \hat{\mathbb{E}}.$$

■

Let $\tau_{\mathcal{N}}$ be the canonical trace on the crossed product $\mathcal{N}$. Using [Con, Lemme 1.4.4] in the second equality, we obtain

$$\big(\mathrm{D}\hat{\varphi} : \mathrm{D}(\tau_{\mathcal{N}} \circ \hat{\mathbb{E}})\big)_t \overset{(2.18)}{=} \big(\mathrm{D}(\hat{\phi} \circ \hat{\mathbb{E}}) : \mathrm{D}(\tau_{\mathcal{N}} \circ \hat{\mathbb{E}})\big)_t = \big(\mathrm{D}\hat{\phi} : \mathrm{D}\tau_{\mathcal{N}}\big)_t = \lambda_t = \big(\mathrm{D}\hat{\varphi} : \mathrm{D}\tau_{\mathcal{M}}\big)_t, \quad t \in \mathbb{R}.$$

By [Str, page 49], we deduce that $\tau_{\mathcal{N}} \circ \hat{\mathbb{E}}$ is the canonical trace $\tau_{\mathcal{M}}$ on the crossed product $\mathcal{M}$. In particular, $\tau_{\mathcal{N}}$ is the restriction of $\tau_{\mathcal{M}}$. From now on, we set $\tau = \tau_{\mathcal{M}}$, and $\tau_{\mathcal{N}} = \tau|_{\mathcal{N}}$

It follows that the inclusion $\mathcal{N} \subset \mathcal{M}$ extends to a topological linear embedding $\mathrm{L}^0(\mathcal{N}, \tau_{\mathcal{N}}) \subset \mathrm{L}^0(\mathcal{M}, \tau_{\mathcal{M}})$ (which respects the operations of $*$-algebras). Using the characterization (2.10) of Haagerup $\mathrm{L}^p$-spaces, it is easy to see that for $1 \leqslant p \leqslant \infty$,

$$\mathrm{L}^p(N, \phi) = \mathrm{L}^p(M, \varphi) \cap \mathrm{L}^0(\mathcal{N}, \tau). \tag{2.20}$$

Note that the external products and Mazur maps in the scale $(\mathrm{L}^p(N, \varphi))$ are inherited from the scale $(\mathrm{L}^p(M, \varphi))$ and that the inclusion $i_p$ of $\mathrm{L}^p(N, \phi)$ into $\mathrm{L}^p(M, \varphi)$ is automatically isometric, in virtue of the formula

$$\|h\|_p = \tau\big(\chi_{[1,+\infty)}(|h|)^p\big)^{\frac{1}{p}}$$

(see [Terp, Ch. II, Lemma 5] for the case $p = 1$). The map $i_p$ is also positive since

$$\mathrm{L}^p(N, \phi)_+ = \mathrm{L}^p(N, \phi) \cap \mathrm{L}^0(\mathcal{N}, \tau)_+ = \mathrm{L}^p(M, \varphi) \cap \mathrm{L}^0(\mathcal{N}, \tau)_+ \subset \mathrm{L}^p(M, \varphi) \cap \mathrm{L}^0(\mathcal{M}, \tau)_+ = \mathrm{L}^p(M, \varphi)_+$$

In fact $i_p$ is completely isometric and completely positive, since for each $n \in \mathbb{N}$, the map $\mathrm{id}_{S_n^p} \otimes i_p\colon S_n^p(\mathrm{L}^p(N, \phi)) \to S_n^p(\mathrm{L}^p(M, \varphi))$ identifies to the inclusion map $\mathrm{L}^p(\mathrm{M}_n(N), \phi_n) \subset \mathrm{L}^p(\mathrm{M}_n(M), \varphi_n)$ where $\phi_n = \mathrm{Tr}_n \otimes \phi$, $\varphi_n = \mathrm{Tr}_n \otimes \varphi$ ($\mathrm{Tr}_n$ being the ordinary trace on the algebra $\mathrm{M}_n$ of $n \times n$ matrices).

On the other hand there is a natural isometric linear injection of $N_*$ into $M_*$, namely the map $\mathbb{E}_*\colon \omega \mapsto \omega \circ \mathbb{E}$. It turns out that in the identification of preduals with $\mathrm{L}^1$-spaces, this maps becomes exactly the inclusion map of $\mathrm{L}^1(N, \phi)$ into $\mathrm{L}^1(M, \varphi)$, that is

$$h_\omega^\phi = h_{\omega \circ \mathbb{E}}^\varphi \tag{2.21}$$

for every $\omega \in N_*$. Indeed, using [Con, Lemme 1.4.4] in the third equality, we have

$$\big(h_{\omega \circ \mathbb{E}}^\varphi\big)^{it} \overset{(2.7)}{=} \big(\mathrm{D}\widehat{\omega \circ \mathbb{E}} : \mathrm{D}\tau\big)_t = \big(\mathrm{D}(\hat{\omega} \circ \hat{\mathbb{E}}) : \mathrm{D}(\tau|_{\mathcal{N}} \circ \hat{\mathbb{E}})\big)_t = \big(\mathrm{D}\hat{\omega} : \mathrm{D}\tau|_{\mathcal{N}}\big)_t \overset{(2.7)}{=} (h_\omega^\phi)^{it}.$$

It follows easily that the functional $\mathrm{Tr}_\varphi$ on $\mathrm{L}^1(M,\varphi)$ and $\mathrm{Tr}_\phi$ on $\mathrm{L}^1(N,\phi)$ coincide on $\mathrm{L}^1(N,\phi)$:

$$\mathrm{Tr}_\varphi\,|\mathrm{L}^1(N,\phi) = \mathrm{Tr}_\phi\,. \tag{2.22}$$

Similarly the restriction map $R\colon M_* \to N_*, \omega \mapsto \omega\big|_N$, identifies to the map $\mathbb{E}_1\colon \mathrm{L}^1(M,\varphi) \to \mathrm{L}^1(N,\phi)$ defined by

$$\mathbb{E}_1(h^\varphi_\omega) = h^\phi_{\omega|_N} = h^\varphi_{\omega\circ\mathbb{E}}$$

for every $\omega \in \mathrm{M}_*$, which means that

$$\mathrm{Tr}\left(\mathbb{E}_1(h)x\right) = \mathrm{Tr}\left(h\mathbb{E}(x)\right), \quad h \in \mathrm{L}^1(M,\varphi), x \in M \tag{2.23}$$

that is, $\mathbb{E}$ is conjugate to $\mathbb{E}_1$ in the duality of the trace functional Tr. Note that in particular $\mathrm{Tr} = \mathrm{Tr}\circ\mathbb{E}_1$ (take $x = I$). It is also easy to see that $\mathbb{E}_1$ is $N$-bimodular, that is

$$\mathbb{E}_1(xhy) = x\mathbb{E}_1(h)y, \quad x, y \in N, h \in \mathrm{L}^1(M,\varphi).$$

Now for $1 < p < \infty$ and $h \in \mathrm{L}^p(M,\varphi)$ we define $\mathbb{E}_p(h)$ as the unique element of $\mathrm{L}^p(N,\phi)$ such that

$$\mathrm{Tr}\left(\mathbb{E}_p(h)k\right) = \mathrm{Tr}(hk), \quad k \in \mathrm{L}^{p^*}(N,\phi) \tag{2.24}$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$. That $\mathbb{E}_p(h)$ exists and is unique stems directly from the fact that $\mathrm{L}^p(N)$ is the conjugate space to $\mathrm{L}^{p^*}(N)$. It is then easy to see that $\mathbb{E}_p$ is a contractive linear projection in $\mathrm{L}^p(M,\varphi)$, which is positive (since the positive cone in $\mathrm{L}^{p^*}(M)$ is polar to that in $\mathrm{L}^p(M)$, [Terp, Ch. II, Prop. 33]). For $h \in \mathrm{L}^p(M,\varphi)$, $k \in \mathrm{L}^{p^*}(M,\varphi)$ we have

$$\mathrm{Tr}\left(\mathbb{E}_p(h)k\right) = \mathrm{Tr}\left(\mathbb{E}_p(h)\mathbb{E}_{p^*}(k)\right) = \mathrm{Tr}\left(h\,\mathbb{E}_{p^*}(k)\right)$$

thus the conjugate of $\mathbb{E}_p$, viewed as a map from $\mathrm{L}^p(M,\varphi)$ into itself, is $\mathbb{E}_{p^*}$. Moreover if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leqslant 1$, $h \in \mathrm{L}^p(M,\varphi)$ and $k \in \mathrm{L}^q(N,\phi)$ it is easily seen that

$$\mathbb{E}_r(hk) = \mathbb{E}_p(h)k \quad \text{and} \quad \mathbb{E}_r(kh) = k\mathbb{E}_p(h). \tag{2.25}$$

In particular with $q = \infty$ we get that $\mathbb{E}_p$ is $N$-bimodular.

Let us point a more abstract way to characterize $\mathbb{E}_p$. For $1 \leqslant p \leqslant \infty$ let $i_p\colon \mathrm{L}^p(N) \to \mathrm{L}^p(M)$ be the embedding associated with the weight $\varphi$ and the $\varphi$-invariant conditional expectation $\mathbb{E}$. Then for $1 \leqslant p \leqslant \infty$ we have

$$\mathbb{E}_p = i_p \circ (i_{p^*})^* \ \text{ if } 1 \leqslant p \leqslant \infty,\ \mathbb{E}_1 = i_1 \circ (i_1)_* \tag{2.26}$$

where $(i_\infty)_*$ means the preconjugate of $i_\infty$.

**Remark 2.5** *In the case where $\varphi$ is a normal bounded linear functional, the above defined conditional expectations $\mathbb{E}_p$ verify the formula:*

$$\mathbb{E}_p\big(h_\varphi^{\frac{1}{2p}} x h_\varphi^{\frac{1}{2p}}\big) = h_\varphi^{\frac{1}{2p}}\mathbb{E}(x)h_\varphi^{\frac{1}{2p}}, \quad x \in M. \tag{2.27}$$

*In particular if $\varphi$ is a normal state, they coincide with the conditional expectations $\mathcal{E}_p$ defined in [JX, sec. 2].*

*Proof* : In this case the 'density operators' $h_\varphi$ and $h_\phi$ associated with the weights $\varphi$, resp. $\phi$ belong respectively to $\mathrm{L}^1(M,\varphi)$ and $\mathrm{L}^1(N,\phi)$; moreover by (2.21) they coincide trivially modulo the embedding of $\mathrm{L}^1(N,\phi)$ into $\mathrm{L}^1(M,\varphi)$ described above (2.20). Thus $h_\varphi \in \mathrm{L}^1(N,\phi)$, hence $h_\varphi^{\frac{1}{2p}} \in \mathrm{L}^{2p}(N,\phi)$ and the equation (2.27) is then an immediate consequence of the bimodularity property (2.25). The map $\mathcal{E}_p$ in [JX, sec. 2] coincides thus with $\mathbb{E}_p$ on the subspace $h_\varphi^{\frac{1}{2p}} M h_\varphi^{\frac{1}{2p}}$, which is dense in $\mathrm{L}^p(M,\varphi)$ by Lemma 2.2. These maps are both linear and contractive (as for $\mathcal{E}_p$ see [JX, Lemma 2.2]), they coincide thus everywhere. ■

It is well known, at least in the 'probabilistic' case, that the maps $\mathbb{E}_p$ are completely positive and completely contractive, see e.g. [JRX, p. 92]. The proof there is by the usual tensorisation argument, and it works as well with our definition of $\mathbb{E}_p$ in the general case. But in our context these properties may be derived directly from the relations (2.26) and the complete positivity and isometry of the maps $i_p$.

## 2.3 Projections on von Neumann algebras

Let $A$ be a C*-algebra. Recall that a linear map $T\colon A \to A$ is a Schwarz map if $T(x)^*T(x) \leqslant T(x^*x)$ for any $x \in A$ [Pal2, Definition 9.9.5]. Note that a Schwarz map is positive. By [Sto2, Corollary 1.3.2], each 2-positive contraction $T\colon A \to A$ on a unital C*-algebra $A$ is a Schwarz map. By [Sto2, Theorem 2.2.2 (2)], if $P\colon A \to A$ is a faithful projection map which is a Schwarz map then $\operatorname{Ran} A$ is a C*-subalgebra of $A$.

**Proposition 2.6** *Let $M$ be a von Neumann algebra. If $P\colon M \to M$ is a normal faithful unital projection which is a Schwarz map, then $P$ is a conditional expectation and* $\operatorname{Ran} P$ *is a von Neumann subalgebra.*

*Proof* : By the above result, the range $\operatorname{Ran} A$ is a (unital) C*-subalgebra of $A$. Since $P$ is normal, $\operatorname{Ran} P = \ker(\mathrm{Id}_M - P)$ is weak* closed and thus $\operatorname{Ran} P$ is a von Neumann subalgebra of $M$. Note that $\|P\| = \|P(1)\| = \|1\| = 1$ since $P$ is positive. Using Tomiyama's theorem [Str, Theorem page 116], we conclude that $P$ is a (normal faithful) conditional expectation. ■

## 2.4 Positive linear maps between noncommutative $\mathrm{L}^p$-spaces

Our main tool will be the following extension of [JRX, Theorem 3.1]. The above lemmas allows us to remove some assumptions in [JRX, Theorem 3.1]. Since the proof of [JRX, Theorem 3.1] contains some gaps and misleading points[2] and since this result is fundamental for the sequel, we give full details.

**Theorem 2.7** *Let $M$ and $N$ be von Neuman algebras. Suppose $1 \leqslant p < \infty$. Let $T\colon \mathrm{L}^p(M) \to \mathrm{L}^p(N)$ be a positive linear map. Let $h$ be a positive element of $\mathrm{L}^p(M)$. Then there exists a unique linear map $v\colon M \to s(T(h))Ns(T(h))$ such that*

$$T\big(h^{\frac12} x h^{\frac12}\big) = T(h)^{\frac12} v(x) T(h)^{\frac12}, \qquad x \in M. \tag{2.28}$$

*Moreover, this map $v$ is unital, contractive, and normal. If $T$ is $n$-positive ($1 \leqslant n \leqslant \infty$), then $v$ is also $n$-positive.*

2. The complete positivity is not proved. The argument for the contractivity is ineffective.

*Proof* : We let $e \overset{\text{def}}{=} s(T(h))$ be the support projection of the element $T(h)$ of $\mathrm{L}^p(N)$. For any $x \in M_+$, since $0 \leqslant x \leqslant \|x\|_\infty$, we have by the part 2 of Lemma 2.1 that $0 \leqslant h^{\frac12} x h^{\frac12} \leqslant \|x\|_\infty h$. Using the positivity of $T$, we obtain that

$$0 \leqslant T\big(h^{\frac12} x h^{\frac12}\big) \leqslant \|x\|_\infty T(h).$$

If $x \neq 0$, note that $s(\|x\|_\infty T(h)) = s(T(h)) = e$. By Lemma 2.3, for any $x \in M_+$, there exists a unique element $z(x) \in eNe$ satisfying

$$T\big(h^{\frac12} x h^{\frac12}\big) = \big(\|x\|_\infty T(h)\big)^{\frac12} z(x) \big(\|x\|_\infty T(h)\big)^{\frac12}.$$

Moreover, we have $0 \leqslant z(x) \leqslant e$. Note that $z(1) = e$ since $e = s((T(h))^{\frac12})$. For any $x \in M_+$, we let $v(x) \overset{\text{def}}{=} \|x\|_\infty z(x)$. So $\|v(x)\|_\infty \leqslant \|x\|_\infty \|z(x)\|_\infty \leqslant \|x\|_\infty \|e\|_\infty \leqslant \|x\|_\infty$. Furthermore, we have

$$T\big(h^{\frac12} x h^{\frac12}\big) = T(h)^{\frac12} v(x) T(h)^{\frac12} \tag{2.29}$$

and $v(1) = e$. Let us show that $v\colon M_+ \to (eNe)_+$ is additive. For any $x, y \in M_+$, we have

$$\begin{aligned} T(h)^{\frac12} v(x+y) T(h)^{\frac12} &\overset{(2.29)}{=} T\big(h^{\frac12}(x+y)h^{\frac12}\big) = T\big(h^{\frac12} x h^{\frac12}\big) + T\big(h^{\frac12} y h^{\frac12}\big) \\ &\overset{(2.29)}{=} T(h)^{\frac12} v(x) T(h)^{\frac12} + T(h)^{\frac12} v(y) T(h)^{\frac12} = T(h)^{\frac12}\big(v(x)+v(y)\big)T(h)^{\frac12}. \end{aligned}$$

We conclude with Lemma 2.2 that $v(x+y) = v(x) + v(y)$. By a standard reasoning, see e.g. [AlT, Lemma 1.26], the map $v$ extends uniquely to a real linear positive map $v\colon M_{\mathrm{sa}} \to (eNe)_{\mathrm{sa}}$. We may extend it to a positive complex linear map from $M$ into $eNe$ by letting $v(x + \mathrm{i}y) = v(x) + \mathrm{i}v(y)$. As a positive and unital map ($v(1) = e$) between $\mathrm{C}^*$-algebras, $v$ is contractive by [Pau, Corollary 2.9]. The equation (2.28) follows by linearity from the case $x \geqslant 0$.

The uniqueness of $v$ is a consequence of part 1 of Lemma 2.2 applied with $T(h)$ and $N$ instead of $h$ and $M$.

Now, we prove that $v$ is normal. Since $1 \leqslant p < \infty$, we may consider $T^*\colon \mathrm{L}^{p^*}(N) \to \mathrm{L}^{p^*}(M)$. We define the element $k \overset{\text{def}}{=} T(h)^p$ of $\mathrm{L}^1(M)_+$ and define a linear map $w\colon k^{\frac12} N k^{\frac12} \to \mathrm{L}^1(M)$ on the dense subspace $k^{\frac12} N k^{\frac12}$ of $\mathrm{L}^1(eNe)$ (Lemma 2.2) by

$$w(k^{\frac12} y k^{\frac12}) \overset{\text{def}}{=} h^{\frac12} T^*\big(k^{\frac{1}{2p^*}} y k^{\frac{1}{2p^*}}\big) h^{\frac12}, \quad y \in N. \tag{2.30}$$

If $x \in M$, we have

$$\begin{aligned} &\mathrm{Tr}\,\big(w(k^{\frac12} y k^{\frac12}) x\big) \overset{(2.30)}{=} \mathrm{Tr}\,\big(h^{\frac12} T^*\big(k^{\frac{1}{2p^*}} y k^{\frac{1}{2p^*}}\big) h^{\frac12} x\big) = \mathrm{Tr}\,\big(T^*\big(k^{\frac{1}{2p^*}} y k^{\frac{1}{2p^*}}\big) h^{\frac12} x h^{\frac12}\big) \\ &= \mathrm{Tr}\,\big(k^{\frac{1}{2p^*}} y k^{\frac{1}{2p^*}} T(h^{\frac12} x h^{\frac12})\big) \overset{(2.29)}{=} \mathrm{Tr}\,\big(k^{\frac{1}{2p^*}} y k^{\frac{1}{2p^*}} T(h)^{\frac12} v(x) T(h)^{\frac12}\big) = \mathrm{Tr}\,\big(k^{\frac12} y k^{\frac12} v(x)\big). \end{aligned} \tag{2.31}$$

It follows that

$$\Big|\mathrm{Tr}\,\big(w(k^{\frac12} y k^{\frac12}) x\big)\Big| = \Big|\mathrm{Tr}\,\big(k^{\frac12} y k^{\frac12} v(x)\big)\Big| \leqslant \big\|k^{\frac12} y k^{\frac12}\big\|_1 \|v(x)\|_\infty \leqslant \big\|k^{\frac12} y k^{\frac12}\big\|_1 \|x\|_\infty.$$

Therefore $w$ extends to a contraction from $\mathrm{L}^1(eNe)$ into $\mathrm{L}^1(M)$. Furthermore, by (2.31) we conclude that $w^* = v$. Hence $v$ is normal.

Assume that $T$ is $n$-positive. Using (2.28) with the positive operator $T^{(n)} = \mathrm{Id}_{S_n^p} \otimes T\colon S_n^p(\mathrm{L}^p(M)) \to S_n^p(\mathrm{L}^p(N))$ and by replacing $h$ with $\mathrm{I}_n \otimes h$ whose projection support is

$s(\mathrm{I}_n \otimes h) = s(\mathrm{I}_n) \otimes s(h) = \mathrm{I}_n \otimes e$, we see that there exists a unique normal completely positive contraction $v_n \colon \mathrm{M}_n(M) \to (\mathrm{I}_n \otimes e)\mathrm{M}_n(N)(\mathrm{I}_n \otimes e) = \mathrm{M}_n(eNe)$ such that for any $[x_{ij}] \in \mathrm{M}_n(M)$ we have

$$T^{(n)}\left((\mathrm{I}_n \otimes h)^{\frac{1}{2}}[x_{ij}](\mathrm{I}_n \otimes h)^{\frac{1}{2}}\right) = \left(T^{(n)}(\mathrm{I}_n \otimes h)\right)^{\frac{1}{2}} v_n\left([x_{ij}]\right)\left(T^{(n)}(\mathrm{I}_n \otimes h)\right)^{\frac{1}{2}}.$$

Note that

$$\begin{aligned}
T^{(n)}\left((\mathrm{I}_n \otimes h)^{\frac{1}{2}}[x_{ij}](\mathrm{I}_n \otimes h)^{\frac{1}{2}}\right) &= T^{(n)}\left(\begin{bmatrix} h & & \\ & \ddots & \\ & & h \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} h & & \\ & \ddots & \\ & & h \end{bmatrix}^{\frac{1}{2}}\right) \\
&= T^{(n)}\left(\begin{bmatrix} h^{\frac{1}{2}}x_{11}h^{\frac{1}{2}} & \cdots & h^{\frac{1}{2}}x_{1n}h^{\frac{1}{2}} \\ \vdots & & \vdots \\ h^{\frac{1}{2}}x_{n1}h^{\frac{1}{2}} & \cdots & h^{\frac{1}{2}}x_{nn}h^{\frac{1}{2}} \end{bmatrix}\right) = \begin{bmatrix} T(h^{\frac{1}{2}}x_{11}h^{\frac{1}{2}}) & \cdots & T(h^{\frac{1}{2}}x_{1n}h^{\frac{1}{2}}) \\ \vdots & & \vdots \\ T(h^{\frac{1}{2}}x_{n1}h^{\frac{1}{2}}) & \cdots & T(h^{\frac{1}{2}}x_{nn}h^{\frac{1}{2}}) \end{bmatrix} \\
&= \begin{bmatrix} T(h)^{\frac{1}{2}}v(x_{11})T(h)^{\frac{1}{2}} & \cdots & T(h)^{\frac{1}{2}}v(x_{1n})T(h)^{\frac{1}{2}} \\ \vdots & & \vdots \\ T(h)^{\frac{1}{2}}v(x_{n1})T(h)^{\frac{1}{2}} & \cdots & T(h)^{\frac{1}{2}}v(x_{2nn})T(h)^{\frac{1}{2}} \end{bmatrix} \\
&= \begin{bmatrix} T(h) & & \\ & \ddots & \\ & & T(h) \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} v(x_{11}) & \cdots & v(x_{1n}) \\ \vdots & & \vdots \\ v(x_{n1}) & \cdots & v(x_{nn}) \end{bmatrix} \begin{bmatrix} T(h) & & \\ & \ddots & \\ & & T(h) \end{bmatrix}^{\frac{1}{2}} \\
&= \left(T^{(n)}(\mathrm{I}_n \otimes h)\right)^{\frac{1}{2}} v_n\left([x_{ij}]\right)\left(T^{(n)}(\mathrm{I}_n \otimes h)\right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently, by unicity, we conclude that $\mathrm{Id}_{\mathrm{M}_n} \otimes v = v_n$. Hence $v$ is $n$-positive. ■

Let $M$ be a von Neumann algebra equipped with a normal faithful state $\varphi$. Let $h_\varphi$ the density operator associated with $\varphi$. If $1 \leqslant p < \infty$, note that by Lemma 2.2 (see also [JX, Lemma 1.1] and [Wat1, Corollary 4]), $h_\varphi^{\frac{1}{2p}} M h_\varphi^{\frac{1}{2p}}$ is a dense subspace of $\mathrm{L}^p(M)$. Suppose that $N$ is another von Neumann algebra equipped with a normal faithful state $\psi$. Consider a unital positive map $T \colon M \to N$ and assume that $\psi(T(x)) = \varphi(x)$ for any $x \in M_+$. Given $1 \leqslant p < \infty$ define

$$\begin{array}{rccc} T_p\colon & h_\varphi^{\frac{1}{2p}} M h_\varphi^{\frac{1}{2p}} & \longrightarrow & h_\psi^{\frac{1}{2p}} N h_\psi^{\frac{1}{2p}} \\ & h_\varphi^{\frac{1}{2p}} x h_\varphi^{\frac{1}{2p}} & \longmapsto & h_\psi^{\frac{1}{2p}} T(x) h_\psi^{\frac{1}{2p}} \end{array} \tag{2.32}$$

By [HJX, Theorem 5.1], the map $T_p$ above extends to a contractive map from $\mathrm{L}^p(M)$ into $\mathrm{L}^p(N)$.

# 3 Projections in noncommutative $\mathrm{L}^p$-spaces

## 3.1 A local description of contractive 2-positive projections.

Suppose $1 \leqslant p < \infty$. Let $\varphi$ be a normal semifinite faithful weight on $M$. Let $P \colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ be a nontrivial 2-positive contractive projection. First consider a non-zero element $h$ of $P(\mathrm{L}^p(M)_+)$. We have $P(h) = h$ and $h \in \mathrm{L}^p(M)_+$. Moreover if $0 \leqslant k \leqslant h$, $k \in \mathrm{L}^p(M)$, then $0 \leqslant Pk \leqslant Ph = h$. By linearity it follows that the linear subspace $\mathcal{I}(h) \overset{\text{def}}{=} \mathrm{Span}\,\{k \in \mathrm{L}^p(M), 0 \leqslant$

$k \leqslant h\} = h^{\frac12} M h^{\frac12}$ is preserved by $P$, and by continuity, so is its closure $\overline{\mathcal{I}(h)} = s(h)\mathrm{L}^p(M)s(h)$. Thus

$$P\big(s(h)\mathrm{L}^p(M)s(h))\big) \subset s(h)\mathrm{L}^p(M)s(h) \tag{3.1}$$

Hence, the restriction $P|_{s(h)\mathrm{L}^p(M)s(h)}$ is a projection from $s(h)\mathrm{L}^p(M)s(h)$ into $s(h)\mathrm{L}^p(M)s(h)$. Recall that $s(h)\mathrm{L}^p(M)s(h)$ may be identified to the space $\mathrm{L}^p(M_h)$, the $\mathrm{L}^p$-space of the reduced von Neumann algebra $M_h = s(h)Ms(h)$. In the sequel, we will show that we can identify the range of this restriction with the noncommutative $\mathrm{L}^p$-space of a suitable von Neumann subalgebra $N_h$ of $M_h$, which is the range of a conditional expectation $\mathbb{E}_h$. Thus $P\big(s(h)\mathrm{L}^p(M)s(h)\big)$ will be a completely order and completely isometric copy of $\mathrm{L}^p(N_h)$ in $\mathrm{L}^p(M_h)$. Moreover the restriction of $P$ will be identified to the conditional expectation $\mathbb{E}_{h,p}$ associated to $\mathbb{E}$ and a suitable weight $\psi_h$ on $M_h$.

By applying Theorem 2.7 to $P$ and the positive element $h$ of $\mathrm{L}^p(M)$, we see that there exists a unique linear map $\mathbb{E}_h\colon M_h \to s(P(h))Ms(P(h)) = M_h$ such that

$$P\big(h^{\frac12} x h^{\frac12}\big) = h^{\frac12}\mathbb{E}_h(x)h^{\frac12}, \qquad x \in M_h. \tag{3.2}$$

Moreover, this map $\mathbb{E}_h$ is unital, contractive, normal and 2-positive. Below, we will show that $\mathbb{E}_h\colon M_h \to M_h$ is a conditional expectation onto a von Neumann subalgebra of $M_h$.

**Lemma 3.1** *The map $\mathbb{E}_h\colon M_h \to M_h$ is faithful.*

*Proof* : Recall that for $1 < p < \infty$, the space $\mathrm{L}^p(M)$ is uniformly convex by [PiX, Cor. 5.2], hence strictly convex by [Meg1, Prop. 5.2.6]. It is easy to deduce that the $\mathrm{L}^p$-norm is strictly monotone[3]. This fact remains trivially true in the case $p = 1$. Now, we will show that if $0 \leqslant k \leqslant h$ and $P(k) = 0$ then $k = 0$. Indeed, from

$$h = P(h) = P(h) - P(k) = P(h-k)$$

we deduce that $\|h\|_p = \|P(h-k)\|_p \leqslant \|h-k\|_p$ by the fact that $P$ is contractive. Since $0 \leqslant h-k \leqslant h$ we infer that $\|h\|_p = \|h-k\|_p$ and finally $k = 0$ by strict monotonicity of the $\mathrm{L}^p$-norm.

It results at once that $\mathbb{E}_h$ is faithful. Indeed, if $x \in M_h^+$ and $\mathbb{E}_h(x) = 0$ we have

$$P\big(h^{\frac12} x h^{\frac12}\big) \overset{(3.2)}{=} h^{\frac12}\mathbb{E}_h(x)h^{\frac12} = 0.$$

Since $h^{\frac12} x h^{\frac12} \leqslant \|x\|_\infty h$ by [Dix, 1.6.9] we see that $h^{\frac12} x h^{\frac12} = 0$ by the first part of the proof. Since $M_h = s(h)Ms(h)$, we conclude that $x = 0$ by Lemma 2.2. ■

**Lemma 3.2** *The map $\mathbb{E}_h\colon M_h \to M_h$ is a normal conditional expectation onto a von Neumann subalgebra $N_h$ of $M_h$.*

*Proof* : For any $x \in M_h$, we have

$$P\big(h^{\frac12} x h^{\frac12}\big) = P^2\big(h^{\frac12} x h^{\frac12}\big) \overset{(3.2)}{=} P\big(h^{\frac12}\mathbb{E}_h(x)h^{\frac12}\big) \overset{(3.2)}{=} h^{\frac12}\mathbb{E}_h^2(x)h^{\frac12}.$$

Using the unicity of $\mathbb{E}_h$ given by Theorem 2.7, we infer that $\mathbb{E}_h^2 = \mathbb{E}_h$, i.e. $\mathbb{E}_h$ is a projection. Moreover $\mathbb{E}_h$ is unital (since $Ph = h$). Since $\mathbb{E}_h$ is in addition a 2-positive contraction, we

3. Suppose $0 \leqslant x \leqslant y$. If $x \neq y$ then we have $\|x\|_p < \|y\|_p$.

deduce by Proposition 2.6 that $N_h \overset{\text{def}}{=} \operatorname{Ran} \mathbb{E}_h$ is a sub-von Neumann algebra of $M_h$ and that $\mathbb{E}_h$ is a conditional expectation from $M_h$ onto $N_h$. ■

Note that $h^p$ belongs to $\mathrm{L}^1(M)$, thus the map $x \mapsto \operatorname{Tr}(h^p x)$ defines a normal positive linear form on $M$ with support $s(h)$ (a normal state if $\|h\|_p = 1$). Let $\psi_h$ be its restriction to the algebra $M_h$.

Now, we prove that $\mathbb{E}_h$ is $\psi_h$-invariant. This property will allows us to extend $\mathbb{E}_h$ in a compatible way to all $\mathrm{L}^q(M_h)$, $1 \leqslant q \leqslant \infty$, as explained in section 2.2.

**Lemma 3.3** *We have* $\psi_h \circ \mathbb{E}_h = \psi_h$.

*Proof* : Let $p^*$ be the exponent conjugate to $p$. Consider first the case where $1 < p < \infty$. Using the contractive dual map $P^* \colon \mathrm{L}^{p^*}(M) \to \mathrm{L}^{p^*}(M)$, we see that

$$\begin{aligned}\|h\|_p^p &\overset{(2.11)}{=} \operatorname{Tr}(h^p) = \operatorname{Tr}(hh^{p-1}) = \operatorname{Tr}\big((P(h)h^{p-1}\big) = \operatorname{Tr}\big(hP^*(h^{p-1})\big)\\ &\leqslant \|h\|_p \big\|P^*(h^{p-1})\big\|_{p^*} \leqslant \|h\|_p \|h^{p-1}\|_{p^*} = \|h\|_p^p.\end{aligned}$$

Thus $\|P^*(h^{p-1})\|_{p^*} = \|h^{p-1}\|_{p^*}$ and $\operatorname{Tr}\big(hP^*(h^{p-1})\big) = \|h\|_p \|h^{p-1}\|_{p^*}$. By [PiX, Cor. 5.2], the Banach space $\mathrm{L}^p(M_h)$ is smooth. Hence, by unicity of the norming functional of $h$ (see [Meg1, Cor. 5.4.3]) we obtain

$$P^*(h^{p-1}) = h^{p-1}. \tag{3.3}$$

For any $k \in \mathrm{L}^p(M)$, it follows that

$$\operatorname{Tr}\big(h^{p-1}P(k)\big) = \operatorname{Tr}\big(P^*(h^{p-1})k\big) \overset{(3.3)}{=} \operatorname{Tr}(h^{p-1}k). \tag{3.4}$$

In particular, for every $x \in M_h$, we have

$$\operatorname{Tr}\big(h^{p-1}(h^{\frac12}\mathbb{E}_h(x)h^{\frac12})\big) \overset{(3.2)}{=} \operatorname{Tr}\big(h^{p-1}P(h^{\frac12}xh^{\frac12})\big) \overset{(3.4)}{=} \operatorname{Tr}\big(h^{p-1}(h^{\frac12}xh^{\frac12})\big)$$

that is $\operatorname{Tr}\big(h^p\mathbb{E}_h(x)\big) = \operatorname{Tr}(h^p x)$ hence $\psi_h\big(\mathbb{E}_h(x)\big) = \psi_h(x)$.

In the case $p = 1$, using the contractivity of $P^* \colon M \to M$, we note that by [Dix, 1.6.9]

$$0 \leqslant P^*(s(h)) \leqslant \big\|P^*(s(h))\big\|_\infty 1 \leqslant \|s(h)\|_\infty \leqslant 1.$$

Hence $0 \leqslant s(h)P^*(s(h))s(h) \leqslant s(h)$. It follows that

$$h^{\frac12}P^*(s(h))h^{\frac12} = h^{\frac12}s(h)P^*(s(h))s(h)h^{\frac12} \leqslant h^{\frac12}s(h)h^{\frac12} = h \tag{3.5}$$

However

$$\operatorname{Tr}\big(h^{\frac12}P^*(s(h))h^{\frac12}\big) = \operatorname{Tr}(hP^*(s(h))) = \operatorname{Tr}(P(h)s(h)) = \operatorname{Tr}(h)$$

hence

$$\|h - h^{\frac12}P^*(s(h))h^{\frac12}\|_1 = \operatorname{Tr}\big(h - h^{\frac12}P^*(s(h))h^{\frac12}\big) = 0$$

and the inequality in (3.5) is in fact an equality, which implies by Lemma 2.2

$$s(h)P^*(s(h))s(h) = s(h). \tag{3.6}$$

Then for every $k \in s(h)\mathrm{L}^1(M)s(h)$ we have

$$\begin{aligned}\operatorname{Tr}(P(k)) &= \operatorname{Tr}(s(h)P(k)) = \operatorname{Tr}(P^*(s(h))k) = \operatorname{Tr}(P^*(s(h))s(h)ks(h))\\ &= \operatorname{Tr}(s(h)P^*(s(h))s(h)k) \overset{(3.6)}{=} \operatorname{Tr}(s(h)k) = \operatorname{Tr}(k).\end{aligned}$$

If $k = h^{\frac{1}{2}} y h^{\frac{1}{2}}$, with $y \in M_h$, we obtain:

$$\psi_h(\mathbb{E}_h y) = \mathrm{Tr}\left(h^{\frac{1}{2}} \mathbb{E}_h y\, h^{\frac{1}{2}}\right) = \mathrm{Tr}(P(h^{\frac{1}{2}} y h^{\frac{1}{2}})) = \mathrm{Tr}(h^{\frac{1}{2}} y h^{\frac{1}{2}}) = \mathrm{Tr}(hy) = \psi_h(y).$$

We deduce that $\psi_h \circ \mathbb{E}_h = \psi_h$ also in this case. ■

Consider a normal semifinite faithful weight $\chi$ on $M$ such that $s(h)$ belongs to the centralizer of $\chi$ and such that the reduced weight $\chi_{s(h)}$ (see subsection 2.1) on $M_h = s(h)Ms(h)$ coincides with $\psi_h$. Then

$$\psi_h(s(h)xs(h)) = \mathrm{Tr}_\varphi(h^p x) = \mathrm{Tr}_\chi(\hat{\kappa}(h)^p x), \quad x \in M \tag{3.7}$$

where $\hat{\kappa}\colon \mathrm{L}^p(M,\varphi) \to \mathrm{L}^p(M,\chi)$ is the canonical map. Note that $\hat{\kappa}(s(h)\mathrm{L}^p(M,\varphi)s(h)) = s(h)L_p(\mathrm{M},\chi)s(h) = \mathrm{L}^p(M_h,\psi_h)$ (since $s(\hat{\kappa}(h)) = \hat{\kappa}(s(h)) = s(h)$ by (2.12)) and the resulting identification of the subspace $s(h)\mathrm{L}^p(M)s(h)$ of $\mathrm{L}^p(M)$ with the noncommutative $\mathrm{L}^p$-space $\mathrm{L}^p(M_h)$ is completely order preserving and completely isometric.

In the sequel, we will use the density operator $h_{\psi_h} \in \mathrm{L}^1(M_h, \chi_{s_h})$ associated with $\psi_h$. We have

$$\mathrm{Tr}_{\psi_h}(h_{\psi_h} x) = \psi_h(x) \overset{(3.7)}{=} \mathrm{Tr}_\chi(\hat{\kappa}(h)^p x) = \mathrm{Tr}_{\psi_h}(\hat{\kappa}(h)^p x), \quad x \in M_h \tag{3.8}$$

(recall that the traces coincide on $\mathrm{L}^1(s(h)Ms(h))$). We conclude that

$$h_{\psi_h} = \hat{\kappa}(h)^p. \tag{3.9}$$

Let $\mathbb{E}_{h,p}\colon \mathrm{L}^p(M_h) \to \mathrm{L}^p(M_h)$ be the conditional expectation on $\mathrm{L}^p(s(h)Ms(h))$ associated with the conditional expectation $\mathbb{E}$ and the weight $\psi_h$. Recall that (by equation (2.27))

$$\mathbb{E}_{h,p}\big(h_{\psi_h}^{\frac{1}{2p}} x h_{\psi_h}^{\frac{1}{2p}}\big) = h_{\psi_h}^{\frac{1}{2p}} \mathbb{E}_h(x) h_{\psi_h}^{\frac{1}{2p}}, \quad x \in M_h.$$

For any $x \in M_h$, we have

$$\begin{aligned} P\hat{\kappa}^{-1}\big(h_{\psi_h}^{\frac{1}{2p}} x h_{\psi_h}^{\frac{1}{2p}}\big) &\overset{(3.9)}{=} P\hat{\kappa}^{-1}\big(\hat{\kappa}(h)^{\frac{1}{2}} x \hat{\kappa}(h)^{\frac{1}{2}}\big) = P\big(h^{\frac{1}{2}} x h^{\frac{1}{2}}\big) \overset{(3.2)}{=} h^{\frac{1}{2}} \mathbb{E}_h(x) h^{\frac{1}{2}} \\ &\overset{(3.9)}{=} \hat{\kappa}^{-1}(h_{\psi_h})^{\frac{1}{2p}} \mathbb{E}_h(x) \hat{\kappa}^{-1}(h_{\psi_h})^{\frac{1}{2p}} = \hat{\kappa}^{-1}\big(h_{\psi_h}^{\frac{1}{2p}} \mathbb{E}_h(x) h_{\psi_h}^{\frac{1}{2p}}\big) \overset{(2.27)}{=} \hat{\kappa}^{-1}\mathbb{E}_{h,p}\big(h_{\psi_h}^{\frac{1}{2p}} x h_{\psi_h}^{\frac{1}{2p}}\big). \end{aligned}$$

Hence, by density,

$$P\hat{\kappa}^{-1}(k) = \hat{\kappa}^{-1}\mathbb{E}_{h,p}(k) \tag{3.10}$$

for all $k \in L^p(M_h, \psi_h)$; we conclude that we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{L}^p(M,\varphi) \arrow[rr, hook, "P"] & & \mathrm{L}^p(M,\varphi) \\ \mathrm{L}^p(M,\chi) \arrow[u, "\hat{\kappa}^{-1}"] & & \mathrm{L}^p(M,\chi) \arrow[u, "\hat{\kappa}^{-1}"'] \\ s(h)\mathrm{L}^p(M,\chi)s(h) = \mathrm{L}^p(M_h,\psi_h) \arrow[u, hook] \arrow[rr, "\mathbb{E}_{h,p}"] & & \mathrm{L}^p(M_h,\psi_h) = s(h)\mathrm{L}^p(M,\chi)s(h) \arrow[u, hook'] \end{tikzcd}$$

Thus $P(\mathrm{L}^p(M_h)) = P\big(s(h)\mathrm{L}^p(M)s(h)\big)$ is completely order and completely isometrically isomorphic to $\mathrm{L}^p(N_h)$.

Since $\mathrm{L}^p(N_h)$ is a $N_h$-bimodule in $\mathrm{L}^p(M_h)$, and the identification of $\mathrm{L}^p(M_h)$ with $s(h)\mathrm{L}^p(M)s(h)$ respects the actions of $M_h$, then $P(s(h)\mathrm{L}^p(M)s(h))$ is a $N_h$-bimodule. Moreover $P$ is $N_h$-bimodular, that is

$$P(xky) = xP(k)y, \quad x, y \in N_h,\ k \in s(h)\mathrm{L}^p(M)s(h).$$

This results simply from the $N_h$-bimodularity of $\mathbb{E}_{h,p}$.

Note that the left supports $s_\ell(k)$, $\sigma_\ell(k)$ of $k \in \mathrm{L}^p(N_h)$, viewed either as an element of $\mathrm{L}^p(M)$ or as an element of $\mathrm{L}^p(N_h)$, do coincide. Indeed we have clearly $\sigma_\ell(k) \geqslant s_\ell(k)$. Moreover if $r \stackrel{\text{def}}{=} \sigma_\ell(k) - s_\ell(k)$, then $0 \leqslant \mathbb{E}_h(r) \leqslant \sigma_\ell(k)$ and $\mathbb{E}_h(r)k = \mathbb{E}_{h,p}(rk) = 0$, which imply $\mathbb{E}_h(r) = 0$ and thus $r = 0$ since $\mathbb{E}_h$ is faithful on $s(h)Ms(h)$. The same coincidence occurs of course for right supports.

Note that the projections of a $\sigma$-finite von Neumann algebra $M$ are exactly the right (or left) supports of elements of $\mathrm{L}^p(M)$. Hence $N_h$ is the von Neumann subalgebra of $M$ generated by the right (resp. left) supports of the elements of $P(s(h)\mathrm{L}^p(M)s(h))$. In particular $N_h$ depends only on $s(h)$, i.e. $s(h_1) = s(h_2)$ implies $N_{h_1} = N_{h_2}$.

## 3.2 From local to global: the $\sigma$-finite case.

Suppose $1 \leqslant p < \infty$. Let $M$ be a $\sigma$-finite (= countaby decomposable) von Neumann algebra and $P\colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ be a positive contractive projection. We define the support $s(P)$ of $\operatorname{Ran} P$ as the supremum in $M$ of the supports of the positive elements in $\operatorname{Ran} P$:

$$s(P) \stackrel{\text{def}}{=} \bigvee_{h \in \operatorname{Ran} P, h \geqslant 0} s(h). \tag{3.11}$$

**Proposition 3.4** *Suppose $1 \leqslant p < \infty$. Let $M$ be a $\sigma$-finite von Neumann algebra and $P\colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ be a positive contractive projection. Then there exists a positive element $h$ of $\operatorname{Ran} P$ such that $s(h) = s(P)$.*

*Proof* : We note first that for every at most countable family $(h_i)_{i\in I}$ of positive elements in $\operatorname{Ran} P$ there is a positive element $h$ in $\operatorname{Ran} P$ such that $s(h) \geqslant s(h_i)$ for all $i \in I$. Indeed assuming $\|h_i\|_p \leqslant 1$ for all $i$, take simply $h = \sum_{i\in I} 2^{-i} h_i$. By [KaR2, Exercice 7.6.46], we can consider a normal faithful state $\psi$ on $M$. We introduce the positive real number

$$a \stackrel{\text{def}}{=} \sup\big\{\psi(s(h)) : h \in \operatorname{Ran} P, h \geqslant 0\big\}.$$

This supremum is attained. Indeed, consider a positive sequence $(h_n)$ of positive elements in $\operatorname{Ran} P$ such that $\psi(s(h_n)) \uparrow a$, then any positive element $h \in \operatorname{Ran} P$ such that $s(h) \geqslant s(h_n)$ for all $n$ satisfies $\psi(s(h)) = a$.

Fix such an element $h \in \operatorname{Ran}(P)$ with $h \geqslant 0$ such that $\psi(s(h)) = a$. We have $s(h) \leqslant s(P)$. If $s(h) \neq s(P)$, there exists $h' \in \operatorname{Ran} P$ with $h' \geqslant 0$ such that $s(h') \not\leqslant s(h)$. This implies that

$$s(h + h') \geqslant s(h) \vee s(h') > s(h).$$

Hence $\psi(s(h + h')) > \psi(s(h))$ which is impossible by maximality of $\psi(s(h))$. Thus $s(h) = s(P)$ as desired. ■

*Proof of parts 1) and 2) of Theorem 1.1 in the $\sigma$-finite case* : Let $\varphi$ be a normal state on $M$. We can suppose $\|h\| = 1$. Clearly $\operatorname{Ran} P = s(P) \operatorname{Ran} P s(P)$ since[4] $P$ is positive. So we have

$$\operatorname{Ran} P = P(\operatorname{Ran} P) = P\big(s(P) \operatorname{Ran} P\, s(P)\big) \subset P\big(s(P)\mathrm{L}^p(M)s(P)\big) \subset \operatorname{Ran} P.$$

Finally, we obtain

$$\operatorname{Ran} P = P\big(s(P)\mathrm{L}^p(M)s(P)\big) = P\big(s(h)\mathrm{L}^p(M)s(h)\big).$$

Note that the map $s(h)Ms(h) \to \mathbb{C}$, $s(h)xs(h) \mapsto \mathrm{Tr}_\varphi(h^p x)$ is a faithful normal state on $s(h)Ms(h)$. Using the procedure (2.16), we can consider a normal faithful state $\chi_h$ on $M$ such that $s(h)$ belongs to the centralizer of $\chi_h$ and the reduced weight $\psi_h \overset{\text{def}}{=} (\chi_h)_{s(h)}$ on $M_h \overset{\text{def}}{=} s(h)Ms(h)$ satisfies

$$\psi_h(s(h)xs(h)) = \mathrm{Tr}_\varphi(h^p x), \quad x \in M. \tag{3.12}$$

If $\hat\kappa\colon \mathrm{L}^1(M,\varphi) \to \mathrm{L}^1(M,\chi)$ is the canonical map, we obtain by (2.14) the equality

$$\psi_h(s(h)xs(h)) = \mathrm{Tr}_\chi(\hat\kappa(h^p x)) = \mathrm{Tr}_\chi(\hat\kappa(h)^p x), \quad x \in M. \tag{3.13}$$

By the results of Subsection 3.1, the range $\operatorname{Ran} P$ identifies to the noncommutative $\mathrm{L}^p$-spaces $\mathrm{L}^p(N_h)$ of the von Neumann subalgebra $N_h$ of $s(h)Ms(h) = s(P)Ms(P)$ generated by the right (resp. left) supports of the elements of $\operatorname{Ran} P$.

Moreover, the restriction of $P$ to $s(P)\mathrm{L}^p(M)s(P)$ identifies to a faithful normal conditional expectation in this reduced $\mathrm{L}^p$-space, with range $\mathrm{L}^p(N_h)$. ■

## 3.3 The non $\sigma$-finite case.

### 3.3.1 The set of supports of elements in the range of $P$ and its associated VNA.

Let $\mathcal{P}(P)$ be the set of all support projections of positive elements in $\operatorname{Ran}(P)$. If $s \in \mathcal{P}(P)$ we define $N_s$ as the von Neumann subalgebra of $sMs$ generated by the projections $e \in \mathcal{P}(P)$ with $e \leqslant s$. The link with the subalgebras $N_h$ defined in section 3.1 is given by the following Lemma.

**Lemma 3.5** *For every $h \in \operatorname{Ran}(P)$ we have $N_{s(h)} = N_h$.*

*Proof* : Let $e \in N_h$ be a projection. We have $e \leqslant s(h)$, hence by (2.2) $s(ehe) = e$. Since $P(s(h)\mathrm{L}^p(M)s(h))$ is $N_h$-bimodular, we have $ehe \in \operatorname{Ran} P$. Thus $e \in \mathcal{P}(P)$ and by $e \leqslant s(h)$, $e \in N_{s(h)}$. Since $N_h$ is generated by its projections, it follows that $N_h \subset N_{s(h)}$. Conversely if $0 \leqslant k \in P(\mathrm{L}^p(M))$ with $s(k) \leqslant s(h)$, then $k = P(k) = P(s(h)ks(h))$, hence $k \in P(s(h)\mathrm{L}^p(M)s(h))$ and thus $s(k) \in N(h)$. Since $N_{s(h)}$ is generated by such projections, it follows that $N_{s(h)} \subset N_h$. ■

**Lemma 3.6** *If $s_1, s_2 \in \mathcal{P}(P)$ then $s_1 \vee s_2$ belongs to $\mathcal{P}(P)$.*

*Proof* : We can write $s_i = s(h_i)$ for some positive element $h_i$ of $\operatorname{Ran} P$. Then the positive element $h = h_1 + h_2$ belongs to $\operatorname{Ran} P$ and $s(h_1 + h_2) = s(h_1) \vee s(h_2)$. ■

We denote by $s(P)$ the supremum of all projections in $\mathcal{P}(P)$ (the support of $\operatorname{Ran} P$) and by $N_P$ the weak* closed $*$-algebra generated by $\mathcal{P}(P)$.

4. If $h \in \operatorname{Ran} P$, we can write $h = h_1 - h_2 + \mathrm{i}(h_3 - h_4)$ with $h_1, h_2, h_3, h_4 \geqslant 0$. Hence $h = P(h_1) - P(h_2) + \mathrm{i}(P(h_3) - P(h_4))$ where $P(h_1), P(h_2), P(h_3), P(h_4) \geqslant 0$.

**Lemma 3.7** *i) If $s \in \mathcal{P}(P)$ and $e \in N_s$ is a projection, then $e \in \mathcal{P}(P)$.*
*ii) if $s_1, s_2 \in \mathcal{P}(P)$ with $s_2 \leqslant s_1$ then $s_2 \in N_{s_1}$ and $N_{s_2} = s_2 N_{s_1} s_2$.*
*iii) for not necessarily comparable $s_1, s_2 \in \mathcal{P}(P)$ we have still $s_2 N_{s_1} s_2 \subset N_{s_2}$.*

*Proof* : i) See the first part of the proof of Lemma 3.5.

ii) It follows from the definition of the algebras $N_s$ that $N_{s_2} \subset N_{s_1}$ whenever $s_2 \leqslant s_1$. Then clearly $N_{s_2} = s_2 N_{s_2} s_2 \subset s_2 N_{s_1} s_2$. Conversely every projection $e$ of the reduced von Neumann algebra $s_2 N_{s_1} s_2$ belongs to $N_{s_1}$, hence to $\mathcal{P}(P)$ by (i), and is majorized by $s_2$, thus $e \in N_{s_2}$. It follows that $s_2 N_{s_1} s_2 \subset N_{s_2}$.

iii) In this case we have $s_2 N_{s_1} s_2 \subset s_2 N_{s_1 \vee s_2} s_2 = N_{s_2}$. ■

It follows from the two preceding lemmas that the family $(N_s)_{s \in \mathcal{P}(P)}$ is nested by inclusion, so that $N_P = \overline{\bigcup_{s \in \mathcal{P}(P)} N_s}^{\mathrm{w}^*}$. Here is a more tractable definition:

**Lemma 3.8** *The algebra $N_P$ is the subset of $s(P)Ms(P)$ consisting of elements $x$ such that $sxs \in N_s$ for every $s \in \mathcal{P}(P)$.*

*Proof* : Since each algebra $N_s$ is included in $s(P)Ms(P)$ and the latter algebra is weak* closed in $M$, the whole algebra $N_P$ is also included in $s(P)Ms(P)$. The set

$$A \stackrel{\text{def}}{=} \big\{x \in s(P)Ms(P) : sxs \in N_s \text{ for all } s \in \mathcal{P}(P)\big\}$$

is weak* closed and contains all the $N_s$'s with $s \in \mathcal{P}(P)$, thus $A$ contains $N_P$. For every $x \in A$ and $s \in \mathcal{P}(P)$, we have $sxs \in N_s \subset N_P$. Then $sxs \to s(P)xs(P)$ weak* when $s \uparrow s(P)$ (as a consequence of the fact that $s\varphi \to s(P)\varphi$ in the norm of $M_*$, for every $\varphi \in M_*$). Thus $x \in N_P$, and $A \subset N_P$. ■

**Lemma 3.9** *The map $P$ is $N_P$-bimodular on $s(P)\mathrm{L}^p(M)s(P)$, and consequently* $\operatorname{Ran} P$ *is a $N_P$-bimodule.*

*Proof* : We have to prove that $P(x_1 h x_2) = x_1 P(h) x_2$ for every $h \in \mathrm{L}^p(M)$ and $x_1, x_2 \in N_P$. Since the maps $x \mapsto xh$ and $x \mapsto hx$ are weak* to weak continuous, we may reduce to the case where $x_1, x_2$ belong to some $N_{s_0}$ with $s_0 \in \mathcal{P}(P)$. Since $s(P) = \bigvee \mathcal{P}(P)$ we have

$$s(P)\mathrm{L}^p(M)s(P) = \overline{\bigcup_{e \in \mathcal{P}(P)} e\mathrm{L}^p(M)e}$$

(norm closure) so by another approximation argument we may assume that $h \in e\mathrm{L}^p(M)e$ with $e \in \mathcal{P}(P)$. Then, setting $s \stackrel{\text{def}}{=} s_0 \vee e$, we have $x_1, x_2 \in N_s$ and $h \in s\mathrm{L}^p(M)s$. Since $P$ is bimodular with respect to $N_s$ on $s\mathrm{L}^p(M)s$, the equation $P(x_1 h x_2) = x_1 P(h) x_2$ follows, as well as the bimodule property. ■

Our goal is now to prove that $\operatorname{Ran} P$ is a suitable copy of $\mathrm{L}^p(N_P)$ inside $s(P)\mathrm{L}^p(M)s(P)$. To this end we define a conditional expectation $\mathbb{E}_P$ from $s(P)Ms(P)$ onto $N_P$ and an $\mathbb{E}_P$-invariant normal semifinite faithful weight on $s(P)Ms(P)$. A preliminary step will be to find a partition of $s(P)$ into projections belonging to $\mathcal{P}(P)$.

**Lemma 3.10** *The set $\mathcal{P}(P)$ is closed under finite or countable joins, under arbitrary meets, and under relative orthocomplements.*

*Proof* : We have seen that if $(h_i)_{i\in I}$ is an at most countable family of normalized positive elements in $\operatorname{Ran} P$, then $\bigvee_{i\in I} s(h_i)$ is the support of $h = \sum_i 2^{-i} h_i$. Thus $\mathcal{P}(P)$ is closed under finite or countable joins.

If $s, e \in \mathcal{P}(P)$ with $e \leqslant s$, then $e$ belongs to $N_s$ by Lemma 3.7 and so does its relative orthocomplement $s - e$. Since $s - e$ is a projection in $N_s$ it belongs to $\mathcal{P}(P)$, by the same Lemma 3.7.

If $s_1, s_2 \in \mathcal{P}(P)$, then $e_1 \stackrel{\text{def}}{=} s_1 \vee s_2 - s_1$ and $e_2 \stackrel{\text{def}}{=} s_1 \vee s_2 - s_2$ belong both to $\mathcal{P}(P)$ by the preceding, and so does $s_1 \wedge s_2 = s_1 \vee s_2 - e_1 \vee e_2$.

If $(s_i)_{i\in I}$ is an arbitrary family in $\mathcal{P}(P)$ and $e \stackrel{\text{def}}{=} \bigwedge_{i\in I} s_i$, then fixing $i_0 \in I$ we have $e = \bigwedge_{i\in I} s'_i$, where $s'_i = s_i \wedge s_{i_0}$. The $s'_i$ belong all to $\mathcal{P}(P)$, and moreover $s'_i \leqslant s_{i_0} \in \mathcal{P}(P)$. By Lemma 3.7 $s'_i \in N_{s_0}$ for every $i \in I$, hence $e \in N_{s_0}$ and this projection belongs to $\mathcal{P}(P)$ by the same lemma. ■

**Theorem 3.11** *There is a family $(s_i)_{i\in I}$ of pairwise disjoint projections in $\mathcal{P}(P)$ such that*

$$s(P) = \bigvee_{i\in I} s_i.$$

*Proof* : Consider a maximal family $(s_i)_{i\in I}$ of pairwise disjoint nonvanishing projections in $\mathcal{P}(P)$. Clearly $s \stackrel{\text{def}}{=} \bigvee_{i\in I} s_i \leqslant s(P)$. If $s \neq s(P)$ there exists $e \in \mathcal{P}(P)$ such that $e \neq e \wedge s$. But $e - e \wedge s = \bigwedge_{i\in I} (e - e \wedge s_i)$ belongs to $\mathcal{P}(P)$ by Lemma 3.10, which contradicts the maximality of the family $(s_i)$. ■

### 3.3.2 Application to the proof of the Main Theorem.

In this subsection we complete the proof of points 1) and 2) of the Main Theorem (Theorem 1.1) in the non-$\sigma$-finite case.

Fix a maximal family $(s_i)_{i\in I}$ of pairwise disjoint projections in $\mathcal{P}(P)$, the existence of which is given by Theorem 3.11. For each $i \in I$ let $h_i$ be a positive element of $\operatorname{Ran} P$ with support $s(h_i) = s_i$. Let $\psi_i$ be the normal positive bounded linear form on $M$ associated with the element $h_i^p$ of $L_1(M, \varphi)$. For each finite subset $J$ of $I$, let us set $s_J \stackrel{\text{def}}{=} \sum_{i\in J} s_i$ , $h_J \stackrel{\text{def}}{=} \sum_{i\in J} h_i$ and let $\psi_J$ be the linear form associated with $h_J^p$. Note that $s(h_J) = s_J$. Moreover, we have

$$P(h_J) = P\Big(\sum_{i\in J} h_i\Big) = \sum_{i\in J} P(h_i) = h_J \quad \text{and} \quad h_J^p = \sum_{i\in J} h_i^p, \quad \text{hence} \quad \psi_J = \sum_{i\in J} \psi_i.$$

Let us also define a n. s. f. weight $\psi$ on $s(P)Ms(P)$ by

$$\psi(x) = \sum_{i\in I} \psi_i(x). \tag{3.14}$$

All the projections $s_i$, $i \in I$ belong to the centralizer of $\psi$, and more generally so do the projections $s_\Gamma \stackrel{\text{def}}{=} w^*\text{-}\sum_{i\in\Gamma} s_i$, $\Gamma \subset I$ ($\Gamma$ finite or not).

By the results of Section 3.1, we have a normal faithful conditional expectation $\mathbb{E}_J \stackrel{\text{def}}{=} \mathbb{E}_{h_J} \colon s_J M s_J \to s_J M s_J$ satisfying

$$\psi_J = \psi_J \circ \mathbb{E}_J \quad \text{and} \quad P\big(h_J^{\frac12} x h_J^{\frac12}\big) = h_J^{\frac12} \mathbb{E}_J(x) h_J^{\frac12} \tag{3.15}$$

Moreover by Lemma 3.5, the range of $\mathbb{E}_J$ does not depend on the choice of the $h_i$ and equals $N_{s_J} = s_J N_P s_J$ (by definition of $N_P$).

In the sequel we denote by $\mathcal{F}(I)$ the set of finite subsets of $I$. This set is naturally ordered by inclusion.

**Lemma 3.12** *The family $(\mathbb{E}_J)_{J\in\mathcal{F}(I)}$ of conditional expectations is compatible, that is if $J_1 \subset J_2$ then $\mathbb{E}_{J_2}|s_{J_1}Ms_{J_1} = \mathbb{E}_{J_1}$. Similarly, we have $\psi_{J_2}|s_{J_1}Ms_{J_1} = \psi_{J_1}$.*

*Proof* : If $J_1 \subset J_2$, then $h_{J_1} = s_{J_1}h_{J_2} = h_{J_2}s_{J_1}$, hence $h_{J_1}^{\frac{1}{2}} = s_{J_1}h_{J_2}^{\frac{1}{2}} = h_{J_2}^{\frac{1}{2}}s_{J_1}$ and for every $x \in s_{J_1}Ms_{J_1}$:

$$h_{J_2}^{\frac{1}{2}}\mathbb{E}_{J_1}(x)h_{J_2}^{\frac{1}{2}} = h_{J_1}^{\frac{1}{2}}\mathbb{E}_{J_1}(x)h_{J_1}^{\frac{1}{2}} \overset{(3.2)}{=} P\big(h_{J_1}^{\frac{1}{2}}xh_{J_1}^{\frac{1}{2}}\big) = P\big(h_{J_2}^{\frac{1}{2}}xh_{J_2}^{\frac{1}{2}}\big) \overset{(3.2)}{=} h_{J_2}^{\frac{1}{2}}\mathbb{E}_{J_2}(x)h_{J_2}^{\frac{1}{2}}$$

which implies $\mathbb{E}_{J_1}(x) = \mathbb{E}_{J_2}(x)$ since both $\mathbb{E}_{J_1}x$, $\mathbb{E}_{J_2}x$ belong to $s_{J_2}\mathrm{L}^p(M)s_{J_2}$. Hence $\mathbb{E}_{J_1}$ is the restriction of $\mathbb{E}_{J_2}$ to $s_1Ms_1$. ■

Next we want to extend the family $(\mathbb{E}_J)$ to a conditional expectation in $s(P)Ms(P)$. To this end we shall make use of the following lemma.

**Lemma 3.13** *A bounded net $(x_\alpha)_{\alpha\in A}$ of elements of $s(P)Ms(P)$ converges $w^*$ iff the reduced nets $(s_Jx_\alpha s_J)_\alpha$, $J \in \mathcal{F}(I)$ are all $w^*$-convergent.*

*Proof* : Note first that the maps $P_J\colon s(P)Ms(P) \to s(P)Ms(P)$, $x \mapsto s_Jxs_J$ are w*-w* continuous projections on $s(P)Ms(P)$, which commute ($P_JP_K = P_KP_J = P_{J\cap K}$, with the convention $P_\emptyset = 0$). Moreover $P_Jx \overset{w^*}{\to} x$ when $J \uparrow I$, which implies that the family $(P_J)_{J\in\mathcal{F}(I)}$ is separating ($P_J(x) = 0$ for all $J \in \mathcal{F}(I)$ implies $x = 0$).

If the net $(x_\alpha)_{\alpha\in A}$ converges w* to $x$, it is clear by the continuity of $P_J$ that $(P_J(x_\alpha))_\alpha$ converges w* (to $P_J(x)$) for each $J \in \mathcal{F}(I)$. Conversely assume that $P_J(x_\alpha) \overset{w^*}{\to} y_J$ for every $J \in \mathcal{F}(I)$). Then if $y$ is a w*-clusterpoint of the net $(x_\alpha)$ (some exists by w*-compactness of the balls), $P_J(y)$ is a w*-cluster point of $(P_Jx_\alpha)$, hence coincides with the w*-limit $y_J$ of the net $(P_Jx_\alpha)_\alpha$. By the fact that the family $(P_J)$ is separating , the w*-cluster point $y$ is unique, and thus it is the w*-limit of $(x_\alpha)$. ■

**Remark 3.14** *As a consequence of the preceding lemma, for every bounded family $(x_J)_{J\in\mathcal{F}(I)}$ such that $x_J \in s_JMs_J$ , and $x_K = P_K(x_J)$ for every $K \subset J \in \mathcal{F}(I)$, then there is a unique $x \in s(P)Ms(P)$ such that $x_K = P_Kx$ for all $J \in \mathcal{F}(I)$.*

*Proof* : Indeed apply the Lemma 3.13 to the nested family $(x_J)_{J\in\mathcal{F}(I)}$. Since for each $K \in \mathcal{F}(I)$ the net $(P_K(x_J))_{J\in\mathcal{F}(I)}$ is stationary (constant for $J \supset K$), the net $(x_J)$ is w*-convergent and its limit $x$ satisfies the relations $P_Kx = w^*\text{-}\lim_J P_Kx_J = x_K$, for all $K \in \mathcal{F}(I)$. ■

**Lemma 3.15** *The compatible set of maps $(\mathbb{E}_J)$ has a unique normal extension $\mathbb{E} : s(P)Ms(P) \to s(P)Ms(P)$. The map $\mathbb{E}$ is a normal faithful conditional expectation with range $N_P$.*

*Proof* : *Existence of* $\mathbb{E}$. Note that if $K \subset J \in \mathcal{F}(I)$ we have

$$P_K\mathbb{E}_JP_Jx = \mathbb{E}_JP_Kx = \mathbb{E}_KP_Kx \tag{3.16}$$

for all $x \in s(P)Ms(P)$ (the first equation by $N_J$-bimodularity of $\mathbb{E}_J$, the second by Lemma 3.12). By Remark 3.14 there exists a unique map $\mathbb{E} : s(P)Ms(P) \to s(P)Ms(P)$ such that:

$$P_K\mathbb{E}x = \mathbb{E}_KP_Kx \tag{3.17}$$

for all $x \in s(P)Ms(P)$. Moreover we have $\mathbb{E}x = w^*\text{-}\lim_J \mathbb{E}_J P_J x$. If $x \in s_K M s_K$ we have $\mathbb{E}_J x = \mathbb{E}_K x$ for all $J \supset K$ and thus $\mathbb{E}x = \mathbb{E}_K x$, i. e. $\mathbb{E}$ extends all the $\mathbb{E}_K$, $K \in \mathcal{F}(I)$. Then by Lemma 3.13 the w*-w* continuity of $\mathbb{E}$ follows from that of the $\mathbb{E}_J$, $J \in \mathcal{F}(I)$. As for the faithfulness, it results from (3.16) and faithfulness of the $\mathbb{E}_J$, since if $x \geq 0$ we have

$$\mathbb{E}x = 0 \implies \forall J \in \mathcal{F}(I),\ \mathbb{E}_J P_J x = P_J \mathbb{E}x = 0 \implies \forall J,\ P_J x = 0 \implies x = 0.$$

It is clear that $\mathbb{E}$ is $s_J M s_J$-bimodular for every $J$. The $N_P$-modularity follows by w*-continuity. Finally $\operatorname{Ran}\mathbb{E}$ is included in $N_P$ and contains $s(P)$. By $N_P$-modularity, $\operatorname{Ran}\mathbb{E} = N_P$.

*Unicity.* If $\mathcal{E}$ is another extension of the $\mathbb{E}_J$'s then by w*-continuity

$$\mathcal{E}x = \mathcal{E}(w^*\text{-}\lim_J P_J x) = w^*\text{-}\lim_J \mathcal{E}P_J x = w^*\text{-}\lim_J \mathbb{E}_J P_J x = w^*\text{-}\lim_J \mathbb{E}P_J x = \mathbb{E}x.$$

■

**Lemma 3.16** *We have $\psi = \psi \circ \mathbb{E}$.*

*Proof* : For $x \in s(P)M_+s(P)$ we have

$$\psi(\mathbb{E}x) = \sum_{i\in I} \psi_i(s_i(\mathbb{E}x)s_i) = \sum_{i\in I} \psi_i(\mathbb{E}_i s_i x s_i) = \sum_{i\in I} \psi_i(s_i x s_i) = \psi(x)$$

■

As explained in § 2.2 there is a canonical inclusion of the $\mathrm{L}^0$ space of the crossed product $N_P \rtimes_{\sigma^\psi} \mathbb{R}$ into that of $s(P)Ms(P) \rtimes_{\sigma^\psi} \mathbb{R}$, with preservation of their canonical traces, which leads to inclusions $i_p$ of the respective $\mathrm{L}^p$-spaces: $\mathrm{L}^p(N_P, \psi_{|N_P}) \subset \mathrm{L}^p(s(P)Ms(P), \psi)$, $1 \leqslant p \leqslant \infty$. The conditional expectation $\mathbb{E}_{\psi,p} \colon \mathrm{L}^p(s(P)Ms(P), \psi) \to \mathrm{L}^p(s(P)Ms(P), \psi)$ was defined there by transposition of $i_{p^*}$(for the Haagerup trace duality).

**Lemma 3.17** *We have the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{L}^p(s(P)Ms(P), \psi) \arrow[rr, "\mathbb{E}_p"] & & \mathrm{L}^p(s(P)Ms(P), \psi) \\
s_J\mathrm{L}^p(s(P)Ms(P), \psi)s_J = \mathrm{L}^p(s_J M s_J) \arrow[u, hook] \arrow[rr, "\mathbb{E}_{J,p}"] & & \mathrm{L}^p(s_J M s_J) = s_J\mathrm{L}^p(s(P)Ms(P), \psi)s_J \arrow[u, hook]
\end{tikzcd}$$

*Proof* : For $h \in \mathrm{L}^p(s_J M s_J, \psi)$ and $k \in \mathrm{L}^{p*}(N_P, \psi)$ we have

$$\begin{aligned}
\mathrm{Tr}_\psi(hk) &= \mathrm{Tr}_\psi(s_J h s_J k) = \mathrm{Tr}_\psi(h s_J k s_J) = \mathrm{Tr}_{\psi_J}(h s_J k s_J) \\
&= \mathrm{Tr}_{\psi_J}((\mathbb{E}_{J,p}h)s_J k s_J) = \mathrm{Tr}_\psi((\mathbb{E}_{J,p}h)s_J k s_J) \\
&= \mathrm{Tr}_\psi(s_J(\mathbb{E}_{J,p}h)s_J k) = \mathrm{Tr}_\psi((\mathbb{E}_{J,p}h)k)
\end{aligned}$$

(using twice the fact that $s_J$ is in the centralizer of $\psi$), hence $\mathbb{E}_{\psi,p}\, h = \mathbb{E}_{J,p}\, h$. ■

Let us consider now a n. s. f. weight $\chi$ on $M$ such that $s(P)$ belongs to the centralizer of $\chi$, and $\chi_{s(P)} = \chi|_{s(P)Ms(P)} = \psi$. Let $\kappa \colon (M \rtimes_{\sigma_i^\varphi} \mathbb{R}, \tau_\varphi) \to (M \rtimes_{\sigma_i^\chi} \mathbb{R}, \tau_\chi)$ be the canonical trace-preserving isomorphism, $\hat{\kappa} \colon \mathrm{L}^0(M \rtimes_{\sigma_i^\varphi} \mathbb{R}, \tau_\varphi) \to \mathrm{L}^0(M \rtimes_{\sigma_i^\chi} \mathbb{R}, \tau_\chi)$ its natural extension, which preserves the respective Haagerup's $\mathrm{L}^p(M)$-spaces. Recall that if we identify $M$ with its canonical image in each crossed product then by (2.12) $\hat{\kappa}$ fixes the points of $M$.

**Proposition 3.18** *We have the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{L}^p(M,\varphi) \arrow[rr, hook, "P"] & & \mathrm{L}^p(M,\varphi) \\
\mathrm{L}^p(M,\chi) \arrow[u, "\hat{\kappa}^{-1}"] & & \mathrm{L}^p(M,\chi) \arrow[u, "\hat{\kappa}^{-1}"'] \\
s(P)\mathrm{L}^p(M,\chi)s(P) = \mathrm{L}^p(s(P)Ms(P),\psi) \arrow[u, hook] \arrow[rr, "\mathbb{E}_{\psi,p}"] & & \mathrm{L}^p(s(P)Ms(P),\psi) = s(P)\mathrm{L}^p(M,\chi)s(P) \arrow[u, hook]
\end{tikzcd}$$

*Proof* : For every $J \in \mathcal{F}(I)$, the projection $s_J$ belongs to the centralizer of $\chi$, since it belongs to the centralizer of $\psi = \chi_{s(P)}$, while $s(P)$ belongs to the centralizer of $\chi$. It follows by (3.10) that

$$P\hat{\kappa}^{-1}(k) = \hat{\kappa}^{-1}\mathbb{E}_{J,p}(k)$$

for all $k \in s_J\mathrm{L}^p(s(P)Ms(P),\psi)s_J = \mathrm{L}^p(s_JMs_J,\psi_J)$. Since by Lemma 3.17, $\mathbb{E}_{J,p}$ is the restriction of $\mathbb{E}_{\psi,p}\colon \mathrm{L}^p(s(P)Ms(P),\psi) \to \mathrm{L}^p(s(P)Ms(P),\psi)$ to $s_J\mathrm{L}^p(s_JMs_J,\psi_J)s_J$ we obtain

$$P\hat{\kappa}^{-1}(k) = \hat{\kappa}^{-1}\mathbb{E}_{\psi,p}(k)$$

for all $k \in \bigcup_J \mathrm{L}^p(s_JMs_J)$. By density of $\bigcup_J \mathrm{L}^p(s_JMs_J)$ in $\mathrm{L}^p(s(P)Ms(P))$ it follows that $P$ coincides with $\hat{\kappa}^{-1} \circ \mathbb{E}_{\psi,p} \circ \hat{\kappa}$ on $s(P)\mathrm{L}^p(M,\varphi)s(P)$. ■

## 3.4 Complement on the case $1 < p < \infty$

Part 3 of Theorem 1.1 is a consequence of a more general fact on contractive projections in non-commutative $L_p$ spaces. For the sake of stating the result we introduce the left and right supports of a projection $P$, that we denote respectively by $s_\ell(P)$ and $s_r(P)$. We set:

$$s_\ell(P) \stackrel{\text{def}}{=} \bigvee_{h\in \operatorname{Ran} P} s_\ell(h), \quad s_r(P) \stackrel{\text{def}}{=} \bigvee_{h\in \operatorname{Ran} P} s_r(h). \tag{3.18}$$

Clearly if $P$ is positive then $s_\ell(P) = s_r(P)$ and both these supports coincide with $s(P)$ defined by formula (3.11). The general result is then:

**Proposition 3.19** *Let $1 < p < \infty$ and $P\colon \mathrm{L}^p(M) \to \mathrm{L}^p(M)$ be a contractive projection. Then for every $h \in \mathrm{L}^p(M)$ we have*

$$P(h) = P(s_\ell(P)hs_r(P)).$$

The proof of this proposition follows the line of classical proofs for the contractive projection in commutative $\mathrm{L}^p$-spaces. Let us first recall some facts about duality mappings.

Recall that a normed linear space $X$ is said to be strictly convex (or rotund) if for any $x, y \in X$ the equalities $\frac{\|x+y\|_X}{2} = \|x\|_X = \|y\|_X$ imply $x = y$.

Let $X$ be a Banach space. For each $x \in X$, we can associate the subset

$$J_X(x) = \left\{x^* \in X^* \ : \ \langle x, x^*\rangle_{X,X^*} = \|x\|_X^2 = \|x^*\|_{X^*}^2\right\}$$

of the dual $X^*$. The multivalued operator $J_X\colon X \to X^*$ is called the normalized duality mapping of $X$. From the Hahn-Banach theorem, for every $x \in X$, there exists $y^* \in X^*$ with

$\|y^*\|_{X^*} = 1$ such that $\langle x, y^*\rangle_{X,X^*} = \|x\|_X$. Using $x^* = \|x\|_X y^*$, we conclude that $J_X(x) \neq \emptyset$ for each $x \in X$. If the dual space $X^*$ is strictly convex (i.e. $X$ is smooth), $J_X$ is single-valued. Indeed, if $x_1^*, x_2^* \in J(x)$ then

$$\|x\|_X \Big\|\frac{1}{2}(x_1^* + x_2^*)\Big\|_{X^*} \geqslant \Big\langle x, \frac{1}{2}(x_1^* + x_2^*)\Big\rangle_{X,X^*} = \|x_1^*\|_{X^*}^2 = \|x_2^*\|_{X^*}^2 = \|x\|_X^2.$$

Since $X^*$ is strictly convex, we conclude that $x_1^* = x_2^*$. When $X$ is a reflexive and strictly convex space with a strictly convex dual space $X^*$, $J_X$ is a singlevalued bijective map and its inverse $J_X^{-1}\colon X^* \to X^{**} = X$ is equal to $J_{X^*}\colon X^* \to X$. Indeed, for $x^* \in X$, by definition of $J_{X^*}(x^*)$ we have

$$\|J_{X^*}(x^*)\|_X = \|x^*\|_{X^*} \quad \text{and} \quad \big\langle J_{X^*}(x^*), x^*\big\rangle_{X,X^*} = \big\langle x^*, J_{X^*}(x^*)\big\rangle_{X^*,X} = \|x^*\|_{X^*}^2.$$

We deduce that $J_X(J_{X^*}(x^*)) = x^*$. Hence $J_X J_{X^*} = \mathrm{Id}_{X^*}$. By symmetry, we get also $J_{X^*} J_X = \mathrm{Id}_X$.

*Proof of Proposition 3.19* : Suppose $1 < p < \infty$. Recall that $\mathrm{L}^p(M)$ is strictly convex and smooth [PiX, Cor. 5.2]. Hence $J_{\mathrm{L}^p(M)}\colon \mathrm{L}^p(M) \to \mathrm{L}^{p^*}(M)$ is a singlevalued bijective map. Moreover, $(J_{\mathrm{L}^p(M)})^{-1} = J_{\mathrm{L}^{p^*}(M)}$. For any $h \in \mathrm{L}^p(M)$ with polar decomposition $h = u|h|$, we have

$$\big\|\|h\|_p^{2-p}|h|^{p-1}u^*\big\|_{p^*} = \|h\|_p^{2-p}\big\||h|^{p-1}\big\|_{p^*} = \|h\|_p^{2-p}\|h\|_p^{p-1} = \|h\|_p$$

and using the fact that $u^*u$ is the support projection of $|u|$ by [Pal2, Theorem 9.1.25], we obtain

$$\big\langle h, \|h\|_p^{2-p}|h|^{p-1}u^*\big\rangle_{\mathrm{L}^p(M),\mathrm{L}^{p^*}(M)} = \|h\|_p^{2-p}\,\mathrm{Tr}\,\big(u|h||h|^{p-1}u^*\big) = \|h\|_p^{2-p}\,\mathrm{Tr}\,\big(|h|^p\big) = \|h\|_p^2.$$

Hence, we infer that the explicit description of $J_{\mathrm{L}^p(M)}$ is given by

$$J_{\mathrm{L}^p(M)}(h) = \|h\|_p^{2-p}|h|^{p-1}u^*. \tag{3.19}$$

From this formula it is clear that the right (resp. left) support of $J_{\mathrm{L}^p(M)}(h) = \|h\|_p^{2-p}|h|^{p-1}u^*$ coincides with the left (resp. right) support of $h$.

On the other hand, from the last line of the proof of [Cal, Th. 1], the map $J_{\mathrm{L}^p(M)}$ sends $\mathrm{Ran}(P)$ onto $\mathrm{Ran}(P^*)$. This remark together with formulae (3.18) imply that $s_r(P^*) = s_\ell(P)$, $s_\ell(P^*) = s_r(P)$. Thus for every $h \in \mathrm{L}^p(M)$, we obtain that

$$\begin{aligned}\big\|P(h)\big\|_{\mathrm{L}^p(M)}^p &= \mathrm{Tr}\,\big(P(h)J_{\mathrm{L}^p(M)}(P(h))\big) = \mathrm{Tr}\,\big(hP^*(J_{\mathrm{L}^p(M)}(P(h)))\big) = \mathrm{Tr}\,\big(hJ_{\mathrm{L}^p(M)}(P(h))\big)\\ &= \mathrm{Tr}\,\big(hs_\ell(P^*)J_{\mathrm{L}^p(M)}(P(h))s_r(P^*)\big) = \mathrm{Tr}\,\big(hs_r(P)J_{\mathrm{L}^p(M)}(P(h))s_\ell(P)\big)\\ &= \mathrm{Tr}\,\big(s_\ell(P)hs_r(P)J_{\mathrm{L}^p(M)}(P(h))\big).\end{aligned}$$

Hence $P(h) = 0$ whenever $hs_r(P) = 0$ or $s_\ell(P)h = 0$. Now, for any $h \in \mathrm{L}^p(M)$, we have

$$\begin{aligned}P(h) =& P\big((1 - s_\ell(P)\big)h\big(1 - s_r(P))\big) + P\big((1 - s_\ell(P))hs_r(P)\big) + P\big(s_\ell(P)h(1 - s_r(P))\big) +\\ &+ P\big(s_\ell(P)hs_r(P)\big) = P\big(s_\ell(P)hs_rP)\big).\end{aligned}$$

■

**Remark 3.20** *It is well known that even in the commutative case and for positive projections the statement of Proposition 3.19 becomes false for $p = 1$.*

**Acknowledgment**. The first named author would like to thank Université of Paris 6 where he had fruitful discussions with the second named author during his visit in 2015.

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