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Alexander Schüler-Meyer, Jenni Ingram, Kirstin Erath

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Twelfth ERME Topic Conference

Language in the Mathematics Classroom



Editors: Alexander Schüler-Meyer, Jenni Ingram, Kirstin Erath

University of Oxford, Department of Education, Great Britain

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Editorial ETC12 Language in the Mathematics Classroom

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Introduction

ETC12 “Language in the Mathematics Classroom” held in Oxford, UK in 2022 was the third conference of the Mathematics and Language TWG of ERME, following previous conferences in Dresden, Germany (ETC4, 2018) and in Montpellier, France (ETC7, 2020). ETC12 was accepted as a Topic Conference by the European Society for Research in Mathematics Education (ERME). Topic conferences on language in the mathematics classrooms attract a group of researchers in the ERME community and beyond that are interested in all aspects of language in mathematics learning and teaching, particularly but not exclusively interaction processes, multilinguality as resource for learning and teaching, developing academic language and classroom discursive practices. The conference was hosted by Jenni Ingram and illustrates her long engagement with promoting research on language in the mathematics classrooms, particularly as chair of the related TWG9 “Mathematics and Language” of CERME conferences.

The international program committee (IPC) for ETC12 consisted of Alexander Schüler-Meyer (Netherlands, Chair); Jenni Ingram (UK, chair local organizing committee); Aurélie Chesnais (France); Kirstin Erath (Germany); Marie Therese Farrugia (Malta); Ingólfur Gíslason (Iceland); Máire Ní Riordáin (Ireland); Núria Planas (Spain); Susanne Prediger (Germany); Frode Rønning (Norway); Marcus Schütte (Germany); and Konstantinos Tatsis (Greece). The local organizing team consisted of Jenni Ingram (UK); Alexander Schüler-Meyer (Netherlands); Kirstin Erath (Germany), supported by Ashley Abbott, Kyla Smith, Gabriel Lee, Cindy Ong, and Gosia Marschall. We thank both the IPC and the local organizers for their work in making this conference a reality. Also, we thank the ERME board and the president of ERME, Carl Winsløw, for approving and supporting this conference.

The themes of ETC12 extend and develop the themes of the previous two ETCs on language in the mathematics classroom, particularly this time also with a focus on how to educate teachers for language responsive teaching, that is, for attending to issues of language in their teaching of mathematics. Furthermore, most European mathematics classrooms are multilingual, that is, have students who speak multiple languages next to the Language of Instruction (LoI). With these developments in mind, the subthemes (ST) of ETC12 were identified to be:

- ST1:** Developments in the study of classroom interaction and discourse;
- ST2:** Language in heterogeneous classrooms, particularly the impact of different linguistic/multilingual contexts on the learning and teaching of mathematics;
- ST3:** Teacher Education for language-responsive learning and teaching of mathematics.

Developments in research on language in the mathematics classroom

In their introduction to the proceedings of the first topic conference on Language in the Mathematics Classroom (ETC4), Marcus Schütte and Núria Planas (2018) identified themes in the CERME conferences and state:

[in past CERMEs, there is] a prevalence of three major objects of study: language of the learner, language of the teacher and the classroom, and language of mathematics. Despite the continuity in the classroom-based orientation and in the prevalence of major objects of study, the ERME domain has changed in terms of complexity in the theories and frameworks implied. (p. 5)

This statement is still true for this topic conference in Oxford. Not only has the complexity of theories increased, but we see more diverse theories being adopted.

Since Schütte and Planas made their statements, further themes seem to have emerged. At this year's conference, teacher education has become a much more relevant theme. At ETC4 in Dresden, research was concerned with classroom interaction and discursive practices, with only a few studies explicitly focusing on teachers' classroom practices, for instance with respect to teachers' use of language in the classroom (Chesnais, 2018), or teachers' support for multiple representations (Kuntze et al., 2018). Four years later, at this ETC12, there is evidence that research is moving towards developing classroom mathematics teaching with respect to language, particularly through developing teachers' expertise for language-responsive teaching. For instance, research adopts the language dimension to enable teacher noticing by facilitating students' literacy practices such that their thinking becomes more visible to teachers (Saego et al., 2023). Other research aims to enable teachers noticing of students discourse practices (Prediger et al., 2023).

We are thankful for having Ewa Bergqvist and Jill Neumeyer DePiper as plenary speakers. Based on her substantial research on the role of textbooks for mathematics learning, Ewa Bergqvist presented ongoing research on the connection of symbolism with written language in textbooks. Rooted in the context of teacher professional development in the US, Jill Neumeyer DePiper presented her wide-scale research on developing teachers' expertise based on video lessons and a focus on rich discourse practices. Both of these talks lead to fruitful discussions and insights and hopefully inspire future research and cooperation. Finally, we are equally thankful to all other contributors for their paper and poster presentations that were equally insightful and inspiring.

Overall, this topic conference on language in the mathematics classroom continues the tradition of attracting researchers from Europe and beyond, also enabled through adopting a hybrid format. Furthermore, it maintains the high scientific standard of previous ETCs and CERME conferences. Finally, ETC12 was one of the first conferences in Mathematics Education being held in person, after two years of the COVID pandemic. Incidentally, the lock-down started only two weeks after ETC7 in Montpellier. While writing this text, we hope that in the future, as with ETC12, we will be able to meet in person and enjoy each other's company in many future topic conferences on language in the mathematics classroom – while we keep the doors open for other participants to join in digitally from all over the world.

Contributions and review

After the call for papers in the three announcements, 30 contributions were received for the conference. ST1 received 9 contributions, ST2 received 8 contributions and ST3 received 5 contributions. Also, there were 8 poster contributions. Two paper contributions and two posters were withdrawn before the conference. The review process was organized by the IPC chair. Each contribution was reviewed by two other contributors, based on a thematic similarity of the contributions. Overall, most reviewer pairs agreed in their evaluation of the papers, and there were no rejections. Accordingly, all contributors had the chance to improve the papers before the conference, which lead to many high-quality contributions. We are thankful for all contributors who also acted as reviewers.

Jill Neumeyer DePiper (WestED, US) and Ewa Bergqvist (Umeå University, Sweden) were invited to give a plenary talk. Further highlights in the program include workshops for young researchers by David Pimm (Simon Fraser University, Canada), Kirstin Erath (Martin Luther University Halle-Wittenberg, Germany) and Alexander Schüler-Meyer (Eindhoven University of Technology, The Netherlands). Following experiences during COVID, ETC12 was held in a hybrid format, which enabled participants unable to travel for various reasons to participate.

Finally, for the first time, the IPC choose to organize the conference in a new format. While previous conferences grouped the contributions according to their subthemes ST1 to ST3, ETC12 grouped contributions based on their thematic similarity. Accordingly, there were timeslots with the topic of teacher PD or equitable mathematics. In this way fruitful discussions and grassroots for future cooperation were enabled.

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Teaching and learning mathematics in a multilingual context: Which language mastery for which mathematics?

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Most Algerian people express themselves by using three languages: spoken Dialect, Arabic for writing and French for specific uses. For many years, this situation (scarcely considered at the institutional level) has become a source of difficulties for teaching and learning mathematics in school and at university, due to students' weak linguistic skills both in Arabic and French. Interviews have been performed on a sample of eight high school teachers to investigate their awareness of the differences between the three languages when expressing some important logical aspects in mathematics and the resulting difficulties for students. Results provide elements to interpret some aspects of the present situation of school teaching of mathematics, in particular a relationship between weak language proficiency and prevailing teaching of procedural mathematics.

Keywords: University mathematics, multi-linguistic context, Arabic, French, Algerian dialect.

Introduction

Algeria is an example of a country where teaching and learning mathematics occur in different languages at the same time and at different levels. The institutional language in all school levels is Arabic, the institutional language at university, for Mathematics, Sciences, and Technology, is French; both languages are different from the mother language and the spoken language (Algerian dialect) of Algerian people. This multilingual challenging context, with a variety of modes of expressing thoughts in different situations, might result in richness and development of linguistic skills. However, in the Algerian educational institutions, this situation impacts negatively on students' learning, particularly in mathematics. This is due, not only to the poor level of students' mastery of both French and Arabic, which became lower and lower over the last two decades but also to substantial differences between Algerian dialect, Arabic, and French regarding some logical aspects that are very important in mathematics (Azrou, 2020). The aspects considered in this study are the conditional statement and negation because they are central in mathematical reasoning and proving, particularly the first one. Moreover, they cannot be expressed correctly if students cannot understand their syntax and semantic in the language they use to talk and write. The differences regarding the expression of conditional statements are as follows:

- In French, like in English, conditional statements usually take the 'if ... then...' (si ... alors ...) form and focus on the mode and the tense of the two verbs to express different situations as regards the realization of the condition (like "if you do this, you get ...", "if you did this, you would get ...", "if you had done this, you would have obtained ...").
- In Arabic, there are many ways (more than ten) with their specific conjunctions (like 'if' and 'then') and grammatical forms for conditional statements. In particular, the conditional statement in Arabic for an unrealized condition ("if he had been in school, he would have been advised that ...") uses different conjunctions from those used to express a possible condition.
- In mathematics teaching, there is only one conventional form for a conditional statement, expressed with (idha kana ... fa inna ...), which is not common in the used (regular) Arabic

language. It has been chosen to be analogous to the French (si ... alors .../ if ... then ...) form. It is usually used for statements like “if a function is derivable on an interval, then it is continuous on that interval”.

- In the Arabic language courses, when conditional statements are taught, the focus is only on the grammar of the two involved verbs. Usually, teaching does not focus on the degree of possibility of realizing the condition (and the resulting conclusion), which in French and in English is expressed by the mode and the tense of the two verbs (keeping the same conjunctions). This is a big difference with Arabic, because in Arabic (as we have seen before in a specific case) there are different conjunctions to express the degree of the possibility of realizing the condition.
- In dialect, there is only one word (ki) for ‘if’ in the case of both a sure and a possible realization of the condition; for an unrealized condition, we use one word derived from Arabic (loukane), used twice (“loukane... loukane...” for “if... then...”).

The differences regarding the negation are as follows:

- In Arabic, we have two different ways to negate (with different meanings), like in English, when we say: all buses do *not* run on Sunday and *not* all buses run on Sunday. In English, we use the same ‘not’. In Arabic there are two different conjunctions (la and laysa).
- In dialect, negation is like in Arabic, but with different conjunctions. Example in Dialect: mashi gaa jaw (not all have come) and majawch gaa (they did not come all).
- In French, the negation is made by ‘*ne*’ verb ‘*pas*’.

These differences prevent many students from making the passage from one language to another in the correct way when thinking, discussing, and writing mathematical reasoning.

If the forms and constructions of one language do not always have exact counterparts in other languages, this may suggest that the thinking processes of the speakers of one language will differ from those of a speaker of any other language (Edmonds-Wathen, Trinick, & Durand-Guerrier, 2012, p. 24).

After my preliminary study (Azrou, 2020), another study to investigate these differences from the point of view of high school secondary teachers is reported in this paper. It concerns their consciousness of the differences because “becoming aware of differences between languages can help teachers and learners avoid confusion as well as enrich the learning environment” (Edmonds-Wathen, Trinick & Durand-Guerrier, 2016, p. 25). It also concerns their interpretation of how students deal with this challenging situation of using different languages when teaching and learning mathematics, and particularly regarding how they interpret resulting students’ difficulties. My aim is also to examine how do these differences impact the learning of mathematics.

The languages of teaching mathematics in Algeria

Politically, Algeria is considered an Arabic country, but it became an Arabic-speaking country little by little over time, right after the Arabic conquest during the seventh century. The autochthones (Berber people) still maintain their proper language called Tamazight, which became official in 2016 and nowadays is taught in the Berber area. Arabic is the official language of school instruction in all the country (including textbooks); it is also used for books, newspapers, and official discourses. The Algerian Dialect is the spoken (but not written) language of the population; it is basically Arabic with

many French words and expressions, and words from Spanish, Italian, Turkish, etc. French is the first foreign language taught from grade three, English is the second foreign language taught from grade seven. After almost a decade after the independence (1962), the government decided to abandon French (the language of colonizers) as the language of instruction in school, despite the good quality of its teaching and learning, to affirm the identity of the country. Even though most Algerian people spoke different dialects and not Arabic, the choice of the Arabic language was motivated by two main reasons: the linguistic homogeneity had to be associated with religion and modernization, and even more importantly, with a genuine identity, which had to be different from the one built by the colonizer and developed through 132 years. The second reason was the fact that linguistic diversity would have been an obstacle against unifying people of different parts of a big country (with different cultural aspects), like Algeria.

The language of teaching mathematics, technology and sciences at university did not change from French after the independence of Algeria. On the contrary, the language of teaching mathematics in school was changed twice. It has changed from French during the sixties and early seventies, to Arabic from the late seventies till 2005. The reform, which took place in 2005, changed the language of writing symbols and formulas to French, while it kept the verbal written expressions and the discussion in the classroom in Arabic. The 2005 reform was aimed at helping students, in their transition from high school to university, to learn mathematics in French at university. This way of teaching mathematics, which requires students and teachers to write both from right to left (Arabic), and from left to right (formulas), is still in effect so far.

In past years, new elements have shaped the linguistic situation regarding mathematics in school and at university: students have lower linguistic skills both in Arabic and French. This lower mastery of the languages of instruction created a linguistic emptiness, which resulted in strengthening the spoken dialect. It is even becoming, nowadays, a written language due to the opportunities offered by technology (SMS and different social media). In the present situation, students, as well as their teachers, use the Algerian dialect in order to go on with the teaching and learning process in the classroom. Consequently, teaching and learning mathematics occur in different languages simultaneously: two in school, Arabic for writing (with formulas in French) and dialect for explanations and discussions (student-student and student-teacher); and three at university – French for writing, Arabic for some oral mathematical terms and expressions, and dialect for explanation and discussions.

Literature review

People in different parts of the world express words, thoughts, and ideas in different languages, which usually show many differences among them, as concerns syntactic and semantic aspects. Using these languages to teach and learn mathematics puts into evidence some of these differences but also possible limitations inherent in the grammatical systems of the languages. Edmons-Wathen et al. (2016) list many examples of differences in logic and reasoning in various languages: the Kpelle language of people of Liberia has two different words for ‘inclusive *or*’ and ‘exclusive *or*’, but it has no easy expression to express the conditional statement ‘if and only if’; the negation in the Maori language (the native language of New Zealand) uses two different words for ‘not’, one for negating non-verbal sentences and one for verbal sentences. In French, when sentences involve an existential quantifier, applying the negation on the verb does not provide a logical negation; in Arabic, when the

negation is inside the sentence, its meaning is not negating the sentence but only negating the verb, which would lead to the opposite meaning if we translate word by word into French. For example: All numbers are not even, in Arabic, the negation is only on even, which would be translated into English (or French) word by word by: all numbers are odd.

The main reference for this study: Cummins' theory and the language of mathematics

Cummins investigated the relationship between second language proficiency and cognition (1979); his studies put into evidence the fact that students need a high mastery of at least one language to reach satisfactory school performance. According to Cummins, students who master two or more languages will outperform and surpass their mates, while those who have a weak mastery of languages will underperform in the school. Concerning Cummins' theory, Clarkson (1992) and Dawe (1983) have examined the relationship between language proficiency and mathematics achievement; their findings show a direct correlation between the level of language proficiency (of the first language or the language of instruction) and the level of achievement in mathematics.

Linguistic skills and mathematical achievement

Research about teaching and learning mathematics in a second or a third language has focused more on school situations and less on situations at university level. Examining the relationship between language proficiency and the achievement of university students is very important, particularly if the language of instruction changes from the one used in the school like in Algeria. The studies of Barton and his collaborators (Barton & Neville-Barton, 2003, 2004t; Barton, Chang, King, Neville-Barton & Sneddon, 2005) concerned the university level; they put into evidence the relationship between mastering a language and performing well in advanced mathematics. The results of the studies showed how second English learners, at the first university year, outperformed their mates who were first English learners, in basic courses of calculus and linear algebra. But in the third year, second English learners showed bigger difficulties with advanced mathematics, where higher language proficiency was required to allow the mastery of linguistically and logically complex mathematical statements and reasoning.

Method

I aimed to check the awareness of high school teachers about:

- The differences between the three languages used in school and at university, as concerns conditional statement and negation in mathematics.
- Students' behavior and possible difficulties resulting from these differences.
- How this would impact students' learning in mathematics.

I have chosen high school teachers because they are those who should prepare students to university. I have designed semi-structured interviews concerning conditional statement and negation; eight teachers from different high schools (urban, public and private), situated in the center of the country, have been interviewed. Teachers had different teaching experiences: One with one year of experience, three with 2 years of experience, two with 5 and 7 years, and two with 10 and 20 years of experience. The interviews lasted around 45 minutes; they were based on main questions and sub-questions. The main questions were the same for all teachers, while during the interviews and according to the answers to the main questions, sub-questions were posed or omitted.

Q1. You know that in high school Mathematics textbooks, and in the official Arabic language for mathematics, there is only one expression for conditional statement: "idha kana ... fa inna ...". But students know other expressions of conditional statement in Arabic, and they currently express conditional statement in Dialect, which also is different from "idha kana ... fa inna ...". Have you observed students' shifting from an expression of conditional statement to another, by using other words than (idha kana ... fa inna ...), or by using expressions in Dialect?

Q1.a For instance how can a student express in Arabic the meaning of the sentence "*si la fonction était derivable, alors elle serait continue*" ("if the function were differentiable it would be continuous")?

Q1b. At university, we observed that several students have big difficulties when moving from their usual conditional statement in Arabic in high school mathematics, to conditional statement in French. Have you any idea about the reasons for these difficulties?

Q1c. What do you think about the possibility of preventing these difficulties in the classroom?

Q2. Do you think that students would have problems when negating mathematical expressions in the translation from Arabic to French or vice versa, like for instance negating "*all the balls are red*"? What kind of difficulties?

Q2a. Does it happen that students use dialect to negate in mathematics?

Analysis of the interviews

R1. All teachers confirmed that students might use other conditional statement, different from 'idha kana ... fa inna ...', in Arabic or Dialect, but only in the oral form, during the discussions with the teacher or among the students. Most teachers (seven) said that possibly students are not aware of the equivalence between different conditional statement, already known in the Arabic language, and the equivalence (or not) between these different statements and the conventional one. They also added that students do not understand clearly the mathematical meaning of the conventional conditional statement: "Even when they write this conventional statement, they do not always understand its structure and its meaning". Teachers have added that the teaching of these statements, in the Arabic language courses, is not coherent with mathematical reasoning as expressing the hypothesis and the thesis (or a condition and a conclusion derived from it).

Students encounter them for the first time in an Arabic language course and deal with them only in a grammatical way. With this teaching, it is not obvious to use them in mathematics. Teachers have to teach them all over and explain the meaning of a conditional statement, how it works, and how it is used, otherwise students would not understand them automatically.

Most teachers argued that if students mastered the mathematical meaning of the conventional conditional form, they would be able to shift from one language to another without difficulty.

R1a. About the conditional statement related to an unrealized condition, teachers said they do not use it and do not have any opportunity to use it, according to them it is irrelevant. Only the required conventional form is used, which expresses a possibility affirmatively. When they write mathematical reasoning on the blackboard by using this form, they go on step by step in deductive reasoning by using 'then'... 'then' Teachers added that even if students might express this conditional

statement by using Dialect, it is not sure that they are aware of the equivalence with the same conditional conventional statement in Arabic.

Every time that we say or write anything in Arabic, students would automatically search for its equivalent in Dialect, the problem is that the equivalence they consider might be not correct, it depends on how they understand the Arabic word or expression.

R1b. teachers expressed clearly their awareness about students' weak mastery of French at all the school levels. "Our French colleagues complain continuously from students' level". "One of my colleagues says that his course became a monologue, he is the only speaker". They added that this phenomenon is not exclusive to French; they also observe students' weak mastery of Arabic and the excessive use of Dialect: "students use Dialect and ask us to speak in Dialect, otherwise they do not understand".

R1c. All teachers agreed that an intervention would be positive, however, they had doubts about its possibility. They mentioned two problems; the first one is the lack of time because the program is very charged with a lot of contents to be covered in a short time. The second one is the necessity to prepare and train teachers for it, because, according to them, it is an important thing to do seriously, and in the present situation, it is not possible, as teachers are not qualified.

R2. Teachers insisted again on the weak mastery of mathematics and the lack of at least one mastered language at the academic level: "These students do not reach the mathematical understanding because they do not have a mastered language". "I have seen some students who master very well Arabic: when they master the mathematical meaning, they would go very far with their mathematical learning, they would even excel".

R2.a. Teachers said that students usually use the negation in the three languages and might find three different meanings. One form of negation in Dialect is total negation (no exceptions). One teacher gave an example to show the linguistic effect on mathematics understanding. He said that students do not understand the terms: 'at least' and 'at most' in Arabic, because they do not exist in Dialect; consequently, students cannot assimilate the negation of 'all' as being 'there exists at least one...'.

Conclusion and discussion

Despite their different professional experiences, all teachers showed their awareness of students who do not learn the multiple conditional statements as expressing a condition and a conclusion derived from it, when learning them in an Arabic language course. Students, whose everyday language is Dialect, are not provided in school with what is needed in Arabic (the language of instruction and textbooks) to deal with the mathematical conditional statement, which is necessary for mathematics. Teachers acknowledged that they should explain the mathematical meaning of the conventional conditional statement, but, according to their answers, they do not realize that it would be necessary to make links and comparisons with the other different (non-conventional) conditional statements in Arabic and Dialect. They neglect them and avoid using and explaining their meaning, thus leaving students in doubt when they use them orally in mathematical discourses. The same situation occurs with negation: students usually translate the standard Arabic expression of negation into Dialect with the risk of getting two sentences with two different meanings. Moreover, it is clear for teachers that students need to use Dialect, and teachers try to help students by using Dialect by themselves, but their responses show they are not aware of students' difficulties in shifting from one language to

another (oral Dialect and written Arabic), which have different structures. This might be a sign of the lack of teachers' awareness of the deep structural differences between the three languages, as concerns the expression of conditional statement and negation. This interpretation is supported by the fact that teachers presented the lack of Arabic mastery and French mastery as the only reasons for students' difficulties with conditional statement and negation in high school and at university, instead of considering the structural linguistic differences. Some teachers insisted on students' weak *mathematical understanding* of the conditional statement and negation, in other words, of how these logical aspects are considered in mathematics (without considering the relevance of students' mastery of linguistic expressions for them). Some other teachers claimed that if students could have strong mathematical understanding and strong language proficiency, they would excel in mathematics. Teachers' answers suggest an important connection between the mastery of mathematics and the mastery of the language of instruction. When the language mastery is weak, it results in a weak mastery of mathematics and when the mastery of mathematics is weak (like in the case of mastery of the mathematical conditional statement) it does not encourage students to draw on the linguistic resources of a language. This results in a vicious circle. In particular, as students did not develop the semantic and syntactic comprehension of the conditional statement in the Arabic language course, they cannot transfer it to mathematics in Arabic, and if students do not develop the sense of the mathematical conditional statement in Arabic, they cannot transfer it to French. Instead, students need to build these first in one language to transfer to another language and to mathematics too (see Cummins, 2000: "conceptual knowledge developed in one language helps to make input in the other language comprehensible"). The same situation occurs when students (to express a thought or to translate it) move from Dialect to Arabic or from Dialect to French; in Dialect, many meanings are either absent or expressed in different forms. A hypothesis, which emerged from teachers' responses, has been suggested by what they said concerning the irrelevance, in mathematics teaching and learning, of the use of the conditional statement in the case of an unrealized condition. As we have seen in the introduction as regards the differences between the three languages, the conjunctions of the expression of "if we had not made the hypothesis that ..., we could not have proved that..." are different in Arabic and in Dialect from those used in the case of "if we make the hypothesis that... we can prove that...". The lack of attention paid to this case *might result* from a style of teaching (very common in high school), which only aims at learning computations, procedures, and proofs to be memorized. Indeed, sentences like "if we had not made that hypothesis ..." usually occur in classroom activities when the teacher promotes reflection on the conditions of validity of a statement and, more generally, on mathematical reasoning, definitions, etc. We claim that language proficiency and mathematics proficiency progress in parallel: low language proficiency matches less cognitively demanding mathematical activities, while the higher the language proficiency, the easier it is to access more cognitively demanding mathematical activities. Thus, teachers' answers concerning the conditional statement in the case of an unrealized condition suggest further investigating the reasons for their opinion about its irrelevance for them.

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Lifeworld Connections – Results on Language Use and Learning Possibilities Arising from Linking Mathematics with Experiences

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Within the mathematics classroom, connections to the everyday world are frequently used with the aim of catching the pupils' attention, fostering their understanding, and building bridges between the familiar 'everyday world' of the pupils and the more formal and abstract 'world of mathematics'. Based on the interpretative paradigm of interpretative classroom research, the presented study carries out interactional analyses of selected passages of mathematics lessons to illustrate how mathematical classroom discourse and language are affected through the use of lifeworld connections and to what extent this could affect the learning of mathematics.

Keywords: Language, discourse, interaction analysis, learning, lifeworld connection.

Lifeworld connection – Linking the learning content with personal experiences

Hearing or reading the term 'lifeworld connection' might evoke different associations, feelings and understandings. This expression seems self-explanatory – a connection to lifeworld. But what is it that the lifeworld is connected to and what is included by this 'lifeworld'? Concerning the existing literature – especially in the field of mathematics education and sociology – tracing back the origin of a widely accepted understanding of the term turns out to be challenging: There exists a number of terms that seem to be used synonymously, like real-world connection, real-world context, contextualization, authentic (modelling) problem, realistic problem, or story problem (Kaiser & Schwarz, 2010; Karakoç & Alacacı, 2015; Larina, 2016; Palm, 2006; Zan, 2017, Zaskis & Liljedahl, 2009). The underlying research uses the term lifeworld connection because it is connectable to its sociological origins: Schütz and Luckmann (1973) describe the term lifeworld as the “unexamined ground of the natural world view” (p. 3) of each individual person. Simultaneously, they underline the significance of social interactions. Following this, lifeworld is an individual and intersubjective sphere of experiences, perceptions and interpretations of a person, developing through and in social interactions. Lifeworld is not solely inter-individually hidden within each person, but also intra-individually meaningful, because it emerges and changes by social interactions, and enables the individual to appropriately interact with others (Schütz & Luckmann, 1973). As the presented research is based in the interpretative paradigm and considers interactions as a driving force for learning processes, this understanding of lifeworld is shared. Therefore, within the frame of this research, the term lifeworld connection is defined as follows:

In situations with lifeworld connection, the (mathematical) learning content is linked or related to a situation, thing, or process of the learners' experiences – based on assumptions¹ of teachers or textbook authors. This relation can be more or less explicit. Both spheres – learning content and

¹ The included addition 'based on assumptions of the teaching person' should underline the fact, that it is impossible to be fully and clearly aware of the learners' real-world experiences. If anything is important and meaningful to the learners, remains to be an interpretation.

real-world experiences – can be intertwined and reciprocally related to each other. The onset of the interactive negotiation can be either the learning content or the real-world experiences of the pupils.

Although this study focusses on lifeworld connections (in the following LWCs) within mathematics lessons in school, this definition can be also useful in other educational settings where a learning content is related to the learners' experiences, like mathematics in nursery school, biology seminars in university, or different areas of adult education. In this regard, the definition is rather a scaffold with the opportunity for changes, modifications, and differentiation than a final statement. Concerning mathematics lessons in school, the empirical data might reveal some characteristic features of LWCs in mathematics education as well as special features of the language use during such situations.

Lifeworld connections in mathematics education

Because there is a lack of a universal specification of the term and there seems to be few high-quality and empirically proved resources for LWCs, mathematics teachers' often select and construct LWCs by their own criteria – mainly guided by their individual beliefs, goals and knowledge (Gainsburg, 2008). Often, teachers use LWCs with the aim of providing a (supposedly easier) access to the more abstract and formal mathematical content by referring to the more personal and sensual experiences of the learners and use them as resources for learning mathematics (Civil, 2007). Further positive aspects – amongst others – are, that LWCs have the potential to illustrate the applicability of the learning content, motivate the learners to participate in the lesson by paying attention to their interests, and to evoke a more positive view on (school) mathematics (e.g. Gainsburg, 2008; Karakoç & Alacacı, 2015; Pongsakdi et al., 2020; Povey, 2013; Verschaffel et al., 2020).

But besides a long list of positive aspects and reasons, it is often neglected that LWCs were mainly posed by the teaching person and with a special didactical purpose, what results in their stereotyped nature and discrepancies between authenticity and mathematical richness. In most cases, a connection to the lifeworld in mathematics education has an unrealistic, numerically clean nature including all necessary data, is always solvable with a single-step in only a single way, has exactly one solution, contains superficial features like key-words, and directly matches with the precede mathematical concept or operation (e.g. Gainsburg, 2008; Greer, 1997; Nesher, 1980; Toom, 1999; Zan, 2017). Additionally, Zan (2017) underlines, that LWCs often end up with a question, that is not posed by the learners but by another person like the teacher or the textbook. In result, the included experiences and contexts rather remains to be a decorative dress for the learning content than a real personal interest of the learners. They have an imaginary character as the included numerical data is seldomly taken from reality and they mix the real with the ideal (Toom, 1999). “What matters in this case is intrinsic consistency and interesting mathematical structure rather than consistency with or importance for everyday life” (Toom, 1999, p. 37). Because of the fact that LWCs often have a stereotyped nature and are being treated in a stereotyped way in classrooms, pupils become trained in solving the included mathematical task exactly in the expected way. Nevertheless, many pupils could not sufficiently identify with the presented LWC, as they had not made compatible experiences (e.g. paying bills and wage slips); or they fail to recognize the purpose of the task and the mathematical content within (e.g. Boaler, 1993; Sullivan et al., 2003; Zan, 2017).

It might also happen that the pieces of information needed to solve the problem are not necessarily consistent from a narrative viewpoint, and if they are inconsistent, they will probably be ignored by those who read in a narrative mode. [...] The role of the context [experiences] is reduced to that of container of necessary (and generally sufficient) data to be able to answer the question. Thus, not surprisingly, the student focuses on the question, while the context is read with relation to that question (in particular by selecting key-words and numerical data) (Zan, 2017, p. 6f.).

Following Leuders and colleagues (Leuders et al., 2011), we can elaborate three characteristics of LWCs that have an influence on fruitful and long-life learning processes: Connectivity, authenticity and mathematical richness. First, an LWC should match the pupils' ways of thinking, speaking and acting and fit their interests and experiences. Second, the pupils should be enabled to rise genuine questions about the real-world context and to learn something about it. Third, the mathematical content of an LWC should be displayed correctly and in all its complexity to engage the pupils to deep explorations of the mathematical content (Leuders et al., 2011).

Learning as an interactional, individual, and emotional process

The presented study is based on the interpretative paradigm of interpretative classroom research (e.g. Bauersfeld, 1994; Krummheuer, 2000). One fundamental assumption of this tradition is that

meaning is negotiated in interactions between several individuals and that social interaction is thus be understood as constitutive of learning processes, speaking about mathematics with others is in itself to be seen as the 'doing' of mathematics and the development of meaning (Schütte et al., 2019, p. 104).

In this regard, social interactions become the center of attention and the 'place', where (mathematical) meaning is negotiated. One fundamental theoretical assumption of this research project is that the use of language – and in result also the assumed opportunities to learn – differs depending on the situation (Moschkovich, 2018). Learning always takes place in a social context and is therefore mainly mediated and visible through language. As it is widely accepted that meaningful contexts are also important for successful (and long-lasting) learning processes (e.g. Kaiser & Schwarz, 2010; Leuders et al., 2011), situations in mathematics lessons involving an LWC should be personal meaningful for the learner', and moreover enable an authentic and rich engagement with the mathematical learning content as well as the real-world experience. In this regard, especially personal feelings and emotions of the learners are meaningful – as they enable learning – while negative emotions and circumstances rather hinder learning processes (Hascher, 2010).

Emotions are, on the one hand, interrelated with the processes of learning and instruction, and, on the other hand, with cognition. [...] Crucial sources of cognition and emotion are the learners themselves, other protagonists of the learning situation like teachers and peers, as well as learning tasks (Hascher, 2010, p. 24f.).

Methodological approach

Most studies on LWCs put the focus on written tasks or typical 'word problems' and seldomly put the focus on language use and learning opportunities that arise by them. Additionally, existing studies mainly seem to have a product-oriented view on the pupils' outcomings and results concerning LWCs (especially regarding to the topic of 'classical' word problems), while this qualitative study

concentrates on the process of the negotiation about an LWC. The presented research mainly considers LWCs during verbally interactions, with a deep qualitative view on language use and the reconstruction of the pupils' opportunities to learn. The following research guiding question can be derived: *What can be stated about the authenticity, mathematical richness, connectivity, and language use in situations with a lifeworld connection in mathematics education and to what extent might these aspects have an impact on the pupils' supposed opportunities to learn?*

To gain a broad impression of the pupils' language use during different situations with a LWCs, mathematics lessons of the several classes (different school types and different class levels as well) in Germany have been video-recorded. In nearly all observed mathematics lessons LWCs could be identified. These, however differ with regard to explicitness, scope, and grade of reality – what becomes visible through language use and might result in different learning opportunities. Selected passages with LWCs were transcribed and later analyzed via interactional analysis (for a detailed description of the approach, its basic concepts and analysis steps see Schütte et al., 2019) to illuminate to what extent LWCs within mathematics lessons have an impact on the pupils' use of language and their learning. The interactional analysis is a qualitative research approach to reconstruct the negotiation of meaning in interactions and consists of the following steps:

First, the sequence is described in general, and structured in scenes. Afterwards, the most important step of the interactional analysis is the detailed sequential interpretation of the individual utterances. This detailed step-by-step analysis helps to look at the transcript in detail and to reconstruct the negotiation process of the interaction and the development of the mathematical content. It is often interwoven with the turn-by-turn analysis. In this analytical step some interpretations of the preceded sequential analysis are regarded as less applicable or less likely interpretations and therefore crossed out. The final step of the interactional analysis – the summary of the interpretation – helps to reduce the diversity of the interpretations that were made before and illustrates only these interpretations, which can best be justified. For reasons of space, the analysis below is limited to the description of the scene and the final summary of the interpretations.

Exemplary lifeworld connection – What is a car?

Of the total sum of 68 video-recorded mathematics lessons (each 45 minutes) 16 lessons were held in a multi-age class in grade 7 and 8. During the following scene, the teacher is sitting with eight 8-graders in a circle in front of the board, while the 7-graders of the class are working on their own.

- | | | |
|----|----------|---|
| 01 | Teacher | Please open page 12 in your textbook. we have already discussed what a function is – a clear assignment. Okay? Example, um ... now an adaption ... I ask you what is a car [in German: <i>Auto</i>]? Okay? |
| 02 | Uldis: | What is a car? |
| 03 | Teacher: | [Nods] What is a car. |
| 04 | Franz: | A function. |
| 05 | Teacher: | No, no. What is a car? You'll try to explain this to me cause I have no idea. |
| 06 | Ulli: | A vehicle |
| 07 | Teacher: | That's not enough. |
| 08 | Ulli: | For locomotion. |
| 09 | Teacher: | That's still not enough. |
| 10 | Ulli: | There can people sit inside. |
| 11 | Paul: | [Loud] A motorized vehicle. |

- 12 Teacher: Car² means translated automobile. You were already right that ‘mobile’ means driving. But for what stands *Auto*?
- 13 Nico: Automobile stands for automatic vehicle ... self ... self-driving.
- 14 Teacher: [Points with his finger in Nicos direction] Yes. You got it. Automatic means that it happens by itself. Automobile – self-driving vehicle. As we do know what it is now, the next question is?
- 15 Uldis: What does it?
- 16 Paul: No. What does a function consist of?
- 17 Teacher: Yes. You would again say, adapted it means: What is it made of? Now you list the individual parts like wheel, engine and this we want to do for a function now. This is not about defining a function. This we have done before – we know a function is a clear assignment. And this chapter now is about the parts of a function, okay? And when you look in your notebook you will see of how much parts a function has?
- 20 Paul: [Looks in his notebook] Three.
- 21 Teacher: [Nods] Namely?
- 22 Paul: [Looks in his notebook] A set of inputs [*Definitionsbereich*], a set of outputs [*Wertebereich*] and a rule [*Zuordnungsvorschrift*]
- 23 Teacher: [Nods] There we have it. And these three parts you can identify in each form of visualization of a function. Okay. Let’s look at them in detail.

During the presented scene the term car (*Auto*) is connected with the definition and the single parts of a mathematical function. The following analysis is limited to the key results, concerning the use of language during the presented scene and the connectivity, authenticity and mathematical richness of the LWC. In the end, some assumed learning possibilities are presented.

Language Use

As the presented transcript is a translated and ‘smoothed’ version of the German original, the linguistic aspects are hard to illustrate. Some key linguistic features like dialectal peculiarities, deictic expressions or the use of sentence-connecting elements like conjunctions play a central role concerning language use but will not be discussed within this paper. Here, the focus is on the existence of technical terms, the grade of (de-)contextualization and the overall existence of everyday or academic language, respectively.

The overall scene is – besides some technical terms like function, adaption, assignment, auto(matic), set of inputs, set of outputs – mainly in the everyday language register (Bitterlich & Schütte, 2021; Moschkovich, 2002). Additionally, we could identify many questions giving the scene the character of a quiz – in this regard we could state that the interaction primarily has an exploratory character. Although the teacher says that they do not want to define function, a definition seems to be central at first – namely the definition of car respectively Auto(mobil). In the end, the teacher explicitly asks the pupils about the technical terms of a function, which they can read in their notebook. It seems, as if these terms were not common to them till now. Compared to other classroom interactions, the overall interaction is decontextualized to a great extent as we do not need a lot of information about the situation to follow the discussion. They speak about a car and a mathematical function, which are not existent within the classroom, neither as object nor as a graphic. Of course, deictic expressions, facial expressions and gesturing do exist in this situation and are important for understanding, but the negotiation is mainly taking place on an abstract-symbolic level.

² In the original language German the teacher says “*Auto heißt übersetzt Automobil*”.

The lifeworld connections' connectivity, authenticity and mathematical richness

Connectivity: The LWC fits to the pupils' experiences as everybody should know what a car is, and to their language use, as the scene is mainly framed as an informal everyday conversation. But in such an everyday and non-mathematical conversation, the question "what is a car" would not evoke mathematical oriented answers. The pupils seem to expect something mathematical to be important during the mathematics lessons and give mathematical-oriented answers (line 4 and 16). But their expectations are disappointed as the teacher seems to frame the situation as something non-mathematical – until line 17, where he explicitly asks for the parts of a function and frames the discussion as something mathematical this time.

Authenticity: It is questionable, if the presented connection between a car and a mathematical function enables the pupils to learn something about cars. Maybe the intense discussion about the meaning of the word *Auto/Automobil* stimulates them to think deeper about the origin of these words. But this might not enlarge their knowledge about cars. The presented LWC between a car and a function respectively between the parts of a car and the parts of a function seems to be rather a decorative dress for the new mathematical learning content than the possibility to an intense consideration with the underlying experiences. We assume, that LWCs are frequently "couched in figurative contexts, often nonsensical, and no-one takes them seriously" (Povey, 2013, p. 17).

Mathematical richness: Concerning the mathematical completeness and complexity of the topic of functions and the parts of a function, the connected experience about cars does match the mathematical topic unsatisfactory. Following the teacher and the pupil's notebook, a function consists of exactly three parts while a car has much more. Additionally, even for a mathematical function we could define more components depending on its realization – a functional equation like $f(x)=3x^2$ displays the 'clear assignment' (using the teacher's words from line 01 and 17) in another way, using linguistical, graphical and mathematical resources other than a function graph.

Discussion

We can assume different goals and intentions of the teacher about why he established the LWC between a car and a function. One possibility is, that he wants to make the abstract and formal mathematical topic of a function and its parts more graspable for the students by linking it to a topic, that they are all familiar to – namely cars. Maybe, he wants to illustrate similarities between a car and a mathematical function – they both consist of several parts, every part has a determined purpose for the whole object, every part is needed to complete the object, a missing part would make the whole object 'unusable'. As the teacher does not communicate explicitly, what he wants to discuss with the pupils, the whole scene has the character of a question game and correct answers seems to be flukes. Nevertheless, the pupils seem to be trained in interpreting the teacher and what he wants them to say.

During this ETC Conference, I want to discuss this example and illustrate other examples which demonstrate that LWCs are frequently posed by the teacher but obviously not reflected in detail. Further, I want to discuss what effects an LWC has on the language use (especially on a discursive level) within classroom interactions, and to what extent situations with an LWC in mathematics create special conditions for learning (both language and mathematics).

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Proof and students' proving in Euclidean geometry: Linguistic aspects and method issues in the rationality perspective

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The aim of this paper is to put into evidence some linguistic aspects and method issues of the student's proving process and of the related proof text in Euclidean geometry, in the perspective of Habermas' rationality. The analysis of three students' productions will show how words and other signs may reveal interesting teleological and epistemic aspects of students' moving from the proving process to its product. In some cases, the interpretation of students' texts may result in rather arbitrary outcomes. We will discuss how suitable didactical choices may allow, on one side, to get less arbitrary interpretations and on the other, to enhance students' awareness of some epistemic and communicative requirements of proving and proof. A related research development will be outlined.

Keywords: Habermas' rationality, proving and proof, linguistic aspects, method issues.

Introduction

In Habermas (1998) the ideal of communicative rationality is expressed through an evocative, condensed but very rich description:

Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1998, p. 315)

The definition of rational mathematical template (RMT) (Boero & Turiano, 2020) as *the couple consisting of a mathematical entity (e.g., definition, proof) and a rational process that is purposefully oriented to produce an instance of that mathematical entity* was an attempt to identify and frame some crucial disciplinary components of rational behavior, whose mastery is needed for the interpersonal (and intra-personal too) productive and enjoyable discursive exchange described by Habermas. In Boero & Turiano (2020) we have shown how the RMT construct may work as a tool to analyze the gradual access of students (under the guide of their teacher) to the mastery of the RMT of definition and to productive classroom discussions on definitions. We have also discussed the relationships of our construct in mathematics education with other constructs, in particular with the “process-oriented routines” and the “product-oriented routines” of Lavie, Steiner and Sfard (2019). In Boero (2022) the analysis of two discussions in the same classroom concerning the RMT of counter-example has allowed to say more about the development of the RMTs in the classroom, by identifying some aspects of the mastery of RMTs on the epistemic and teleological sides, which are relevant for the development of students' rationality. In this paper we focus on the relationships among the components of the RMT of proof (rational process of proving, and proof as its product) and on the development of students' communicative rationality as key requirement not only for the mastery of the RMT of proof, but also for teachers' reliable interpretations of students' productions,

which are necessary to orient didactical choices. We will tackle two questions (the second one emerged from the study regarding the first one):

- What are the relationships between the requirements of rationality of the process, and the requirements of rationality of the product, particularly as concerns the epistemic and the communicative aspects?
- How to deal with the method issues related to the (possibly arbitrary) interpretation of the signs through which students express themselves in the approach to the rationality of proof?

Theoretical framework

Basic notions concerning Habermas' rationality are resumed here. Habermas (1998) proposed the construct of rationality to deal with discursive practices that are characterized by *awareness* when checking the truth of statements and the validity of reasoning according to shared criteria (*epistemic rationality*), evaluating strategies to attain the aim of the activity (*teleological rationality*), and choosing suitable communication tools to reach others in a given social context (*communicative rationality*), the three components being strictly interconnected. In past years, researchers both in our group and outside it have attempted to adapt Habermas' construct to mathematics teacher education and to plan teaching aimed at developing and analyzing students' rational behaviors (see Boero & Planas, 2014, for a general account about it and a presentation of five studies). Our elaboration of the RMT construct was developed in the last four years; it was aimed at dealing with specific disciplinary aspects of rationality in mathematics and in other domains as well.

The Rational Mathematical Template of proof

RMT of proof is characterized by specific epistemic, teleological and communicative aspects: the process is aimed at producing a text with the specific logical and communicative requirements of proof, according to the different methods of proof (direct, by contradiction, by contraposition, by induction...). The process of proving may be considered "rational" when its different phases (exploration, construction of the reasoning, writing the proof text – not necessarily in this linear order) are consciously developed and evaluated according to the aim of the activity, attention being paid to epistemic and communicative requirements inherent in the product. Awareness is a distinctive character of the rational process that brings from students' taking in charge of the statement to the construction of its proof. Its importance at every school levels is motivated not only by the need of promoting students' critical control skills and of getting a well-structured and easy-to-understand proof (without logical gaps and unjustified propositions), but also to avoid those phenomena of subordination to the presumed expectations of the teacher for reasons inherent in the didactic contract, which have been put into evidence by Azrou and Khelladi (2019) even at the university level.

Method

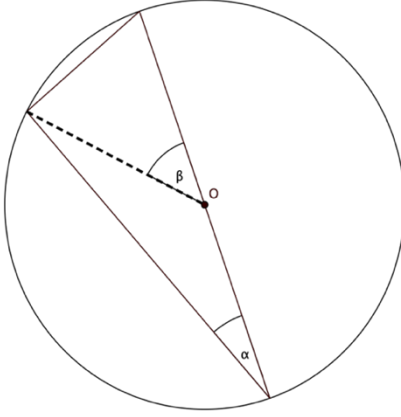
We will consider a teaching experiment on Euclidean proof, which involved two 10th grade classes of scientific and technological oriented high school, with 19 and 25 students each. The activities were performed in the period November, 17, 2017 - May, 11, 2018, with two hours each week, for most of the school weeks in the period, for a total of 36 hours, in parallel with other activities on algebra, analytic geometry and probability. Previous preliminary activities in plane geometry (in grade IX, with the same teacher, and at the beginning of grade X) had concerned the nature of definitions, and some statements of theorems already met by students in comprehensive school, with a few easy proofs

utilizing them. In this paper we will focus on a situation of conjecturing and proving (and related activities) and on the productions of three students (we will name Mario, Fabio and Lucia) with different personal characteristics and different levels of mastery of the three components of rationality of proving. The activity took place in March, 2018. The general design of the sequence of activities on the approach to Euclidean proof in Geometry took into account the fact that geometric constructions (with related theoretical justifications) and theorems alternate in Euclid's Elements. The classroom activities (a couple of tasks for each two hours) included, from the beginning, tasks of individual geometric construction, with related verbal description. Each of them was followed by oral (through a classroom discussion) or written individual revision of constructions produced by some schoolfellows and selected by the teacher. Revisions included checking the generality of the construction and identifying lacking details and erroneous verbal expressions. Tasks of theoretical, written individual justifications of the construction (based on known statements) were proposed for each construction. They were followed by individual comparison and/or individual revision and/or classroom discussion of theoretical justifications produced by some schoolmates. This choice allowed a smooth approach to generality and precision of the discourse on geometric figures and to proving. Students' acquired familiarity with geometric constructions allowed them to produce suitable geometric drawings for conjecturing and for proving tasks. Conjecturing and proving activities related to geometric figures, and then proving activities of statements proposed by the teacher, started at the beginning of March, 2018 (the three proof texts reported below concerns the first activity of this kind). Like for the other activities, systematic individual and/or classroom revisions, comparisons, discussions of proof texts followed each individual proving activity, attention being paid to the key elements of the produced statements and proofs (particularly as concerns the expression of the hypothesis and the thesis, and the necessity of a complete and not redundant proof text). Other activities were proposed, starting from January, 2018: individual cloze activities (followed by a classroom discussion) to complete a theoretical justification (provided by the teacher) of a construction or a statement, by choosing the kind of justification of some steps (by construction; by hypothesis; by definition of...; by theorem...); identification of the proof strategy in the proof text of a schoolmate, with search for possible lacks and mistakes. The alternation of individual productions (or revisions) and classroom comparisons and discussions was aimed at developing the students' RMT of proof as a mediator between the students, the students and the teacher, and the students and the culture (see Boero & Turiano, 2020, p. 145).

In the following analysis of the students' texts some weaknesses on the epistemic and communicative ground will be put into evidence by the use of *italic*.

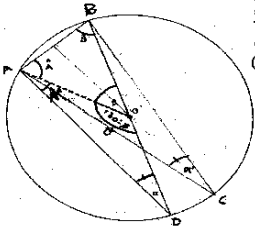
Mario's text and its analysis

Mario was one of the students who moved from a low level of performances at the beginning of the sequence (and in Mathematics in general), to an over the average level at the end.



The task: Observe the figure in order to find which relationship exists between α and β . Then prove the relationship that you have found.

Osserva la figura allo scopo di trovare quale relazione esista tra gli angoli α e β . Poi dimostra la relazione trovata.



PARTE 1

$$\hat{O} = \hat{A} + \hat{B} \rightarrow \beta = 180^\circ - \hat{O} \rightarrow \beta = 180^\circ - \hat{A} - \hat{B}$$

$$\alpha = 180^\circ - \hat{O} - \hat{A}' - \hat{A}$$

$$\hat{A} + \alpha + 180^\circ - \hat{B} - \hat{A}$$

$$2\alpha = \beta$$

PARTE 2

OSSEVANDO LA FIGURA HO NOTATO CHE L'ANGOLO β POTREBBE ESSERE IL DOPIO DELL'ANGOLO α .
 COME PRIMA COSA HO RIPRODOTTO L'ANGOLO α IN MODO CHE FOSSE IN LINEA CON β , INVOIPRANDO DUE TRIANGOLI EQUILATERI ABO E ABC. DI QUESTI DUE TRIANGOLI SAPRAMO CHE HANNO LA BASE IN COMUNE (AB).
 DAL DISEGNO SI PUÒ GIÀ NOTARE COME GLI ANGOLI ALLA BASE DEL TRIANGOLO ABC SONO PIÙ AMPI DI QUELLI DEL TRIANGOLO ABO, DA CÌÒ POSSIAMO DEDURRE CHE L'ANGOLO α' (CHE È UGUALE AD α) È MENO AMPIO DELL'ANGOLO β PER DIFFERENZA DI ANGOLI INTERNI DI UN TRIANGOLO.

PARTE 3

RITORNANDO AD ESSI AI TRIANGOLI INTERNI DEL DISEGNO SI NOTA CHE AOD È UN TRIANGOLO ISOSCELE E CHE QUINDI $\hat{A}' = \hat{D} = \alpha$.
 NOI SUPPONIAMO CHE $\alpha = \frac{1}{2}\beta$ PERCÌO L'ANGOLO \hat{O} DEL TRIANGOLO AOD DOVE È UGUALE ALLA SOMMA DEGLI ANGOLI \hat{A} E \hat{B} DEL TRIANGOLO AOB PERCÌO $\hat{O} = \hat{A} + \hat{B}$.

(PART 1) By observing the figure, I noticed that angle β might be the double of angle α . As first thing, I reproduced the angle α in such a way that it was aligned with β , by finding two equilateral triangles ABO and ABC. These two triangles have their base in common (AB). From the drawing, we may already notice how the angles adjacent to the base of the triangle ABC are wider than those of the triangle ABO, from which we may deduce that the angle α' (that is equal to α) is less wide that the angle β by difference of internal angles of a triangle.

(PART 2) Now, by coming back to the initial triangles of the figure, we notice how AOD is an isosceles triangle and then $\hat{A}' = \hat{D} = \alpha$. We suppose that $\alpha = \frac{1}{2}\beta$ thus the angle \hat{O} of the triangle AOD must be equal to the sum of the angles \hat{A} and \hat{B} of the triangle AOB, hence $\hat{O} = \hat{A} + \hat{B}$.

(PART 3)

$$\hat{O} = \hat{A} + \hat{B} \rightarrow \beta = 180^\circ - \hat{O}$$

$$\rightarrow \beta = 180^\circ - \hat{A} - \hat{B}$$

$$\alpha = 180^\circ - \hat{B} - \hat{A} - \hat{A}'$$

$$\hat{A}' + \alpha = 180^\circ - \hat{B} - \hat{A}$$

$$2\alpha = \beta$$

Figure 1: The task (on the left), and Mario’s text, with the translation of its Part 1 and Part 2

From the teleological point of view Mario looks aware of the different phases of his conjecturing and proving process (the spatial organization of the text and their labels PARTE 1, PARTE 2, PARTE 3 shows three distinguished steps; within the third step Mario puts the core of the proof into evidence, like in the above quote). Mario moves from an initial, possible conjecture (“the angle β might be the double of the angle α ”; the initial writing was “the angle β is the double of the angle α ”) to an exploration of the situation. We may notice a *communicative mistake* (“equilateral triangles” instead of “isosceles triangles”) and the *lack of justification* of *isosceles triangles*. Then Mario exploits the familiarity with geometric constructions to get a suitable figure, and finally he gets the justification of a weaker statement ($\alpha < \beta$) through visual evidence, a theoretical justification (“by difference of

internal angles of a triangle”) *implicitly* based on the theorem that the sum of the internal angles is the same for any triangle, and an *unjustified claim* ($\alpha' = \alpha$). In the second part of his reasoning, by exploring the original figure of the worksheet, Mario notices that the angle α is equal to the angle \hat{A}' (by a theoretical, explicit reason related to the fact that the triangle AOD is *isosceles*); however, the *theoretical justification of it is lacking* – only visual evidence is put on the fore. At that point he foresees how to get the proof: he comes back to the initial possible conjecture, that now is expressed as a *hypothesis to derive what follows*, but probably plays the role of a *hypothesis* to be verified, which results in an *abduction*. This is the starting point of a piece of text of difficult interpretation (at the end of part 2 and at the beginning of part 3), in particular it is not clear the meaning of the two arrows. Mario seems to feel the need to work on the angle \hat{O} of the triangle AOD, which *must be* equal to the sum of the angles \hat{A} and \hat{B} in order to find some relationships that are needed to get the proof. Difficulties of interpretation might depend on Marios' difficulties to organize elements derived from an abduction into components of a deductive chain (see Pedemonte, 2007). It is clear that Mario works on already considered properties of the triangles (the sum of the internal angles, and the congruence of the angles of isosceles triangles) but *explicit justifications are lacking*. This phase seems to play a heuristic role to get the underlined formula: $\underline{\beta = 180^\circ - \hat{A} - \hat{B}}$. At that point Mario starts a sequence of algebraic expressions that bring to the conclusion. From the surrounding line it is clear that Mario considers what is inside as the proof. The *lack of verbal comments and of some intermediate algebraic expressions* (e.g. the recall of $\underline{\beta = 180^\circ - \hat{A} - \hat{B}}$ and of $\hat{A}' = \alpha$) do not prevent the reader from interpreting Mario's reasoning, also thanks to the spatial disposition of the lines.

Fabio's text and its analysis

Fabio was a student who engaged in classroom activities in all the disciplines the minimum that was needed to pass to the next school level (however in that year he did not succeed in it). In some similar cases he tried to take profit from his mates' drawings and words to tackle his tasks. He was supported by a sufficiently good intuition when new and not very complex tasks were proposed. His text well represents his potential and limitations. Fabio's initial exploration starts from a visual intuition of the relationship between β and α , but exploring does not provide him with elements to move from the conjecture to its validation. In fact, naming some individual elements of the given figure (see Figure 1, on the left side) as “ray”, “arc”, “diameter” is far from establishing relationships among them – which would be necessary to move towards validation of the conjecture. This phase ends with a self-declaration of stand-still condition. Then a constructive phase of the process starts and quickly results in a substantially correct proof. Was the starting point derived from some hints received from schoolmates' figures (possibly from marks for congruent sides) or from an autonomous noticing related to the shape of the figure? Let us consider what needs an interpretative effort by the reader in Fabio's text. C^2 and E' are equal to K *looks as* “the sum of C^2 and E' is equal to K ”. “Due to the fact that the sum of the internal angles must be 180° , C^2 and E' are equal to K ” *lacks the justification*: “since $180^\circ = C^2 + E' + \beta = K + \beta$ then $C^2 + E' = K$ ”. “ α and C^1 are equal and their sum is β ” *lacks a similar justification for the equality* $\alpha + C^1 = \beta$. The fact that the first justification might have been obvious at Fabio's eyes may justify the lack of the second justification. If interpreted this way, Fabio's proof satisfies the epistemic requirements of proof; the only weaknesses would concern the communicative requirements. The need of an interpretation like the above one suggests the necessity of an intervention in order to develop Fabio's awareness on the epistemic requirements of proof (in

particular, to fill the gaps of the proof text), and to make him aware of the communicative requirements of the proof text.

	<p>According to me the angle β is the double of the angle α.</p> <p>The side CO is the ray of the circle because O is the center of the circle. The line CD is an arc of the circumference, while the side ED is the diameter.</p> <p>I do not succeed to find a proof suitable to support my theory; but COD is an isosceles triangle. I see.</p> <p>Due to the fact that the sum of the internal angles must be 180°, C2 and E' are equal to K; given that the triangle COD is isosceles α and C1 are equal. By knowing that α and C1 are equal and their sum is β I can state that β is the double of α. $\beta/2 = \alpha$</p>
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Figure 2: Fabio's figure (on the left) and translated text

Lucia's text and its analysis

<p>1. Osserva la figura allo scopo di trovare quale relazione esiste tra gli angoli α e β. Poi dimostra la relazione trovata.</p> <p>1. Ci sono 2 triangoli isosceli, uno COA e l'altro COB. 2. I due triangoli hanno uguali i lati: OA e OB perché sono tutti e 2 i raggi della circonferenza, dato che questi due passano per O. Se mettiamo il lato OC è uguale a CO e di conseguenza essendo in comune OC con i 2 triangoli, OC è uguale BO e di conseguenza BO è uguale a OA. 3. La somma degli angoli interni dei 2 triangoli è 180° per ognuno. 4. Il triangolo COA ha 180° somma degli angoli, si sommano α e ω che sono gli angoli adiacenti alla base e quindi sono uguali, questi si sommano all'angolo β, la somma è 180°, però l'angolo β e ϕ formano allo stesso tempo un angolo di 180°. Quindi la relazione tra l'angolo β e α è che i 2 angoli sommano lo stesso angolo.</p>	<p>There are two isosceles triangles, one of them is COA and the other is COB. The triangles have equal sides OA and OB because both of them are rays of the circle, because the sides pass through O. If we observe the side OA it is equal to CO, and consequently, since OC is in common of the two triangles, OC is equal to BO and then BO is equal to OA.</p> <p>The sum of the internal angles of the two triangles is 180° for each of them. The triangle COA has 180° as sum of its angles, we sum α and ω that are adjacent angles to the basis then they are equal. These angles sum up with the angle γ, the sum is 180°, but γ and β make at the same time an angle of 180°. Thus the relationship between the angle β and the angle α is that the two angles sum up the same angle.</p>
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Figure 3: Lucia's text (on the left) and its translation

Lucia was a student who usually engaged a lot in dealing with mathematical tasks. We may consider her text as the transcript of a process that progressively moves from some hints derived from the observation of the figure, to a substantially well justified statement. The initial phase includes an apparently useless argumentation. We may consider it as an exploratory step that provides Lucia with a solid base for the following phases of her process. This interpretation is suggested by the fact that the equal sides of the triangles COA and COB are marked in a careful way according to each considered triangles. We may consider what follows (note the spatial gap between the first and the second part of Lucia's text) as an organized exploration that at the same time builds the statement of the theorem and the reasons for its truth, which are organized in a consequential chain. On the epistemic side there are no gaps (in particular, no unjustified intermediate statements), provided that we integrate what is written by Lucia with some additional or substitutive words:

If these angles sum up with the angle y , the sum is 180° , but y and β make at the same time an angle of 180° . The relationship between the angle β and the angle α results from the fact that the two congruent angles α and ω sum up the angle β . The conclusion is that $2\alpha = \beta$

Lucia's text requires a relevant effort by the reader in order to interpret it, and suggests the need of further developing Lucia's rationality in proving, particularly as concerns the awareness of the communicative requirements of the proof text.

Theoretical, methodological and educational implications and developments

The analysis of the three students' texts suggests to consider the relationships between the proving process and the proof text, the difficulty of getting reliable interpretations of the students' productions, and how to choose further, suitable activities for their approach to the RMT of proof. As concerns the first issue, both Mario's and Lucia's productions suggest to distinguish rationality criteria for the exploration phase and for the construction of the proof text. In particular, in the exploration phase epistemic requirements need to be looser than in the construction phase. Analogy and abduction as well as visual evidence, which are not acceptable in a proof, may play a crucial role in order to get the elements that will be organized in a deductive chain ("cognitive unity of theorems"). Gaps and lack of logical connections (like in the case of Mario's first two "parts") are acceptable during the exploratory phase of conjecturing, or to access the situation represented by a statement to be proven. As concerns the difficulty of getting reliable interpretation of students' productions, let us consider the case of the proof texts: the less explicit the connections between different proof steps and their justifications are, the less reliable is the interpretation of the proof text not only by the researcher, but more importantly by the teacher (in the perspective of her making subsequent, effective didactical choices). Students' development of the awareness of the epistemic, teleological and, specially, communicative requirements of rationality would provide the teacher (and the researcher too!) with the opportunity of accessing students' productions in a more reliable way. The double role of the teacher as interpreter of students' productions during the classroom implementation of the didactical sequence and as promoter of the development of students' mastery of the RMT of proof puts into evidence the necessity of identifying criteria for a careful balance between the need of a design of the teaching sequence and the need of its adaptation to students' difficulties emerged through the interpretation of students' productions. The diversity among students as concerns the mastery of the different components of the RMT of proof provides the teacher with the opportunity to enhance (in particular) the students' awareness of the requirements of proof by comparing and discussing their proof texts. In this perspective the analysis of the three students' productions shows different levels of awareness of different aspects of the proving process and its product, depending on the didactical choices already implemented in their classroom (see the first subsection of Method) and on each student's personal sensitivity to them. We may say that while Mario's awareness of the teleological components of conjecturing and proving is well developed, Fabio and Lucia only show some initial signs in that direction (mainly through the spatial organization of their texts). On the contrary, in spite of some specific points that need a non-immediate interpretation, the awareness of the epistemic and communicative requirements of proof (the reference to known properties and the use of appropriate words to accompany the construction of the proof) looks higher in the case of Lucia and even Fabio. A delicate issue concerns the kind of interventions that are appropriate to improve the communicative quality of the student's writings during the proving process, in favor of the student

herself (when she needs to reflect on her attempts or previous steps of reasoning), without imposing standardized rules over her personal style of communication with herself. At present, classroom comparison of those writings looks as reasonably good solution that encourages the student to be more clear when explaining her thoughts to the schoolmates (and consequently to herself). Again, the crucial role of the teacher as responsible of the students' approach to the RMT of proof may rely upon the variety of students' productions and the comparison of them. The necessity of re-thinking the construct of cognitive unity of theorems in the perspective of the development of the mastery of the RMT of proof results from the relevance of awareness as a crucial requirement of the process that brings to proof as its product. A hypothesis, to be dealt with both theoretically and practically, is that awareness (when sufficiently developed) can allow students to overcome the difficulties, put into evidence by Pedemonte (2007), to build proofs based on elements identified through explorations.

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Explorative participation in the context of a classroom discourse

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We aim to explore the characterization of explorative participation of prospective mathematics teachers and lecturers on propositions in the context of classroom discourse. The data sources for examining the teacher's and student's discourses included classroom observations. The data were analyzed in terms of participants' and lecture's characterization of explorative participation from a commognitive perspective. The results indicated that the lecturer had prompted the explorative questions for classroom discussion to obtain explorative engagement where the actions were aligned with the goal of the lecturer, and they were flexibly applied in a logical structure.

Keywords: Explorative participation, propositions, prospective mathematics teachers.

Introduction

Mathematical statements comprise a basis for mathematical thinking that enables communication about mathematical objects. Mathematical statements also help students think about mathematical concepts and improve their reasoning skills and logic. In examining current literature, mathematical statements are primarily encountered in issues such as proof and reasoning and the scope of proof validation and refuting (Barak, 2018; Bleiler et al., 2013; Epp, 2003; Selden & Selden, 1995). In addition to these issues, existing literature also focuses on the understanding or interpretation of mathematical statements of in-service/pre-service teachers or undergraduate and below students (Buchbinder & Zaslavsky, 2013; Hawthorne & Rasmussen, 2015; Piatek-Jimenez, 2010; Saban et al., 2014; Zeybek Simsek, 2021). Students have difficulties understanding, creating, and expressing mathematical statements. This difficulty may negatively affect students' ability to comprehend and prove advanced mathematical ideas (Dubinsky et al., 1998; Saban et al., 2014).

In the study of Buchbinder and Zaslavsky (2013), the role of examples given by students in proving mathematical statements was examined in 10th-grade students in group discussions, and the findings showed that there were inconsistencies in students' understanding of statements. A framework was developed by Hawthorne and Rasmussen (2015) for characterizing students' thinking about logical statements while working with undergraduate math and computer science primary students conducting individual and class discussions. In the study of Levenson et al. (2012) with in-service secondary school teachers focus on the development of teachers' knowledge interactively in a course that is related to mathematical statements and suitable proof techniques; meanwhile, it shows the relation with the mathematical statement, the instructor, and teachers in class discussions. Saban et al. (2014) determined students' perceptions of mathematical statements containing one or more quantifiers in individual interviews with pre-service elementary mathematics teachers. Similarly, in another study on how undergraduate students interpreted mathematical statements containing quantifiers, it was stated that it was difficult for students to interpret statements with multiple quantifiers, especially when the existential quantifier preceded the universal quantifier (Piatek-Jimenez, 2010). However, the difficulties encountered in perceiving existential and universal quantifiers in these two studies are the opposite Saban et al. (2014) stated that, as a reason for this,

the difficulties that can be experienced might differ according to the study group. Another study conducted by Zeybek Simsek (2021) and interviewed individually pre-service middle school mathematics teachers on how they decided on the validity of a proposition and how they evaluated student arguments. As a result of this study, pre-service teachers struggle to assess the tasks containing students' ideas, create counterexamples to prove or disprove the mathematical statements, and construct arguments to justify their decisions regarding the mathematical statements (Zeybek Simsek, 2021). In the study by Buchbinder and Zaslavsky (2007), in which they examined how high school students decide on the validity of mathematical statements, an environment was created for students to demonstrate the truth of the given proposition, such that students had to produce and support their arguments and share their mathematical ideas with the classroom environment. Due to this learning environment, the researchers had the chance to identify where the students had difficulties or strengths while interpreting the mathematical statements. In addition, it is said that the correct interpretation of open propositions containing one or more quantifiers plays an important role both in learning the conceptual structure of mathematics and in transferring mathematical knowledge (Saban et al., 2014).

Most of the existing studies stress the difficulties students encountered with mathematical statements. Most of the studies are based on cognitive perspectives, and some of them explore classroom interaction through sociocultural perspectives. In this study, we focus on the propositions, a mathematical statement in the classroom discourse, to analyze the understanding of learners and teachers. We aim to explore the characterization of explorative participation of prospective mathematics teachers and lecturers on propositions in the context of classroom discourse. Our study uses a commognitive perspective because it highlights the interaction in a natural classroom setting and enables us to analyze the exploration of learners and teachers. We address the following question: How do the characterization of explorative participation of prospective mathematics teachers and lecturers on propositions in the context of classroom discourse?

Theoretical Framework

Sfard (2008) defines discourse as a “special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions” (p. 297). Routines are the “set of metarules defining a discursive pattern that repeats itself in certain types of situations” (Sfard, 2008, p. 301). Routines define how and when a process is performed (Sfard, 2008). The how of a routine “which is a set of metarules that determine the course of patterned discursive performance; when of a routine, which is a collection of metarules that determine, or just constrain, those situations in which the discussant would deem this performance as appropriate” (Sfard 2008, p. 208). Closure of the routines comprises how routines end (Sfard, 2008). Routines are known patterns of action one recalls in a task situation (Lavie et al., 2019). There are two different types of routines as rituals and explorations. Rituals are process-oriented, and explorations are goal-oriented (Sfard, 2008).

As a sociocultural approach, the commognitive perspective addressed learning as participating in a particular discourse (Lavie et al., 2019; Sfard, 2008). One of the main assumptions in commognitive theory is not about changing the cognitive structures of participants; it is more about changing routines of participation that are involved in a specific discourse (Heyd-Metzuyanım et al., 2019). Researchers distinguish two different types of participation ritual and exploration (Lavie et al., 2019). Ritual participation is process-oriented and involves imitating the teacher but has a vital role in

entering a specific discourse (Lavie et al., 2019). On the other hand, explorative participation is goal-oriented and can be posed as self-oriented or, by the actor, flexible and logically coherent (Heyd-Metzuyanim et al., 2019). Explorative participation requires the production of a new narrative and endorsed it (Lavie et al., 2019). The learner can participate in discourse with initial encounters in a ritualized way, then learners' participation can turn into fully-fledged explorations (Lavie et al., 2019). The development from ritual to explorative participation could be seen both in the task selection and its implication and in the orchestrating of mathematical discussions (Heyd-Metzuyanim et al., 2019).

The explorative task makes way for new truths about mathematical objects that enable the production of new narratives about mathematical objects (Lavie et al., 2019). For rituals, tasks and the procedure are one; performers focus on the process they need to proceed with. Performers can only have social reasons like avoiding punishment or honoring a teacher for ritual participation (Sfard, 2008).

Nachlieli & Tabach (2019) provide a methodological lens about ritual-enabling and exploration-requiring opportunities to learn in their study. Ritual-enabling opportunities to learn were defined as applying a procedure that learners have known before. In contrast, the term exploration-requiring opportunities to learn were explained as students cannot be achieved a task just by performing a ritual but rather just by participating exploratively in producing mathematical narratives focusing on expected outcomes (Nachlieli & Tabach, 2019). In the explorative participation, a teacher will prompt the words such as what, why, find, and explain.

Methodology

The data for this study was conducted in the context of a “basic mathematical concepts” course in a mathematics education department in Turkey. We collected data from classroom observations conducted in the context of a “basic mathematical concepts” course for 20 seniors studying at a mathematics education department. The lecturer is a professor who has a PhD degree in mathematics and works in the department of mathematics education. Prospective mathematics teachers (PMT) take mathematics education content courses (such as calculus, discrete mathematics, and linear algebra), mathematics education courses (such as geometry education, algebra education, material design, and technology in mathematics education), and pedagogical courses (such as developmental psychology, classroom management, approaches and theories of teaching and learning).

In the context of a “basic mathematical concepts” course, PMTs analyze and discuss basic mathematical concepts (such as equations, inequalities, polygon, vector, function, and transformation). PMTs work in groups of four people in this course. PMTs investigate the origins, meaning, and history of the specific mathematical concepts, and then they analyze and categorize the definitions of the particular concept in the literature. After prospective teachers examine the specific mathematical concepts, each group presents one clear mathematical concept in the classroom and comprehensively discusses the definitions. Each group justifies their ideas and supports their mathematical thinking about the definitions of specific mathematical concepts. In the end, the presenting group made their definition of the particular mathematical concept they had discussed. In this study, we just focused on the concept of propositions.

Classroom observations collected through a video camera were transcribed in participants' native language and then translated from Turkish into English. The transcripts of the classroom observations

included participants’ utterances and their visual mediators and actions. The data were analyzed regarding participants’ and lecturers’ characterization of explorative participation (Sfard, 2008). As discussed in the theoretical framework section, we adopted a methodological lens that Nachlieli & Tabach (2019) provide about ritual-enabling and exploration-requiring opportunities to learn. We interpreted explorative-requiring opportunities to learn as explorative participation and analyzed data on *how* (procedure) and *when* (initiation and closure) explorative participation was actualized as given in Table 1.

Explorative participation		
Initiation	What is the question that the teacher poses (raises)?	What is it I want to get?
Procedure	How is the procedure of the routine determined?	Students are expected to choose from alternative procedures. They are expected to make independent decisions.
Closure	What type of answer does the teacher expect?	Indicating the new narrative produced.

Table 1: Methodological lens: Explorative Participation (Nachlieli & Tabach, 2019)

Results

In this section, we provide the analysis of classroom discourse in the context of “basic mathematical concepts.” We examine the characterization of explorative participation of prospective mathematics teachers and lecturers on propositions in classroom discourse. The group with four participants presents the propositions’ origins, meaning, and history. Then, they investigated the definition of open propositions and explored 42 sources. After categorizing these definitions into three themes, as given below, they comprehensively discussed the definitions of the open propositions. We characterize explorative participation on *how* (procedure), and *when* (initiation and closure) were actualized. Prospective mathematics teachers participated exploratively in the classroom discourse by producing mathematical narratives focusing on expected outcomes. We explored that the lecturer has initiated the words such as what, why, find, and frequently explain, which enable explorative participation for PMTs.

- Theme 1: Propositions that contain at least one variable and declare true or false judgments according to the variable's value are called open propositions.
- Theme 2: Given a set of X . If a proposition denoted $p(x)$ is determined for each x element of the set X , then p is called a propositional function (open proposition) defined on the set X . The proposition function takes either a true or false value for each variable’s value.
- Theme 3: An expression that contains the unknown and is true or false according to the values that the unknown will take.

The group has presented three different definitions of open propositions in the classroom. Then, each group has indicated their ideas about which definition is more appropriate for themselves. They have put in order the themes from more meaningful for them to not significant. Each group has listed their decisions on the definitions of the open propositions. Here are the groups’ decisions: 1st Group: 2-1-3; 2nd Group: 2-1-3; 3rd Group: 1-2-3; 4th Group: 2-1-3.

Initiation-1: The lecturer asks PMTs why they have listed the third definition at the end. Below is the dialogue between the lecturer and the PMTs.

Lecturer: As far as I can see now, groups considered the third theme at the end. Let's reread the third definition and know why they have decided like that.

Procedure-1:

Ada: It should say propositions that make a judgment; I think it is the wrong usage. It's incomplete.

Güneş: It was said that the unknown in Theme 2 should have used a variable instead of the unknown and a judgmental proposition instead of a critical expression, right?

Kiraz: In Theme 3, it said, "an expression that contains the unknown and is true or false according to the values that the unknown will take." For example, I said, "the best meal is x " there is also an unknown in it, and since it does not use the word definite judgment, its accuracy or inaccuracy may vary from person to person. For example, instead of that x , you can put eggplant dish and lentil soup. Since that expression does not say definitive judgment, it can be interpreted that way.

Closure-1: All of the groups agreed that the definition given on the third theme is intuitional, not mathematical. Most groups think that the utterance of "value of unknown" is not clear and explicit. Some groups recommended using variables rather than the "value of unknown," and others recommended using variables.

Below is given a classroom discourse as second explorative participation, which includes initiation, procedure, and closure.

Initiation-2: The lecturer asked PMTs why they found the second theme more acceptable than the third one. Here is the dialogue between PMTs and the lecturer.

Procedure-2:

Hacer: Actually, the 2nd definition seemed more mathematical compared to 1st definition. For example, we use at least one variable, our variable is x , and according to the value of the variable, namely $p(x)$, we decide whether it is right or wrong. So, 2nd definition made more sense.

Can: There is a domain given with x ; we know where the x we have chosen comes from.

Lecturer: Well, the propositional function didn't confuse you? It defined something like that; that is, for each x element of the set X , a proposition denoted by $p(x)$ is determined. Is this expression correct? So, is it a meaningful expression?

Class: No answer.

Then, the lecturer asked, "Propositions that state true or judgments according to the variable's value. So, what's the difference between Theme 1 and Theme 2? The 3rd group already said Theme 1, but why did the 3rd group bring Theme 1 forward?" Below is the dialogue between the lecturer and PMTs.

Selma: Teacher, I think the definition of Theme 1 is sufficient; there is no need to express it as a propositional function as in Theme 2.

Sevgi: It means the same thing, but the second theme is more mathematical. Other propositions, $p(x)$'s, and such sets are determined.

Lecturer: The following is missing in Theme 1; of course, there will be a variable; it's like what the variable represents is missing.

Ali: For example, it says the variable's value; we cannot even understand that it is a numerical thing [he is talking about Theme 1].

Lecturer: Do you think it is suitable for a variable to take a value? As a discourse, does your mathematics culture that you have created throughout your student life so far if you consider what you are trying to learn, is variable take a value? [There is humming in the classroom]

Lecturer: Or would you substitute it? To represent this, it means that taking a value is one thing. You choose the representative. It represents a group, you substitute it with elements from that group, and you look at it. Substitution is not taking value. Then to sum it up, it turns out something like this, once the set X is given, X being a different set from the empty set, an open proposition on the set X , yes, we can say this. When the proposition $p(x)$ is obtained, we call p the open proposition on x . Of course, this proposition may be true or false. Is not it? We do not have any comments regarding its accuracy or inaccuracy. But that certainly is; whatever I get from the representatives instead of every x , we should end up with a proposition. Since we will get the proposition in the end, it has to contain a variable. Then there are two essential things: a set and a representative of a set; you can get a proposition with each set representative. That statement becomes an open proposition. So, what did we say here?

Closure-2:

Damla: Teacher, while we were doing this study, the definition that came closest to us was the 1st definition: “Propositions that contain at least one variable and that declare true or false judgments according to the variable’s value are called open propositions.”

Then, after discussing all three themes on the definition of open propositions, the lecturer asks explorative questions to clarify the meaning of the open proposition. The lecturer has stressed the two critical constructs to define an open proposition as a group's set and representative. The lecturer and PMTs discussed the concept of propositions comprehensively. We investigate characteristics of explorative participation that generate a new historical fact or a new truth about propositions that is an outcome-oriented routine in the form of new, transformed, or rearranged objects. We explored that the lecturer frequently prompts questions to PMTs to produce their narratives and enable their production of mathematical thinking. Most mathematics teachers expect their students to participate in their classes exploratively; however, pure explorative participation is a rare occurrence (Lavie et al., 2019). There are different forms of explorative participation characterized by initiation, procedures, and closures (Nachielli & Tabach, 2019).

Conclusion and Discussion

In this study, we have explored the characterization of explorative participation of prospective mathematics teachers and lecturers on propositions in the classroom discourse. The main discussion in this classroom is driven by the definitions of the available propositions found in the literature. PMTs provides three different themes on definitions of open propositions. The lecturer has orchestrated the classroom discourse by exploring their decisions on the definitions. Lecturer asked explorative questions to PMTs when they were explaining their ideas. Each group has provided logical arguments to defend their decision-making process. This finding agrees with Buchbinder and Zaslavsky’s (2007) results, showing that students can produce and support their opinions about a given proposition and share their mathematical ideas with the classroom environment. However, the current study results do not support the previous research (Zeybek Simsek, 2021) that states pre-service teachers have challenges constructing arguments to justify their decisions regarding the validity of the mathematical statements.

Heyd-Metzuyanım et al. (2019) indicated the core elements of ritual to explorative participation as the practice of task selection and the practice of orchestrating mathematical discussions. The main task in this study is to analyze the definitions of propositions that have the flexibility and logical

structure to understand the mathematical statements given. Align with the literature; explorative questioning forms a basis for explorative participation, so the task selection and implication process of the task is a crucial point for explorative participation (Heyd-Metzuyanim et al., 2019). The lecturer has prompted the explorative questions for classroom discussion to obtain explorative engagement where the actions were aligned with the goal of the lecturer and were flexibly applied in a logical structure (Heyd-Metzuyanim et al., 2019).

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Analyzing word classes and word meanings as a way to identify accessible language, illustrated by logical connectives in proofs

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Systemic functional linguistics considers language and variations between everyday, academic, and technical language. This paper describes how one aspect of systemic functional linguistics (changes in word classes) can be applied to identify compact words and word phrases and ways they may be unfolded by researchers and teachers for understanding. Changes in word classes are used to show how insights based on word classes and the mathematical structure of learning content can be used together to help researchers and teachers to identify accessible words and word phrases that might support teachers and improve students' understanding. The approach is illustrated by logical connectives, which express logical relations in proofs. In this case, offering logical connectives such as conjunctions in mathematics classrooms might be helpful for students to become aware of all logical relations and enable the expression of logical relations before prepositions are used.

Keywords: Logical connectives, proofs, systemic functional linguistics, variation of word classes.

Introduction

Academic and mathematical language have been described as challenging for many students due to their particular linguistic features, which allow speakers to express highly compacted ideas (Pimm, 1987). To familiarise students with these linguistic features in mathematics classrooms, features need to be identified and their impact on mathematical ideas must be examined by researchers.

In this paper, one linguistic feature is examined: a change in word class for words with the same potential meaning, which was described by the systemic functional linguist Halliday (2004). By qualitatively analysing students' articulations of logical relations in proof texts, I disentangle how different word classes are used for different degrees of linguistic compaction. The qualitative analysis illustrates how one part of systemic functional linguistics can be applied to analyse the language of mathematical learning content. Finally, the consequences of enhancing students' learning are discussed.

Theoretical background

Language registers in school

Halliday (1996) describes different language registers as “functional variation in language” (Halliday, 1996, p. 323). More precisely, a language register is the situation-related choice of language (words, structures) in verbal and written language production within different contexts to express meanings (Halliday, 1996). Compared to other language registers (everyday and technical), the school academic register is an especially important learning medium (Lambert & Cobb, 2003). Language as a learning medium resonates with the epistemic role of language, which has been widely discussed (Pimm, 1987; Vygotsky, 1962). Mathematical meaning should become accessible through language (Barwell, 2014; Morgan, 2014). For instance, Prediger and Zindel (2017) use the theory of language registers as a basis for systematic language variation. Teachers can use different language registers to foster students' understanding with helpful unfolding words and word phrases. This paper focuses on

variation in word classes with almost the same meaning. A functional perspective on language is applied to identify linguistic obstacles and opportunities to foster mathematical teaching and learning.

Variation of word classes from the perspective of systemic functional linguistics

Halliday (2004) describes different language registers within systemic functional linguistics. The author addresses the narrow relation between (language) forms and (potential) meanings by describing a phenomenon in which the meaning of language remains almost the same, but the word class changes from everyday language to academic language to technical language.

Halliday (2004) describes this shift of language in words (lexis) and word classes (grammar) as expressing meaning with different word classes. In other words, a *process* that is expressed in everyday language with a *verb* becomes a *thing* in academic language with a *noun* (e.g., “to assume” becomes “assumption”). As a result of this shift, it is much more challenging to understand the meaning of the words; meaning no longer correlates with word class, as it does in everyday language.

Halliday and Matthiessen (1999) describe a change in word class for different meanings (e.g., quality, process, and relator). When this change occurs, the word classes of everyday language help one to figure out the meaning of the words (such as conjunctions express relations in everyday language) (Halliday & Matthiessen, 1999). Variation in word classes also has an impact on the word classes of words in the same sentence, causing different parts of the sentence to change their role which is named “syndrome” by Halliday & Matthiessen (1999). For instance, “*the group decided yesterday*” becomes “*yesterday’s decision by the group*” in academic language (Halliday & Matthiessen, 1999, p. 251). In this example, the process (“decided”, a verb) becomes a thing (“decision”, a noun), the participants (“the group”, an article noun) become a quality (“by the group”, a preposition article noun), and the circumstances (“yesterday”, an adverb) become an expansion of the thing (“yesterday’s”, possessive case of a noun) (Halliday & Matthiessen, 1999).

There are different degrees of compaction. For instance, “glass cracks more quickly the harder you press on it” can be compacted in the academic register “the rate of crack growth depends on the magnitude of the applied stress”, or more compacted “glass crack growth rate is associated with applied stress magnitude” (shortened example from Halliday & Matthiessen, 1999, p. 258). When more compaction occurs, the semantic relations are expressed less explicitly, and the semantic information conveyed through word class is increasingly lost (Halliday & Matthiessen, 1999). The word class no longer makes clear what meaning class (e.g., relation, process, and participant) is expressed through the words.

Halliday and Matthiessen (1999) also describe strategies to unfold shifted language so that relations and hidden aspects become accessible again. These strategies include reverting word classes (i.e., to starting points, such as nouns) and telling the almost same meaning to children of different ages. Accessible word classes, as in everyday language, should be provided to learners by adults to give opportunities for understanding and learning, as parents usually practice intuitively (Halliday, 2004; O’Halloran, 2000). On the one hand, this strategy is crucial for (mathematical) learning; on the other hand, understanding word classes and their change is a considerable challenge for most learners (e.g., Martin, 1999; Schleppegrell, 2004). For this reason, this special aspect of the differences between language registers is considered in this paper. Systemic functional linguistics is already applied in mathematics education. For instance, Morgan (2014) broadly describes how systemic functional

linguistics can be used to analyse discourses in mathematics education practices and take communicative interactions into account to a greater extent than concrete vocabulary.

The shift of word classes is mainly applied in mathematics education for nominalisation (O’Halloran, 2000). Based on the theoretical framework, this paper aims to analyze word classes in combination with an analysis of expressed mathematical content. In this way, words and word phrases can be identified, which are helpful in the learning process. Finally, this paper uses an example in which logical relations in proofs are the potential meaning expressed with logical connectives.

Logical connectives as an example of varying word classes and derived research questions

Connectives are words for relations. Gardner (1975) listed different words used for logical relations in the sciences. Dawe (1983) emphasises that understanding logical connectives is crucial for understanding proofs. Within systemic functional linguistics, different degrees of compaction have been described for the meaning of “relation” (Halliday, 2004). Relations such as “so” or “because” (conjunctions) are compressed to things (nouns, such as “cause” or “reason”) or to prepositions (connectors, such as “due to”). In German, this is equivalent to “weil” or “denn” (conjunction), “laut” or “gemäß” (preposition), and “Grund” (reason) (noun). Shifted expressions for relations are used in technical language and in mathematics, especially in the context of proofs. They are therefore a linguistic challenge that appears late in students’ school life (Halliday, 2004). In mathematical texts, logical relations between mathematical objects are rarely or often not expressed in language (O’Halloran, 2000). Most students find it challenging to understand relations in mathematical theorems, although doing so is important (e.g., Selden & Selden, 1995).

The mathematical meaning of logical relations can be described as follows. The Toulmin model (1958) is widely used in mathematics education to describe logical structures. Here, this model is applied to describe one step of the proof and the logical relation between the elements of the Toulmin model (see Figure 1 below). Logical relations, especially between premise and warrant (named “Vérification des conditions” by Duval, 1991, p. 235), are important for learners to understand the structural change from informal reasoning to proof (e.g., Duval, 1991).

The student’s task was to prove the alternate interior theorem with the corresponding angle theorem and the vertical angle theorem. In the presented proof step, the student applies the corresponding angle theorem for the proof of the alternate interior theorem. Figure 1 presents both the logical structure of one step and the mathematical content of the analysed proof step (logical relations are represented as arrows).

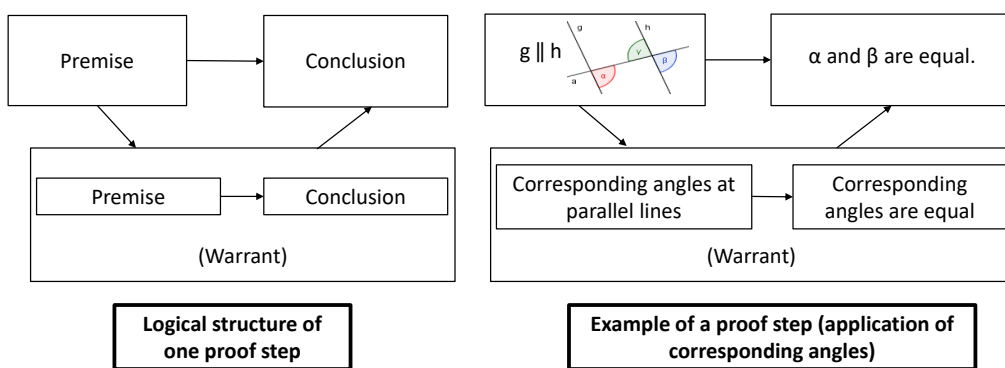


Figure 1: Logical structure of one step (abstract and mathematical content)

Based on these previous considerations and the task, the research questions are as follows:

RQ1: Which word classes express logical relations, and what linguistic impact does this have on other word classes in the proof step?

RQ2: Which mathematical aspects of the proof step are expressed with different logical connectives? (Relations between word classes and logical structures)

Methodological framework

Data collection

The analysed data stem from the project MuM Proving (see Hein, 2021; Prediger & Hein, 2017). Data were collected during design experiments, as described by Cobb et al. (2003), focusing on proofs of angle theorems. The overall data corpus consists of transcribed videos from design experiments with 24 pairs of students in grades 8–12 and their written proofs. The data corpus for this paper comprises 63 proof texts from students. This paper focuses on a more linguistic view of possible logical connectives in one proof step, the word classes of these logical connectives, and how the logical structures of one proof step are expressed with logical connectives. Similar examples of words representing logical relations are presented in Hein (2019).

Methods of qualitative data analysis

The written proof texts are qualitatively coded in four steps. First, each proof step is coded with respect to the explicitly addressed logical structures. For this purpose, the logical elements and logical relations are identified and presented graphically based on Toulmin's (1958) model of argumentation (Figure 1). Second, the logical connectives are identified in each proof step. Third, the word classes of the logical connectives and other words are identified. Fourth, following systemic functional linguistics, versions of the logical connectives are graded relative to their degree of compaction (Halliday & Matthiessen, 1999, p. 249). The compacted version (prepositions) is presented above, and the more unfolded versions are presented below (see Figure 2).

Empirical study

Analysis: Variant degrees of expressing logical relations with different logical connectives

Figure 2 lists three typical cases of how students articulate the same step (application of the corresponding angle theorem) in the proof of the alternate angle theorem. The original German text is translated and analysed based on the word classes (in **bold**), and different logical elements and relations are expressed, showing different degrees of compaction.

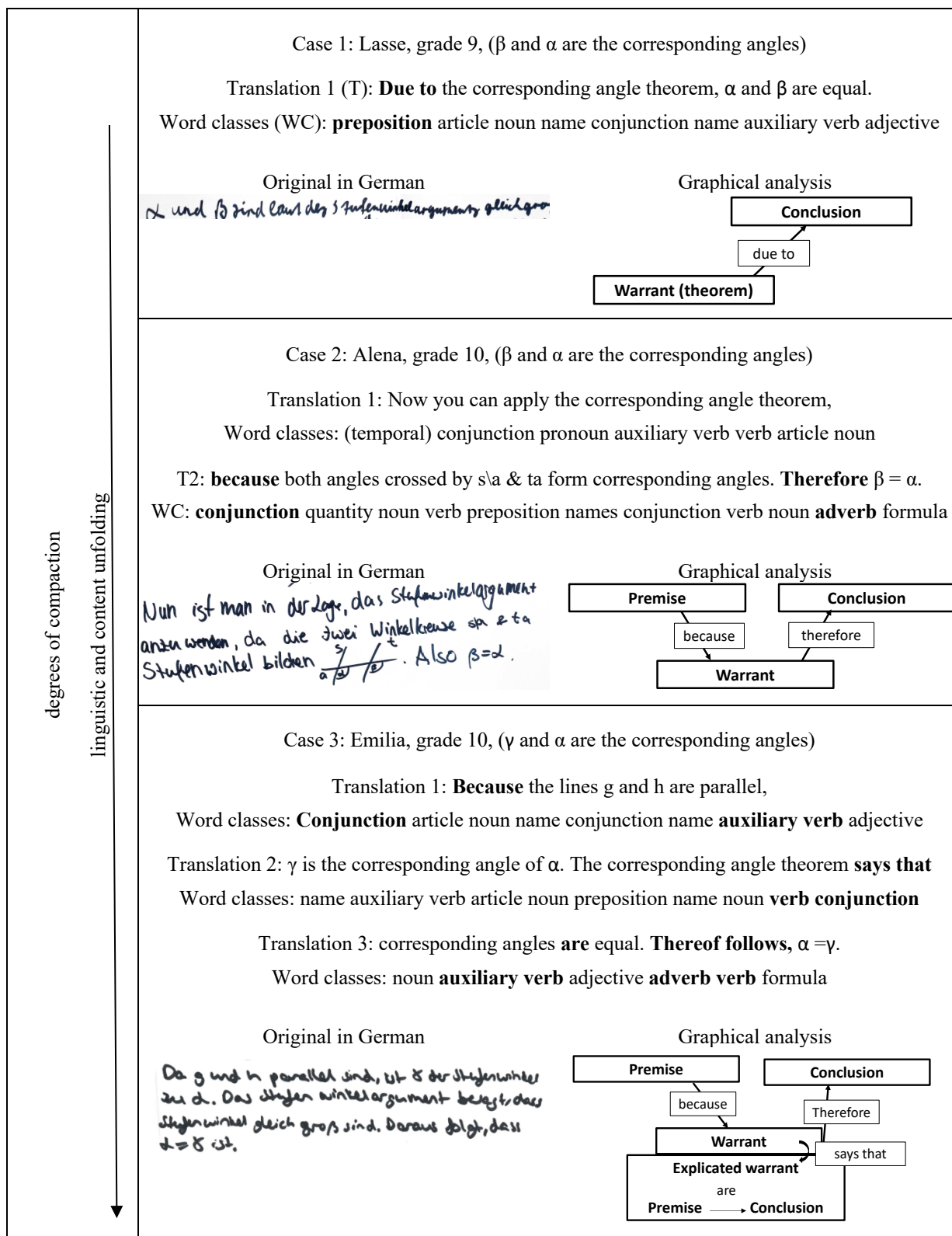


Figure 2: Variant degrees of word classes and expressed logical relations in one proof step

Considerations from systemic functional linguistics based on the analysis

Logical relations are expressed through conjunctions, adverbs, and prepositions. More logical relations between the premise and warrant are stated with conjunctions and adverbs than with prepositions. This pattern occurs, which occurs in the three cases presented above and in other texts

analysed previously (Hein, 2019), and can be explained by systemic functional linguistics (Halliday, 2004). The warrant is stated only with the name (“corresponding angle theorem”) and the preposition (“due to”). Change of the conjunction to a preposition occurs with the nominalisation of the theorem. From a purely linguistic perspective, this is consistent because prepositions belong to nouns. In line with Halliday and Matthiessen (1999), the parallel appearance of these different word class changes can be described as “syndrome”, whereby the change from the conjunction to a preposition demands that the warrant be a noun because prepositions need nouns. It is linguistically difficult to express other logical relations than the relation from the warrant to the conclusion with prepositions. However, with conjunctions (“because”) or adverbs (“thereof”), the expression of these logical structures seems possible. Furthermore, based on the data, additional words and word phrases, such as “says that”, are necessary to introduce the explication of the structure and content of the warrant. Here, this occurs as follows: “that corresponding angles at parallel lines (premise) are always equal in size (conclusion)”.

Considerations regarding the expressed logical structures based on the analysis

The analysis shows that the logical relation between warrant and conclusion is primarily explicated with compressed logical connectives (prepositions). As the preposition goes along with compression and hides other important aspects of the mathematical learning content, such as the premise and logical relation from premise to warrant, the use of prepositions hinders awareness of structures. Explication of the content of the warrant, in addition to the name of the warrant, is challenging with prepositions. This conflicts with the fact that logical relations are important to see; although they are often hidden, they are crucial for understanding formal proofs (Duval, 1991). Based on this, conjunctions or adverbs seem necessary to make the logical relations in a proof visible, and prepositions have to be avoided to ensure mathematical understanding within the first contact of students with proofs.

Conclusions

Summary

In line with Halliday (2004), the analysis shows that logical relations in students’ proof texts are expressed with different logical connectives, namely conjunctions (e.g., “because”), adverbs (e.g., “thereof”), and prepositions (e.g., “due to”). From a combined perspective integrating linguistics and mathematics education, prepositions are not only a challenging word class but also occur alongside with nominalisation of the warrant and are not suitable to express the crucial logical relation between premise and warrant. Some meta-language is necessary to unfold the warrant with its content and logical structure. This paper shows how different word classes of logical connectives relate to the theory of language registers and different degrees of compaction (Halliday & Matthiessen, 1999). The different degrees of word classes are consistent with Martin (1999), who describes the challenging change of word classes between different language registers, which must be mastered in upper grades. In combination with their meaning, word classes can be a starting point for researchers and teachers for figuring out which words are more accessible for understanding. In this way, the words of mathematics must be unfolded before learners and teachers can use the compact versions of the words.

In mathematics classrooms, conjunctions (“because”) and adverbs (“thereof”) should be provided to foster an understanding of logical structures and linguistic accessibility. In addition, meta-language, such as “the argument says that...”, should be given by teachers to unfold the warrant and make its structure visible. Prepositions such as “due to” should be used later, when the students understand the logical structures and it is no longer necessary to make the logical relations visible. Teachers can unfold the language of textbooks and their expressions during teaching. When teachers evaluate their students’ oral and written language, they have to figure out whether the students see the logical structures or if the students use compacted language without understanding what it means. More generally, teachers’ awareness of accessible word classes should be fostered so that the teachers have a tool to identify possible obstacles and words that might help the students understand the words and learn the different language registers.

The considerations here are only theoretical and based on written products; nothing can be said about the students’ learning process without further information. In this paper, the analysis of word classes is based only on one type of learning content and a small number of students. Because of the different needs related to unfolding words and word phrases for different learning content (depending on crucial aspects of mathematical structures, particular words, and word phrases), this analysis has to be repeated for other learning content. Additionally, even if this phenomenon of shifts in word classes occurs in English and German, the findings may not be generally transferable to other languages or mathematical content. The relevance of word classes has to be carefully reviewed in each case.

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The quality and quantity of student discourse across countries

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There has been extensive research advocating for teachers to support students in contributing to classroom interactions in meaningful ways. There have also been numerous small scale studies illustrating ways in which teachers can support these student contributions both through case studies of practice and through specifically designed professional development. However, the nature of interactions in classroom is also culturally shaped. In this paper, I report on an analysis of a large-scale video observation study of mathematics classrooms across eight countries or jurisdictions, focusing on the video analysis that included measures of both the quantity of student discourse and the quality of student discourse. This analysis reveals cultural differences in both the quantity and quality of interactions in everyday mathematics classrooms.

Keywords: TALIS Video Study, nature of discourse, questioning, student thinking.

Introduction

There has been extensive research advocating for teachers to support students in contributing to classroom interactions in mathematically meaningful ways (Cazden, 2001; Ingram, 2021; Morgan et al., 2014). This research has often focused on the opportunities students have to participate (e.g., Burns & Myhill, 2004; Emanuelsson & Sahlström, 2008), how teachers can support different types of participation through the questions they ask (Franke et al., 2009), as well as the nature of this participation, e.g., by focusing on student explanations (Erath et al., 2018). These studies have all built a picture of the ways in which students' participation in mathematics classroom interactions and dialogue can support their learning. However, there are other forms of interaction and communication that are more challenging to research or are less visible in classrooms (such as listening) that could also support students' learning of mathematics.

Other studies across countries have also shown that classroom interaction and communication is a cultural practice (Xu & Clarke, 2019). The Learner's Perspective Study, for example, included an analysis of classroom teaching across at least ten lessons with three 'expert' teachers from twelve countries, including China and Germany. This analysis included a focus on student participation, patterns of interaction within and across countries, as well as contrasting student participation across countries. A lot of the existing research in mathematics education focused on classroom discourse and interaction has come from western contexts, particular the USA, Australasia, and Europe and as Clarke and Xu point out, the arguments about the benefits of classroom interactions may not extend to other cultural contexts including East Asian contexts such as China and Japan.

Large scale studies of classroom interaction have often focused on the quantity of student participation, for example by counting the number of times a teacher or student speaks during a lesson (e.g., An et al., 2021; Xu & Clarke, 2019). A few studies have focused on coding the nature of what students have said (e.g., Erath et al., 2018; Howe et al., 2019; Vrikki et al., 2019) in order to make distinctions between different forms of dialogue and their influence on student learning. These studies show that in general teachers dominate classroom interactions, but also that many of the forms of interaction that are widely argued to support learning occur relatively rarely when considering

students' contributions to these interactions, in particular students rarely engage in mathematical reasoning or build on others' ideas. Consequently, arguments around how student contributions can support their learning have largely focused on detailed case studies or analyses of interactions where these contributions occur. In this paper I report on an analysis of a publicly available international video study of mathematics teaching in secondary schools which incorporated several measures relating to discourse and interaction to not only examine the quantity of students' contributions to classroom interactions but also the quality of these contributions and the similarities of these across different countries and jurisdictions.

Methods

The data is taken from the TALIS Video study, part of the Organisation for Economic Co-operation and Development (OECD), Global Teaching InSights programme (<https://www.oecd.org/education/school/global-teaching-insights.htm>). In this study, two videos of mathematics lessons focused on the teaching and learning of quadratic equations were analysed for around 85 teachers in each of the 8 countries or jurisdictions that participated. The videos were analysed using higher inference coding (components) and lower inference coding (indicators). In this paper I focus on six components from two of the Domains that were higher inference and were rated for each 16-minute segment of a lesson. Three of these components come from the domain of discourse as measured in the study (Nature of Discourse, Questioning, and Explanations) and three components from the domain of assessing and responding to students' thinking (Eliciting Student Thinking, Teacher feedback, and Aligning Instruction). Each of these components focuses on either the quantity or the quality of student discourse in the lesson segment, and is rated on a scale of 1 to 4, with 1 representing the lowest rating and 4 representing the highest possible rating.

In the final OECD report from the study, averages for each of these components within each country or jurisdiction were reported (Bell et al., 2020). These average ratings were relatively low across all the participating countries and jurisdictions. However, classroom interactions can serve a variety of purposes and we would not necessarily expect the average ratings on each of these components to be high as this would suggest there was little variety in the nature of discourse across a lesson. Instead, in the analysis below I focus on the maximum rating average across the two raters for each component achieved within a lesson. The analysis is consequently at the lesson level, rather than the teacher level or country level. All the analyses were completed using R version 3.6.2 (R Core Team, 2019).

It is important to note that the sample of teachers for the TALIS Video study varied across countries and jurisdiction meaning that in the majority of contexts the sample cannot be considered to be representative of mathematics teaching in each country or jurisdiction. On the other hand, the codes were developed in collaboration with experts from each context and the coding of the videos was completed by each country using master raters using standardised training materials, with routine calibration and validation activities across all the participating countries and jurisdictions meaning that the scoring scales meant the same thing within and across countries and jurisdictions (Bell, 2020).

Findings and Discussion

The proportion of lessons achieving the highest average ratings of 3.5 at some stage of the lesson for each country or jurisdiction are included in Table 1. These ratings were averaged across raters for each segment and then the maximum rating across all segments within a lesson was recorded. As is

evident in this table in many cases there are variations in the quality and the quantity of classroom discourse across the different cultural contexts, but there are also many similarities.

Table 1: Proportion of lessons with the highest ratings for components in the Discourse and the Assessment of and responding to Students' Thinking domains

Context	Total number of Lessons	Proportion of lessons with average rating of 3.5 or above at some point in the lesson (%)					
		Nature of discourse	Questioning	Explanations	Eliciting student thinking	Feedback loops	Aligning instruction
B-M-V						4	42
Chile	196	16	12	8	15		
Colombia	166	6	2	4	17	5	24
England	167	22	8	6	31	5	73
Germany	100	64	30	15	48	7	79
K-S-T						7	61
Japan	177	34	28	2	40		
Madrid	169	27	8	8	20	7	38
Mexico	206	14	13	5	23	6	39
Shanghai	170	3	0	9	61	1	28

In all countries and jurisdictions, the discourse in lessons was predominantly teacher-directed. Only in Germany did the majority of lessons (64%) include some interactions between teachers and students that were characterised by detailed student contributions. In Japan (34%), Madrid (27%) and England (22%) there were several lessons where these types of interactions did occur, but in Shanghai (3%) and Colombia (6%) there were very few lessons where the discourse could be characterised by detailed student contributions even though there were detailed student contributions at some point in the majority of lessons.

In all countries and jurisdictions, the amount of student thinking that was elicited was moderate or higher in the vast majority of lessons (see Figure 1). In Shanghai, this student thinking often went beyond students describing their answers or the procedures they used to include ideas and concepts in the majority of lessons (61%). So while the discourse in lessons was predominately teacher-directed in the lessons in Shanghai, the students' contributions to this discourse were often higher quality than in other countries and jurisdictions. This supports the analysis of classrooms in Seoul, Shanghai and Tokyo by Xu and Clarke (2013) that illustrate differences in the nature and function of students' participation in cultural contexts where there are relatively few opportunities for students to participate in classroom interactions. In Xu and Clarke's study the teachers in Shanghai built the

conclusions of the lesson on the students’ contributions whereas the teachers in Japan emphasised opportunities for students to share their own understandings. In Germany, Japan and England more than a quarter of the lessons included evidence of student thinking that included ideas and concepts.

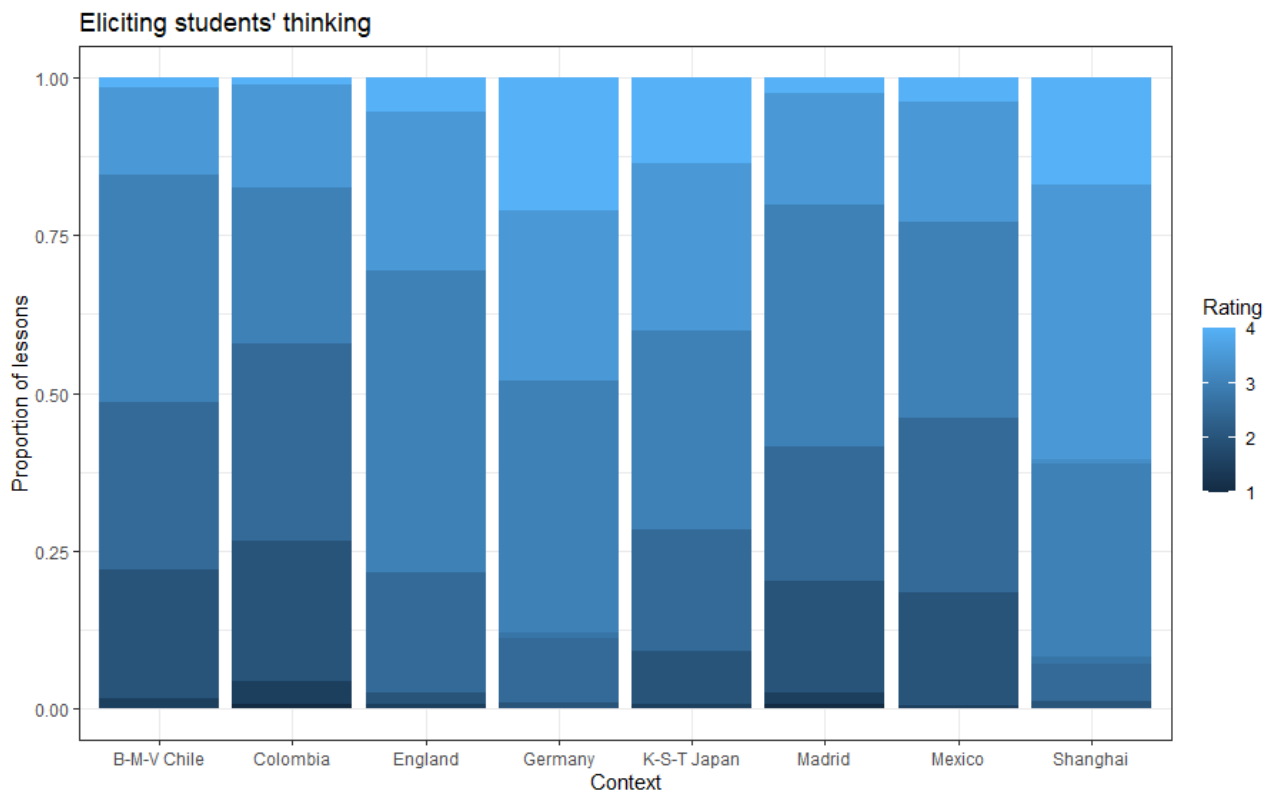


Figure 1: The proportion of lessons in each context and the maximum rating for eliciting student thinking

Looking at only those segments of the lesson that could be characterised by detailed student contributions, in Japan slightly more than half of these lesson segments (56%) also included student thinking about ideas and concepts and not just answers, procedures or the steps needed to solve a problem. In Germany and England less than half of these lesson segments (43% and 29% respectively) that could be characterised as included student thinking about ideas and concepts.

The aligning instruction component included both teachers making use of student contributions and how teachers handled students’ difficulties or errors. In England, Germany and Japan the majority of lessons included some time where the teachers were frequently using students’ contributions or providing cues or hints to support students when they made errors or were struggling (73%, 79% and 61% respectively) (see Figure 2). In all three of these contexts, the majority of the segments that could be characterised by students’ contributions (nature of discourse) also included the highest ratings for aligning instruction (87%, 62% and 67% respectively), though a much smaller proportion of those segments with the highest ratings for aligning instruction also included the highest rating for nature of discourse (24%, 39% and 29% respectively), suggesting that teachers were commonly building on students’ contributions. The large difference in England between lesson with the highest ratings for nature of discourse also included the highest ratings for aligning instruction, and segments with the highest ratings for aligning instruction that also including the highest ratings for nature of discourse, suggests more of an emphasis on supporting students who make errors or struggle mathematically in

contrast to Germany and Japan which suggests more lesson segments where teachers were using or building on students’ thinking.

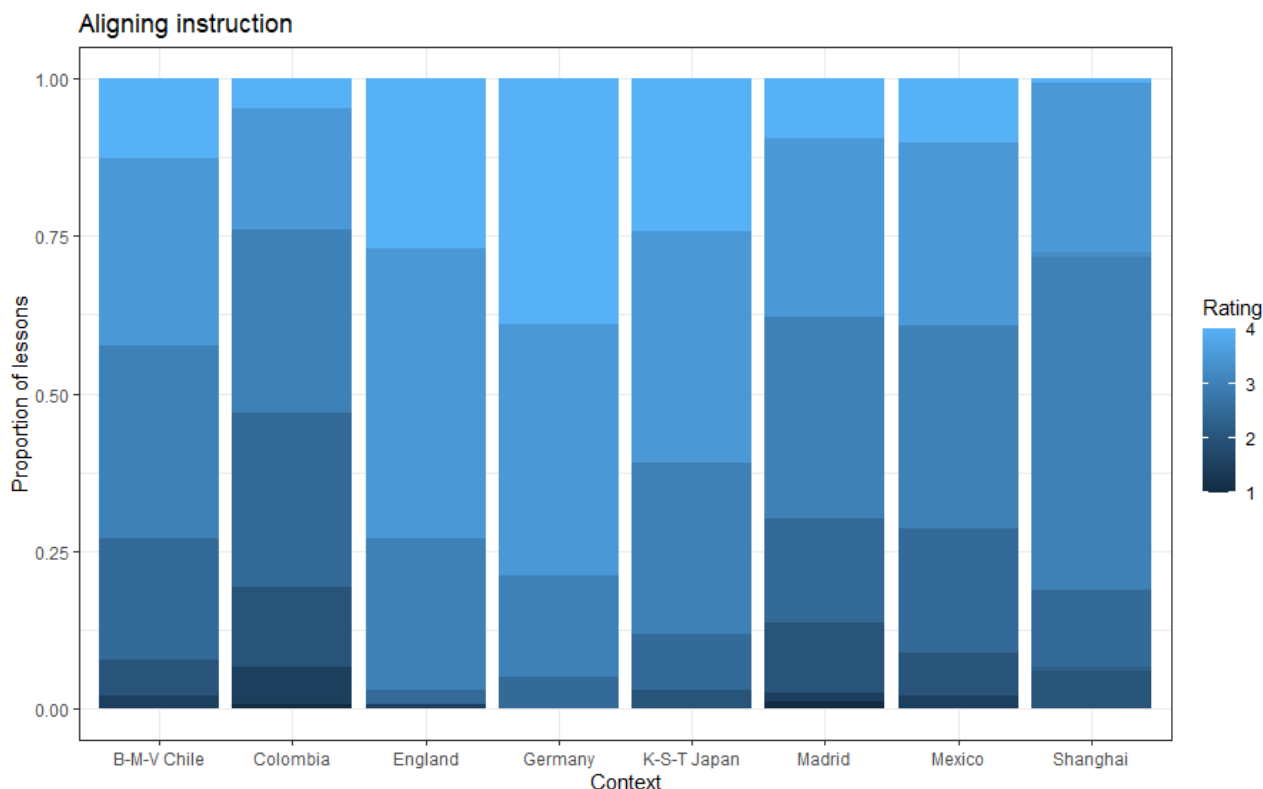


Figure 2: The proportion of lessons in each context and the maximum rating for aligning instruction to present student thinking

There were also differences in the patterns of questioning within the different countries and jurisdictions. In all countries except Colombia the majority of questions teachers asked requested students to summarise, explain, classify, or apply rules, process or formulae. In German and Japan there were also several lessons (30% and 28% respectively) where at some stage in the lesson there were periods where the emphasis of the questioning was asking students to analyse, synthesise, justify or conjecture. Less than 10% of lessons in Colombia, England, Madrid and Shanghai asked students these types of questions.

The component focused on explanations during the lessons measured both quality and quantity (in terms of length) of the explanations within the same code and it is not possible to separate these out using the data publicly available. In all countries and jurisdictions there were longer explanations or explanations that focused on the deeper features of the mathematics in focus in the majority of lessons. In Germany (15%) and Japan (26%) there were also a noticeable number of lessons that included segments where all the explanations were longer or focused on the deeper features of mathematics but generally teachers in all countries combined a mixture of brief or superficial explanations and longer or deeper explanations.

The final component, Teacher Feedback, focused on teacher-student exchanges that occurred over several turns, indicating a dialogue or interaction that went beyond the common place IRF sequence. Very few lessons in all the countries and jurisdictions included these types of interactions in ways

that dealt with the mathematics in a complete manner but in the majority of the countries these exchanges did occur at some point in the lessons. Only in Germany did over three-quarters of the lessons include these exchanges but these were a mixture of exchanges that addressed the mathematics in a limited and complete manner.

Conclusion

In this paper, I examined what we could learn about the quantity and quality of student contributions in their everyday mathematics lessons. The video analysis from the TALIS Video Study was focused on a particular topic and did not encourage or suggest particular ways of interacting in the lessons, giving us some insight into what interactions students participate in every day in their mathematics lessons. The analysis shows that in Germany and Japan there is more of an emphasis on students contributing both more often and in more complex ways, focusing on the deeper mathematics than in many of the other countries. In Shanghai we still see this focus on the deeper mathematics but the students have fewer opportunities to contribute suggesting that when students do contribute in these lessons it is often in a mathematically meaningful ways. In contrast, students in the England lessons have more opportunities to contribute and participate, but these contributions less frequently focus on the deeper aspects of the mathematics. However, the teachers participating in Germany, Japan and Shanghai were selected from a narrower range of schools than the teachers in the other countries so these differences may be indicative of what is considered ‘higher quality’ teaching in each of these countries, whereas the sample in England is more likely to be representative of the range of teaching within the country.

Cross-cultural studies of interaction can offer opportunities to consider possible alternative forms and functions of student participation, as well as opportunities to understand more deeply the similarities across these different cultures. However, we must also be careful about not only valuing what can be measured in these large studies. Classroom interactions are complex and ephemeral making them challenging to characterise in this way. The wider findings of the TALIS Video study found no clear associations between discourse or assessments of and responses to students’ thinking and any of the outcome measures of the study, including student attainment. Japan and Shanghai recorded some of the highest scores on the study pre-test and the study post-test but the analysis here illustrate the very different patterns of classroom interactions and the nature of these interactions in each of these contexts. This suggests that there are a variety of ways in which students’ contributions to classroom interactions can support their learning over time, rather than specific types of interaction or contribution across different contexts.

Analysis at a large-scale such as that in TALIS Video Study requires the use of fairly broad codes that can be used reliably across different cultural contexts. These codes can only tell us so much about the quality and quantity of student participation in mathematics lessons. This type of analysis can give us insight into the prevalence of student contributions and specific types of contribution, but they tell us little about how these contributions support their learning. Further research is needed to better understand the nuance and variation within the patterns identified in such large-scale international studies of classrooms, particularly across culturally distinctive contexts. This is particularly important within mathematics classrooms where the nature of explanation and argument contrasts with those that students experience in other classroom contexts.

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Evaluating Minecraft as a mathematical language resource

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In discussions about supporting students with language diverse backgrounds, the use of different resources has been highlighted as important. Nevertheless, very little attention has focused on digital resources, even though these are often touted as valuable for all mathematics learners. By analysing interactions between students in grade 7 about measurement problems, it is possible to identify the kinds of support that Minecraft made available for students' mathematical argumentation. The results show that experiences of using digital games at home and learning mathematics at school affected the possibilities that Minecraft made available, including what languages were supported.

Keywords: Minecraft, multilingual classrooms, multiplicative thinking, mathematical argumentation.

Introduction

In discussions about mathematics in multilingual classrooms, the need to utilise students' strengths and interests has been recognised as important (see for example Domínguez, 2011) for providing "high-quality learning opportunities" (de Araujo & Smith, 2022, p. 67). This has led to a focus on teaching resources that utilise students' strengths and interests. Although digital resources are promoted as a way of transforming mathematics education (Trigueros et al., 2014), in multilingual classrooms they are usually considered only in limited ways, such as providing translations (Meaney & Rangnes, 2022) or for individual instruction in the home language (Le Pichon et al. 2021). In this paper, we investigate if and how Minecraft facilitated Grade 7 students' mathematical argumentation in a multilingual classroom, when completing volume and capacity tasks. Minecraft is a digital sandbox game in which participants build their own worlds with virtual 1 m³ blocks and is played by millions, across the world (Hewett et al., 2020). The popularity of Minecraft, with its possibility to change languages, makes its use in multilingual mathematics classrooms potentially valuable. However, the use of such a digital game, not designed specifically for mathematics teaching, may not be unproblematic (see Nebel et al., 2016).

Although how Minecraft can facilitate mathematics learning has been investigated to some degree (Jarvoll, 2018), research in multilingual classrooms is more ad hoc. For example, in a small study about one teacher's views about using a Minecraft task about designing a house to fit certain dimensions, in her multilingual classroom, she described how she:

Encouraged the use of Arabic by a recently arrived refugee student with another student to support their understandings about perimeter, area and volume. She also provided a word list with the prepositions in Swedish and asked the students to provide Arabic translations. However, by still requiring them to present their finding to the rest of the class, she supported them to use Swedish (the language of instruction). (Meaney & Pajic, 2018, p. 185)

Minecraft was chosen because many students expressed interest in it and those who were not familiar with it, such as the refugee student, were also motivated to use it, in this way connecting to their interests and strengths. Using Minecraft provided the possibility for using and interpreting different

representations, such as the visualisations of volume and perimeter. Similarly, in a Norwegian study, Jarvoll (2018) found that students switched the language in Minecraft to English so they could learn more English while completing mathematical tasks. This suggests the tasks using Minecraft could be mathematically-rich and also language-rich.

However, concerns have been noted with the use of Minecraft in educational settings. Nebel et al. (2016) identified that experienced players could dominate classroom tasks or become frustrated when the educational version did not provide the tools they used at home. They also used shortcuts to produce results, that bypassed the teacher's learning intention. Experienced players could feel that the core principle of the game, connected to identifying and solving problems of interest to themselves, was violated when they were required to complete someone else's task. Jarvoll (2018) found that some students were more interested in finishing mathematics tasks as quickly as possible so they could play the game. Callaghan (2016) also highlighted that some students rejected the idea that learning could occur through playing video games, thus reducing their willingness to use Minecraft in school settings. Similarly, in Jarvoll's (2018) study, some students did not recognise they were learning, but rather thought they were just building.

As a consequence, even though Minecraft seems to have the potential to provide mathematical and language learning opportunities, these possibilities may not be realised. In this study, we investigate the use of Minecraft by two groups of students in a multilingual mathematics class, to determine what affected the realisation of its potential as a resource for learning.

Methodology

The data was collected in a 7th grade Norwegian classroom. As part of a compulsory assignment to have students engage in mathematical conversations and use digital tools, a group of first year preservice teachers implemented a series of tasks, around Minecraft. Two groups of students, one with two students and the other three students, were video recorded over two sessions. A video camera focussed on the students and an app recorded what happened on the screen. The sessions began with worksheet tasks, that used screenshots from Minecraft. Later tasks had the students engage in a virtual world. The second author was in the classroom as the students completed the tasks. The version of Minecraft the students used was in English, while the tasks and introduction were in Norwegian.

We analysed the interactions between the students, with an adjusted version of de Araujo and Smith's (2022) guiding questions. De Araujo and Smith (2022) developed the questions from research about English language learners' needs. They then used the questions to examine algebra learning materials. We considered these questions to be relevant for students in Norwegian multilingual mathematics classrooms. The guiding questions were:

1. Is there an explicit focus on vocabulary? If so, how is vocabulary treated?
2. Are students encouraged to use their full linguistic repertoire?
3. Are there connections to students' out of school experiences?
4. What is the cognitive demand of the task?
5. Are there opportunities for the student to explain their thinking?
6. Does the task include multiple representations? (p. 70)

De Araujo and Smith (2022) examined algebra text-based learning materials, while our study was on the contribution of Minecraft to interactions between students to do with mathematical argumentation.

Nevertheless, in examining the usefulness of Minecraft, we were also interested in how language was promoted by using this digital game as well as the connections to students' interests and strengths to develop their mathematical argumentation.

As de Araujo and Smith (2022) had developed their questions from previous research in classrooms, we considered that their questions were still useful for our analysis. Nevertheless, some adjustments were made. Although neither the tasks nor Minecraft presented vocabulary to be learnt, in the interactions there were times when the students showed uncertainty about specific Norwegian words to do with measuring three-dimensional shapes. We, therefore, classified these parts of the interactions as being about vocabulary. The students predominantly used Norwegian, but integrated some English words and expressions into their talk, suggesting they were supported to use at least one of their other languages. At times, the students also used gestures to describe what they saw on the screen, as part of their linguistic repertoire for meaning making. Some students used their home experiences of playing Minecraft to solve problems, alongside their in-school experiences of doing mathematics tasks. We determined the cognitive demands of the tasks by considering whether the students were prompted to engage in mathematical argumentation about procedures or about concepts. We also identified what prompted the students to explain their thinking. Multiple representations were connected to how Minecraft produced visualisations of three-dimensional shapes and to students' presentations of their results in symbolic form.

As each interaction could often be related to more than one question, we chose to describe the groups' interactions with two tasks and the ways in which Minecraft affected the students' mathematical argumentation. The two sets of interactions were chosen because they showed differences between how the different groups used Minecraft to solve the tasks. All the exchanges were in Norwegian, with English utterances in italics. The interactions have been translated into English for this paper.

Example 1: The swimming pool problems

Minecraft provided opportunities to engage in solving problems that were unusual, compared with typical textbook problems. In the worksheet task for day 1, the students had to determine the capacity of three swimming pools from screenshots from Minecraft (see Figure 1).

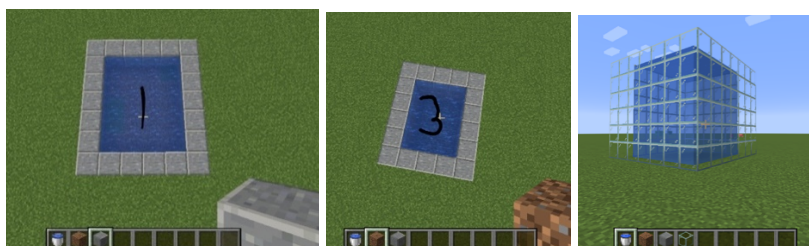


Figure 1: The students had to find the capacity of the pools where the numbers indicate the depth

In the first group, the confusion over what to measure resulted in an explicit discussion, in which they involved the teacher.

- Student1: Find out how many litres there are in the pool. This is 1 block deep.
 Student2: Is that the height then?
 Student1: Yes. 1, 2, 3, 4, 5. Five blocks long. 1, 2, 3, 4 blocks. Twenty. Then it's 1000. One block was 1000, twenty blocks. It will be 20 000?
 Student2: Yes.
 Student1: 20 000 litres. This is 1, 2, 3, 4. 1, 2, 3. Twelve times three.

- Student2: Thirty-six
Student1: It will be 36 000 litres. 1, 2... wait, I can take the width first. Teacher? Can I ask you something? [Turns to the teacher]
Student2: I think it's five. [Looking at the figure on the worksheet]
Student1: Yes. [Turns to teacher] Because, I was wondering, since inside here, it's in a way not corners, so they're not really going to be included, are they? ...
Teacher: No. (You are) completely right.
Student1: I only counted it inside, not the corners.
Teacher: Yes. It will not be quite the same as with volume then. [making the outline of a rectangular prism with his hands.]
Student1: 1, 2, 3, 4, 5. It is five times 1, 2, 3, 4, 5, since we are not counting the corners. So, it becomes five times five is twenty-five.
Student 2: Times
Student 1: Times 1, 2, 3, 4, 5, 6. Twenty-five times six. If you take twenty times two. No, twenty times six is 120 [Student2 says "mm" for yes]. So, 5 times 6 is 30. So, 150.

In contrast, Group 2 which had a similar issue with determining what to measure seemed to come to an implicit agreement that their first calculations needed to include the border.

- Student3: Problem 1. Six times one. Six times seven. No, that's wrong.
Student4: Yes, it is six times seven.
Student4: You have to find out what's inside. Six times seven, what is it?
Student3: Here's the first one. Here, it is six times seven cubic meters in three dimensions.

Vocabulary

The Minecraft screenshots led some of the students to check their understanding of terms to do with measuring three-dimensional objects. In the interaction, Student2 asked "is that the height then?" after Student1 referred to one block deep. In Norway, textbook problems about three-dimensional shapes usually refer to height. However, in out-of-school experiences, such as with swimming pools, depth is more appropriate. The Minecraft screenshots provided insights into the relationship between the terms. Later, the term "corner" when discussing what should be included in the calculation was not clarified, either by the teacher or the student, although they seemed to understand each other. In research reviewed by de Araujo and Smith (2022), attention to language in multilingual classrooms was often reduced to vocabulary, isolated from the mathematics tasks. The screenshots did provide opportunities to discuss measurement vocabulary in context, but these were not always taken up.

Use their full linguistic repertoire

In this task, the students used Norwegian, supplemented by pointing to the Minecraft screenshots, particularly when counting the blocks in the perimeter around the swimming pools, and with gestures. The teacher responded to the Student1's question by using a gesture to show a rectangular prism when talking about volume. Therefore, Minecraft could provide students with opportunities to use multiple aspects of their linguistic repertoire, but expectations about Norwegian being the language to discuss mathematics may have reduced the students' willingness to use other languages.

Out-of-school experiences

Minecraft, being a digital game played at home, connected the students to an out-of-school context. Minecraft's virtual world also provided contexts for the tasks, such as swimming pool, which students may have had familiarity with from outside school. In Group 1's interaction, it was clear that Student1 had played with Minecraft at home and this may have motivated him to solve the tasks. Student2,

who had not played Minecraft at home, took a more passive role. Nebel et al. (2016) had noted concerns about how those who were familiar with Minecraft dominated the solving of tasks.

Cognitive demand

The screenshots of the swimming pools (Figure 1), both below and above ground, seemed to challenge the students in Group 1 to do more than just calculate volume and/or capacity. For example, they made sense of how the different measurements contributed to determining the capacity of the pools. However, the students in Group 2 did not reflect on what they were doing. They calculated the volume of the pool in cubic metres, based on counting the individual blocks and then multiplying the amounts. Minecraft seemed to have the potential to challenge the students' thinking, but expectations about finding the right answer quickly did not facilitate the students in Group 2 to go beyond this.

Explain their thinking

Mathematical argumentation includes students explaining their thinking, describing not just what was done but also justifying why it was done (de Araujo & Smith, 2022). The tasks resulted in both groups describing, in more or less detail, what they were doing, with Student1 also providing input about why they were doing it, in relationship to not counting the “corners” and with his implicit use of the distributive law to work out the volume for the final pool example. This may have been because explaining their thinking was part of the classroom norms, although some students, especially in group 2, took this to mean just explaining the “what”, rather than also the “why”.

Multiple representations

The tasks required the student to interpret Minecraft screenshots which presented diagrams of three-dimensional shapes in two dimensions and to write their computations and answers in symbolic form. Although as described earlier, the representations of the virtual world were not easily interpreted, Minecraft did provide ways for students to use different representations to illustrate their ideas.

Example 2

On day 2, one task required the students to enter the virtual world and use three blocks of TNT to blow a hole in the ground. They had to do this three times and calculate the volume/capacity to determine whether the TNT produced the same size hole each time. Although the task could have been solved by the students identifying rectangular prisms in the holes and calculating their volume, Group 1 worked out the volume/capacity by filling the hole and counting each block (see Figure 2).

- Student2: Can we not then dig down and check?
 Student1: Yes. *We have to find TNT-T-T-T* {said in English}. Then we find a
 Student2: I think you need to dig it down too.
 Student1: I just have to find, here. [Chose a flint and steel from the menu]
 Student 2: Are you going to set it alight?
 Student 1. Yes [sets TNT alight] and one here [sets TNT alight on a new spot.]
 Student2: That is bigger than the other, I think.
 Student1: Shall we fill it in? [Counted individual blocks to 115 as he filled the hole]. Then it is the next hole. [Counted to 14]. I can do it like this then [Double-clicked so he filled in two blocks at a time and counted to 93]. Then it is underground.

The students described the process as boring to each other and to the teacher. No alternatives were suggested to counting individual or pairs of blocks.

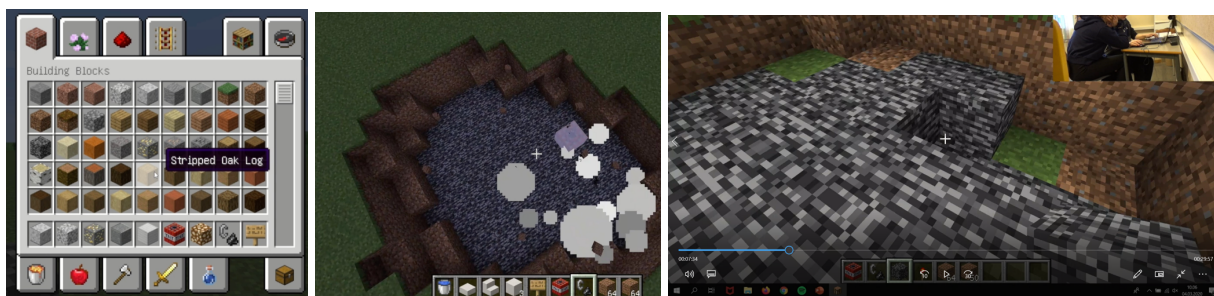


Figure 2: Group 1 using TNT to make holes

In contrast, in Group 2, one student squared off the sides with individual blocks and then calculated the volume of the remaining, empty rectangular prism, by connecting this task to the swimming pool task from the previous session.

- Student3: Okay. How much have we blown up now? I know an easy method. If we take [counted to 16]
 Student4: Should it be the swimming pool?
 Student3: The swimming pool yes. [Counted to 33]. Then we can only calculate 1, 2, 3, 4, four times five is twenty, times three, which is sixty. Then we write ninety-three.

For this example, we only discuss the questions of de Araujo and Smith (2022) which provide other insights about the usefulness of Minecraft than was shown in the previous example. As there was no confusion about vocabulary, we do not discuss that question here.

Use their full linguistic repertoire

In these interactions, the students used similar pointing gestures to those in the previous example. However, Student1 also interspersed English when looking at the menu to find TNT. The menu being in English might have supported the switch in language. In the transcripts, English words or phrases, including swear words, were used. The connection to English as the main language for online playing may have supported these students like the ones in Jarvoll's (2018) study to view English as a viable alternative language. However, this switch to English did not extend to discussing mathematics, which was done exclusively in Norwegian.

Out-of-school experiences

Blowing things up in the virtual world was similar to activities the students might engage with Minecraft at home. Although the holes were perhaps larger than they might usually make, because of the requirement to use three lots of TNT, the filling in the hole block by block may have been a typical game strategy. However, even when they identified this way as boring, the students did not consider alternative strategies, which could have used their mathematics knowledge and been more efficient. The out-of-school experiences of playing Minecraft may have inhibited them making connections to multiplication for Group1.

Cognitive demand

The task was not a typical textbook task because the students were the ones creating the three-dimensional shape and seeing if the three lots of TNT produced the same size hole each time. The Minecraft game aspect of blowing things up engaged the students' interest. However, Group1's block-by-block approach to finding an answer was not cognitively demanding and, therefore, not engaging. Unlike Group 1, Student3 in Group 2 did identify a way to simplify the task by identifying

a rectangular prism in the centre of the hole. Group 2 by discussing the task as being about a swimming pool may have contributed to the students making connections to the worksheet tasks and so supported them to find a more efficient method of determining the volume. Understanding the usefulness of identifying rectangular prisms raised the level of cognitive demand because the students had to do more than just use a pre-determined calculation to solve the problem.

Explain their thinking

As Student1 merely counted blocks to determine the volume of the hole, there was no need to describe what they did or why. On the other hand, Student3 in Group 2 described how he worked out the solution, using both words but also actions of filling in parts of the hole to square of the sides and provide a rectangular prism, possibly because he felt the need to convince his group of the value of his strategy. In Domínguez' (2011) terms, Minecraft encouraged this student “to do things with words” (p. 310). Thus, like the previous example, Minecraft provided opportunities for students to explain their thinking, but the Minecraft task alone did not produce this result.

Multiple representations

The dynamic nature of Minecraft's virtual world meant that the students had to reinterpret what they were seeing and make sense of it, including how to use the menu. For those, who had previous experience playing Minecraft, navigating around the game and its virtual world provided them with more options to see things in the game.

Conclusion

Digital tools have rarely been considered as learning resources in multilingual classrooms. In this study, we investigated if and how Minecraft could act as a learning resource, using de Araujo and Smith's (2022) guiding questions for analysing learning resources. Analysis of interactions of both groups' solving two sets of problems showed that Minecraft had the possibility to support students to engage in mathematics tasks. Minecraft appear to motivate the students, as it had in Meaney and Pajic's (2018) study, especially those who played it at home. However, as had been noted by others (Nebel et al., 2016), home experiences allowed some students, such as Student1 to dominate the problem solving. It may also be that online playing led to the use of other languages, than Norwegian, being restricted to English and to non-mathematical aspects. Other ways of integrating Minecraft into the classroom would be needed to support students to use their other languages, especially for explaining their thinking, if this was not a norm in the classroom. Therefore, digital tools, such as Minecraft do have potential as a learning resource, which could be more useful than translation tools (Meaney & Rangnes, 2022), but for this potential to be realised, other classroom actions are required.

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Structured partner work for multilingual learners in mathematics

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Equitable mathematics learning environments include meaningful access to concepts and practices and attention to positive mathematics identities, particularly through opportunities for students to share and discuss their mathematical ideas. To enact these elements with multilingual learners (i.e., those identified as “English learners” in the U.S. as they develop fluency in English, the language of instruction in the U.S.), this study investigated the role of structured partner work and the learning opportunities that it affords multilingual learners. As part of a larger randomized-control trial with 24 northeastern U.S. grade 6 mathematics classes, we analyzed student written reflections for themes related to the role of partner work in learning opportunities. Understanding the structure of partner work and related themes from student reflections has implications for instruction and for future research about strategies that support mathematical access and identity for multilingual learners.

Keywords: Classroom discourse, middle grades math, English learners, instructional strategies.

Principles for equitable learning environments for multilingual learners.

Two key elements of equitable and robust learning environments that are designed to support the strengths and needs of each and every learner, are: 1) meaningful access to concepts and practices for all students, and 2) opportunities for constructing positive mathematical identities by presenting, discussing, and refining ideas (e.g., Schoenfeld, 2018). This paper explores how structured partner interactions in the mathematics classroom facilitate participation in line with both of these elements for multilingual learners (MLs) in U.S. classrooms (i.e., students identified as English learners who are developing fluency in English as the language of instruction).

The first element, focusing on equitable access to content, means involving learners in meaningful ways, inviting their active engagement in content and practices, and including structures and routines to do so. Giving MLs meaningful access to content and practices involves the use of strategies that make tasks and assignments possible for all students to complete. In addition to clear language and explanations, access involves inviting all students to actively participate, providing scaffolds, and specifically engaging MLs in rich discourse practices (e.g., Erath et al., 2021). Both types of access for MLs are critical so that their language differences are not positioned as deficits and so that all students can participate actively in meaning-making and problem-solving. The second element, focusing on student identity and agency, involves providing students opportunities to contribute to conversations, build on others' ideas, and have other students build on their ideas. Students must see themselves as doers of mathematics in order to encourage their engagement and problem solving. It is important for teachers to position students as offering important contributions and to elicit and value their ideas, and instructional strategies are necessary to establish a community where all students are positioned as knowing and doing mathematics (Chval et al., 2021). Teachers facilitate student' agency and identity development when students are engaged in mathematics and when their mathematical ideas are central, for example, when they explain their reasoning and are the drivers of the conversation, responding to and building on others' ideas. Attention to student agency is critical

for MLs, who too often experience instruction that over-emphasizes lower-level content (Varley Gutiérrez et al., 2011) and lacks opportunities for rigorous mathematics (e.g., Torff & Murphy, 2020).

Facilitating access to content and student agency for MLs during partner work.

Partner participation structures are one way that teachers can actively support meaningful mathematical participation for MLs. Two strategies that may support equitable access to content and student agency and identity within partner work are student use of and analysis of diagrams in solving problems and instructional strategies that support language and communication.

For MLs, diagrams can facilitate understanding and communication of mathematics (Driscoll et al., 2012), reinforce learning of concepts, processes, language, and mathematical communication norms (Moschkovich, 2002), and support mathematics achievement (de Araujo et al., 2018). Representing mathematical relationships with diagrams engages MLs in mathematics while addressing linguistic challenges and facilitates their communication of ideas that they may be less able to share through other representation systems (e.g., Erath et al., 2021; Stylianou, 2010). MLs' diagrams can serve as artifacts for class discussions and when teachers call attention to MLs' diagrams and ideas, they position MLs as competent mathematical thinkers (Turner et al., 2013). Furthermore, student analyses of examples that show a mathematical approach such as diagramming can support students to learn strategies to then apply to other problem-solving situations (Atkinson et al., 2000; Booth et al., 2013), particularly when they are combined with opportunities for students to provide mathematical explanations for those examples (Woodward et al., 2012).

In addition, to meet the strengths and needs of MLs in their mathematics classrooms, mathematics teachers must attend to students' English language learning and integrate support for language and communication given that English is the main language of instruction in the U.S. and given the language and literacy demands embedded in U.S. Common Core State Standards of Mathematics (Bunch, 2013). Language access strategies, such as scaffolded reading strategies, support MLs to learn about the context of tasks and ask questions to ensure their understanding of words about mathematics processes (e.g., justify); words about context (e.g., soil); and words that may be common across subjects, but have subject-specific meanings (e.g., base) or different meanings (e.g., property) (Driscoll et al., 2012). Language strategies can also provide MLs with opportunities to respond to questions and communicate mathematically about ideas, arguments, and conclusions, using both academic and non-academic vocabulary (Erath et al., 2021; Driscoll et al., 2016). Strategies such as sentence starters and opportunities for individual reflection can support ML language production by facilitating students' framing of ideas (Driscoll et al., 2016; Maldonado et al., 2009). All of these strategies can be embedded in different classroom activities, including partner work.

Need for research focused on MLs and strategies for mathematics class.

There is, however, limited empirical evidence supporting mathematics teachers with practices for MLs (de Araujo et al., 2018; Erath et al., 2021), and additional research is needed to better understand exactly *the ways in which* the strategies related to access and agency facilitate MLs' learning in mathematics (Celedón-Pattichis et al., 2018). Explicitly noting what MLs learn through these strategies can further support identifying changes in student learning and participation. Identifying these changes are particularly critical in middle grades mathematics because of the lack of research in this area (de Araujo et al., 2018; Turner et al., 2013) and the high-stakes nature of middle grades

mathematics. Furthermore, as we learn about how MLs are positioned in mathematical discussions—for example, how their ideas are taken up by others, how others respond to them, and how they participate—we can learn more about setting up learning experiences support MLs’ agency, and “(re)structure ELs’ participation in school mathematics in ways that over time can support the development of positive mathematical identities” (Turner et al., 2013, p. 205).

Analyzing Diagrams: Supports for English Learners (ADSEL) project and unit

The Analyzing Diagrams: Supports for English Learners (ADSEL) project investigated student learning related to strategies for access and agency in middle grades mathematics classes. The project involved two years of development of a grade 6 fraction division unit with input from teachers, mathematics coaches and leaders, and students, followed by a cluster randomized trial with 23 grade 6 teachers from northeastern U.S. to understand the impact of the lessons (e.g., Neumayer DePiper & Nikula, 2021). This paper describes one study from this project, which examined how partner work structures from the ADSEL instructional unit supported MLs’ access and agency and in what ways.

The ADSEL instructional unit is a two-week Grade 6 fraction division unit with rigorous fraction division tasks and embedded lesson design features that encourage the use of diagrams and support language access and communication for MLs. Lessons focused on students’ conceptual understanding of fraction division content and their skills with diagramming as a problem-solving tool. Fraction computation is core content in Grade 6 that is linked to learning of later mathematics (Siegler et al., 2012) and diagrams are an important tool for learning and problem solving related to fraction division because constructing and using number lines and area models can help students visualize and make sense of the relationships among quantities (Jitendra et al., 2011; Terwel et al., 2009). Diagrams can help students build on existing intuitions about partitioning and equal sharing to aid in structuring and organizing ideas when solving problems with fraction division (e.g., Murata, 2008). The lesson materials include teacher resources as research on teachers’ understandings of representations found that many teachers grapple with how to integrate diagrams into their instruction (Stylianou, 2010). The lesson materials include: detailed lesson plans for all eight lessons; teacher instructions that include a lesson overview, task description and objectives, and a review of language strategies; a teacher guide with detailed descriptions about and rationale for diagrams, fraction division content, and language strategies; and a student book with sample responses for the teacher to review.

Each lesson begins with a launch task with a reading access strategy and prompts to start creating or analyzing a diagram. Then, students engage in partner work to discuss their problem-solving and the diagrams they created or analyzed, and to revise their approaches as needed. Students had access to sentence starters, such as “This diagram shows...” as a way to begin their discussions. Each lesson had a second opportunity for diagram creation or analysis with partner interactions, followed by a closing discussion and individual reflection. Across the unit, multiple embedded instructional strategies related to diagramming and to language and mathematical communication seek to foster student access and agency. This paper focuses on one strategy in particular—structured partner work.

Research design

Within a larger study on the impact of the ADSEL lessons (e.g., Neumayer DePiper & Nikula, 2021), this sub-study of students’ perspectives on their learning was guided by the following research question: *What learning experiences did partner work provide for multilingual learners?*

Participants and setting.

Based on publicly available data, we identified school districts in a New England state with at least 15% students identified as MLs, then shared information with district leaders, principals, and teachers to recruit Grade 6 teachers for the study. We randomized our teacher sample to establish a treatment and control group; we randomly assigned pairs of teachers from the same school to the two conditions, and randomly assigned the remaining group of teachers to the two conditions, resulting in a control group with 11 teachers who used their “business-as-usual” fraction division lessons during the study, and a treatment group with 12 teachers who used the ADSEL fraction division unit. One class per teacher participated in data collection. For these analyses, the analytic sample included 78 ML students across 9 treatment teachers who wrote reflections about partner work, as described below.

Data sources and analysis.

To learn about students’ perspectives and reflections on their learning from partner work, we analyzed students’ written responses to prompts embedded in the ADSEL lessons. Student reflection prompts were included at the end of each lesson, for example, about creating diagrams or about working with a partner on problem solving. The prompts were designed as an instructional tool to support student reflections on learning and as formative assessment for teachers, but they also serve this research as the responses offer students’ insights into how elements of instructional strategies were important to their mathematics participation and learning. One hundred MLs across 11 treatment classes wrote responses to at least one prompt. For this sub-study, we analyzed responses across three prompts that focused on partner work (in lessons 1, 3, and 8 in the 8-lesson unit) (e.g., “How did working with your partner help you solve fraction division tasks?”). In total, 78 MLs across 9 treatment classrooms wrote 114 responses to these prompts. Most students wrote responses in English (i.e., the language of instruction), but responses that included other languages were translated for analysis. Examples in this paper are written how the student wrote them, including any alternative spellings, etc.

To analyze responses, we identified broad coding categories through open coding and reviewing literature on partner work and classroom discussions for MLs (e.g., Chval et al., 2021; Turner et al., 2013). We refined the coding system to establish thirteen codes to use related to partner work. We consensus coded 45% of the responses related to partner work (responses from 35 students), and a single coder coded the remaining responses. We applied more than one code when a student wrote about more than one idea. After coding, we reviewed the set of coded responses and identified emergent themes on learning opportunities highlighted related to partner work (Miles et al., 2014).

Findings on partner work structures as supports for MLs’ learning experiences

Analysis of the written responses highlighted themes related to ML learning during partner work. We identified a total of 8 categories related to how partner work helped students; with some students’ responses included in multiple categories, such that percentages indicated below add up to more than 100%. Some students noted that work with a partner was helpful and did not add additional detail (15%)—e.g., writing “working with my partner helped.” Others noted that working with a partner generally helped understand what to do, without specifics about how the partner work helped with that understanding (10%), for example, noting that working with a partner on fraction division tasks helps me to “learn what to do.” In addition to these two general categories of support, the remaining eight categories can be characterized through two themes of significance, as described below.

Theme 1: Opportunity to compare solutions or problem solve and discuss together.

More than one-fourth of students (26%) said working with partners offered an opportunity to compare solutions or work on a solution with a partner. Working collaboratively included responses such as, “[Partner work] helped me because they were explaining to me how to do that” and “Sometimes I get the wrong answer then when you look at partner diagram and they get the answer they talk that how I got this. It helps me.” Students described sharing ideas and comparing steps, and they did not suggest that partner work was a chance to merely copy answers. Rather, they were talking about their solutions and learning together. Similarly, some students (19%) remarked on learning a new strategy, sharing that working with a partner helped “learn a new stragieg when I don't how to solve something. My partner will help me understand.” Some students (13%) explicitly emphasized how working with a partner supported their communication about mathematics, for example, “We discussed different ways to solve the problem.” By supporting student communication, partner work likely provided learning opportunities that are not always available to MLs in whole group settings. Taken together, more than half of the students who responded to the partner prompt emphasized how partner work provided an opportunity for problem solving and building understanding together.

Theme 2: Opportunity to complete work efficiently.

Another large group of students (22%) said that working with a partner helped them “do my work” or “get it faster and better.” Students felt more efficient and productive when they had a partner to work with on tasks. Their responses suggest that partner work supported their learning by increasing both comfort and productivity. A small group of students (5%) noted how partner work helped them understand the task (e.g., working with a partner helps me to “think about the question”), and being able to get started on the task may have contributed to that efficiency. Similarly, some students (10%) also noted specifically that partners helped with finding the answer, for example noting that the partner helps to “learn and know more about the answers.” As mathematics tasks in the ADSEL unit did have explicit correct answers, solving the task was a stated but perhaps obvious goal.

Improved perceptions of partner support.

In addition to the themes noted above, we also investigated whether students reported partner work being helpful or not, regardless of how they said it was helpful. Across students who wrote reflections about partner work, 88% only had positive reflections to share. Another subset of students (8%) wrote during an early lesson that partner work was not helpful, but then during a later lesson indicated a way that partner work was helpful for building understanding or providing an opportunity to compare solutions. For example, after Lesson 1, one student wrote, “Well, my partner do nothing but he learned from mine,” but after Lesson 8, that same student noted that working with a partner helped him to “understand what I did wrong on my number line and help me get my answer with a clear explanation.” Only three students (4%) responded that their partner did “nothing” or “did not help me at all” without providing a second response saying that partner work supported their learning.

Summary and Discussion

In summary, MLs who participated in ADSEL fraction division lessons during this study and reflected on their learning in relation to partner work in response to lesson prompts identified a variety of ways that partner work supported their participation and learning. Most students who responded to the prompts indicated that partner work helped them, and they described general help, support for

collaborating and/or communicating to work on solving problems, and support for completing work efficiently and finding answers. It is valuable to understand these nuances of partner work structures from the perspectives of MLs and in particular the values of partner work in order to create language-responsive mathematics classrooms where language practices engage students in rigorous mathematics. Furthermore, these analyses suggest ways in which partner work operationalizes ideas of access and agency in mathematics classrooms, particularly when seeking to scaffold student-to-student discourse through partner conversations. The specific strengths and needs of different students who are MLs vary widely, and thus, the ways that partner work will benefit each may vary as well; the findings reported here represent the variety of ways that MLs may perceive structured partner work as supporting their work—not every student will be supported in the same ways.

Implications

The results of this study, which focused on the perspectives of students who are MLs in the U.S. on the affordances of partner work during mathematics class, have implications for instruction and for future research. Curriculum developers, teachers, and coaches and professional development providers who focus on mathematics instruction for MLs may find it instructive to consider students' perspectives on how partner work, if structured as it was here, can facilitate MLs' collaborative problem solving, communication, and learning of new approaches. The partner work that these students experienced also incorporated features to consider for instruction such as: diagrams as artifacts of mathematical thinking to support problem solving and communication; sentence starters that were specific to that diagram work to scaffold conversation; a routine for using diagrams during partner work (sharing initial ideas after individual work then working to build on or revise initial diagrams); and reading strategies to make sense of a word problem.

Future research could build on this study by investigating MLs' perspectives on other strategies and structures in the mathematics classroom. This study is based on reflections written in response to prompts that were designed for instructional purposes, not research purposes, and represent ways that partner work can be helpful for at least some MLs. Future research could interview students and teachers about strategies to get more in-depth information and from multiple perspectives, guided by the initial findings reported here, as well as use observations of teacher moves to understand other dimensions of classroom practice in language-responsive classrooms (Prediger & Neugebauer, 2021). To understand more about how to facilitate these outcomes for MLs, next steps for this ADSEL research include analyzing additional ML reflections and interviews to investigate other lesson design elements, and continuing to seek out ways to research and facilitate student access and agency.

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The mathematical language of videos produced by students

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Communication involves drawing upon various modes of meaning-making. Very often, these modes can be verbal, vocal, or visual. In the last two decades, particular attention has been given to how different modes of communication, such as the home language, algebraic notations, and diagrammatic representations, play a role in teaching and learning mathematics. This study investigates the ways in which two high school students from the South of Brazil explain their working out of a linear system in a pre-recorded video. Particular attention will be given to head nods, pointing gestures, symbolism (Algebraic notations), and home language as these two students explain their working out concerning solving linear systems.

Keywords: Gestures, videos, multimodality, communication, mathematical language.

Introduction

Mathematics is a discipline that has its own register. It consists of symbols, images and people use their own home languages (English, Portuguese, Italian, Spanish or others) to explain or introduce problems. O'Halloran (1998, 2005) opines that mathematics can be represented in three forms: home language, symbolism, and visual representations.

Often, in the classroom, there is a combination of these registers, that could be verbal or no-verbal, having what it is called as multimodality. A question arises as to how these modalities are perceived in videos? How do students use these modes to express content? Few studies investigate the modes in mathematics videos. For example, Oechsler and Borba (2020) investigated how can video production become a teaching and learning tool for students. Domingues and Borba (2021) presented how video production impacts the classroom, mathematics students and teachers in Brazil, and mathematics knowledge production developed with this media.

In this work, an analysis of a video produced by high school students in a public school in Brazil will be shown. We will examine how different modes of meaning making are used in students' discourse and how this multimodality helps to systematize mathematical explanation.

Mathematical language and cinematographic language

When we produce videos, it is common to mix common languages from the explained content with the cinematographic language. Authors such as Martin (2005) argue that cinema has its own language, which aims to communicate and inform, in addition to being an art. Martin (2005) argues that the cinematographic language combines image (which includes moving images and, secondly, graphic annotations, such as subtitles and other inscriptions) and sound (dialogue, music and noise). Aumont (2002) describes that the basic elements of the film are images, sound, and graphic inscriptions. All authors explore the issue of image and sound (which does not necessarily need to exist, as there is

silent cinema or pantomime). A point on which all authors agree is that the specific material of cinematographic language is the moving image (Aumont, 2002; Martin, 2005).

Mathematics, as pointed out by O'Halloran, makes use of the home language, symbolism and visual representation to express its content. We realized that each of the modes (verbal, writing or visual representation) has its potential to communicate a mathematical content. O'Halloran (2005, p.94) summarizes that each mode:

[...] has a particular contribution or function within mathematical discourse. Language is often used to introduce, contextualize and describe the mathematics problem. The next step is typically the visualization of the problem in graphical or diagrammatic form. Finally, the problem is solved using mathematical symbolism through a variety of approaches.

However, these potentials can be expanded, or, as Lemke (2010) would say, their meaning can be multiplied, if these modes are used together, characterizing multimodality.

In videos with mathematical content, we can unite the mathematical language (home language, symbolism, and visual representation) with the cinematographic language (moving images, sound, among others). To explain, in more detail, some steps of the count, we can use gestures with symbolism and home language (explaining in oral voice what we are doing in the resolution). By doing so, we can explain mathematics in videos, using several modes, which can contribute to the expansion and production of meaning in mathematics classes. This work seeks to analyze how the modes used in the videos can help to systematize mathematical explanation. The video shows the following modes: head nods and pointing gestures (which are an integral part of cinematographic language – moving images) and symbolism and home language (which are indeed part of the mathematical language).

Head nods:

Nodding is a culturally given and socially shaped resource for meaning-making whose interpretation varies cross-culturally. For example, while in most European and Latin American countries, "yes" is typically associated with head movement from back to front, in Bulgaria, it is denoted by a head-shake from left-to-right, while in the Abyssinian cultural group by upward head motion and raised brows (Poyatos, 1992). The head is rocked from side to side in India, called the Head Wobble to denote yes. This very act is perceived to be confusing for Westerners and Europeans, who use this gesture to communicate 'maybe yes – maybe no'. Interestingly enough, in Japan, head-nodding does not necessarily mean 'yes, I agree' – it usually means 'yes I hear you' (Pease & Pease, 2006).

Pointing gestures:

Gestures are an integral part of communication. Whenever we talk, it is very likely that we employ gestures. The term 'gesture' has many definitions; for example, McNeill (1992) defines the term gesture as the movements of the arm(s) and hands that are closely synchronized with the flow of its verbal counterparts. Kendon refers to the spontaneous hand movements produced while talking as 'gesticulation'. On a similar line, Sfard (2009, p.194) defines a gesture as a "body movement fulfilling communicational function". Teachers' gestures are perceived to be of high value in education and educational research, not only in terms of teaching, but also when assessing students' understandings (Farsani, 2015a). Furthermore, gestures students produce can convey pertinent mathematical

information (Farsani et al., 2020). Pointing is a subset of gesturing (McNeill, 1992) whereby a gesticulator can point to objects that are either present or non-present in the environment. Gesturing can be done by a hand (often an index finger), or using an extension (e.g., a ruler or laser pointing).

Symbolism (Algebraic notations):

According to O'Halloran (2005), mathematical symbolism was developed as a feature that had clearly defined functions. Among these functions we can cite the description of patterns of relationships and the reordering of these relationships to create models of the physical world, allowing problems to be solved and predictions made.

Home language

In all the videos, students explain, with words, what they are pointing or solving. For that, they use their home language: Portuguese. The use of the home language intends to explain what is written in symbolic language, making the resolution process clearer. Often, students have difficulties understanding the steps described in symbolic language, and, for that, they add an oral explanation of what they are doing to these steps.

The use of all these modes in video seeks to convey a message. According to Oechsler and Borba (2020), “through video production, they show their understanding of the content”.

Data

The video analyzed in this work was produced by two high school students from a public school in Gaspar, south of Brazil. Students were learning about linear systems and the teacher asked them to produce a video solving a system of equations with more than two unknowns. The activity was part of the class of the first author. In this study, the ethical consent forms were collected from the students.

Video production process took place in four stages, which were recorded by the teacher through audio recording. Each interaction between the groups and the teacher was recorded by audio and, together with the videos, constitute the research data.

Initially, students were introduced to various types of videos (animation, software recording, whole class video, role play, etc.). They could get to know various formats and choose what they felt most comfortable to record and explain the content.

After this presentation, the teacher explained to the students that they should make a video explaining the resolution of a linear system. At this stage, the students conducted research on which system they would explore and set up the video script.

After preparing the script, students began recording the scenes using, for this, various devices: animation software, whiteboard, chemistry laboratory (to explain the balancing of the chemical reactions through linear systems) among others.

Once the recording process was completed, students edited the scenes, and the video was shown to the whole class in one of the mathematics classes. In total, 11 videos were produced. In the next section, a video will be presented using multimodal transcripts (Farsani, 2015b) and consequently analyzed. This particular video is chosen because different modes of meaning making were used both to accentuate the mathematical problem and also guiding the viewers into the prospective steps in solving a system of equations.

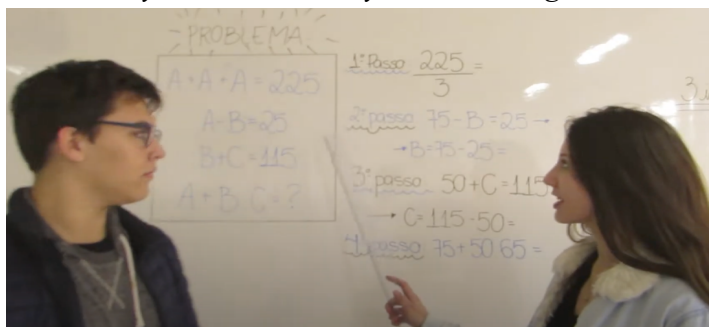
Multimodal transcription

Multimodal transcripts are used to denote various modes of meaning making in this video. It is important to note that the transcript below accounts for only 30 seconds of the entire video³. We will pay particular emphasis on how nodding, gestures and the verbal language is used to convey the mathematical meaning.

- 1 Amanda: E aí galera, meu nome é Amanda.
Hey guys, my name is Amanda.
- 2 Carlos: E aí galera, meu nome é Carlos Eduardo.
Hey guys, my name is Carlos Eduardo.
- 3 Amanda: E nós estamos aqui para ajudar o nosso amigo Gabriel que está com dificuldade para resolver um problema. E para ajudar você que está
And we are here to help our friend Gabriel, who is having trouble to solving a problem. And to help you who are
assistindo esse vídeo e está com dificuldades em sistemas lineares.
watching this video and are having difficulties with linear systems.



- 4 Amanda: Bom, o problema já está aqui escrito para vocês verem certinho.
The problem is already written here so you can see right.



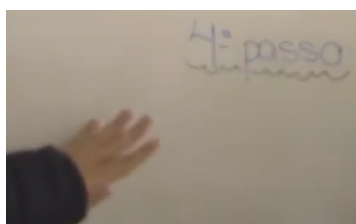
- 5 Carlos Bom, o problema de Gabriel como a Amanda mencionou, pode ser resolvido
Gabriel's problem, as Amanda mentioned, can be solved
com sistema linear. (...) *with a linear system*



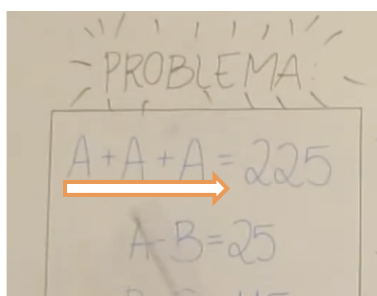
- 6 Carlos Em apenas quatro partes podemos ensinar vocês e explicar para vocês como
In just four parts we can teach you and explain to you how

³ The link for the full video can be found at: <https://www.youtube.com/watch?v=Srj2WJfY5cA>

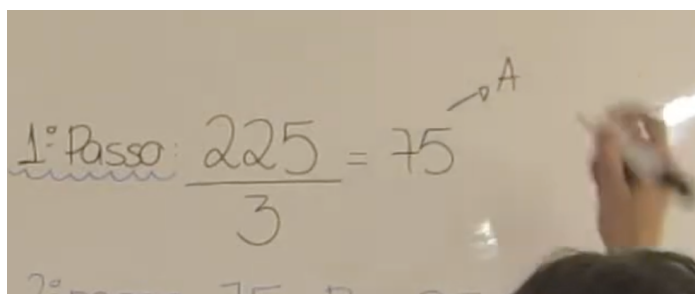
resolver essa questão.
to resolve this question.



- 7 Amanda: Então, o primeiro passo nos podemos ver que nessa soma de três incógnitas, The first step we can see that in this sum of three unknowns que é a letra A, e a soma delas é igual a 225. which is the letter A, and the sum of them is equal to 225.



- 8 Amanda Então podemos pegar 225 e dividir por três que é igual a 75. Temos o valor So we can take 225 and divide by three which is equal to 75. We have the value de A. of A.



Data analysis

At the beginning of the video (lines 1-4), students introduce themselves and explain that they will help their colleague Gabriel solve a problem. When Amanda says that student Carlos makes a head movement (up and down), emphasizing Amanda's speech. This head movement is often used when we want to say something. We usually speak the affirmative using the home language (lines 3 and 4), and, at the same time, we nod our heads in an affirmative gesture, corroborating what we say (line 4). This is a classic combination of two modes (head gesture and home language) to expand meaning.

After ensuring that they will solve the problem, students present the problem on the board (steps 4-5), combining the home language with the explanation, symbolic mathematics with the representation of the problem, and gestures when indicating their fingers steps of the problem.

In lines 6, we observe the use of the home language in Carlos' speech, combined with Amanda's head movement (up and down confirming what Carlos is saying and looking towards Carlos as he explains what they will do).

Lines 6 to 8 shows the use of the home language to explain the solution of the first step of the system combined with the gesture, which points to the equation to be solved, and the symbolic language, in which the students solve the problem, which requires finding the value 75 for unknown A. Gestures in all the videos are used to show and emphasize a step of the solving. For example, in Figure 2, Amanda points to the inscriptions on the whiteboard (it is a general pointing). Moreover, in Figure 4, four fingers are shown to emphasize the four steps they will use to solve the problem. Gestures are used together home language to explain the calculus.

Usually, when solving a linear system, students have many difficulties. The question is usually: where do I start? This was also the difficulty of the students in the video. They brought up the problem and asked for help in solving it:

- Carlos: we wanted to see how to solve by linear system
Teacher: Here: $A + A + A$ is 225. So, $3A$ is 225. Then A is $225/3$.
Carlos: Yes.
Professora: Knowing the value of A you substitute in the other sentence and find the letter B , knowing the value of B you substitute and find the one of C . This is your system.
Carlos: But isn't it bound to be that big deal? (referring to the scaling system resolution method). (Teacher's conversation with the group during video production).

Realizing that this type of problem could be solved by the substitution method, the students decided to systematize the way they solve the problem, separating them in a step by step (emphasized in speech - line 13 and in gesture - line 14) In line 20, we see the explanation of the 1st step of the calculation: the value of the unknown A , using the mathematical symbology.

During the solving process, the students systematized the way they understood the resolution of the problem, through 4 steps. And to explain each of the steps, they combined several modes. In a written test, there are few limited ways (modes) students would use to find the value of the unknowns, usually symbolic language. Nevertheless, what are the students thinking when they are doing the calculation? Although this might be a rhetorical question, the employment of different modes of communication is visible in students' explanations (home language, pointing, gestures and, head nods). Moreover, exploiting these different modes is possible because of the video, which permits the analysis of several modes of meaning-making and not just mathematical language. The students resort to other modes in the video to explain how they solve them, using gestures to point to which equation starts solving, which operation to use, and what value is found. This combination of modes seeks to facilitate the viewers' understanding of the resolution of the problem and helps explain mathematical content.

Conclusions and practical implications

Thinking about using these modes in the classroom to enhance systematizing mathematical explanation, we agree with O'Halloran (2005). She, in the pedagogical discourse, in addition to specific modes of mathematics, such as writing, symbology, and visual representation, integrate into

the other systems of meaning, such as gesture, body movement, voice intonation, among others, which seek to enhance the meaning produced during the speech in the classroom.

We noticed that students used several of these ways to communicate their understanding of the content during the video production process, using their language, as well as aspects of audiovisual language (such as perceptions of the plan, framing, design). These modes of meaning-making enabled them to build a link between what they understand and a symbolic mathematics. Furthermore, we can research this video production process to understand the ways in which students systematized mathematical contents and how *potentially* this explanation can be their sign of learning (Oechsler & Borba, 2020); because they explain what they understood, which in return enable us to observe how they understand mathematics. Furthermore, we strongly believe that regardless of a students' grades and experiences and exposure in mathematics, it is always worth questioning the forms, styles and quality of the messages conveyed verbally and nonverbally in their explanations. Raising awareness to these very subtle and silent nonverbal messages can have a direct positive impact by better understanding students' intent as conveying their mathematical understanding (Farsani et al., 2020; Rosa & Farsani, 2021).

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Second Language Approaches and the Teaching of Mathematics

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The link between language and developing conceptual knowledge and understanding in mathematics is an integral part of mathematics teaching and learning. This paper describes a search for more effective ways to teach mathematics, on the basis that the description of mathematics is consistent with Halliday's Systemic Functional Linguistic Theory of a language. Mathematics can be regarded as a language and as it cannot be a first language for anyone, teaching mathematics using Second Language Teaching Approaches may present opportunities to improve student learning outcomes in mathematics. This paper presents a representational summary of the findings of a systematic literature review of second language teaching approaches, summarised in a concept map. The concept map highlights three approaches that are deemed the most appropriate on which to base a set of teaching strategies that aim to improve student learning outcomes in mathematics.

Keywords: Second language approaches, lexical approach, communication approach, usage-based approach, teaching strategies.

Theoretical framework

The mathematics language used in its teaching and learning has two purposes, namely communicative and epistemic, which are intertwined (Pimm, 1987) and co-dependent. The inter-related complexity of mathematical language, the use of natural (e.g. English, French) language to teach mathematics, the relationship between students' language proficiency and the development of conceptual knowledge and understanding in mathematics have therefore been a subject of concern for researchers for several decades. Recent poor performance of students in mathematics has been researched and explained from the point of view of students' lack of proficiency in the language of instruction (Adler, 2001; Moschkovich, 2010; Schleppegrell, 2007). The role of language, classroom talk, in the development of mathematical concepts has been examined, linking language use to cognition, the development of mathematical concepts, and the ability to communicate mathematically (Pimm, 1987). Students' difficulties with mathematical texts, for example word problems, serve as reminders of the links between the language of mathematics and the language of instruction.

A consensus amongst researchers is that language is pivotal to the development of conceptual understanding in mathematics (Erath et al., 2021). Extensive research addressing language in mathematics and ways to better support the teaching of mathematics resulted in the development of the 'language-responsive mathematics teaching' framework, known as 'five points planning' for mathematics teaching (Erath et al., 2021; Prediger, 2019), advocating that teachers must pre-empt the language to be used in the lesson. Language responsive teaching involves the explicit foregrounding of language, supporting students to develop language proficiency (Prediger, 2019). Contributing to the importance of language in the conceptualisation in mathematics, Planas (2018) examined language as a verbal resource, and lexicalization has the potential to provide mathematics teachers with the resources to explain and communicate specialised mathematical meaning and concepts.

Acknowledging the importance of language in mathematics and the complexities of mathematics language have led to this PhD research project. The study uses a Design-Based Research Methodology (Plomp, 2013) to examine the characteristics and outcomes of a teaching intervention in primary schools in both Australia and Denmark. The teaching of mathematics for this study is informed by Second Language Approaches [SLA] that emerged as a result of the systematic literature review that was undertaken. When combined with mathematics and mathematics teaching strategies, this approach is named Second Language Mathematics [SLM] and is the framework developed by the researcher for this study.

Mathematics as a language

Halliday's contribution to this discussion is rooted in the analysis of mathematics language registers, its structures, and the function of registers in a language. Halliday's conceptual framework is based on his theory of social semiotic, also known as Systemic Functional Linguistic (SFL). Using this theory, Halliday and Matthiessen (2004) demonstrated that language is a resource for the creation and dissemination of meaning, and that the grammar of the language serves the purpose imposed on it by the function of the language. Similarly, mathematics has its own register and structures that serve the function of the language enabling precise and concise meaning making. Mathematics, therefore, is more than new vocabulary and words that may change meaning when used for mathematical purposes. It is a new way of using words, phrases, and symbols to create new ways to argue and reason.

Using Halliday's SFL approach, O'Halloran (2005; 2010) explored the language of mathematics concluding that, like other languages, mathematics uses words, symbols, and visual imagery (a multi-semiotic language) for creating, organising, and making meaning. Mathematics does not use long descriptions, instead the grammar of mathematical symbolism is rearranged and simplified enabling its use in mathematical arguments (O'Halloran, 2010). The use of spoken language together with visual images, concrete objects, gestures, and actions, known as inter-semiotics, contribute to the grammarisation of mathematics (O'Halloran, 2010). Consequently, mathematics has developed as a precise and concise language enabling accurate explanation, prediction, generalisation, proofs, and modelling of phenomena that contribute to the construction, innovation and organisation of our societies (Kharde, 2016; O'Halloran, 2005). Learning mathematics thus requires learners to engage in this formal mathematical discourse.

The complexity of the linguistic structures of mathematics, coupled with the students' lack of proficiency in the language of instruction, made it difficult for second language learners in America to form mathematical concepts (Schleppegrell, 2007). Prediger (2019) reported similar findings with German students, though she argued that both First and Second Language speakers experienced similar problems and concluded that proficiency in the language of instruction is a big contributor to this issue. Research to date has targeted the students' first language proficiency and the complexity of the mathematical language, and the close relationship between them, and their impact on students' learning of mathematics. However, researchers have yet to conduct empirical research that interrogates language and mathematics learning and teaching from the perspective of mathematics as language. Given the close relationship between mathematics and language, researchers have yet to investigate whether the teaching of mathematics could gain from using Second Language Approaches (SLA) to inform teaching strategies for mathematics.

'Language' is the principal method of human communication, consisting of words and symbols used in a structured way and conveyed by speech, writing or gestures, while grammar provides the rules and syntax organises the symbols (Helmenstine, 2019). Mathematics shares these characteristics, using words, symbolic notations, syntax, and grammar to construct and communicate meaning. However, often words pertaining to mathematics hold different meanings to those in natural language, because mathematics holds on to original meanings, while natural languages are always evolving (O'Halloran, 2005). Mathematics uses symbolic notations evolved from natural language and used as semiotic resources enabling mathematicians to communicate and encode human experiences of the world into abstract and logical mathematical texts (O'Halloran, 2005). These symbolic notations were created by mathematicians to serve the purpose of mathematics and science, which O'Halloran (2005, p. 68) described as "designed semiotic resource", a resource that is not available in natural languages. Nevertheless, neuroscientists have demonstrated that the relationship between mathematics and language is very strong and humans construct meaning for number and language in the same way (Danesi, 2016).

This study positions mathematics as a multi-semiotic language and investigates the outcomes of mathematics learning when taught using teaching strategies designed and informed by second language learning theories. This approach is premised on the view that mathematics is not a natural language (i.e nobody's first language), but as a language mathematics could benefit from the use of second language approaches in its teaching. This paper also provides insights gained from conducting a systematic literature review of SLA, highlighting relationships between the different approaches. The knowledge gained from this review has been used to inform teaching strategies to be trialled and evaluated for the teaching of mathematics as the next stage of this study.

Second language approaches

Second Language teaching is considered an interdisciplinary area incorporating linguistics, cognitive science, sociology, and anthropology (Bybee, 2010; Christiansen & Chater, 2008; Ellis, 2002; Eskildsen, 2009; Król-Markefka, 2014; Lantolf et al., 2015; Menezes, 2013). With a growing interest in Second Language learning, due to globalisation and current mass migration, coupled with new knowledge about mind and language, cognitive scientists, applied linguists and others have joined this already crowded arena in the search for effective ways to teach second languages. Researchers are pushing boundaries and language learning theories have been revisited and revised, making the task of reviewing SLA more complex.

To gain an understanding of SLA and the theories underpinning them, a systematic literature review was conducted, which captured 162 pieces of data from 2010 to 2021 from three data sources comprising empirical research and theories. These data also provided the general learning theories, and perspectives on language learning that underpin SLA. Information gained from the three data sets, together with general learning theories, enabled the creation of a concept map (Figure 1). As can be seen from the concept map the Approaches have been organised under those that prioritise cognition and others that prioritised socio-cultural aspects of language (coloured arrows). The arrows show how the Approaches are sometimes related to, or evolved from one another, such as the Lexical Approach being derived from the Communication Approach, while the texts between arrows explain shared characteristics. The emergence of new Approaches depends on many contributing factors, for example, new theories, beliefs in education, digital technologies, and so on. Digital technologies have

enabled cognitive scientists to try to explain the role on the mind of language, with a lot of this work culminating to the Usage-based Approach (Ellis, 2002). School curricula and/or teachers, however, often hold on to an approach even when a more recent version is added to their repertoire. A detailed review of this topic is beyond the scope of this paper.

Approaches to Second Language

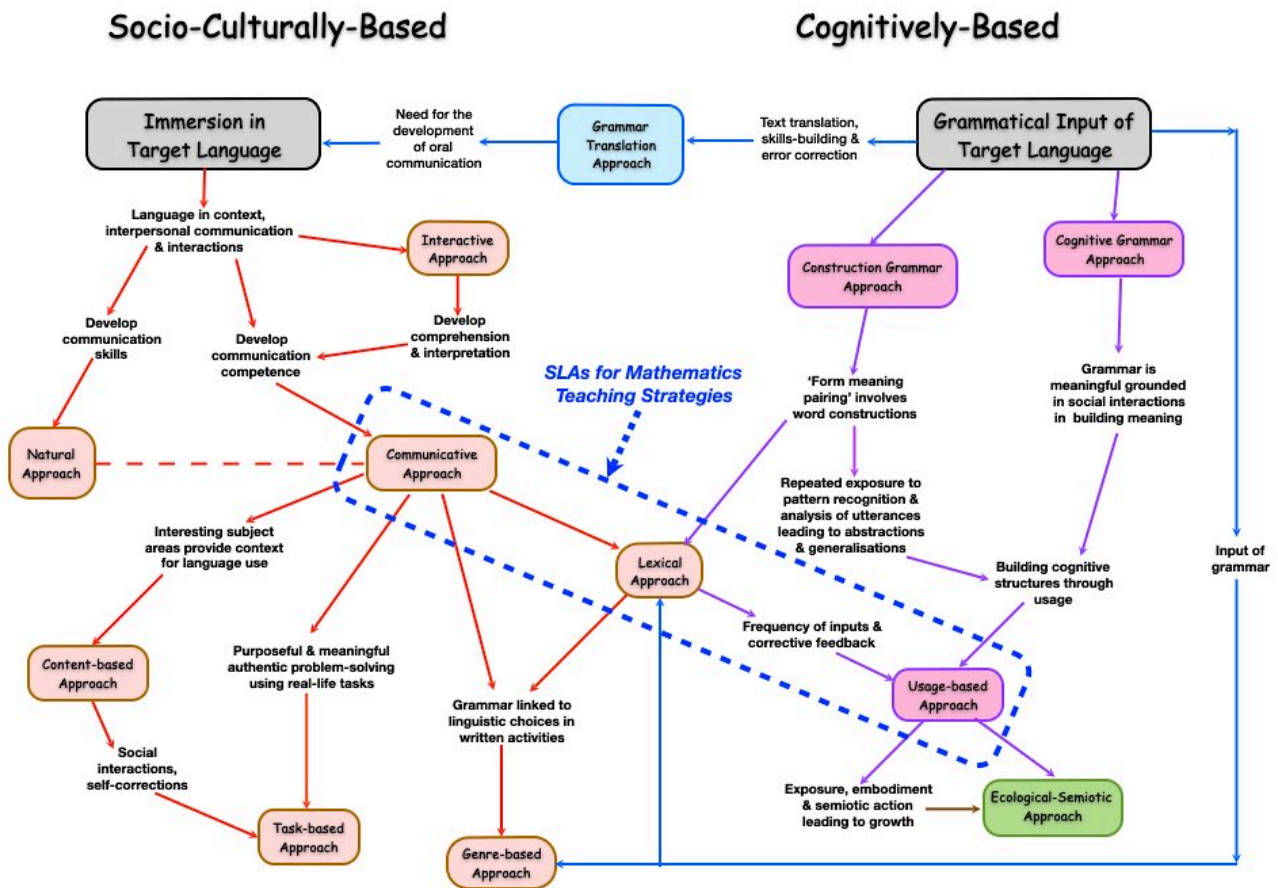


Figure 1. Concept map showing the Second Language Approaches

Second language approaches to the teaching and learning of mathematics

Knowledge gained from the review of the nature of mathematics, its teaching and learning, together with insights from the systematic review of SLA, informed the selection of the three SLA used to guide the design of teaching strategies to be applied in the research interventions for mathematics teaching in primary schools.

The *Lexical Approach* is based on the theory that language comprises lexical units, which can be a single word or word combinations known as lexical ‘chunks’. Chunks are stored in our brain as mental lexicons and later used to create new and creative utterances, the belief being that language is recycled (Lewis, 1993). Examples of ‘chunks’ are: “What is the cost of...” and “I totally forgot...”. While some chunks can stand alone, others provide the beginnings of utterances that the learner must complete. The learner uses ‘chunks’ as ‘raw data’ which, when appropriately combined using the language pattern, leads to the construction of meaningful structures (Moudraia, 2001). Teachers can

use students' known lexical chunks as springboards to teach grammatical structures and develop the students' ability to recognise patterns, structures and lexical units to create meaningful and coherent texts (Moudraia, 2001; Ramirez, 2012).

The *Communicative Approach* evolved from the Grammar Translation Approach and is underpinned by Halliday's Systemic Grammar. It is based on the understanding that learning a language implies learning to meaningfully communicate in the language and entails developing communicative, sociological, and grammatical competence, all of these competences being linked to social contexts (Alamri, 2018). Although the goal of teaching a language is to simultaneously develop the four language skills (listening, speaking, reading, and writing), to be competent in a language the learner also needs to develop social and cultural competency, which involves understanding the impact that culture, context and social settings have on language use and meaning.

The *Usage-based Approach* is rooted in cognitive linguistics (Ellis & Wulff, 2015) and combines Cognitive Grammar (Langacker, 2013) and Construction Grammar. This Approach is based on the understanding that humans have the capacity to acquire language using 'constructional templates' (Hoffman, 2017). Such templates are used to construct linguistic understanding of grammar involving 'form-meaning pairing' and requires the learner to be actively engaged in the construction of the language. This approach includes acquisition of word combinations (word phrases), memorised through continual use, for later use. As a dynamic and emergent approach, the Usage-based Approach developed from an understanding that usage is crucial in language acquisition, and language use supports language development enabling pattern recognition, analysis, generalisation and use within social contexts (Bybee, 2010; Ellis et al., 2013; Eskildsen & Cadierno, 2015).

Connecting second language to mathematics

Learning mathematics requires cognitive engagement with the mathematical discourse along with the ability to use the symbolic notations and visual images of mathematics for encoding, explaining, and sharing meaning as well as for constructing viable arguments (O'Halloran, 2005). The multi-semiotic nature of mathematics and its unique grammatical strategies, though pertinent for problem solving, are not usually learned through casual interactions in everyday life, but need to be explicitly taught (O'Halloran, 2005). Therefore, certain characteristics of the Lexical, Communicative and Usage-based Approaches have been excluded.

Both the Lexical and Communicative Approaches recognise the importance of grammar, although neither advocate learning grammar for its own sake (see concept map). Grammar is learnt in context and is linked to already known meanings, words, and word phrases within the structure of the language to create meaningful texts, thus making them relevant to the development of mathematics. Pattern identification and use of lexical units to construct meaningful grammatical structures involves conscious listening, talking, reading and writing, requiring the learner to cognitively engage with, and use the language, as well as listening to others talk. Mathematical symbolism uses lexicogrammatical strategies to encode mathematical meaning (O'Halloran, 2005).

The mathematical lexis comprises words and word phrases (lexical 'chunks') as well as a specialised register of technical words, such as 'vertex'. Mathematics also uses words from natural language, but often with different meanings that are rooted in historical languages, for example 'volume'. Similarly, as in the English language, the meaning of mathematical words can be derived from using the base

words, for example ‘triangle’. ‘Word phrases’ are also included in mathematics language, such as ‘a half of ...’, and can also include ‘number bonds’. Inclusion of these word phrases, along with known facts and knowledge, enables the creation of a knowledge base onto which further mathematical knowledge can be constructed. ‘Chunks’ are easily memorised and can be easily built upon, although extending the discussion on ‘chunks’ in mathematics is beyond the scope of this paper. In addition to knowing mathematical symbolisms, it is important to understand that some symbols can have different roles in mathematics. For example, in the equation $3a(\sqrt[3]{81})$ the symbol '3' has two different meanings and purposes, both of which are different to its numerical sense that students encounter in everyday activities.

The Usage-based Approach emphasises the cognitive perspective of language learning, embracing constructivism, connectionism, communication together with usage and frequency of use. Usage helps to explicitly connect the social and cognitive engagement with mathematical discourse while constructing grammatical structures using mathematical lexis, symbolism, and visual images (O'Halloran, 2005; 2010). Participating in conversations facilitates the formulation of mathematical ideas and provides others the opportunity to give ‘corrective feedback’. The frequency of use and the quality of the engagement provides opportunities for construction and co-construction of mathematical language.

Central to the Communicative Approach is the communicative competence (oral and written communication and genre) along with the grammatical, sociological, and strategic competences. While genre impacts on written and oral communications in natural languages, in mathematics the formalised discourse developed by mathematicians often eliminates contextual meaning-making, rendering the language abstract and defined, even though mathematics evolved from cultural contexts and the language of the mathematics register is still rooted in sociological linguistics. Mathematics language must take social contexts into consideration when writing for specific audiences (conferences, journals, reports) and in ‘worded problems’ where students learn to extract information from the context, which is manipulated and then put back into context.

Using several characteristics from these three SLAs, a set of teaching strategies have been developed and are being trialled in two South Australian primary schools and one Danish school. The teaching strategies being trialled involve conferencing, rehearsing, communicative competences, building mathematically rich and language-rich learning classroom environments, developing mathematical language through use and frequency of use, ‘languaging’ to mathematising, and use of real-life situations. While there are some positive feedbacks, it is too early in the research process to provide any conclusive outcomes.

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Beyond vocabulary: Enabling mathematics teachers to focus on rich discourse practices for language-responsive mathematics teaching

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When mathematics teachers start with language-responsive teaching practices, they often focus only on isolated vocabulary rather than rich discourse practices. This restricted orientation is often documented and raises the need to be widened in professional development (PD) programs. In our PD design research project, we developed activities inviting teachers to re-focus on rich discourse practices. This paper reports on theoretically and empirically derived potentials of PD activities.

Keywords: Language-responsive mathematics teaching, specifying language demands, PD activity.

Need for professional development for language-responsive mathematics teaching

By language-responsive mathematics teaching, we mean practices directed at enhancing students' language for mathematics learning (Erath et al., 2021). Based on well-established research on the epistemic role of academic language for mathematics learning (Moschkovich, 2015), mathematics education researchers and curriculum designers have developed design principles and curriculum materials for language-responsive mathematics teaching (Erath et al., 2021). In controlled trials, these approaches have been shown to be effective for developing students' conceptual understanding in mathematics, when enacted productively by the teachers (ibid.). However, studies document huge differences in teachers' enactment of language-related teaching practices (Barwell, 2020; Lucas & Villegas, 2013). That is why well-designed professional development (PD) programs on language-responsive teaching are required. In this paper, we focus on one particular challenge for PD that has often been documented: Many mathematics teachers who start language-responsive mathematics teaching have a restricted *orientation about what counts as relevant language demand* in mathematics classrooms and focus exclusively on isolated vocabulary without their functional use in discourse practices (Bunch, 2013; Turner et al., 2019), such as reporting procedures or explaining meanings which are the much more relevant language unit as shown in many classroom studies (Moschkovich, 2015). Thus, the PD design research study presented in this paper pursued the following design research question:

How can PD activities contribute to widen mathematics teachers' language perspective from vocabulary orientation towards rich discourse practices when starting language-responsive teaching?

Background: Definition of discourse practices and findings on mathematics teachers' restricted vocabulary orientation

Discourse practices are considered as key units of language that evolved socio-culturally and are interactively co-constructed in mathematics classroom discourses (Moschkovich, 2015; Erath et al., 2021). To substantiate what we mean by *rich discourse practices*, Figure 1 provides examples of discourse practices for "magic multiplication", a graphical version of the multi-digit multiplication

procedure (later treated in one of the PD activities). The discourse practice of *reporting how* this procedure works ensures the transmission of procedural knowledge, whereas *explaining why* this procedure works can contribute to consolidating conceptual understanding of place values (the interplay of numbers and digits) and multiplication (as unitizing, i.e., counting in bundles). Whereas *reporting-how* only requires sequential connectives (first, second, finally), *explaining-why* requires integrative connectives (such as corresponds to, as, that’s why) and more complex sentence structures. Thus, *explaining-why* is considered as richer, mathematically *and* discursively (Moschkovich, 2015).

When teachers (implicitly or explicitly) specify language demands for explaining mathematical aspects, they tend to neglect the discourse practices (and the vocabulary needed to enact them) and focus on vocabulary in isolated ways (Turner et al., 2019). Teachers’ implicit specifications become explicit in other teaching practices, e.g., in providing language support by scaffolding only (mostly technical) vocabulary (e.g., digit, orthogonal lines), or in formatively assessing students’ language only with respect to vocabulary while neglecting the discourse practices in view (Brunch, 2013). Empirical evidence for this was provided by Prediger et al. (2019) with the diagnostic PD activity in Figure 2. It elicited the diagnostic categories that 223 mathematics teachers chose to formatively assess three students’ written texts.

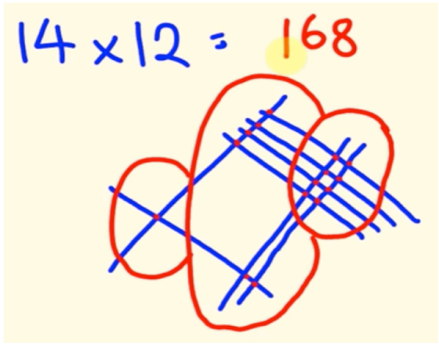
<p>Magic multiplication</p>  <p>(Source YouTube video tutorial http://www.youtube.com/watch?v=_AJvshZmYPs)</p>	<p>Reporting how the procedure of magic multiplication works</p> <p>For multiplying 14 x 12, you first draw 1 and 4 parallel lines for 14 and, orthogonally to these, 1 and 2 parallel lines for 12.</p> <p>Second, you count the intersections of the 4 and 2 lines on the right, resulting in 8 as ones-digit. The intersections in the middle are 6, they stand for the tens-digit. On the left there is 1 crossing, standing for hundreds digit.</p> <p>Finally, you write the digits from right to left next to each other, this results in 168.</p>	<p>Explaining why the procedure of magic multiplication works</p> <p>Drawing the lines corresponds to decomposing the numbers into its tens and ones. When counting the crossings for the ones, you see four bundles of twos, this corresponds to 4 x 2. The crossings of the lines for ones and tens represent six bundles of tens, and counting in bundles corresponds to multiplying 6 x 10 = 60. The crossings of the lines for tens represent ten sets of tens, thus 10 x 10 = 100. Altogether, writing the three digits next to each other corresponds to the product: 100 + 60 + 8 is composed into 1 6 8.</p>
<p>Possible vocabulary needed</p>	<ul style="list-style-type: none"> ▪ Digit ▪ orthogonal and parallel lines ▪ intersection 	<ul style="list-style-type: none"> ▪ de-/composing numbers ▪ lines represent tens/ones ▪ crossings represent bundles of ones/tens

Figure 1. Example for two discourse practices: Reporting procedures and explaining why they work

<p>Diagnostic Activity</p> <p>The Grade 8 class has introduced the meaning and the formula for the slope of linear functions. The homework task (formulated as on the right) requested students to write summaries.</p> <p>Analyze the three students' texts using four self-defined categories, two for mathematical aspects and two for language aspects. Evaluate using 0,1, or 2 points and justify your decisions.</p>		<p>Explain what $m = \frac{y_2 - y_1}{x_2 - x_1}$ means and what you can describe with it.</p>
<p>Ali</p> <p>You want to calculate the slope. First you chose two points, e.g. $x=3, y=1$ and $x=5, y=8$. Second, you evaluate: $m = \frac{8-1}{5-3} = \frac{7}{2}$.</p> <p>Done.</p>	<p>Suleika</p> <p>The slope saying, how much growing the function per x-step. Thus, how much get y more per how much get x more. The PER makes the DIVIDED.</p>	<p>Tom</p> <p>Iff you for exempel have $y=0.2+10$ of the mobile phone tarif. For exempel at 10 minutes it is 12 €, at 30 minutes it is 16 €. Thus, distance 20, price more 4 € and than $\frac{4 \text{ €}}{20 \text{ min}} = \frac{0.2 \text{ €}}{1 \text{ min}}$.</p> <p>Thenn, costs per minute my price 0.20 € more costly.</p>

Figure 2. Diagnostic PD activity for mathematics teachers in the beginning of a PD program (Prediger et al., 2019, p. 108; translated from German with students' errors preserved)

The analysis reveals that from the teachers' self-chosen language-related diagnostic categories, only 14% of the teachers focused on discourse practices, but 29% on vocabulary (30% on grammar and 48% on orthography). Up to this point, these percentages replicate findings of Turner et al. (2019). However, what we have also seen in the analyses is that teachers' *mathematical* diagnostic categories for these texts included 79% discourse practices (Prediger et al., 2019).

This very important additional finding shows that mathematics teachers indeed focus on how students report or explain, but before the PD on language-responsive teaching, they do not subsume these discourse practices under their individual conceptualization of language. The analysis also revealed that those teachers who already activated discourse practices as language-related categories had accepted their responsibility to enhance students' language in mathematics classrooms to a significantly higher degree than those who conceptualize language only on word and sentence level (Prediger et al., 2019, p. 111). In the following, we report how we used these findings to improve our PD design by starting from these identified teacher resources to widen their vocabulary orientation on language into a more comprehensive orientation, with a clear focus on discourse practices as relevant language units.

Methodological framework of content-related PD design research

Content-related PD design research as methodology

For pursuing our design research question, we chose the methodology of content-related PD design research (Prediger, 2019). In general, content-related design research combines two aims, designing learning opportunities and providing empirical insights into the initiated learning processes on the learning content in view. In our case, the PD learning content in view is the focus on discourse practices as mathematical relevant language units. The four working areas of content-related PD design research comprise (a) structuring the PD content (here, the need to go beyond the vocabulary focus is related to the identified starting points in teachers' mathematical focus on discourse practices), (b) developing the PD design, (c) conducting and analyzing the design experiments, and (d) contributing to theorizing on teachers' learning pathways and refined PD content specifications.

General PD design in the overarching project

The whole PD program on language-responsive mathematics teaching comprises a series of four PD sessions of 3-6 h each, with teachers' intermediate classroom experimentations (Prediger, 2019). It developed from early attempts to offer inert knowledge about the role of language in mathematics learning to more situated approaches that are based on the content-related model of teacher expertise (Prediger, 2019, adapted from Schoenfeld, 2010). The model of content-related expertise starts from teachers' typical jobs (in the case of language-responsive teaching these are (1) IDENTIFYING mathematically relevant language demands, (2) DEMANDING students' language production, (3) NOTICING students' language resources and challenges, (4) SUPPORTING students' language production, and (5) successively DEVELOPING students' language proficiency). PD activities (like in Figure 2) are developed as simulations to approximate these jobs (Grossmann et al. 2009), followed by reflections with the teachers on the underlying pedagogical content knowledge (PCK) categories and orientations needed to master these jobs productively. By this inductive PD approach, we work with what teachers collectively bring in, and we systematize and connect their intuitive ideas, sometimes complementing them by informative deductive inputs.

Methods of data gathering for this paper

In this paper, we focus on the PD activities in the first PD session that were successively optimized to better reach our specified learning goal. Design experiments were conducted with repeated trials of the PD session in view, first by the authors as PD facilitators, later also by other PD facilitators that we qualified for this task. In sum, 23 PD design experiments of the first PD session were conducted with more than 400 mathematics teachers from secondary schools in six iterative design experiment cycles. 18 of the PD sessions were audiorecorded or videorecorded and partly transcribed. Additionally, teachers' written products were collected.

Methods of qualitative data analysis for this paper

To optimize the PD activities iteratively, we qualitatively investigated the recorded teachers' processes while completing and reflecting on the PD activities. For this, teachers' utterances and self-reported practices for mastering the jobs addressed in the PD activities were analyzed to infer the underlying orientations and PCK categories they implicitly or explicitly activated in their practices for mastering the simulated jobs.

Insights into the iterative design experiments

Design Experiment Cycles 1-2: Offering inert knowledge without reaching the orientations

After a brief introduction on language disparities in mathematics achievement, the first PD session starts with an introductory PD activity that addresses the job IDENTIFYING mathematically relevant language demands (see Figure 3 left). When teachers are asked to collect typical language difficulties on the pinboard, it is typical that 70% of them refer to vocabulary and about 60% to word problems.

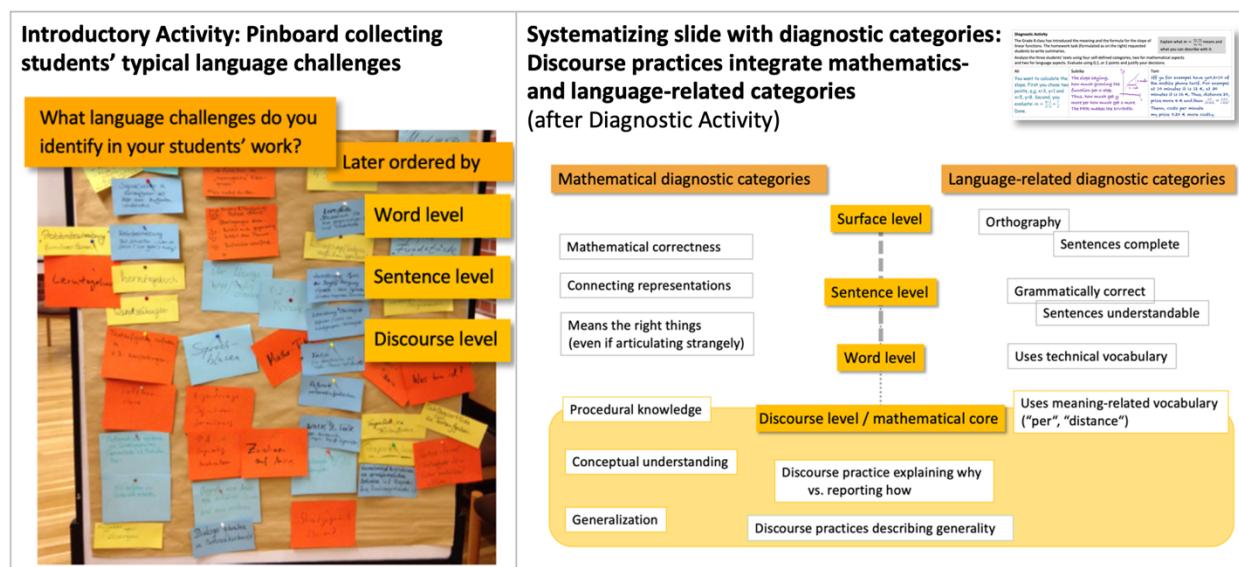


Figure 3. Introductory activity and slide after diagnostic activity from Figure 2

Only under 10% of the teachers already refer to discourse practices. The reflection phase systematizes the collected language demands and introduces the PCK categories word level, sentence level, and discourse level hinting to this imbalance. However, we observed in the early design experiments that the introduced PCK categories remained only inert knowledge. That means, the participating teachers did not use them for the next PD activity, NOTICING students' language (Figure 2). From this observation, we inferred the need for more intensive reflections, systematizing the mathematical and language-related diagnostic categories that the teachers used for the diagnostic activity. This reflective systematization process (for which the outcome is depicted in Figure 3 right) turned out to be fruitful and necessary to deepen teachers' knowledge about language levels and show that the discourse practices help to integrate mathematics and language as they are situated in the mathematical core and, at the same time, on the discourse level of language. Nevertheless, most teachers still did not consciously use their knowledge about discourse practices when approximating the next job, SUPPORTING students' language when asked to write down useful phrases for a writing task (analyzed in Prediger, 2019). We realized that the explicit knowledge about word level, sentence level and discourse level and the explicit activation of the PCK category discourse practice rather than vocabulary seems to be guided by a more unconscious element of teachers' expertise, their orientation. Following Schoenfeld (2010, p. 29), we define orientations as beliefs, values or pedagogical attitudes about particular aspects of mathematics (and language) teaching and learning that implicitly or explicitly guide teachers' perception and prioritization of jobs, and that can hinder the activation of knowledge for productive practices.

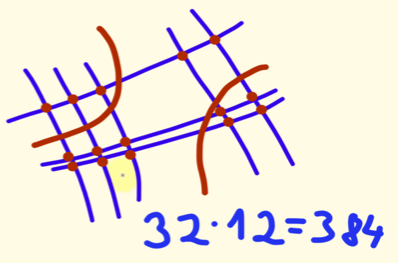
Design Experiment Cycle 3-4: Introducing and refining the self-experiment

The change of orientations cannot be reached only by introducing new knowledge. It requires substantial personal experiences to reach deeper affective levels that will serve as the anchor for a set of alternative orientations toward teaching (Schoenfeld, 2010). From the beginning of the design experiment cycles, we included a PD activity to increase mathematics teachers' acceptance of their responsibility to enhance students' language proficiency, the first of all orientations that teachers need to adopt before engaging in language-responsive teaching (Lucas & Villegas, 2013). In this PD activity

(Figure 4), we ask the participating teachers to conduct a self-experiment of writing and explaining the magic multiplication (Figure 1) in a foreign language. This PD activity is not an approximation of typical teacher jobs, but it puts the teacher into the role of students with language difficulties. This PD activity was very effective for sensitizing teachers to the communicative and epistemic role of language and to the affective consequences of feeling unable to fulfill the DEMANDED language productions without suitable SUPPORT.

Why is language so crucial? Self-experiment on magic multiplication

Self-experiment: Think, write and explain in your foreign language, **why** this procedure works. Don't switch to your home language German!

<p>Helpful words</p> <p>Line Intersection Digit Multiply, Add up Count Place values Digits Ones Tens Carry (Übertrag)</p>	<p>Vocabulaire</p> <p>la Ligne une Intersection le Chiffre Multiplier, additionner Compter le Valeur de position le Chiffre des Unités le Chiffre des Dizaines le Chiffre des Centaines</p>	
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(Lists of supporting words occur only after some minutes) Magic Multiplication (You-Tube-Video, 1:44 min, played without sound)
http://www.youtube.com/watch?v=_A1vshZmYPs
 (Idea adapted from Ingrid Gogolin)

Figure 4. Self-experiment for sensitizing teachers to focus on discourse practices

In reaction to the identified requirement of further PD opportunities to extend teachers' vocabulary orientation in Cycle 2, we refined this activity in Cycle 3 into one that allows also the reflection on the discourse practices. From Cycle 3 on, the activity was presented with a support of isolated vocabulary (see Figure 4), which is offered some minutes after DEMANDING the written text. Although most teachers in our PDs have the conceptual understanding underlying the magic multiplication procedure, many of them do not refer to it and do not articulate the demanded discourse practice of explaining-why (left side of Figure 1 instead of right side). Being heavily occupied with writing in a foreign language, they restrict to reporting-how. Their own surprise about this shortcoming is a very good culmination point for the reflection on these different discourse practices: Usually, the reflection starts with the affective experiences ("I felt under pressure of this task which was unsolvable for me, with my bad English.") and then proceeds to the affective support provided by the list of isolated vocabulary ("I felt really relieved when I got this list and started writing."). But teachers also reflect the limited potential of the isolated vocabulary support for their writing attempts ("I cannot use only nouns, I searched for relational phrases such as 'this represents that' or 'this stands for that', 'it is orthogonal to'.") and conclude that chunks with relational phrases (connectives) might be more important. In this way, they feel the relevance of the distinction between word level and sentence level themselves, creating an affective anchor of this difference. Only then, we ask for the discourse practices they realized ("Who has really explained why, who only reported how the procedure works?", "I haven't, I was too busy with articulating how it works", "Why haven't you done what you were asked for?"; all quoted utterances transcribed from an audio-record of Cycle 3). During these conversations, most participants become aware that the difference between *reporting how* and *explaining why* is the most critical language unit in play. As this was experienced personally, with own affective

involvement, this PD activity increases the chance to have sustainable impact with regard to the underlying orientations (shift from vocabulary orientation to focus on discourse practices).

Design Experiment Cycle 5/6: Exploring relevant aspects as a task for facilitation of discussion

As the reflection of the self-experiment seemed to bear potentials, we investigated them more systematically in Cycle 6 to find teachers' utterances that give starting points for shifts in orientations. Indeed, Figure 5 provides examples of teachers' self-reflective statements that repeatedly occurred.

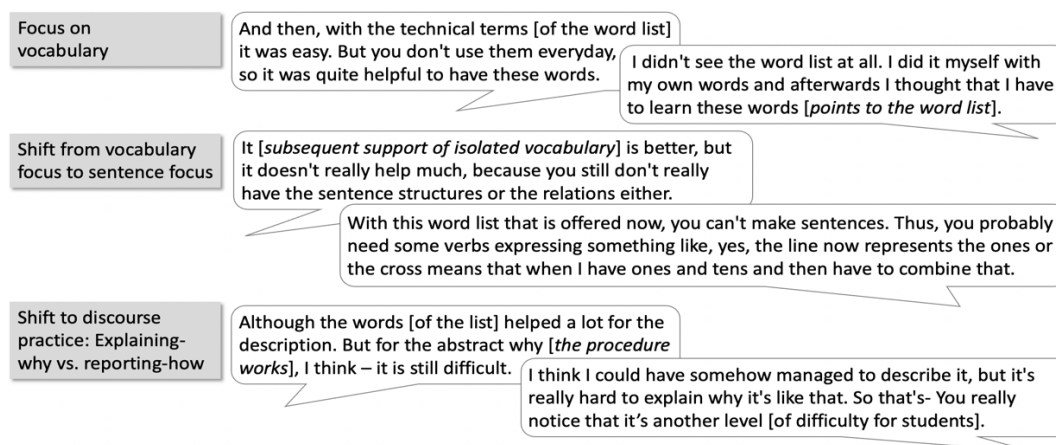


Figure 5. Aspects around the focused orientation raised in the reflection of the self-experiment

Although these self-reflective utterances occurred in all videorecorded PDs, the discussions around them varied strongly in their success of making use of the self-reflection for this important PD learning goal. A requirement for the potential success of the shift from vocabulary orientation to focus on discourse practices within the reflection seems to be that the facilitator notices these starting points and picks them up by confirming, strengthening or expanding them towards the direction of a stabilized shift in teachers' focus of attention. We will need further analytic steps focusing on facilitators' moderation practices to disentangle conditions of success.

Discussion

Preparing mathematics teachers for language-responsive teaching is often called for, but not yet always targeted to most urgent PD content aspects (Lucas & Villegas, 2013; Turner et al., 2019). By our long-term PD design research project (Prediger et al., 2019) we could identify *the shift from an isolated vocabulary orientation to a focus on the functional use of vocabulary in discourse practices* as one critical content aspect that does not only require teachers' knowledge about language levels, but a deeper orientation that needs active reflection in order to develop in the productive direction (Schoenfeld, 2010). In this paper, we present three PD activities that can contribute to extend mathematics teachers' vocabulary orientation about language towards a wider focus on rich discourse practices when starting language-responsive teaching. They combine the establishment of knowledge about language levels with their situated use not only for IDENTIFYING language demands, but also for NOTICING and SUPPORTING students' language. When teachers adopt the role of language learning students, this experience can build a more affective anchor. When the experience is reflected with respect to the teachers' surprising failure to engage in the demanded discourse practices, this can have an impact on their orientations. In future research, we will deepen these insights

by (a) analyzing the facilitator's moderation practices in more detail, (b) by tracing teachers' shifts in orientations also with quantitative measures.

Acknowledgement

The PD design research has grown in the MuM research group (mathematics learning in language diverse contexts, funded by German Ministry of Education and Research, current grant FachBiss, 01JI2001E to S. Prediger) and the DZLM network (funded by Leibniz Gemeinschaft).

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The role of the teacher in stimulating pupils' language

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This paper is based on experiences from a collaboration project between researchers at the university and teachers in primary school. I will show how close collaboration between teachers and researchers with the aim to create classroom sessions based on dialogic teaching with clearly defined learning goals, also can contribute to the professional development of teachers. A crucial feature of this professional development is a careful analysis of the mathematical content of the topics to be worked with in the classroom sessions before the actual planning of the sessions.

Keywords: Theory of Didactical Situations, multiplicative structures, teacher professional development, dialogic teaching.

Introduction

The project Language Use and Development in the Mathematics Classroom (LaUDiM) was an intervention study carried out in collaboration between researchers at the university and two primary schools in the period 2014-2018. A central part of the project was to design and implement teaching sequences in close collaboration between researchers and teachers. The teachers were responsible for the implementation in their respective classes. The design of a teaching sequence was guided by principles from *The Theory of Didactical Situations, TDS* (Brousseau, 1997). A teaching sequence initially consisted of a planning meeting between teachers and researchers followed by two (sometimes three) classroom sessions. The pre-analysis (see Figure 1) was added later. Between the classroom sessions there were reflection sessions to discuss what happened in the classroom and, if necessary, revise the plans. After the classroom sessions teachers and researchers met to watch and discuss video recordings from the classroom, first with focus on the pupils' work (Video Session 1) and then with focus on the teachers (Video Session 2).

In this paper I will use experiences from the LaUDiM project to discuss how a research and development project like this can function as part of teachers' professional development. In particular, I will discuss the role of deep mathematical knowledge for orchestrating classroom sessions for young learners and the role of language in the process of focusing on crucial elements of the tasks at hand.

The design of a teaching sequence

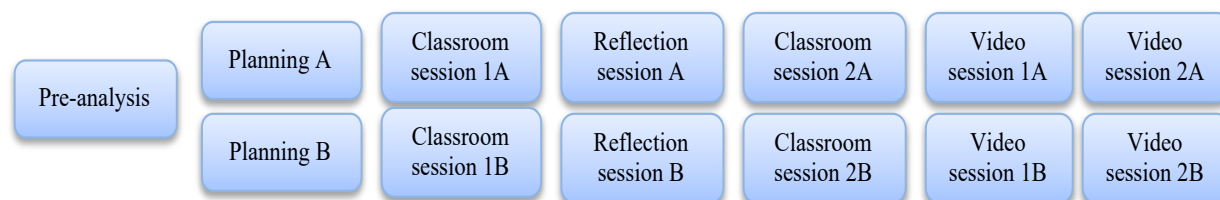


Figure 1: Model of an intervention sequence (Adapted from Nilssen & Høyenes, 2020, p. 295)

In the project, two classes, one from each of the two schools (A and B), were followed from Grade 2 to Grade 5. In total, seven interventions were done, covering different mathematical topics. The idea

was to implement the interventions with the same content and close in time at both schools, but for practical reasons this was not always possible. Figure 1 shows the different components of an intervention (teaching) sequence.

The pre-analysis

This part of an intervention sequence was not planned for at the beginning of the project. The original idea was to start with the planning meeting, with the goal to design activities to be implemented in the classroom sessions. After the planning meeting, each teacher would then work on the detailed planning of the activities in her classroom. After the third intervention, where the topic was *Properties of polygons*, the project group experienced a need for going deeper into the mathematical content. The classroom sessions on properties of polygons (Grade 2) revealed some language issues connected to the parts of a polygon which both teachers and researchers realised they had not been sufficiently prepared for (see Rønning & Strømskag, 2017). This led to the decision add a meeting (pre-analysis) to study and discuss deeper the mathematical topic to be addressed in the classroom sessions.

In Grade 3, the topic of the first intervention was planned to be multiplication and division. Now the planning meeting was preceded by the pre-analysis session, led by one of the researchers, with a theoretical discussion of multiplicative situations, or multiplicative structures. As a basis for this discussion, all participants had read the chapter by Greer (1992) on classification of multiplicative situations. Some aspects of multiplicative situations that were discussed in this session were:

- Different models for multiplication (equal groups, multiplicative comparison, rate, Cartesian product) and division (partitive and quotative)
- Different roles of the multiplicand and multiplier, and of the dividend and the divisor
- Commutative and non-commutative situations
- Limitations of the models, e.g., when extending from integers to fractions

The pre-analysis, which included *an epistemological analysis* (Strømskag, 2017) of the concept, had the aim to create awareness of the complexity of a mathematical concept. The idea was that the teachers should be aware of this complexity already when introducing the concept to pupils, preparing the ground for later work with the concept in more advanced settings.

Commutative and non-commutative situations

As an example of the importance of an epistemological analysis, I will use commutativity of multiplication. It is common knowledge that multiplication is a commutative operation but there may not be sufficient awareness of distinguishing between the *situation* in which multiplication is used and the *computational procedure*. Figure 2 shows a task given to the pupils.

The class will arrange a party, and they are going to make muffins for this party. According to the recipe they need four eggs for one portion. The pupils have figured out that they will make twelve portions. How many eggs do they have to buy?

Figure 2: The muffins task

In this case, one may argue that the computation $12 + 12 + 12 + 12 = 48$ gives the quickest way to finding the correct answer, but from the context it does not make immediate sense to add 12 four times. The number 12 represents the number of portions, and the final answer, 48, represents the

number of eggs. Adding eggs would give the computation $4 + 4 + \dots + 4$, twelve times. This is computationally more challenging, with more possibilities for making errors. To distinguish between the commutativity of the operation and the commutativity of the situation was seen as an important part of the teachers' professional knowledge. The awareness of this difference could better prepare the teachers for possible questions like “can I instead do $12 + 12 + 12 + 12$?”, and if so, why?

From a language point of view, it was emphasised that the situation four 12s is different from twelve 4s, thereby making a distinction between the role of the multiplicand and the multiplier. These terms were used in discussions with the teachers, but not introduced to the pupils. However, the convention that the notation $4 \bullet 12$, read as “four times twelve” or “twelve four times”, should always mean “four 12s”, was agreed on with the pupils. The different role of the numbers was continued into division, discussing partitive and quotative division.

Partitive and quotative division

Partitive division is a model which is restricted because it makes sense only when the divisor is a positive integer. Introducing both partitive and quotative models, and distinguishing between them, at an early stage when still working only with positive integers, was done with the assumption that it would ease the transition to other number sets at a later stage. Fishbein et al. (1985) claimed that partitive division is the intuitive model for division. Experiences from LaUDiM indicate that the quotative model may be equally intuitive. The task in Figure 3 is used to show this.

The class will invite the parents to an “open day” at the school. The parents will bring bread rolls for the café. They brought 96 bread rolls, and they will be placed on plates. On each plate there is space for 12 bread rolls. How many plates are needed?

Figure 3: Distributing bread rolls

In Figure 4, a solution to this task is shown in the picture to the left. The picture shows plates with 12 rolls on each plate, and a counting procedure $12 - 24 - 36 - 48 - 60 - 72 - 84 - 96$, followed by the final result, 8 (plates). The picture to the right corresponds to a similar situation, starting with 48 and a repeated subtraction of 6 is carried out until there is nothing left. Then the number of subtractions is counted, giving 8 as the result. The quotative model, represented in two different ways in Figure 4, is also helpful to see multiplication and division as inverse operations. Figure 4, to the left, shows $96 : 12 = 8$ as well as $8 \bullet 12 = 96$.

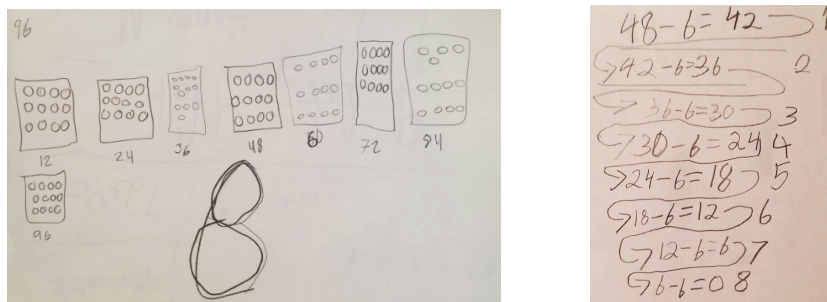
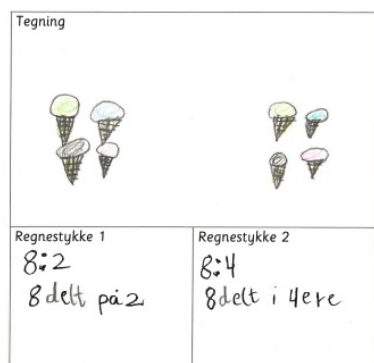


Figure 4: Pupils' solutions to a quotative situation

In the work with the teachers, distinguishing between partitive and quotative division was emphasised. This led to an initiative from one of the teachers to introduce a precise language to use for distinguishing between the two models. For example, in the situation with the bread rolls, the



teacher would say “we divide the 96 rolls *in twelves* (deler i tolv), i.e., in groups of twelve. For the partitive situation, she would say “divided *on*”. An example: “We have 78 pencils to distribute equally between 6 boxes. How many pencils will there be in each box?” Then the teacher would say “78 divided on 6” (delt på 6). For the pupils, the phrase “divided in” would then represent quotative division and the phrase “divided on” would represent partitive division. Note also that the form of the number word representing the divisor is different in the two situations: “divided *in twelves*” for quotative and “divided *on six*” for partitive.

Figure 5: Two models for division

Figure 5 shows a pupil’s solution to a task starting with a picture showing two groups of four ice cream cones in each group. The pupils were asked to express this as a division in two different ways. This pupil has given the two possibilities $8 : 2$ expressed as “8 divided on 2”, and $8 : 4$ expressed as “8 divided in 4s”.

The Theory of Didactical Situations

The classroom sessions were designed following the Theory of Didactical Situations in mathematics, TDS. I will mention a few important concepts from this theory. For a more complete exposition, I refer to Brousseau (1997). In TDS, the particularity of the knowledge taught (the target knowledge) plays a significant role. Working towards the target knowledge is seen as consisting of different situations (phases), adidactical and didactical. In the adidactical phases, the teacher should, ideally, not intervene in the pupils’ work. In these phases, the teacher has (temporarily) transferred responsibility for solving the problem to the pupils. The milieu that is created, should, again ideally, provide sufficient feedback to the pupils for them to realise whether their responses are adequate with respect to obtaining the target knowledge. It has been shown (Strømskag, 2017) that this is not always easy to accomplish. Situations of action, formulation, and validation are (intentionally) adidactical situations, whereas the situation of institutionalisation is a didactical phase. The situation of *institutionalisation* is where the teacher connects the knowledge built by the pupils to the scholarly and decontextualised forms of knowledge aimed at by the institution. I will later show how this phase can be crucial for language support and development. Introducing the teachers to the principles of TDS was also seen as part of their professional development.

The role of the teacher in the different phases of TDS

The different phases in a session designed according to the principles of TDS provide different opportunities for language stimulation. In the adidactical phases, the teacher should step back and let the internal logic of the problem guide the pupils’ work. When pupils are left to themselves, there is a danger that the language used by the pupils focuses mainly on practical matters with few mathematical terms being used. It has been pointed out, e.g., by Pimm that “merely increasing the amount of pupil talk in mathematics classes should not be seen as an end in itself” (Pimm, 1987, p.

48). Pimm continues by stating that “[i]t is important that the conversations be task-focused and the style and level of explicitness of the talk are both important” (Pimm, 1987, p. 48). This resonates well with the fundamental idea of TDS that the target knowledge of the activities should be clearly defined, making the epistemological analysis of the concept at stake important. This analysis is an important part of the teachers’ preparation for issues that may come up during the pupils’ work with the given problem.

In the project, communicating through arguing and reasoning was an important way of working. The teachers acted as role models for the pupils in reasoning and arguing, and the phrase “to work like mathematicians” became a slogan for how to work with mathematics. Working like a mathematician meant e.g., putting discoveries and ideas into words and arguing their merit as well as working with explanations and discussion of strategies (Nilssen & Klemp, 2020, p. 73). An example of what it could mean to “work like a mathematician” can be seen in the work with geometrical shapes. Here the idea of a definition was central, e.g., when classifying polygons (Rønning & Strømskag, 2017). In this work, the teachers emphasised to the pupils that definitions are not a priori given, they represent something that *the mathematicians have agreed on*.

It is acknowledged that keeping the focus on the task and on the target knowledge in an didactical situation is challenging (Strømskag, 2017). It may therefore be necessary for the teacher to intervene, and knowing when this is appropriate is also part of the teacher’s professional knowledge. An example can be seen from the work with the task shown in Figure 6.

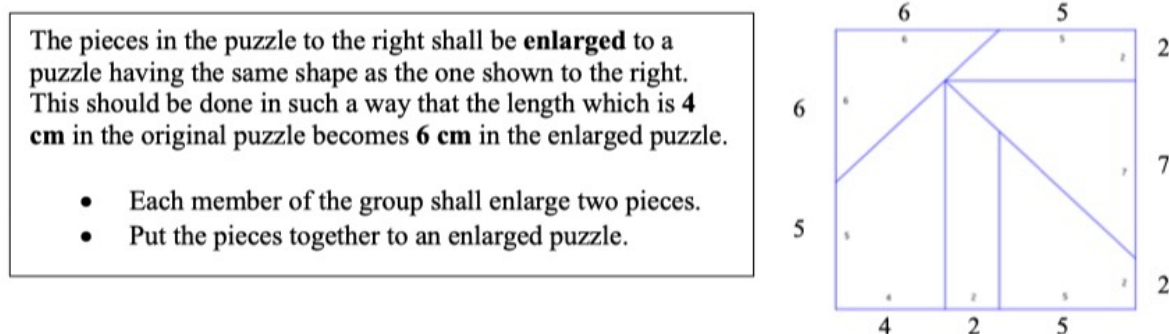


Figure 6: The enlargement task

The main challenge with this task is to recognise the situation as multiplicative. The first attempt from all the pupils was to add 2 cm to all the lengths in the puzzle, with the result that the pieces no longer fitted together to a square. The pupils obviously realised that what they had done was not correct, and many of them created very elaborate additive strategies to try to solve the problem, again with no success (Rønning, 2020). To help them further, the teacher gave the additional information that the length 5 cm should be 7.5 cm in the enlarged puzzle. After this, the teacher observed in one group a strategy with some potential for multiplicative thinking. Later, the teacher used the strategy from this group and provided further language support to stimulate multiplicative thinking. Nora presented for the whole class the strategy her group had developed:

Nora: We thought first that four should be six. So then we added two. Then we found that on the other we should add two point five. So we added one half more on the next. Then we tried to do that upwards on the other numbers.

Here, the utterance “one half more on the next” is crucial, and the teacher uses this utterance as a basis for the discussion. Then also other pupils engage in the conversation.

- Teacher: Do you see any connection between the lengths in the original puzzle and how much we should add?
 Frances: I think we have to add a half each time.
 Teacher: What half then, do you think? What is it half of?
 Mary: Perhaps when it is eight, we have to add half of eight.

In the dialogue above, the teacher introduces a shift in the language use concerning ‘half’ when she says “What half then, do you think? What is it half of?” Introducing the word *of* changes the role of *half* from being a number to being an operator. Mary picks up on this shift by saying “Perhaps when it is eight, we have to add half of eight.” It is reasonable to think that for Mary the number eight does not play a special role and that she would be able to generalise into something like “when we start with a number, we add half of that number”.

In the institutionalisation phase the teacher could draw on episodes and utterances that she had observed when pupils worked together in the didactical phases, bring these up before the whole class and encourage the pupils to argue and justify their solutions. One example of this was shown above with the enlargement task. Another example is from the pupils’ work on combinatorial problems (see Figures 7 and 8). Combinatorial problems are examples of a multiplicative structure characterised by Vergnaud as a *Product of Measures* situations. These situations involve a structure with a mapping from a product of two measure spaces into a third measure space, $M_1 \times M_2 \rightarrow M_3$ (Vergnaud, 1983, p. 134), where the measure space M_3 is not initially present. The counting unit of the situation lives in the space M_3 and since this is in a sense unknown, it is not obvious when to stop counting, i.e., it is not clear when the problem has been solved (English, 1991).


How many different gingerbread biscuits can we make if we have cutters in these four shapes  and we have white, green and red icing?

Figure 7: Combinatorial task 1

Ms. Hall has 3 pairs of trousers and 5 sweaters. The trousers are in the colours blue, black, and grey. The sweaters are in the colours blue, red, black, green and purple. She will use one pair of trousers and one sweater each day, and she will combine different pairs of trousers with different sweaters. How many days in a row can Ms. Hall wear different outfits?

Figure 8: Combinatorial task 2

It turned out that the pupils struggled with solving Task 2. In Task 1, they developed representations and language structures that showed clear evidence of multiplicative thinking, whereas this was to a much smaller extent the case with Task 2 (Rønning, 2022). It was also striking that although the tasks were worked with on two days in the same week, the pupils did not seem to connect them. In the whole class session, the teacher wanted to focus on the similarities between the tasks. Building on the experience from Task 1, where all groups had represented the solution in a more or less regular rectangular pattern, the teacher invited the pupils to look for similarities. She started with a written

explanation by two pupils of a version of Task 1 with six shapes and three colours: “Our method is that we draw all the shapes and then draw two more below. Then we colour the shapes in the colours we are supposed to use. Then we count all the shapes. And in this task, it was 18 shapes.” This was generalised by another pupil as “So many colours as there are, so many rows of shapes do we need”, and the teacher introduced the notation $3+3+3+3+3+3 = 18$, or $6 \cdot 3 = 18$. When discussing similarities, the following exchange took place.

- Fabian: One shape can be one sweater. And that shape you can take three times. And the sweater can be used three times since there are three pairs of trousers. There were three colours on the biscuits also.
- Teacher: Any more similarities?
- Roger: Both times it was a “times-task”.
- Nadia: It was almost the same question, how many different ... and we used the same method ...
- Teacher: The first time we asked how many different *biscuits* we can make and today we asked how many different *outfits* we can make. And we could use almost the same method, Nadia said.

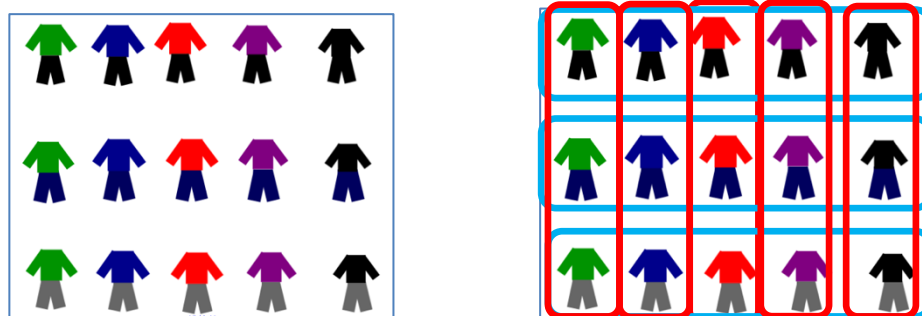


Figure 9: The teacher's representation of task 2, with indications of multiplicative structure

To further focus on the similarities, the teacher presented her representation of Task 2, see Figure 9, left. Then she said: “Look at my drawing. What do you think I thought when I drew this?” After a brief discussion, Filipa says that she can see a pattern and she comes to the board to indicate the same colour of trousers along the rows and the same colour of sweaters along the columns. This leads to a discussion of where in the picture one can see five times three, and where one can see three times five. One pupil marks the rows to indicate three fives (blue marking in Figure 9 to the right) and another pupil marks the columns to indicate five threes (red marking in Figure 9 right).

Summary

In this paper I have shown how a careful analysis of the mathematical content, using multiplicative structures as an example, can provide teachers with tools to support pupils in their language development. In the project, multiplicative structures came up as a recurring theme over several years. Already in Grade 3, the difference between e.g., four 12s and twelve 4s was emphasised, and that for a given situation, it was possible that only one of these expressions made immediate sense. However, in situations like the combinatorial tasks, both three fives and five threes were equally meaningful. The teachers expressed in interviews that the pre-analysis provided them with a mathematical foundation that increased their ability to deal with contingent situations in the classroom. The theoretical aspects concerning e.g., multiplicative structures provided a language for the teachers to talk in greater precision about mathematical concepts. They were also challenged to find ways of transposing this language in ways that would make sense to the pupils. I have shown examples of this

in one of the teacher's usage of the terms *divided in* and *divided on*, to distinguish between the two models for division. The use of *half of* in the enlargement task is another example. Here the teacher is able to, in a subtle way, to switch the role of *half* from being a number to being an operator.

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Design heuristic for generating conceptual learning opportunities through multiple languages – Exemplified for algebra

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Multilingual mathematics classrooms that harness students' multiple languages as epistemic resources are likely to contribute to students' understanding of mathematics, when learning activities are designed that harness this epistemic resource. In this paper, we propose a design heuristic for learning activities that enhance deeper conceptual understanding through language comparisons. The theoretical background to this design heuristic is first discussed before two algebra tasks are exemplify the heuristics for the development of such tasks. Next necessary steps for empirically investigating their epistemic potential are then discussed.

Keywords: Algebra, design research, language as resource, multilingualism.

Introduction: Epistemic potential of connecting multiple languages.

When mathematics tasks are designed to harness students' multiple languages, they can enhance students' mathematics learning (Moschkovich, 2002; Planas, 2018). We propose that multilinguality offers an *epistemic potential*, particularly when individual languages offer different, yet mutually supporting perspectives on a mathematical concept. This epistemic potential supports multilingual students to use their language repertoires to develop a deeper understanding of mathematical concepts (Prediger, et al., 2019). In mathematics classrooms with shared multilinguality (e.g., English/Spanish in California), harnessing multiple languages as epistemic resource can be realized, e.g., by including relevant mathematical problems where students can use their home languages for thinking mathematically. However, until recently, this epistemic potential of multiple languages has only occasionally been identified in non-interventionist learning processes (e.g., Moschkovich, 2002).

To address this gap, design research studies have identified two promising aspects for systematically exploiting the *epistemic potential* of multiple languages: reflecting language-related conceptual differences with students and supporting the activation of home languages by conceptual unfolding, e.g., through students' translation or explaining activities (Prediger & Uribe, 2021). In this paper, we elaborate on a heuristic for designing topic-specific learning opportunities that utilizes the possibility to compare languages for deepening mathematical understanding. Our focus is on the mathematical topic of algebra, which has been shown to be challenging for students with diverse language backgrounds (Stephens et al., 2022). Thus, we investigate the following design research question:

How can learning opportunities be designed that harness the epistemic potential of connecting multiple languages to deepen students' conceptual understanding?

Design approach: Comparing meaning-related phrases in multiple languages

Different practices vary in their potential for deepening students' understanding: While *reporting procedures* in technical language is a dominant practice in many classrooms, the practice of *explaining meanings* is more relevant for developing and deepening conceptual understanding (Prediger & Uribe, 2021). With respect to language, for explaining, meaning-related phrases are more important than technical phrases. For instance, the technical terms “multiplier, multiplicand, and product” do not provide the same information about what 3×5 means, in comparison to meaning-related phrases such as “three lots of five” which articulate multiplicative structures for unitizing (Götze, 2019).

Realizing meaning-related phrases is dependent on how different languages describe mathematical concepts and the nuances connected to them. For instance, Turkish and German conceptualize fractions differently, namely thinking from the part to the whole (German, English: three fifths) vs. from the whole to the part (Turkish, other Asian languages: five-therein three). During their learning processes, bilingual students were found to connect the Turkish- and German-related nuances across languages in a unified repertoire to deepen their conceptual insights (Prediger, et al., 2019).

These empirical findings led Prediger and Uribe (2021) to develop a design approach for activities in which students were asked to reflect on conceptual nuances, through comparing and connecting meaning-related phrases in multiple languages, even if these languages were not shared by all students in class. To harness the epistemic potential of such multilingual comparing and connecting activities, the learning activities had to be designed to be cognitively demanding (Setati, et al., 2008). Overall, these design concerns must be essential components if the inclusion of multiple languages in the mathematics classroom is to be more than a tokenistic inclusion effort.

A design heuristic for learning activities to harness the epistemic potential of comparing meaning-related phrases in multiple languages

In a classroom situation with multiple, non-shared languages, harnessing language as a resource in such a way requires a careful design of tasks so that teachers can prepare the comparison and connection of meaning-related phrases across multiple languages in advance. Particularly, harnessing language as a resource requires the identification of language-specific nuances, specific to the concept being discussed, as the basis for comparing languages. In the following sections, we illustrate how the design heuristic contributed to the development of learning activities for realizing this goal. Our focus is on algebraic concepts, namely (a) equivalence and equations and (b) multiplicative thinking. Based on the work of Prediger and Uribe (2021), the design heuristic consists of three steps:

Step 1. Identify key facets of the mathematical concept, by investigating this concept's informal and formal representations, standard interpretations, and typical student difficulties.

Step 2. Find language-related nuances of this concept by investigating the concept facets' linguistic realizations in multiple languages.

Step 3. Devise learning activities where these language nuances are intended to lead to conceptual insights, for instance in activities of comparing or connecting language-related nuances.

Exemplifying the design heuristic for equations

In this section, we provide an example of how the proposed design heuristic was used to develop an activity for the concept of equivalence.

Step 1. Identify key facets of the concept of equivalence

Equivalence and equations are key concepts in algebra education for which several relevant facets were identified from empirical research: Firstly, understanding equivalence requires students to understand the equal sign as a relation sign and to leave behind operational notions (e.g., Kieran, 1979). Secondly, equations can be regarded as both, a means to determine an unknown number, as in $2(x + 3) = 8$, and as a statement about the equivalence of two quantities, such as $x + 3 = y$ or $5 + 3 = 2 + 6$ (Weigand, et al., 2022). Thirdly, $x + 3 = y$ can also be interpreted as a co-variative relation between x and y , which students often develop in pattern generalization tasks (e.g., Carraher et al., 2008). Moving from operational notions of the equal sign towards a relational understanding is demanding for many students, also for students with diverse backgrounds (Stephens et al., 2022).

Step 2. Find language-related nuances of the concept of equivalence

Table 1 illustrates language-related nuances in Dutch and English (two languages in multilingual Dutch classrooms besides e.g., Arabic, Turkish, Berber).

Table 1: Language-related nuances for the concept equation

	English:	Dutch:
	Equations as statements about equality	Equations as comparisons of quantities
Typical language use	equation solve an equation equal 1. same quantity, amount (like in quality /nature/status); 2. identical in logical value (equivalence) informal talk to conceptualize equivalence as comparison of two sets: (amount x) is the same as (amount y)	Vergelijking (equation, nominalization of vergelijken), Vergelijking oplossen (solve an equation) gelijk (equal, similar, the same), vergelijken (to compare; seems not to be used together with equations in textbooks) een vergelijking van de lijn (a function-equation, literal: „an equation of the line”)
Language-related conceptual nuance	implies investigating a statement with respect to equivalence	implies comparing two quantities to find out whether they are equal (vergelijken) or to find the unknown such that the amounts are equal (gelijk)

The brief linguistic analysis of common word use suggests that, if we assume this language reflects students’ “thinking-language”, there are differences in how equations are conceptualized by them. Hence, even these two languages show relevant differences and highlight two language-related conceptual nuances, namely equations as statements and equations as comparisons of two quantities. It can be expected that further differences can be found when more languages are considered.

Step 3. Devise learning activities that utilize differences in language-related nuances

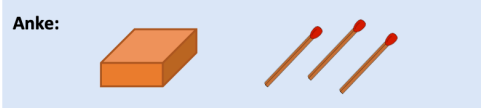
The learning activities presented here are designed for the students to work in language-homogeneous small groups where they share at least one additional language other than the Language of Instruction.

Task 1a. Discuss and solve the following situation in your home languages


How many matchsticks have to be in a box, so that Anke and Martijn have the same number of matchsticks?

Situation A

Anke:



Martijn:

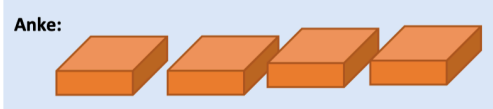


Task 1b. Discuss and solve the following situations in your home languages

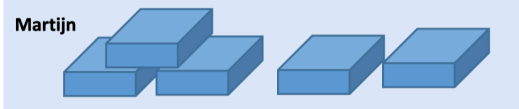
The students again compare their boxes. Boxes of the same color have the same number of matchsticks inside.

Anke and Martijn have the same number of Matchsticks.

Anke:

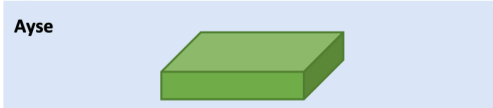


Martijn:

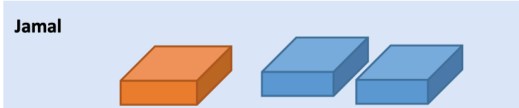


Ayse and Jamal also have the same number of matchsticks.

Ayse:

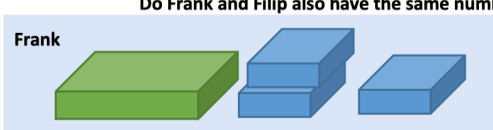


Jamal:

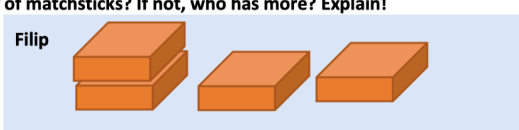


Do Frank and Filip also have the same number of matchsticks? If not, who has more? Explain!

Frank:



Filip:



Task 1c. Write in one language other than the language of instruction (Dutch) and translate back to Dutch

- Describe your strategy of finding the number of matchsticks in Task 1a in a written form.
- Write down your explanations for comparing Frank & Filip's boxes in Task 1b.
- Highlight the most relevant phrases in each text. Which phrases are common for describing both situations? You can also add other sentences, if you think they are suitable in these tasks.
- Translate your writings with google translate to Dutch. Use your phrase list from (c) to decide whether these translations are adequate. Give reasons why the translation is adequate or not.

Figure 1: Tasks for harnessing the epistemic potential of multiple languages

Accordingly, in a classroom, it is proposed that several small groups work simultaneously in different languages on the tasks. The tasks, originally written in the language of instruction (Dutch) (Figure 1) are designed to facilitate learners' conceptual understanding by engaging them in comparing and contrasting the identified language nuances for equations.

Task 1a emphasizes the nuance of comparing two quantities, while Task 1b emphasizes the nuance of equations as statements, where students are asked to investigate whether the last arrangement represents an equal number of matchsticks – accordingly, this arrangement is an implicit statement about the equivalence of two quantities. In Task 1c, students describe their strategies and explanations in their home languages and investigate the meaning-related language used in both situations.

Through this comparison of meaning-related phrases students might recognize differences in the conceptualizations of equations: In Task 1a, students describe removing (subtraction) or grouping matchsticks (division) on both sides. In Task 1b, students describe ways of systematically comparing quantities, which can also entail grouping, but in this case for the purpose of substitution. Finally, students are asked to translate their explanations. It may be that the translation tools make students descriptions/explanations less precise as they do not properly translate mathematical language. This can contribute to students gaining awareness that direct or loose translations may not describe adequately their ideas and so raises a discussion about the conceptual differences in Table 1.

Exemplifying the design heuristic for multiplicative thinking

Another example of using the design heuristic is in the development of a task about multiplicative thinking in relation to describing an algebraic pattern.

Step 1. Identify key facets of multiplicative structures for generalising algebraic patterns

Multiplicative thinking has been identified as an important component for understanding the abstract operations in algebra. As Callingham and Siemon (2021) stated “it is reasonable to think that algebraic reasoning would be strongly associated with multiplicative thinking through the notion of generalised arithmetic” (p. 3). Similarly, Booker and Windsor (2010) identified the need for students to move from additive to multiplicative thinking in order to generalise algebraic patterns. Consequently, being able to discuss multiplicative structures of equal grouping and how they contribute to describing algebraic patterns can be considered a key mathematical focus.

Step 2. Find language-related nuances of multiplication for generalising algebraic patterns

Multiplication can be described in many ways that can highlight different aspects such as how many times (multiplier) an amount (multiplicand) is increased by to determine a result (product). Understanding the different ways of expressing multiplication can contribute to a deeper understanding of the relationship between the parts. For example, Hurst (2017) stated that “being able to understand and articulate that four groups of seven is quite a different multiplicative situation to seven groups of four, demonstrates more powerful knowledge” (p. 7). Different languages emphasize the multiplier, the multiplicand or both in different ways in how they describe the multiplication.

However, many languages have multiple ways to discuss multiplication which also can highlight different facets, through different grammatical structures. For example, in English, it is possible to talk about “doubling, tripling and quadrupling”. In German, even a generalization of this is possible, “verdreifachen, vervierfachen, ... vervielfachen”. In Spanish this structure can be extended to indicate multiplication by numbers until 13 and by 100. Similarly in Italian and Korean (Sum & Kwon, 2018), the pattern can be extended to 5, 6 and upwards, although not all possibilities are commonly used. Other expressions are “double”, “twice” and “thrice”, which is similar to Russian. In English, the terms two-fold, three-fold are used rarely these days. Norwegian has an equivalent to three-fold and four-fold, with terms “tredobbelt” and “fireddobbelt”, where “dobbelt” describes how much something is multiplicatively increased by (as with “-fold”). In these words, the multiplier is separate from the process, either by the hyphen in the case of “three-fold” or the word formation in “tredobbelt”.

Another way to express multiplication is to have a separate word for the process, such as “multiple” or “times” which does not have a direct connection to the multiplier. When the process itself is high-

lighted, there is an opportunity to generalise from individual examples, focused on the resulting products from the specific kind of increasing being done. Götze (2019) showed that for a low-achieving student a focus on language which emphasised the general process of identifying “Vierer” (“set of four”, linguistically constructed by the number word “Vier” for four and the suffix -er) allowed the student to see the multiplicative structure of equal grouping as a general process. Nevertheless, even within one language, there are a variety of ways to describe the multiplication relationship. For example, in Polish, “pięć razy pięć” means “five multiplied with five”, “pięciokrotnie pięć” means five times five, “pomnożyć pięć, pięć razy” means multiply five, five times and “po pięć razy pięć” which literally means I take five with five but which is interpreted as five times five.


For pattern generalization, there needs to be an emphasis on equal groupings. The examples above illustrate the differences in linguistic constructions that emphasize the unitizing process between languages. Accordingly, these differences open up opportunities to deepen knowledge about the underlying structure of multiplication in describing generalisable algebraic patterns.

Step 3. Devise learning activities that utilize differences in language-related nuances

The learning activity is designed for students in pairs who may have different home languages. It would be formulated in the common language of the classroom. The activity is designed to support the students to share the ways that the languages that they are most familiar with in describing the multiplication process and how different ways could contribute to describing a generalisable pattern.

The results of the task would then be shared in a whole class session which could open up a discussion about what aspects of multiplication are highlighted in different languages.

In the country of Guest-hospitality, if someone turns up at your door, then you must feed them. Every family, therefore, keeps lots of tables and chairs so they can accommodate people for a meal on any specific day. Tables must be placed end to end because it is considered rude to put people on one table away from others. Generally, each rectangular table can accommodate four chairs along the long sides and one chair at each of the short sides, as shown in the diagram.



(1) Describe a rule in your own words for the relationship between the number of tables and the number of chairs, so that every family will know how many chairs they need.

(2) This problem was given to another class and Sandra described the rule as “to work out the number of chairs needed, double the number of tables and add four”. However, her partner Jan didn’t think this was correct. In Mandarin, he said the rule would be “椅子的数量是桌子数量的4倍再加2”, (literally: “The chair number is the number of tables 4 times plus 2”). Which student do you think is correct and why?

(3) In how many different ways and languages could you write the rule? Which one is the easiest to understand? Why do you think that?

Figure 2: Task for comparing language expressions for multiplicative structures of equal grouping

Discussion and outlook

In this paper, we build on the work of Prediger and Uribe (2021), to systematically design learning opportunities that can harness the epistemic potential of multiple languages, something that has been

recommended but empirically identified only occasionally (Barwell, 2018; Moschkovich, 2002, Planas, 2018). We suggest such a heuristic should be based on research where language is exploited for facilitating conceptual understanding (Prediger et al., 2019) and illustrate it with two examples from the topic of algebra. This heuristic emphasizes the importance of identifying the key facets of the concept in focus and the relevance of having the expertise for scanning different languages to investigate these concept facets' linguistic realizations. The language nuances are rarely provided by automated word by word translations, but by looking deeply in the way concepts are articulated in the respective languages, as the linguistic structures of these articulations might carry different nuances worthy of being compared and reflected upon.

The tasks presented here exemplify the design heuristic and illustrate different ways to include previously identified language-related nuances into the task-design for facilitating students to compare meaning-related phrases. In the first example, Tasks 1a and 1b address two different facets of a concept, which are differently highlighted by the *lexical* choices in the involved languages. Task 1c makes concrete use of languages as opportunity for comparing and contrasting key conceptual facets of the concept. The second example starts with another language (Mandarin) and opens to scrutinizing also the *syntactical* structure of the meaning-related phrases in further languages.

While these extract from our ongoing work provides examples for applying the design heuristic to task design, further design research will be needed in the future to investigate the reflections and learning processes initiated in the classes, also for other mathematical concepts and languages.

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Design of asynchronous teacher professional development and its effects on mathematics teacher learning

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To best meet student needs, mathematics teachers need professional learning opportunities that connect language, representations, and discourse in coordination with key mathematics content. In this paper, we share the design and effects on teacher learning of a set of two-hour asynchronous online mathematics professional learning modules that aim to impact teachers' mathematical knowledge for teaching linear functions and their instructional practices, with particular attention to mathematical discourse. We investigate teacher learning across three facilitation formats: project staff-facilitated, district leader-facilitated, or structured independent. Analysis of teacher learning, as evident in their written reflections, found teachers demonstrated learning of key practices related to teacher questioning and mathematical discourse; there were not significant differences by facilitation format. Implications for design of asynchronous professional learning are discussed.

Keywords: Algebra and algebraic thinking, classroom discourse, mathematical representations, professional development, teacher knowledge.

Affordances and impacts of asynchronous online professional development

High-quality professional learning is widely accepted as critical to effective teaching and learning and meaningful school reform (Koellner et al., in press). Online teacher professional development (PD) has the potential to increase teacher access to engaging, meaningful, practice-based professional learning opportunities. Teachers have stated their interests and enthusiasm for online PD, and in rigorous research studies comparing online offerings and traditional face-to-face PD, online PD has led to increases in teachers' attitude and self-efficacy as well as high satisfaction and relatively high levels of information sharing (Yoon et al., 2020). What is less clear are the key elements of the design of rigorous, practice-based asynchronous mathematics teacher PD when seeking to support teacher learning of practices that lie at the intersection of mathematics and language learning goals.

Theoretical framework

To meet the learning needs of mathematics teachers, it is critical that teacher professional learning opportunities are designed using research on effective teacher PD *in coordination with* research on effective teaching principles that meet the diversity of student needs. Hallmarks of effective mathematics teacher PD include attention to content and students, connections to instruction, active engagement, and collaboration with colleagues and experts (e.g., Darling-Hammond et al., 2017). Additionally, to meet student learning needs, PD needs to facilitate mathematics teacher learning of language-responsive mathematics teaching, particularly as related to mathematical discourse (Prediger & Neugebauer, 2021). Thus, we propose that mathematics teacher PD needs to attend to connecting language, representations, and discourse in coordination with specific mathematics content in order to meet the needs of all students, and the design of both instruction and PD needs to

integrate mathematics teaching practices with language-responsive learning environments (Erath et al., 2021). This highlights the intersection between mathematical tasks that promote reasoning and multiple entry points and effective teaching practices such as using and connecting mathematical representations, facilitating meaningful mathematics discourse, posing purposeful questions, and eliciting and using evidence of student thinking (NCTM, 2014).

Video in the Middle (VIM) project

The Video in the Middle (VIM) project is video-based PD that includes 40 two-hour online, asynchronous PD modules designed to expand teachers’ mathematical knowledge and their instructional practice related to linear functions. Specifically, the modules seek to facilitate teachers’ mathematical knowledge for teaching (MKT) (Ball & Bass, 2002) and teacher learning related to National Council of Teachers of Mathematics (2014) mathematics teaching practices (MTPs), a research-driven “core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics” (p. 9) that focus on mathematics teaching and learning and attend to mathematical discourse. Each module follows a common set of structured activities, where a video clip serves as the centerpiece of the professional learning experience as teachers engage online in mathematical problem solving, video analysis of classroom practice, and pedagogical reflection (Seago et al., 2018; figure 1).


Pre-video Activities	Video	Post-video Activities
<ol style="list-style-type: none"> 1. Introduction to learning goals 2. Explore math task and reflect in journal 3. Share your work on the math task 4. Consider other solutions and mathematical perspectives 	<ol style="list-style-type: none"> 5. Review the context of the lesson 6. Watch video and reflect 7. Reflect on the lesson graph and solution methods 8. Annotate video transcript 9. Watch video with math educator commentary 	<ol style="list-style-type: none"> 10. Module reflection 11. Bridge to practice

Figure 1: Set of activities in each VIM module

All modules are designed to provide teachers with repeated opportunities to examine and compare multiple mathematical representations around linear pattern tasks. Activities provide teachers the opportunity to construct meaning for new language when moving between the various representations and language registers (von Kügelgen, 1994; Prediger, 2022); this can in turn support related student instruction. For example, teachers examine and compare various solutions to linear pattern tasks across visual, tabular, symbolic and meaning-related phrases and use color-coding to note corresponding relationships across the multiple registers; they then consider how to facilitate students to examine and discuss similar relationships.


In addition, all activities and materials are explicitly rooted in teacher practice through the use of authentic, unedited videos of classroom interactions, and representation of a practice-based theory of professional learning (Ball & Bass, 2002). To varying degrees, the modules create opportunities for teachers to learn about each of the following mathematics teaching practices: implementing tasks that promote reasoning and problem solving; using and connecting mathematical representations; facilitating meaningful mathematical discourse; posing purposeful questions; and eliciting and using evidence of student thinking. For example, the VIM module *Lindsey’s Question: Connecting Geometry to a Rule* (figure 2), teachers begin by exploring a linear pattern task that promotes reasoning, facilitates meaningful connections from representations to mathematical registers, and encourages varied solution strategies. Teachers solve the task and examine each other’s solution

methods on an online Community Wall, then discuss and compare two solution methods with corresponding mathematical language, symbolic, and visual representations (figure 3). During the video analysis phase of the module, teachers make sense of two student methods and the language and corresponding representations used to connect to the geometry of the visual pattern (figure 3). In their journals, teachers consider and respond to prompts related to student mathematical thinking and language: *How does the teacher respond to Nick’s explanation of why the equation includes +2 and then Lindsey’s suggestion that adding 4 makes more sense? How does the teacher help students see the connections between the numeric pattern and the geometry of the triangle? How can the teacher support Lindsey to construct meanings for new language when she moves between the numeric pattern and the geometry of the triangle?* The final activity of the module asks participants to examine how two teachers use the *Triangles* task differently and how they posed focusing questions or “funneling” questions and then to write through their learnings about questioning practices.

VIM 3: Lindsey’s Question: Connecting Geometry to a Rule

Polygons

If I line up (sharing one side) 100 regular triangles in a row, what will the perimeter be?

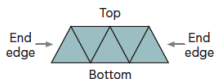


Can you create a rule for finding the perimeter for any number of triangles?

Learning Goals	Video clip Description
<ul style="list-style-type: none"> Make sense of how two different approaches to the general rule for the task connect to its geometry. Examine how a teacher responds to students’ ideas and questions. Consider how you might purposefully plan your questioning in order to elicit student thinking. 	<p>During the 7/8th grade whole class discussion of the triangle problem, Kristen says that the perimeter for any number of triangles would be the number of triangles plus 2.</p> <p>The teacher writes $t + 2 = p$ on the overhead. She asks the class why the rule says we’re only adding $t + 2$ when every time we add a triangle, we are adding 3 edges.</p> <p>Nick responds that two sides get closed off. Chris says that you have the top and bottom and you add two for the ends.</p> <p>Lindsey asks, “Why isn’t it plus 4?”</p>


Figure 2: VIM module task, learning goals, and video clip description

2 Top, bottom, and 2 end edges: $P = n + 2$



n is the number of triangles.
Each triangle contributes 1 side along either the top or bottom, so (tops + bottoms) = n .
There is also 1 side on each end.
The perimeter equals the total number of triangle tops + bottoms + 2.

3 Whole – shared sides: $P = 3n - 2(n - 1)$



$3n$ is the perimeter of each triangle times the number of triangles.
 $2(n - 1)$ is the number of shared sides, i.e., sides of the triangles not contributing to the perimeter.
 P is the total perimeter of each triangle minus the number of shared sides.

Figure 3: Solution methods representations

Research design

The VIM project provides an opportunity to learn more about if and how asynchronous online PD supports teacher learning, specifically when deliberately designed and with defined differences in facilitation formats and attention to integrating key instructional practices that relate to language and mathematical representations. The guiding research question of this study is: *How does VIM participation support mathematics teacher learning related to mathematical teaching practices, and how does teacher learning differ by facilitation format?*

Intervention and facilitation formats

The VIM project experience consisted of four sequenced modules (including VIM 3 described in detail above), where participants spent about two hours on each module, for a total of approximately 8 hours of professional development over the course of 12 weeks in Spring 2020. The four VIM modules emphasized key mathematics teaching practices and shared a common structure, activities, and set of design principles, as described above. The set of four modules were offered in three formats: (1) project staff-facilitated, (2) district leader-facilitated, and (3) structured independent.

All three facilitation formats reflect what is known about effective teacher PD (Darling-Hammond et al., 2017) and particularly mathematics PD (Heck et al., 2019; Hill et al., 2019). Key features of

effective PD were embedded in all conditions, and we conceive of these features as intersecting and connected. Activities and tasks are focused on content, connected to teacher practice, and encourage collective participation. Specifically, all three formats include a content focus, attention to bringing what they learned into practice, and the same asynchronous opportunities for sharing their solution methods and written reflections with colleagues. Facilitation formats differed in two ways: 1) pace, as teachers in the project-staff and district-facilitated formats were asked to complete one module per week, and those in the structured independent format worked at their own pace, and 2) the role of a facilitator. As each VIM module had a well-defined structure, reflecting research on teacher PD, attention to student thinking, and the importance of teacher reflection, the role of the facilitator was focused on responding to teacher posts and online discussions. The structured independent format was not monitored by a facilitator, and those teachers did not receive comments on their work or facilitator encouragement.

Participating teachers

Participating teachers taught middle school math, Algebra 1, or first-year high school math. Teachers in the district leader-facilitated condition were recruited by mathematics leaders from each of two school districts. Both leaders then served as the facilitator for their district group. Additional teachers were recruited from districts across California and randomized into either the structured independent condition or the project staff-facilitated condition. Where multiple teachers were recruited from the same district, teachers were randomly split between the two conditions. Where single teachers were recruited from a site, singleton teachers were matched by similar location or demographics; matched pairs were then randomized into the two conditions. Teachers in each of the two district leader-facilitated cohorts were from the same district, while teachers in the other groups were from different districts. Of the 68 teachers who began the study, 56 teachers (82%) completed all or nearly all activities across the four modules; completion rates were 80%, 80%, and 83%, for the project staff-facilitated, district leader-facilitated, and structured independent format, respectively.

Facilitators

There were two district-based facilitators and two project-based facilitators. All four had previous experience in facilitating face-to-face professional development, and the district-based facilitators also had knowledge of school and district contexts and goals as well as knew at least some, if not all, of the teachers participating in their format. In January 2020, all facilitators participated in a 90-minute video-conference orientation with other project staff. The orientation included an overview of the study and timeline, VIM module structure, and online tools. Facilitators also had access to a web-based facilitator guide and video tutorial demonstrating how to respond to participants.

Data and analysis

Across the project, multiple measures were used to gather impact data on teachers, including teachers' pre-post analysis of student work, their work on the mathematics tasks, module reflections, and post-study interviews. To understand more about teacher learning, we analyzed data from teachers' responses to end-of-module reflection prompts. Each VIM module concluded with two end-of-module reflection prompts: *What did you learn from this module? What new ideas do you intend to take/use from this professional learning?* Although the prompts were originally designed as a PD activity to support teacher learning and not a research measure, they offer insights into how teachers

made sense of their learning, how the VIM modules supported teachers' MKT, and what mathematics teaching practices *most* resonated with them.

Sixty-one teachers, across the three formats (18 district-leader facilitated, 17 project-staff facilitated, 26 structured independent) responded to at least one of the eight prompts, resulting in 446 end-of-module reflections, and 54 to 59 teachers responding to each prompt. Responses were loaded into MAXQDA and coded using the eight mathematics teaching practices (MTPs) (NCTM, 2014); in addition to being a valuable set of mathematics teaching practices and skills, the MTPs offer a framework for conceptualizing and identifying teachers' MKT growth and intended shifts in classroom practice. Coding for MTPs was a means to identify evidence of and differences in teachers' MKT across conditions. Two coders, blind to teacher condition, coded responses in batches of 10 to 15 teachers, detailing the coding document and reaching consensus for coding of all responses.

Findings

Across conditions, teachers showed evidence of learning consistent with the VIM modules. Over 92% of the 61 teachers who responded to the reflection questions gave at least one response indicating meaningful learning related to an MTP. This is notable given the open-ended nature of the prompts and that the prompts were written and designed as PD activity and not as part of a research instrument.

Evidence of learning of mathematics teaching practices

High percentages of teachers across conditions showed evidence of learning related to posing purposeful questions (MTP 5) (67.2%), implementing tasks that promote reasoning and problem solving (MTP 2) (46.7%), using and connecting representations (MTP 3) (39.3%), facilitating meaningful mathematical discourse (37.1%) and eliciting and using student mathematical thinking (MTP 8) (32.8%), areas that were emphasized in the VIM modules.

Teachers' responses highlight their attention to learning related to mathematical discourse practices specifically. For example, teachers across facilitation formats noted the importance of posing different types of questions (MTP 5) and being intentional about when to ask—and when to answer—certain questions:

I really examined better ways to pose questions to students. This module made me reflect on questions for information gathering, probing, making mathematics visible and reflection and justification of ideas. Being deliberate in your questioning, not necessarily asking questions to get the "right" answer can inform me as a teacher the process in which my students are thinking and problem-solving. (project-staff-facilitated; 425)

Participants noted the importance to mathematical discourse, suggesting “that it is more meaningful to have students discuss vocabulary than just teach it” (district-facilitated;121). Participants also made connections from questioning to discourse:

I learned the importance of allowing a student to explain their reasoning. Probing questions are essential to use to guide students to fully explain their reasoning. Without these questions, students may give responses that may not fully explain their reasoning... Promoting higher order thinking for students also can be difficult without prompting. We need have an idea of the questions we want to ask students before going into our lessons. (structured-independent; 527)

Similarly, as related to using and connecting representations (MTP 3), teachers highlighted their learning about color coding, noting patterns, and connecting to equations: “I learned that students developing their own diagrams and incorporating colors into not only the drawing but also the equations can be a very useful tool in making connections and clarifying” (district-facilitated -541). Representations were also connected to equations and mathematical discourse through clarifying.

MTPs across facilitation formats

Initial analysis of the percent of teachers in each facilitation format that showed evidence of MTPs in their responses demonstrated little variation of MTPs by teacher condition; for example, the percent of teachers that demonstrated evidence of practices related to facilitating meaningful mathematical discourse (MTP 4), ranged only from 30.8 to 41.2%, which translates to 7 or 8 teachers in each group. Similarly, the percent of teachers that demonstrated evidence of posing purposeful questions, which was the practice that was most referenced by teachers across groups, ranged only from 65.4% to 70.6%. Analyses of these differences by responses by format using chi-square tests were completed when the chi-square test assumption of minimum number of expected values in all cells was met (MTPs 2, 3, 4, 5, and 8). Results showed that the differences across groups were not statistically significant for these outcomes (e.g., implementing tasks that promote problem-solving and reasoning (MTP 2), $\chi^2 = 1.89$, $p = 0.39$; eliciting and using student thinking (MTP 8), $\chi^2 = 3.38$, $p = 0.16$).

Discussion

The evidence of MTP-related learnings after VIM participation, as designed and hypothesized, emphasizes how the online asynchronous VIM modules supported all teachers in learning about multiple teaching practices *across conditions*. Statistical analyses also do not show differences in evidence of MTPs by condition. Thus, while the percent of teachers who evidenced learning about a particular MTP varied across facilitation formats, these differences were not statically significant and suggest that at this time there was no differential impact for one facilitation format over another.

In particular, teachers evidenced learning focused on posing purposeful questioning (MTP 5), more than others, even as this practice was not the focus of more than one VIM module and other practices were also included in the VIM modules. This finding is notable as prompts were designed to elicit discussion of the learning that teachers found most salient, not necessarily to probe for evidence of learning related to *specific MTPs*. Furthermore, although these teaching practices are commonly viewed as overlapping and interrelated (NCTM, 2014), few responses across all three conditions were coded as clear evidence of learning related to more than one MTP.

Teachers’ attention to one specific practice, particularly on questioning, may be the result of a variety of factors. While each module was designed to stress one or two MTPs, many MTPs are interrelated and modules thus touch on several MTPs; depending on their background and context, some teachers may be in position to attend more closely to a practice that is not necessarily the primary focus of a module. As related to teacher questioning, this practice may be key component of eliciting student thinking, connecting representations, and facilitating student discourse, and thus it emerged as most salient to them. Teacher’s purposeful questioning supports students’ engagement in rich discourse practices such as explaining meanings, describing relations, and justifying decisions (Prediger, 2022).

Conclusion and implications

The VIM PD modules and the three facilitation formats were designed and structured following research-based structures and design principles and in particular focused on teacher learning of mathematics instructional practices. The trends in analyses of the MTPs in teachers' written responses show promising preliminary evidence of teacher learning related to MTPs and emphasize the strength of all three asynchronous online facilitation formats. This analysis provides initial impact evidence of independent, asynchronous PD, when it is well designed and structured. Responses also offer opportunities for further analyses of trends and additional themes, as teachers' responses from each condition were detailed, while varied. Future prompts designed to elicit evidence of learning related to particular practices may yield different results.

In addition to the positive findings across asynchronous online facilitation formats, these findings also prompt further discussion about what practices could be the focus in professional development to facilitate teacher learning of instructional practices related to mathematical discourse and leverage connections to mathematical language. The VIM modules emphasized mathematical discourse and representations, and the connections across representations and registers, and therefore can provide insight into further professional development that is explicitly focused on systematically developing student language use. For example, as teachers evidenced attention to questioning after the VIM modules, posing purposeful questions may be a practice that teachers readily take up when seeking support related to integrating language, representations, and the mathematics of linear patterns and functions. While teachers may find some mathematics teaching practices, such as building procedural fluency from conceptual understanding, less concrete, and some practices may be harder to define in a single short classroom video clip, the practice of posing purposeful questions may be one that teachers find accessible and visible, and thus perhaps may be a practice that can leverage teacher professional learning of other key instructional practices related to language and representations. Questioning practices may be a key area of focus in teacher PD that seeks to facilitate teacher implementation of new practices at the intersections of language, mathematics, and discourse, and particularly may be a practice to focus on in language-responsive mathematics teacher education in asynchronous PD.

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Interactive role of prosody in multilingual children's utterances in a New Zealand primary classroom

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This paper explores 9 to 11-year-old multilingual children's use of prosody (e.g., pitch, the loudness of voice, among others) as they engage in classroom interactions during geometry lessons on shapes and their properties in a multilingual classroom context. Transcribed data of one episode from six audiovisually recorded lessons is provided. The episode is analysed using a Conversation Analysis. The analysis suggests that multilingual children use the prosodic features of their multiple languages as interactional tools while engaging in interactions during geometry lessons to display their understanding of geometric shapes and their properties. Moreover, the analysis also provides insights into the fluid nature of emotional stances that children may hold about their knowledge claims. The paper suggests that children's use of prosodic features is a subtle yet powerful aspect of classroom interactions that influence children's learning in mathematics classrooms.

Keywords: multilingual, geometric shapes, primary school, conversation analysis.

Introduction

Research on classroom interactions has been majorly informed by studies from Conversation Analysis (CA) perspective (see, Mushin et al., 2013; Tainio & Laine, 2015). Studies from the CA perspective explore the overall interactional structure of the classroom talk, including the sequential organisation of conversation (Schegloff, 2007), as well as the construction of participants' utterances (Drew, 2013), by exploring the intonation, stress, volume and silence in utterances. Exploration of sequences of talk in mathematics classrooms has often reported a three-step interactional pattern of Initiation-Response-Evaluation/Feedback, abbreviated as IRE/F (Mehan, 1979) as part of the formal talk in the classroom. It is an interactional pattern, whereby the teacher initiates a sequence of talk, students respond, and then the teacher evaluates the students' responses. Research has also suggested that the rewording or reformulation of the statements by the teacher can be useful in providing students with appropriate feedback. Mushin et al. (2013), in their study on the role of language in Year 1 (5 to 6 years) oral assessments in an Australian Indigenous community school, found that the subtle factors – repetition or reformulation of phrases, word choice, falling and rising intonation patterns in teachers' utterances – could have a significant impact on the student's interpretations of the tasks which may influence the student's capacity to demonstrate their understanding of mathematical tasks. Additionally, research from the field of sociolinguistics informs us that the same intonation patterns may perform a variety of interactional functions. For example, in English, the use of high-rising terminal intonation at the end of an utterance is often interpreted as a sign for questioning (Ward, 2019). However, in the New Zealand context, a high rising terminal (HRT) intonation pattern may indicate the speaker's intention to check if the listener is following the speaker or as a way to develop communicational solidarity (Warren, 2016). Additionally, Tainio and Laine (2015) argued that the emotions and affective stances of teachers and students often differ in a classroom, even though the main purpose is to promote students' learning. They videotaped ten mathematics lessons in Grade 6 (11 to 12 years) classrooms in Finnish schools in Helsinki, Finland.

Using CA techniques, they studied the verbal as well as non-verbal aspects of classroom conversations during moments when students gave incorrect answers to teachers' questions, with a focus on the emotional states of students. They too suggested that when responding to students' incorrect responses "teachers, consciously or accidentally, display their stance towards student mistakes" (p. 84). If these stances are charged with embarrassment, as in the case of student mistakes, these utterances may negatively affect the student's emotion. Recently, and outside mathematics education research, Kamiloğlu et al. (2020) reviewed 108 published studies on prosody and its connection with positive emotions in English language, and argued that loud voice, pitch, and speech rate might indicate a variety of positive emotions. They argued, for example, that a higher pitch may indicate emotions of amusement, interest or relief, and a low pitch may indicate admiration.

In mathematics education research, with few exceptions (e.g., Tainio & Laine, 2015), the link between prosody and emotions in a multilingual context appears to be unexplored. This paper aims to explore the interactional role of prosody in children's utterances as well as to uncover the emotional stances embedded in children's utterances in a multilingual context of a New Zealand English-medium classroom.

Theoretical Framework

This paper draws its theoretical framework on the principles of Discursive Psychology perspective. Discursive Psychology views language-in-use (talk and text) as a domain of action in its own right rather than construing it as an outcome of mental states and cognitive processes (Potter, 2012). Thus, the focus of study from this perspective is on how people construct, understand, and display their understanding of the world around as they interact in "everyday and institutional situations" (Potter, 2012, p. 113). Within the Discursive Psychology perspective, Ethnomethodology (Garfinkel, 1967) informed this study. Ethnomethodology is the study of how reasoning and everyday activities are organised within a culture as identifiable events and occurrences. The ethnomethodological description provides a detailed description of how members make sense of any activity as it unfolds in its everyday manner. Barwell (2019) argued that it is the ordered nature of indexicality, that is the aspect of language that helps in interpretation of "what" is being said by focusing on "how" it is being said in a local context of particular activities, group membership and situations. Thus, the following research question guided this study: *How do 9 to 11-year-old children interact to construct their understanding of 2D shapes, 3D shapes, and their properties in a New Zealand multilingual primary classroom?*

Research Design

A qualitative research was designed for this study. The study took place in a Year 5/6 class (9 to 11-year-olds) at a New Zealand English-medium primary school. Informed and voluntary consent to participate in the study was sought from the participants following the ethical approval gained from the University of Waikato Division of Education Ethics Committee. Participants included fifteen children with their mathematics teacher (Pseudonym: Jenny). Nine of fifteen students were multilingual (01 Somali, 02 Tongan, 04 Māori, 01 Chinese, and 01 Filipino). Data was gathered using fieldnotes, audiovisual recording of lessons, and relevant children's work samples. Six geometry lessons on shapes and their properties were observed. Fieldnotes were taken for each of the lesson observed. Field notes are detailed descriptions of observations and interactions in the field that are

kept as a chronological log, and include a fuller description of settings and events along with the researchers’ analytic ideas, inferences, memos, personal feelings and reflections. All six geometry lessons were also audiovisually recorded. Audiovisual recording makes it easier to revisit the “close to reality” (Orel-Cass, 2018, p. 100) episodes of research sites to access the delicate, minute yet crucial details for examination and thus makes it a valuable data gathering tool. Data gathered through fieldnotes, audiovisual recording and children’s work samples enabled me to provide a holistic and rich account of the data for this study, which is crucial for establishing reliability of the findings (Onwuegbuzie & Leech, 2007). Participants’ utterances were considered as the unit of analysis. The presented episode was analysed using selected features of CA. In CA research, the term “turn” is used for participants’ utterances. The analysis of turn design explores how participants construct their utterance using words and prosodic features (which include pitch, volume of voice, silence) to convey the intended meaning and action. The audiovisually recorded episode from lesson was transcribed using a few selected features of Jefferson’s (2004) transcript convention for both linguistic (such as words, umms, gaps) along with prosodic features (such as high/low pitch, loud voice, creak voice). Overlapping talk is denoted by “[]”. Transcription in double brackets “(())” describes the gestures, body language that participants displayed. Prosodic features in children’s utterances were interpreted using insights from sociolinguistics research on English speakers and speakers of other languages. The analysis, in this way, allowed analysis of prosody in multilingual children’s utterances. Peer consultation was also sought to receive comment and feedback on the interpretation and analysis from the other CA practitioners and Māori colleagues to ensure the reliability of the findings. Additionally, research outside mathematics education has reported that prosodic patterns display different emotional stances of participants; therefore analysis of intonation patterns in participants’ utterances also allowed identification of the emotional stances that children held about their knowledge claims (Cresswell & Sullivan, 2020).

Analysis

This episode is taken from the audiovisual data of the second lesson. In this lesson, the teacher had provided children with playdough or sticks with adhesive to glue sticks together. During this episode, Zara (female 9-year-old Māori-English bilingual Māori child) claimed that the shape that she made was a “perfect square” (see the circled shape in Figure 1).

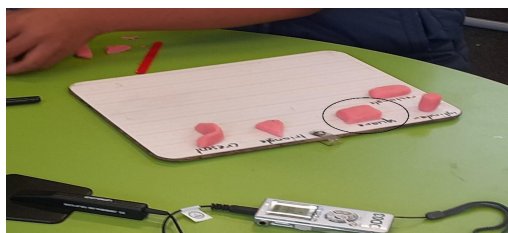


Figure 1. Zara's playdough shapes

The transcribed data show the classroom conversation that followed.

- 189 Zara >look whaea Jenny:< (1.0) whaea Jenny (.) a
- 190 perfect square ((shows the shape by holding it in her hands))
- 191 Teacher is it perf (.) why is it a perfect square? zara
- 192 Zara I dun↓no
- 193 Teacher what makes it a perfect square(2.0)>come on zara ↑I
- 194 need< to ↑kno:w(0.5)because you said its perfect so

- 195 what makes a perfect square a perfect square
 196 (1.0)=
 197 Matiu a ↑s[quare
 198 Teacher =[↑anyone ↑know why a perfect squa:re a perfect
 199 square
 200 Matiu becoz its a square?
 201 Garry (h)(h)
 202 Teacher yeah because its a square doesnt tell me
 203 much(1.0) ELIE what do you think
 204 Elie becau::se um: [if you have to= (2.0)
 205 Zara [you put on ((rolled her eyes))
 206 Elie =um: because um::(1.0) if you have the right type
 207 of shape. or if you (1.0)if or if you have(.2)
 208 havin:g a right(0.5)type of equipment (.) °you can
 209 have°
 210 Teacher °okay°
 211 Elie so:: if you are trying to make square of that
 212 one (1.0) you can roll into a ball then you
 213 start pressing it down the other side >the
 214 other side and you can [get square<
 215 Teacher [oh thank thank you Elie
 216 (0.5)↑can any↑one ↑tell me why a perfect square
 217 might be (0.2)might be perfect square using geometry
 218 language
 219 Zara [um ((looks at the roof trying to figure out how to say what she wants to
 220 say))
 221 Matiu [um: °its got°
 222 Teacher Matiu
 223 Matiu because the face °no:(0.2)the si::des°(2.5) nah
 224 °I dun know°
 225 Teacher yeah you re on the right track. the si:des what
 226 (.) what would the sides be here
 227 Matiu perfectly:: aligned? with each other?=
 228 Teacher =aligned with each other?
 229 Matiu ah(1.0) perfectly the same?
 230 Teacher perfectly the sa:me the sides ↑are perfectly
 231 the same

Zara self-selected and claimed that the shape that she had made of playdough was a perfect square (line 189-190). To her claim, the teacher responded with a question, thus initiating the IRE/F pattern in the classroom. The teacher initially structured her question to ask if the shape was a perfect square (line 191); however, in the same turn, she rearticulated her question as “why is it a perfect square?”, thereby leaving more space for children’s explanations. Zara, then (line 192), stated she did not know, by saying “I dunno”. She initially used a flat pitch and then lowered her pitch (denoted by↓). Ward (2019) has shown that this type of construction, using a flat and low pitch, is often made to signal the listener that the speaker has given in and is not able to provide any further explanation. Thus, it may be interpreted that Zara initially intended to show the shape (line 189) that she made was perfect in terms of the physical appearance of the shape as smooth and flat, and she was not expecting a question from the teacher about the shape. The teacher, in her next utterance (line 193), rephrased the question and emphasised “what” to encourage Zara to think about the shape’s properties. The teacher used longer pauses of two seconds (denoted as (2.0)) and one second (denoted as (1.0)), in line 193 and 196, in the same utterance to allow Zara to bring some explanation of her claim.

In the next utterance, Matiu (male 11-year-old Māori-English bilingual child) self-selected and stated that being a square makes it a perfect square (line 197). In the following utterance (line 198), the teacher appeared to have ignore Matiu's utterance. This may be because the teacher required children to raise their hands before speaking (Fieldnotes). The teacher used high pitch (denoted by ↑) at the beginning of "anyone" and "know" to open the floor for all children to respond (line 198). Ward (2019) showed that high onset is often used for initiating a new topic. This time, the teacher looked at Matiu and provided him with her consent to speak. Matiu responded that a perfect square is perfect because it is a square, as he stressed the word "square" (line 200) by using HRT (denoted by ?). The use of the HRT in English spoken in New Zealand often implies the speaker's intention to check if the listener follows what the speaker is trying to say (Warren, 2016). Thus, Matiu's use of HRT at the end of his utterance may be interpreted as his way to check with the teacher whether she agrees with his response. In the next utterance, it seems that Garry (male 11-year-old Filipino-English bilingual Philippines child) might not have understood the use of this specific intonation pattern in New Zealand English as he laughed at Matiu's response (line 201). Jefferson et al. (1987) showed that laughter within talk-in-interaction sometimes signals trouble, as the recipient engages in laughter to embarrass the speaker. In this case, it seems that Garry might have evaluated Matiu's response as wrong and redundant.

In the following utterance, it appears that the teacher did not accept Matiu's response as she said that "Being a square doesn't tell me much" (line 202). Matiu (line 200) had used HRT to seek approval from the teacher. The teacher selected Elie as the next speaker to answer "why a perfect square is a perfect square". In lines 204, 206-209, Elie used "um", stretches and pause of one second (denoted as (1.0)) to construct her utterance. These features are often a mark of a non-response (Sacks, 1987). Thus, it may be that she was not sure of what the teacher wanted her to comment on about the square, which may suggest Elie's emotional stance of doubt or lack of confidence. It appears that because Elie had not used geometry-specific language, the teacher did not accept her response but thanked her for her attempt (line 215). The teacher's utterance may be interpreted as an implicit rejection of Elie's response. The teacher did not overtly evaluate her response as incorrect. It seems that the explicit negative evaluation by the teacher was considered dispreferred. Moreover, the teacher rephrased her question (line 216) and stressed the words "geometry language" to direct the children's attention to the geometry-specific features of the shape that made it a perfect square. Following this cue, Zara and Matiu self-selected. However, Zara used "um" as a filler and started looking at the ceiling of the classroom in an attempt to recall the shape (line 219). Matiu (line 221) used "um" to hold the floor, and then he used his low tone (whispering) to state his utterance. The teacher selected Matiu as the next speaker (line 222). He attempted to answer (line 223) by emphasising the word "face", but then he changed the term "face" to "sides". He used his whispering tone for his utterance. Ward (2019) reported that speakers often use whispering at the end of their utterances to signal diffidence. Thus, Matiu's use of whispering tone (denoted by °) and pauses of 2.5 seconds (line 223) may suggest Matiu's emotional stance of lack of confidence or doubt about his knowledge claim. And as he was uncertain, Matiu, in this utterance, self-initiated a repair (a conversational mechanism to correct the use of the wrong word) (Kitzinger, 2013). He realised that he might be wrong, and therefore he stated that he did not know. In the following turn (line 225), the teacher provided positive feedback and again stressed the word "side" as she stretched it (denoted by :) and used slightly high volume to emphasize (denoted by underline) to children that the answer she was looking for was

related to the properties of a square in terms of equal sides. In doing so, she clearly showed her intent for children to use geometry-specific language by explicitly asking about the property of sides in the square. After receiving positive feedback from the teacher, Matiu responded that sides needed to be perfectly aligned with each other (line 227). However, this time as compared to his previous utterance (line 223), Matiu used HRT to check if the teacher agreed with him. It seems that the teacher acknowledged that Matiu might have been looking for agreement as he used HRT; thus, in the following utterance (line 228), she responded with a question to Matiu to let him reconsider his response. She used HRT at the end of her utterance, probably to signal the partial correctness of Matiu's response. It seems that Matiu (line 229) picked the teacher's cue about his answer as partially correct, and that he needed to restructure his response to meet the teacher's expectation. Thus, he used a filler and paused for one second to hold the floor while looking for the right word (line 229). He again used HRT with his utterance "perfectly the same" (line 229) to check with the teacher. This time, the teacher stretched the word "same" to emphasise its use (line 230). She used a slightly high pitch along with stretching the first syllable. Moreover, she reiterated the phrase "sides are perfectly the same" three times in her following utterance (lines 230-233).

The analysis of the presented episode revealed few important findings. Firstly, multilingual children may use prosodic features from their repertoire of multiple languages. Secondly, the analysis suggests that native and non-native speakers of English may perceive the same intonation feature differently, such as HRT. For example, in this episode, Matiu used HRT to check if the teacher agreed with him, rather than by asking a question. This intonation was apparently interpreted by Garry as a question or marker of uncertainty (Ward, 2019) in Matiu's response, which is probably why he laughed in the following utterance. Thirdly, the analysis shows that prosodic features may also suggest different emotional stances embedded in children's utterances.

Discussion and Conclusion

The analysis of the presented episode, first, suggest that prosodic features play a crucial interactional role in displaying children's understanding of geometric concepts. This finding is in line with the previous research. Mushin et al. (2013) too suggested that the intonation patterns in teacher feedback on student responses, and found that teachers often use different prosodic patterns with the same words used in the students' responses to achieve different interactive results. That is, the teacher may use slightly high pitch (or mid-level pitch) with the same words to indicate incompleteness of students' responses and to provide supportive feedback. This was evident in the teacher's response (lines 225 and 228) to Matiu's response in the presented episode.

Second, the analysis also seems to suggest that multilingual children to make use of a variety of prosodic features from across their repertoire of multiple languages. The study, therefore, suggests that the multilinguality of the classrooms like the one observed in this study with English as the language of instruction may be realised in the linguistic resources (including prosody) used by multilingual children. Moreover, multilingual children may perceive these prosodic cues differently from English-speaking children and vice versa. Therefore, the study also suggests that prosody may act as a demand for both monolingual and multilingual children in a multilingual interactional context such as one presented in this study.

Finally, the analysis suggests that the children displayed doubt, confidence, authority, and an inclination to give up a discussion when they lost interest in continuing the interaction. From a Discursive Psychology perspective, these emotional stances are observable indicators and practices that children use to show their emotions. The present episode suggests that the children's emotional stances were in flux and were constructed within the interactional moment. Thus, negative emotional stances, such as doubt or lack of confidence may also be constructed during interactions. Tainio and Laine (2015) have claimed that a teacher's display through prosody of dispreference for a student's incorrect response may lead to a negative emotional stance in students. The analysis in this paper suggests that prosody plays a subtle yet crucial interactional role in displaying children's understanding about geometry concepts. Thus, the paper suggests that it is important for teachers to be aware of the subtleties that have an impact on the construction of meanings and understanding about geometry concepts in a multilingual context, even when English is the language of instruction and language of majority.

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Multimodal Participation in Mathematical Negotiation Processes

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This paper contributes an expansion of a methodological approach for reconstructing possibilities for participating in mathematical negotiation processes. The production design based on Krummheuer (2007) was expanded by modifying it on the basis of current mathematics education research on multimodal communication in inclusive mathematics education. In this paper, a participation analysis is successfully conducted by using the expanded production design for the reconstruction of mathematical learning processes through an increasingly autonomous participation.

Keywords: Participation analysis, expanded production design, interactional theory, learning through negotiation.

Interactional theory of learning

From an interactionist perspective, processes of learning mathematics take place within social interaction. The participants of the interaction continuously interpret the utterances, gestures and actions with objects brought forth within the interaction and coordinate their mathematical interpretations of the respective situation they are in through mathematical negotiation processes. The negotiation processes can take the form of collective argumentations (Jung, 2018), leading to the mutual production of an interpretation that is considered as taken-as-shared meaning or an *interpretive interim* (Tewes & Schütte, in press; Schütte et al., 2021). This social act of interpretation, in which meanings are constructed through interactive negotiation processes, is understood as learning mathematics (Krummheuer & Brandt, 2001). The social interaction, and therefore the participation in such negotiation processes, constitute thus the starting point of mathematical learning processes (Friesen et al., 2019; Jung, 2018; Jung & Schütte, 2018). Participation as a fundamental concept within the interactional theory of learning mathematics (Krummheuer, 2007) is the focus of this paper.

Why participating in collective negotiation processes within the classroom is seen as enabling or promoting learning mathematics is explained by Krummheuer (2007) using Bruner's (1983) concept of *format*. Within a format, which Bruner characterizes as a standardized pattern of interaction, the individual is able to take increasing (autonomous) responsibility in producing subject-related contributions in collective negotiation processes (Jung & Schütte, 2018). Accordingly, increasingly autonomous participation in collective negotiation processes is an essential characteristic for learning mathematics (Jung & Schütte, 2018; Tewes & Schütte, in press). By reconstructing how students participate in mathematical negotiation processes, learning processes can be described. In order to reconstruct students' participation more precisely, this paper focuses on further developing a methodological tool used for analyzing the participation in social interaction. First, however, we will focus on multimodal communication and its significance for learning mathematics.

Relevance of multimodal communication for learning mathematics

Recent work and research highlight that participation in mathematical negotiation processes is multimodal (Fetzer & Tiedemann, 2017; Huth, 2014). This is based on the fact that communication⁴ in general refers not only to the use of oral or written language but also takes place via paralinguistic aspects, which include for example intonation, speed and emphasis of speech, gestures that we express with our face, hands or other body parts (Pimm, 2021), as well as actions with objects. In order to reconstruct learning processes more completely and comprehensively, the multimodality of communication should be incorporated more into mathematics education research. Intonation, speed and emphasis of speech are often included in research focusing on language, but what about gestures and actions with objects? What has research within mathematics education shown so far in relation to the use of gestures and actions with objects in social interaction?

For the analysis of social interaction, the distinction between gestures and actions with objects can be difficult, as both are mainly performed with arms and hands, as they can be interwoven and as objects can also be used for pointing gestures (Vogel & Huth, 2020). For this paper, gestures and actions with objects are to be viewed as separate, as the transition from one to the other could reflect learning processes. Actions with objects in this paper always refer to the use or modification of objects, whereas gestures refer to motoric simulation of actions, which can, for example, be gestures depicting actions or shapes, or gestures indicating locations (Salle, 2020). What is viewed as a gesture and how the meaning of the gestures is interpreted depends on the situation and is negotiated in the interaction by the participants (Kendon, 2004; Mead, 1934; Pimm, 2021). Gestures accompanying speech can communicate the same or the opposite or they can give additional information. In any case, they refer to one another (Pimm, 2021) and should therefore be included when analyzing a social interaction. In mathematics classrooms, gestures can provide additional information by, for example, specifying the “where”, the “what” and the “how” of a linguistic utterance (Krause, 2016; Huth, 2014) and can enable children to express information which they are not yet able to communicate verbally (Goldin-Meadow & Singer, 2003). Fetzer and Tiedemann (2017) show that objects also contribute to the mathematical learning processes. On the one hand, objects can help students express themselves more easily, and on the other hand, they can also challenge students to find new ways of expressing themselves and therefore encourage growth in both mathematical thinking and language use. In cooperative settings as well, objects can contribute to the process of negotiation by making individual interpretations more explicit and therefore more accessible for others (Fetzer, 2019). This research indicates that the use of gestures and objects can lead to different opportunities for students’ participation in social interaction and can possibly support mathematical learning processes. Therefore, analyzing which roles gestures and objects play in social interaction of mathematics classrooms is beneficial for understanding these learning processes and describing them more

⁴ In interactionist research within the field of mathematics education, the term *interaction* is often used instead of the term *communication* (Krummheuer & Brandt, 2001). In this paper however, the term (*social*) *interaction* is used synonymously with the term *communication* following (Mead, 1934), as both mean a mutual interacting process in which meanings are negotiated (for an overview see Böckmann & Tewes, in press).

accurately. In order to include the three modes – language⁵, gestures and actions with objects – in the reconstruction of students’ participation, the production design of the participation analysis by Krummheuer (2017) was expanded.

Expanded production design

With the help of the participation analysis, the syntactic or semantic responsibilities which individuals assume in their utterances and actions can be reconstructed (Krummheuer, 2007). This is done by analyzing who is responsible for the idea and for the formulation for each utterance. In order to be able to describe the responsibilities for an utterance and thus to reconstruct the participation in the mathematical negotiation process more precisely, multimodal communication was integrated into the production design⁶ of the participation analysis through which an *expanded production design* (figure 1) was devised (Böckmann & Tewes, in press). In order to develop the expanded production design, some changes needed to be made to the previously used production design. A first conceptual change is to include the view that communication is multimodal and therefore that individuals are able to participate in different modes. A person who is responsible for an utterance can communicate something not only linguistically but also gesturally (e.g., indicating silence with the index finger held in front of the closed mouth) or by actions with objects (e.g., sorting shapes). Therefore, the distinction is no longer made between speaking and non-speaking but between *communicators* and *responsibility with non-communicators*.

communicators							Responsibility lies with non-communicators					
		emergence	formulation			content			formulation			content
			language	gesture	action				language	gesture	action	
author	linguistic	+	+	-	-	+	-					
	gestural		-	+	-							
	acting		-	-	+							
paraphraser	linguistic	+	+	-	-	-	sponsor	linguistic	-			+
	gestural		-	+	-							
	acting		-	-	+							
ghostee	linguistic	+	-			+	ghostor	linguistic	+	-	-	-
	gestural							-	+	-		
	acting							-	-	+		
relayer	linguistic	+	-			-	deviser	linguistic	+	-	-	+
	gestural							-	+	-		
	acting							-	-	+		

Figure 1: expanded production design (see also Böckmann & Tewes, in press)

Furthermore, the categories used to describe this participation are henceforth called *categories of participation* instead of speaker categories. The categories of participation can now also include objects for non-communicators which, based on Fetzer and Tiedemann (2017), can be actors in the interaction. Objects can take responsibility for content (e.g., material that lies on the table in a certain predetermined structure) or even formulation (e.g., in the form of a text on a worksheet). Since a

⁵ In this mode, we include not only spoken language expressions but also sign language and written language.

⁶ The production design of participation analysis was originally elaborated by Krummheuer and Brandt (2001) based on *the participation format* of Goffmann (1981) and Levinson (1988). For a closer look to the previously used production design read Krummheuer & Brandt (2001) and Krummheuer (2007).

person can also assume responsibility for several functions (emergence, formulation and/or content), a total of seven categories of participation can be distinguished (figure 1). To better circumscribe the seven categories of participation with the broader theoretical view of multimodal communication, a conceptual change was also made in the naming of one category. The participation status *spokesman* (Krummheuer, 2007) is named *paraphraser* as this term better describes the fact that gestures and actions with objects can also be used to communicate. Someone is called a *paraphraser* if the individual is the communicator of an utterance and is responsible for its formulation, but the content of this utterance is taken from someone else. The individual whose content was reformulated is a non-communicator and has the status of a *sponsor*⁷. In order to be able to describe the complexity and multimodality of learning mathematics in interaction and more specifically the participation of learners in collective negotiation processes, it is furthermore significant to consider the three modes – *language*, *gestures* and *actions (with objects)* – in the participation analysis. The expanded production design therefore includes these three different modes of formulations (figure 1 – highlighted in gray). Whenever a person is involved in the interaction (as communicator or non-communicator) and is responsible for the formulation, the different modes of formulations are reconstructed more specifically. This leads to a specification of the participant categories, for example, an author is then a *gestural author*, *linguistic author*, or *acting author*. In addition, if a content is communicated, for example, by using language and an action, then both are marked as “*linguistic-acting author*”.

Participation Analysis using the expanded production design

In this section, we exemplify a participation analysis⁸ using the expanded production design to trace its possibilities for multimodal participation more specifically. In the analyzed scene (figure 2)⁹, Jella, who is in first grade, and the teacher are playing a game. The game consists of a cork board, various templates with different geometric shapes and the matching geometric shapes made of wood. According to the game instructions, the wooden shapes are to be placed on the matching shapes on the template sheet and then fixed with a nail.

Participation Analysis

In line 20, the teacher takes a red square from the box and shows it to Jella. She identifies the color (red) and names the wooden shape (square). In addition, she runs her finger along the edges of the square, drawing attention to the outer shape. Here, the teacher assumes the status of a linguistic-acting author, since she is responsible both for the idea of taking a square out of the box and classifying it on the basis of the outer shape, as well as for the formulation. Jella takes the square in her hand (l. 21) and the teacher then points to a red square on the template sheet (l. 22). She thus gesturally

⁷ A more detailed presentation of the different categories of participation can be found in Krummheuer (2007) and Böckmann & Tewes (in press).

⁸ The participation analysis is based on the interpretations reconstructed in the interactional analysis (Schütte et al., 2019) which is not included in the text and can be found in summary in Tewes (2020).

⁹ The transcript is derived from the data of Ann-Kristin Tewes' dissertation project which focuses on support systems in inclusive mathematics education.

indicates that the shape in Jella's hand should be placed on that spot. Here, the teacher is a gestural author, as she introduces the idea that shapes which are congruent and have the same color should be matched with each other. She formulates this idea with a pointing gesture. Jella transforms the teacher's pointing gesture into an action in line (l. 23). Jella, therefore, has the status of an acting paraphraser, with the teacher being the gestural sponsor (l. 22). With the expanded production design, it is possible at this point to define Jella's responsibility for the acting formulation more precisely and thus to make Jella's co-responsibility in the negotiation process more visible.

Figure 2: Transcript with Participation Analysis

After Jella and the teacher nail the red square to the template sheet, the teacher asks Jella in line 43

	person	utterance	categories of participation
20	T	here is a red square [<i>takes a red square and holds it up with the left hand</i>] this is a square [<i>traces the edges of the square with the right index finger and moves the square in Jella's direction</i>]	author (linguistic-acting)
21	Jella	< [<i>takes the square with her hand</i>]	
22	T	< [<i>taps with the right index finger on the red square on the template sheet at the top left</i>]	author (gestural)
23	Jella	[<i>place the square where teacher pointed it at the top left of the template</i>]	paraphraser (acting) T is sponsor (gestural) <22>
<i>Jella and the teacher are fixing the square on the template sheet</i>			
43	T	then take [<i>runs the left index finger on the right where the shapes are</i>] then take	author (linguistic-gestural)
44	Jella	[<i>reaches for the geometric shapes</i>] there / [<i>smiles and looks at the teacher and takes a yellow triangle and holds it up</i>]	paraphraser (acting) T is sponsor (linguistic-gestural) <43> author (acting)
45	T	what is that/	paraphraser (linguistic) T is sponsor (linguistic-acting) <20>
46	Jella	[<i>takes the triangle and puts it with one corner to her forehead</i>] angly	paraphraser (linguistic-acting) T is sponsor (linguistic-acting) <20>
47	T	a yellow triangle	paraphraser (linguistic) Jella is sponsor (linguistic-acting) <46>
48	Jella	hmm [<i>looks at the template sheet and alternately takes the triangle in the left and right hand, then takes the right hand a little forward, looks at the triangle and then at the template, then takes the triangle in the left hand and places it on the template sheet on a yellow square</i>]	ghostee (acting) Jella is ghostor (acting) <23> T is ghostor (gestural) <22>

to take a new shape and supports this with a pointing gesture. Here, the teacher is a linguistic-gestural author. Jella then reaches for the shapes and takes a yellow triangle from the box, again translating the teacher's gesture into an action that she performs herself (l. 44). Jella therefore has the status of an acting paraphraser and the teacher that of a linguistic-gestural sponsor (l. 43). This utterance again shows the importance of integrating the different modes into the production design, as it is now possible to reconstruct Jella's responsibility for the acting formulation and to show which formulation (linguistic and gestural) she has transformed into a new mode. However, Jella takes a new shape from the box and is thus responsible for choosing a yellow triangle. Thus, in addition, she can be here described as an acting author (l. 44) for the idea of taking a triangle. In line 45, the teacher expresses

the idea of naming the shape but uses a different linguistic formulation. Therefore, she is both linguistic paraphraser and linguistic-acting sponsor (l. 20). Jella takes the yellow triangle in line 46 with an edge to her forehead, thus feeling the outer shape of the triangle, and names it. Here, she adopts the teacher's idea of naming the figure based on its outer shape (l. 20) but expresses this in a different linguistic and acting formulation. With the expanded production design, Jella is a linguistic-acting paraphraser and the teacher the linguistic-acting sponsor (l. 20). The teacher then also names the shape but uses a different linguistic formulation and is thus a linguistic paraphraser (l. 47), while Jella is a linguistic-acting sponsor (l. 46). In the last line of the transcript, Jella matches the yellow triangle to a yellow square on the template sheet. She performs the same action as before (l. 23) but is responsible for a new idea for matching the wooden shape and the shapes on the template. Jella can therefore be called a ghostee since she transfers the same action to a new figure and is responsible for the new match. With this match, she shows the idea that the properties angular and yellow belong together and that the shapes do not necessarily have to be congruent. Both Jella and the teacher are responsible for the original formulation and are therefore ghostors – Jella is an acting ghostor (l. 23) and the teacher a gestural ghostor (l. 22).

Conclusion

The participation analysis with the expanded production design shows that it enables more precise statements about the possibilities of participation. Through this more detailed analysis, it can be for example shown that Jella transforms gestures, which were shown by the teacher, into actions with objects (ls. 23;44). However, by using the expanded production design with its subdivisions into linguistic, gestural, and acting formulations, it becomes apparent that Jella uses different modes of formulations than the teacher and thus is a paraphraser. She can therefore be attributed more responsibility for an utterance. With the reconstruction of the different modes of formulation, a more precise picture of Jella's learning process, and thus an increasingly autonomous participation in the mathematical learning processes, can be drawn. In summary, using the participation analysis with the expanded production design and by distinguishing between the modes of formulation, one's responsibilities for an utterance can be better revealed and thus more autonomy in participation becomes visible. This is of significant importance especially with regard to the interactionist learning theory, which sees the increasingly autonomous participation in mathematical negotiation processes as a characteristic of learning mathematics (Jung & Schütte, 2018; Krummheuer, 2007). With reference to the elaborated theoretical foundations on the importance of gestures and actions with objects in mathematics education, we also believe that the expanded production design is suitable for tracing learning processes in inclusive mathematics education. Especially for students who need language support or express themselves primarily through gestures and object-related actions, the expanded production design can be used to reconstruct more specific responsibilities for utterances and thus an increasingly autonomous participation in mathematical learning processes.

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Language as a resource or as a problem? The case of Sofia

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Motivated by an interest in understanding what language means to teachers in the mathematics classroom, a qualitative case study was carried out to surface the existing state of one secondary teacher's knowledge, orientations, and dilemmas in relation to language (in particular, to the mathematics register). Two semi-structured interviews were designed and conducted to gather data on what the teacher noticed as possible language-related dilemmas in relation to the teaching and learning of mathematics, and the corresponding actions she may take to overcome these problems. Through a discussion of her language-related dilemmas and actions, I was able to glean a better understanding of her perspectives of language – as a resource or as a problem – in her mathematics classrooms. Consequently, her knowledge and orientations in relation to language seemed to be key in explaining the actions she would take in managing or overcoming these dilemmas.

Keywords: Mathematics register, teachers' knowledge, orientations, teaching dilemmas.

As a teacher, I have always been interested in understanding the role of language in the teaching and learning of mathematics. Other than noticing the increasing emphasis on communication with mathematical language in curriculum reforms, it is also intriguing how often I heard the comment “it is a language problem”, in discussions with colleagues regarding why students were not able to understand mathematical concepts or problems. This interest was piqued further when I was introduced to the notion of the mathematics register in the book *Speaking mathematically* by David Pimm (1987). Through discussing the “often-heard claim that ‘mathematics is a language’” (p. 2), Pimm brought attention to how the *mathematics register* (first coined by Halliday, 1975, p. 65), may more accurately represent what is commonly referred to as the mathematical language. In particular, the *mathematics register* is deemed to serve the function of thinking about (and communicating in spoken or written forms) mathematical ideas and meanings (Pimm, 1987), i.e., the mathematics register can be considered a tool for thinking and communicating mathematics (Vygotsky, 1934/1986), and thus be viewed as an important resource for mathematics education.

Yet, research seems often to have framed language as a problem, rather than a resource, in mathematics education, echoing my personal interactions with language as a teacher. Considering how language (framed by the mathematics register) may constitute both a resource and a problem in mathematics education, I am keen to understand what language means to teachers by exploring their knowledge, orientations, and dilemmas in relation to language in mathematics classrooms.

Teachers' knowledge, orientations, and dilemmas in relation to language

With the mathematics register being positioned as an important tool for mathematical thinking, it is consequently a rich resource for mathematics teaching (and learning) with appropriate mediation. Yet the register will remain a theoretical notion if teachers do not understand “the forms and the meanings and ways of seeing enshrined in the mathematics register” (Pimm, 1987, p. 207), i.e., have the necessary (and sufficient) knowledge of the register, or certain orientations towards seeing language as a resource rather than an add-on or even a problem.

In particular, Lane and colleagues (2019) have argued that the mathematics register is an important aspect of pre-service teachers' mathematical knowledge for teaching. They adapted the Knowledge Quartet (Rowland et al., 2005) – originally used to analyse pre-service teachers' mathematical content knowledge – and proposed a framework which attends to the mathematics register as the key aspect of knowledge. In particular, they redefined how teachers' knowledge of the mathematics register can be analysed and discussed in relation to the four dimensions of the Knowledge Quartet: (a) the *foundation dimension* focusing on teachers' knowledge and understanding of the register and their awareness of differences between everyday language and the register; (b) the *transformation dimension* focusing on teachers' planning and actual teaching, in terms of how they plan for mathematical language and use representations and analogies to elicit mathematical meaning; (c) the *connection dimension* focusing on teachers' consistency in the use of the register within and between lessons, and across different mathematics topics, coupled with an awareness of students' difficulties; (d) the *contingency dimension* focusing on teachers' abilities to interpret students' register in line with the mathematics register and facilitate an adherence to the register during classroom interactions (see Table 1 in Lane et al., 2019, p. 793, for the complete mathematics register knowledge quartet).

On a similar note, Prediger and colleagues (2019) discussed how teachers' language-related orientations are likely to lead to different focus or treatment, in terms of pedagogical approaches and actions, of language as a resource for mathematics teaching. Consequently, they identified five language-related orientations as being crucial in influencing teachers' practices and actions. These include the extent to which mathematics teachers assume responsibility for *language learning as a goal* in their classrooms; *strive for pushing rather than reducing language* in relation to language demands within their classrooms; *focus on the discourse level rather than on word level only* in learning the mathematical language; have *integrative perspectives instead of additives only* to learning language in their classrooms; focus on *conceptual understanding before procedures* which necessitates the use of language (i.e. the mathematics register) as a resource for mathematics teaching.

Other than having the necessary knowledge and orientations, having an awareness of language-related issues or dilemmas that will arise in the mathematics classroom will also be important for teachers to make more informed pedagogical decisions which tap on language as a resource to develop mathematical thinking. Notably, the existence of certain language-related teaching issues or dilemmas was first suggested by Jill Adler (2002) in her work, primarily in the contexts of multilingual classrooms. She identified three specific language-related dilemmas as commonly faced by teachers in mathematics classrooms: (a) the *dilemma of code-switching* where teachers need to decide whether to change the language of instruction to develop students' mathematical understanding and compromise the learning of the mathematics language; (b) the *dilemma of mediation* where teachers need to decide whether to intervene to validate students' meanings during group discussions and compromise students' opportunities to develop mathematical communicative competence; (c) the *dilemma of transparency* where teachers need to decide whether to teach the mathematics language explicitly and compromise the development of mathematical understanding. While these dilemmas were surfaced through her work in multilingual classrooms, she suggested that they can be similarly faced by any teacher who attempts to use the mathematics register as a resource for teaching and learning in the mathematics classroom.

A small exploration

As a preliminary attempt to address my research interest, I chose to explore what language means to a particular secondary mathematics teacher in her mathematics classrooms within this paper. With reference to how teachers' knowledge, orientations, and dilemmas in relation to language (with attention given to the mathematics register) can influence their perspectives of language as a resource or a problem in mathematics teaching (and learning) as discussed above, I drew on aspects of these three theoretical constructs in my exploration and analysis. My initial hypothesis was that a discussion of her language-related dilemmas will lead me to understand the language problems she faces in her mathematics classrooms; and her knowledge and orientations will be key in explaining the actions she takes in managing or overcoming these problems. Correspondingly, this will provide insights to her perspectives of language – as a resource or as a problem – in her mathematics classrooms.

The data for this exploration was gathered through two semi-structured interviews with Sofia (pseudonym), an experienced English-speaking high school mathematics teacher, via Zoom. As a student in a Mathematics Education PhD program, she is interested in, and has been exposed to, many discussions on language in mathematics education, which made her an interesting case as a pilot study. In order to analyse her knowledge, orientations and dilemmas towards the use of language (framed by the mathematics register) in mathematics teaching and learning, I needed to elicit these two sets of information from Sofia: her background, beliefs and experience in teaching and learning mathematics which will also contribute to her knowledge of and orientation towards the use of language in her teaching practice; her experience with and in managing language-related dilemmas, including Adler's three types of dilemmas, though not exclusively.

Initially, only one interview was planned. However, as the initial questions turned out to be rather broad, the data from the first interview did not seem focused and lacked evidence to support my hypothesis. Hence, a second interview was conducted to clarify interpretations which arose from the first interview; and to collect more focused responses with the use of specific reflection tasks, which would mirror the site of teaching practice – the mathematics classroom – without observing Sofia in action. The tasks were presented in the form of hypothetical classroom-based dialogues with elements of possible language-related dilemmas embedded. These dialogues – which I wrote after considering my personal teaching experiences and the literature (e.g. Pimm, 1987) – served as triggers in the interview to enable a deeper reflection about her own experience with the use of language as a resource (or not). Data for analysis was subsequently created with an initial careful listening for key ideas which led to decisions made to transcribe relevant segments of the interviews in detail as specific evidence for the entire analysis. These key ideas and transcribed segments were eventually analysed using the constructs of teachers' knowledge (Lane et al., 2019), orientations (Prediger et al., 2019) and dilemmas (Adler, 2002) in relation to language. Due to the size of this paper, the findings will only be supported using short quotations from the interview transcripts where necessary.

The case of Sofia: Her backgrounds, beliefs, and experiences with language

Sofia is a mathematics teacher who has been teaching in both elementary and secondary mathematics classrooms for more than twenty years. She comes with a diverse (not fully mathematics) educational background, though becoming a mathematics teacher has always been her aspiration. Language was never an issue for her in her own learning of mathematics as she has always been able to make

connections in terms of the specific mathematics vocabulary to the everyday language in her own way. Teaching in the context of Canada with a high proportion of immigrants, her classrooms are typically multilingual with the presence of international students (i.e., English Language Learners). In contrast, she is effectively monolingual and has a very good command of English.

Sofia is not the most traditional teacher among her peers as she believes and adapts learner-centred pedagogies in her classrooms, though she thinks that some teacher-directed teaching is still necessary. Hence, one is likely to see a blend of both pedagogical approaches in her classrooms. With a strong belief that students should be taking the lead and working with each other to learn mathematics, she has been taking time and effort to inculcate a thinking classroom culture. Unfortunately, she is typically the only one adapting such pedagogies within her schools and most students are not used to learning mathematics in a learner-centric environment. Moreover, she is concerned with how students tend to see mathematics as an academic subject which they need to pass and has no other purpose in their lives, thus leading them to have certain fixed ideas and expectations of mathematics and how it is usually taught. Thus, she highly advocates the need for students to appreciate and respect the subject for what it is beyond meeting only curriculum needs. In addition, she always aims to make the learning of mathematics both meaningful and accessible to different students based on their individual interests, needs and goals in life so as to help students make individual connections with mathematics as possible and to reduce the fear or anxiety they have for the subject.

Her knowledge of the mathematics register

Sofia described the register as “words from the language we speak that are used particularly for math” and that “they are a subset that have special meaning”. She also recognises that it is based in language as she said, “there’s the language, it’s word-based, it’s you know, it doesn’t include symbols”. Sofia’s mathematics register proficiency was further analysed with the dimensions of the mathematics register knowledge quartet (Lane et al., 2019). Within the *foundation dimension*, Sofia clearly demonstrates knowledge of the various specialised terms with mathematical meanings – such as *asymptote* or *denominator* – residing in the mathematics register. She is also cognizant of the differences in interpretations of existing words from the everyday language in English, in a metaphorical sense, such as *stretch* or *rationalise*. Moreover, she shows an understanding in and fluency with the use of composite words and phrases in the mathematics register, such as rationalise the denominator or vertical compression. This is evident from how she has used these words and phrases to convey certain mathematical meanings during the interviews.

In terms of the *transformation dimension*, Sofia shared that she often customises how she explains mathematical concepts based on her students’ age and their readiness and familiarity with the mathematics register. She also utilises stories to help her students access and make sense of both the mathematical concepts and the mathematics register. Specifically, she mentioned how “there is explicitness in the stories [...] and you eventually give them a place to build their knowledge on”.

Although it did not surface explicitly in the interviews that Sofia plans for the use of the mathematics register in her classrooms, her responses seem to indicate that the register is incorporated into her daily pedagogical strategies and teaching practices. Her deliberate choices and usage of examples, explanations (and gestures) to help students connect mathematical concepts, mathematics register and everyday register are also representative of her knowledge-in-action (Rowland et al., 2005) with

respect to the register, within the *connection dimension*. Sofia shared how she always tries to make connections between “the common terms” to “the proper registration” in her classrooms. Her emphasis on these connections seems to be attributed to her awareness of the difficulties students often face with the mathematics register – a key aspect in the *connection dimension*. She elaborated by explaining how students may be confused between the mathematical meaning and the everyday understanding of an English word, e.g. *stretch*, as “it doesn’t mean just getting bigger” in mathematics contexts all the time. As such, she is aware of and consistent in her use of the mathematics register between lessons and allows sufficient time for her students to make sense of it and develop proficiency. In addition, Sofia is likely cognizant of the relationships or connections across topics with respect to the mathematics register, as she was able to relate the process of rationalising the denominator with the notion of rational and irrational numbers.

Lastly, for the *contingency dimension*, Sofia is certainly able to understand and relate students’ use of everyday language with the appropriate mathematics register. She even uses it as her pedagogical strategy to facilitate students’ development of the mathematics register by deliberately intertwining it with everyday language in her interactions with her students. In particular, she mentioned how she is “continuously using what they say, fatter or skinnier, but intertwining it with what does it mean, when something is fatter, in the case of quadratic, it’s being vertically compressed”. In essence, Sofia demonstrates strong mathematics register proficiency within all four dimensions of the knowledge quartet and can meaningfully transform that knowledge into practice.

Her orientations towards language in mathematics classrooms

From what Sofia shared, I was able to obtain a glimpse of her language-related orientations (Prediger et al., 2019). Firstly, it is evident that *language competency is a learning goal* in Sofia’s mathematics classrooms. She sees it as her responsibility as a mathematics teacher to help develop students’ language abilities. She will introduce and discuss words or phrases, e.g., *approaches*, in the mathematics register that also have their everyday meaning and usage. She also views language as an important and valuable aspect of mathematics, beyond the mathematical understanding she wants her students to develop in her classes. In particular, she relates the value of language to communication and to thinking when she highlighted “from commognitive terms, communicating is thinking”.

Moreover, Sofia believes in *pushing for language rather than reducing* it although she is cognizant of the “weak language, weak language structures” that deter many of her students. This is evident from how she maintains the stand that students are expected to use the mathematics register in her mathematics classrooms. Another motivation for Sofia’s emphasis on language in her classrooms is probably the long-term utility of language. To her, the notion of register is not unique to mathematics and students will always need to learn and acquire the different registers in various disciplines.

From how she values language learning as a part of her mathematics classrooms, it is no surprise that Sofia adopts an *integrative* rather than *additive* approach in doing so. For instance, she thinks it is more relevant to incorporate her school’s literacy goals into her mathematics classrooms through the functional use of the mathematics textbook, rather than teaching it as an add-on. She will ask her students to read the textbook and use it as a resource to develop both their understanding of the mathematical concepts and their ability to become “discerning readers”. Sofia also prefers to integrate the teaching of the mathematics register in her classroom discussions by talking about them when the

situations arise, emphasising both the word and meaning. Moreover, she does not insist that her students not to “use a word” and “only use the word” though she will certainly highlight the importance of using the mathematics register to think about and communicate the appropriate mathematical ideas and meaning. Such evidence suggests that her focus is likely *on the discourse level, rather than on the word level only* in terms of the use of the mathematics register as a resource in her teaching practices. While her orientation towards the use of the register is likely towards the development of *conceptual understanding before procedures*, there was no specific discussion in the interviews which provided evidence for this orientation.

Her experience with and in managing language-related dilemmas

Sofia’s reflections of her teaching experiences indicate that she does have encounters with the different language-related dilemmas (Adler, 2002) and faced tensions while managing them in her mathematics classes. Interestingly, for Sofia, code-switching across languages did not surface as a dilemma even though she teaches mathematics in a multilingual context. In cases when students have difficulties understanding what is taught due to language issues, she shared how help in translating will be offered by the peers. Moreover, the teacher as the translator in such cases is probably not an option as she is monolingual. Instead, the *dilemma of code-switching* between everyday language and the mathematics register (Zazkis, 2000) created slight tensions for Sofia. In managing this dilemma, her actions primarily resided upon students’ readiness and needs. By talking about the need for a “transitional language”, with a corresponding strategy of “talking normally” so that the students do not “feel bad”, it suggests that she will use the everyday language mainly to garner students’ attention first before using the mathematics register. However, she also mentioned the importance of deliberately code-switching between the mathematics register and the everyday language frequently in her teaching as they are “intertwined” so there is a need to be “always moving the two together”.

The *dilemma of mediation* appears to be the most apparent for Sofia in her mathematics classrooms. When asked to think about what she will do when she happens to hear that students are not using the mathematics register in group discussions and her related concerns with such an issue, she says she will likely mediate in such situations, either as a class or with specific students. While her decisions to mediate depend very much on whether these students are making sense even though they may not be using the appropriate register during the discussions, she seems to struggle in terms of when and what to mediate. The timing and extent of mediation appears to depend on the students’ level of understanding, e.g., whether they are at the phase when they are still developing understanding with the use of “transitional language”. It may also depend on the needs of specific students (e.g., someone who is aiming to be an actress vs. an engineer). Yet, when I clarified about the kind of actions she would take in meeting the needs of the students, she further added how she is “balancing *it* because they are doing the traditional tests” as “the non-traditional stuff and register that they (students) use can influence their marks but it’s not what everything is based on”. This suggests that, within the dilemma of mediation, her greater struggle seemed to be between the type of students she is working with and the traditional assessment that they must take at the end of the day. While she thinks that she does not need to “force” students who are not planning to study science and engineering to use the mathematics register, she needs to, in some way, ensure that these students still understand, and at least be familiar with, the minimum necessary so that they can do examinations. She further elaborated on the tension she faced specifically in the dilemma of mediation with students who have

weak language proficiency. On the one hand, she needs to help these students to understand mathematical meaning by using more everyday language. On the other hand, she sees potential in how tapping on the mathematics register may help build their language abilities. Specifically, she shared how “taking the language in math and going outwards can help them learn *approaches* is a really good word to use normally, and in essay, and in any subject”.

As for the *dilemma of transparency*, it did not seem to create any tension to her as she expressed that she does not believe in teaching the mathematics register explicitly when asked if she does so in her teaching. Instead, she sees the modelling of the use of the mathematics register as a crucial pedagogical move in the teaching and learning of mathematics and consciously does that explicitly in her classrooms. As such, she may have unknowingly managed the dilemma of transparency through her modelling of the language use.

Overall, while Sofia has had experiences with language-related dilemmas, language did not seem to be framed as a major problem in her mathematics classrooms. This is probably attributed to her keen interest in language in mathematics education which has inevitably led to an increased awareness of the role of language in mathematics classrooms, thus attributing towards the type of language-related orientations she has as a mathematics teacher. Specifically, in her interviews, she, more than once, highlights *the role of language as a resource* for mathematical thinking and communication; and the importance of emphasising both the word and the word meaning to help students make sense of the mathematical concepts. Moreover, as she has a strong knowledge and understanding of the mathematics register, she can fluently switch between the mathematics register and everyday language to help students make the necessary connections. As such, the dilemma of code-switching did not seem to bother her to the extent of becoming a language problem.

In comparison, she certainly faces more tension in terms of the dilemma of mediation, in relation to the use of the mathematics register in her classes. Driven by her motivation of helping students appreciate and respect mathematics for what it is beyond the curriculum, she tends to deliberate over how and what to mediate in terms of the use of the mathematics register. Consequently, she is likely to have to mediate differently to help different students form different connections, depending on their individual needs and goals. Yet she also needs to ensure that students acquire a minimum proficient level of language (though not specifically in terms of the register), which she deems as an important tool for thinking and communication.

Finally, her indifference towards the dilemma of transparency, as a possible language problem, may probably be explained by her integrative rather than additive language-related orientation (Prediger et al., 2019) in her teaching practices. She does not see a need to explicitly teach the mathematics register since she is naturally inclined to integrate (the teaching of) the register into appropriate contexts so that students can better relate to and not learn the register in isolation. Her orientation of having language as a learning goal in her mathematics classrooms may also have led to her implicit move in making language a visible resource in the teaching and learning of mathematics through her conscious modelling of the register, even though she may argue that that is not explicit teaching.

Some concluding thoughts

This exploration has provided me with a better understanding of the perspective Sofia has regarding the role of language in her mathematics classes. Through a discussion of her language-related

dilemmas, it suggests that these dilemmas still exist in her mathematics classrooms even though Adler's work was published more than two decades ago. However, influenced by her background, experiences and beliefs about teaching and learning mathematics, these dilemmas have different nuances in terms of whether language is framed as a resource or a problem in her mathematics classrooms. In particular, the knowledge and orientations that Sofia holds in relation to language in mathematics education have likely shaped and influenced her perspective of language to some extent.

However, I am cognizant that some claims I have made about Sofia's perspective of language in her mathematics classrooms will require more evidence, e.g., her actual teaching practice in the classroom, which I could not gather due to constraints of the pandemic. A finer analysis which considers the intricate interactions between the three theoretical constructs – teachers' knowledge, orientations, and dilemmas in relation to language – may also be necessary to strengthen the current discussion. Moving forward, I intend to consider how Sofia's perception of *language as a resource for teachers* relates to her perception of *language as a resource for students* as I may have been overtly focused on the former, rather than the latter in the analysis for this paper.

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What meaning-related phrases do language learners need to develop functional understanding? Analysis for the bottle-filling activity

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Students with diverse language proficiencies can successfully engage in meaningful, rich discourse practices, but only if their meaning-related language is supported and developed. But which meaning-related phrases do language learners need for explaining meanings or articulating connections between multiple representations? We pursued this question in a topic-specific design research study on the well-known bottle-filling activities for introducing functions and the technical language of graphs. The qualitative analysis of design experiments with 14 students reveals that highly condensed statements about matches between representations are easy to verbalize, whereas unfolding the underlying concept elements require more particular phrases to articulate the contextual interpretation of the dependent variable. And exactly these meaning-related phrases require discursive and lexical support that has so far been neglected in many designs.

Keywords: Functions, meaning-related phrases, second language learners, design research.

Specifying the required meaning-related phrases as an empirical task

Already forty years ago, Bell and Janvier (1981) promoted the design principle of engaging students in articulating and discussing mathematical ideas for developing their conceptual understanding and they realized it for functions and graphs (e.g., by the bottle-filling activity, Figure 1). Since then, many studies have indeed shown that eliciting students' intuitive everyday language and connecting it to the technical language of graphs can contribute to developing their understanding of functions as well as their technical language of graphs (e.g., Thompson & Carlson, 2017). However, engaging students in discussions and eliciting their existing language resources are only sufficient for some privileged students.

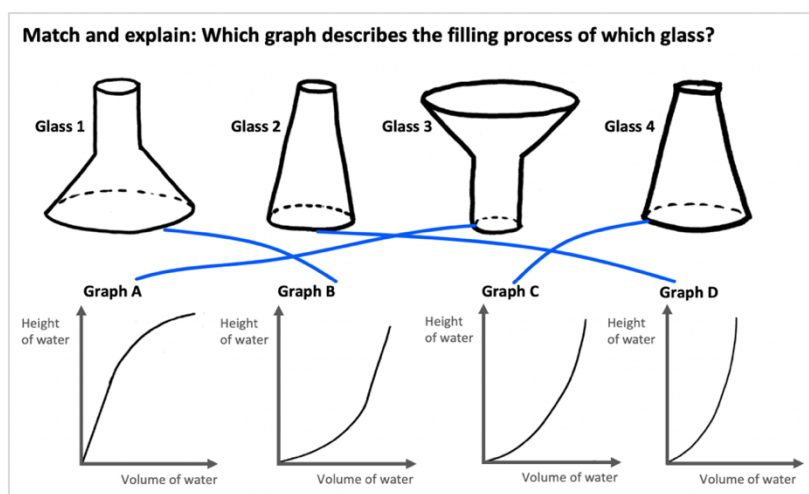


Fig. 1: Bottle-filling activity (adapted from Bell & Janvier, 1981)

Many monolingual and multilingual students do not yet bring in all resources in the (academic) language of instruction with the required conciseness and explicitness needed for such meaning-

making processes (Moschkovich, 1996; Smit et al., 2016). This group of so-called *language learners* requires further language learning opportunities to amplify their language repertoire so that it can serve its epistemic role in subtle meaning-making processes (Moschkovich, 1996; Barwell, 2018). We call this intermediate language between everyday language and technical language the *meaning-related language* and focus mainly on oral discourse practices (Prediger & Zindel, 2017). Within the last years, empirical evidence was provided for various mathematical topics (e.g., fractions, percentages, algebraic expressions) that also language learners can productively engage in rich discourse practices if the needed meaning-related language curriculum is (a) specified in topic-specific design research projects, (b) supported by the curriculum material and teachers, and then (c) successively developed by the students (summarized in Prediger, 2022 for various studies from the MuM research group, and similar in Smit et al., 2016).

For each new mathematical topic, this state of research raises the empirical task to specify which meaning-related phrases are really needed for engaging in discourse practices like explaining meanings and connections of multiple representations (Prediger & Zindel, 2017). Our topic-specific design research study aims at such an empirical specification for the particular learning environment of the bottle-filling activity, pursuing the following research question: *What meaning-related language do language learners use and need in rich learning situations for explaining the connection of graphs and glasses within the bottle-filling activity?*

Background: Developing functional understanding with bottle-filling activities

Since the pioneering work of Bell and Janvier (1981), the context of filling bottles has often been used to assess or promote covariational thinking (e.g., Thompson & Carlson, 2017): Students successively fill bottles or glasses with water portions and measure the increasing height of water. The hands-on activity of documenting the height covarying with the volume of water is a well-established first access to understanding function graphs and the technical language of graphs (see Figure 1).

According to Vollrath (1989), students need to develop three perspectives on functions, (a) correspondence perspectives (pointwise assignments, e.g., when 50 cl are filled in, the height is 10 cm), (b) covariation perspectives (considering, e.g., how the water height changes with the filled-in water volume), and (c) holistic perspectives (taking into consideration the function as a whole). In empirical studies, students' covariational understanding regarding the bottle-filling activities was further disentangled (e.g., Thompson & Carlson, 2017). We build upon these studies to distinguish *seven concept elements of functional understanding* relevant to the bottle-filling situation (in seven lines in the first two columns in Figure 2). Whereas the holistic perspective reveals only highly condensed statements of a match of graphical and geometrical representations, the explanation of the match requires unfolding the very condensed statements (see examples in the three last columns of Figure 2) into more detailed concept elements. The seven concept elements are listed by a decreasing degree of compaction that students have to unfold within their processes of understanding and explaining.


Concept elements	Adopted perspectives within this functional view	Geometric phrases on shape of glass	Meaning-related phrases for connecting graph and glass	Technical phrases in language of graphs
Holistic Description	Holistic match of graph with geometric shapes	like a rectangle like a triangle		straight, like a curve, like a wavy line
Holistic Comparison	Holistic comparison of two graphs in qualitative terms		the water height in Graph A grows more slowly than in Graph B	Graph A rises more steeply than Graph B
Holistic Qualitative Change	Covariational-holistic description of qualitative change in several intervals (2 nd derivative)		the height grows faster and faster / always grows at the same speed	the graph rises flatly and more flatly / rises evenly
Interval Qualitative Change	Covariational qualitative comparison of change in one or two intervals (1 st derivative)	wide / narrow always the same width gets narrower	the water height grows slowly / the water height in the 1 st interval grows more slowly than in 2 nd	the graph has flatter slope in the 1 st interval than in the 2 nd the graph rises steeply
Qualitative Differences	Covariational description of differences in change in qualitative terms		the height always grows by the same / by a larger amount of water	Graph C always rises by the same / by a bigger and bigger interval
Quantified Differences	Covariational-correspondence description of discrete changes with concrete values		the water height grows from ... cm to ... cm / growing by ... cm	Graph D rises from ... to ... Graph D rises by ...
Pointwise Description	Correspondence description of pairs of values		at ... ml of water the water height is ... cm	at ... ml the graph is at ... cm

Figure 2: Seven concept elements for functional understanding in the bottle-filling situation and possible phrases used to articulate them (grey: advance organizer on the empirical outcome)

Already Bell and Janvier (1981) emphasized that the hands-on activities only contribute to developing conceptual understanding when being accompanied by “speaking meaningfully [...] in [...] a rich environment in which the links between the language and the situational facts are diversified and numerous” (Bell & Janvier, 1981, p. 40). But what exactly does “speaking meaningfully” entail? Herbel-Eisenmann (2002) showed the relevance of a bridging, intermediating language that we will further refine. The last column in Figure 2 lists examples of *technical phrases in the language of graphs* (e.g., regarding the slope of the graph) applied to articulate each of the seven concept elements, with an increasing degree of unfolding. This rich technical language is to be developed by the students, but especially the most complex phrase structures for the covariational changes need active processes of meaning construction to be concisely understood and used. Students’ typical language resources are shown in the third column: They often bring in *geometric language* which turned out to be sufficient for addressing the glasses, but we will show in the empirical part that these phrases were not sufficient for describing the graphs or explaining the connection between them and the glasses. The grey column contains *meaning-related phrases of the context* that we have identified as necessary to bridge between technical language and the matching glasses. This column serves as an advance organizer for the analytic outcome of this paper. In the original design of the activity sheet, we provided language support for the geometric and technical phrases but did not put sufficient emphasis on this column of meaning-related language which turned out to be crucial. In the following, we present how we reached this analytic outcome in pursuing the research question mentioned above.

Methodological framework and research design of the study

Research context

The episodes presented are part of a larger design research project (Gravemeijer & Cobb, 2006) on language use in students’ development of understanding of functional relationships (continuing Prediger & Zindel, 2017). It combines two goals: (a) empirically grounded design of language-

responsive teaching-learning arrangements for language learners and (b) empirical investigation of students' learning pathways on concepts and language.

Methods of data collection.

The design experiments were conducted in small groups of recently immigrated adolescents in two cycles with 2 or 3 sessions of 70 to 90 minutes each, all videotaped (more than 1900 minutes of video material) and partially transcribed. The selected episodes focus on two recently immigrated teenagers, Hania (17 years old, 2nd year in Germany, language level A2 in the Common European Framework of Reference for Languages) and Emir (18 years old, 2nd year in Germany, language level A2–B1), while working on the task presented in Figure 1. The comparison of these two reveals typical phenomena from the larger data set.

Methods of data analysis.

The qualitative analysis of the transcripts was carried out to trace the students' learning pathways. Every utterance was coded by two raters (variations were negotiated later): In Step 1, the involved concept elements (lines in Figure 2) were inferred. As students made strong use of gestures and deixis, they expressed rich ideas in their emerging German language. In Step 2, the language means the students used for discussing the graphical and geometric representation and their connection (columns in Figure 2) were coded. In Step 3, students' utterances were located in the navigation space derived from Figure 2, and the sequence of their utterances was documented graphically as navigation pathways through the navigation space (see Figures 3 and 5).

Empirical insights into students' language use while explaining the matches

Episode 1: Meaning-related language as a mediator

Emir has worked on explaining a constant slope for a different filling situation before. But he still struggles with explaining its meaning in the language of instruction. The episode starts after he matches Glass 3 and Graph A (Figure 3) and explains the match of the upper part of the glass:

75 Emir: [...] If I fill in water, it does go up always steep upwards.
At the top,
it becomes more slowly again,
a bit flatter.

76 Teacher: Why?

77 Emir: Because up there,
more space, is wide [*holding hands wide apart*]

...
79 You have more and more slowly at the top.

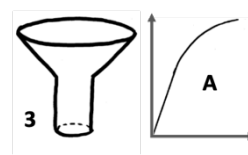


Fig. 3: Graphs and glasses to be matched

Emir starts describing the change of the water height in the lower part of the glass as always rising constantly (“always steep” used as ‘equally steep’ in Turn 75, line a) and compares it to the slower growth at the upper part of the glass (Turn 75, line b). This contextual description of the growth of the filling height is then linked back to Graph A (Turn 75, line c). Up to this point, Emir compares broad features of the glass and graph in the upper and bottom parts, while starting to use meaning-related language for explaining the match. Then, he describes the change of the filling height in the upper part of the glass in more detail and with deictic means (“More slowly, a bit more slowly. Like that.” - referring to the graph) and finally reformulates his statement in a very precise and (in German)

more condensed way in Turn 79 (“more and more slowly”). In between, Emir explains the growth of the filling height by the shape of the glass (“wide”, Turn 77, line c) and refers to the space for water inside (“more space”, Turn 77, line b), not to the specific shape that becomes wider and wider.

Emir’s visualized pathway through the navigation space (Figure 4) illustrates how he makes sense of the second derivative (Holistic Qualitative Change) by first explaining the match in a more unfolded functional view (Interval Qualitative Change, Turns 75–77) and then successively compacting his explanation to the functional view required (Turns 75, line e – 79). Emir mainly uses meaning-related language about the filling height for connecting the other representations (Turn 75, lines b–d; Turn 77, lines a–b). With these rich resources, he fills the graph with meaning and explains the match.



Functional view	Geometric phrases on shape of glass	Meaning-related phrases for connecting graph and glass	Technical phrases in language of graphs
Holistic Description			
Holistic Comparison			
Holistic Qualitative Change	76 Teacher	79 Emir: You have more and more slowly at the top.	
Interval Qualitative Change	77a Emir: Because up there,	77b Emir: more space 75e Emir: Yes, more slowly, a bit more slowly. Like that.	75d Emir: a bit flatter 75a Emir: go up always steep upwards
	75b Emir: At the top	75c Emir: it becomes more slowly again,	

Fig. 4: Emir’s pathway through the navigation space (dotted lines: prompt not taken up)

In this example of a very productive pathway, we discovered the high potential of meaning-related language about the water height as a mediator within the bottle-filling activity to explain the matches of graphs and glasses. Meaning-related statements about the water height allow students like Emir to think about change and even change of change and provide support regarding the conceptual demand of connecting the graphic with the iconic representation of the glasses.

Episode 2: Incomplete unfolding due to missing contextual interpretation

Hania and her partner are working on Glasses 2/4 and the matching Graphs C/D (Figure 5). At this point of the activity, her partner has already talked about Graph A and correctly matched it to Glass 3 (Figure 3). For *explaining* this match, the two students holistically viewed broad characteristics of Graph A and linked them to the bottom part of the glass. Building upon this strategy, the teacher asks the students to now describe the differences between Graph C and D (non-printed Turn 57).

- 58 Hania: This here is bigger [*Graph C*] and that here is smaller [*Graph D*].
- 59 Teacher: What do you think when saying that? What does it mean?
- 60 Hania: This is bigger and that is smaller [*Graph C, then D*].
- 61 Teacher: Hmm, wider and narrower, the glass, you mean? [*holding hands with a wide gap, then narrow gap*]
- 62 Hania: Yes!
- 63 Teacher: How can you see that [*in the graph*]?
- 64 Hania: Yeah, from here [*pointing at Graph C*].

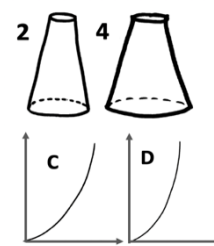


Fig. 5: Graphs and glasses to be matched

- ...
 75 Teacher: Yeah, okay, but how do you know that then *here* [pointing at Graph C & D]
 76 Hania: Because they are different, the glasses.
 77 Teacher: But how do you see that here [pointing at graphs]? How do you know, this [pointing at Graph D] is the narrow one? What do the graphs mean?
 78 Hania: Ah, I see. Because here is the line is a bit bigger here [Graph C] than that line [Graph D].

Hania starts holistically describing a bigger and a smaller graph (Turn 58), very condensed utterances, and, with the help of the teacher, matches them to the correct glasses (Turns 59-62). Although the teacher asks several times for a conceptual explanation of why the glasses and graphs match (Turns 59b, 63, 75, and more explicitly again in Turn 77), Hania doesn't pick up the prompts and closely sticks to her holistic description by using geometric everyday language. With this, she mainly concentrates on broad features of the graph but doesn't show any conceptual thinking about the change in the filling height. Quite the opposite, Hania seems to take the superficial description of the graphs as a sufficient explanation. Additionally, most of the teacher's attempts to get Hania to unfold her statements into less condensed conceptual elements are unsuccessful (Turns 59/60, 63/64, 77/78).

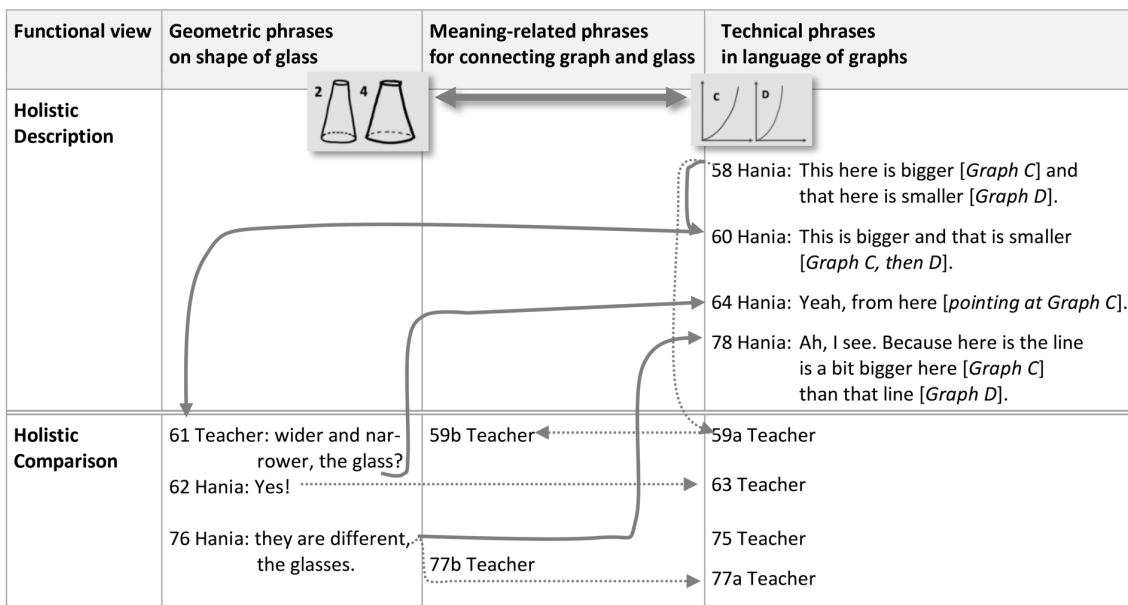


Fig. 6: Hania's pathway through the navigation space and phrases used

The graphical summary of Episode 2 illustrates several aspects that replicate existing findings and their topic-specific substantiation: First, asking for the meaning of the graphs (more or less explicitly) does not support Hania to use meaning-related language and thereby develop understanding. However, other episodes with her and further students showed that explicitly asking for what is happening to the filling height and offering meaning-related language (3rd column that stays empty in Figure 6) supports some students in a better way.

Second, the meaning-related bridging language use cannot only be characterized by individually applied phrases but by whole discourse practices (e.g., explaining connections between different representations) regarding what students say and what they consider necessary for further unfolding and justifying. This becomes visible in the graphical summary in Figure 6 that documents Hania's attempts of explaining the match as well as the teacher's mostly unsuccessful prompts: Hania doesn't

follow the teacher's invitation to describe or compare the graphs but stays with descriptions of the glasses or highly condensed utterances on graphs, while she receives no sufficient support to address the contextual interpretation (referring to the depending variable of water height). Third, Hania's challenges also seem to be shaped by the fact that she does not unfold her utterances into more detailed concept elements which is necessary for meaning-making. The analysis indicates that the support of unfolding concept elements and the offer of meaning-related contextual phrases might influence each other.

Discussion and Outlook

Returning to the research question, we have shown that, in Emir's case, the rich experimental environment mobilized discourse practices and language means to produce an explanation of meanings because he referred to the speed of the water height several times on his own. This resonates with Bell's & Janvier's (1981) optimism that a rich environment is sufficient: "No lists of words are referred to ..., no catalogs of grammatical forms. But the contact with his environment ... makes it happen" (p. 40). However, some of our language learners acted like Hania for whom the rich demands were not sufficient. Unlike Emir, these students did not start to talk about the contextual situation (more concisely, the dependent variable of water height) which is essential (not only for Emir) for really justifying their matches of glasses and graphs. The qualitative analysis of Hania's pathways (and those of 14 students in total) reveals that highly condensed statements of matches between representations, classified as the individual Holistic Description, are linguistically rather easy to manage, whereas the students seem to require more particular phrases to articulate the contextual interpretation of the dependent variable and thereby unfold the underlying concept elements. Comparing Emir's and Hania's pathways helps to specify more concisely the concrete meaning-related language needed for the bottle-filling activity, namely the attention to the context and the corresponding phrases to articulate the changes in the filling height. In total, the analysis of the larger data set reveals that the meaning-related phrases listed in the fourth column of Figure 2 seem to reveal the communicative and epistemic tools to construct and articulate a situation model of the bottle-filling context and its mathematical structures (Bell & Janvier, 1981; Prediger & Zindel, 2017). The productive use of these meaning-related phrases as bridging language (Herbel-Eisenmann, 2002) also needs discursive support (Barwell, 2018). As a consequence for our design, both kinds of support have been refined. Beyond these concrete cases, the analysis exemplifies how meaning-related language (used to articulate the relevant concept elements for the topic and context in view) can support students. We substantiate earlier projects (Herbel-Eisenmann, 2002; Prediger, 2022) showing how meaning-related language can serve as an epistemically (relevant) mediator between representations and identified lexical and discursive support for unfolding the complex concept of functional relationships within the bottle-filling activity. We assume that this is equally relevant for students with more elaborate language proficiency in German.

As the design experiments were conducted with recent language learners, the qualitative analysis cannot delineate whether the discourse and conceptual challenges would have equally occurred in the students' family languages. However, as we intend to support also their language learning, the students' limitations in German are important to identify to provide learning opportunities for the most relevant discourse practices and phrases. Furthermore, we focused on concept elements adapted

to one special task, the bottle-filling activity, so future research should be extended and transferred to other tasks for dealing with functional understanding.

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