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(m,n)-QUASITILTED AND (m,n)-ALMOST HEREDITARY ALGEBRAS

EDSON RIBEIRO ALVARES, DIANE CASTONGUAY, PATRICK LE MEUR, AND TANISE CARNIERI PIERIN

ABSTRACT. Motivated by the study of (m,n)-quasitilted algebras, which are the piecewise hereditary algebras obtained from quasitilted algebras of global dimension two by a sequence of (co)tiltings involving $n-1$ tilting modules and $m-1$ cotilting modules, we introduce (m,n)-almost hereditary algebras. These are the algebras with global dimension $m+n$ and such that any indecomposable module has projective dimension at most m, or else injective dimension at most n. We relate these two classes of algebras, among which $(m,1)$-almost hereditary ones play a special role. For these, we prove that any indecomposable module lies in the right part of the module category, or else in an m-analog of the left part. This is based on the more general study of algebras the module categories of which admit a torsion-free subcategory such that any indecomposable module lies in that subcategory, or else has injective dimension at most n.

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INTRODUCTION

Quasitilted algebras were defined in [7] as the opposite algebras of endomorphism algebras of tilting objects of Hom-finite, Krull-Schmidt hereditary abelian categories. These algebras feature several properties and characterisations which explain their relevance. First, by [7, Chapter II, Theorem 2.3], an Artin algebra over an Artin ring is quasitilted if and only if it has global dimension at most two and, for any indecomposable module X,

$$\operatorname{pd} X \leqslant 1 \text{ or else } \operatorname{id} X \leqslant 1\,. \tag{0.1}$$

Next, for any quasititled algebra A, the following decomposition holds (see [7, Chapter II, Proposition 1.6])

$$\operatorname{ind} A = \mathcal{L}_A \cup \mathcal{R}_A, \tag{0.2}$$

where $\mathcal{L}_A$ and $\mathcal{R}_A$ are the left and right parts of the module category of A, respectively. These subcategories are efficient tools to classify algebras and study their representation theory. In particular (see [7, Theorem 1.14]),

$$A \text{ is quasitilted if and only if } A \in \operatorname{add} \mathcal{L}_A; \tag{0.3}$$

we refer the reader to [1] for a survey on these subcategories. Finally, by general theory of derived equivalences arising from tilting complexes, they are piecewise hereditary, that is, the bounded derived category of their module categories are triangle equivalent to those of hereditary abelian categories.

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Recall that, for any piecewise hereditary algebra, there is a sequence of algebras with first term a quasitilted algebra, with last term the given algebra and such that each algebra of the sequence is the opposite algebra of the endomorphism algebra of a splitting tilting or cotilting module over the preceding algebra (see [6], or Section 2 for a reminder). In many examples, that given algebra often has homological properties like (0.1) depending on the number of tilting modules and the number of cotilting modules involved in the sequence.

Given positive integers m and n, we introduce (m, n)-quasitilted algebras, defined as the algebras for which there is a sequence as above such that the number of involved cotilting modules is $m - 1$, the number of involved tilting modules is $n - 1$ and the global dimensions of the algebras of the sequence increase strictly. The aim of this article is to show that these algebras have homological properties like (0.1) and to derive consequences relative to their representation theory in terms of specific subcategories, like (0.2).

For this purpose, we define the (m, n)-almost hereditary algebras as the algebras with global dimension $m + n$ and such that, for any indecomposable module X,

$$\operatorname{pd} X \leqslant m \text{ or else } \operatorname{id} X \leqslant n\,. \tag{0.4}$$

Note that (0.4) alone implies that the global dimension is at most $m + n + 1$ (see Lemma 2.3).

Our results establish that any (m, n)-quasitilted algebra is (m, n)-almost hereditary (Corollary 2.5); although this is an equivalence when $(m, n) = (1, 1)$, the converse implication does not hold in general. Moreover, it appears that any (m, n)-quasitilted algebra is $(m + n - 1, 1)$- or else $(1, m + n - 1)$-almost hereditary (Lemma 2.8). This shows the relevance of $(m, 1)$- and $(1, m)$-almost hereditary algebras among the ones considered previously. For these algebras, we prove in Theorem 3.6 that, if A is $(m, 1)$-almost hereditary, then

$$\operatorname{ind} A = \mathcal{L}_A^m \cup \mathcal{R}_A, \tag{0.5}$$

where $\mathcal{L}_A^m$ denotes the class of indecomposable modules such that the predecessors in $\operatorname{ind} A$ of which have projective dimension at most m. This result is obtained as a consequence of a more general one: we prove in Proposition 3.4 that if $\mathcal{C}$ is a torsion-free class in the module category of A such that any indecomposable module lies in $\mathcal{C}$ or else has injective dimension at most n, then

$$\operatorname{ind} A = \mathcal{L}_{\mathcal{C}} \cup \mathcal{R}_A^m, \tag{0.6}$$

where the definition of $\mathcal{R}_A^m$ is dual to that of $\mathcal{L}_A^m$ and the definition of $\mathcal{L}_{\mathcal{C}}$ is given in 3.1. These results raise the question to determine when an algebra is $(m, 1)$-almost hereditary. We give a partial answer to this question in Proposition 3.9 (compare with (0.3)),

$$A \text{ lies } \mathcal{L}_A^m \text{ and has global dimension } m + 1 \Rightarrow A \text{ is } (m, 1)\text{-almost hereditary.}$$

The article is therefore organised as follows. Base material is collected in Section 1. Section 2 introduces (m, n)-quasitilted algebras and (m, n)-almost hereditary ones, and relates them. Section 3 studies $(m, 1)$-almost hereditary algebras in terms of $\mathcal{L}_A^m$ and $\mathcal{R}_A$. Finally, Section 4 studies the behaviour of $(m, 1)$-almost hereditary algebras under taking one point extensions.

Throughout the text, k denotes an Artin commutative ring and A denotes an Artin k-algebra.

1. Preliminaries

We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules and by $\operatorname{ind} A$ a full subcategory consisting of exactly one representative from each isomorphism class of indecomposable A-modules. For a subcategory $\mathcal{C}$ of $\operatorname{mod} A$ we write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$.

If $T \in \operatorname{mod} A$, then $\operatorname{add} T$ denotes the full subcategory of $\operatorname{mod} A$ whose objects are the direct sums of direct summands of T. Given an A-module M, we denote by $\operatorname{pd}_A M$ and $\operatorname{id}_A M$, respectively, its projective and injective dimensions. The global dimension of A is denoted by $\operatorname{gl.dim} A$.

The space of morphisms from an object X to an object Y in $\mathcal{D}^b(\operatorname{mod} A)$ is denoted by $\operatorname{Hom}(X,Y)$. Whenever $T \in \mathcal{D}^b(\operatorname{mod} A)$ is a tilting complex, that is, there is no nonzero morphism $T \to T[i]$ in $\mathcal{D}^b(\operatorname{mod} A)$ for all $i \in \mathbb{Z}\backslash\{0\}$ and $\mathcal{D}^b(\operatorname{mod} A)$ is the smallest triangulated subcategory of $\mathcal{D}^b(\operatorname{mod} A)$ containing T and stable under taking direct summands, the module categories of $\operatorname{mod} A$ and $\operatorname{mod} B$ have equivalent bounded derived categories, where B is the endomorphism algebra of T in $\mathcal{D}^b(\operatorname{mod} A)$. In such a situation, $\operatorname{mod} B$ is identified with the following subcategory of $\mathcal{D}^b(\operatorname{mod} A)$,

$$\{X \in \mathcal{D}^b(\operatorname{mod} A) \mid (\forall i \in \mathbb{Z}\backslash\{0\})\ \operatorname{Hom}(T, X[i]) = 0\}\,.$$

In particular, whenever $X, Y \in \operatorname{mod} A \cap \operatorname{mod} B$ and $i \in \mathbb{Z}$, then $\operatorname{Ext}^i_A(X,Y)$ is naturally identified with $\operatorname{Ext}^i_B(X,Y)$.

For further background on the representation theory of A, we refer the reader to [3], [2], [4].

1.1. **Paths, left and right parts.** Given $M, N \in \operatorname{ind} A$, a *path* from M to N (denoted by $M \rightsquigarrow N$) is a sequence of nonzero morphisms

$$(*)\ \ M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_t} M_t = N$$

where $M_i \in \operatorname{ind} A$ for all i. In this case N is called *successor* of M and M *predecessor* of N.

Following [5], the *left part* $\mathcal{L}_A$ of $\operatorname{mod} A$ is the full subcategory whose objects are those $M \in \operatorname{ind} A$ such that every predecessor of M in $\operatorname{ind} A$ has projective dimension at most one. Clearly $\mathcal{L}_A$ is closed under predecessors. The *right part* $\mathcal{R}_A$ is defined dually and has dual properties.

1.2. **A basic fact on short exact sequences.** The following lemma is used several times in this article (see [11, Lemma 1.2, part (ii)] for a similar statement).

Lemma 1.1. *Let $0 \to X \xrightarrow{f} E \xrightarrow{g} Y \to 0$ be a non-split exact sequence.*

- (a) *If X is indecomposable then each coordinate morphism of g is nonzero.*
- (b) *If Y is indecomposable then each coordinate morphism of f is nonzero.*

Proof. We only prove (a) since the proof of (b) can be obtained dually. Suppose that $E = E_1 \oplus E_2$, where E_1 is indecomposable, and $g_1 = 0$. Consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E_1 \arrow[r, "1"] \arrow[d] & E_1 \arrow[d, "\binom{1}{0}"] & & \\
0 \arrow[r] & X \arrow[r, "\binom{f_1}{f_2}"] \arrow[d] & E_1 \oplus E_2 \arrow[r, "(0\ g_2)"] \arrow[d, "(0\ 1)"] & Y \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_2 \arrow[r] \arrow[d] & E_2 \arrow[r, "g_2"] \arrow[d] & Y \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

It follows that f_1 is a split epimorphism and, because X is indecomposable, f_1 is an isomorphism. Then the exact sequence $0 \to X \to E_1 \oplus E_2 \to Y \to 0$ splits, which is a contradiction. $\square$

1.3. **A short review of tilting theory.** Recall that ${}_AT \in \operatorname{mod} A$ is called a *tilting module* provided the following three conditions are satisfied:

(1) $\operatorname{pd}_A T \leq 1$,
(2) $\operatorname{Ext}^1_A(T,T) = 0$
(3) the number of pairwise non-isomorphic indecomposable direct summands of T equals the rank of the Grothendieck group, $K_0(A)$.

Given a tilting module ${}_AT$, let $B = (\operatorname{End}_A T)^{\mathrm{op}}$. Recall that ${}_AT_B$ induces torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ on $\operatorname{mod} A$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ on $\operatorname{mod} B$, where $\mathcal{T}(T) = \{X \in \operatorname{mod} A; \operatorname{Ext}^1_A(T,X) = 0\}$, $\mathcal{F}(T) = \{X \in \operatorname{mod} A; \operatorname{Hom}_A(T,X) = 0\}$, $\mathcal{X}(T) = \{X \in \operatorname{mod} B; T \otimes_B X = 0\}$ and $\mathcal{Y}(T) = \{X \in \operatorname{mod} B; \operatorname{Tor}^B_1(T,X) = 0\}$. Due to the Brenner - Butler Theorem ([9]), $\operatorname{mod} A$ and $\operatorname{mod} B$ are related as follows: restriction of functor $\operatorname{Hom}_A(T,-) : \operatorname{mod} A \to \operatorname{mod} B$ to $\mathcal{T}(T)$ is an equivalence $\mathcal{T}(T) \to \mathcal{Y}(T)$ and restriction of $\operatorname{Ext}^1_\Lambda(T,-)$ to $\mathcal{F}(T)$ gives rise to an equivalence $\mathcal{F}(T) \to \mathcal{X}(T)$. A tilting module ${}_AT$ is called *splitting* if the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ on $\operatorname{mod} B$ splits, that is, if each indecomposable B-module lies in $\mathcal{X}(T)$, or else in $\mathcal{Y}(T)$. According to Hoshino [12], if ${}_AT$ is a tilting module, then ${}_AT$ is splitting if and only if $\operatorname{id}_A X \leq 1$ for every $X \in \mathcal{F}(T)$.

Dually, it is possible to define a *cotilting* module. We will use but not state explicitly the dual results and properties which hold true for cotilting modules. For further definitions and results on tilting theory, we refer the reader to [9], [13].

2. (m,n)-Quasitilted algebras

The purpose of the section is to relate (m,n)-quasitilted algebras to (m,n)-almost hereditary algebras. We say that A is (m,n)-*quasitilted* if there exists a sequence of triples $(A_i, T_i, A_{i+1} = (\operatorname{End}_{A_i} T_i)^{\mathrm{op}})$ such that: A_0 is a quasitilted algebra of global dimension two; $A = A_{m+n-2}$; each T_i is a *stair* splitting tilting or cotilting A_i-module, that is a splitting tilting or cotilting module with the property that $\operatorname{gl.dim} A_i < \operatorname{gl.dim} A_{i+1}$; and $n-1$ modules among the T_i are tilting whereas the $m-1$ remaining ones are cotilting. Any (m,n)-quasitilted algebra is piecewise hereditary. Actually, it is proved in [6] that any piecewise hereditary algebra A may be obtained from some quasitilted algebra by a sequence of tilting or cotilting processes, where each involved (co)tilting module is splitting. And, when k is a field and A is derived equivalent to a finite dimensional hereditary algebra, then A may be obtained from hereditary algebra by a sequence of tilting processes, where each involved tilting module is splitting (see [8]).

By definition, the $(1,1)$-quasitilted algebras are the quasitilted algebras of global dimension two; accordingly a $(1,1)$-quasitilted algebra A satisfies *(i)* $\operatorname{gl.dim} A = 2$ and *(ii)* $\operatorname{pd}_A X \leq 1$ or else $\operatorname{id}_A X \leq 1$, for each indecomposable A-module X ([7]). Below, we prove that any (m,n)-quasitilted algebra A has the two following homological properties.

$(Q1)$ $\operatorname{gl.dim} A = m+n$;
$(Q2)$ for each indecomposable A-module X, $\operatorname{pd}_A X \leq m$ or else $\operatorname{id}_A X \leq n$.

We call (m,n)-*almost hereditary* an algebra which satisfies $(Q1)$ and $(Q2)$, where m and n are positive integers. Note that condition $(Q2)$ is not a consequence of condition $(Q1)$ as shown in the following example.

Example 2.1. Let A be the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[rr] \arrow[dr, "\alpha"'] & & 2 \\ & 3 \arrow[ur, "\beta"'] & \end{tikzcd}$$

bound by the relation $\beta\alpha = 0$; the indecomposable A-module $\begin{smallmatrix} & 2 & \\ 1 & & \end{smallmatrix}$ has projective and injective dimension equal to two, while global dimension of A is two.

Note also that condition $(Q1)$ cannot be obtained from condition $(Q2)$ as shown in the following example.

Example 2.2. Consider the radical square zero algebra A given by the quiver

$$1 \to 2 \to \cdots \to m+n+2\,,$$

where m, $n \geq 1$. In this case, $\mathrm{gl.dim} A = m+n+1$, whereas each indecomposable A-module has projective dimension at most m or injective dimension at most n. Of course, A is a (a,b)-almost hereditary algebra for all positive integers a, b such that $a+b = m+n+1$.

In fact, it is possible to verify that an algebra which satisfies condition $(Q2)$ and with global dimension greater than $m+n-1$ is a (m,n)-almost hereditary or a $(m+1,n)$-almost hereditary algebra, as stated in the following lemma.

Lemma 2.3. *Let m and n be positive integers and let A be an algebra such that $\mathrm{pd}_A X \leq m$ or $\mathrm{id}_A X \leq n$ for each indecomposable A-module X. Then $\mathrm{gl.dim} A \leq m+n+1$. In particular, if $\mathrm{gl.dim} A > m+n-1$ then A is a (m,n)-almost hereditary or a $(m+1,n)$-almost hereditary algebra.*

Proof. Let $M \in \mathrm{ind}\, A$. Then, any indecomposable direct summand of $\Omega^{n+1} M$ has injective dimension at least $n+1$, and hence has projective dimension at most m. Therefore, $\mathrm{pd}_A\, \Omega^{n+1} M \leqslant m$. Accordingly, $\mathrm{pd}_A\, M \leqslant m+n+1$. □

A first step to relate (m,n)-quasitilted algebras to (m,n)-almost hereditary ones is to investigate how the former behave under tilting (or, dually, under cotilting). The relationship is then obtained as a corollary.

Theorem 2.4. *Let m, n be positive integers. Let A be an algebra satisfying $(Q2)$. Let T be a splitting tilting A-module. Denote $(\mathrm{End}_A T)^{\mathrm{op}}$ by B. Then, any indecomposable B-module has projective dimension at most m, or else injective dimension at most $n+1$. In particular, if A is (m,n)-almost hereditary and T is stair, then B is $(m,n+1)$-almost hereditary.*

Proof. We only prove the first statement because the second one follows from the first one and from the definition of stair tilting modules. For all subcategories $\mathcal{A}$, $\mathcal{B}$ of $\mathrm{mod}\, A$ and $i \in \mathbb{Z}$, denote by $\mathrm{Ext}^i_A(\mathcal{A},\mathcal{B})$ the collection $\{\mathrm{Ext}^i_A(X,Y); X \in \mathcal{A}, Y \in \mathcal{B}\}$. Also, for a full subcategory $\mathcal{A}$ of $\mathrm{mod}\, A$, we denote $\sup\{\mathrm{id}_A X; X \in \mathcal{A}\}$ by $\mathrm{id}_A\, \mathcal{A}$. First, $\mathrm{id}_B\, \mathcal{X}(T) \leqslant 1$. Indeed, using that $\mathrm{id}_A\, \mathcal{F}(T) \leqslant 1$, we have

$$\begin{array}{rcl} \mathrm{Ext}^2_B(\mathcal{X}(T),\mathcal{X}(T)) & = & \mathrm{Ext}^2_A(\mathcal{F}(T),\mathcal{F}(T)) = 0 \\ \mathrm{Ext}^1_B(\mathcal{Y}(T),\mathcal{X}(T)) & = & \mathrm{Ext}^2_A(\mathcal{T}(T),\mathcal{F}(T)) = 0\,. \end{array}$$

Next, consider the decomposition

$$(\mathrm{ind}\, \mathcal{Y}(T) =)\mathrm{ind}\, \mathcal{T}(T) = \mathcal{C}_1 \cup \mathcal{C}_2\,,$$

where $\mathcal{C}_1 = \{X \in \mathrm{ind}\, \mathcal{T}(T); \mathrm{pd}_A X \leqslant m\}$ and $\mathcal{C}_2 = \{X \in \mathrm{ind}\, \mathcal{T}(T); \mathrm{id}_A X \leqslant n\}$. Since

$$\begin{array}{rcl} \mathrm{Ext}^{m+1}_B(\mathcal{C}_1,\mathcal{Y}(T)) & = & \mathrm{Ext}^{m+1}_A(\mathcal{C}_1,\mathcal{T}(T)) = 0 \\ \mathrm{Ext}^{m+1}_B(\mathcal{C}_1,\mathcal{X}(T)) & = & \mathrm{Ext}^{m+2}_A(\mathcal{C}_1,\mathcal{F}(T)) = 0\,, \end{array}$$

it follows that $\mathrm{pd}_B\, \mathcal{C}_1 \leqslant m$. Also, $\mathrm{id}_B\, \mathcal{C}_2 \leqslant n+1$ because

$$\begin{array}{rcl} \mathrm{Ext}^{n+2}_B(\mathcal{X}(T),\mathcal{C}_2) & = & \mathrm{Ext}^{n+1}_A(\mathcal{F}(T),\mathcal{C}_2) = 0 \\ \mathrm{Ext}^{n+2}_B(\mathcal{Y}(T),\mathcal{C}_2) & = & \mathrm{Ext}^{n+2}_A(\mathcal{T}(T),\mathcal{C}_2) = 0\,. \end{array}$$

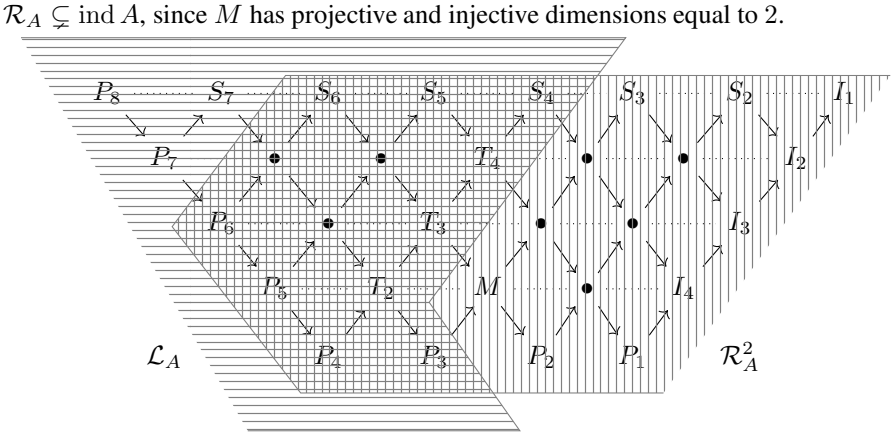
This proves the theorem. □

Now, here is the announced relationship between (m,n)-quasitilted algebras and (m,n)-almost hereditary ones.

Corollary 2.5. *If A is a (m,n)-quasitilted algebra, then A is (m,n)-almost hereditary.*

Proof. Let A be a (m,n)-quasitilted algebra. By definition, there exists a sequence of triples $(A_i, T_i, A_{i+1} = (\mathrm{End}_{A_i} T_i)^{\mathrm{op}})$ such that A_0 is strict quasitilted, each T_i is a stair and splitting tilting or cotilting A_i-module and $A = A_{m+n-2}$, where $m-1$ and $n-1$ are the numbers of cotilting and tilting processes, respectively. As mentioned before, A_0 is $(1,1)$-almost hereditary and therefore, according to Theorem 2.4, A is (m,n)-almost hereditary. □

Example 2.6. Let A be the path k-algebra given by the bound quiver $1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_7} 8$, with relations $\alpha_{i+4} \cdots \alpha_i = 0$, where $1 \leq i \leq 3$. Consider the tilting A-module $T = P_4 \oplus T_2 \oplus T_3 \oplus T_4 \oplus S_4 \oplus P_3 \oplus P_2 \oplus P_1$. Note that $B = (\mathrm{End}_A T)^{\mathrm{op}}$ is a tilted algebra, which yields that A is $(1,2)$-quasitilted, since $A \simeq \mathrm{End} T_B$. According to Corollary 2.5, A is also $(1,2)$-almost hereditary. It is an easy verification that $\mathcal{L}_A$ and $\mathcal{R}_A^2$ consist, respectively, of those modules in horizontal and vertical lines patterned areas in the illustration below. We point out that $\mathcal{L}_A \cup \mathcal{R}_A \subsetneq \mathrm{ind}\, A$, since M has projective and injective dimensions equal to 2.



Note that the converse of the previous corollary is not true in general. Here is a counter-example.

Example 2.7. Let A be the path k-algebra given by the bound quiver $1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{10}} 11 \xrightarrow{\alpha_{11}} 12$, with relations $\alpha_{i+6} \cdots \alpha_i = 0$, where $1 \leq i \leq 5$. It follows from [10] that A is not piecewise hereditary algebra, which entails that A is not $(1,2)$- nor $(2,1)$-quasitilted (or even (m,n)-quasitilted for any m and n). However, it is easily seen that A is $(1,2)$- and $(2,1)$-almost hereditary.

Later, the article concentrates on $(m,1)$- and $(1,m)$-almost hereditary algebras. The reason is the following result which states that, although an algebra of finite global dimension may not be (m,n)-almost hereditary for any m,n, this property becomes true for the algebra obtained from it by a (co)tilting using a stair splitting (co)tilting module.

Lemma 2.8. *Let d be a positive integer. Let A be an algebra with $\mathrm{gl.dim} A = d$, T an A-module and $B = (\mathrm{End}_A T)^{\mathrm{op}}$.*

(i) If T is a stair splitting tilting module, then B is $(d,1)$-almost hereditary.
(ii) If T is a stair splitting cotilting module, then B is $(1,d)$-almost hereditary.

In particular, any (m,n)-quasitilted algebra is $(m+n-1,1)$-almost hereditary, or else $(1,m+n-1)$-almost hereditary.

Proof. (i) Since $\mathrm{id}_A \mathcal{F}(T) \leq 1$ we can use the same considerations as in the proof of Theorem 2.4 and conclude that $\mathrm{id}_B \mathcal{X}(T) \leq 1$. Also, $\mathrm{pd}_B \mathcal{Y}(T) \leq d$ because $\mathrm{gl.dim} A = d$.

Now, assume that B is (m,n)-quasitilted. Let (A,T,B) be the resulting last triple appearing in the definition of (m,n)-quasitilted algebras. Then $\mathrm{gl.dim} A = m+n-1$, and hence B is $(m+n-1,1)$- or $(1,m+n-1)$-quasitilted according to whether T is a tilting or cotilting module, respectively. □

From now on we focus on $(m,1)$-almost hereditary algebras. The $(1,m)$-almost hereditary algebras may be treated using dual considerations.

3. $(m,1)$-Almost hereditary algebras

The purpose of this section is to prove (0.5) whenever A is $(m,1)$-almost hereditary, (see Theorem 3.6). In order to prove this theorem, preparatory material is first established in 3.1 on algebras with nice small homological properties related to $(Q1)$ and $(Q2)$. And the theorem is proved with some consequences in 3.2.

3.1. **On algebras with small homological dimensions.** The proof of (0.5) when A is $(m,1)$-almost hereditary is mainly based on the fact that $\mathcal{L}_A^m$ is a torsion-free class of A, which is true because $\mathrm{gl.dim}\, A = m+1$. Hence, this subsection is devoted to the investigation of (0.5) in the more general situation where

- the class of A-modules with projective dimension at most m is replaced by a torsion-free class $\mathcal{C}$ of $\mathrm{mod}\, A$;
- $\mathcal{L}_A^m$ is replaced by the class of indecomposable A-modules such that all the predecessors in $\mathrm{ind}\, A$ of which lie in $\mathcal{C}$, this class is denoted by $\mathcal{L}_{\mathcal{C}}$.

For a given positive integer n, this investigation establishes (0.6) whenever every indecomposable A-module lies in $\mathcal{C}$ or else has injective dimension at most n (see Proposition 3.4). The first step of this investigation is to show that there are no nonzero morphisms from any indecomposable A-module not lying in $\mathcal{C}$ to any indecomposable A-module with a large injective dimension. This is done in the two following lemmas.

Lemma 3.1. *Let $\mathcal{C}$ be a torsion-free class of $\mathrm{mod}\, A$. Consider a nonzero morphism $f\colon U \to V$, where U and V are indecomposable modules such that $U \notin \mathcal{C}$ and $V \in \mathcal{C}$, and such that $\ell(U)+\ell(V)$ is minimal for these properties. Then $\mathrm{Im}\, f \in \mathcal{C}$, $\mathrm{Ker}\, f \notin \mathcal{C}$ and $\mathrm{Ker}\, f$ is indecomposable.*

Proof. Since $V \in \mathcal{C}$ and $\mathcal{C}$ is stable under taking submodules, it follows that

$$\mathrm{Im}\, f \in \mathcal{C}. \tag{3.1}$$

Using the exact sequence $0 \to \mathrm{Ker}\, f \to U \to \mathrm{Im}\, f \to 0$, since $U \notin \mathcal{C}$, $\mathrm{Im}\, f \in \mathcal{C}$ and $\mathcal{C}$ is stable under extensions, then

$$\mathrm{Ker}\, f \notin \mathcal{C} \text{ and } f \text{ is not a monomorphism.} \tag{3.2}$$

Therefore, there exists an indecomposable direct summand K of $\mathrm{Ker}\, f$ such that $K \notin \mathcal{C}$. Consider the push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \mathrm{Im}\, f \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & K \arrow[r] & U' \arrow[r] & \mathrm{Im}\, f \arrow[r] & 0.
\end{tikzcd}$$

Notice that $U' \notin \mathcal{C}$, because $K \notin \mathcal{C}$ and $\mathcal{C}$ is closed under submodules. In this case, there exists an indecomposable direct summand U'' of U' such that $U'' \notin \mathcal{C}$. Consider the composite morphism

$$U'' \to \mathrm{Im}\, f \hookrightarrow V.$$

This morphism is nonzero for the following reasons. Should the exact sequence $0 \to K \to U' \to \mathrm{Im}\, f \to 0$ split, then the split epimorphism $\mathrm{Ker}\, f \twoheadrightarrow K$ would factor through U, and hence there would exist a split epimorphism $U \to K$, which would entail that $\mathrm{Ker}\, f = U = K$, a contradiction to f being nonzero; therefore the exact sequence $0 \to K \to U' \to \mathrm{Im}\, f \to 0$ does not split; accordingly, the coordinate morphism $U'' \to \mathrm{Im}\, f$ is nonzero, and hence nor is $U'' \to V$.

By construction, $\ell(U'')+\ell(V) \leqslant \ell(U)+\ell(V)$. By minimality of $\ell(U)+\ell(V)$ and because $U'' \notin \mathcal{C}$, it follows that $\ell(U'') = \ell(U)$. Accordingly, $U'' = U' = U$, and hence $K = \mathrm{Ker}\, f$. Thus $\mathrm{Ker}\, f$ is indecomposable. □

Lemma 3.2. *Let n be a positive integer. Let $\mathcal{C}$ be a torsion-free class of $\mathrm{mod}\, A$ such that, for all $X \in \mathrm{ind}\, A$,*

$$X \in \mathcal{C}, \text{ or else } \mathrm{id}_A X \leqslant n.$$

If U, $V \in \mathrm{ind}\, A$ such that $U \notin \mathcal{C}$ and $\mathrm{id}_A V > n$, then $\mathrm{Hom}_A(U,V) = 0$.

Proof. Let $U, V \in \operatorname{ind} A$ be such that $U \notin \mathcal{C}$ and $\operatorname{id}_A V > n$. By absurd, suppose that there exists a nonzero morphism $f\colon U \to V$. Assume that $\ell(U) + \ell(V)$ is minimal for these properties.

According to Lemma 3.1, $\operatorname{Im} f \in \mathcal{C}$, $\operatorname{Ker} f \notin \mathcal{C}$ and $\operatorname{Ker} f$ is indecomposable. Consequently, $\operatorname{id}_A \operatorname{Ker} f \leqslant n$.

Next, considering the injective dimensions of the modules in the following exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \operatorname{Im} f \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Im} f \arrow[r] & V \arrow[r] & \operatorname{Coker} f \arrow[r] & 0 \end{tikzcd}$$

yields that $\operatorname{id}_A \operatorname{Im} f \leqslant n$ and $\operatorname{id}_A \operatorname{Coker} f > n$. This permits to prove that $\operatorname{Coker} f$ is indecomposable. Indeed, the dual considerations of those following (3.2) in the proof of Lemma 3.1 may be applied here to the exact sequence

$$0 \to \operatorname{Im} f \to V \to \operatorname{Coker} f \to 0$$

instead of to the exact sequence $0 \to \operatorname{Ker} f \to U \to \operatorname{Im} f \to 0$ provided that $\mathcal{C}$ is replaced by $\{M \in \operatorname{mod} A; \operatorname{id}_A M \leqslant n\}$, which is stable under extensions. Accordingly, $\operatorname{Coker} f$ is indecomposable.

Finally, consider any exact sequence

$$0 \to \operatorname{Coker} f \to I_0 \to \cdots \to I_{n-2} \to C \to 0,$$

where $I_0, \ldots, I_{n-2}$ are injective modules and C is the $(n-1)$-th cosyzygy of $\operatorname{Coker} f$; in the particular case $n = 1$, just take $C = \operatorname{Coker} f$ and discard the exact sequence. From the long exact sequence obtained upon applying $\operatorname{Hom}_A(C, -)$ to the exact sequence

$$0 \to \operatorname{Ker} f \to U \to \operatorname{Im} f \to 0,$$

there results an exact sequence (recall that $\operatorname{id}_A \operatorname{Ker} f \leqslant n$)

$$\operatorname{Ext}^n_A(C, U) \to \operatorname{Ext}^n_A(C, \operatorname{Im} f) \to 0.$$

Accordingly, there exist $N, N_0, \ldots, N_{n-2} \in \operatorname{mod} A$ fitting into a commutative diagram as follows, where the rows are exact and the leftmost square is cocartesian

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & N_{n-2} \arrow[r] \arrow[d] & C^1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Im} f \arrow[r] & V \arrow[r] & I_0 \arrow[r] & \cdots \arrow[r] & I_{n-2} \arrow[r] & C \arrow[r] & 0; \end{tikzcd}$$

in the particular case $n = 1$, take the lower row equal to the short exact sequence $0 \to \operatorname{Im} f \to V \to C = \operatorname{Coker} f \to 0$ and take the sequence $N_0, \ldots, N_{n-2}$ to be void.

Consider the following commutative diagram with exact rows and columns, whatever the value of n,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{Ker} f^1 \arrow[r] \arrow[d] & \operatorname{Ker} f \arrow[d] & & \\
0 \arrow[r] & U \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & \operatorname{Coker} f^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Im} f \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & \operatorname{Coker} f \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Notice that the exact sequence

$$0 \to \operatorname{Ker} f \to N \to V \to 0 \tag{3.3}$$

does not split since, otherwise, the composite morphism $\operatorname{Ker} f \to U \to N$ would be a section and hence the indecomposable U would equal $\operatorname{Ker} f$, a contradiction to f being nonzero; similarly, the exact sequence

$$0 \to U \to N \to \operatorname{Coker} f \to 0 \tag{3.4}$$

does not split.

Now, consider the exact sequence

$$0 \to U \to \operatorname{Im} f \oplus N \to V \to 0.$$

On one hand, since $\mathcal{C}$ is stable under taking submodules and $U \notin \mathcal{C}$, it follows that $\operatorname{Im} f \oplus N \notin \mathcal{C}$; and since $\operatorname{Im} f \in \mathcal{C}$, there exists an indecomposable direct summand N' of N such that $N' \notin \mathcal{C}$. On the other hand, because $\operatorname{id}_A U \leqslant n$ and $\operatorname{id}_A V > n$, it follows that $\operatorname{id}_A \operatorname{Im} f \oplus N > n$; and since $\operatorname{id}_A \operatorname{Im} f \leqslant n$, there exists an indecomposable direct summand N'' of N such that $\operatorname{id}_A N'' > n$.

Because the exact sequences (3.3) and (3.4) do not split and $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are indecomposable modules, the coordinate morphisms with indecomposable domain and codomain

$$N' \to V \text{ and } U \to N''$$

are nonzero. Moreover, they feature the following properties,

- $N' \notin \mathcal{C}$ and $\operatorname{id}_A V > n$; and
- $U \notin \mathcal{C}$ and $\operatorname{id}_A N'' > n$.

By assumption on $\mathcal{C}$, the indecomposable modules N' and N'' are not isomorphic, and hence $N' \oplus N''$ is a direct summand of N; accordingly,

$$\ell(N') + \ell(N'') \leqslant \ell(N) < \ell(U) + \ell(V).$$

Therefore, $\ell(N') < \ell(U)$ or else $\ell(N'') < \ell(V)$. In the former case, $\ell(N') + \ell(V) < \ell(U) + \ell(V)$; and, in the latter case, $\ell(U) + \ell(N'') < \ell(U) + \ell(V)$. Both cases contradict the minimality of $\ell(U) + \ell(V)$. □

The second step in proving (0.6) consists in gathering information on indecomposable A-modules which have successors in $\operatorname{ind} A$ with a large injective dimension. This is done in the following lemma.

Lemma 3.3. *Let n be a positive integer. Let $\mathcal{C}$ be a torsion-free class in $\operatorname{mod} A$ such that, for all $X \in \operatorname{ind} A$,*

$$X \in \mathcal{C}, \text{ or else } \operatorname{id}_A X \leqslant n.$$

Then

(1) for every path $Y \to U \to V$ in $\operatorname{ind} A$ such that $\operatorname{id}_A V > n$, there exists $Z \in \operatorname{ind} A$ such that $\operatorname{id}_A Z > n$ and $\operatorname{Hom}_A(Y, Z) \neq 0$;
(2) for every $Y \in \operatorname{ind} A$ which has a successor in $\operatorname{ind} A$ with injective dimension greater than n, there exists $Z \in \operatorname{ind} A$ such that $\operatorname{id}_A Z > n$ and $\operatorname{Hom}_A(Y, Z) \neq 0$;
(3) if $X \in \operatorname{ind} A$ is such that $\operatorname{id}_A X > n$, then every predecessor of X in $\operatorname{ind} A$ lies in $\mathcal{C}$.

Proof. (1) Let $Y \to U \to V$ be a path in $\operatorname{ind} A$ such that $\operatorname{id}_A V > n$. By absurd, assume that $\operatorname{Hom}_A(Y, Z) = 0$ for all $Z \in \operatorname{ind} A$ such that $\operatorname{id}_A Z > n$. Denote by $\mathcal{C}'$ the following torsion-free class of $\operatorname{mod} A$

$$\mathcal{C}' = \{M \in \operatorname{mod} A; \operatorname{Hom}_A(Y, M) = 0\}.$$

By assumption, $U \notin \mathcal{C}'$, $\operatorname{id}_A V > n$ and the following holds for all $X \in \operatorname{ind} A$,

$$X \in \mathcal{C}', \text{ or else } \operatorname{id}_A X \leqslant n.$$

Apply Lemma 3.2 to U and V for the torsion-free class $\mathcal{C}'$; then $\operatorname{Hom}_A(U, V) = 0$, a contradiction to $U \to V$ being nonzero. Thus, there exists $Z \in \operatorname{ind} A$ such that $\operatorname{id}_A Z > n$ and $\operatorname{Hom}_A(Y, Z) \neq 0$.

(2) Using (1), an induction on ℓ shows that, for every $Y \in \operatorname{ind} A$, if there exists a path of length ℓ in $\operatorname{ind} A$ starting in Y and ending in a module with injective dimension greater than n, then there exists $Z \in \operatorname{ind} A$ such that $\operatorname{id}_A Z > n$ and $\operatorname{Hom}_A(Y, Z) \neq 0$.

(3) Let $X \in \operatorname{ind} A$ be such that $\operatorname{id}_A X > n$. Let Y be a predecessor of X in $\operatorname{ind} A$. Following (2), there exists $Z \in \operatorname{ind} A$ such that $\operatorname{id}_A Z > n$ and $\operatorname{Hom}_A(Y, Z) \neq 0$. Lemma 3.2 entails that $Y \in \mathcal{C}$. □

Now, it is possible to prove (0.6). Recall that, for every torsion-free class $\mathcal{C}$ of $\operatorname{mod} A$, the piece of notation $\mathcal{L}_{\mathcal{C}}$ denotes the class of $X \in \operatorname{ind} A$ such that every predecessor of X in $\operatorname{ind} A$ lies in $\mathcal{C}$.

Proposition 3.4. *Let n be a positive integer. Let $\mathcal{C}$ be a torsion-free class of $\operatorname{mod} A$ such that, for all $X \in \operatorname{ind} A$,*

$$X \in \mathcal{C}, \text{ or else } \operatorname{id}_A X \leqslant n.$$

Then $\operatorname{ind} A = \mathcal{L}_{\mathcal{C}} \cup \mathcal{R}_A^n$.

Proof. Let $X \in \operatorname{ind} A \backslash \mathcal{R}_A^n$. Then there exists a successor Z of X in $\operatorname{ind} A$ such that $\operatorname{id}_A Z > n$. Part (3) of Lemma 3.3 entails that $Z \in \mathcal{L}_{\mathcal{C}}$. Thus $X \in \mathcal{L}_{\mathcal{C}}$. □

The previous proposition does not apply to (m, n)-almost hereditary algebras for general m and n because, when $\operatorname{gl.dim} A = m + n$, the class of A-modules with projective dimension at most m need not be torsion-free. However, Proposition 3.4 may be applied to certain algebras satisfying $(Q2)$ as the following result shows.

Theorem 3.5. *Let m, n be positive integers. Assume that*

(a) $\operatorname{gl.dim} A = \max\{m, n\} + 1$; and
(b) for all $X \in \operatorname{ind} A$, then $\operatorname{pd}_A X \leqslant m$, or else $\operatorname{id}_A X \leqslant n$.

Then $\operatorname{ind} A = \mathcal{L}_A^m \cup \mathcal{R}_A^n$.

Proof. Assume first that $m \geqslant n$. Denote by $\mathcal{C}$ the class

$$\mathcal{C} = \{M \in \operatorname{mod} A; \operatorname{pd}_A M \leqslant m\};$$

since $\operatorname{gl.dim} A = m + 1$, this is a torsion-free class in $\operatorname{mod} A$. By definition, $\mathcal{L}_A^m = \mathcal{L}_{\mathcal{C}}$. Hence, the conclusion follows from Proposition 3.4.

When $n \geqslant m$, the conclusion follows from dual considerations, using the dual version of Proposition 3.4. □

3.2. **Applications to $(m,1)$-almost hereditary algebras.** Now, we can prove the main result of this section.

Theorem 3.6. *Let A be a $(m,1)$-almost hereditary algebra. Then* $\operatorname{ind} A = \mathcal{L}_A^m \cup \mathcal{R}_A$.

Proof. The theorem now follows from Theorem 3.5. □

It is a consequence of Theorem 3.6 that the subcategories $\mathcal{L}_A^m$ and $\mathcal{R}_A$ determine a trisection in the sense of [7, Chapter II, Section 1, p. 36] in $\operatorname{ind} A$, when A is a $(m,1)$-almost hereditary algebra, as stated in the following corollary.

Corollary 3.7. *Let A be an algebra which satisfies conditions (i) and (ii) as described in Theorem 3.5. Then $(\mathcal{L}_A^m \setminus \mathcal{R}_A^n, \mathcal{L}_A^m \cap \mathcal{R}_A^n, \mathcal{R}_A^n \setminus \mathcal{L}_A^m)$ is a trisection in* $\operatorname{ind} A$. *In particular, if A is $(m,1)$-almost hereditary, then $(\mathcal{L}_A^m \setminus \mathcal{R}_A, \mathcal{L}_A^m \cap \mathcal{R}_A, \mathcal{R}_A \setminus \mathcal{L}_A^m)$ is a trisection in* $\operatorname{ind} A$.

To end this section, we discuss sufficient conditions for an algebra to be $(m,1)$-almost hereditary. It is proved in [7], then an algebra A is $(1,1)$-almost hereditary if and only if all indecomposable projective modules belong to $\mathcal{L}_A$. When replacing $\mathcal{L}_A$ and $(1,1)$ by $\mathcal{L}_A^m$ and $(m,1)$, respectively, part of the equivalence may be proved. In order to do so, it is again convenient to first replace the class of modules with projective dimension at most m by a torsion-free class.

Proposition 3.8. *Let $\mathcal{C}$ be a torsion-free class of* $\operatorname{mod} A$. *If $A \in \operatorname{add} \mathcal{L}_{\mathcal{C}}$, then for all $X \in \operatorname{ind} A$,*

$$X \in \mathcal{C}, \text{ or else } \operatorname{id}_A X \leqslant 1.$$

Proof. Let $X \in \operatorname{ind} A$ be of injective dimension at least 2. Then $\operatorname{Hom}_A(\tau^{-1}X, A) \neq 0$. Therefore, $\tau^{-1}X \in \mathcal{L}_{\mathcal{C}}$, and hence $X \in \mathcal{C}$. □

Now, here is a sufficient condition for an algebra to be $(m,1)$-almost hereditary.

Proposition 3.9. *Let m be a positive integer. Assume that $A \in \operatorname{add} \mathcal{L}_A^m$. Then*

(1) $\operatorname{gl.dim} A \leqslant m+1$ *and, for all $X \in \operatorname{ind} A$, then* $\operatorname{pd}_A X \leqslant m$ *or else* $\operatorname{id}_A X \leqslant 1$*;*
(2) if, moreover, $\operatorname{gl.dim} A = m+1$*, then A is $(m,1)$-almost hereditary.*

Proof. (1) Since $A \in \mathcal{L}_A^m$, the syzygy of any A-module has projective dimension at most m. Accordingly, $\operatorname{gl.dim} A \leqslant m+1$. Denote by $\mathcal{C}$ the class

$$\mathcal{C} = \{M \in \operatorname{mod} A \mid \operatorname{pd}_A M \leqslant m\};$$

since $\operatorname{gl.dim} A \leqslant m+1$, this is a torsion-free class of $\operatorname{mod} A$. The rest of the statement of (1) therefore follows from Proposition 3.8.

(2) follows directly from (1). □

We make the following conjecture.

Conjecture 3.10. Any $(m,1)$-almost hereditary algebra A is such that $A \in \operatorname{add}\ \mathcal{L}_A^m$.

The corresponding statement for (m,n)-almost hereditary does not hold when $n > 1$ as shown in the following example.

Example 3.11. Let A be the radical square zero path algebra given by the quiver $m+n+1 \to \cdots \to 2 \to 1$.
Then A is (m,n)-almost hereditary and $P_{m+n+1} \notin \mathcal{L}_A^m$.

4. One-point extensions of $(m,1)$-almost hereditary algebras

From now on, we assume that k is a field and A is a finite dimensional k-algebra. The purpose of this section is to investigate how $(m,1)$-almost hereditary algebras behave under one-point extension process. In particular, conditions for the one-point extension to be $(m,1)$-almost hereditary are presented. First, here is a necessary condition for a one-point extension to be a $(m,1)$-almost hereditary algebra.

Proposition 4.1. *Assume that k is a field. Let B be a finite dimensional k-algebra with $\mathrm{gl.dim}B = m+1$ and assume that $A = B[M]$ for some B-module. If A is a $(m,1)$-almost hereditary algebra, then B is $(m,1)$-almost hereditary.*

Proof. When $m = 1$, the proposition is proved in [7, Chapter III, Proposition 2.3]. Up to replacing the inequality "$\mathrm{pd} > 1$" by "$\mathrm{pd} > m$", the proof given there works here in the general case. □

We point out that the hypothesis on $\mathrm{gl.dim}B$ above is necessary to conclude that B is $(m,1)$-almost hereditary for some positive integer m as shown in the following example.

Example 4.2. Let B be the path algebra of the quiver

$$\begin{tikzcd} 5 \arrow[r] \arrow[rr, bend left] & 4 \arrow[r] \arrow[rr, bend left] & 3 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 1, \end{tikzcd}$$

which has $\mathrm{gl.dim}B = 2$ but is not $(1,1)$-almost hereditary. However, by letting $M = S_5$ we obtain the $(2,1)$-almost hereditary algebra $A = B[M]$ given by the quiver

$$\begin{tikzcd} 6 \arrow[r] \arrow[rr, bend left] & 5 \arrow[r] \arrow[rr, bend right] & 4 \arrow[r] & 3 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 1. \end{tikzcd}$$

For $m = 1$, it is well known that the converse of Proposition 4.1 does not hold true. This is also the case for each positive integer $m \neq 1$, as we can see in the following example.

Example 4.3. Let B be the path algebra given by the quiver

$$\begin{tikzcd} m+3 \arrow[r] & m+2 \arrow[r] \arrow[rr, bend left] & m+1 \arrow[r] \arrow[rr, bend right] & m \arrow[r] & \cdots \arrow[r] & 3 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 1. \end{tikzcd}$$

where all the paths of length two with source in $\{1,\dots,m+2\}$ are relations. It can be easily checked that B is $(m,1)$-almost hereditary. Now, taking $M = I_{m+3} = S_{m+3}$, the one-point extension algebra $A = B[M]$ is such that $\mathrm{gl.dim}A = \mathrm{gl.dim}B$, but it is not $(m,1)$-almost hereditary, since $\mathrm{pd}_A S_{m+2} = m+1$ and $\mathrm{id}_A S_{m+2} = 2$.

Besides Example 4.3, if $M \in \mathrm{add}\ \mathcal{L}_B^m$ then property $(Q2)$ of the definition of (m,n)-almost hereditary algebra is satisfied by those indecomposable $B[M]$-modules of the shape $(0,X,0)$, as stated in the following proposition.

Proposition 4.4. *Assume that k is a field. Let B be a $(m,1)$-almost hereditary algebra oer k and let $A = B[M]$ for some B-module M. If $M \in \mathrm{add}\ \mathcal{L}_B^m$, then:*

(i) $\mathrm{gl.dim}A = m+1$,
(ii) if $(0,X,0)$ is an indecomposable A-module, then $\mathrm{pd}_A(0,X,0) \leq m$, or else $\mathrm{id}_A(0,X,0) \leq 1$.

Proof. When $m = 1$, the proposition is proved in [7, Chapter III, Lemma 2.5]. The proof given there may be adapted to prove the general case by replacing the inequality "$\mathrm{pd} > 1$" by "$\mathrm{pd} > m$". □

In order to present a sufficient condition for $B = A[M]$ to be a $(m,1)$-almost hereditary algebra, the following technical lemma is needed.

Lemma 4.5. *Assume that k is a field. Let B be a finite dimensional k-algebra with $\mathrm{gl.dim}B = m+1$ and assume that $A = B[M]$ for a B-module M. If (Y,X,f) is an A-module, then $\mathrm{pd}_A(Y,X,f) \leq m$ if and only if:*

(1) $\mathrm{pd}_B \operatorname{Ker} f \leq m-1$,
(2) $\mathrm{Ext}_B^{m-1}(\operatorname{Ker} f, -) \xrightarrow{\theta} \mathrm{Ext}_B^{m+1}(\operatorname{Coker} f, -) \to 0$ *is an exact sequence, where θ is naturally induced by f.*

Proof. Let (Y,X,f) be an A-module. Since $(0, P_{\operatorname{Coker} f}, 0) \oplus (Y, M \otimes_k Y, 1_{M\otimes Y})$ is a projective cover of (Y,X,f), it is possible to construct the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & M\otimes_k Y & \xrightarrow{1_{M\otimes Y}} & M\otimes_k Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K & \longrightarrow & M\otimes_k Y \oplus P_{\operatorname{Coker} f} & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

In this case, $\mathrm{pd}_A(Y,X,f) \leq m$ if and only if $\mathrm{pd}_A(0,K,0) \leq m-1$. By the snake Lemma we get the exact sequences $0 \to \operatorname{Ker} f \to K \to P_{\operatorname{Coker} f} \to \operatorname{Coker} f \to 0$ and $0 \to \operatorname{Ker} f \to K \to \Omega^1(\operatorname{Coker} f) \to 0$, and from the latter we obtain that

$$\mathrm{pd}_B \operatorname{Ker} f \leq max\{\mathrm{pd}_B K, \mathrm{pd}_B \Omega^1(\operatorname{Coker} f) - 1\}\,. \tag{4.1}$$

Since $\mathrm{gl.dim}B = m+1$, it follows from (4.1) that $\mathrm{pd}_B \operatorname{Ker} f \leq m-1$ if $\mathrm{pd}_A(Y,X,f) \leq m$, which proves (1). Now, by applying $\mathrm{Hom}(_,-)$ to $0 \to \operatorname{Ker} f \to K \to \Omega^1(\operatorname{Coker} f) \to 0$ we get the exact sequence $\mathrm{Ext}_B^{m-1}(\operatorname{Ker} f, -) \xrightarrow{g} \mathrm{Ext}_B^m(\Omega^1(\operatorname{Coker} f), -) \to 0$. The connecting morphism $\delta : \mathrm{Ext}_B^m(\Omega^1(\operatorname{Coker} f), -) \to \mathrm{Ext}_B^{m+1}(\operatorname{Coker} f, -)$ is an isomorphism, hence $\mathrm{Ext}_B^{m-1}(\operatorname{Ker} f, -) \xrightarrow{\theta} \mathrm{Ext}_B^{m+1}(\operatorname{Coker} f, -) \to 0$ is an exact sequence, where $\theta = \delta g$.

Conversely, assume (1) and (2). In order to prove that $\mathrm{pd}_A(Y,X,f) \leqslant m$, it is enough to prove that $\mathrm{pd}_B K \leq m-1$. Applying $\mathrm{Hom}(_,-)$ to

$$0 \to \operatorname{Ker} f \to K \to \Omega^1(\operatorname{Coker} f) \to 0$$

gives the exact sequence $\mathrm{Ext}_B^{m-1}(\operatorname{Ker} f, -) \xrightarrow{g} \mathrm{Ext}_B^m(\Omega^1(\operatorname{Coker} f), -) \to \mathrm{Ext}_B^m(K,-) \to \mathrm{Ext}_B^m(\operatorname{Ker} f, -) = 0$. It follows from (2) that $\theta = \delta g$ is an epimorphism, and so is g. Hence $\mathrm{Ext}_B^m(K,-) = 0$, that is, $\mathrm{pd}_B K \leq m-1$. □

Now, here is a sufficient condition for a one-point extension to be $(m,1)$-almost hereditary.

Theorem 4.6. *Assume that k is a field. Let B be a $(m,1)$-almost hereditary algebra and assume that $A = B[M]$ for a projective B-module M. If, for every indecomposable A-module (k^t, X, f), one has $\mathrm{pd}_B X \leq m$ or else $\mathrm{id}_B X \leq 1$, then A is $(m,1)$-almost hereditary.*

Proof. It is proved in [7, Chapter III, Proposition 1.3] that $\mathrm{gl.dim}A = max\{\mathrm{gl.dim}B, \mathrm{pd}_B M + 1\}$, which allows us to conclude that $\mathrm{gl.dim}A = m+1$. Let (k^t, X, f) be an indecomposable A-module. Assume first $t = 0$. In this case, $\mathrm{pd}_B X \leq m$ or else $\mathrm{id}_B X \leq 1$, since B is $(m,1)$-almost hereditary and X is an indecomposable B-module. Therefore $\mathrm{pd}_A(0,X,0) \leq m$ or else $\mathrm{id}_A(0,X,0) \leq 1$.

Suppose now $t \neq 0$. According to [7, Chapter III, Lemma 2.2], if $\mathrm{id}_B X \leq 1$, then $\mathrm{id}_A(k^t, X, f) \leq 1$, because $\mathrm{Ext}_B^1(M,X) = 0$. Finally, assume that $\mathrm{pd}_B X \leq m$ and consider the following short exact sequences

$$\begin{cases} 0 \to \operatorname{Ker} f \to M \otimes_k k^t \to \operatorname{Im} f \to 0 \\ 0 \to \operatorname{Im} f \to X \to \operatorname{Coker} f \to 0\,. \end{cases} \tag{4.2}$$

Lemma 4.5 will be applied to conclude that $\mathrm{pd}_A(k^t, X, f) \leq m$. For this purpose we first prove that $\mathrm{pd}_B \operatorname{Ker} f \leq m-1$. Applying $\mathrm{Hom}(_, -)$ to (4.2) gives rise to the exact sequences

$$\begin{cases} 0 = \mathrm{Ext}_B^m(M^t, -) \to \mathrm{Ext}_B^m(\operatorname{Ker} f, -) \to \mathrm{Ext}_B^{m+1}(\operatorname{Im} f, -) \to \mathrm{Ext}_B^{m+1}(M^t, -) = 0 \\ 0 = \mathrm{Ext}_B^{m+1}(X, -) \to \mathrm{Ext}_B^{m+1}(\operatorname{Im} f, -) \to \mathrm{Ext}_B^{m+2}(\operatorname{Coker} f, -) = 0\,, \end{cases} \tag{4.3}$$

from where we get that $\mathrm{Ext}_B^m(\operatorname{Ker} f, -) = 0$. Thus $\mathrm{pd}_B \operatorname{Ker} f \leq m-1$.

Now, since $\mathrm{pd}_B X \leq m$ there are exact sequences $\mathrm{Ext}_B^{m-1}(\operatorname{Ker} f, -) \xrightarrow{\alpha} \mathrm{Ext}_B^m(\operatorname{Im} f, -) \to 0$ and $0 \to \mathrm{Ext}_B^m(\operatorname{Im} f, -) \xrightarrow{\beta} \mathrm{Ext}_B^{m+1}(\operatorname{Coker} f, -) \to \mathrm{Ext}_B^{m+1}(X, -) = 0$. Therefore

$$\mathrm{Ext}_B^{m-1}(\operatorname{Ker} f, -) \xrightarrow{\beta\alpha} \mathrm{Ext}_B^{m+1}(\operatorname{Coker} f, -) \to 0$$

is an exact sequence. By Lemma 4.5, $\mathrm{pd}_A(k^t, X, f) \leq m$, which finishes the proof. □

Corollary 4.7. *Assume that k is a field. Let B be a $(m, 1)$-almost hereditary algebra and let M be a projective B-module such that*

$$\mathrm{Hom}_B(M, -)|_{\mathcal{R}_B \setminus \mathcal{L}_B^m} = 0.$$

Then $B[M]$ is $(m, 1)$-almost hereditary.

Proof. Let t be a natural integer. Notice that if (k^t, X, f) is an indecomposable A-module, then any indecomposable direct summand of X is a successor in $\operatorname{ind} B$ of an indecomposable direct summand of M^t. Since there is no nonzero morphism from M to $\mathcal{R}_B \setminus \mathcal{L}_B^m$ we get that $X \in \operatorname{add} \mathcal{L}_B^m$, hence $\mathrm{pd}_B X \leq m$. The conclusion therefore follows from Theorem 4.6. □

REFERENCES

[1] Ibrahim Assem, Flávio U. Coelho, Marcelo Lanzilotta, David Smith, and Sonia Trepode. Algebras determined by their left and right parts. In *Algebraic structures and their representations*, volume 376 of *Contemp. Math.*, pages 13–47. Amer. Math. Soc., Providence, RI, 2005.

[2] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[3] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[4] Dieter Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[5] Dieter Happel and Idun Reiten. Hereditary abelian categories with tilting object over arbitrary base fields. *J. Algebra*, 256(2):414–432, 2002.

[6] Dieter Happel, Idun Reiten, and Sverre Smalø. Piecewise hereditary algebras. *Arch. Math.*, 66(3):182–186, 1996.

[7] Dieter Happel, Idun Reiten, and Sverre O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.*, 120(575):viii+ 88, 1996.

[8] Dieter Happel, Jeremy Rickard, and Aidan Schofield. Piecewise hereditary algebras. *Bull. London Math. Soc.*, 20(1):23–28, 1988.

[9] Dieter Happel and Claus Michael Ringel. Tilted algebras. *Trans. Amer. Math. Soc.*, 274(2):399–443, 1982.

[10] Dieter Happel and Uwe Seidel. Piecewise hereditary Nakayama algebras. *Algebr. Represent. Theory*, 13(6):693–704, 2010.

[11] Dieter Happel and Dan Zacharia. A homological characterization of piecewise hereditary algebras. *Math. Z.*, 260(1):177–185, 2008.

[12] Mitsuo Hoshino. On splitting torsion theories induced by tilting modules. *Commun. Algebra*, 11:427–439, 1983.

[13] Claus Michael Ringel. *Tame algebras and integral quadratic forms*, volume 1099 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

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