HAL
open science

ENDOSCOPIC CONGRUENCES MODULO ADJOINT L-VALUES FOR GSp(4)

Francesco Lemma, Tadashi Ochiai

To cite this version:

Francesco Lemma, Tadashi Ochiai. ENDOSCOPIC CONGRUENCES MODULO ADJOINT L-VALUES FOR GSp(4). 2023. hal-03991987

HAL Id: hal-03991987

https://hal.science/hal-03991987

Preprint submitted on 16 Feb 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1806.02135v5 [math.NT] 31 Oct 2019

ENDOSCOPIC CONGRUENCES MODULO ADJOINT L-VALUES FOR GSp(4)

FRANCESCO LEMMA AND TADASHI OCHIAI

Abstract. We establish the existence of congruences between a fixed endoscopic, globally generic, cuspidal automorphic representation Π of GSp(4) of square-free conductor and stable cuspidal automorphic representations of the same weight modulo certain prime factors of the value at 1 of the adjoint L-function of Π normalized by a suitable period.

Contents

1. Introduction 1
2. Notation and conventions 5
2.1. The algebraic group GSp(4) and its algebraic representations 5
2.2. Measures 6
2.3. Representations of K_∞ and discrete series classification 7
2.4. Hecke algebras 10
3. Adjoint L-value and Petersson norm after Ichino 12
4. Integral Betti cohomology of Siegel threefolds 14
4.1. Integral local systems on Siegel threefolds 14
4.2. Hecke action on integral cohomology 15
4.3. Isotypical component of cohomology 17
4.4. Poincaré duality 20
5. Definition of the periods 27
6. Discriminant and adjoint L-values 30
7. The congruence criterion 39
References 42

1. Introduction

In modern number theory, the study of congruences modulo a prime number p between automorphic forms plays a central role. It gives a basis of the theory of p-adic families of automorphic forms and the theory of p-adic families of p-adic Galois representations and it also gives us a deeper understanding of some important arithmetic objects. For example, since the prime number 691 divides the numerator of the rational number $\frac{\zeta(12)}{\pi^{12}}$, it divides the constant term of the unique normalized Eisenstein series E_{12} of weight 12. This yields a congruence modulo 691 between E_{12} and the Ramanujan cuspform Δ of weight 12 and the existence of such a congruence helps us to understand the action of $\mathrm{Gal}(\mathbb{Q}(\mu_{691})/\mathbb{Q})$

on the 691-part of the class group of $\mathbb{Q}(\mu_{691})$ (see [Ri76]).

Instead of the Eisenstein series E_{12}, we consider an automorphic representation Π of $\mathrm{GSp}(4,\mathbb{A})$, where $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$, which is cuspidal but whose functorial lift to $\mathrm{GL}(4,\mathbb{A})$ is not cuspidal. Such an automorphic representation Π is called endoscopic. In this article, we investigate congruences between Π and cuspidal automorphic representation Π' of $\mathrm{GSp}(4,\mathbb{A})$ whose functorial lift to $\mathrm{GL}(4,\mathbb{A})$ remains cuspidal. Such automorphic representations Π' are called stable.

In [Hi81a], Hida considers a holomorphic cuspform f of $\mathrm{GL}(2,\mathbb{A})$ and establishes the existence of congruences between f and other cuspforms of the same level and weight as f modulo certain prime numbers dividing a value of the adjoint L-function of f, normalized by a suitable period. Let us recall Hida's theorem. For a primitive cuspform $f \in S_\kappa(\Gamma_1(M))$, $Z(\kappa, f)$ denotes the product

$$Z(\kappa, f) = \prod_{\sigma: E \to \mathbb{C}} L(\kappa, \mathrm{Sym}^2({}^\sigma f)) = \prod_{\sigma: E \to \mathbb{C}} L(1, \pi({}^\sigma f), \mathrm{Ad})$$

where E denotes the number field generated by the Fourier coefficients of f, $\pi({}^\sigma f)$ denotes the cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ attached to ${}^\sigma f$ and $L(s, \pi({}^\sigma f), \mathrm{Ad})$ denotes the imprimitive adjoint L-function attached to ${}^\sigma f$. Let $c(f)$ denote the real number defined as

$$c(f) = \varepsilon(M)^\rho((\kappa-1)! \cdot 2^{-(\kappa+2)})^\rho (M\varphi(M))^\rho \frac{Z(\kappa, f)}{u(f)\pi^{(\kappa+1)\rho}} \tag{1}$$

where ρ denotes $[E:\mathbb{Q}]$, where $\varepsilon(M) = 12, 4$ or 1 according as $M = 1, 2$ or ≥ 3 and where $u(f)$ denotes the period defined in [Hi81a, (6.6_c)]. It follows from [Hi81a, Theorem 6.2] that $c(f)^2 \in \mathbb{Z} \setminus \{0\}$.

The following theorem is a particular case of the main theorem of [Hi81a] stated assuming that the Nebentypus of f is trivial.

Theorem 1.1 (Hida)**.** *Let $f \in S_\kappa(\Gamma_1(M))$ be a primitive cuspform, with $\kappa \geq 2$ and with trivial Nebentypus. Then, if a prime p such that $p > \kappa - 2$ and $p \nmid 6M$ divides $c(f)^2$, there exists a congruence between f and another normalized eigen cuspform $g \in S_\kappa(\Gamma_1(M))$ which is not conjugate to f modulo a prime ideal above p in $\overline{\mathbb{Q}}$.*

As we announced earlier, we will prove an extension of Theorem 1.1 to congruences between a fixed endoscopic, globally generic, cuspidal automorphic representation Π of $\mathrm{GSp}(4)$ of square-free conductor and stable cuspidal automorphic representations. As we need it in the proof of our main result, we use a variant of Theorem 1.1 proved by Ghate [Gh02].

Let $\Gamma(s, \mathrm{Ad}(f))$ denote the Γ-factor of $L(s, \mathrm{Ad}(f))$ and let $W(f)$ be the complex constant that arises in the functional equation of the standard L-function attached to f (see [Gh02] for the precise definitions).

Theorem 1.2 (Ghate)**.** *Let $f \in S_\kappa(\Gamma_1(M))$ be a primitive cuspform, with $\kappa \geq 2$ and with trivial Nebentypus. Let K denote the number field which appears in* [Gh02, Theorem 1].

Let us fix a prime $\mathfrak{p}$ of K over a prime $p > \kappa - 2$ and such that $p \nmid 6M$ and let us denote by $\mathcal{O}_{(\mathfrak{p})}$ the ring of integers of K localized at $\mathfrak{p}$. Let $c'(f) \in \mathcal{O}_{(\mathfrak{p})}$ be as follows

$$c'(f) = W(f)\Gamma(1, \mathrm{Ad}(f))\frac{L(1, \pi(f), \mathrm{Ad})}{\omega(f,+)\omega(f,-)} \tag{2}$$

where $\omega(f,+), \omega(f,-) \in \mathbb{C}^\times/\mathcal{O}_{(\mathfrak{p})}^\times$ are complex periods given in [Gh02]*. Then, if $\mathfrak{p}$ divides $c'(f)$, there exists a congruence between f and another normalized eigen cuspform $g \in S_\kappa(\Gamma_1(M))$ different from f modulo $\mathfrak{p}$.*

To state our result, let us consider a cuspidal automorphic representation $\Pi \simeq \Pi_\infty \otimes \Pi_f$ which satisfies the conditions (i)-(v) of Section 3. In particular Π_∞ is a generic discrete series of $\mathrm{GSp}(4, \mathbb{R})$ of Harish-Chandra parameter $(k+3, k'+1)$ for two integers $k \geq k' \geq 0$ and Π has square-free paramodular conductor N. The conditions on Π imply that Π_f is defined over its rationality field, which is a number field $E(\Pi_f)$ of degree r over $\mathbb{Q}$. Let us fix a prime number p and assume that $p \notin S_{N,3} \cup S_{\mathrm{weight}} \cup S_{\mathrm{tors}} \cup S'_{\mathrm{tors}}$ where $S_{N,3}$, S_{tors}, S'_{tors}, S_{weight} are defined by (15), (16), (23) and (24) respectively. Considering the Betti cohomology of Siegel threefolds, we define a free $\mathbb{Z}_{(p)}$-module $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ of finite rank $2r$ endowed with a natural bilinear form

$$\langle\, ,\, \rangle : L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \times L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}$$

and a period $\Omega(\Pi_f) \in \mathbb{R}^\times / \mathbb{Z}_{(p)}^\times$ (see equation (34) and Remark 5.6). Let $\delta_1, \ldots, \delta_{2r}$ be a $\mathbb{Z}_{(p)}$-basis of $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ and let $d(\Pi_f) = \det(\langle \delta_i, \delta_j \rangle)_{1 \leq i,j \leq 2r}$ be the discriminant of $\langle\, ,\, \rangle$. This is an element of $\mathbb{Z}_{(p)}$ whose image in $\mathbb{Z}_{(p)}/(\mathbb{Z}_{(p)}^\times)^2$ does not depend on the choice of the basis $\delta_1, \ldots, \delta_{2r}$. Let $C_{k,k'} = \dfrac{(-1)^{k+k'}(k+k'+4)!(k+k'+5)!}{3^3 \cdot 5}$ and $C_N = \prod_{l|N}(l + l^{-1})^{-1}(l^2+1)^{-1}$. For $x, y \in \mathbb{R}$ we write $x \sim y$ if there exists $s \in (\mathbb{Z}_{(p)}^\times)^2$ such that $x = sy$.

Theorem 1.3 (Theorem 6.8)**.** *Let $\Pi \simeq \Pi_\infty \otimes \Pi_f$ be a cuspidal automorphic representation given above. Then we have:*

$$d(\Pi_f) \sim \left(\left(\frac{2^{k+k'+13} C_{k,k'} C_N \pi^{3k+k'+12}}{k+k'+5} \right)^r \Omega(\Pi_f)^{-1} \prod_{\sigma : E(\Pi_f) \to \mathbb{C}} L(1, {}^\sigma\Pi, \mathrm{Ad}) \right)^2.$$

We shall prove Theorem 1.3 at the end of Section 6. A direct consequence of Theorem 1.3 is that the real number

$$\left(\left(\frac{2^{k+k'+13} C_{k,k'} C_N \pi^{3k+k'+12}}{k+k'+5} \right)^r \Omega(\Pi_f)^{-1} \prod_{\sigma : E(\Pi_f) \to \mathbb{C}} L(1, {}^\sigma\Pi, \mathrm{Ad}) \right)^2$$

is in fact a non-zero rational number which is a p-integer. Moreover, the assumption that Π is endoscopic implies that it is obtained as a Weil lifting of two primitive classical elliptic modular forms f_1 and f_2 of weights $k_1 = k+k'+4$ and $k_2 = k-k'+2$ and level N_1 and N_2, with $N_1 N_2 = N$. Given a prime ideal $\mathfrak{P}$ above p in $\overline{\mathbb{Q}}$ and another cuspidal automorphic representation Π' of $\mathrm{GSp}(4, \mathbb{A})$, we write $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$ if Π' is congruent to Π modulo $\mathfrak{P}$ (see Definition 7.1).

Theorem 1.4 (Theorem 7.2)**.** *In addition to the setting given above, assume further that p is prime to N and is outside the finite set S''_{tors} which is defined by* (26)*. Assume the following conditions:*

(a) *the residual* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representations* $\overline{\rho}_{f_1}$ *and* $\overline{\rho}_{f_2}$ *of* f_1 *and* f_2 *are irreducible,*
(b) *the prime* $\mathfrak{p}$ *does not divide* $c'(f_1)$ *nor* $c'(f_2)$ *for any prime* $\mathfrak{p}$ *above* p *in* $\overline{\mathbb{Q}}$*,*
(c) *the prime* p *divides*

$$\left(\left(\frac{2^{k+k'+13}C_{k,k'}C_N\pi^{3k+k'+12}}{k+k'+5}\right)^r \Omega(\Pi_f)^{-1} \prod_{\sigma:E(\Pi_f)\to\mathbb{C}} L(1,{}^\sigma\Pi,\mathrm{Ad})\right)^2,$$

Then, there exists a prime divisor $\mathfrak{P}$ *of* p *in* $\overline{\mathbb{Q}}$ *and a cuspidal representation* $\Pi' \simeq \bigotimes'_v \Pi'_v$ *of* $G(\mathbb{A})$ *such that*

(1) *the non archimedean part* Π'_f *of* Π' *satisfies* $(\Pi'_f)^{K_N} \neq 0$*,*
(2) *the representation* Π'_∞ *is a discrete series with the same parameter as* Π_∞*,*
(3) *the cuspidal representation* Π' *is stable,*
(4) *we have* $\Pi' \not\simeq {}^\sigma\Pi$ *for all* $\sigma \in \mathrm{Aut}(\mathbb{C})$*,*
(5) *we have* $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$*.*

We shall prove Theorem 1.4 at the end of Section 7.

Note that by the work of Lan-Suh [LS13], the primes in S_{tors} and S'_{tors} can be shown to be smaller than and explicit bound when the weight is sufficiently regular. On the other hand S''_{tors} is a subset of the prime numbers dividing the torsion in the cohomology of the boundary and hence is more delicate.

Before our work, Hida's theorem has been extended to cuspforms of GL(2) over an imaginary quadratic field by Urban [U95], to Hilbert modular cuspforms by Ghate [Gh02] and Dimitrov [Di05], to cuspforms of GL(2) over an arbitrary number field by Namikawa [N15] and to cuspforms of GL(n) over an arbitrary number filed by Balasubramanyam and Raghuram [BaR17]. Congruences between holomorphic Yoshida lifts and stable cusp forms on GSp(4) has also been discussed in [Ka08], [BDSP12] and [AK13] but the methods of these papers are entirely different from ours.

In the proof of Theorem 1.3, the work of Chen-Ichino [CI19, Theorem 1.1] relating the Petersson norm of a suitably normalized cuspform $\varphi \in \Pi$ to $L(1,\Pi,\mathrm{Ad})$ plays an important role ([CI19] is a refinement of an unpublished preprint [I08]). The assumptions (i)-(iv) on Π stated above are all already present as hypothesis of loc. cit. Theorem 1.1, which also covers the case where Π is stable. Assumption (v) says that Π is endoscopic.

One of our important contributions for Theorem 1.3 is to compute the discriminant $d(\Pi_f)$ in terms of the explicit constant $C_{k,k'}$, the period $\Omega(\Pi_f)$ and the Petersson norms of the normalized cuspforms ${}^\sigma\varphi \in {}^\sigma\Pi$. Note that in [BaR17], the rational number analogous to $C_{k,k'}$ is not explicitly computed as in the present work. It seems difficult to perform a similar explicit computation for higher rank reductive groups. Furthermore, let us emphasize that this work is the first application of Hida's ideas to a reductive group different from GL(n) and in particular to the study of endoscopic congruences. Moreover this is the first work considering endoscopic congruences for generic, hence non-holomorphic, cuspforms

on GSp(4).

Aknowledgments. The first-named author aknowledges support of the ANR Formules des Traces Relatives, Périodes, Fonctions L et Analyse Harmonique [ANR-13-BS01-0012]. The second-named author is supported by KAKENHI (Fostering Joint International Research: Garant Number 16KK0100, Fund for the Promotion of Joint International Research and Grant-in-Aid for Scientic Research (B): Grant Number 26287005) of JSPS. This work progressed during the stay of the second-named author at IMJ-PRG and Université Paris 7 in the year 2017-2018. The second-named expresses his gratitude to IMJ-PRG and Université Paris 7 for their hospitality. We would like also to thank Atsushi Ichino, Giovanni Rosso and Jacques Tilouine for encouragements and feedback.

2. Notation and conventions

2.1. **The algebraic group** GSp(4) **and its algebraic representations.** Let I_2 be the identity matrix of size two and let $\mathcal{J}$ be the symplectic form whose matrix in the canonical basis of $\mathbb{Z}^4$ is

$$\mathcal{J} = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}.$$

The symplectic group GSp(4) is defined as

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) \,|\, {}^t g \mathcal{J} g = \nu(g)\mathcal{J},\, \nu(g) \in \mathbb{G}_m\}.$$

This is a $\mathbb{Z}$-group scheme that we will denote by G in what follows. Then $\nu : G \to \mathbb{G}_m$ is a character and the derived group of G is $\mathrm{Sp}(4) = \mathrm{Ker}\,\nu$. We denote by $T \subset G$ the diagonal maximal torus defined as

$$T = \{\mathrm{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) |\, \alpha_1, \alpha_2, \nu \in \mathbb{G}_m\}$$

and by $B = TU$ the standard Borel subgroup of upper triangular matrices in G where U is the unipotent radical

$$U = \left\{ \begin{pmatrix} 1 & x_0 & & \\ & 1 & & \\ & & 1 & \\ & & -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \middle| \, x_0, x_1, x_2, x_3 \in \mathbb{G}_a \right\}$$

We identify the group $X^*(T)$ of algebraic characters of T to the subgroup of $\mathbb{Z}^2 \oplus \mathbb{Z}$ of triples (k, k', c) such that $k + k' \equiv c \pmod 2$ via

$$\lambda(k, k', c) : \mathrm{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) \longmapsto \alpha_1^k \alpha_2^{k'} \nu^{\frac{c-k-k'}{2}}.$$

Let $\rho_1 = \lambda(1, -1, 0)$ be the short simple root and $\rho_2 = \lambda(0, 2, 0)$ be the long simple root. Then the set $R \subset X^*(T)$ of roots of T in G is

$$R = \{\pm\rho_1, \pm\rho_2, \pm(\rho_1 + \rho_2), \pm(2\rho_1 + \rho_2)\}$$

and the subset $R^+ \subset R$ of positive roots with respect to B is

$$R^+ = \{\rho_1, \rho_2, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}.$$

Then, the set of dominant weights is the set of $\lambda(k, k', c)$ such that $k \geq k' \geq 0$. For any dominant weight λ, there is an irreducible algebraic representation $V_{\lambda,\mathbb{Q}}$ of $G_{\mathbb{Q}}$ in a

finite dimensional $\mathbb{Q}$-vector space, of highest weight λ, unique up to isomorphism, and all isomorphism classes of irreducible algebraic representations of $G_{\mathbb{Q}}$ are obtained in this way. We will be interested in the specific realization of $V_{\lambda,\mathbb{Q}}$ obtained by Weyl's construction as follows. Let $\mathrm{Std}_{\mathbb{Q}} \simeq V_{\lambda(1,0,1),\mathbb{Q}}$ be the standard 4-dimensional representation of $G_{\mathbb{Q}}$, let $\lambda = \lambda(k, k', c)$ be a dominant weight, let $d = k + k'$. If $d \geq 2$, for two integers $1 \leq p, q \leq d$, let $\Psi_{p,q} : \mathrm{Std}_{\mathbb{Q}}^{\otimes d} \to \mathrm{Std}_{\mathbb{Q}}^{\otimes(d-2)}$ denote the map defined by

$$\Psi_{p,q} : v_1 \otimes \ldots \otimes v_d \mapsto {}^t v_p \mathcal{J} v_q v_1 \otimes \ldots \otimes \hat{v}_p \otimes \ldots \otimes \hat{v}_q \otimes \ldots \otimes v_d \tag{3}$$

Let $\mathrm{Std}_{\mathbb{Q}}^{\langle d\rangle} \subset \mathrm{Std}_{\mathbb{Q}}^{\otimes d}$ be the intersection of the kernels of all the $\Psi_{p,q}$. Let $t = \frac{c-k-k'}{2}$. Then

$$V_{\lambda,\mathbb{Q}} = \left(\mathrm{Std}_{\mathbb{Q}}^{\langle d\rangle} \cap \mathbb{S}_\lambda(\mathrm{Std}_{\mathbb{Q}})\right) \otimes \nu^{\otimes t}$$

where $\mathbb{S}_\lambda(\mathrm{Std}_{\mathbb{Q}}) \subset \mathrm{Std}_{\mathbb{Q}}^{\otimes d}$ is the image of an explicit element $c_\lambda \in \mathbb{Z}[\mathcal{S}_d]$ acting on $\mathrm{Std}_{\mathbb{Q}}^{\otimes d}$ (see [FH91, Theorem 17.11]). Here $\mathcal{S}_d$ denotes the symmetric group of order $d!$. By replacing $\mathrm{Std}_{\mathbb{Q}}$ by the standard representation $\mathrm{Std}_{\mathbb{Z}}$ of G over $\mathbb{Z}$, we obtain a free $\mathbb{Z}$-module $V_{\lambda,\mathbb{Z}}$ endowed with a linear action of G such that $V_{\lambda,\mathbb{Z}} \otimes \mathbb{Q} = V_{\lambda,\mathbb{Q}}$.

2.2. **Measures.** Consider the unitary group $\mathrm{U}(2) = \{g \in \mathrm{GL}(2,\mathbb{C}) \,|\, {}^t\overline{g}g = I_2\}$ where $\overline{g}$ denotes the complex conjugate of g. The map $\kappa : \mathrm{U}(2) \longrightarrow \mathrm{Sp}(4,\mathbb{R})$ defined by

$$g = A + iB \longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B denote the real and imaginary parts of g, identifies $\mathrm{U}(2)$ with a maximal compact subgroup K_∞ of $\mathrm{Sp}(4,\mathbb{R})$. Let K'_∞ denote the subgroup $\mathbb{R}_+^\times K_\infty$ of $G(\mathbb{R})$, where $\mathbb{R}_+^\times$ denotes the connected component of the center of $G(\mathbb{R})$. Let dk_∞ be the unique Haar measure on K_∞ such that $\mathrm{vol}(K_\infty, dk_\infty) = 1$. Let $d\mu$ be a left translation invariant measure on $G(\mathbb{R})_+/K'_\infty$. Given a measurable function $f : G(\mathbb{R})_+/\mathbb{R}_+^\times \to \mathbb{C}$, the function $\overline{f} : G(\mathbb{R})_+/K'_\infty \to \mathbb{C}$ defined as

$$\overline{f}(\overline{g}_\infty) = \int_{K_\infty} f(g_\infty k) dk_\infty$$

where g_∞ is a lift of $\overline{g}_\infty$ by the canonical projection $G(\mathbb{R})_+/\mathbb{R}_+^\times \to G(\mathbb{R})_+/K'_\infty$, is well defined. We define the left translation invariant measure dg_∞ on $G(\mathbb{R})_+/\mathbb{R}_+^\times$ by the formula

$$\int_{G(\mathbb{R})_+/\mathbb{R}_+^\times} f(g_\infty) dg_\infty = \int_{G(\mathbb{R})_+/K'_\infty} \overline{f} d\mu. \tag{4}$$

Let

$$\mathcal{H}_+ = \{Z = X + iY \in \mathfrak{gl}_{2,\mathbb{C}} \,|\, {}^t Z = Z, Y > 0\} = G(\mathbb{R})_+/K'_\infty$$

be Siegel upper half-plane of genus 2.

Lemma 2.1. *The following statements hold.*

1. *The measure $dXdY/\det(Y)^3$ on $\mathcal{H}_+$ is left translation invariant by $G(\mathbb{R})_+$.*
2. *For any measure $d\mu$ on $\mathcal{H}_+$ which is left translation invariant by $G(\mathbb{R})_+$, then there exists $c \in \mathbb{R}_+^\times$ such that $d\mu = cdXdY/\det(Y)^3$.*

Proof. The first statement is a particular case of [An87, Proposition 1.2.9]. Let us prove the second statement. Let $d\mu$ be a measure on $\mathcal{H}_+ = G(\mathbb{R})_+/K'_\infty$ which is left translation invariant by $G(\mathbb{R})_+$. Let $d\nu$ denote the measure $dXdY/\det(Y)^3$. By the construction (4) above, we associate to $d\mu$ and $d\nu$ Haar measures $dg_{\infty,\mu}$ and $dg_{\infty,\nu}$ on the Lie group $G(\mathbb{R})_+/\mathbb{R}_+^\times$. There exists $c \in \mathbb{R}_+^\times$ such that $dg_{\infty,\mu} = c dg_{\infty,\nu}$. Let $s : G(\mathbb{R})_+/\mathbb{R}_+^\times \to G(\mathbb{R})_+/K'_\infty$ denote the canonical projection. By an easy computation we have $s_*(dg_{\infty,\mu}) = \mathrm{vol}(K_\infty, dk_\infty)d\mu = d\mu$ and similarly $s_*(dg_{\infty,\nu}) = d\nu$. As a consequence $d\mu = cd\nu$ as claimed. $\square$

2.3. **Representations of K_∞ and discrete series classification.** Let $\mathfrak{k}$ and $\mathfrak{k}'$ denote the Lie algebra of K_∞ and K'_∞ respectively. Let $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{k}'_\mathbb{C}$ denote their complexifications. The differential of κ induces an isomorphism of Lie algebras $d\kappa : \mathfrak{gl}_{2,\mathbb{C}} \simeq \mathfrak{k}_\mathbb{C}$. Let $\mathfrak{sp}_4$ denote the Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$ and by $\mathfrak{sp}_{4,\mathbb{C}}$ its complexification. A compact Cartan subalgebra of $\mathfrak{sp}_4$ is defined as $\mathfrak{h} = \mathbb{R}T_1 \oplus \mathbb{R}T_2$ where

$$T_1 = d\kappa\left(\begin{pmatrix} i & \\ & \end{pmatrix}\right) = \begin{pmatrix} & & & 1 \\ & & & \\ -1 & & & \\ & & & \end{pmatrix},$$

$$T_2 = d\kappa\left(\begin{pmatrix} & \\ & i \end{pmatrix}\right) = \begin{pmatrix} & & & \\ & & & 1 \\ & -1 & & \\ & & & \end{pmatrix}.$$

Define a $\mathbb{C}$-basis of $\mathfrak{h}_\mathbb{C}^*$ by $e_1(T_1) = i, e_1(T_2) = 0, e_2(T_1) = 0, e_2(T_2) = i$. The root system Δ of the pair $(\mathfrak{sp}_{4,\mathbb{C}}, \mathfrak{h}_\mathbb{C})$ is $\Delta = \{\pm 2e_1, \pm 2e_2, \pm(e_1 \pm e_2)\}$. We denote by Δ_c, respectively Δ_{nc}, the set of compact, respectively non-compact roots in Δ. We have $\Delta_c = \{\pm(e_1 - e_2)\}$ and $\Delta_{nc} = \Delta - \Delta_c$. We choose the set of positive roots as $\Delta^+ = \{e_1 - e_2, 2e_1, e_1 + e_2, 2e_2\}$. Then, the set of compact, respectively non-compact, positive roots is $\Delta_c^+ = \Delta_c \cap \Delta^+$, respectively $\Delta_{nc}^+ = \Delta_{nc} \cap \Delta^+$. For each symmetric matrix $Z \in \mathfrak{gl}_{2,\mathbb{C}}$, define the element $p_\pm(Z)$ of $\mathfrak{sp}_4$ by

$$p_\pm(Z) = \begin{pmatrix} Z & \pm iZ \\ \pm iZ & -Z \end{pmatrix}.$$

Let us consider the generator

$$p_+\left(\begin{pmatrix} 1 & \\ & \end{pmatrix}\right) \wedge p_+\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \wedge p_+\left(\begin{pmatrix} & \\ & 1 \end{pmatrix}\right) \wedge p_-\left(\begin{pmatrix} 1 & \\ & \end{pmatrix}\right) \wedge p_-\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \wedge p_-\left(\begin{pmatrix} & \\ & 1 \end{pmatrix}\right)$$

of the one-dimensional $\mathbb{C}$-vector space $\bigwedge^6 \mathfrak{sp}_{4,\mathbb{C}}/\mathfrak{k}_\mathbb{C}$. To this generator is associated a non-zero left translation invariant measure $d\mu$ on $\mathrm{Sp}(4,\mathbb{R})/K_\infty = G(\mathbb{R})_+/K'_\infty$ in a standard way. Let $c \in \mathbb{R}_+^\times$ be the constant given by the second assertion of Lemma 2.1. Let us denote by $\sqrt[6]{c}$ the positive sixth root of c in $\mathbb{R}_+^\times$. Let $X_{(\alpha_1,\alpha_2)} \in \mathfrak{sp}_{4,\mathbb{C}}$ be defined as

$$X_{\pm(2,0)} = \sqrt[6]{c}p_\pm\left(\begin{pmatrix} 1 & \\ & \end{pmatrix}\right), X_{\pm(1,1)} = \sqrt[6]{c}p_\pm\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right), X_{\pm(0,2)} = \sqrt[6]{c}p_\pm\left(\begin{pmatrix} & \\ & 1 \end{pmatrix}\right).$$

It follows from an easy computation that $X_{(\alpha_1,\alpha_2)}$ is a root vector corresponding to the non-compact root $(\alpha_1,\alpha_2)=\alpha_1 e_1+\alpha_2 e_2$. If we set

$$\mathfrak{p}^{\pm}=\bigoplus_{\alpha\in\Delta_{nc}^{+}}\mathbb{C}X_{\pm\alpha}, \tag{5}$$

we have the Cartan decomposition $\mathfrak{sp}_{4,\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}^{+}\oplus\mathfrak{p}^{-}$. Furthermore the inclusion $\mathfrak{sp}_{4\,\mathbb{C}}\subset\mathfrak{g}_{\mathbb{C}}$ induces a canonical isomorphism $\mathfrak{sp}_{4\,\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}=\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}''_{\mathbb{C}}$. In particular $\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}''_{\mathbb{C}}=\mathfrak{p}^{+}\oplus\mathfrak{p}^{-}$.

Integral weights are defined as the $(k,k')=ke_1+k'e_2\in\mathfrak{h}^{*}_{\mathbb{C}}$ with $k,k'\in\mathbb{Z}$ and an integral weight is dominant for Δ_c^{+} if $k\geq k'$. Assigning its highest weight to a finite dimensional irreducible complex representation τ of K_∞, we define a bijection between isomorphism classes of finite-dimensional irreducible complex representations of K_∞ and dominant integral weights, whose inverse will be denoted by $(k,k')\longmapsto\tau_{(k,k')}$. Let (k,k') be a dominant integral weight and let $d=k-k'$. Then $\dim_{\mathbb{C}}\tau_{(k,k')}=d+1$. More precisely, there exists a basis $(\boldsymbol{v}_s)_{0\leq s\leq d}$ of $\tau_{(k,k')}$, such that

$$\tau_{(k,k')}\left(d\kappa\begin{pmatrix}1&\\&\end{pmatrix}\right)\boldsymbol{v}_s=(s+k')\boldsymbol{v}_s, \tag{6}$$

$$\tau_{(k,k')}\left(d\kappa\begin{pmatrix}&\\&1\end{pmatrix}\right)\boldsymbol{v}_s=(-s+k)\boldsymbol{v}_s, \tag{7}$$

$$\tau_{(k,k')}\left(d\kappa\begin{pmatrix}&1\\&\end{pmatrix}\right)\boldsymbol{v}_s=(s+1)\boldsymbol{v}_{s+1}, \tag{8}$$

$$\tau_{(k,k')}\left(d\kappa\begin{pmatrix}&\\1&\end{pmatrix}\right)\boldsymbol{v}_s=(d-s+1)\boldsymbol{v}_{s-1} \tag{9}$$

which we call a standard basis of $\tau_{(k,k')}$. In the identities above, we agree to use the convention $\boldsymbol{v}_{-1}=\boldsymbol{v}_{d+1}=0$. Note that $(\mathrm{Ad},\mathfrak{p}^{+})$ is equivalent to $\tau_{(2,0)}$ and that $(\mathrm{Ad},\mathfrak{p}^{-})$ is equivalent to $\tau_{(0,-2)}$. Under the identification $(\mathrm{Ad},\mathfrak{p}^{+})\simeq\tau_{(2,0)}$, the basis $(\boldsymbol{v}_2,\boldsymbol{v}_1,\boldsymbol{v}_0)=(X_{(2,0)},X_{(1,1)},X_{(0,2)})$ is a standard basis and the under the identification $(\mathrm{Ad},\mathfrak{p}^{-})\simeq\tau_{(0,-2)}$, the basis $(\boldsymbol{v}_2,\boldsymbol{v}_1,\boldsymbol{v}_0)=(X_{(0,-2)},-X_{(-1,-1)},X_{(-2,0)})$ is a standard basis. As the weights of $\bigwedge^2\mathfrak{p}^{+}\otimes_{\mathbb{C}}\mathfrak{p}^{-}$ are the sums of two distinct weights of $\mathfrak{p}^{+}$ and of a weight of $\mathfrak{p}^{-}$, as $\mathbb{C}[K_\infty]$-modules we have

$$\begin{aligned}\bigwedge^2\mathfrak{p}^{+}\otimes_{\mathbb{C}}\mathfrak{p}^{-}&=\tau_{(3,-1)}\oplus\tau_{(2,0)}\oplus\tau_{(1,1)},\\ \mathfrak{p}^{+}\otimes_{\mathbb{C}}\bigwedge^2\mathfrak{p}^{-}&=\tau_{(1,-3)}\oplus\tau_{(0,-2)}\oplus\tau_{(-1,-1)}.\end{aligned}$$

Lemma 2.2. (1) *The following is a standard basis of* $\tau_{(3,-1)} \subset \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^-$*:*

$$\begin{aligned}
\boldsymbol{w}_4 &= X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)},\\
\boldsymbol{w}_3 &= -X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)} + 2X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\\
\boldsymbol{w}_2 &= X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)} - 2X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)} + X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\\
\boldsymbol{w}_1 &= 2X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)} - X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)},\\
\boldsymbol{w}_0 &= X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)}.
\end{aligned}$$

(1) *The following is a standard basis of* $\tau_{(2,0)} \subset \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^-$*:*

$$\begin{aligned}
\boldsymbol{x}_2 &= X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)} + 2X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\\
\boldsymbol{x}_1 &= -2X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)} + 2X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\\
\boldsymbol{x}_0 &= -2X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)} - X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}.
\end{aligned}$$

(1) *The following is a standard basis of* $\tau_{(1,1)} \subset \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^-$*:*

$$\boldsymbol{y}_0 = X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)} + X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)} + X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)}.$$

Lemma 2.3. *In the basis*

$$X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)},\ X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)},\ X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)},$$

$$X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\ X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)},\ X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)},$$

$$X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\ X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)},\ X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)}$$

of $\bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^-$ *the matrix of the projection* $p : \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^- \to \tau_{(3,-1)}$ *is*

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 & 0\\
0 & -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -1 & 0\\
0 & 0 & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

We will denote by W_{K_∞} the Weyl group of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. According to the classification theorem [Kn86, Thm. 9.20], as W_{K_∞} has 4 elements, we have:

Proposition 2.4. *Let* $G(\mathbb{R})_+$ *be the identity component of* $G(\mathbb{R})$*, let* ξ *be a character of* $\mathbb{R}^\times$ *and let* $(k, k') \in \mathfrak{h}_{\mathbb{C}}^*$ *be an integral weight. Assume* $k \geq k' \geq 0$*. Then, there exist* 4 *isomorphism classes* $\Pi_\infty^{3,0}$*,* $\Pi_\infty^{2,1}$*,* $\Pi_\infty^{1,2}$*,* $\Pi_\infty^{0,3}$ *of irreducible discrete series representations of* $G(\mathbb{R})_+$ *with Harish-Chandra parameter* $(k+2, k'+1)$ *and central character* ξ*. Furthermore, the restrictions of these representations to* K_∞ *contain as minimal* K_∞*-types the irreducible representations* $\tau_{(k+3,k'+3)}$*,* $\tau_{(k+3,-k'-1)}$*,* $\tau_{(k'+1,-k-3)}$*,* $\tau_{(-k'-3,-k-3)}$ *respectively and these occur with multiplicity* 1*.*

In the proposition above, the discrete series $\Pi_\infty^{3,0}$ is holomorphic, $\Pi_\infty^{2,1}$ and $\Pi_\infty^{1,2}$ are generic, which means that they have a Whittaker model and $\Pi_\infty^{0,3}$ is antiholomorphic. In what follows, we will denote by Π_∞^H and Π_∞^W the discrete series of $\mathrm{GSp}(4,\mathbb{R})$ defined as

$$\Pi_\infty^H = \mathrm{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_\infty^{3,0} = \mathrm{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_\infty^{0,3},$$
$$\Pi_\infty^W = \mathrm{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_\infty^{2,1} = \mathrm{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_\infty^{1,2}.$$

In particular, we have

$$\Pi_\infty^H|_{G(\mathbb{R})_+} = \Pi_\infty^{3,0} \oplus \Pi_\infty^{0,3},$$
$$\Pi_\infty^W|_{G(\mathbb{R})_+} = \Pi_\infty^{2,1} \oplus \Pi_\infty^{1,2}.$$

2.4. **Hecke algebras.** Let l be a prime number and let $K_l \subset G(\mathbb{Q}_l)$ be a compact open subgroup. Let $\mathcal{H}_l^{K_l}$ be the Hecke algebra of $\mathbb{Z}$-valued compactly supported functions on $G(\mathbb{Q}_l)$, which are invariant by translation on the left and on the right by K_l, with product given by the convolution product with respect to the Haar measure dx_l on $G(\mathbb{Q}_l)$ giving measure 1 to the maximal compact subgroup $G(\mathbb{Z}_l)$ of $G(\mathbb{Q}_l)$. If $K = \prod_l' K_l \subset G(\mathbb{A}_f)$ is a compact open subgroup, we denote by $\mathcal{H}^K$ the restricted tensor product $\mathcal{H}^K = \bigotimes_l' \mathcal{H}_l^{K_l}$. For any ring R, we shall denote by $\mathcal{H}_R^K$ the R-algebra $\mathcal{H}^K \otimes R$. Let Π_l denote a smooth admissible representation of $G(\mathbb{Q}_l)$. Then $\Pi_l^{K_l}$ is endowed with the action of $\mathcal{H}_\mathbb{C}^{K_l}$ as follows. For $f \in \mathcal{H}_\mathbb{C}^{K_l}$ and $\chi \in V_{\Pi_l}$ a vector invariant by K_l, we define $f.\chi$ as

$$f.\chi = \int_{G(\mathbb{Q}_l)} \Pi_l(x)(\chi) f(x) dx_l.$$

Lemma 2.5. *Let $K_l \subset G(\mathbb{Q}_l)$ be a compact open subgroup. Then, for any $g \in G(\mathbb{Q}_l)$ we have*

$$[K_l : gK_lg^{-1} \cap K_l] = [K_l : g^{-1}K_lg \cap K_l].$$

Proof. Let $1_{gK_lg^{-1}\cap K_l}$ and $1_{g^{-1}K_lg\cap K_l}$ denote the characteristic functions of the subgroups $gK_lg^{-1} \cap K_l$ and $g^{-1}K_lg \cap K_l$ of K_l respectively. We have

$$\begin{aligned}
\mathrm{vol}(gK_lg^{-1} \cap K_l, dx_l) &= \int_{G(\mathbb{Q}_l)} 1_{gK_lg^{-1}\cap K_l}(x_l) dx_l \\
&= \int_{G(\mathbb{Q}_l)} 1_{gK_lg^{-1}\cap K_l}(gx_lg^{-1}) dx_l \\
&= \int_{G(\mathbb{Q}_l)} 1_{g^{-1}K_lg\cap K_l}(x_l) dx_l \\
&= \mathrm{vol}(g^{-1}K_lg \cap K_l, dx_l)
\end{aligned}$$

where the second equality follows the change of variable $x_l \to g^{-1}x_lg$ and from the equality $d(gx_lg^{-1}) = dx_l$ which follows from the unimodularity of $G(\mathbb{Q}_l)$ (see [R10, Proposition V.5.4]). The conclusion follows. □

Let dx^{Tam} denote the Tamagawa measure on $G(\mathbb{A})$. Given two cusp forms φ_1 and φ_2 on $G(\mathbb{A})$ with trivial central character, let $\langle \varphi_1, \varphi_2 \rangle$ denote the Petersson inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi_1(x)\overline{\varphi_2(x)} dx^{\mathrm{Tam}}.$$

Proposition 2.6. *Assume that φ_1 and φ_2 have trivial central character and are invariant by right translation by K_l. Let $g \in G(\mathbb{Q}_l)$, let $T_g \in \mathcal{H}^{K_l}$ be the characteristic function of $K_l g K_l$ and let $T_{\nu(g)g^{-1}}$ be the characteristic function of $K_l \nu(g) g^{-1} K_l$. Then*

$$\langle T_g \varphi_1, \varphi_2 \rangle = \langle \varphi_2, T_{\nu(g)g^{-1}} \varphi_2 \rangle.$$

Proof. Let n denote the integer

$$n = [K_l : gK_l g^{-1} \cap K_l] = [K_l : g^{-1} K_l g \cap K_l] \tag{10}$$

where the second equality follows from Lemma 2.5. According to equality (10) and to [DS05, Lemma 5.5.1 (c)] , there exists $\beta_1, \ldots, \beta_n \in G(\mathbb{Q}_l)$ such that

$$K_l g K_l = \bigsqcup_{j=1}^{n} \beta_j K_l = \bigsqcup_{j=1}^{n} K_l \beta_j. \tag{11}$$

Note that [DS05, Lemma 5.5.1 (c)] is proved in a global situation and for SL_2 but that the proof is purely group theoretical and hence works when applied to our situation once Γ in loc. cit. is replaced by $G(\mathbb{Z}_l)$ and α in loc. cit. is replaced by g. Hence we have

$$\begin{aligned}
\langle T_g \varphi_1, \varphi_2 \rangle &= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_l)} \varphi_1(xh_l) \mathbf{1}_{K_l g K_l}(h_l) \overline{\varphi_2(x)} dh_l dx^{\mathrm{Tam}} \\
&= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{j=1}^{n} \int_{K_l} \varphi_1(x\beta_j h_l) \overline{\varphi_2(x)} dh_l dx^{\mathrm{Tam}} \\
&= \mathrm{vol}(K_l, dh_l) \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{j=1}^{n} \varphi_1(x\beta_j) \overline{\varphi_2(x)} dx^{\mathrm{Tam}} \\
&= \mathrm{vol}(K_l, dh_l) \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{j=1}^{n} \varphi_1(x) \overline{\varphi_2(x\beta_j^{-1})} dx^{\mathrm{Tam}} \\
&= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi_1(x) \sum_{j=1}^{n} \int_{K_l} \varphi_2(x\beta_j^{-1} h_l) dh_l dx^{\mathrm{Tam}} \\
&= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi_1(x) \sum_{j=1}^{n} \int_{K_l} \varphi_2(x\nu(g)\beta_j^{-1} h_l) dh_l dx^{\mathrm{Tam}}
\end{aligned}$$

where the first equality is the definition of the action of T_g, the second follows from the first equality in (11), the third follows from the fact that φ_1 is right translation invariant by K_l, the fourth follows from an obvious change of variable in the integral, the fifth is similar as the third and the sixth follows from the fact that φ_2 has trivial central character. Note that (11) implies that $K_l \nu(g) g^{-1} K_l = \bigsqcup_{j=1}^{n} \nu(g)\beta_j^{-1} K_l$. As a consequence

$$\langle T_g \varphi_1, \varphi_2 \rangle = \langle \varphi_1, T_{\nu(g)g^{-1}} \varphi_2 \rangle$$

as claimed. □

3. Adjoint L-value and Petersson norm after Ichino

For the convenience of the reader, let us recall a particular case of the main result of [CI19] expressing the value at 1 of the adjoint L-function of some cuspidal automorphic representations of $G(\mathbb{A})$ in terms of a Petersson norm.

Let $k \geq k' \geq 0$ be two integers. Let Π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let us fix an isomorphism

$$\Pi \simeq \bigotimes_v{}' \Pi_v. \tag{12}$$

We make the following assumptions on Π:

(i) the central character of Π is trivial,
(ii) Π is globally generic,
(iii) the paramodular conductor $N \in \mathbb{Z}_{>0}$ of Π (see [RoS07]) is square-free,
(iv) $\Pi_\infty|_{G(\mathbb{R})_+} = \Pi_\infty^{2,1} \oplus \Pi_\infty^{1,2}$,

where $\Pi_\infty^{2,1}$ and $\Pi_\infty^{1,2}$ are the discrete series representations with Harish-Chandra parameter $(k+2, k'+1)$ and respective minimal K_∞-types the irreducible representations $\tau_{k+3,-k'-1}$ and $\tau_{k'+1,-k-3}$ (see Proposition 2.4). According to [AS06], the automorphic representation Π has a functorial lift Σ to $\mathrm{GL}(4,\mathbb{A})$ and we say that Π is endoscopic if Σ is not cuspidal. From now on, we assume that

(v) Π is endoscopic.

Let $\eta = \otimes_v \eta_v$ be the standard additive character of $\mathbb{Q}\setminus\mathbb{A}$, so that $\eta_\infty(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$. By abuse of notation, we denote again by η the character of $U(\mathbb{Q})\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ defined by

$$\begin{pmatrix} 1 & x_0 & & \\ & 1 & & \\ & & 1 & \\ & & -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \mapsto \eta(-x_0 - x_3)$$

Let Π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. The Whittaker function of a cusp form $\psi \in \Pi$ is

$$W_\psi(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \psi(ug)\overline{\eta(u)}du$$

for $g \in G(\mathbb{A})$, where du is the Tamagawa measure on $U(\mathbb{A})$. Let us consider the unique element $\varphi = \bigotimes_v{}' \varphi_v \in \Pi$ normalized as in [CI19, §1] in the following way:

(a) for every prime l not dividing N, the vector φ_l is invariant by $G(\mathbb{Z}_l)$,
(b) for every prime l dividing N, the vector φ_l is invariant by the paramodular subgroup

$$K_l = \left\{ g \in G(\mathbb{Q}_l) \,|\, \nu(g) \in \mathbb{Z}_l^\times, g \in \begin{pmatrix} \mathbb{Z}_l & \mathbb{Z}_l & l^{-1}\mathbb{Z}_l & \mathbb{Z}_l \\ l\,\mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ l\,\mathbb{Z}_l & l\,\mathbb{Z}_l & \mathbb{Z}_l & l\,\mathbb{Z}_l \\ l\,\mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \end{pmatrix} \right\}$$

of level l,

(c) the archimedean component φ_∞ is a lowest weight vector of the minimal U(2)-type of $\Pi_\infty^{1,2}$,

(d) we have $W_{\varphi_l}(1) = 1$ for any finite prime l,

(e) we have

$$W_{\varphi_\infty}(1) = e^{-2\pi} \int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi^3)^{(-s_1+k+4)/2} (4\pi)^{(-s_2-k'-1)/2}$$
$$\times \Gamma\left(\frac{s_1+s_2-2k'-1}{2}\right) \Gamma\left(\frac{s_1+s_2+1}{2}\right) \Gamma\left(\frac{s_1}{2}\right) \Gamma\left(\frac{s_2}{2}\right)$$

where $c_1, c_2 \in \mathbb{R}$ satisfy $c_1 + c_2 + 1 > 0$ and $c_1 > 0 > c_2$.

Let $\langle \varphi, \varphi \rangle$ denote the Petersson norm

$$\langle \varphi, \varphi \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\varphi(x)|^2 dx^{\mathrm{Tam}}.$$

The following statement is a particular case of Theorem 2.1 in [CI19].

Theorem 3.1. *We have*

$$\langle \varphi, \varphi \rangle = 2^{k+k'+13} \cdot 3^3 \cdot 5\pi^{3k+k'+9} (k+k'+5)^{-1} \prod_{l|N} (l+l^{-1})^{-1} L(1, \Pi, \mathrm{Ad}). \tag{13}$$

Proof. The statement is a copy of the statement of Theorem 2.1 in [CI19]. However, we need to explain the change of parameters from those used in [CI19] to those used in our paper. The right hand of our equality (13) is $2^c \dfrac{L(1,\Pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \prod_v C(\pi_v)$ in [CI19]. As explained in [CI19], we find that $c = 2$ since our representation π is endoscopic and we have

$$\frac{2^c}{\Delta_{\mathrm{PGSp}_4}} = 2^2 \zeta(2)^{-1} \zeta(4)^{-1} = 2^2 \frac{6}{\pi^2} \frac{90}{\pi^4} = \frac{2^4 \cdot 3^3 \cdot 5}{\pi^6}.$$

The factor $C(\pi_\infty)$ at $v = \infty$ is given by

$$C(\pi_\infty) = 2^{\lambda_1-\lambda_2+5} \pi^{3\lambda_1-\lambda_2+5} (1+\lambda_1-\lambda_2)^{-1}$$

where (λ_1, λ_2) are Blattner parameter corresponding to π_∞. We note that λ_1, λ_2 are presented as $\lambda_1 = k+3$ and $\lambda_2 = -k'-1$. Thus we have

$$C(\pi_\infty) = 2^{k+k'+9} \pi^{3k+k'+15} (k+k'+5)^{-1}.$$

As for $C(\pi_l)$, the value is equal to 1 when $l \nmid N$. When $l|N$,

$$C(\pi_l) = \frac{1}{l} \zeta_l(2)^{-1} \zeta_l(4) = \frac{1}{l} \left(1 - \frac{1}{l^2}\right) \frac{1}{1-\frac{1}{l^4}} = (l+l^{-1})^{-1}.$$

This completes the proof. □

4. Integral Betti cohomology of Siegel threefolds

4.1. **Integral local systems on Siegel threefolds.** Siegel threefolds are the Shimura varieties associated to the group G. Let us briefly recall their definition. Let $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ be the Deligne torus and let $\mathcal{H}$ be the $G(\mathbb{R})$-conjugacy class of the morphism $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ given on $\mathbb{R}$-points by

$$x + iy \longmapsto \begin{pmatrix} x & & y & \\ & x & & y \\ -y & & x & \\ & -y & & x \end{pmatrix}.$$

The pair $(G, \mathcal{H})$ verifies the axioms of Deligne-Shimura (see [L05, Lemme 2.1]). Let K_N denote the compact open subgroup of $G(\mathbb{A}_f)$ defined as

$$K_N = \prod_{l \in S(N)} K_l \times \prod_{l \notin S(N)} G(\mathbb{Z}_l).$$

In this work, we are interested in the level K_N, but as K_N is not neat we need to consider smaller congruence subgroups. Let $L_{\mathbb{Z}} \subset \mathbb{Q}^4$ be a $\mathbb{Z}$-lattice such that K_N stabilizes $L_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$ and let $K_N(3) = \{g \in K_N \,|\, (g-1)L_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}} \subset 3L_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}\}$. Then $K_N(3)$ is a normal compact open subgroup of K_N which is neat. Let $S_{K_N(3)}$ the Siegel threefold of level $K_N(3)$. This is a smooth quasiprojective $\mathbb{Q}$-scheme such that, as complex analytic varieties, we have $S_{K_N(3)} \simeq G(\mathbb{Q})\backslash(\mathcal{H} \times G(\mathbb{A}_f)/K_N(3))$, and carrying a universal abelian surface $a : A \to S_{K_N(3)}$ whith a polarization of degree N^2 and a principal level 3 structure.

Let $\lambda(k, k', c)$ be a dominant weight. We associate to the representation $V_{\lambda,\mathbb{Z}}$ of G a local system of $\mathbb{Z}$-modules for the classical topology on $S_{K_N(3)}$ as follows. To the standard representation $\mathrm{Std}_{\mathbb{Z}}$ we associate the first relative homology group $\mathcal{T}_{\mathbb{Z}} = \underline{\mathrm{Hom}}(R^1 a_* \mathbb{Z}, \mathbb{Z})$ of a. The polarization on A induces a pairing $\Psi : \mathcal{T}_{\mathbb{Z}} \otimes \mathcal{T}_{\mathbb{Z}} \to \mathbb{Z}$. If $d \geq 2$, let $\mathcal{T}_{\mathbb{Z}}^{\langle d \rangle} \subset \mathcal{T}_{\mathbb{Z}}^{\otimes d}$ be the intersection of the kernels of all the contractions $\mathcal{T}_{\mathbb{Z}}^{\otimes d} \to \mathcal{T}^{\otimes d-2}$ defined anlogously as the $\Psi_{p,q}$ (3) using Ψ instead of $\mathcal{J}$ and let $\tilde{V}_{\lambda,\mathbb{Z}} = \mathcal{T}_{\mathbb{Z}}^{\langle d \rangle} \cap \mathbb{S}_\lambda(\mathcal{T}_{\mathbb{Z}}) \otimes (2\pi i)^t\, \mathbb{Z}$, where $\mathbb{S}_\lambda(\mathcal{T}_{\mathbb{Z}})$ denotes the image of $c_\lambda \in \mathbb{Z}[\mathcal{S}_d]$ acting on $\mathcal{T}_{\mathbb{Z}}^{\otimes d}$ and where $t = \frac{c-k-k'}{2}$. Then $\tilde{V}_{\lambda,\mathbb{Z}}$ is a local system of $\mathbb{Z}$-modules on S_N. From now on the representation $V_{\lambda,\mathbb{Z}}$ and the local system associated to it $\tilde{V}_{\lambda,\mathbb{Z}}$ will be denoted by the same symbol $V_{\lambda,\mathbb{Z}}$.

We are interested in the Betti cohomology $H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$, in the Betti cohomology with compact support $H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ and in the inner cohomology $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ defined as

$$H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) = \mathrm{Im}(H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \to H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})).$$

These are finitely generated $\mathbb{Z}$-modules. To prove that they are endowed with a natural action of $\mathcal{H}^{K_N(3)}$, we shall make use of the formalism of Grothendieck's functors $(f^*, f_* f_!, f^!)$ on the derived categories of sheaves of $\mathbb{Z}$-modules on locally compact topological spaces (see [Ay08], [Ay10] and [KaS94]).

4.2. **Hecke action on integral cohomology.** Let $g \in G(\mathbb{A}_f)$. Then we have the Hecke correspondence

$$\begin{tikzcd} & S_{K_N(3)\cap gK_N(3)g^{-1}} \arrow[dl, "c_1(g)"'] \arrow[dr, "c_2(g)"] & \\ S_{K_N(3)} & & S_{K_N(3)} \end{tikzcd}$$

where $c_1(g)$ is induced by the inclusion $K_N(3) \cap gK_N(3)g^{-1} \subset K_N(3)$ and $c_2(g)$ is induced by the morphism $K_N(3) \cap gK_N(3)g^{-1} \hookrightarrow K_N(3)$ defined by $k \mapsto gkg^{-1}$. This diagram depends only on the class of g in $K_N(3)\backslash G(\mathbb{A}_f)/K_N(3)$.

Lemma 4.1. *There exists a canonical morphism*

$$\varphi_* : c_1(g)^* V_{\lambda,\mathbb{Z}} \to c_2(g)^* V_{\lambda,\mathbb{Z}}.$$

Proof. It follows from the modular description of $c_1(g)$ and $c_2(g)$ which, in our case, is similar to the one given at p. 9 of [L05] that we have an isogeny $\varphi : c_1(g)^*A \to c_2(g)^*A$ over $S_{K_N(3)\cap gK_N(3)g^{-1}}$. Taking homology, we obtain a morphism $\varphi_* : c_1(g)^*\mathcal{H}_\mathbb{Z} \to c_2(g)^*\mathcal{H}_\mathbb{Z}$ which induces the morphism of the statement. □

Note that as $c_i(g)$ is finite and étale for $i = 1, 2$, we have canonical isomorphisms of functors $c_i(g)_! \simeq c_i(g)_*$ and $c_i(g)^! \simeq c_i(g)^*$. Let us denote by $\bullet$ the topological space reduced to a point and let $p_{K_N(3)} : S_{K_N(3)} \to \bullet$ the canonical continuous map. Then $H^*(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ is the cohomology of the complex $p_{K_N(3)*}V_{\lambda,\mathbb{Z}}$ and $H^*_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ is the cohomology of the complex $p_{K_N(3)!}V_{\lambda,\mathbb{Z}}$.

Definition 4.2. *The endomorphism*

$$T_g : H^*(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \to H^*(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$$

is defined as the endomorphism induced by the map

$$\begin{aligned} p_{K_N(3)*}V_{\lambda,\mathbb{Z}} &\longrightarrow p_{K_N(3)*}c_1(g)_*c_1(g)^*V_{\lambda,\mathbb{Z}} \longrightarrow p_{K_N(3)*}c_2(g)_*c_1(g)^*V_{\lambda,\mathbb{Z}} \\ &\xrightarrow{\varphi_*} p_{K_N(3)*}c_2(g)_*c_2(g)^*V_{\lambda,\mathbb{Z}} \longrightarrow p_{K_N(3)*}c_2(g)_!c_2(g)^!V_{\lambda,\mathbb{Z}} \longrightarrow p_{K_N(3)*}V_{\lambda,\mathbb{Z}} \end{aligned}$$

in the derived category of $\mathbb{Z}$-modules, where the first arrow is induced by the adjunction morphism, the second arrow is induced by the isomorphism of functors

$$p_{K_N(3)*}c_1(g)_* \simeq (p_{K_N(3)} \circ c_1(g))_* \simeq (p_{K_N(3)} \circ c_2(g))_* \simeq p_{K_N(3)*}c_2(g)_*,$$

the third arrow is induced by the canonical morphism φ_* of Lemma 4.1, the fourth arrow is induced by the isomorphism of functors $c_2(g)_*c_2(g)^* \simeq c_2(g)_!c_2(g)^!$ and the last arrow is induced by trace map. The endomorphism

$$T_{g,c} : H^*_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \to H^*_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$$

is defined similarly as the endomorphism induced by

$$\begin{aligned} p_{K_N(3)!}V_{\lambda,\mathbb{Z}} &\longrightarrow p_{K_N(3)!}c_1(g)_*c_1(g)^*V_{\lambda,\mathbb{Z}} \longrightarrow p_{K_N(3)!}c_2(g)_!c_1(g)^*V_{\lambda,\mathbb{Z}} \\ &\xrightarrow{\varphi_*} p_{K_N(3)!}c_2(g)_!c_2(g)^*V_{\lambda,\mathbb{Z}} \longrightarrow p_{K_N(3)!}c_2(g)_!c_2(g)^!V_{\lambda,\mathbb{Z}} \longrightarrow p_{K_N(3)!}V_{\lambda,\mathbb{Z}}. \end{aligned}$$

Proposition 4.3. *For any $g \in G(\mathbb{A}_f)$ the diagram*

$$\begin{array}{ccc} H^*_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) & \longrightarrow & H^*(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \\ \Big\downarrow{\scriptstyle T_{g,c}} & & \Big\downarrow{\scriptstyle T_g} \\ H^*_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) & \longrightarrow & H^*(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \end{array}$$

where the maps are defined above is commutative and depends only on the double class $K_N(3)gK_N(3)$.

Proof. The statement follows from the commutativity of the following diagram in the derived category of $\mathbb{Z}$-modules

$$\begin{array}{ccc} p_{K_N(3)\,!}V_{\lambda,\mathbb{Z}} & \longrightarrow & p_{K_N(3)\,*}V_{\lambda,\mathbb{Z}} \\ \Big\downarrow & & \Big\downarrow \\ p_{K_N(3)\,!}c_1(g)_*c_1(g)^*V_{\lambda,\mathbb{Z}} & \longrightarrow & p_{K_N(3)\,*}c_1(g)_*c_1(g)^*V_{\lambda,\mathbb{Z}} \\ \Big\downarrow{\scriptstyle \sim} & & \Big\downarrow{\scriptstyle \sim} \\ p_{K_N(3)\,!}c_2(g)_!c_1(g)^*V_{\lambda,\mathbb{Z}} & \longrightarrow & p_{K_N(3)\,*}c_2(g)_*c_1(g)^*V_{\lambda,\mathbb{Z}} \\ \Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle \varphi_*} \\ p_{K_N(3)\,!}c_2(g)_!c_2(g)^*V_{\lambda,\mathbb{Z}} & \longrightarrow & p_{K_N(3)\,*}c_2(g)_*c_2(g)^*V_{\lambda,\mathbb{Z}} \\ \Big\downarrow & & \Big\downarrow \\ p_{K_N(3)\,!}c_2(g)_!c_2(g)^!V_{\lambda,\mathbb{Z}} & \longrightarrow & p_{K_N(3)\,*}c_2(g)_!c_2(g)^!V_{\lambda,\mathbb{Z}} \\ \Big\downarrow & & \Big\downarrow \\ p_{K_N(3)\,!}V_{\lambda,\mathbb{Z}} & \longrightarrow & p_{K_N(3)\,*}V_{\lambda,\mathbb{Z}}, \end{array}$$

where the vertical lines are defined in Definition 4.2 and where the first, the second, the fifth and the sixth horizontal arrows are induced by the morphism of functors $p_{K_N(3)\,!} \to p_{K_N(3)\,*}$ and the third and the fourth horizontal arrows are induced by the morphism of functors

$$p_{K_N(3)\,!}c_2(g)_! \longrightarrow p_{K_N(3)\,!}c_2(g)_* \longrightarrow p_{K_N(3)\,*}c_2(g)_*.$$

The commutativity of all but the second square follows from functoriality. Let us prove the commutativity of the second square. According to [Ay08, Proposition 1.7.3], the following diagram of functors

$$\begin{array}{ccc} p_{K_N(3)\,!}c_1(g)_! & \longrightarrow & p_{K_N(3)\,*}c_1(g)_* \\ \Big\downarrow{\scriptstyle \sim} & & \Big\downarrow{\scriptstyle \sim} \\ (p_{K_N(3)} \circ c_1(g))_! & \longrightarrow & (p_{K_N(3)} \circ c_1(g))_* \end{array}$$

where the upper horizontal map is the composite

$$p_{K_N(3)\,!}c_1(g)_! \longrightarrow p_{K_N(3)\,!}c_1(g)_* \longrightarrow p_{K_N(3)\,*}c_1(g)_*$$

is commutative. As the first map above is an equality, the diagram

$$\begin{array}{ccc} p_{K_N(3)!}c_1(g)_* & \longrightarrow & p_{K_N(3)*}c_1(g)_* \\ \downarrow\sim & & \downarrow\sim \\ (p_{K_N(3)}\circ c_1(g))_! & \longrightarrow & (p_{K_N(3)}\circ c_1(g))_* \end{array}$$

is commutative. Furthermore, the diagram

$$\begin{array}{ccc} (p_{K_N(3)}\circ c_1(g))_! & \longrightarrow & (p_{K_N(3)}\circ c_1(g))_* \\ \| & & \| \\ (p_{K_N(3)}\circ c_2(g))_! & \longrightarrow & (p_{K_N(3)}\circ c_2(g))_* \\ \downarrow\sim & & \downarrow\sim \\ p_{K_N(3)!}c_2(g)_! & \longrightarrow & p_{K_N(3)*}c_2(g)_* \end{array}$$

where the last horizontal map is defined as above, is commutative. By combining the commutativity of the two previous diagrams, we complete the proof. □

As each function $f \in \mathcal{H}^{K_N(3)}$ is a finite linear combination with $\mathbb{Z}$-coefficients of characteristic functions of double cosets $K_N(3)gK_N(3)$ with $g \in G(\mathbb{A}_f)$, by the previous result we define a ring homomorphism

$$\mathcal{H}^{K_N(3)} \to \mathrm{End}_{\mathbb{Z}}(H^*_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})) \tag{14}$$

by sending the characteristic function $1_{K_N(3)gK_N(3)}$ to the endomorphism of $H^*_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ induced by T_g.

4.3. **Isotypical component of cohomology.** For any $\mathbb{Z}$-algebra R let us denote by $V_{\lambda,R}$ the representation of G_R deduced from V_λ by extending the scalars to R. By abuse of notation, we will denote by the same symbol the local system of R-modules on $S_{K_N(3)}$ naturally associated to $V_{\lambda,R}$. By the universal coefficients theorem, we have $H^*_!(S_{K_N(3)}, V_{\lambda,\mathbb{C}}) = H^*_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \otimes \mathbb{C}$. Let

$$P(V_{\lambda,\mathbb{C}}) = \{\Pi^{3,0}_\infty, \Pi^{2,1}_\infty, \Pi^{1,2}_\infty, \Pi^{0,3}_\infty\}$$

be the discrete series L-packet attached to the complex representation $V_{\lambda,\mathbb{C}}$. If $\lambda = \lambda(k, k', c)$ with $k \geq k' \geq 0$ and $c \equiv k + k' \pmod 2$ then the set $P(V_{\lambda,\mathbb{C}})$ is the set of discrete series of $G(\mathbb{R})_+$ with Harish-Chandra parameter $(k+2, k'+1)$ and central character $x \mapsto x^{-c}$ given by Proposition 2.4. Let us denote by $m(\Pi)$ the cuspidal multiplicity of an automorphic representation Π of $G(\mathbb{A})$.

Proposition 4.4. *There is a canonical isomorphism of $\mathcal{H}^{K_N(3)}_{\mathbb{C}}$-modules*

$$H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{C}}) = \bigoplus_{\Pi=\Pi_\infty\otimes\Pi_f} H^3(\mathfrak{g}_{\mathbb{C}}, K_\infty, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty)^{\oplus m(\Pi)} \otimes_{\mathbb{C}} \Pi_f^{K_N(3)}$$

where the sum is indexed by isomorphism classes of cuspidal automorphic representations $\Pi \simeq \Pi_\infty \otimes \Pi_f$ of $G(\mathbb{A})$ such that $\Pi_\infty|_{G(\mathbb{R})_+} \in P(V_{\lambda,\mathbb{C}})$.

Proof. This is well known and follows for example from [Le17] (8) and (9). □

In order to avoid difficulties arising in the presence of torsion in the cohomology, we are going to work with $\mathbb{Z}_{(p)}$ coefficients instead of $\mathbb{Z}$ coefficients for sufficiently big prime numbers p. Let

$$S_{N,3} = \{p \text{ prime}, p | \# K_N/K_N(3)\}. \tag{15}$$

Let $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})_{\text{tors}}$ denote the torsion subgroup of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$. Let

$$S_{\text{tors}} = \{p \text{ prime}, p | \# H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})_{\text{tors}}\} \tag{16}$$

and fix a prime number p such that $p \notin S_{N,3} \cup S_{\text{tors}}$.

The finite group $K_N/K_N(3)$ acts on $S_{K_N(3)}$ by automorphisms and hence acts on the cohomology $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$. The action of $\mathcal{H}^{K_N(3)}$ on $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ induces an action of $\mathcal{H}^{K_N}$ on the submodule $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ defined as

$$\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) = H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})^{K_N/K_N(3)}$$

via the natural map $\mathcal{H}^{K_N} \to (\mathcal{H}^{K_N(3)})^{K_N/K_N(3)}$. Note that the fact that $p \notin S_{N,3}$ implies that $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ is a direct factor of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$. Let $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{C}})$ denote $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{C}$. It follows from the decomposition of Proposition 4.4 that we have a $\mathcal{H}^{K_N}$-equivariant decomposition

$$\tilde{H}^3_!(S_N, V_{\lambda,\mathbb{C}}) \simeq \bigoplus_{\Pi_i = \Pi_{i,\infty} \otimes \Pi_{i,f}} H^3(\mathfrak{g}_{\mathbb{C}}, K_\infty, V_{\lambda,\mathbb{C}} \otimes \Pi_{i,\infty})^{\oplus m(\Pi_i)} \otimes_{\mathbb{C}} \Pi_{i,f}^{K_N} \tag{17}$$

indexed as above. We will denote by $\overline{\mathcal{H}}^{K_N}_{\mathbb{Z}_{(p)}}$ the subring of $\mathrm{End}_{\mathbb{Z}_{(p)}}(\tilde{H}^3_!(S_N, V_{\lambda,\mathbb{Z}_{(p)}}))$ defined as the image of the homomorphism defined above $\mathcal{H}^{K_N}_{\mathbb{Z}_{(p)}} \to \mathrm{End}_{\mathbb{Z}_{(p)}}(\tilde{H}^3_!(S_N, V_{\lambda,\mathbb{Z}_{(p)}}))$. For any $\mathbb{Z}_{(p)}$-algebra R, we denote by $\overline{\mathcal{H}}^{K_N}_R$ the base change $\overline{\mathcal{H}}^{K_N}_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} R$. Note that as $\tilde{H}^3_!(S_N, V_{\lambda,\mathbb{Z}_{(p)}})$ is torsion free, the $\mathbb{Z}_{(p)}$-algebra $\overline{\mathcal{H}}^{K_N}_{\mathbb{Z}_{(p)}}$ is torsion free and hence $\overline{\mathcal{H}}^{K_N}_R$ is canonically isomorphic to the image of $\mathcal{H}^{K_N}_R$ in $\mathrm{End}_R(\tilde{H}^3_!(S_N, V_{\lambda,\mathbb{Z}_{(p)}}) \otimes_{\mathbb{Z}_{(p)}} R)$.

Let $\Pi = \Pi_\infty \otimes \Pi_f$ be the cuspidal automorphic representation of $G(\mathbb{A})$ appearing in the statement of Theorem 3.1. As $\Pi_\infty|_{G(\mathbb{R})_+} \in P(V_{\lambda,\mathbb{C}})$, the $\mathcal{H}^{K_N}_{\mathbb{C}}$-module $\Pi_f^{K_N}$ appears in the decomposition (17). As N denotes the paramodular conductor of Π, the space of fixed points $\Pi_f^{K_N}$ is one-dimensional. Let $E(\Pi_f)$ be the rationality field of Π_f. By definition $E(\Pi_f)$ is a subfield of $\mathbb{C}$ and it follows from [BHR94, Theorem 3.2.2] that $E(\Pi_f)$ is a number field. Let $\theta_\Pi : \overline{\mathcal{H}}^{K_N}_{\mathbb{Z}_{(p)}} \subset \overline{\mathcal{H}}^{K_N}_{\mathbb{C}} \to \mathbb{C}$ the character giving the action of $\overline{\mathcal{H}}^{K_N}_{\mathbb{Z}_{(p)}}$ on $\Pi_f^{K_N}$. As $\Pi_f^{K_N}$ is one-dimensional, it follows from [Wa85, Lemme I.1] that Π_f is defined over $E(\Pi_f)$ and so, the character θ_Π has values in $E(\Pi_f)$.

Lemma 4.5. *Let $\Pi = \Pi_\infty \otimes \Pi_f$ be a cuspidal automorphic representation of $G(\mathbb{A})$ of conductor N. Then for the character θ_Π of $\overline{\mathcal{H}}^{K_N}_{\mathbb{Z}_{(p)}}$, $\mathrm{Im}\,\theta_\Pi \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is isomorphic to $E(\Pi_f)$.*

Proof. We consider the decomposition of the $\mathbb{C}$-algebra $\overline{\mathcal{H}}_{\mathbb{C}}^{K_N} = \prod \overline{\mathcal{H}}_{i,\mathbb{C}}^{K_N}$ corresponding to the decomposition (17) of the $\overline{\mathcal{H}}_{\mathbb{C}}^{K_N}$-module $\tilde{H}_!^3(S_N, V_{\lambda,\mathbb{C}})$. Since Π has conductor N, the component $\overline{\mathcal{H}}_{i_0,\mathbb{C}}^{K_N}$ corresponding to Π is isomorphic to $\mathbb{C}$. Thus $\operatorname{Im}\theta_\Pi \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ must be a number field because it is an artinian subalgebra of $\mathcal{H}_{\mathbb{C},i_0}^{K_N} \cong \mathbb{C}$. By the definition of the coefficient field the number field $E(\Pi_f)$, $\operatorname{Im}\theta_\Pi \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is isomorphic to $E(\Pi_f)$. $\square$

Let us denote by $M(\Pi_f, V_{\lambda,\mathbb{Q}})$ the finite dimensional $\mathbb{Q}$-vector space

$$M(\Pi_f, V_{\lambda,\mathbb{Q}}) = \left(\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) \otimes_{\overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N}} \overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N} / \ker\theta_\Pi \right) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}\,.$$

This is a direct factor of the $\mathbb{Q}$-vector space $\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Q}})$. Let

$$p_M : \tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Q}}) \to M(\Pi_f, V_{\lambda,\mathbb{Q}}) \tag{18}$$

denote the projection and let $M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ be defined as

$$M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) = p_M\left(\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})\right). \tag{19}$$

Let $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ denote the $\mathbb{Z}_{(p)}$-lattice of $M(\Pi_f, V_{\lambda,\mathbb{Q}})$ defined as

$$L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) = M(\Pi_f, V_{\lambda,\mathbb{Q}}) \cap \tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}). \tag{20}$$

It is clear that $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ is a sub $\mathbb{Z}_{(p)}$-module of $M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$. For any $\mathbb{Z}_{(p)}$-algebra R, we will denote by $L(\Pi_f, V_{\lambda,R})$ the R-module $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \otimes_{\mathbb{Z}_{(p)}} R$. According to Corollary 4.5, we have

$$L(\Pi_f, V_{\lambda,\mathbb{C}}) = M(\Pi_f, V_{\lambda,\mathbb{C}}) = \bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} \tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) \otimes_{\theta_\Pi^\sigma} \mathbb{C}$$

where for any embedding $\sigma : E(\Pi_f) \to \mathbb{C}$ the character $\theta_\Pi^\sigma : \overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N} \to \mathbb{C}$ is the composite $\theta_\Pi^\sigma : \overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N} \to E(\Pi_f) \xrightarrow{\sigma} \mathbb{C}$.

Theorem 4.6. *Let $\Pi = \Pi_\infty \otimes \Pi_f$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and such that $\Pi_\infty|_{G(\mathbb{R})_+} \in P(V_{\lambda,\mathbb{C}})$. Then the following statements hold for any $\sigma \in \operatorname{Aut}(\mathbb{C})$.*

1. *There exists an irreducible cuspidal automorphic representation ${}^\sigma\Pi \simeq \bigotimes_v' \Pi_{\sigma,v}$ of $G(\mathbb{A})$ with trivial central character such that $\Pi_{\sigma,\infty}|_{G(\mathbb{R})_+} \in P(V_{\lambda,\mathbb{C}})$ and whose non-archimedean component is equivalent to ${}^\sigma\Pi_f$.*
2. *If one assumes that Π is globally generic and endoscopic then so is ${}^\sigma\Pi$.*

Proof. Statement (1) follows from the proof of [GaR13, Proposition 2.4] combined with the fact that for Siegel threefolds, L^2-cohomology coincides with cuspidal cohomology (see [MT02, Proposition 1]). As $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$, statement (2) is a particular case of [GaR13, Theorem 10.1 and Theorem 9.5] . $\square$

The decomposition (17) and the remark at the end of section 2.4 imply that we have a canonical isomorphism

(21)
$$L(\Pi_f, V_{\lambda,\mathbb{C}}) \simeq \bigoplus_{\sigma: E(\Pi_f)\to\mathbb{C}} \bigoplus_{\Pi_\infty|_{G(\mathbb{R})_+}\in P(V_{\lambda,\mathbb{C}})} H^3(\mathfrak{g}_\mathbb{C}, K_\infty, V_{\lambda,\mathbb{C}}\otimes\Pi_\infty)^{\oplus m(\Pi_\infty\otimes{}^\sigma\Pi_f)} \otimes_\mathbb{C} {}^\sigma\Pi_f^{K_N}.$$

Furthermore, for any r, s, the $\mathbb{C}$-vector spaces $H^3(\mathfrak{g}_\mathbb{C}, K_\infty, V_{\lambda,\mathbb{C}}\otimes\Pi_\infty^{r,s})$ are one-dimensional (see for example [Le17, Proposition 3.7]). As a consequence, we have

$$\dim_\mathbb{C}\Big(L(\Pi_f, V_{\lambda,\mathbb{C}})\otimes_{\mathbb{Z}_{(p)}}\mathbb{C}\Big) = 2\sum_{\sigma: E(\Pi_f)\to\mathbb{C}} m(\Pi_\infty^H\otimes{}^\sigma\Pi_f) + m(\Pi_\infty^W\otimes{}^\sigma\Pi_f). \tag{22}$$

Recall that we say that a globally generic cuspidal automorphic representation Π of $G(\mathbb{A})$ is endoscopic if its functorial lift to $\mathrm{GL}(4,\mathbb{A})$ is not cuspidal.

Proposition 4.7. *Let $\Pi = \Pi_\infty\otimes\Pi_f$ be a globally generic irreducible unitary endoscopic cuspidal automorphic representation of $G(\mathbb{A})$. Then*

1. *we have $m(\Pi) = 1$,*
2. *we have $m(\Pi_\infty^H\otimes\Pi_f) = 0$.*

Proof. The first statement follows from the main result of [JSo07]. According to [AS06, Proposition 2.2 (a)], Π is obtained as a Weil lifting from $\mathrm{GSO}(2,2,\mathbb{A})$. As Π is assumed to be globally generic, for every prime l the representation Π_l has a Whittaker model. Hence, applying [We09, Theorem 5.2 (4)] to Π we obtain

$$m(\Pi_\infty^H\otimes\Pi_f) = m(\overline{\Pi}_\infty^H\otimes\Pi_f) = \frac{1}{2}(1+(-1)^{\epsilon_\infty})$$

where $\epsilon_\infty = 0$ or 1 if Π_∞^H has or has not a Whittaker model. But it is well known that Π_∞^H is the archimedean component of an automorphic representation associated to a cuspidal Siegel modular form, which does not have a Whittaker model. As a consequence we obtain $m(\Pi_\infty^H\otimes\Pi_f) = m(\overline{\Pi}_\infty^H\otimes\Pi_f) = 0$. □

Corollary 4.8. *Let the notation and assumptions be as in the previous result. Then*

$$\mathrm{rk}_{\mathbb{Z}_{(p)}} L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) = 2[E(\Pi_f):\mathbb{Q}].$$

Proof. This follows from the fact that $\mathrm{rk}_{\mathbb{Z}_{(p)}} L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) = \dim_\mathbb{C} L(\Pi_f, V_{\lambda,\mathbb{C}})$ and from (22) combined with Theorem 4.6 and Proposition 4.7. □

4.4. **Poincaré duality.** Let $\check{V}_{\lambda,\mathbb{Z}_{(p)}}$ denote the dual local system $\check{V}_{\lambda,\mathbb{Z}_{(p)}} = \underline{\mathrm{Hom}}(V_{\lambda,\mathbb{Z}_{(p)}}, \mathbb{Z}_{(p)})$. Let us define

$$S'_{\mathrm{tors}} = \{p \text{ prime }, p|\#H_c^4(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})_{\mathrm{tors}}\}. \tag{23}$$

Lemma 4.9. *Assume that $p\notin S'_{\mathrm{tors}}$. Then we have a canonical isomorphism*

$$H^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}}) \simeq \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)}).$$

Proof. Recall that $p_3 : S_{K_N(3)} \to \bullet$ denotes the canonical continuous map from $S_{K_N(3)}$ to the topological space reduced to a point. According to [KaS94, Proposition 3.1.10] we have a canonical isomorphism

$$R\underline{\mathrm{Hom}}(Rp_{3\,!}V_{\lambda,\mathbb{Z}_{(p)}}, \mathbb{Z}_{(p)}) \simeq Rp_{3\,*}R\underline{\mathrm{Hom}}(V_{\lambda,\mathbb{Z}_{(p)}}, p_3^!\,\mathbb{Z}_{(p)})$$

in the derived category of abelian groups. Hence, we have two spectral sequences

$$\begin{aligned} E_2^{p,q} &= \mathrm{Ext}^p_{\mathbb{Z}_{(p)}}(H_c^{-q}(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)}) \quad \Longrightarrow \quad E_\infty^{p+q}, \\ E_2'^{\,p,q} &= H^p(S_{K_N(3)}, \underline{\mathrm{Ext}}^q(V_{\lambda,\mathbb{Z}_{(p)}}, p_3^!\,\mathbb{Z}_{(p)})) \quad \Longrightarrow \quad E_\infty'^{\,p+q} \end{aligned}$$

where $E_\infty^{p+q} \simeq E_\infty'^{\,p+q}$. According to [KaS94, Proposition 3.3.2 (i)] , as $S_{K_N(3)}$ is smooth of real dimension 6, we have $p_3^!\,\mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[6]$. As a consequence, using the fact that $V_{\lambda,\mathbb{Z}_{(p)}}$ is a sheaf of free $\mathbb{Z}_{(p)}$-modules of finite type, we have $\underline{\mathrm{Ext}}^q(V_{\lambda,\mathbb{Z}_{(p)}}, p_3^!\,\mathbb{Z}_{(p)}) = 0$ for $q \neq -6$ and $\underline{\mathrm{Ext}}^{-6}(V_{\lambda,\mathbb{Z}_{(p)}}, p_3^!\,\mathbb{Z}_{(p)}) = \check{V}_{\lambda,\mathbb{Z}_{(p)}}$. This implies immediately $E_2'^{\,3,-6} \simeq E_\infty'^{\,-3}$.

On the other hand, as $\mathbb{Z}_{(p)}$ is a discrete valuation ring, we have $E_2^{p,q} = 0$ for any $p \geq 2$ and any $q \in \mathbb{Z}$. Hence $E_2^{0,-3} = E_\infty^{0,-3}$ and $E_\infty^{2,-5} = E_\infty^{3,-6} = 0$. The assumption that p is outside S'_{tors} implies that $E_2^{1,-4} = E_\infty^{1,-4} = 0$. Hence $E_2^{0,-3} = E_\infty^{-3}$. As $E_\infty^{-3} \simeq E_\infty'^{\,-3}$ we have a canonical isomorphism $E_2^{0,-3} \simeq E_2'^{\,3,-6}$ which proves the claim. □

By the previous Lemma, assuming that $p \notin S'_{\mathrm{tors}}$, we have a non-degenerate pairing

$$\langle\,,\,\rangle : H^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}}) \times H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}\,.$$

By composing with the natural inclusion $H_!^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}}) \hookrightarrow H^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}})$ on the first factor, we obtain a pairing

$$\langle\,,\,\rangle : H_!^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}}) \times H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}\,.$$

Lemma 4.10. *For any $x \in H_!^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}})$ and for any $y \in \ker(H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}))$ we have $\langle x, y\rangle = 0$.*

Proof. Let us denote by $\langle\,,\,\rangle_{\mathbb{C}} : H_!^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{C}}) \times H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{C}}) \to \mathbb{C}$ be the pairing obtained from $\langle\,,\,\rangle$ by base change to $\mathbb{C}$. Let $x_{\mathbb{C}}$ denote the image of x in $H_!^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{C}})$ and let $y_{\mathbb{C}}$ denote the image of y in $H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{C}})$. It is enough to prove that $\langle x_{\mathbb{C}}, y_{\mathbb{C}}\rangle_{\mathbb{C}} = 0$. By the comparison isomorphism between Betti and de Rham cohomology and by compatibility of the Poincaré duality in Betti and de Rham cohomology, it is enough to prove the following statement: for any $x'_{\mathbb{C}} \in H^3_{dR,!}(S_{K_N(3)}, \check{\mathcal{V}}_\lambda)$ and any $y'_{\mathbb{C}} \in \ker(H^3_{dR,c}(S_{K_N(3)}, \mathcal{V}_\lambda) \to H^3_{dR}(S_{K_N(3)}, \mathcal{V}_\lambda))$, we have $\langle x'_{\mathbb{C}}, y'_{\mathbb{C}}\rangle_{dR,\mathbb{C}} = 0$ where $\mathcal{V}_\lambda$ is the complex vector bundle associated with the local system $V_{\lambda,\mathbb{C}}$, $\check{\mathcal{V}}_\lambda$ is the dual complex vector bundle and $\langle\,,\,\rangle_{dR,\mathbb{C}}$ denotes the Poincaré duality pairing in de Rham cohomology. Let us abusively denote by $x'_{\mathbb{C}}$ and $y'_{\mathbb{C}}$ closed differential forms in the cohomology class of $x'_{\mathbb{C}}$ and $y'_{\mathbb{C}}$ respectively. Because $x \in \mathrm{Im}(H_c^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}}) \to H^3(S_{K_N(3)}, \check{V}_{\lambda,\mathbb{Z}_{(p)}}))$, there exists a compactly supported closed differential 3-form x'' and a differential 2-form u such that $x'_{\mathbb{C}} = x'' + du$ and because $y \in \ker(H_c^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}))$ there exists a differential 2-form

y'' such that the compactly supported differential form $y'_{\mathbb{C}}$ satisfies $y'_{\mathbb{C}} = dy''$. We need to prove $\int_{S_{K_N(3)}} x'_{\mathbb{C}} \wedge y'_{\mathbb{C}} = 0$. But

$$x'_{\mathbb{C}} \wedge y'_{\mathbb{C}} = x'' \wedge dy'' + du \wedge y'_{\mathbb{C}} = d(-x'' \wedge y'' + u \wedge y'_{\mathbb{C}}).$$

As the differential form $-x'' \wedge y'' + u \wedge y'_{\mathbb{C}}$ is compactly supported, the statement follows from Stokes theorem. □

Lemma 4.11. *Let $\lambda = \lambda(k, k', 0)$ be a dominant weight with trivial central character. Assume that $k + k' + 3 \leq p$. Then, there exists a non-degenerate $G_{\mathbb{Z}_{(p)}}$-equivariant pairing*

$$[\,,\,] : V_{\lambda,\mathbb{Z}_{(p)}} \times V_{\lambda,\mathbb{Z}_{(p)}} \to \mathbb{Z}_{(p)}$$

where $V_{\lambda,\mathbb{Z}_{(p)}} \times V_{\lambda,\mathbb{Z}_{(p)}}$ is endowed with the diagonal action and $\mathbb{Z}_{(p)}$ with the trivial action.

Proof. Let us denote by $V_{\lambda,\mathbb{F}_p}$, resp. $\check{V}_{\lambda,\mathbb{F}_p}$ the reduction modulo p of $V_{\lambda,\mathbb{Z}_{(p)}}$, resp. of $\check{V}_{\lambda,\mathbb{Z}_{(p)}}$. It follows from the Lemma in section 1.9 of [PT02] that $V_{\lambda,\mathbb{F}_p}$ is irreducible. Furthermore as isomorphism classes of irreducible representations of $G_{\mathbb{F}_p}$ are determined by their highest weight, we have $V_{\lambda,\mathbb{F}_p} \simeq \check{V}_{\lambda,\mathbb{F}_p}$. Let r be the composition of the canonical projection $V_{\lambda,\mathbb{Z}_{(p)}} \to V_{\lambda,\mathbb{F}_p}$ and of the isomorphism $V_{\lambda,\mathbb{F}_p} \simeq \check{V}_{\lambda,\mathbb{F}_p}$. Let us fix $v \in V_{\lambda,\mathbb{Z}_{(p)}}$ such that $r(v) \neq 0$, let $w \in \check{V}_{\lambda,\mathbb{Z}_{(p)}}$ be a lifting of $r(v)$ and let $i : V_{\lambda,\mathbb{Z}_{(p)}} \to \check{V}_{\lambda,\mathbb{Z}_{(p)}}$ be the unique $G_{\mathbb{Z}_{(p)}}$-equivariant map sending v to w. This map i is well defined because $V_{\lambda,\mathbb{Z}_{(p)}}$ is irreducible by Nakayama's lemma. The reduction modulo p of i is the isomorphism an isomorphism $V_{\lambda,\mathbb{F}_p} \simeq \check{V}_{\lambda,\mathbb{F}_p}$ considered above. Hence i is an isomorphism by Nakayama's lemma. □

From now on, we fix a dominant weight $\lambda = \lambda(k, k', 0)$ with trivial central character. Let S_{weight} denote the finite set

$$S_{\text{weight}} = \{p \text{ primes} \mid p < k + k' + 3\} \tag{24}$$

and assume from now on that $p \notin S_{\text{weight}}$. The non-degenerate bilinear form of the previous lemma allows to identify $V_{\lambda,\mathbb{Z}_{(p)}}$ and $\check{V}_{\lambda,\mathbb{Z}_{(p)}}$. According to Lemma 4.10, there exists a unique pairing

$$\langle\,,\,\rangle : H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \times H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}\,.$$

such that the diagram

$$\begin{array}{ccc} H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \times H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) & \xrightarrow{\langle\,,\,\rangle} & \mathbb{Z}_{(p)} \\ \| & \downarrow & \| \\ H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \times H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) & \xrightarrow{\langle\,,\,\rangle} & \mathbb{Z}_{(p)} \end{array} \tag{25}$$

commutes. Let us define

$$S''_{\text{tors}} = \{p \text{ prime }, p | \#(H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})/H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}))_{\text{tors}}\} \tag{26}$$

Corollary 4.12. *Let $p \notin S_{\text{weight}} \cup S'_{\text{tors}} \cup S''_{\text{tors}}$. Then the natural map*

$$H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)})$$

induced by the pairing above is an isomorphism.

Proof. The commutative diagram (25) induces the commutative diagram

$$\begin{tikzcd} H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)}) \\ H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \arrow[u, hook] \arrow[r, hook] & \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)}) \arrow[u, hook] \end{tikzcd} \tag{27}$$

where the upper horizontal map is an isomorphism according to Lemma 4.9. In particular, the lower horizontal map is injective. Let's prove the surjectivity of this map. The diagram obtained by tensoring over $\mathbb{Z}_{(p)}$ with $\mathbb{Q}$ the diagram above is

$$\begin{tikzcd} H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Q}}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Q}}(H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Q}}), \mathbb{Q}) \\ H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Q}}) \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Q}}(H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Q}}), \mathbb{Q}). \arrow[u, hook] \end{tikzcd} \tag{28}$$

Note that the lower horizontal map is an isomorphism because it is an injection between two $\mathbb{Q}$-vector spaces of the same finite dimension. Let $x \in \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)})$ and let $x' \in H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ denote the image of x by the composite of the right hand vertical map and of the inverse of the upper horizontal map of (27). The image $x'_{\mathbb{Q}}$ of x' in $H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Q}})$ is in fact an element of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Q}})$. As a consequence, the image x'' of x' in $H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})/H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ is torsion. By our assumption $p \notin S''_{\mathrm{tors}}$, this implies $x'' = 0$. Hence there exists $y \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ which maps to x by the lower horizontal map of (27). □

Recall that by definition we have $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) = H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})^{K_N/K_N(3)}$.

Corollary 4.13. *Let $p \notin S_{N,3} \cup S_{\mathrm{weight}} \cup S'_{\mathrm{tors}} \cup S''_{\mathrm{tors}}$. Then the $\mathbb{Z}_{(p)}$-bilinear map*

$$\langle\ ,\ \rangle : \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) \times \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)} \tag{29}$$

obtained by restricting the bilinear map of the lower horizontal line of diagram (25) *is non-degenerate.*

Proof. For any $g \in K_N/K_N(3)$ and any $x, y \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$, we have $\langle g^*x, g^*y\rangle = \langle x, y\rangle$ as g acts as an automorphism of the $\mathbb{Q}$-scheme $S_{K_N(3)}$ and hence is orientation preserving. Let us prove that the map

$$\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathrm{Hom}_{\mathbb{Z}\,(p)}(\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)}) \tag{30}$$

defined as $x \mapsto (z \mapsto \langle x, z\rangle)$ is injective. Let $x \in \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$. Then, for any $g \in K_N/K_N(3)$ and any $y \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$, we have $\langle x, y\rangle = \langle g^*x, g^*y\rangle = \langle x, g^*y\rangle$. By summing over all $g \in K_N/K_N(3)$ and dividing by $|K_N/K_N(3)|$, we obtain

$$\langle x, y\rangle = |K_N/K_N(3)|^{-1}\langle x, \sum_{g\in K_N/K_N(3)} g^*y\rangle. \tag{31}$$

Assume that $\langle x, z\rangle = 0$ for any $z \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})^{K_N/K_N(3)}$. Then for any $y \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ we have $\langle x, y\rangle = |K_N/K_N(3)|^{-1}\langle x, \sum_{g\in K_N/K_N(3)} g^*y\rangle = 0$ because

$\sum_{g\in K_N/K_N(3)} g^*y$ is an element of $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$. According to Corollary 4.12 this implies that $x=0$. Hence (30) is injective.

To prove its surjectivity let $\chi \in \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)})$. Let us denote by $\chi' \in \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}), \mathbb{Z}_{(p)})$ the element $\chi' = \chi\circ\rho$ of where ρ is the projection $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \to \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$. By Corollary 4.12 there exists an element $x \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ such that $\langle x, y\rangle = \chi'(y)$ for any $y \in H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$. In particular for any element $z \in \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ we have $\langle x, z\rangle = \chi'(z) = (\chi\circ\rho)(z) = \chi(z)$. But for any such z we have $\chi(z) = \langle x, z\rangle = \langle \rho(x), z\rangle$ by (31). Hence (30) is surjective. □

For $g \in G(\mathbb{A}_f)$, we introduced the Hecke operator $T_g : H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \to H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ in definition 4.2. The dual Hecke operator $T_g^* : H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}}) \to H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}})$ is deduced from the sequence of maps defining T_g by applying the Verdier duality functor

$$\mathbb{D}(X) = R\underline{\mathrm{Hom}}(X, p_3^!\,\mathbb{Z})$$

and using the fact that $\mathbb{D}f_* = f_!\mathbb{D}$ and $\mathbb{D}f^* = f^!\mathbb{D}$. Let us denote again by T_g and T_g^* the endomorphisms deduced after extending scalars to $\mathbb{Z}_{(p)}$. Then we have $\langle T_g x, y\rangle = \langle x, T_g^* y\rangle$ for any $g \in G(\mathbb{A}_f)$ and any $x \in H^3(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$, $y \in H^3_c(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$.

Lemma 4.14. *Assume that $p \notin S_{N,3} \cup S_{\mathrm{weight}} \cup S_{\mathrm{tors}} \cup S'_{\mathrm{tors}} \cup S''_{\mathrm{tors}}$ where $S_{K_N/K_N(3)}$, S_{tors}, S'_{tors}, S_{weight} and S''_{tors} are defined by* (15), (16), (23), (24) *and* (26) *respectively. Then we have a $\mathbb{Z}_{(p)}$-linear pairing*

$$L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \times L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}\,. \tag{32}$$

Proof. Recall that in particular, p does not divide the order of $K_N/K_N(3)$, and so the $\mathbb{Z}_{(p)}$-module $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ is a direct factor of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$. As $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ is a $\mathbb{Z}_{(p)}$-torsion free quotient of $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ by definition, it is also a direct factor of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$. Let $\check{\Pi}_f$ be the contragredient of Π_f. Note that, as the representation $V_{\lambda,\mathbb{Z}_{(p)}}$ has trivial central character, the representation $\check{\Pi}_f$ contributes to the cohomology with coefficients $V_{\lambda,\mathbb{Z}_{(p)}}$. Similarly as before, the $\mathbb{Z}_{(p)}$-module $M(\check{\Pi}_f, V_{\lambda,\mathbb{Z}_{(p)}})$ is a direct factor of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$. Hence the $\mathbb{Z}_{(p)}$-module $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \times L(\check{\Pi}_f, V_{\lambda,\mathbb{Z}_{(p)}})$ is a direct factor of $H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}}) \times H^3_!(S_{K_N(3)}, V_{\lambda,\mathbb{Z}_{(p)}})$ and by restricting the pairing (25) we obtain a pairing $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \times L(\check{\Pi}_f, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}$. One of the assumptions on Π in section 3, is that the central character of Π is trivial. According to [We05, Lemma 1.1], this implies that $\Pi \simeq \check{\Pi}$ where $\check{\Pi}$ is the contragredient of Π. This concludes the proof. □

The following results will be useful in the next section. Let T' be the maximal compact subtorus of $\mathrm{Sp}(4,\mathbb{R})$ defined by

$$T' = \left\{ \begin{pmatrix} x & & y & \\ & x' & & y' \\ -y & & x & \\ & -y' & & x' \end{pmatrix} \mid x^2 + y^2 = x'^2 + y'^2 = 1 \right\}.$$

The Lie algebra of T' is the compact Cartan subalgebra of $\mathfrak{sp}_4$ that we denoted by $\mathfrak{h}$ in section 2.3. Let $\mathbb{R}_+^\times$ be the identity component of the center of $G(\mathbb{R})$. For integers n, n', c

such that $n+n' \equiv c \pmod 2$, let $\lambda'(n,n',c) : \mathbb{R}_+^\times T' \longrightarrow \mathbb{C}^\times$ denote the character defined by

$$\begin{pmatrix} x & & y & \\ & x' & & y' \\ -y & & x & \\ & -y' & & x' \end{pmatrix} \longmapsto (x+iy)^n (x'+iy')^{n'} (x^2+y^2)^{\frac{c-n-n'}{2}},$$

and by $\lambda'(n,n')$ the restriction of $\lambda'(n,n',c)$ to T'. Note that the simple root $e_1 - e_2$, respectively $2e_2$, defined in section 2.3, coincides with the differential at the identity matrix of the restriction to T' of the character $\lambda'(1,-1,0)$, respectively $\lambda(0,2,0)$. Let $J \in \mathrm{Sp}(4,\mathbb{C})$ be the matrix

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & i & \\ & 1 & & i \\ i & & 1 & \\ & i & & 1 \end{pmatrix}.$$

Lemma 4.15. *Let $w \in V_{\lambda,\mathbb{C}}$ be a vector of weight $\lambda(u,u',c)$ for the action of the algebraic torus T. For the action of the torus $\mathbb{R}_+^\times T'$, the vector $v = Jw$ has weight $\lambda'(u,u',c)$ and the vector $\overline{v} = \overline{J}w$ has weight $\lambda'(-u,-u',c)$.*

Proof. The first statement follows from [Le17, Lemma 4.25]. The second statement follows from the fact that

$$\begin{aligned} \overline{J}^{-1} \begin{pmatrix} x & & y & \\ & x' & & y' \\ -y & & x & \\ & -y' & & x' \end{pmatrix} \overline{J} &= J \begin{pmatrix} x & & y & \\ & x' & & y' \\ -y & & x & \\ & -y' & & x' \end{pmatrix} \overline{J} \\ &= \begin{pmatrix} x-iy & & & \\ & x'-iy' & & \\ & & x+iy & \\ & & & x'+iy' \end{pmatrix} \end{aligned}$$

and hence

$$\begin{aligned} \begin{pmatrix} x & & y & \\ & x' & & y' \\ -y & & x & \\ & -y' & & x' \end{pmatrix} \overline{J}w &= \overline{J} \begin{pmatrix} x-iy & & & \\ & x'-iy' & & \\ & & x+iy & \\ & & & x'+iy' \end{pmatrix} w \\ &= (x-iy)^{-k}(x'-iy')^{k'}(x^2+y^2)^{\frac{c+k-k'}{2}} \overline{J}w \\ &= (x+iy)^{k}(x'+iy')^{-k'}(x^2+y^2)^{\frac{c-k+k'}{2}} \overline{J}w \\ &= \lambda'(k,-k',c) \left(\begin{pmatrix} x & & y & \\ & x' & & y' \\ -y & & x & \\ & -y' & & x' \end{pmatrix} \right) \overline{J}w. \end{aligned}$$

□

Lemma 4.16. *Let $w \in V_{\lambda,\mathbb{C}}$ be a vector of weight $\lambda(-k,k',0)$, let $[\,,\,]_{\mathbb{C}} : V_{\lambda,\mathbb{C}} \otimes V_{\lambda,\mathbb{C}} \to \mathbb{C}$ denote the pairing obtained from $[\,,\,]$ after extending scalars to $\mathbb{C}$. Then*

$$[Jw, \overline{J}w] \neq 0.$$

Proof. The pairing $[\,,\,]_{\mathbb{C}}$ is $G_{\mathbb{C}}$-equivariant. In particular, given two vectors v_1 and v_2 of weights $\lambda'(u_1,u_1',0)$ and $\lambda'(u_2,u_2',0)$, we have

$$[v_1,v_2]_{\mathbb{C}} \neq 0 \iff u_1+u_2 = u_1'+u_2' = 0.$$

The weight $\lambda(-k,k',0)$ belongs to the orbit under the action of W of the dominant weight $\lambda(k,k',0)$, hence has multiplicity one in $V_{\lambda,\mathbb{C}}$. Then it follows from Lemma 4.15 that $\lambda'(-k,k',0)$, which is the weight of Jw, has multiplicity one in $V_{\lambda,\mathbb{C}}$ and a similar argument applies to $\lambda(k,-k',0)$, which is the weight of $\overline{J}w$. As a consequence, if $[Jw,\overline{J}w]_{\mathbb{C}} = 0$, then $[Jw,w']_{\mathbb{C}} = 0$ for any $w' \in V_{\lambda,\mathbb{C}}$. This contradicts the fact that $[\,,\,]_{\mathbb{C}}$ is non-degenerate. Hence $[Jw,\overline{J}w] \neq 0$. □

Let

$$X_{(1,-1)} = d\kappa\left(\begin{pmatrix} & 1 \\ & \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} & 1 & & -i \\ -1 & & -i & \\ & i & & 1 \\ i & & -1 & \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$$

and let

$$X_{(-1,1)} = d\kappa\left(\begin{pmatrix} & \\ 1 & \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} & 1 & & i \\ -1 & & i & \\ & -i & & 1 \\ -i & & -1 & \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$$

These are root vectors corresponding to the positive (resp. negative), compact root. Let us denote $v = Jw$ and $\overline{v} = \overline{J}w$.

Lemma 4.17. *For any $i,j \in \mathbb{Z}$ we have*

$$\left[X^i_{(1,-1)}v, X^j_{(-1,1)}\overline{v}\right]_{\mathbb{C}} = \begin{cases} 0 & \text{if } i \neq j \\ (-1)^i \frac{i!(k+k')!}{(k+k'-i)!}[v,\overline{v}] & \text{if } i = j. \end{cases}$$

Proof. The first statement follows from weight reasons. Let us prove the second one. As $[\,,\,]_{\mathbb{C}}$ is compatible with the action of the Lie algebra $\mathfrak{k}_{\mathbb{C}}$, we have

$$\left[X^i_{(1,-1)}v, X^i_{(-1,1)}\overline{v}\right]_{\mathbb{C}} = (-1)^i\left[v, X^i_{(1,-1)}X^i_{(-1,1)}\overline{v}\right]_{\mathbb{C}}$$

The irreducible sub $\mathbb{C}[K_\infty]$-module of $V_{\lambda,\mathbb{C}}$ generated by the vector $\overline{v}$ is equivalent to $\tau_{(k',-k)}$ and $\overline{v}$ is a highest weight vector of it. By an easy inductive application of (8) and (9) we obtain $X^i_{(1,-1)}X^i_{(-1,1)}\overline{v} = \frac{i!(k+k')!}{(k+k'-i)!}\overline{v}$ and the conclusion follows. □

5. Definition of the periods

For $\Pi_\infty \in P(V_{\lambda,\mathbb{C}})$ we have

$$H^3(\mathfrak{g}_\mathbb{C}, K_\infty, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty) = \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty\right)$$

and it is known that this $\mathbb{C}$-vector space is one-dimensional (see [Le17, Proposition 3.7] for example). The character λ has the form $\lambda = \lambda(k, k', c)$ for $k \geq k' \geq 0$ and $k + k' \equiv c \pmod 2$. Recall that $\Pi_\infty^W|_{G(\mathbb{R})_+} = \Pi_\infty^{2,1} \oplus \Pi_\infty^{1,2}$.

Lemma 5.1. *Let $\phi \in \Pi_\infty^{2,1}$ be a lowest weight vector of the minimal K_∞-type, let $w \in V_{\lambda,\mathbb{C}}$ be a vector of weight $\lambda(-k, k', c)$ and let $v = Jw \in V_{\lambda,\mathbb{C}}$.*

(1) *There exists a unique non-zero element*

$$[\phi, v] \in \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^W\right)$$

such that

$$[\phi, v](X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) = \sum_{i=0}^{k+k'} (-1)^i X_{(1,-1)}^i v \otimes X_{(1,-1)}^{k+k'+4-i}\phi$$

where $X_{(2,0)}$, $X_{(1,1)} \in \mathfrak{p}^+$ and $X_{(0,-2)} \in \mathfrak{p}^-$ are the root vectors defined in section 2.3.

(2) *The map $[\phi, v]$ factors through the canonical projection*

$$\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C} \to \bigwedge^2 \mathfrak{p}^+ \otimes_\mathbb{C} \mathfrak{p}^- \to \tau_{(3,-1)}$$

where $\tau_{(3,-1)}$ is the irreducible sub $\mathbb{C}[K_\infty]$-module of $\bigwedge^2 \mathfrak{p}^+ \otimes_\mathbb{C} \mathfrak{p}^-$ generated by the highest weight vector $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$.

Proof. To prove the existence part of statement (1), note that $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$ is a highest weight vector of weight $\lambda'(3, -1, 0)$ of the $\mathbb{C}[K_\infty]$-module $\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}$. Hence, as we know that $\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^W\right)$ has dimension 1, to define a morphism in this space it is enough to define the image of $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$, under the condition that this image is a vector of $V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^W$ which is a highest weight vector and which has the same weight as $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$. According to Lemma 4.15, the vector v has weight $\lambda'(-k, k', c)$ and hence the vector $X_{(1,-1)}^i v$ has weight $\lambda'(-k+i, k'-i, c)$. On the other hand according to Proposition 2.4, the vector $\phi \in \Pi_\infty^W$ has weight $\lambda'(-k'-1, k+3, -c)$ and hence the vector $X_{(1,-1)}^{k+k'+4-i}\phi$ has weight $\lambda'(k+3-i, -k'-1-i, -c)$. As a consequence, the vector $\sum_{i=0}^{k+k'} (-1)^i X_{(1,-1)}^i v \otimes X_{(1,-1)}^{k+k'+4-i}\phi$ has the same weight as $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$. Furthermore

$$X_{(1,-1)}\left(\sum_{i=0}^{k+k'} (-1)^i X_{(1,-1)}^i v \otimes X_{(1,-1)}^{k+k'+4-i}\phi\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{k+k'} (-1)^i X_{(1,-1)}^{i+1} v \otimes X_{(1,-1)}^{k+k'+4-i} \phi + \sum_{i=0}^{k+k'} (-1)^i X_{(1,-1)}^{i} v \otimes X_{(1,-1)}^{k+k'+5-i} \phi \\
&= \sum_{i=0}^{k+k'-1} (-1)^i X_{(1,-1)}^{i+1} v \otimes X_{(1,-1)}^{k+k'+4-i} \phi + \sum_{i=1}^{k+k'} (-1)^i X_{(1,-1)}^{i} v \otimes X_{(1,-1)}^{k+k'+5-i} \phi \\
&= \sum_{i=1}^{k+k'} (-1)^{i-1} X_{(1,-1)}^{i} v \otimes X_{(1,-1)}^{k+k'+5-i} \phi + \sum_{i=1}^{k+k'} (-1)^i X_{(1,-1)}^{i} v \otimes X_{(1,-1)}^{k+k'+5-i} \phi \\
&= 0.
\end{aligned}
$$

This means that $\sum_{i=0}^{k+k'} (-1)^i X_{(1,-1)}^{i} v \otimes X_{(1,-1)}^{k+k'+4-i} \phi$ is a highest weight vector. As a consequence the element $[\phi, v]$ of statement (1) of the Lemma exists. Its unicity follows from the fact that $\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}'_{\mathbb{C}}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^W\right)$ has dimension 1. Statement (2) is a direct consequence of the 1-dimensionality of $\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}'_{\mathbb{C}}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^W\right)$ and of the construction of $[\phi, v]$. □

Let $\Pi = \Pi_\infty \otimes \Pi_f$ be as in section 3. Let Π_f^0 a model of Π_f over the rationality field $E(\Pi_f)$. Assume in addition that Π has level 1. For any embedding $\sigma : E(\Pi_f) \to \mathbb{C}$, let us define the representation ${}^\sigma\Pi_f = \Pi_f^0 \otimes_\sigma \mathbb{C}$. Let us introduce the $\mathbb{C}$-linear map ${}^\sigma\omega_{\phi,v}$ defined by

$$
{}^\sigma\omega_{\phi,v} : {}^\sigma\Pi_f^{K_N} \to L(\Pi_f, V_{\lambda,\mathbb{C}}),\ \psi \mapsto [\phi, v] \otimes \psi. \tag{33}
$$

Definition 5.2. *Let $M(\Pi_f, V_{\lambda,\mathbb{R}})$ denote the sub $\mathbb{R}$-vector space of $M(\Pi_f, V_{\lambda,\mathbb{C}})$ of vectors which are fixed under the involution defined on each factor*

$$
H^3(\mathfrak{g}_{\mathbb{C}}, K_\infty, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty) \otimes {}^\sigma\Pi_f^{K_N} = \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}'_{\mathbb{C}}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty \otimes {}^\sigma\Pi_f^{K_N}\right)
$$

of the decomposition (21) *by $\overline{h} : X \mapsto \overline{h(\overline{X})}$.*

Remark 5.3. *Note that this action is well defined because for any cusp form $\psi \in \Pi_\infty \otimes {}^\sigma\Pi_f^{K_N}$, the cusp form $\overline{\psi}$ still belongs to $\Pi_\infty \otimes {}^\sigma\Pi_f^{K_N}$ via the identification of $\Pi_\infty \otimes {}^\sigma\Pi_f$ with its contragredient via the Petersson inner product.*

Lemma 5.4. *Let* $\mathrm{Re} : L(\Pi_f, V_{\lambda,\mathbb{C}}) \to L(\Pi_f, V_{\lambda,\mathbb{R}})$ *be the $\mathbb{R}$-linear projection defined as $h \mapsto \frac{1}{2}(h + \overline{h})$.*

(1) For any ϕ, v as above, the composition

$$
\bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} {}^\sigma\Pi_f^{K_N} \hookrightarrow L(\Pi_f, V_{\lambda,\mathbb{C}}) \overset{\mathrm{Re}}{\twoheadrightarrow} L(\Pi_f, V_{\lambda,\mathbb{R}}),
$$

is an isomorphism of $\mathbb{R}$-vector spaces of dimension $2[E(\Pi_f) : \mathbb{Q}]$ where the first map is $\bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} {}^\sigma\omega_{\phi,v}$.

(2) *For any embedding* $\sigma : E(\Pi_f) \to \mathbb{C}$*, let* ${}^\sigma\varphi_\infty \in {}^\sigma\Pi_\infty$ *and* ${}^\sigma\varphi_f \in {}^\sigma\Pi_f^{K_N}$ *be vectors such that* ${}^\sigma\varphi = {}^\sigma\varphi_\infty \otimes {}^\sigma\varphi_f$ *via the isomorphism* (12)*. Let* $\sigma_1, \dots, \sigma_r$ *denote the embeddings of* $E(\Pi_f)$ *in* $\mathbb{C}$*. Then*

$$\left(\mathrm{Re}\,{}^{\sigma_1}\omega_{\varphi_\infty,v}({}^{\sigma_1}\varphi_f), \dots, \mathrm{Re}\,{}^{\sigma_r}\omega_{\varphi_\infty,v}({}^{\sigma_r}\varphi_f), \mathrm{Re}\,{}^{\sigma_1}\omega_{\varphi_\infty,v}(\sqrt{-1}\,{}^{\sigma_1}\varphi_f), \dots, \mathrm{Re}\,{}^{\sigma_r}\omega_{\varphi_\infty,v}(\sqrt{-1}\,{}^{\sigma_r}\varphi_f)\right)$$

is a basis of the $\mathbb{R}$*-vector space* $L(\Pi_f, V_{\lambda,\mathbb{R}})$*.*

Proof. The second statement is an direct consequence of the first. To prove the first, as the dimensions of the $\mathbb{R}$-vector spaces $\bigoplus_{\sigma:E(\Pi_f)\to\mathbb{C}} {}^\sigma\Pi_f^{K_N}$ and $L(\Pi_f, V_{\lambda,\mathbb{R}})$ are finite and equal to $2[E(\Pi_f) : \mathbb{Q}]$, it is enough to prove the surjectivity of the $\mathbb{R}$-linear map of the statement. To this end, let us fix a non-zero vector ${}^\sigma\psi \in {}^\sigma\Pi_f^{K_N}$ for any σ. The vectors ${}^\sigma\psi$ and $\sqrt{-1}\,{}^\sigma\psi$ form a basis of the $\mathbb{R}$-vector space underlying ${}^\sigma\Pi_f^{K_N}$. Furthermore by the isomorphism (21), Theorem 4.6, Proposition 4.7 and the remark at the beginning of Section 5 we have

$$L(\Pi_f, V_{\lambda,\mathbb{C}}) \simeq \bigoplus_{\sigma:E(\Pi_f)\to\mathbb{C}} \left(\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{2,1}\right) \oplus \right.$$
$$\left. \mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{1,2}\right)\right) \otimes {}^\sigma\Pi_f^{K_N}$$

and the complex conjugation exchanges the subspace $\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{2,1}\right)$ and the subspace $\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{1,2}\right)$. As a consequence, the vectors

$$\left\{[\phi, v] \otimes {}^\sigma\psi,\ \sqrt{-1}[\phi, v] \otimes {}^\sigma\psi,\ \overline{[\phi, v]} \otimes {}^\sigma\psi,\ \sqrt{-1}\,\overline{[\phi, v]} \otimes {}^\sigma\psi\right\}_{\sigma:E(\Pi_f)\to\mathbb{C}}$$

form a basis of the $\mathbb{R}$-vector space underlying $L(\Pi_f, V_{\lambda,\mathbb{C}})$. This implies that the vectors

$$\left\{([\phi, v] + \overline{[\phi, v]}) \otimes {}^\sigma\psi,\ ([\phi, v] + \overline{[\phi, v]}) \otimes \sqrt{-1}\,{}^\sigma\psi\right\}_{\sigma:E(\Pi_f)\to\mathbb{C}}$$

form a basis of the $\mathbb{R}$-vector space $L(\Pi_f, V_{\lambda,\mathbb{R}})$. For any $\sigma : E(\Pi_f) \to \mathbb{C}$ the vector $([\phi, v] + \overline{[\phi, v]}) \otimes {}^\sigma\psi$ (resp. $([\phi, v] + \overline{[\phi, v]}) \otimes \sqrt{-1}\,{}^\sigma\psi$) is the image of ${}^\sigma\psi$ (resp. of $\sqrt{-1}\,{}^\sigma\psi$) by the the composite map

$$\bigoplus_{\sigma:E(\Pi_f)\to\mathbb{C}} {}^\sigma\Pi_f^{K_N} \hookrightarrow L(\Pi_f, V_{\lambda,\mathbb{C}}) \overset{\mathrm{Re}}{\twoheadrightarrow} L(\Pi_f, V_{\lambda,\mathbb{R}})$$

of the statement. This proves the surjectivity of this map as claimed. □

Remark 5.5. *For any* $1 \le i \le r$*, the vectors* $\mathrm{Re}\,{}^{\sigma_i}\omega_{\varphi_\infty,v}({}^{\sigma_i}\varphi_f)$ *and* $\mathrm{Re}\,{}^{\sigma_i}\omega_{\varphi_\infty,v}(\sqrt{-1}\,{}^{\sigma_i}\varphi_f)$ *do not depend on the choice of* ${}^{\sigma_i}\varphi_\infty$ *and of* ${}^{\sigma_i}\varphi_f$ *and only depend on* ${}^{\sigma_i}\varphi = {}^{\sigma_i}\varphi_\infty \otimes {}^{\sigma_i}\varphi_f$ *and* v*.*

According to Lemma 4.16, the pairing $[v, \overline{v}]$ is a non-zero complex number. As a consequence we can normalize v in such a way that $[v, \overline{v}] = 1$. Following [Hi81a, §6] , we can introduce the period of interest in this work. Let us choose a basis $(\delta_1, \dots, \delta_{2r})$ of the free $\mathbb{Z}_{(p)}$-module $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$. Let us denote by $(\omega_1, \dots, \omega_{2r})$ the basis of the second assertion

in the previous lemma and let $U \in \mathrm{GL}_{2r}(\mathbb{R})$ be such that $(\delta_1, \ldots, \delta_{2r})U = (\omega_1, \ldots, \omega_{2r})$. Then we define

$$\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r})) = \det(U). \tag{34}$$

Remark 5.6. *Under the above normalization of v, the vector v is unique up to multiplication by ± 1. Furthermore, if we change the basis $(\delta_1, \ldots, \delta_{2r})$ by another basis $(\delta'_1, \ldots, \delta'_{2r})$ the period $\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r}))$ is changed by an element of $\mathbb{Z}_{(p)}^\times$. Hence, the image of the real number $\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r}))$ in $\mathbb{R}^\times / \mathbb{Z}_{(p)}^\times$ is independent of the choice of v normalized as above and of the basis $(\delta_1, \ldots, \delta_{2r})$. In what follows, the image of $\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r}))$ in $\mathbb{R}^\times / \mathbb{Z}_{(p)}^\times$ will be denoted by $\Omega(\Pi_f)$.*

6. Discriminant and adjoint L-values

In what follows Π denotes an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ satisfying the assumptions of section 3. Let $\mathbf{1}$ denote the generator of the one-dimensional $\mathbb{C}$-vector space $\bigwedge^6 \mathfrak{sp}_{4,\mathbb{C}}/\mathfrak{k}_\mathbb{C}$ defined as

$$\mathbf{1} = X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,2)} \wedge X_{(-2,0)} \wedge X_{(-1,-1)} \wedge X_{(0,-2)}.$$

This determines a left translation invariant measure $d\mu$ on $\mathrm{Sp}(4,\mathbb{R})/K_\infty = G(\mathbb{R})_+/K'_\infty$ in a standard way. By our normalization of the vectors $X_{(2,0)}, X_{(1,1)}, X_{(0,2)}, X_{(-2,0)}, X_{(-1,-1)}$ and $X_{(0,-2)}$ (see Section 2.3), this measure coincides with the standard invariant measure $dXdY/\det(Y)^3$ via the isomorphism $G(\mathbb{R})_+/K'_\infty \simeq \mathcal{H}_+$. Let dg_∞ be the left invariant measure on $G(\mathbb{R})_+/\mathbb{R}_+^\times$ attached to $d\mu$ by the construction (4). For every prime number p, let dg_p be the unique translation invariant measure on $G(\mathbb{Q}_p)$ such that $\mathrm{vol}(G(\mathbb{Z}_p), dg_p) = 1$ and let dg be the measure on $Z(\mathbb{A})\backslash G(\mathbb{A})$ defined by $dg = \prod_v dg_v$.

Proposition 6.1. *The pairing obtained from* (32) *after extending coefficients from $\mathbb{Z}_{(p)}$ to $\mathbb{C}$ is given by*

$$\begin{aligned}
&\left(\bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} \bigoplus_{\Pi_\infty \in P(V_{\lambda,\mathbb{C}})} H^3(\mathfrak{g}_\mathbb{C}, K'_\infty, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty)^{\oplus m(\Pi_\infty \otimes {}^\sigma\Pi_f)} \otimes_\mathbb{C} {}^\sigma\Pi_f^{K_N} \right) \\
&\quad \otimes \left(\bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} \bigoplus_{\Pi_\infty \in P(V_{\lambda,\mathbb{C}})} H^3(\mathfrak{g}_\mathbb{C}, K'_\infty, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty)^{\oplus m(\Pi_\infty \otimes {}^\sigma\Pi_f)} \otimes_\mathbb{C} {}^\sigma\Pi_f^{K_N} \right) \to \mathbb{C}
\end{aligned} \tag{35}$$

where (35) *is induced by the composite*

$$\begin{aligned}
&\left(\mathrm{Hom}_{K'_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty \right) \otimes {}^\sigma\Pi_f^{K_N} \right) \otimes \left(\mathrm{Hom}_{K'_\infty}\left(\bigwedge^3 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes \Pi'_\infty \right) \otimes {}^{\sigma'}\Pi_f^{K_N} \right) \\
&\quad \to \mathrm{Hom}_{K'_\infty}\left(\bigwedge^6 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, V_{\lambda,\mathbb{C}} \otimes V_{\lambda,\mathbb{C}} \otimes \Pi_\infty \otimes \Pi'_\infty \right) \otimes {}^\sigma\Pi_f^{K_N} \otimes {}^{\sigma'}\Pi_f^{K_N} \\
&\quad\quad \to \mathrm{Hom}_{K'_\infty}\left(\bigwedge^6 \mathfrak{g}_\mathbb{C}/\mathfrak{k}'_\mathbb{C}, \Pi_\infty \otimes \Pi'_\infty \right) \otimes {}^\sigma\Pi_f^{K_N} \otimes {}^{\sigma'}\Pi_f^{K_N} \to \mathbb{C}
\end{aligned}$$

where the first map is the exterior product, the second map is induced by $[\,,\,]_{\mathbb{C}}$ *and the third is the composition of the evaluation at* $\mathbf{1}$ *followed by* $\int_{\mathbb{R}_+^\times G(\mathbb{Q})\backslash G(\mathbb{A})/K_N} dg$.

Proof. This follows from the compatibility of the Poincaré duality with the comparison isomorphism (21) between Betti and de Rham cohomology and the description given in [Bo81, (5)] of the Poincaré duality in de Rham cohomology in terms of $(\mathfrak{g}_{\mathbb{C}}, K'_\infty)$-cohomology. □

For any $\mathbb{Z}_{(p)}$-algebra R, let us denote by

$$\langle\,,\,\rangle_R : L(\Pi_f, V_{\lambda,R}) \times L(\Pi_f, V_{\lambda,R}) \to R$$

the pairing deduced from (32) after extending scalars to A. Then, we have the commutative diagram

$$\begin{array}{ccc} L(\Pi_f, V_{\lambda,\mathbb{R}}) \times L(\Pi_f, V_{\lambda,\mathbb{R}}) & \xrightarrow{\langle\,,\,\rangle_{\mathbb{R}}} & \mathbb{R} \\ \downarrow & & \downarrow \\ L(\Pi_f, V_{\lambda,\mathbb{C}}) \times L(\Pi_f, V_{\lambda,\mathbb{C}}) & \xrightarrow{\langle\,,\,\rangle_{\mathbb{C}}} & \mathbb{C} \end{array} \tag{36}$$

where the vertical arrows are the natural inclusions.

Lemma 6.2. *There exists* $\epsilon_\lambda \in \mathbb{Z}$ *such that for any* $v, w \in L(\Pi_f, V_{\lambda,\mathbb{C}})$ *we have*

$$\langle w, v\rangle_{\mathbb{C}} = (-1)^{\epsilon_\lambda}\langle v, w\rangle_{\mathbb{C}}.$$

Proof. In the statement of Proposition 6.1, the first map in the sequence defining (35) is alternate as it is defined as the exterior product of differential forms of degree 3. The second map is induced by the $G_{\mathbb{C}}$-invariant bilinear form $[\,,\,]_{\mathbb{C}}$ on $V_{\lambda,\mathbb{C}} \otimes V_{\lambda,\mathbb{C}}$. As $V_{\lambda,\mathbb{C}}$ is irreducible, it is an easy consequence of Schur Lemma that such a bilinear form is unique up to a scalar. Let $[\,,\,]'_{\mathbb{C}}$ be the bilinear form on $V_{\lambda,\mathbb{C}} \otimes V_{\lambda,\mathbb{C}}$ defined by $[v, w]'_{\mathbb{C}} = [w, v]_{\mathbb{C}}$. There exists $\lambda \in \mathbb{C}$ such that $[v, w]'_{\mathbb{C}} = \lambda[v, w]_{\mathbb{C}}$. Let $v, w \in V_{\lambda,\mathbb{C}}$ such that $[v, w]_{\mathbb{C}} \neq 0$. Then

$$\begin{aligned} [v, w]_{\mathbb{C}} &= \lambda[v, w]'_{\mathbb{C}} \\ &= \lambda[w, v]_{\mathbb{C}} \\ &= \lambda^2[w, v]'_{\mathbb{C}} \\ &= \lambda^2[v, w]_{\mathbb{C}}. \end{aligned}$$

Hence $\lambda = \pm 1$ and $[\,,\,]_{\mathbb{C}}$ is either symmetric or alternate. The conclusion now follows from the fact that the last map in the sequence defining (35) is symmetric. □

Lemma 6.3. *As measures on* $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$, *we have*

$$dg = \frac{\pi^3}{270} dg^{\mathrm{Tam}}.$$

Proof. By Weil conjecture on Tamagawa numbers proved by Kottwitz (see [Ko88]), we have

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} dg^{\mathrm{Tam}} = 1.$$

On the other hand, as $\mathrm{vol}(K_\infty, dg) = \mathrm{vol}(G(\mathbb{Z}_p), dg) = 1$ we have

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} dg = \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty G(\widehat{\mathbb{Z}})} dg.$$

It follows from the definition of dg that there is an isomorphism of measured spaces

$$(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty G(\widehat{\mathbb{Z}}), dg) \simeq (\mathrm{PSp}(4,\mathbb{Z})\backslash \mathcal{H}_+, dXdY/\det(Y)^3)$$

and in particular

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty G(\widehat{\mathbb{Z}})} dg = \int_{\mathrm{PSp}(4,\mathbb{Z})\backslash \mathcal{H}_+} dXdY/\det(Y)^3.$$

Let $\xi(s)$ denotes the complete Riemann zeta function $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. According to [Sie43, Theorem 11] the last displayed integral is equal to $2\xi(2)\xi(4)$. The conclusion now follows from the well known equalities $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. □

Proposition 6.4. *Let $\sigma : E(\Pi_f) \to \mathbb{C}$ be an embedding, let ${}^\sigma\varphi_f \in {}^\sigma\Pi_f$ be the normalized vector, let ${}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \in L(\Pi_f, V_{\lambda,\mathbb{C}})$ be defined by (33). For any $0 \le i \le k+k'$, any $0 \le r \le 4$, any $0 \le u \le r$ such that $i - u \ge 0$, let us denote*

$$\begin{aligned} r^{k,k'}_{i,u,r} &= \frac{(k+k'+u-i)!(k+k'+4-i)!(i+r-u)!}{(i-u)!(k+k'-i)!(k+k'+4-i-r+u)}, \\ s^{k,k'}_{i,u} &= \frac{(i-u)!}{(k+k'-i+u)!}, \\ t^{k,k'}_{i,u,r} &= \frac{(k+k'+4+u-r-i)!}{(i+r-u)!} \end{aligned}$$

let us denote $a_0 = -1, a_1 = -\frac{1}{4}, a_2 = \frac{1}{72}, a_3 = -\frac{1}{72}, a_4 = -\frac{1}{576}$, let us define $C'_{k,k'} \in \mathbb{Q}$ by

$$C'_{k,k'} = \frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2} \sum_{r=0}^{4} \sum_{i=0}^{k+k'} \sum_{\substack{0\le u,u'\le r \\ 0\le i-u}} (-1)^{r+u+u'} a_r \binom{r}{u}\binom{r}{u'} r^{k,k'}_{i,u,r} r^{k,k'}_{i-u+u',u',r} s^{k,k'}_{i,u} t^{k,k'}_{i,u,r} \tag{37}$$

Then we have

$$\langle {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f), \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}\rangle_{\mathbb{C}} = \frac{\pi^3}{3^3\cdot 5} C'_{k,k'} \prod_{l|N} (l^2+1)^{-1} \langle {}^\sigma\varphi, {}^\sigma\varphi\rangle. \tag{38}$$

Proof. We need to calculate the image of the vector ${}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}$ by the sequence of maps defining (35). Let us introduce the following notation: $e_1 = X_{(2,0)}, e_2 = X_{(1,1)}, e_3 = X_{(0,2)}, e_4 = X_{(-2,0)}, e_5 = X_{(-1,-1)}, e_6 = X_{(0,-2)}$. Then, by definition of the exterior product, we have

$$({}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \wedge \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)})(\mathbf{1})$$

$$= \frac{1}{3!^2} \sum_{\sigma \in \mathcal{S}_6} \epsilon(\sigma)({}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)})(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)} \otimes e_{\sigma(4)} \otimes e_{\sigma(5)} \otimes e_{\sigma(6)})$$
$$= \frac{1}{3!^2} \sum_{\sigma \in \mathcal{S}_6} \epsilon(\sigma){}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(e_{\sigma(4)} \otimes e_{\sigma(5)} \otimes e_{\sigma(6)})$$

According to the second statement of Lemma 5.1 we have ${}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = 0$ whenever the image of $e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ by the projection

$$\bigotimes^3 \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}'_{\mathbb{C}} \to \bigwedge^3 \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}'_{\mathbb{C}} \to \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^- \to \tau_{(3,-1)}$$

is zero. The weight vectors

$$X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)},\ X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)},\ X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)},$$

$$X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\ X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)},\ X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)},$$

$$X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)},\ X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)},\ X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)}$$

form a basis of $\bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^-$. Computing the signatures, we have

$$({}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \wedge \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)})(\mathbf{1})$$

$$\begin{aligned}
&= \frac{1}{3!^2}\Big({}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(0,2)} \otimes X_{(-1,-1)} \wedge X_{(-2,0)}) \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(0,2)} \otimes X_{(0,-2)} \wedge X_{(-2,0)}) \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(0,2)} \otimes X_{(0,-2)} \wedge X_{(-1,-1)}) \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(1,1)} \otimes X_{(-1,-1)} \wedge X_{(-2,0)}) \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(1,1)} \otimes X_{(0,-2)} \wedge X_{(-2,0)}) \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(1,1)} \otimes X_{(0,-2)} \wedge X_{(-1,-1)}) \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(2,0)} \otimes X_{(-1,-1)} \wedge X_{(-2,0)}) \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(2,0)} \otimes X_{(0,-2)} \wedge X_{(-2,0)}) \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}(X_{(2,0)} \otimes X_{(0,-2)} \wedge X_{(-1,-1)})\Big).
\end{aligned}$$

As a consequence, using that $\overline{\mathfrak{p}^+ \otimes_{\mathbb{C}} \bigwedge^2 \mathfrak{p}^-} = \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^-$ and $\overline{X}_{(r,s)} = X_{(-r,-s)}$, we have

$$
\begin{aligned}
&({}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \wedge \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)})(\mathbf{1}) \\
&= \frac{1}{3!^2}\Big(- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)})} \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)})} \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)})} \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)})} \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)})} \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)})} \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)})} \\
&+ {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)})} \\
&- {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)}) \otimes \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)})}\Big).
\end{aligned}
$$

A standard basis of $\tau_{(3,-1)}$ is computed in Lemma 2.2 and the matrix of the projection $p : \bigwedge^2 \mathfrak{p}^+ \otimes_{\mathbb{C}} \mathfrak{p}^- \to \tau_{(3,-1)}$ is computed in Lemma 2.3. From these two results we deduce the following equalities which give the image by p of the basis vectors in terms of the highest weight vector $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$ of $\tau_{(3,-1)}$:

$$
\begin{aligned}
&p(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) = X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}, \\
&p(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-1,-1)}) = -\frac{1}{2}\mathrm{Ad}_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}), \\
&p(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(-2,0)}) = \frac{1}{12}\mathrm{Ad}^2_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}), \\
&p(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(0,-2)}) = \frac{1}{4}\mathrm{Ad}_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}), \\
&p(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}) = -\frac{1}{6}\mathrm{Ad}^2_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}), \\
&p(X_{(2,0)} \wedge X_{(0,2)} \otimes X_{(-2,0)}) = \frac{1}{24}\mathrm{Ad}^3_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \ \otimes X_{(0,-2)}), \\
&p(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(0,-2)}) = \frac{1}{12}\mathrm{Ad}^2_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}), \\
&p(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-1,-1)}) = -\frac{1}{12}\mathrm{Ad}^3_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \ \otimes X_{(0,-2)}), \\
&p(X_{(1,1)} \wedge X_{(0,2)} \otimes X_{(-2,0)}) = \frac{1}{24}\mathrm{Ad}^4_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \ \otimes X_{(0,-2)}).
\end{aligned}
$$

Using these equalities, we find

$$({}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f) \wedge \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)})(\mathbf{1})$$

$$= \frac{1}{3!^2} \sum_{0\le i,j\le k+k'} (-1)^{i+j} \Big\{ - \Big(X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} {}^{\sigma}\varphi \Big) \otimes \Big(X^j_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j}_{(-1,1)} \overline{{}^{\sigma}\varphi} \Big)$$
$$- \frac{1}{4} \mathrm{Ad}_{X_{(-1,1)}} \Big(X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} {}^{\sigma}\varphi \Big) \otimes \mathrm{Ad}_{X_{(1,-1)}} \Big(X^j_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j}_{(-1,1)} \overline{{}^{\sigma}\varphi} \Big)$$
$$+ \frac{1}{72} \mathrm{Ad}^2_{X_{(-1,1)}} \Big(X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} {}^{\sigma}\varphi \Big) \otimes \mathrm{Ad}^2_{X_{(1,-1)}} \Big(X^j_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j}_{(-1,1)} \overline{{}^{\sigma}\varphi} \Big)$$
$$- \frac{1}{72} \mathrm{Ad}^3_{X_{(-1,1)}} \Big(X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} {}^{\sigma}\varphi \Big) \otimes \mathrm{Ad}^3_{X_{(1,-1)}} \Big(X^j_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j}_{(-1,1)} \overline{{}^{\sigma}\varphi} \Big)$$
$$- \frac{1}{576} \mathrm{Ad}^4_{X_{(-1,1)}} \Big(X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} {}^{\sigma}\varphi \Big) \otimes \mathrm{Ad}^4_{X_{(1,-1)}} \Big(X^j_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j}_{(-1,1)} \overline{{}^{\sigma}\varphi} \Big) \Big\}.$$

We have

$$\mathrm{Ad}^r_{X_{(-1,1)}} \Big(X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} {}^{\sigma}\varphi \Big) = \sum_{u=0}^{r} \binom{r}{u} r^{k,k'}_{i,u,r} X^{i-u}_{(1,-1)} v \otimes X^{k+k'+4-i-(r-u)}_{(1,-1)} {}^{\sigma}\varphi,$$

where we use the convention that $X^{i-u}_{(1,-1)} v = 0$ if $i-u<0$, and similarly

$$\mathrm{Ad}^r_{X_{(1,-1)}} \Big(X^j_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j}_{(-1,1)} \overline{{}^{\sigma}\varphi} \Big) = \sum_{u=0}^{r} \binom{r}{u} r^{k,k'}_{j,u,r} X^{j-u}_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j-(r-u)}_{(-1,1)} \overline{{}^{\sigma}\varphi}.$$

with the convention that $X^{j-u}_{(-1,1)} \overline{v} = 0$ if $j-u<0$. Then we have

$$({}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f) \wedge \overline{{}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f)})(\mathbf{1}) = \frac{1}{3!^2} \sum_{r=0}^{4} \sum_{0\le i,j\le k+k'} \sum_{\substack{0\le u,u'\le r \\ 0\le i-u \\ 0\le j-u'}}$$
$$(-1)^{i+j} a_r \binom{r}{u} \binom{r}{u'} r^{k,k'}_{i,u,r} r^{k,k'}_{j,u',r} (X^{i-u}_{(1,-1)} v \otimes X^{k+k'+4-i-(r-u)}_{(1,-1)} {}^{\sigma}\varphi) \otimes (X^{j-u'}_{(-1,1)} \overline{v} \otimes X^{k+k'+4-j-(r-u')}_{(-1,1)} \overline{{}^{\sigma}\varphi}).$$

Let us denote by $[({}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f) \wedge \overline{{}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f)})(\mathbf{1})]$ the element of ${}^{\sigma}\Pi \otimes {}^{\sigma}\overline{\Pi}$ defined as the image of $({}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f) \wedge \overline{{}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f)})(\mathbf{1})$ by the map induced by the pairing $[\,,\,]_{\mathbb{C}}$. Then, thanks to Lemma 4.17, we have

$$[({}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f) \wedge \overline{{}^{\sigma}\omega_{\varphi_\infty,v}({}^{\sigma}\varphi_f)})(\mathbf{1})] = \frac{(k+k')!}{3!^2} \sum_{r=0}^{4} \sum_{i=0}^{k+k'} \sum_{\substack{0\le u,u'\le r \\ 0\le i-u}}$$
$$(-1)^{j-u} a_r \binom{r}{u} \binom{r}{u'} r^{k,k'}_{i,u,r} r^{k,k'}_{i-u+u',u',r} s^{k,k'}_{i,u} X^{k+k'+4-i-(r-u)}_{(1,-1)} {}^{\sigma}\varphi \otimes X^{k+k'+4-i-(r-u)}_{(-1,1)} \overline{{}^{\sigma}\varphi}.$$

As the pairing ${}^{\sigma}\Pi^{K_N} \otimes {}^{\sigma}\overline{\Pi}^{K_N} \to \mathbb{C}$ defined as

$$\varphi \otimes \psi \mapsto \int_{\mathbb{R}^\times_+ G(\mathbb{Q})\backslash G(\mathbb{A})/K_N} \varphi(g)\psi(g)dg$$

is a morphism of $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules we have

$$\int_{\mathbb{R}_+^\times G(\mathbb{Q})\backslash G(\mathbb{A})/K_N} X^i_{(1,-1)}\varphi(g)X^j_{(-1,1)}\overline{\varphi}(g)dg$$
$$= \begin{cases} 0 & \text{if } i \neq j \\ (-1)^i \frac{i!(k+k'+4)!}{(k+k'+4-i)!}\int_{\mathbb{R}_+^\times G(\mathbb{Q})\backslash G(\mathbb{A})/K_N} \varphi(g)\overline{\varphi}(g)dg & \text{if } i = j. \end{cases}$$

As a consequence

$$\langle {}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f), \overline{{}^\sigma\omega_{\varphi_\infty,v}({}^\sigma\varphi_f)}\rangle_{\mathbb{C}}$$
$$= \frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2}\sum_{r=0}^{4}\sum_{i=0}^{k+k'}\sum_{\substack{0\leq u,u'\leq r\\ 0\leq i-u}}$$
$$(-1)^{r+u+u'}a_r\binom{r}{u}\binom{r}{u'}r^{k,k'}_{i,u,r}r^{k,k'}_{i-u+u',u',r}s^{k,k'}_{i,u}t^{k,k'}_{i,u,r}\int_{\mathbb{R}_+^\times G(\mathbb{Q})\backslash G(\mathbb{A})/K_N} {}^\sigma\varphi(g)\overline{{}^\sigma\varphi}(g)dg$$

We have

$$\begin{aligned}\int_{\mathbb{R}_+^\times G(\mathbb{Q})\backslash G(\mathbb{A})/K_N} |{}^\sigma\varphi(g)|^2dg &= \mathrm{vol}(K_N, dg)^{-1}\int_{\mathbb{R}_+^\times G(\mathbb{Q})\backslash G(\mathbb{A})} |{}^\sigma\varphi(g)|^2dg \\ &= 2\mathrm{vol}(K_N, dg)^{-1}\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |{}^\sigma\varphi(g)|^2dg \\ &= 2\mathrm{vol}(K_N, dg)^{-1}\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |{}^\sigma\varphi(g)|^2dg \\ &= \frac{\pi^3}{135}\prod_{l|N}(l^2+1)^{-1}\langle {}^\sigma\varphi, {}^\sigma\varphi\rangle\end{aligned}$$

where the last equality follows from Lemma 6.3, the definition of $\langle {}^\sigma\varphi, {}^\sigma\varphi\rangle$ and [RoS07, Lemma 3.3.3]. □

Namikawa indicated the authors a conjecture on the simplification of the factor $C'_{k,k'}$ by the constant $C_{k,k'}$ below according to some numerical calculations and Yasuda pointed out the following proof based on a formula on Pochhammer symbols (41) stated below. We thank to Namikawa for the conjecture and to Yasuda for indicating us the formula and its proof.

Lemma 6.5. *The constant $C'_{k,,k'}$ is equal to the constant $C_{k,k'}$ defined as follows:*

$$C_{k,k'} = \frac{(-1)^{k+k'}(k+k'+4)!(k+k'+5)!}{3^3\cdot 5}.$$

Proof. Let us recall the Pochhammer symbol as follows:

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \frac{(x+n-1)!}{(x-1)!}$$

where x is a non-negative integer and n is a natural number. First, we can check that

$$r^{k,k'}_{i,u,r} r^{k,k'}_{i-u+u',u',r} s^{k,k'}_{i,u} t^{k,k'}_{i,u,r} = \frac{(k+k'-i+1)_4(k+k'-i+u-u'+1)_4(i-u+1)_r}{(k+k'-i+u+1)_{4-r}}. \tag{39}$$

The right-hand side of (37) is equal to

$$\frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2} \sum_{r=0}^{4} \sum_{i=0}^{k+k'} \sum_{u=0}^{r} \sum_{u'=0}^{r} (-1)^{r+u+u'} a_r \binom{r}{u} \binom{r}{u'} \frac{(k+k'-i+1)_4(k+k'-i+u-u'+1)_4(i-u+1)_r}{(k+k'-i+u+1)_{4-r}}. \tag{40}$$

Here we note that the sum $\sum_{\substack{0\le u,u'\le r\\ 0\le i-u}}$ in the original expression was replaced by $\sum_{u=0}^{r}\sum_{u'=0}^{r}$ since the symbol $(i-u+1)_r$ in the numerator is zero when $i-u<0$. Now we recall that the following formula holds for any integer b and any integers l, m satisfying $0 \le l \le m$:

$$\sum_{a=0}^{l} (-1)^a \binom{l}{a} (b-a)_m = \binom{m}{l} l!(b)_{m-l}. \tag{41}$$

In fact, we can show this formula on the induction wit respect to l for the value $\mathrm{LHS}(b,l,m)$ (resp. $\mathrm{RHS}(b,l,m)$) on the left-hand side (resp. the right-hand side). For $l=0$, the equality $\mathrm{LHS}(b,l,m) = \mathrm{RHS}(b,l,m)$ is trivially true. The induction argument for $l>0$ proceeds since we have $\mathrm{LHS}(b,l,m) = \mathrm{LHS}(b,l-1,m) - \mathrm{LHS}(b-1,l-1,m)$ and $\mathrm{RHS}(b,l,m) = \mathrm{RHS}(b,l-1,m) - \mathrm{RHS}(b-1,l-1,m)$.

By applying (41) with $a=u'$, $b=k+k'-i+u+1$, $l=r$ and $m=4$, the expression (40) is simplified as follows:

$$\frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2} \sum_{r=0}^{4} \sum_{i=0}^{k+k'} \sum_{u=0}^{r} (-1)^{r+u} a_r \binom{r}{u} \binom{4}{r} r!(k+k'-i+1)_4(i-u+1)_r. \tag{42}$$

By applying (41) again with $a=u$, $b=i+1$, $l=m=r$, the expression (42) is further simplified as follows:

$$\frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2} \sum_{r=0}^{4} \sum_{i=0}^{k+k'} (-1)^r a_r \binom{4}{r} (r!)^2 (k+k'-i+1)_4. \tag{43}$$

Since we have $\sum_{r=0}^{4} (-1)^r a_r \binom{4}{r} (r!)^2 = \frac{4}{3}$, the expression (43) is equal to:

$$\frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2} \frac{4}{3} \sum_{i'=0}^{k+k'} (i'+1)_4. \tag{44}$$

by putting $i' = k + k' - i$. Finally, by noting that

$$\sum_{i'=0}^{k+k'} (i'+1)_4 = \frac{1}{5} \sum_{i'=0}^{k+k'} \big((i'+1)_5 - (i')_5\big) = \frac{(k+k'+1)_5}{5},$$

the expression (44) is equal to:

$$\frac{(-1)^{k+k'}(k+k'+4)!(k+k'+5)!}{3^3 \cdot 5}. \tag{45}$$

This completes the proof. □

Recall that we have fixed a basis $(\delta_1, \ldots, \delta_{2r})$ of $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$.

Definition 6.6. *The discriminant $d(\Pi_f)$ of the pairing* (32) *is defined as follows:*

$$d(\Pi_f) = \det(\langle \delta_i, \delta_j \rangle_{\mathbb{Z}_{(p)}})_{1 \le i,j \le 2r}.$$

It is an element of $\mathbb{Z}_{(p)}$ whose image in $\mathbb{Z}_{(p)}/(\mathbb{Z}_{(p)}^\times)^2$ is independent of the choice of the $\mathbb{Z}_{(p)}$-basis $(\delta_i)_{1 \le i \le 2r}$ of $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$.

Definition 6.7. *For $x, y \in \mathbb{R}$ we write $x \sim y$ if there exists $s \in (\mathbb{Z}_{(p)}^\times)^2$ such that $x = sy$.*

Recall that $C_N = \prod_{l|N} (l + l^{-1})^{-1}(l^2+1)^{-1}$.

Theorem 6.8. *Assume that $p \notin S_{N,3} \cup S_{\text{weight}} \cup S_{\text{tors}} \cup S'_{\text{tors}}$ where where $S_{N,3}$, S_{tors}, S'_{tors}, S_{weight} are defined by* (15), (16), (23) *and* (24) *respectively. Then we have:*

$$d(\Pi_f) \sim \left(\left(\frac{2^{k+k'+13} C_{k,k'} C_N \pi^{3k+k'+12}}{k+k'+5} \right)^r \Omega(\Pi_f)^{-1} \prod_{\sigma : E(\Pi_f) \to \mathbb{C}} L(1, {}^\sigma\Pi, \mathrm{Ad}) \right)^2.$$

Proof. Recall that $(\omega_1, \ldots, \omega_{2r})$ denotes the $\mathbb{R}$-basis of $L(\Pi_f, V_{\lambda,\mathbb{R}})$ normalized in the second assertion of Lemma 5.4. Let T denote the matrix $T = (\langle \delta_i, \delta_j \rangle_{\mathbb{Z}_{(p)}})_{1 \le i,j \le 2r}$ and let S denote the matrix $S = (\langle \omega_i, \omega_j \rangle_{\mathbb{R}})_{1 \le i,j \le 2r}$. We have

$$d(\Pi_f) \sim \det T \sim \Omega(\Pi_f)^{-2} \det S$$

where the first equality is the definition of the discriminant, and the second from the definition of $\Omega(\Pi_f)$ (see (34)). Hence we have to compute the pairings $\langle \omega_i, \omega_j \rangle_{\mathbb{R}}$. By commutativity of the diagram (36) we have $\langle \omega_i, \omega_j \rangle_{\mathbb{R}} = \langle \omega_i, \omega_j \rangle_{\mathbb{C}}$. Because of the vanishing of the Petersson inner product, we have $\langle \omega_i, \omega_j \rangle_{\mathbb{C}} = 0$ for any $1 \le i, j \le 2r$ such that $j \ne i$, $j \ne i + r$ and $i \ne j + r$. Furthermore, as the Poincaré duality pairing is a morphism of Hodge structures, for any $1 \le i \le r$ we have

$$\langle {}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f), {}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f) \rangle_{\mathbb{C}} = \langle \overline{{}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f)}, \overline{{}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f)} \rangle_{\mathbb{C}} = 0.$$

As a consequence, for $1 \leq i \leq r$ we have

$$\begin{aligned}
\langle \omega_i, \omega_i \rangle_{\mathbb{C}} &= \begin{cases} 2^{-1}\langle {}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f), \overline{{}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f)} \rangle_{\mathbb{C}} & \text{if } \epsilon_\lambda \text{ is even,} \\ 0 & \text{if } \epsilon_\lambda \text{ is odd,} \end{cases} \\
\langle \omega_i, \omega_{i+r} \rangle_{\mathbb{C}} &= \begin{cases} 0 & \text{if } \epsilon_\lambda \text{ is even,} \\ -2^{-1}\sqrt{-1}\langle {}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f), \overline{{}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f)} \rangle_{\mathbb{C}} & \text{if } \epsilon_\lambda \text{ is odd,} \end{cases} \\
\langle \omega_{i+r}, \omega_{i+r} \rangle_{\mathbb{C}} &= \begin{cases} 2^{-1}\langle {}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f), \overline{{}^{\sigma_i} r_\infty({}^{\sigma_i}\varphi_f)} \rangle_{\mathbb{C}} & \text{if } \epsilon_\lambda \text{ is even,} \\ 0 & \text{if } \epsilon_\lambda \text{ is odd.} \end{cases}
\end{aligned}$$

The statement now follows from Proposition 6.4 and Theorem 3.1. □

7. The congruence criterion

Let $\overline{\mathcal{H}}^N_{\mathrm{sph}} \subset \overline{\mathcal{H}}^{K_N}$ denote the spherical Hecke algebra outside N over $\mathbb{Z}$. Let $\Pi' \simeq \Pi'_\infty \otimes \Pi'_f$ be a cuspidal representation which contributes to $\tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{C}})$. This means that $(\Pi'_f)^{K_N}$ is non-zero (but not necessarily one-dimensional) and $\Pi'_\infty \in P(V_{\lambda,\mathbb{C}})$. Let $E(\Pi'_f)$ be the rationality field of Π'_f and let $\theta_{\Pi'} : \overline{\mathcal{H}}^N_{\mathrm{sph}} \to E(\Pi'_f)$ be the character such that for any $g \in \bigotimes'_{v \nmid N\infty} (\Pi'_v)^{G(\mathbb{Z}_v)}$ and any $h \in \overline{\mathcal{H}}^N_{\mathrm{sph}}$ we have $hg = \theta_{\Pi'}(h)g$. Let $\mathcal{O}_{E(\Pi'_f)}$ be the ring of integers of $E(\Pi'_f)$. According to Corollary 4.5 and to the fact that $\overline{\mathcal{H}}^{K_N}$ is a $\mathbb{Z}$-module of finite type we have $\operatorname{Im}\theta_{\Pi'} \subset \mathcal{O}_{E(\Pi'_f)}$.

Definition 7.1. *Let $\mathfrak{P}$ be a prime ideal of $\overline{\mathbb{Q}}$. The cuspidal representation Π' is congruent to Π modulo $\mathfrak{P}$ if there exists a number field E containing $E(\Pi'_f)$ and $E(\Pi_f)$, with ring of integers $\mathcal{O}_E$ such that the following diagram commutes*

$$\begin{tikzcd}
 & \mathcal{O}_{E(\Pi_f)} \arrow[r, hook] & \mathcal{O}_E \arrow[dr] & \\
\overline{\mathcal{H}}^N_{\mathrm{sph}} \arrow[ur, "\theta_\Pi"] \arrow[dr, "\theta_{\Pi'}"'] & & & \kappa \\
 & \mathcal{O}_{E(\Pi'_f)} \arrow[r, hook] & \mathcal{O}_E \arrow[ur] &
\end{tikzcd}$$

where $\kappa = \mathcal{O}_E/\mathfrak{P} \cap \mathcal{O}_E$. In this case we write $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$.

Recall that Π is a cuspidal representation of $G(\mathbb{A})$, which, among other things is assumed to have trivial central character, to be globally generic and endoscopic. This means that the functorial lift Σ of Π to $\mathrm{GL}(4, \mathbb{A})$ is not cuspidal. In particular, according to [AS06, Proposition 2.2], there exists σ_1 and σ_2 two inequivalent unitary cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with trivial central characters such that Σ is the isobaric sum $\Sigma = \sigma_1 \boxplus \sigma_2$ and such that Π is obtained as a Weil lifting from (σ_1, σ_2). According to [We09, Corollary 4.2] the cuspidal representations σ_1 and σ_2 correspond to primitive cuspforms f_1 and f_2 of respective weights $k_1 = k + k' + 4$ and $k_2 = k - k' + 2$ and of respective levels N_1 and N_2. For $i = 1, 2$ let $c(f_i)$ denote Ghate's congruence number attached to f_i as

defined in the introduction. We say that the cuspidal automorphic representation attached to a holomorphic Siegel modular form F is stable if F is not a Saito-Kurokawa lift and its functorial lift to $\mathrm{GL}(4,\mathbb{A})$ (see [PSS14] for its construction) is cuspidal.

Theorem 7.2. *Let us assume that the conductor N of the automorphic representation $\Pi \simeq \Pi_\infty \otimes \Pi_f$ is prime to p and that $p \notin S_{N,3} \cup S_{\mathrm{weight}} \cup S_{\mathrm{tors}} \cup S'_{\mathrm{tors}} \cup S''_{\mathrm{tors}}$ where $S_{N,3}$, S_{tors}, S'_{tors}, S_{weight} and S''_{tors} are defined by (15), (16), (23), (24) and (26) respectively. Assume the following conditions:*

(a) *the residual $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations $\overline{\rho}_{f_1}$ and $\overline{\rho}_{f_1}$ of f_1 and f_2 are irreducible,*
(b) *the prime $\mathfrak{p}$ does not divide $c'(f_1)$ nor $c'(f_2)$ for any prime $\mathfrak{p}$ above p in $\overline{\mathbb{Q}}$,*
(c) *the prime p divides*

$$\left(\left(\frac{2^{k+k'+13}C_{k,k'}C_N\pi^{3k+k'+12}}{k+k'+5}\right)^r \Omega(\Pi_f)^{-1} \prod_{\sigma:E(\Pi_f)\to\mathbb{C}} L(1,{}^\sigma\Pi,\mathrm{Ad})\right)^2,$$

Then, there exists a prime divisor $\mathfrak{P}$ of p in $\overline{\mathbb{Q}}$ and a cuspidal representation $\Pi' \simeq \bigotimes'_v \Pi'_v$ of $G(\mathbb{A})$ such that

(1) *the non archimedean part Π'_f of Π' satisfies $(\Pi'_f)^{K_N} \neq 0$,*
(2) *we have $\Pi'_\infty \in P(V_{\lambda,\mathbb{C}})$,*
(3) *the cuspidal representation Π' is stable,*
(4) *we have $\Pi' \not\simeq {}^\sigma\Pi$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*
(5) *we have $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$.*

For the proof of this theorem, we need the following result.

Lemma 7.3. *Let $\langle\,,\,\rangle_{\mathbb{Q}}$ denote the bilinear form deduced from (32) after base change from $\mathbb{Z}_{(p)}$ to $\mathbb{Q}$. Let*

$$L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})^* = \{x \in L(\Pi_f, V_{\lambda,\mathbb{Q}}) \,|\, \forall y \in L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}), \langle x,y\rangle_{\mathbb{Q}} \in \mathbb{Z}_{(p)}\}.$$

be the lattice dual of $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$. Then as lattices in $L(\Pi_f, V_{\lambda,\mathbb{Q}})$, we have

$$M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) = L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})^*.$$

Proof. We would like to apply [Hi81a, (4.6)] so we verify the assumptions of this result. According to Corollary 4.13, the $\mathbb{Z}_{(p)}$-lattice $L = \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ of the $\mathbb{Q}$-vector space $V = \tilde{H}^3_!(S_{K_N}, V_{\lambda,\mathbb{Q}})$ is self-dual for the bilinear form deduced from (29) by base change from $\mathbb{Z}_{(p)}$ to $\mathbb{Q}$, that we also denote by $\langle\,,\,\rangle_{\mathbb{Q}}$. Let W_1 denote the subspace $W_1 = M(\Pi_f, V_{\lambda,\mathbb{Q}})$ of V and let W_2 denote the kernel of the projection $p_M : V \to W_1$ (18). Then $V = W_1 \oplus W_2$. By the proof of Lemma 4.5, there exists $e \in \overline{\mathcal{H}}^{K_N}_{\mathbb{Q}}$ an idempotent such that $W_1 = eV$. Then $W_2 = (1-e)V$. We claim that W_1 is orthogonal to W_2 for $\langle\,,\,\rangle$. In order to prove this, it is enough to prove that $W_1 \otimes_{\mathbb{Q}} \mathbb{C}$ and $W_2 \otimes_{\mathbb{Q}} \mathbb{C}$ are orthogonal for $\langle\,,\,\rangle_{\mathbb{C}}$. For any $v, w \in V \otimes_{\mathbb{Q}} \mathbb{C}$, we have $\langle ev, (1-e)w\rangle_{\mathbb{C}} = \langle v, e^*(1-e)w\rangle_{\mathbb{C}} = 0$ where the first equality follows from Proposition 2.6 and Proposition 6.1. As e is the projector on the Π_f isotypical component, e^* is the projector on the $\check{\Pi}_f$ isotypical component, where $\check{\Pi}_f$ denoted the contragredient of Π_f. But as Π_f has trivial central character, we have $\Pi_f \simeq \check{\Pi}_f$ and so $e^* = e$. As a consequence $\langle ev, (1-e)w\rangle_{\mathbb{C}} = 0$. By definition, we have $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) = L \cap W_1$

and $M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ is the projection of L on W_1 along W_2. Hence the statement of the lemma follows from [Hi81a] (4.6). $\square$

Proof of Theorem 7.2. Recall that, for any $\mathbb{Z}_{(p)}$-algebra R, we denote by $\overline{\mathcal{H}}_R^{K_N}$ the algebra $\overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N} \otimes_{\mathbb{Z}_{(p)}} R$ where $\overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N}$ denotes the image of the abstract Hecke algebra $\mathcal{H}_{\mathbb{Z}_{(p)}}^{K_N}$ with $\mathbb{Z}_{(p)}$-coefficients in $\mathrm{End}_{\mathbb{Z}_{(p)}}(\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}))$. Let Y denote the orthogonal complement of $L(\Pi_f, V_{\lambda,\mathbb{R}})$ in $\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{R}})$. As $L(\Pi_f, V_{\lambda,\mathbb{R}})$ is stable by the action of $\overline{\mathcal{H}}_{\mathbb{R}}^{K_N}$, it follows from Proposition 2.6 that Y is also stable by $\overline{\mathcal{H}}_{\mathbb{R}}^{K_N}$ so that the decomposition $L(\Pi_f, V_{\lambda,\mathbb{R}}) \oplus Y = \tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{R}})$ holds as $\overline{\mathcal{H}}_{\mathbb{R}}^{K_N}$-modules. Define the finitely generated $\mathbb{Z}_{(p)}$-module M_Y by the equation $M_Y = p_Y(\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}))$ where $p_Y : \tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{R}}) \to Y$ is the canonical projection. The finitely generated $\mathbb{Z}_{(p)}$-module L_Y is defined as $L_Y = Y \cap \tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$. These $\mathbb{Z}_{(p)}$-modules are stable by the action of $\overline{\mathcal{H}}_{\mathbb{Z}_{(p)}}^{K_N}$ on $\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}})$ and we have $L_Y \subset M_Y$. According to Theorem 6.8 the prime p divides $d(\Pi_f)$. It follows from [Hi81a, Proposition 4.3] that $|d(\Pi_f)| = [L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})^* : L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})]$. Hence, it follows from Lemma 7.3 that p divides the index $[M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) : L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})]$. By replacing in the proof of [Hi81a, Theorem 7.1] the symbols L, L_f, M_f and R by the symbols $\tilde{H}_!^3(S_{K_N}, V_{\lambda,\mathbb{Z}_{(p)}}), L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}), M(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}})$ and $\overline{\mathcal{H}}_{\mathrm{sph},\mathbb{Z}_{(p)}}^N$ respectively, we obtain a prime divisor $\mathfrak{P}$ of p in $\overline{\mathbb{Q}}$ and a $\overline{\mathcal{H}}_{\mathrm{sph},\mathbb{Z}_{(p)}}^{K_N}$-module contributing to Y which is congruent to Π modulo $\mathfrak{P}$. This amounts to the existence of a cuspidal automorphic representation $\Pi' \simeq \bigotimes_v' \Pi'_v$ such that $(\Pi'_f)^{K_N}$ is non zero, such that $\Pi'_\infty \in P(V_{\lambda,\mathbb{C}})$, such that $\Pi' \not\simeq {}^\sigma\Pi$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ and such that $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$.

In the rest of the proof, we show that Π' is stable under the conditions (a) and (b). In order to show this with contradiction, let us assume that Π' is not stable. According to the classification of the discrete spectrum of GSp(4) conjectured in [Ar04], proved in [GeT18], this means that Π' is either of Yoshida type, Saito-Kurokawa type, Howe Piatetski-Shapiro type or one-dimensional type. The mod $\mathfrak{P}$ semi-simple Galois representation $\overline{\rho}_\Pi$ attached to Π is given by $\overline{\rho}_\Pi \simeq \overline{\rho}_{f_1} \oplus \overline{\rho}_{f_2}(-1-k')$. Hence the last three types mentionned above are excluded by our condition (a). Now, let us assume that Π' is of Yoshida-type. By [We09, Corollary 4.2], there exists a pair (σ'_1, σ'_2) of cuspidal representations of $\mathrm{GL}(2, \mathbb{A})$ which correspond to normalized eigen elliptic cuspforms f'_1 and f'_2 of weight $k'_1 = k + k' + 4$ and $k'_2 = k - k' + 2$ respectively such that $\overline{\rho}_{\Pi'} \simeq \overline{\rho}_{f'_1} \oplus \overline{\rho}_{f'_2}(-1-k')$. Since we have the congruence $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$, we have

$$\overline{\rho}_\Pi \simeq \overline{\rho}_{\Pi'}. \tag{46}$$

As

$$\overline{\rho}_\Pi \simeq \overline{\rho}_{f_1} \oplus \overline{\rho}_{f_2}(-1-k'), \quad \overline{\rho}_{\Pi'} \simeq \overline{\rho}_{f'_1} \oplus \overline{\rho}_{f'_2}(-1-k'). \tag{47}$$

As $\overline{\rho}_{f_1}$ and $\overline{\rho}_{f_2}$ are irreducible, the equation (46) implies either $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f'_1}$ or $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f'_2}(-1-k')$. We will prove that these equalities never happen. For $i = 1, 2$ let N_i be the conductor of f_i and let N'_i be the conductor of f'_i. We have $N_1 N_2 = N$, in particular $(N_1, N_2) = 1$ and N_1, N_2 are square-free. Moreover, as Π' has level N, we have $N'_1 N'_2 | N_1 N_2$.

We will prove that the first case $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_1'}$ is excluded by contradiction assuming that $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_1'}$ holds. When the conductor N_1' of f_1' divides the conductor N_1 of f_1, this contradicts to the assumption that $\mathfrak{p}$ does note divide $c'(f_1)$ because the existence of such f_1' by assumption, must imply that a prime $\mathfrak{p}$ above p in $\overline{\mathbb{Q}}$ divides $c'(f_1)$ according to Theorem 1.2. When N_1 divides N_1', the conductor N_2' of f_2' must divide the conductor N_2 of f_2 because we have $N_1'N_2'|N_1N_2$. Hence a similar argument as above shows that this contradicts to the assumption that no prime $\mathfrak{p}$ above p in $\overline{\mathbb{Q}}$ divides $c'(f_2)$. Finally suppose that none of N_1 nor N_1' divides the other. Then the equality $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_1'}$ implies that the analytic conductor of the mod p Galois representation $\overline{\rho}_{f_1}$ is a strict divisor of N_1. Hence the Serre conjecture proved by Khare-Wintenberger [KhWI09], [KhWII09] implies that there must exist a normalized eigen cuspform f' congruent to f_1 modulo p whose conductor is a strict divisor of N_1. Because $p \notin S_{\text{weight}}$ the modular form f' has the same weight as f_1. This is again a contradiction to the assumption that no prime $\mathfrak{p}$ above p in $\overline{\mathbb{Q}}$ divides $c'(f_1)$.
In order to exclude the second case $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_2'}(-1-k')$, let us recall about the mod p Galois representation $\overline{\rho}_f$ of an elliptic cusp form f of weight $k < p-2$ and of level N prime to p. When f is ordinary at p, $\overline{\rho}_f$ is reducible when restricted to the decomposition group D_p at p and the restriction to the inertia subgroup is given as

$$\overline{\rho}_f\big|_{I_p} \simeq 1 \oplus \omega^{1-k}. \tag{48}$$

When f is non ordinary at p, the restriction to the inertia subgroup of $\overline{\rho}_f$ is given as

$$\overline{\rho}_f\big|_{I_p} \simeq \omega_2^{k-1} \oplus (\omega_2')^{k-1}, \tag{49}$$

where ω_2 and ω_2' are fundamental characters of level 2 such that ω_2 and ω_2' are both of order p^2-1 and we have $\omega_2' = (\omega_2)^p$. Suppose that f_1 and f_2' are both ordinary. The fact that $p \notin S_{\text{weight}}$ implies that $\overline{\rho}_{f_2'}(-1-k')$ does not contain the trivial character and hence the second case $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_2'}(-1-k')$ is excluded. When one of f_1 and f_2' is ordinary and the other is non ordinary, the seconde case $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_2'}(-1-k')$ is excluded again since it is not difficult to see that a representation of the type (48) is never isomorphic to a twist of a representation of the type (49). This is a contradiction. Finally when f_1 and f_2' are both non ordinary, the second case $\overline{\rho}_{f_1} \simeq \overline{\rho}_{f_2'}(-1-k')$ is excluded by the assumption $p \notin S_{\text{weight}}$.

By the above discussion Π' is either stable. □

References

[AK13] M. Agarwal, K. Klosin, *Yoshida lifts and the Bloch-Kato conjecture for the convolution L-function*, J. Number Theory 133 (2013), no. 8, 2496-2537.

[An87] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundlehren der Mathematischen Wissenschaften, 286, Springer-Verlag, Berlin, 1987. xii+374.

[Ar04] J. Arthur, *Automorphic representations of GSp(4)*, Contributions to automorphic forms, geometry and number theory, John Hopkins Univ. Press, Baltimore, MD, (2004), 65-81.

[AS06] M. Asgari, F. Shahidi, *Generic transfer from* GSp(4) *to* GL(4), Compositio Math. 142, (2006), 541-550.

[ASch01] M. Asgari, R. Schmidt, *Siegel modular forms and representations*, Manuscripta Math. 104, (2001), 173-200.

[Ay08] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique I*, Astérisque no. 314, (2007), x+466.

[Ay10] J. Ayoub, *Note sur les opérations de Grothendieck et la réalisation de Betti*, J. Inst. Math. Jussieu 9, no. 2, (2010), 225-263.

[BDSP12] S. Böcherer, N. Dummigan, R. Schulze-Pillot, *Yoshida lifts and Selmer groups*, J. Math. Soc. Japan 64 (2012), no. 4, 1353-1405.

[BHR94] D. Blasius, M. Harris, D. Ramakrishnan, *Coherent cohomology, limits of discrete series and Galois conjugation*, Duke Math. Journ. 73 (1994), no. 3, 647-685.

[BaR17] B. Balasubramanyam, A. Raghuram, *Special values of adjoint L-functions and congruences for automorphic forms on GL(n) over a number field*, Amer. Journ. of Math 139, No.3, (2017), 641-679.

[Bo81] A. Borel, *Stable real cohomology of arithmetic groups II*, in Manifolds and Lie groups (Notre Dame, Ind., 1980), pp. 21-55, Progr. Math., 14, Birkhäuser, Boston, Mass., (1981).

[CI19] S. Chen, A. Ichino, *On Petersson norms of generic cusp forms and special values of adjoint L-functions for* GSp_4, preprint, arXiv:1902.06429 [math.NT].

[DS05] F. Diamond, J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005. xvi+436.

[Di05] M. Dimitrov, *Galois representations modulo p and cohomology of Hilbert modular varieties*, Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), no. 4, 505–551.

[FH91] W. Fulton, J. Harris, *Representation theory. A first course*, GTM 129, Springer-Verlag, New-York, (1991).

[GaR13] W. T. Gan, A. Raghuram, *Arithmeticity for periods of automorphic forms*, Automorphic representations and L-functions, Tata Inst. Fundam. Res. Stud. Math., 22, Mumbai, (2013), 187-229.

[GeT18] T. Gee, O. Taïbi, *Arthur's multiplicity formula for* GSp_4 *and restriction to* Sp_4, preprint, arXiv:1807.03988 [math.NT].

[Gh02] E. Ghate, *Adjoint L-values and primes of congruence for Hilbert modular forms*, Compositio Math. 132 (2002), no. 3, 243–281.

[Hi81a] H. Hida, *Congruence of cusp forms and special values of their zeta functions*, Invent. Math. 63, (1981), no. 2, 225-261.

[Hi81b] H. Hida, *On congruence divisors of cusp forms as factors of the special values of their zeta functions*, Invent. Math. 64 (1981), no. 2, 221-262.

[I08] A. Ichino, *On critical values for adjoint L-functions for GSp(4)*, preprint, (2008), available at https://www.math.kyoto-u.ac.jp/∼ichino/ad.pdf.

[JSo07] D. Jiang, D. Soudry, *The multiplicity one theorem for generic automorphic forms of GSp(4)*, Pacific Journ. of Math. Vol. 229, no. 2, (2007), 381-388.

[KaS94] M. Kashiwara, P. Schapira, *Sheaves on manifolds*, A Series of Comprehensive Studies in Mathematics, 292, Springer-Verlag (1994), x+512.

[Ka08] H. Katsurada, *Congruence of Siegel modular forms and special values of their standard zeta functions*, Mathematische Zeitschrift, Volume 259, Issue 1, (2008), pp. 97–111.

[KhWI09] C. Khare, J.-P. Witenberger, *Serre's modularity conjecture I*, Invent. Math. 178 (2009), no. 3, 485-504.

[KhWII09] C. Khare, J.-P. Witenberger, *Serre's modularity conjecture II*, Invent. Math. 178 (2009), no. 3, 505-586.

[Kn86] A. W. Knapp, *Representation theory of semisimple Lie groups. An overview based on examples*, Princeton Mathematical Series 36, Princeton University Press, Princeton, NJ, (1986), 1-774.

[Ko88] R. E. Kottwitz, *Tamagawa numbers*, Annals of Math., Second Series, Vol. 127, No. 3, (1988), 629-646.

[LS13] K.-W. Lan, J. Suh, *Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties*, Adv. Math. 242 (2013), 228-286.

[L05] G. Laumon, *Fonctions zêtas des variétés de Siegel de dimension* 3, in Formes Automorphes (II): le cas du groupe GSp(4), Astérisque No. 302, (2005), 1-66.

[Le17] F. Lemma, *On higher regulators of Siegel threefolds II: the connection to the special value*, Compositio Math. 153, no. 5, (2017), 889-946.

[MT02] F. Mokrane, P. Polo, J. Tilouine, *Cohomology of Siegel varieties*, Astérisque 280, (2002), 1-135.

[N15] K. Namikawa, *On a congruence prime criterion for cusp forms on GL_2 over number fields*, J. reine angew. Math. 707, (2015), 149-207.

[Pe15] D. Petersen, *Cohomology of local systems on the moduli of principally polarized abelian surfaces*, Pacific J. Math. 275 (2015), no. 1, 39–61.

[PSS14] A. Pitale, A. Saha, R. Schmidt, *Transfer of Siegel cuspforms of degree* 2, Mem. Amer. Math. Soc. (2014), 232 (1090).

[PT02] P. Polo, J. Tilouine, *Besnstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_{(p)}$ for representations with p-small weights*, Astérisque 280, (2002), 97-135.

[R10] D. Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés, SMF, Paris, 2010. vi+332.

[Ri76] K. A. Ribet, *A modular construction of unramified p-extensions of* $\mathbb{Q}(\mu_p)$, Invent. Math. 34 (1976), no. 3, 151–162.

[Ri83] K. A. Ribet, *Mod p Hecke operators and congruences between modular forms*, Invent. Math. 71 (1983), no. 1, 193-205.

[RoS07] B. Roberts, R. Schmidt, *Local newforms for* GSp(4), Lecture Notes in Mathematics, 1918. Springer, Berlin, 2007. viii+307 pp.

[Sie43] C. L. Siegel, Symplectic geometry, Amer. J. of Math., 65, (1943), 1-86.

[U95] E. Urban, *Formes automorphes cuspidales pour* GL_2 *sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences*, Compositio Math. 99 (1995), no. 3, 283–324.

[Wa85] J.-L. Waldspurger, *Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2)*, Compositio Math. 54, no. 2, (1985), 121-171.

[We05] R. Weissauer, *Four dimensional Galois representations*, Formes automorphes II. Le cas du groupe GSp(4), Astérisque No. 302, (2005), 67-150.

[We09] R. Weissauer, *Endoscopy for* GSp(4) *and the cohomology of Siegel modular threefolds*, Lecture Notes in Mathematics 1968, Springer-Verlag, Berlin, (2009), xviii+368.

Univ Paris Diderot, Institut mathématique de Jussieu-Paris Rive Gauche, UMR 7586, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France.

E-mail address: `francesco.lemma@imj-prg.fr`

Graduate School of Science, Osaka University, 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan

E-mail address: `ochiai@math.sci.osaka-u.ac.jp`