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HIGHER REGULATORS OF SIEGEL SIXFOLDS AND NON-CRITICAL VALUES OF SPIN L-FUNCTIONS

ANTONIO CAUCHI, FRANCESCO LEMMA, AND JOAQUÍN RODRIGUES JACINTO

Abstract. We construct classes in the middle degree plus one motivic cohomology of Siegel sixfolds and we compute their image by Beilinson higher regulator in terms of Rankin-Selberg type automorphic integrals. Using results of Pollack and Shah, we relate the integrals to non-critical special values of the degree 8 Spin L-functions. Along the way, by defining and studying complexes of tempered currents on smooth projective complex varieties endowed with a normal crossings divisor, we provide a new description of Deligne–Beilinson cohomology for any Shimura variety. This is particularly useful for computations of higher regulators and fills a gap in the literature on the subject.

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1. Introduction

Beilinson conjectures are vast generalisations of the analytic class number formula. They relate special values of L-functions of motives to motivic cohomology classes via higher regulators. Very few cases of these conjectures are known [2], [3], [13], [14], [43], [6], [29], [34], and they remain one of the main open problems in arithmetic geometry.

The purpose of the present article is to construct classes in the middle degree plus one motivic cohomology of Siegel sixfolds and to prove a formula relating their image under the Beilinson regulator to certain adelic integrals of Rankin-Selberg type. These integrals are known to compute non-critical special values of L-functions of automorphic forms for GSp_6, and hence our results give further evidence for Beilinson conjectures. The main technical innovation of the present work, which is also of independent interest, is the introduction and study of complexes of tempered currents on smooth projective complex varieties endowed with a normal crossings divisor. Using these complexes, we give a new explicit description of Deligne–Beilinson cohomology for an arbitrary Shimura variety. This is particularly useful for computations and provides what we believe to be the natural setting for computing higher regulators of Shimura varieties.

1.1. Main results.

Let $\mathbf{G} = \mathrm{GSp}_6$ be the symplectic similitude group of rank 3. Denote by $\mathrm{Sh}_{\mathbf{G}}$ the six dimensional Shimura variety associated to $\mathbf{G}$. These Shimura varieties and their cohomology play a prominent role in the study of arithmetic aspects of cuspidal automorphic representations of $\mathbf{G}(\mathbf{A})$ and their associated Galois representations.

Fundamental objects used in most of the approaches to Beilinson conjectures are modular units. These are elements of the motivic cohomology groups $H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}, \overline{\mathbf{Q}}(1)) \cong \mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2})^\times \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ of the modular curves $\mathrm{Sh}_{\mathrm{GL}_2}$, which can be seen as motivic realizations of Eisenstein series. Indeed, by the second Kronecker limit formula, their logarithm is related to limiting values of some real analytic Eisenstein series. Using these modular units, and inspired by the work of Pollack and Shah [42], we construct natural cohomology classes

$$\mathrm{Eis}_{\mathcal{M}} \in H^7_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)),$$

where $U \subseteq \mathbf{G}(\mathbf{A}_f)$ denotes an appropriate level structure. The construction goes as follows. Let F denote a real étale quadratic $\mathbf{Q}$-algebra, i.e. F is either a totally real quadratic extension of $\mathbf{Q}$ or $\mathbf{Q} \times \mathbf{Q}$. Denote by $\mathrm{GL}^*_{2,F}/\mathbf{Q}$ the subgroup scheme of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F}$ sitting in the Cartesian diagram

$$\begin{tikzcd} \mathrm{GL}^*_{2,F} \arrow[r, hook] \arrow[d] & \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F} \arrow[d, "\det"] \\ \mathbf{G}_m \arrow[r, hook] & \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_{\mathrm{m},F}. \end{tikzcd}$$

Let $\mathbf{H}$ denote the group

$$\mathbf{H} := \mathrm{GL}_2 \boxtimes \mathrm{GL}^*_{2,F} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}^*_{2,F} \mid \det(g_1) = \det(g_2)\}.$$

Then one has an embedding $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$. This embedding induces a closed embedding

$$\iota : \mathrm{Sh}_{\mathbf{H}} \to \mathrm{Sh}_{\mathbf{G}}$$

of codimension 3. Letting $\mathrm{pr}_1 : \mathbf{H} \to \mathrm{GL}_2$ denote the projection to the first factor, and $u \in H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}, \overline{\mathbf{Q}}(1))$ be a modular unit, one defines $\mathrm{Eis}_{\mathcal{M}} := \iota_*(\mathrm{pr}_1^*(u))$.

Recall the existence of the Beilinson regulator map

$$r_{\mathcal{D}} : H^7_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)) \to H^7_{\mathcal{D}}(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathbf{R}(4)) \otimes_{\mathbf{R}} \overline{\mathbf{Q}},$$

where $H^*_{\mathcal{D}}(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathbf{R}(-))$ denote the Deligne–Beilinson cohomology groups of $\mathrm{Sh}_{\mathbf{G}}(U)$. According to Beilinson conjectures, if the latter cohomology group is non-zero, one expects to be able to construct non-zero motivic cohomology classes which are related to special values of L-functions.

Now let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character, for which π_∞ is a discrete series of Hodge type $(3,3)$ and such that π_f has a non-zero vector fixed by U. Associated to a cusp form $\Psi = \Psi_\infty \otimes \Psi_f \in \pi_\infty \otimes \pi_f^U$ such that Ψ_∞ is a highest weight vector of one minimal K_∞-type of π_∞, there is a harmonic differential form ω_Ψ on $\mathrm{Sh}_{\mathbf{G}}(U)$ of type $(3,3)$. Via a careful study of Deligne–Beilinson cohomology, we define a natural pairing

$$\langle\ , \omega_\Psi\rangle : H^7_{\mathcal{D}}(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathbf{R}(4)) \otimes_{\mathbf{R}} \overline{\mathbf{Q}} \to \mathbf{C} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}.$$

The following is our first main result.

Theorem 1.1 (Theorem 5.3)**.** *We have*

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}), \omega_\Psi\rangle = C_U \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} E(h_1, 0)(A.\Psi)(h)dh,$$

where $E(h,s)$ is a real analytic Eisenstein series on GL_2, the operator A is an element of $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ (defined precisely in §5.2), the constant C_U is a volume factor depending on $U \cap \mathbf{H}(\mathbf{A}_f)$ and dh is a fixed Haar measure on $\mathbf{H}(\mathbf{A})$.

The main theorem of [42] gives a Rankin-Selberg formula for the Spin L-function of certain automorphic representations of $\mathbf{G}$. As a consequence of this and Theorem 1.1, we get the following result, verifying instances of Beilinson conjectures.

Theorem 1.2 (Theorem 5.17)**.** *Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character such that π_∞ is a discrete series of Hodge type $(3,3)$. Let Σ be a finite set of primes containing ∞ and the bad primes for π. Let $\Psi = \Psi_\infty \otimes \Psi_f$ be a factorizable cusp form in π which is unramified outside Σ and which supports a certain Fourier coefficient of type $(4\,2)$. Then*

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}), \omega_\Psi\rangle = C_U \lim_{s\to 0} \left(I_\Sigma(\Phi_\Sigma, A.\Psi_\Sigma, s) L^\Sigma(\pi, \mathrm{Spin}, s)\right),$$

where $L^\Sigma(\pi, \mathrm{Spin}, s) = \prod_{p\notin\Sigma} L(\pi_p, \mathrm{Spin}, s)$ and $I_\Sigma(\Phi_\Sigma, A.\Psi_\Sigma, s)$ is the integral over the finite set of places Σ as defined in Equation (15).

Some remarks are in order.

Remark 1.3. One can show (cf. Corollary 5.18) that there exist a cusp form $\widetilde{\Psi} = \Psi_\infty \otimes \widetilde{\Psi}_f \in \pi$, with Ψ and $\widetilde{\Psi}$ coinciding outside Σ, and a Schwartz function defining the Eisenstein series, such that the finite integral $I_\Sigma(\Phi_\Sigma, A.\widetilde{\Psi}_\Sigma, s)$ is equal to the archimedean integral $I_\infty(\Phi_\infty, A.\Psi_\infty, s)$, implying that

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}), \omega_{\widetilde{\Psi}}\rangle = C_U \lim_{s\to 0} \left(I_\infty(\Phi_\infty, A.\Psi_\infty, s) L^\Sigma(\pi, \mathrm{Spin}, s)\right).$$

We expect the value $\lim_{s\to 0}\big(I_\infty(\Phi_\infty, A.\Psi_\infty, s)L^\Sigma(\pi, \mathrm{Spin}, s)\big)$ to be equal, up to some non-zero constant, to the leading term of the Taylor expansion of $L^\Sigma(\pi, \mathrm{Spin}, s)$ at $s = 0$ (cf. Remark 5.19 for a more detailed discussion). We would like to point out that, thanks to [16, Proposition 12.1], this archimedean integral can be made non-zero at arbitrary $s = s_0$ if one has some freedom on the choice of Φ_∞ and Ψ_∞. This implies that the archimedean integral does not vanish identically, however the main crux to improve our formula is to calculate it when Ψ_∞ is the highest weight vector in the minimal K_∞-type of π_∞.

Remark 1.4. The classes $\mathrm{Eis}_{\mathcal{M}}$ were used in [10], where the first and third authors showed that their étale realisations could be assembled into a norm-compatible tower of cohomology classes for some p-level subgroups, giving rise to an element of the Iwasawa cohomology of the local p-adic Galois representation associated to the automorphic representation of GSp_6. The above theorem shows that the p-adic L-functions constructed in [10] are indeed related to special values of complex L-functions in the spirit of Perrin-Riou conjectures.

According to the taste of the reader, one could also be interested in the non-vanishing of the motivic cohomology group in which our class lives.

Corollary 1.5 (Corollary 5.21)**.** *Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ and Ψ a cuspidal form in π as in Theorem 1.2. Suppose that*

$$\lim_{s\to 0}\big(I_\infty(\Phi_\infty, A.\Psi_\infty, s)L^\Sigma(\pi, \mathrm{Spin}, s)\big) \neq 0.$$

Then $H^7_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4))$ is non-zero.

Let us now make some remarks on and explain the strategy for proving Theorem 1.1. The main difficulty in proving this result resides in the fact that the differential form ω_Ψ does not necessarily (to our knowledge) extend by zero to the boundary of a smooth compactification of $\mathrm{Sh}_{\mathbf{G}}(U)$. This problem only appears when one works with automorphic representations whose archimedean component is not the holomorphic discrete series. As it was pointed out to us by A. Pollack and S. Shah, a first version of this manuscript contained a fatal mistake. Essentially, the problem was coming from using a description of Deligne–Beilinson cohomology in terms of currents (cf. [26] or [8]) on a smooth compactification of the open Shimura variety, i.e. linear forms of differential forms which extend to the boundary. The key idea was to approximate the rapidly decreasing differential form ω_Ψ by compactly supported ones in order to apply these currents and, by using our hypothesis on the Hodge type of ω_Ψ, get the desired integral formula. However the main problem of this approach is that there is no control on the Hodge type of the compactly supported differential forms that approximate ω_Ψ. Moreover, it turned out that a similar gap appears in other works on complex regulators in different settings, such as [29], [33], [34], [41].

Our novel idea is to give a description of Deligne–Beilinson cohomology in terms of *tempered currents*, i.e. sheaves of continuous linear forms on rapidly decreasing differential forms. The basic ingredient of our result is the construction of various de Rham complexes à la Deligne, formed by slowly increasing differential forms and tempered currents, that calculate the Hodge structure of the cohomology of a smooth complex variety together with its underlying real structure. We work this out in full generality because we believe it is of independent interest and also because our methods are flexible enough and translate verbatim to correct the aforementioned gap in the literature. Let us describe these results in more detail.

1.2. **Hodge structures on the cohomology of a smooth complex variety.** Let $\overline{X}$ be a proper smooth complex variety, $D \subseteq \overline{X}$ be a normal crossing divisor and let $X = \overline{X} - D$.

In his seminal work [12], Deligne constructed a complex $\Omega^*_{\overline{X}}(\log D)$ of differential forms with logarithmic singularities along D, which is equipped with Hodge and weight filtrations F and W and he showed that $(\Omega^*_{\overline{X}}(\log D), W, F)$ is a mixed Hodge complex inducing the canonical mixed Hodge structure on the cohomology of X. The results of Deligne have been consequently extended by Navarro [39] and Burgos Gil [7], who showed that the mixed Hodge structure on the cohomology can be calculated by means of complexes of analytic and smooth differential forms with logarithmic singularities at the boundary. We will denote by $\mathscr{A}^*_{\overline{X}}(\log D)$ the complex of smooth differential forms with logarithmic singularities along D constructed in [7]. This complex has also the advantage of capturing the underlying real structure that is not seen in $\Omega_{\overline{X}}(\log D)$, as well as to provide a resolution by fine sheaves, as they admit partitions of unity.

In the study of the arithmetic of non-holomorphic automorphic forms, one is naturally led to study various types of singularities along the boundary divisor to consider complexes of differential forms which are slowly increasing or rapidly decreasing along the divisor D. In the context of coherent cohomology, this has been extensively used by Harris-Phong [23], Harris [21] and Harris-Zucker [24]. In this article, we show that one can calculate the Hodge structure of the cohomology of such a variety using this kind of complexes.

We now describe our result in detail. Locally around any point, one can find a coordinate system $(z_1, \dots, z_k, z_{k+1}, \dots z_d)$ such that $\overline{X}$ is isomorphic to a polydisc Δ^d_r of dimension d and radius $r > 0$ and that the normal crossing divisor D is given by the equations $z_1 \dots z_k = 0$. Slowly increasing (resp. rapidly decreasing) functions on X are then defined locally by asking that

$$|f(z)| \leq C \left(\prod_{i=1}^{k} |\log |z_i|| \right)^N$$

for some $N \geq 0$ (resp. for all $N \leq 0$) and some constant C, and similar conditions for certain derivatives of f (cf. Definition 4.1). Then one defines in an analogous way (cf. Definition 4.3) complexes $\mathscr{A}^*_{si}$, resp. $\mathscr{A}^*_{rd}$, of sheaves on $\overline{X}$ of slowly increasing and rapidly decreasing differential forms. These are complexes of fine sheaves equipped with a Hodge structure (given as usual by the type of a differential form) and with a real structure (given by real valued smooth differential forms). A new object introduced in this article is the complex $\mathscr{D}^*$ of sheaves on $\overline{X}$ of tempered currents. Rapidly decreasing differential forms are naturally equipped with a Fréchet topology and we define $\mathscr{D}^{p,q}$ as the sheaves $U \mapsto \Gamma_c(U, \mathscr{A}^{d-p,d-q}_{rd})^*$ of continuous linear forms on compactly supported sections on U of rapidly decreasing differential forms, where $U \subseteq \overline{X}$ is an open set. It is also a complex of fine sheaves and it is equipped with a Hodge filtration as well as with a natural real structure.

Theorem 1.6 (Proposition 4.7, Proposition 4.8 and Theorem 4.20)**.** *The natural inclusions*

$$(\Omega^*_{\overline{X}}(\log D), F) \to (\mathscr{A}^*_{\overline{X}}(\log D), F) \to (\mathscr{A}^*_{si}, F) \to (\mathscr{D}^*, F)$$

are filtered quasi-isomorphisms. Moreover, the last two quasi-isomorphisms are compatible for the corresponding real structures.

The first quasi-isomorphism is one of the main results of [7]. The proof of the quasi-isomorphism $(\Omega^*_{\overline{X}}(\log D), F) \to (\mathscr{A}^*_{si}, F)$ can be more or less extracted from the work of Harris and Zucker [24] and Kato, Matsubara and Nakayama [28]. The last one does not seem to have been yet considered in the literature and is the one that we will need for our

calculations.

Immediately from Theorem 1.6 we obtain the following description of Deligne–Beilinson cohomology. Denote by $\mathscr{D}^*(\overline{X})$ the complex of global sections of $\mathscr{D}^*$, let $\mathscr{D}^*_{\mathbf{R}(p-1)}$ be the complex of sheaves of $\mathbf{R}(p-1) = (2\pi i)^{p-1}\mathbf{R}$-valued currents and denote by $\mathscr{D}^*_{\mathbf{R}(p-1)}(\overline{X})$ the corresponding complex of global sections.

Corollary 1.7 (Theorem 4.23)**.** *We have*

$$\mathbf{R}(p)_{\mathcal{D}} = \operatorname{cone}\left(F^p\mathscr{D}^*(\overline{X}) \to \mathscr{D}^*_{\mathbf{R}(p-1)}(\overline{X})\right)[-1].$$

In particular, we have

$$H^n_{\mathcal{D}}(X, \mathbf{R}(p)) = \frac{\{(S,T) : dS = 0, dT = \pi_{p-1}(S)\}}{d(\widetilde{S}, \widetilde{T})},$$

where $(S,T) \in F^p\mathscr{D}^n(\overline{X}) \oplus \mathscr{D}^{n-1}_{\mathbf{R}(p-1)}(\overline{X})$ *and* $d(\widetilde{S}, \widetilde{T}) = (dS, dT - \pi_{p-1}(S))$.

This explicit description has the pleasant property of being covariant with respect to proper morphisms, which allows us to explicitly describe the image under the Gysin maps of Siegel units as tempered currents, which can naturally be paired against the rapidly decreasing differential form ω_Ψ. This, together with the second Kronecker's limit formula, allow us to conclude the proof of Theorem 1.1.

1.3. **Organisation of the article.** In Section 2 we introduce the relevant reductive groups and the attached Shimura varieties and we explain the construction of the motivic cohomology classes. In Section 3 we recall some classical results on the cohomology of locally symmetric spaces and we describe the construction of the harmonic differential form associated to a cusp form on $\mathrm{GSp}_6(\mathbf{A})$. Section 4 is the technical heart of the article and gives an explicit description of the Deligne–Beilinson cohomology of a smooth variety in terms of tempered currents. We also define natural pairings between elements in Deligne–Beilinson cohomology and certain closed rapidly decreasing differential forms. Finally, in section 5, we prove Theorem 1.1, expressing the value of this linear form on the archimedean realisation of our motivic classes in terms of Rankin-Selberg type integrals and, using the main result of [42], we relate these integrals to non-critical values of the Spin L-function of the fixed cuspidal automorphic representation of $\mathrm{GSp}_6(\mathbf{A})$.

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2. Classes in motivic cohomology

This first section is devoted to the construction of the classes in the motivic cohomology groups of Siegel sixfolds which will be the main characters of the article.

2.1. **Groups.** Let GSp_6 be the group scheme over $\mathbf{Z}$ whose R-points, for any commutative ring R with identity, are described by

$$\mathrm{GSp}_6(R) = \{A \in \mathrm{GL}_6(R) \mid {}^tAJA = \nu(A)J,\ \nu(A) \in \mathbf{G}_m(R)\},$$

where J is the matrix $\left(\begin{smallmatrix} 0 & I_3 \\ -I_3 & 0 \end{smallmatrix}\right)$, for I_3 denoting the 3×3 identity matrix.

2.1.1. *Subgroups.* Let F be a totally real étale quadratic $\mathbf{Q}$-algebra. Denote by $\mathrm{GL}^*_{2,F}/\mathbf{Q}$ the subgroup scheme of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F}$ sitting in the Cartesian diagram

$$\begin{tikzcd} \mathrm{GL}^*_{2,F} \arrow[r, hook] \arrow[d] & \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F} \arrow[d, "\det"] \\ \mathbf{G}_m \arrow[r, hook] & \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_{\mathrm{m},F}. \end{tikzcd}$$

For instance, when $F = \mathbf{Q} \times \mathbf{Q}$, we have

$$\mathrm{GL}^*_{2,F} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g_1) = \det(g_2)\}.$$

Consider F^2 with its standard F-alternating form $\langle\ ,\ \rangle_F$. We fix the standard symplectic F-basis $\{e_1, f_1\}$ and define $\langle\ ,\ \rangle_{\mathbf{Q}}$ to be $\mathrm{Tr}_{F/\mathbf{Q}} \circ \langle\ ,\ \rangle_F$. Then, by definition $\mathrm{GL}^*_{2,F} \subset \mathrm{GSp}(\langle\ ,\ \rangle_{\mathbf{Q}})$. The extensions $F/\mathbf{Q}$ are parametrised by $a \in \mathbf{Q}^\times_{>0}/(\mathbf{Q}^\times_{>0})^2$, and we identify $F = \mathbf{Q} \oplus \mathbf{Q}\sqrt{a}$, for a representative a of the corresponding class in $\mathbf{Q}^\times_{>0}/(\mathbf{Q}^\times_{>0})^2$. Fixing the $\mathbf{Q}$-basis of F^2 given by

$$\{\tfrac{1}{2\sqrt{a}}e_1, \tfrac{1}{2}e_1, \sqrt{a}f_1, f_1\}$$

gives an isomorphism $\mathrm{GSp}(\langle\ ,\ \rangle_{\mathbf{Q}}) \simeq \mathrm{GSp}_4$. Indeed, such a basis represents the alternating form $\langle\ ,\ \rangle_{\mathbf{Q}}$ as given by J. Thus we have an embedding

$$\mathrm{GL}^*_{2,F} \hookrightarrow \mathrm{GSp}(\langle\ ,\ \rangle_{\mathbf{Q}}) \simeq \mathrm{GSp}_4. \tag{1}$$

Let V_3 be the standard representation of GSp_6 with symplectic basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$. We will consider the embedding

$$\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4 \hookrightarrow \mathrm{GSp}_6 \tag{2}$$

induced by the decomposition $V_3 = \langle e_1, f_1\rangle \oplus \langle e_2, e_3, f_2, f_3\rangle$.

By composing the maps of (1) and (2), we construct the embedding

$$\iota : \mathbf{H} := \mathrm{GL}_2 \boxtimes \mathrm{GL}^*_{2,F} \hookrightarrow \mathbf{G}$$

2.2. **Shimura varieties.** Keep the notation of the previous section and denote by $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{m/\mathbf{C}}$ the Deligne torus. Denote by $X_{\mathbf{H}}$ the $\mathbf{H}(\mathbf{R})$-conjugacy class of

$$h : \mathbf{S} \longrightarrow \mathbf{H}_{/\mathbf{R}}, \quad x + iy \mapsto \left(\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right), \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right), \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)\right).$$

The pair $(\mathbf{H}, X_{\mathbf{H}})$ defines a Shimura datum whose reflex field is $\mathbf{Q}$. Denote by $\mathrm{Sh}_{\mathbf{H}}$ the corresponding Shimura variety of dimension 3. If $U \subseteq \mathbf{H}(\mathbf{A}_f)$ is a fibre product (over the similitude characters) $U_1 \times_{\mathbf{A}_f^\times} U_2$ of sufficiently small subgroups, we have

$$\mathrm{Sh}_{\mathbf{H}}(U) = \mathrm{Sh}_{\mathrm{GL}_2}(U_1) \times_{\mathbf{G}_m} \mathrm{Sh}_{\mathrm{GL}^*_{2,F}}(U_2),$$

where $\times_{\mathbf{G}_m}$ denotes the fibre product over the zero dimensional Shimura variety of level $D = \det(U_1) = \det(U_2)$

$$\pi_0(\mathrm{Sh}_{\mathrm{GL}_2})(D) = \hat{\mathbf{Z}}^\times / D.$$

We recall the reader that the complex points of $\mathrm{Sh}_{\mathbf{H}}(V)$ are given by

$$\mathrm{Sh}_{\mathbf{H}}(V)(\mathbf{C}) = \mathbf{H}(\mathbf{Q})\backslash\mathbf{H}(\mathbf{A})/\mathbf{Z}_{\mathbf{H}}(\mathbf{R})K_{\mathbf{H},\infty}V,$$

where $\mathbf{Z}_{\mathbf{H}}$ denotes the center of $\mathbf{H}$ and $K_{\mathbf{H},\infty} \subseteq \mathbf{H}(\mathbf{R})$ is the maximal compact defined as the product $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

Notice that the embedding $\iota : \mathbf{H} \to \mathbf{G}$ induces another Shimura datum $(\mathbf{G}, X_{\mathbf{G}})$ of reflex field $\mathbf{Q}$. For any neat open compact subgroup U of $\mathbf{G}(\mathbf{A}_f)$, denote by $\mathrm{Sh}_{\mathbf{G}}(U)$ the associated Shimura variety of dimension 6. We also write $\iota : \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \hookrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$ the closed embedding of codimension 3 induced by the group homomorphism $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$.

2.3. **Motivic cohomology classes for GSp_6.** We now define the cohomology classes we want to study in this article.

2.3.1. *Modular units and Eisenstein series.* The input of our construction are the modular units already considered by Beilinson and Kato, which are related to real analytic Eisenstein series by the second Kronecker limit formula.

Let $\mathbf{T}_2$ denote the diagonal maximal torus of GL_2 and let $\mathbf{B}_2$ denote the standard Borel. Define the algebraic character $\lambda : \mathbf{T}_2 \to \mathbf{G}_m$ by $\lambda(\mathrm{diag}(t_1, t_2)) = t_1/t_2$. Let $\mathcal{S}(\mathbf{A}^2, \overline{\mathbf{Q}})$ denote the space of $\overline{\mathbf{Q}}$-valued Schwartz-Bruhat functions on $\mathbf{A}^2$. Given $\Phi \in \mathcal{S}(\mathbf{A}^2, \overline{\mathbf{Q}})$, denote by

$$f(g, \Phi, s) := |\det(g)|^s \int_{\mathrm{GL}_1(\mathbf{A})} \Phi((0,t)g)|t|^{2s} d^\times t$$

the normalised Siegel section in $\mathrm{Ind}_{\mathbf{B}_2(\mathbf{A})}^{\mathrm{GL}_2(\mathbf{A})}(|\lambda|^s)$ and define the associated Eisenstein series

$$\mathrm{E}(g, \Phi, s) := \sum_{\gamma \in \mathbf{B}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{Q})} f(\gamma g, \Phi, s). \tag{3}$$

Fix the Schwartz-Bruhat function Φ_∞ on $\mathbf{R}^2$ defined by $(x, y) \mapsto e^{-\pi(x^2+y^2)}$ and, for each $\overline{\mathbf{Q}}$-valued function $\Phi_f \in \mathcal{S}(\mathbf{A}_f^2, \overline{\mathbf{Q}})$, the smallest positive integer N_{Φ_f} such that Φ_f is constant modulo $N_{\Phi_f}\widehat{\mathbf{Z}}^2$. Finally, denote $\mathcal{S}_0(\mathbf{A}_f^2, \overline{\mathbf{Q}}) \subset \mathcal{S}(\mathbf{A}_f^2, \overline{\mathbf{Q}})$ the space of elements Φ_f such that $\Phi_f((0,0)) = 0$. We now state the following (classical) result, which relates modular units to values of the adelic Eisenstein series defined in (3).

Proposition 2.1. *Let $\Phi_f \in \mathcal{S}_0(\mathbf{A}_f^2, \overline{\mathbf{Q}})$ with $N_{\Phi_f} \geq 3$, then there exists*

$$u(\Phi_f) \in \mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2}(K(N_{\Phi_f})))^\times \otimes \overline{\mathbf{Q}}$$

such that for any $g \in \mathrm{GL}_2(\mathbf{A})$ we have

$$E(g, \Phi, s) = \log|u(\Phi_f)(g)| + O(s),$$

where $\Phi = \Phi_\infty \otimes \Phi_f$.

Proof. This is the second statement of [41, Corollary 5.6], where ν_1 is taken to be the trivial character. □

Example 2.2. When $\Phi_f = \mathrm{char}((0,1) + N\widehat{\mathbf{Z}}^2)$ for $N \geq 4$, the corresponding $u(\Phi_f) \in \mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2}(K(N)))^\times \otimes \mathbf{Q}$ is given by $\prod_{b\in(\mathbf{Z}/N\mathbf{Z})^\times} g_{0,b/N}^{\varphi(N)}$, where $g_{0,\star/N}$ is the Siegel unit as in [27, §1.4]. Indeed, $\mathrm{Sh}_{\mathrm{GL}_2}(K(N))$ is a disjoint union of connected components all isomorphic to the modular curve $Y(N)$, which are indexed by the class of $[\det(k)] \in \widehat{\mathbf{Z}}^\times/(1+N\widehat{\mathbf{Z}})$, for $k \in \mathrm{GL}_2(\widehat{\mathbf{Z}})$. Choose a system of representatives given by the elements $k_d = \left(\begin{smallmatrix} 1 & \\ & d \end{smallmatrix}\right)$, as d varies in $\widehat{\mathbf{Z}}^\times/(1+N\widehat{\mathbf{Z}})$. Then, a point in $\mathrm{Sh}_{\mathrm{GL}_2}(K(N))(\mathbf{C})$ is represented by a pair (z, k_d), with $z \in Y(N)$. By [41, Corollary 5.6], as function on $\mathrm{Sh}_{\mathrm{GL}_2}(K(N))(\mathbf{C})$,

$$u(\Phi_f)(z,k_d) = \prod_{b\in(\mathbf{Z}/N\mathbf{Z})^\times} g_{(0,b/N)\cdot k_d^{-1}}^{\varphi(N)}(z) = \prod_{b\in(\mathbf{Z}/N\mathbf{Z})^\times} g_{0,br_d/N}^{\varphi(N)}(z),$$

where r_d denotes the inverse of d modulo N, φ is Euler's totient function, and $g_{0,\star/N}$ is the Siegel unit as in [27, §1.4]. Thus, $u(\Phi_f)$ descends to an element of $\mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2}(K_1(N)))^\times \otimes \mathbf{Q}$, as each $g_{0,b/N}$ does.

2.3.2. *The construction.* Let

$$u : \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}}) \to H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}, \overline{\mathbf{Q}}(1)) \simeq \mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2})^\times \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$$

be the $\mathrm{GL}_2(\mathbf{A}_f)$-equivariant map defined by $\Phi_f \mapsto u(\Phi_f)$, where $H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}, \overline{\mathbf{Q}}(1))$ denotes $\varinjlim_V H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}(V), \overline{\mathbf{Q}}(1))$ and $\mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2})^\times \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ denotes $\varinjlim_V (\mathcal{O}(\mathrm{Sh}_{\mathrm{GL}_2}(V))^\times \otimes_{\mathbf{Z}} \overline{\mathbf{Q}})$, the limits being taken over all neat compact open subgroups $V \subset \mathrm{GL}_2(\mathbf{A}_f)$.

Let

$$V_1 \subset \mathrm{GL}_2(\mathbf{A}_f), V_2 \subset \mathrm{GL}^*_{2,F}(\mathbf{A}_f)$$

denote neat compact open subgroups such that the images of V_1 and V_2 by the similtude characters are the same. Taking the fiber products over the similitude character, we obtain a compact open subgroup $V = V_1 \times_{\mathbf{A}_f^\times} V_2$ of $\mathbf{H}(\mathbf{A}_f)$. Let $U \subset \mathbf{G}(\mathbf{A}_f)$ be a neat compact open subgroup such that the embedding ι induces a closed embedding $\mathrm{Sh}_{\mathbf{H}}(V) \hookrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$ of codimension 3. As a consequence, we have an induced map on motivic cohomology

$$\iota_* : H^1_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{H}}(V), \overline{\mathbf{Q}}(1)\big) \to H^7_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)\big).$$

The projection on the first factor of $\mathrm{Sh}_{\mathbf{H}}(V)$ is a morphism $p_1 : \mathrm{Sh}_{\mathbf{H}}(V) \to \mathrm{Sh}_{\mathrm{GL}_2}(V_1)$. Hence we have the sequence of morphisms

$$\begin{aligned} \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}})^{V_1} &\xrightarrow{\ u\ } H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}(V_1), \overline{\mathbf{Q}}(1)) \\ &\xrightarrow{\ p_1^*\ } H^1_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{H}}(V), \overline{\mathbf{Q}}(1)\big) \\ &\xrightarrow{\ \iota_*\ } H^7_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)\big). \end{aligned}$$

Definition 2.3. *We define* $\mathrm{Eis}_{\mathcal{M}} : \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}})^{V_1} \to H^7_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)\big)$ *to be the composite of these morphisms.*

Remark 2.4. The notation $\mathrm{Eis}_{\mathcal{M}}$ is slightly abusive as these morphisms depend also on U, V and the data entering in the definition of ι.

3. Cohomology of locally symmetric spaces

In this chapter, we recollect some classical general facts that will be used later or that serve as motivation for our constructions. We also describe the construction of the differential form ω_Ψ that will be used in the statement of the main result.

3.1. **Representation theory.** We set the notations for the representation theory background needed to describe the component at infinity of the automorphic representations under consideration.

3.1.1. *Cartan decomposition.* The maximal compact subgroup K_∞ of $\mathrm{Sp}_6(\mathbf{R})$ is described as

$$K_\infty = \{\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mid AA^t + BB^t = 1, AB^t = BA^t\}.$$

It is isomorphic to $U(3)$ via the map $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mapsto A + iB$ and its Lie algebra is

$$\mathfrak{k} = \{\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mid A = -A^t, B = B^t\}.$$

Letting

$$\mathfrak{p}_{\mathbf{C}}^{\pm} = \left\{\left(\begin{smallmatrix} A & \pm iA \\ \pm iA & -A \end{smallmatrix}\right) \in M(6, \mathbf{C}) \mid A = A^t\right\},$$

one has a Cartan decomposition

$$\mathfrak{sp}_{6,\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}^{+} \oplus \mathfrak{p}_{\mathbf{C}}^{-}.$$

3.1.2. *Root system.* For $1 \leq j \leq 3$, let D_j be the square matrix of size 3 with entry 1 at position (j, j) and 0 elsewhere. Define

$$T_j = -i\left(\begin{smallmatrix} 0 & D_j \\ -D_j & 0 \end{smallmatrix}\right).$$

Then $\mathfrak{h} = \oplus_j \mathbf{R} \cdot T_j$ is a compact Cartan subalgebra of $\mathfrak{sp}_{6,\mathbf{C}}$. We let $(e_j)_j$ denote the basis of $\mathfrak{h}_{\mathbf{C}}^*$ dual to $(T_j)_j$. A system of positive roots for $(\mathfrak{sp}_{6,\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ is then given by

$$\begin{aligned} 2e_j, &\qquad 1 \leq j \leq 3, \\ e_j + e_k, &\qquad 1 \leq j < k \leq 3, \\ e_j - e_k, &\qquad 1 \leq j < k \leq 3. \end{aligned}$$

The simple roots are $e_1 - e_2, e_2 - e_3, 2e_3$. We note that $\mathfrak{p}_{\mathbf{C}}^{+}$ is spanned by the root spaces corresponding to the positive roots of type $2e_j$ and $e_j + e_k$. We denote $\Delta = \{\pm 2e_j, \pm(e_j \pm e_k)\}$ the set of all roots, $\Delta_{\mathrm{c}} = \{\pm(e_j - e_k)\}$ the set of compact roots and $\Delta_{\mathrm{nc}} = \Delta - \Delta_{\mathrm{c}}$ the non-compact roots. Finally, we note $\Delta^+, \Delta_{\mathrm{c}}^+$ and Δ_{nc}^+ the set of positive, positive compact and positive non-compact roots, respectively.

The corresponding root vectors for each root space are given as follows:

- For $1 \leq j \leq 3$, the element $X_{\pm 2e_j} = \left(\begin{smallmatrix} D_j & \pm iD_j \\ \pm iD_j & -D_j \end{smallmatrix}\right)$ spans the root space of $\pm 2e_j$.
- For $1 \leq j < k \leq 3$, letting E_{jk} be the matrix with entry 1 at positions (j, k) and (k, j) and zeroes elsewhere, the elements $X_{\pm(e_j+e_k)} = \left(\begin{smallmatrix} E_{jk} & \pm iE_{jk} \\ \pm iE_{jk} & -E_{jk} \end{smallmatrix}\right)$ spans the root space for of $e_j + e_k$.
- Finally, for $1 \leq j < k \leq 3$, letting $F_{j,k}$ be the matrix with entry 1 at position (j, k), -1 at position (k, j) and zeroes elsewhere, the element $X_{\pm(e_j-e_k)} = \left(\begin{smallmatrix} \pm F_{jk} & -iE_{jk} \\ iE_{jk} & \pm F_{jk} \end{smallmatrix}\right)$ spans the root space of the compact root $\pm(e_j - e_k)$.

3.1.3. *Weyl groups.* Recall that the Weyl group of Sp_6 is given by $\mathfrak{W}_{\mathrm{Sp}_6} = \{\pm 1\}^3 \rtimes \mathfrak{S}_3$. The reflection σ_j in the orthogonal hyperplane of $2e_j$ simply reverses the sign of e_j while leaving the other e_k fixed. The reflection σ_{jk} in the orthogonal hyperplane of $e_j - e_k$ exchanges e_j and e_k and leaves the remaining e_ℓ fixed. The Weyl group $\mathfrak{W}_{K_\infty}$ of $K_\infty \cong U(3)$ is isomorphic to $\mathfrak{S}_3$ and, via the embedding into $\mathbf{G}$, identifies with the subgroup of $\mathfrak{W}_{\mathbf{G}}$ generated by the σ_{jk}. With the identification $\mathfrak{W}_{\mathrm{Sp}_6} = N(T)/Z(T)$, an explicit description of $\mathfrak{W}_{\mathrm{Sp}_6}$ and $\mathfrak{W}_{K_\infty}$ is given as follows. The matrices corresponding to the reflections σ_{jk} are

$$\left(\begin{smallmatrix} S_{jk} & 0 \\ 0 & -S_{jk} \end{smallmatrix}\right),$$

where S_{jk} is the matrix with entry 1 at places (ℓ,ℓ), $\ell \neq j,k$, (k,j) and (j,k) and zeroes elsewhere. The matrices corresponding to the reflection σ_j in the hyperplane orthogonal to $2e_j$ are of the form

$$\left(\begin{smallmatrix} 0 & T_j \\ -T_j & 0 \end{smallmatrix}\right),$$

where T_j denotes the diagonal matrix with -1 at the place (j,j) and ones at the other entries of the diagonal.

3.1.4. *K_∞-types.* We previously defined the maximal compact subgroup $K_\infty \simeq U(3)$ of $\mathrm{Sp}_6(\mathbf{R})$, with Lie algebra $\mathfrak{k}$, and we considered the Cartan decomposition $\mathfrak{sp}_{6,\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+_{\mathbf{C}} \oplus \mathfrak{p}^-_{\mathbf{C}}$. Recall that $\mathfrak{p}^\pm_{\mathbf{C}} = \bigoplus_{\alpha\in\Delta^+_{\mathrm{nc}}} \mathbf{C}X_{\pm\alpha}$.

Denote by $(k_2,k_2,k_3) = k_1e_1 + k_2e_2 + k_3e_3$, with $k_i \in \mathbf{Z}$ the integral weights. Integral weights are dominant for our choice of Δ^+_{c} if $k_1 \geq k_2 \geq k_3$. Recall that there is a bijection between isomorphism classes of finite dimensional irreducible complex representations of K_∞ and dominant integral weights, given by assigning to the representation $\tau_{(k_1,k_2,k_3)}$ its highest weight (k_1,k_2,k_3).

3.2. **Lie algebra cohomology.** Let $A_{\mathbf{G}} = \mathbf{R}^\star_+$ denote the identity component of the center of $\mathbf{G}(\mathbf{R})$ and let $K_{\mathbf{G}} = A_{\mathbf{G}}K_\infty \subset \mathbf{G}(\mathbf{R})$. The embedding $\mathfrak{sp}_{6,\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ induces an isomorphism $\mathfrak{sp}_{6,\mathbf{C}}/\mathfrak{k} \simeq \mathfrak{g}_{\mathbf{C}}/(\mathrm{Lie}(K_{\mathbf{G}}))_{\mathbf{C}}$. By [5, II. Proposition 3.1], for any discrete series π_∞ associated to the trivial representation (cf. §3.3 below), we have

$$H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_\infty) = \mathrm{Hom}_{K_\infty}(\bigwedge^6 \mathfrak{sp}_{6,\mathbf{C}}/\mathfrak{k}, \pi_\infty).$$

By using the Cartan decomposition above, we get

$$\bigwedge^6 \mathfrak{sp}_{6,\mathbf{C}}/\mathfrak{k} = \bigoplus_{p+q=6} \bigwedge^p \mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^q \mathfrak{p}^-_{\mathbf{C}}.$$

One can easily decompose each term of the sum above in its irreducible constituents (if treated as a K_∞-representation via the adjoint action). This will be helpful for writing explicit elements in $H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_\infty)$ according to the minimal K_∞-type of π_∞. Indeed, $\mathfrak{p}^+_{\mathbf{C}}$ (resp. $\mathfrak{p}^-_{\mathbf{C}}$) is the six dimensional irreducible representation of K_∞ of weight $(2,0,0)$ (resp. $(0,0,-2)$). Using Sage package for Lie groups, one can see the following.

Lemma 3.1. *We have the following decompositions in ireducible components of the following K_∞-representations.*

$$\begin{aligned}
\bigwedge^6 \mathfrak{p}^+_{\mathbf{C}} &= \tau_{(4,4,4)}\\
\bigwedge^5 \mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \mathfrak{p}^-_{\mathbf{C}} &= \tau_{(4,2,2)} \oplus \tau_{(4,3,1)} \oplus \tau_{(4,4,0)}\\
\bigwedge^4 \mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^2 \mathfrak{p}^-_{\mathbf{C}} &= \tau_{(2,1,1)} \oplus \tau_{(2,2,0)} \oplus 2\tau_{(3,1,0)} \oplus 2\tau_{(3,2,-1)} \oplus \tau_{(3,3,-2)} \oplus \tau_{(4,0,0)} \oplus\\
&\quad \oplus \tau_{(4,1,-1)} \oplus \tau_{(4,2,-2)}\\
\bigwedge^3 \mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^3 \mathfrak{p}^-_{\mathbf{C}} &= 2\tau_{(0,0,0)} \oplus \tau_{(1,1,-2)} \oplus 2\tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)} \oplus \tau_{(2,1,-3)} \oplus \tau_{(2,2,-4)} \oplus\\
&\quad \oplus 4\tau_{(2,0,-2)} \oplus \tau_{(3,-1,-2)} \oplus 2\tau_{(3,0,-3)} \oplus \tau_{(4,-2,-2)}\\
\bigwedge^2 \mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^4 \mathfrak{p}^-_{\mathbf{C}} &= \tau_{(-1,-1,-2)} \oplus 2\tau_{(1,-2,-3)} \oplus \tau_{(1,-1,-4)} \oplus \tau_{(2,-3,-3)} \oplus \tau_{(2,-2,-4)} \oplus\\
&\quad \oplus \tau_{(0,-2,-2)} \oplus 2\tau_{(0,-1,-3)} \oplus \tau_{(0,0,-4)}\\
\mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^5 \mathfrak{p}^-_{\mathbf{C}} &= \tau_{(-2,-2,-4)} \oplus \tau_{(-1,-3,-4)} \oplus \tau_{(0,-4,-4)}\\
\bigwedge^6 \mathfrak{p}^-_{\mathbf{C}} &= \tau_{(-4,-4,-4)}.
\end{aligned}$$

It will be useful to have some explicit description of the components $\tau_{(2,2,-4)}$ and $\tau_{(4,-2,-2)}$ of $\bigwedge^3 \mathfrak{p}^+_{\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^3 \mathfrak{p}^-_{\mathbf{C}}$.

Lemma 3.2. *The vector*

$$X_{(2,2,-4)} := (X_{2e_1} \wedge X_{2e_2} \wedge X_{e_1+e_2}) \otimes (X_{-e_1-e_3} \wedge X_{-e_2-e_3} \wedge X_{-2e_3})$$

is a highest weight vector of $\tau_{(2,2,-4)}$. Analogously, a highest weight vector of $\tau_{(4,-2,-2)} \subseteq \tau_{(4,1,1)} \otimes \tau_{(0,-3,-3)}$ is given by

$$X_{(4,-2,-2)} := (X_{2e_1} \wedge X_{e_1+e_2} \wedge X_{e_1+e_3}) \otimes (X_{-e_2-e_3} \wedge X_{-2e_2} \wedge X_{-2e_3}).$$

Proof. We have a decomposition of K_∞ representations $\bigwedge^3 \mathfrak{p}^+_{\mathbf{C}} = \tau_{(3,3,0)} \oplus \tau_{(4,1,1)}$, $\bigwedge^3 \mathfrak{p}^-_{\mathbf{C}} = \tau_{(-1,-1,-4)} \oplus \tau_{(0,-3,-3)}$. Since each of the four summands have multiplicity-free weights (i.e. every weight space has dimension at most one), then one can easily check that the vector

$$X_{(2,2,-4)} := (X_{2e_1} \wedge X_{2e_2} \wedge X_{e_1+e_2}) \otimes (X_{-e_1-e_3} \wedge X_{-e_2-e_3} \wedge X_{-2e_3})$$

is a highest weight vector of $\tau_{(2,2,-4)} \subseteq \tau_{(3,3,0)} \otimes \tau_{(-1,-1,-4)}$. Indeed $\tau_{(2,2,-4)}$ is the Cartan component of the tensor product and each of the terms in the tensor product defining $X_{(2,2,-4)}$ is a highest weight vector of its corresponding representation. Analogously, one shows that $X_{(4,-2,-2)}$ is a highest weight vector of $\tau_{(4,-2,-2)} \subseteq \tau_{(4,1,1)} \otimes \tau_{(0,-3,-3)}$. Observe finally that we can pass from $\tau_{(2,2,-4)}$ to $\tau_{(4,-2,-2)}$ by the action of the matrix inducing complex conjugation. □

3.3. **Discrete series *L*-packets and test vectors.** We recall some standard facts on discrete series. For any non-singular weight $\Lambda \in \Delta$, define

$$\Delta^+(\Lambda) := \{\alpha \in \Delta \ : \ \langle \alpha, \Lambda \rangle > 0\}, \quad \Delta^+_c(\Lambda) = \Delta^+(\Lambda) \cap \Delta_c,$$

where $\langle\ ,\ \rangle$ is the standard scalar product on $\mathbf{R}^3$. Let λ be a dominant weight for Sp_6 (with respect to the complexification $\mathfrak{h}_{\mathbf{C}}$ of the compact Cartan algebra $\mathfrak{h}$) and let $\rho =$

$\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = (3,2,1)$. As $|\mathfrak{W}_{\mathrm{Sp}_6}/\mathfrak{W}_{K_\infty}| = 8$, the set of equivalence classes of irreducible discrete series representations of $\mathrm{Sp}_6(\mathbf{R})$ with Harish-Chandra parameter $\lambda+\rho$ contains 8 elements. More precisely, let us choose representatives $\{w_1,\dots,w_8\}$ of $\mathfrak{W}_{\mathrm{Sp}_6}/\mathfrak{W}_{K_\infty}$ of increasing length and such that for any $1\le i\le 8$, the weight $w_i(\lambda+\rho)$ is dominant for K_∞. Then for any $1\le i\le 8$ there exists an irreducible discrete series π_∞^Λ, where $\Lambda = w_i(\lambda+\rho)$, of Harish-Chandra parameter Λ and containing with multiplicity 1 the minimal K_∞-type with highest weight $\Lambda+\delta_{\mathrm{Sp}_6}-2\delta_{K_\infty}$ where δ_{Sp_6}, resp. δ_{K_∞}, is the half-sum of roots, resp. of compact roots, which are positive with respect to the Weyl chamber in which Λ lies, i.e.,

$$2\delta_{\mathrm{Sp}_6} := \sum_{\alpha\in\Delta^+(\Lambda)}\alpha,\quad 2\delta_{K_\infty} := \sum_{\alpha\in\Delta_c^+(\Lambda)}\alpha.$$

Moreover, for $i\neq j$, $\Lambda = w_i(\lambda+\rho)$, $\Lambda' = w_j(\lambda+\rho)$, the representations π_∞^Λ and $\pi_\infty^{\Lambda'}$ are not equivalent and any discrete series of $\mathrm{Sp}_6(\mathbf{R})$ is obtained in this way ([30, Theorem 9.20]). We define the discrete series L-packet $P(V^\lambda)$ associated to λ to be the set of isomorphism classes of discrete series of $\mathrm{Sp}_6(\mathbf{R})$ whose Harish-Chandra parameter is of the form $\Lambda = w_i(\lambda+\rho)$, for some $1\le i\le 8$.

Lemma 3.3. *There exist two irreducible discrete series representations $\pi_\infty^{3,3}$ and $\overline{\pi}_\infty^{3,3}$ of $\mathrm{Sp}_6(\mathbf{R})$ with Harish-Chandra parameter $(2,1,-3)$ and $(3,-1,-2)$, and trivial central character whose minimal K_∞-types are $\tau_{(2,2,-4)}$ and $\tau_{(4,-2,-2)}$.*

Proof. As explained in section 3.1.3, the reflections σ_i in the orthogonal hyperplane of the long roots $2e_i$ for $1\le i\le 3$ generate a system of representatives of $\mathfrak{W}_{\mathrm{Sp}_6}/\mathfrak{W}_{K_\infty}$. But these elements do not necessarily meet the condition that $\sigma_i\rho$ is dominant for K_∞. In order to find representatives satisfying this condition, we have to multiply these elements by elements of $\mathfrak{W}_{K_\infty}$ to put the coordinates of $\sigma_i\rho$ in decreasing order. We find the representatives defined by their action on ρ as follows: $w_1(3,2,1) = (3,2,1)$, $w_2(3,2,1) = (3,2,-1)$, $w_3(3,2,1) = (3,1,-2)$, $w_4(3,2,1) = (2,1,-3)$, $w_5(3,2,1) = (3,-1,-2)$, $w_6(3,2,1) = (2,-1,-3)$, $w_7(3,2,1) = (1,-2,-3)$, $w_8(3,2,1) = (-1,-2,-3)$.

For each $\Lambda = w_i\rho$, $1\le i\le 8$, observe that $\delta_{\mathrm{Sp}_6} = w_i\rho$ and hence the minimal K_∞-type of the discrete series π_∞^Λ is given by the formula

$$\Lambda+\delta_{\mathrm{Sp}_6}-2\delta_{K_\infty} = 2w_i\rho - 2\delta_{K_\infty}.$$

Using this formula, one easily checks that the minimal K_∞-types corresponding to each of representative w_i, $i=1,\dots,8$ described above are, respectively, $\tau_1 = \tau_{(4,4,4)}, \tau_2 = \tau_{(4,4,0)}, \tau_3 = \tau_{(4,2,-2)}, \tau_4 = \tau_{(2,2,-4)}, \tau_5 = \tau_{(4,-2,-2)}, \tau_6 = \tau_{(2,-2,-4)}, \tau_7 = \tau_{(0,-4,-4)}$ and $\tau_8 = \tau_{(-4,-4,-4)}$. The result follows by considering $i=4,5$. □

Lemma 3.4. *Let $\pi = \pi_\infty\otimes\pi_f$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with archimedean component π_∞ the discrete series such that $\pi_\infty|_{\mathrm{Sp}_6(\mathbf{R})} \simeq \pi_\infty^{3,3}\oplus\overline{\pi}_\infty^{3,3}$ and such that $\pi_f^U\neq 0$. Let $\Psi = \Psi_\infty\otimes\Psi_f$ be a cusp form in the space of π such that Ψ_∞ is a highest weight vector of the minimal K_∞-type $\tau_{(2,2,-4)}$ of $\pi_\infty^{3,3}$ and Ψ_f is a non-zero vector in π_f^U. Let $X_{(2,2,-4)}$ be the highest weight vector in the K_∞-type $\tau_{(2,2,-4)}\subset\bigwedge^3\mathfrak{p}_{\mathbf{C}}^+\otimes_{\mathbf{C}}\bigwedge^3\mathfrak{p}_{\mathbf{C}}^-$ of Lemma 3.2. Then there exists a unique non-zero harmonic $(3,3)$ differential form*

$$\omega_\Psi\in\mathrm{Hom}_{K_\infty}\left(\bigwedge^3\mathfrak{p}_{\mathbf{C}}^+\otimes_{\mathbf{C}}\bigwedge^3\mathfrak{p}_{\mathbf{C}}^-,\pi_\infty^{3,3}\right)\otimes\pi_f^U$$

on $\mathrm{Sh}_{\mathbf{G}}(U)$ such that $\omega_\Psi(X_{(2,2,-4)}) = \Psi$.

Proof. The results follows immediately from the fact that the minimal K_∞-type $\tau_{(2,2,-4)}$ of $\pi_\infty^{3,3}$ appears with multiplicity one, cf. [5, Theorem II.5.3]. □

3.4. **Archimedean L-functions and Deligne cohomology.** We end up this chapter by recalling some classical results on the relation between Deligne cohomology groups and the L-function of a motive, which explains when one expects to have non-trivial motivic cohomology. This section has only motivational interest and is not relevant to the main calculations of this text.

3.4.1. *Hodge decomposition.* In this section, we describe the Hodge decomposition of the "interior motive" $M(\pi_f)$ defined from the π_f-isotypical component of the interior cohomology

$$M(\pi_f) \otimes \pi_f = H^6_!(\mathrm{Sh}_{\mathbf{G}}, \mathbf{Q})[\pi_f],$$

which will allow us to describe Γ-factor of its L-function and its simple poles. Recall from [15], [37] that $M(\pi_f)$ is pure of weight 6. The Hodge weights of $M(\pi_f)$ lie in the set of pairs

$$(0,6),(1,5),(2,4),(3,3),(4,2),(5,1),(6,0).$$

When π_f is sufficiently nice, more precisely stable at infinity, all the weights (p,q) appear.

3.4.2. *Archimedean L-functions and Deligne cohomology.* We recall now, following [46], the definition of the Γ-factor of $M(\pi_f)$. The simple poles of this factor indicate the set of Tate twists of the motive $M(\pi_f)$ for which the Deligne cohomology is of positive dimension. It is precisely for these twists that one expects to construct non-trivial motivic cohomology classes.

The Betti realisation of $M(\pi_f)$ admits a Hodge decomposition

$$M_B(\pi_f) \otimes \mathbf{C} = \bigoplus_{p+q=6} H^{p,q}$$

which is equipped with an involution σ such that $\sigma(H^{p,q}) = H^{q,p}$. Denote $h^{p,q} = \dim_{\mathbf{C}} H^{p,q}$ the Betti numbers. For any p, write $H^{p,p} = H^{p,+} \oplus H^{p,-}$, where $H^{p,\pm} = \{x \in H^{p,p} | \sigma(x) = \pm(-1)^p x\}$, and let $h^{p,\pm 1} = \dim_{\mathbf{C}} H^{p,\pm}$. Let $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ be the real and complex Gamma factors, so that we have $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$. The archimedean factor of the L-function of $M(\pi_f)$ is then defined as

$$\Gamma(M(\pi_f), s) = \prod_{p<q} \Gamma_{\mathbf{C}}(s-p)^{h^{p,q}} \prod_p \Gamma_{\mathbf{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbf{R}}(s-p+1)^{h^{p,-}}.$$

Since the Gamma function $\Gamma(s)$ has simple poles at $s = -n$, $n \in \mathbf{N}$, a simple calculation shows that the order of the pole of $\Gamma(M(\pi_f), s)$ at $s = m$, $m \in \mathbf{Z}$, is given by

$$\sum_{m \le p < q} h^{p,q} + \begin{cases} 0 & \text{if } m > 3 \\ h^{3,(-1)^{m-3}} & \text{otherwise} \end{cases}.$$

Immediately from this formula, we get:

Lemma 3.5. *The function $\Gamma(M(\pi_f), s)$ has a pole of order $h^{3,+}$ at $s = 3$.*

Finally, recall the following result:

Proposition 3.6 ([45, §2, Proposition])**.** *We have*

$$\dim_{\mathbf{R}} H^1_{\mathcal{D}}(M(\pi_f)(7-m)) = \begin{cases} \mathrm{ord}_{s=m} L(M(\pi_f), s) & \text{if } m < 3 \\ \mathrm{ord}_{s=m} L(M(\pi_f), s) - \mathrm{ord}_{s=m+1} L(M(\pi_f), s) & \text{if } m = 3. \end{cases}$$

In this text, we are interested in the study of these Deligne cohomology groups for $m = 3$. The crucial hypothesis of Lemma 3.4 and §5.1 on the existence of a non-trivial test vector of the right Hodge type translates into asking that $h^{3,+} \neq 0$. Using the functional equation of the completed L-function for $M(\pi_f)$ and the fact that $\Gamma(M(\pi_f), s)$ is non-zero at $s = 3+1$, this translates into the equality

$$h^{3,+} = \mathrm{ord}_{s=3} L(M(\pi_f), s) - \mathrm{ord}_{s=4} L(M(\pi_f), s).$$

In view of Proposition 3.6, this in turn implies that $H^1_{\mathcal{D}}(M(\pi_f)(4))$ is of positive dimension. According to Beilinson conjectures (cf. [45, §5]), one then expects to construct non-trivial motivic cohomology classes in $H^1(M(\pi_f)(4))$.

Thanks to the results of [42], we can relate the archimedean realisation of our motivic class to a non-critical special value of the degree 8 Spin L-function. Indeed, under certain technical hypotheses, the L-function of $M(\pi_f)(3)$ agrees with the (partial) Spin L-function $L(\pi, \mathrm{Spin}, s)$ (cf. [31, Proposition 13.1]), which are both centered at $s = 1/2$. In this case, Beilinson conjectures predict an isomorphism

$$(H^7_{\mathcal{M},!}(\mathrm{Sh}_{\mathbf{G}}, \mathbf{Q}(4)) \otimes \mathbf{R}) \oplus (N^3(\overline{\mathrm{Sh}}_{\mathbf{G}}) \otimes \mathbf{R}) \xrightarrow{(r_{\mathcal{D}}, c_B)} H^7_{\mathcal{D},!}(\mathrm{Sh}_{\mathbf{G}}/\mathbf{R}, \mathbf{R}(4)),$$

where $r_{\mathcal{D}}$ denotes the archimedean regulator, $N^3(\overline{\mathrm{Sh}}_{\mathbf{G}})$ is the space of codimension 3 cycles (up to homological equivalence) in a smooth compactification $\overline{\mathrm{Sh}}_{\mathbf{G}}$, and c_B is the (Betti) cycle class map. In this article, we focus our attention to the cuspidal automorphic forms which contribute to the motivic part of the conjecture, i.e. precisely those for which $h^{3,+1} = \mathrm{ord}_{s=0} L(\pi, \mathrm{Spin}, s) > 0$. Our main result, Theorem 5.17, gives striking evidence towards the conjecture for these automorphic forms. The forms which contribute to the other term, in the spirit of Tate's conjecture, are instead the ones of Hodge type $(3,3)$, for which $\mathrm{ord}_{s=1} L(\pi, \mathrm{Spin}, s) = -1$. These automorphic forms are lifts from the split G_2 under the exceptional theta correspondence (cf. [42, Theorem 1.3]). In [9], we address this part of Beilinson conjecture.

4. Deligne–Beilinson cohomology and tempered currents

Let X denote a complex analytic variety which is smooth, quasi-projective and of pure dimension d. Let $\overline{X}$ be a smooth compactification of X such that $D = \overline{X} - X$ is a simple normal crossing divisor. We denote by $j : X \to \overline{X}$ the open embedding. We will assume that X is defined as the analytification of the base change to $\mathbf{C}$ of a smooth, quasi-projective $\mathbf{R}$-scheme. The complex conjugation F_∞ is an antiholomorphic involution on X. For $p \in \mathbf{Z}$, let $\mathbf{R}(p)$ denote the subgroup $(2\pi i)^p \mathbf{R}$ of $\mathbf{C}$. We will denote by the same symbol the constant sheaf with value $\mathbf{R}(p)$ on X.

We recall the definition of Deligne–Beilinson cohomology (DB-cohomology for short) of X. Let Ω^*_X be the sheaf of holomorphic differential forms on X and let $\Omega^*_{\overline{X}}(\log D)$ be the sheaf of holomorphic differential forms on X with logarithmic poles along D (see [12, §3.1]). The Hodge filtration on $\Omega^*_{\overline{X}}(\log D)$ is defined as $F^p \Omega^*_{\overline{X}}(\log D) = \bigoplus_{p' \geq p} \Omega^{p'}_{\overline{X}}(\log D)$. There are natural quasi-isomorphisms of complexes $Rj_* \mathbf{C} \to Rj_* \Omega^*_X$ and $\Omega^*_{\overline{X}}(\log D) \to Rj_* \Omega^*_X$ (see [12] or [26] for the basic facts used here). For any $p \in \mathbf{Z}$, the DB-cohomology groups $H^n_{\mathcal{D}}(X, \mathbf{R}(p))$ of X with coefficients in $\mathbf{R}(p)$ are defined as the hypercohomology groups of the complex

$$\mathbf{R}(p)_{\mathcal{D}} := \mathrm{cone}(Rj_* \mathbf{R}(p) \oplus F^p \Omega^*_{\overline{X}}(\log D) \to Rj_* \Omega^*_X)[-1], \tag{4}$$

where the arrow is given by the difference of the natural maps. Let $\overline{F^*_\infty} = F^*_\infty \otimes c$ be the de Rham involution given by the action of the complex conjugation on X and on the

coefficients. The real DB-cohomology groups are defined as

$$H^n_{\mathcal{D}}(X/\mathbf{R},\mathbf{R}(p)) = H^n_{\mathcal{D}}(X,\mathbf{R}(p))^{\overline{F^*_\infty}=1}.$$

The purpose of this chapter is to give some useful new explicit description of DB-cohomology that will be suitable for the computations of the next chapter. In order to do this, we will explain how the Hodge filtration and the real structure on the cohomology groups of X are induced from Hodge complexes of differential forms with growth conditions and tempered currents.

4.1. **Function spaces.** If $r>0$, we denote by $\Delta^*_r = \{z\in\mathbf{C}\ :\ 0<|z|\leq r\}$ the punctured disc of of radius r and $\Delta_r=\{z\in\mathbf{C}\ :\ 0\leq|z|\leq r\}$ its closure. If $m,n\in\mathbf{N}$, we will denote by $z=(z_1,\dots,z_m,z_{m+1},\dots,z_{m+n})$ the coordinates of $(\Delta^*_r)^m\times(\Delta_r)^n$.

Definition 4.1 ([24, §2.2])**.** *Let $N\in\mathbf{N}$. A $\mathcal{C}^\infty$-function $f(z)$ on $(\Delta^*_r)^m\times(\Delta_r)^n$ is said to have logarithmic growth of degree N, resp. logarithmic decay of degree N, if*

$$|f(z)|\leq C\Big(\prod_{j=1}^m\big|\log|z_j|\big|\Big)^N,$$

resp. if

$$|f(z)|\leq C\Big(\prod_{j=1}^m\big|\log|z_j|\big|\Big)^{-N},$$

for some constant $C>0$. We say that f is slowly increasing if, $\forall a,b\in\mathbf{N}^m, c,d\in\mathbf{N}^n$, the function

$$\left(\prod_{i=1}^m\big(z_i\frac{\partial}{\partial z_i}\big)^{a_i}\big(\overline{z}_i\frac{\partial}{\partial\overline{z}_i}\big)^{b_i}\prod_{j=m+1}^{n+m}\big(\frac{\partial}{\partial z_j}\big)^{c_j}\big(\frac{\partial}{\partial\overline{z}_j}\big)^{d_j}\right)(f) \tag{5}$$

is of logarithmic growth for some $N\in\mathbf{N}$. The function f is uniformly slowly increasing if the same N works for all. We say that f is rapidly decreasing if f and all the terms as in (5) are of logarithmic decay N for all $N\in\mathbf{N}$.

Remark 4.2. The condition (5) on the derivatives comes from asking that all the exterior higher derivatives of f will satisfy the same growth conditions. This will assure in particular that the complexes of differential forms defined below are well defined. We also point out that a slowly increasing functions is equivalent to a smooth function on Δ^{m+n} equipped with the log structure given by the normal crossing divisor $\{z_1\dots z_m=0\}$, cf. [28, Lemma 3.4(2)]. We also point out that this is the usual condition for defining rapidly decreasing functions taking into account the logarithmic change of variables on polar coordinates.

Let $\overline{X}$ be a smooth compact complex manifold, $D\subseteq\overline{X}$ a normal crossing divisor and $X=\overline{X}-D$. This means that each point $x\in\overline{X}$ has a coordinate neighbourhood $U=\Delta^d_r$ with coordinates $(z_1,\dots,z_d)$ for which $x=(0,\dots,0)$ and such that there is an integer k, $0\leq k\leq d$, so that

$$X\cap U=(\Delta^*_r)^k\times(\Delta_r)^{d-k}=\{(z_1,\dots,z_d)\mid z_1\dots z_k\neq 0\}.$$

We note $j:X\to\overline{X}$ the natural inclusion.

We denote by $\mathcal{O}_{\overline{X}}$ the structural sheaf of holomorphic functions on $\overline{X}$ and we denote by $\Omega^*_{\overline{X}}$ the holomorphic de Rham complex on $\overline{X}$. This is a complex of locally free $\mathcal{O}_{\overline{X}}$-module of finite type. Recall also that $\Omega^*_{\overline{X}}(\log D)$ is defined to be the sub-$\mathcal{O}_{\overline{X}}$-algebra of $j_*\Omega^*_X$ locally generated by the sections $\frac{dz_i}{z_i}$, $1\leq i\leq k$ and dz_i $k<i\leq d$. We shall denote by $\mathscr{A}^0_{\overline{X}}$

the sheaf of smooth functions on $\overline{X}$ and $\mathscr{A}^*_{\overline{X}}$ the complex of sheaves of smooth differential forms. The complex $\mathscr{A}^*_{\overline{X}}(\log D)$ of smooth differential forms on X with logarithmic growth at D is defined (cf. [7, §2]) to be the $\mathscr{A}_{\overline{X}}$-algebra subsheaf of $j_*\mathscr{A}^*_X$ locally generated by the sections

$$\frac{dz_i}{z_i},\ \frac{d\overline{z}_i}{\overline{z}_i},\ \log|z_i| \quad (1 \leq i \leq k),$$
$$dz_i,\ d\overline{z}_i \quad (k < i \leq d).$$

Definition 4.3. *We denote by $\mathscr{A}^0_{si}$ (resp. $\mathscr{A}^0_{rd}$) the sheaf on $\overline{X}$ whose sections at $U \subseteq \overline{X}$ are given by complex valued functions on $U \cap X$ which are locally at each point of U slowly increasing (resp. rapidly decreasing). We define the complex $\mathscr{A}^*_{si}$ of sheaves of slowly increasing differential forms to be the subcomplex of $j_*\mathscr{A}^*_X$ of $\mathscr{A}^0_{si}$-modules locally generated by $\frac{dz_i}{z_i}$, $\frac{d\overline{z}_i}{\overline{z}_i}$ for $1 \leq i \leq k$ and dz_i, $d\overline{z}_i$ for $k < i \leq d$. The complex $\mathscr{A}^*_{rd}$ of rapidly decreasing differential forms is defined analogously.*

Remark 4.4. Observe that, even if it is not reflected in the notation, the sheaves $\mathscr{A}^*_{si}$ and $\mathscr{A}^*_{rd}$ depend a priori on the divisor D.

Precisely, for any open subset $U \subseteq \overline{X}$, a differential form $\omega \in j_*\mathscr{A}^*_X(U)$ lies in $\mathscr{A}^*_{si}(U)$ (resp. $\mathscr{A}^*_{rd}(U)$) if it can locally be written as

$$\omega = \sum_{I,I',J,J'} \alpha_{I,I',J,J'} \frac{dz_I}{z_I} \wedge dz_J \wedge \frac{d\overline{z}_{I'}}{\overline{z}_{I'}} \wedge d\overline{z}_{J'},$$

where $I, I' \subseteq \{1, \ldots, k\}$, $J, J' \subseteq \{k+1, \ldots, d\}$ and $\alpha_{I,I',J,J'}$ is a function in $\mathscr{A}^0_{si}(U)$ (resp. $\mathscr{A}^0_{rd}(U)$) for every I, I', J, J', and where, for $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$, we have used the usual notation $\frac{dz_I}{z_I} = \frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_m}}{z_{i_m}}$, and idem for the other terms.

Observe that there are natural inclusions

$$\mathscr{A}^*_{rd} \subseteq \mathscr{A}^*_{\overline{X}}(\log D) \subseteq \mathscr{A}^*_{si} \subseteq j_*\mathscr{A}^*_X.$$

They are all are complexes of fine sheaves since their terms are modules over $\mathscr{A}^0_{\overline{X}}$ and hence admit partitions of unity. We denote by $A^*_{\overline{X}}(\log D)$, $\mathscr{A}^*_{rd}(\overline{X})$ and $\mathscr{A}^*_{si}(\overline{X})$ the corresponding complexes of global sections. Moreover, the complex structure on $\overline{X}$ induces compatible bigradings $\mathscr{A}^n_{rd} = \bigoplus_{p+q=n} \mathscr{A}^{p,q}_{rd}$, $\mathscr{A}^n_{\overline{X}}(\log\ D) = \bigoplus_{p+q=n} \mathscr{A}^{p,q}_{\overline{X}}(\log\ D)$, $\mathscr{A}^n_{si} = \bigoplus_{p+q=n} \mathscr{A}^{p,q}_{si}$, with corresponding Hodge filtrations $F^p\mathscr{A}^n_{rd} = \bigoplus_{p' \geq p} \mathscr{A}^{p',q}_{rd}$, etc. Finally, we denote by $\mathscr{A}^*_{si,\mathbf{R}} \subseteq \mathscr{A}^*_{si}$, $\mathscr{A}^*_{rd,\mathbf{R}} \subseteq \mathscr{A}^*_{rd}$, $\mathscr{A}^*_{\overline{X},\mathbf{R}}(\log\ D) \subseteq \mathscr{A}^*_{\overline{X}}(\log\ D)$ the subcomplexes of sheaves of $\mathbf{R}$-valued differential forms. The complex $\Omega^*_{\overline{X}}(\log\ D)$ is endowed with a Hodge filtration F^* defined by

$$F^p\Omega^*_{\overline{X}}(\log\ D) = \bigoplus_{p' \geq p} \Omega^{p'}_{\overline{X}}(\log\ D).$$

Proposition 4.5 ([7, Corollary 2.2])**.** *The natural map $(\Omega^*_{\overline{X}}(\log\ D), F) \to (\mathscr{A}^*_{\overline{X}}(\log\ D), F)$ is a filtered quasi-isomorphism and $Rj_*\mathbf{R} \to \mathscr{A}^*_{\overline{X},\mathbf{R}}(\log\ D)$ is a quasi-isomorphism.*

4.2. $\overline{\partial}$-Poincaré lemma with growth conditions.

Lemma 4.6. *There is a long exact sequence*

$$0 \to \Omega^p_{\overline{X}}(\log\ D) \to \mathscr{A}^{p,0}_{si} \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1}_{si} \to \ldots \to \mathscr{A}^{p,d}_{si} \xrightarrow{\overline{\partial}} 0.$$

Proof. One first reduces to the local situation with coordinates $(z_1, \dots, z_d) \in (\Delta_r^*)^k \times \Delta_r^{d-k}$. Then, the exactness at the middle follows from [23, Lemme 1] (cf. also [21, (3.1.1) Lemma] or [24, 2.1.3]), where they show, in the case of dimension 1, that if $g \in \mathscr{A}_{si}^0$ is a function which has logarithmic growth of order N then there exists $f \in \mathscr{A}_{si}^0$ of logarithmic growth $N+1$ such that $\overline{\partial} f = g(z)\frac{d\overline{z}}{\overline{z}}$. This implies that f is slowly increasing (resp. rapidly decreasing) whenever g is. The proof for higher dimension is similar, but we also refer to [28, §3.7] for a proof with complete details.

The exactness at the first term is [23, Lemme 2] and it is also shown in [28, 3.5]. We recall the proof for commodity of the reader. Since $\mathscr{A}_{si}^{p,q} = \mathscr{A}_{si}^{0,q} \otimes_{\mathcal{O}_{\overline{X}}} \Omega_{\overline{X}}^p(\log\, D)$, it suffices to prove the result for $p = 0$. If $f \in \mathscr{A}_{si}^{0,0}$ is such that $\overline{\partial} f = 0$, then f is holomorphic on X and slowly increasing. It follows from definition that, locally around any point, the product $(\prod_{i=1}^k z_i) f(z)$ is a holomorphic function on Δ^d vanishing at the boundary $D = \{z_1 \dots z_k = 0\}$, and hence divisible by $\prod_{i=1}^k z_i$, which shows that f is actually holomorphic on Δ^d. This implies that $f \in \mathcal{O}_{\overline{X}}$. The case for rapidly decreasing forms is treated similarly. □

Proposition 4.7. *There is a filtered quasi-isomorphism*

$$(\Omega_{\overline{X}}^*(\log\, D), F) \to (\mathscr{A}_{si}^*, F).$$

Proof. This is basically [24, Proposition 5.8]. In order to show that the natural inclusion $\Omega_{\overline{X}}^*(\log\, D) \subseteq \mathscr{A}_{si}^*$ induces a filtered quasi-isomorphism, one needs to check that it is a filtered morphism and that the corresponding graded pieces of the complexes are quasi-isomorphic. The first condition is satisfied by construction, since the Hodge filtration on $\mathscr{A}_{si}^*$ induces that of $\Omega_{\overline{X}}^*(\log\, D)$. For the last assertion, we have

$$\mathrm{Gr}_F^p \Omega_{\overline{X}}^*(\log\, D) = \Omega_{\overline{X}}^p(\log\, D),$$

$$\mathrm{Gr}_F^p \mathscr{A}_{si}^* = [\mathscr{A}_{si}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{A}_{si}^{p,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathscr{A}_{si}^{p,d}].$$

So the result follows from Lemma 4.6. □

Proposition 4.8. *The natural inclusion* $(\mathscr{A}_{\overline{X}}^*(\log D), F) \to (\mathscr{A}_{si}^*, F)$ *is a filtered quasi-isomorphism compatible with the real structures.*

Proof. The inclusion map obviously respects the Hodge filtration by construction. To show that the map is a filtered quasi-isomorphism, it suffices to show that there is a quasi-isomorphism

$$\mathrm{Gr}_F^p \mathscr{A}_{\overline{X}}^*(\log D) \cong \mathrm{Gr}_F^p \mathscr{A}_{si}^*.$$

By the proof of [7, Proposition 2.3], we know that

$$\mathrm{Gr}_F^p \mathscr{A}_{\overline{X}}^*(\log D) \cong \Omega_{\overline{X}}^p(\log\, D).$$

The result follows then from the proof Proposition 4.7. □

Remark 4.9. Proposition 4.8 could also be deduced directly from Proposition 4.7 and [7, Theorem 2.1].

4.3. **The Bochner-Martinelli kernel and rapidly decreasing differential forms.** We now introduce the Bochner-Martinelli operator on rapidly decreasing differential forms which will be used later to show the $\overline{\partial}$-Poincaré lemma for tempered currents. Our calculations are very much inspired in the work of Harris and Phong [23], where the case of dimension 1 is treated for showing that the complexes of rapidly decreasing and slowly increasing differential forms calculate the cohomology of X. By standard arguments, one reduces to working locally around a point of $\overline{X}$, i.e. on a domain of the form $(\Delta^*)^m \times \Delta^n$, with

$\Delta = \Delta_{1/2}$. Since all the computations we will make in this section carry over in the same way, we will assume $n = 0$ to simplify the exposition.

Let η be a smooth differential form on $(\Delta^*)^m = (\Delta^*_{1/2})^m$ of type $(0, q)$. Consider the operator induced by the Bochner-Martinelli formula

$$(K\eta)(z) = \int_{\Delta^m} k(z, w) \wedge \eta(w),$$

where, if one sets

$$\Phi_i(\zeta) = (-1)^{i-1}\zeta_i d\zeta_1 \wedge \ldots \wedge \widehat{d\zeta_i} \wedge \ldots \wedge d\zeta_d,$$
$$\Phi(\zeta) = d\zeta_1 \wedge \ldots \wedge d\zeta_d,$$

the kernel $k(z, w)$ of the operator is given by

$$k(z, w) = \frac{(m-1)!}{(2\pi i)^m} \frac{\sum_{i=1}^m \overline{\Phi_i(z-w)} \wedge \Phi(w)}{|z-w|^{2m}}.$$

For generalities on this operator, we refer the reader to [19] and [32]. The Bochner-Martinelli kernel is a locally integrable differential form on $\mathbf{C}^m \times \mathbf{C}^m \backslash \{z = w\}$ that decomposes as

$$k(z, w) = \sum_{q'=1}^{m} k^{q'}(z, w),$$

where $k^{q'}(z, w)$ is of type $(0, q'-1)$ on the variably z and of type $(m, m-q')$ on the variable w. Then integral defining the operator K needs not converge but we will show that it does whenever η is rapidly decreasing.

Let $I \subseteq \{1, \ldots, m\}$ with $|I| = q$ and let $\eta(w) = f(w)\frac{d\overline{w}_I}{\overline{w}_I}$ be a rapidly decreasing form of type $(0, q)$ on $(\Delta^*)^m$. For every $N \in \mathbf{N}$ we have

$$|f(w)| \leq C_{\eta,N} |\log |w||^{-N} (:= C_{\eta,N} \prod_{j=1}^{m} |\log |w_j||^{-N}) \tag{6}$$

for some constant $C_{\eta,N} > 0$ and the same estimate holds (with variable constant) for all exterior derivatives of η (cf. Definition 4.1). We have by definition

$$(K\eta)(z) = \int_{(\Delta^*)^m} k(z, w) \wedge \eta(w) = \frac{(m-1)!}{(2\pi i)^m} \sum_{i=1}^{m} \int_{(\Delta^*)^m} \frac{\overline{\Phi_i(z-w)} \wedge \Phi(w)}{\|z-w\|^{2m}} \wedge \eta(w).$$

Observe that $\int_{(\Delta^*)^m} \frac{\overline{\Phi_i(z-w)} \wedge \Phi(w)}{\|z-w\|^{2m}} \wedge \eta(w) = 0$ unless $i \in I$. Let $i \in I$, then

$$\int_{(\Delta^*)^m} \frac{\overline{\Phi_i(z-w)} \wedge \Phi(w)}{\|z-w\|^{2m}} \wedge \eta(w) = \left(\int_{(\Delta^*)^m} \frac{(\overline{z}_i - \overline{w}_i) f(w)}{\|z-w\|^{2m} \overline{w}_I} dw \wedge d\overline{w} \right) d\overline{z}_{I-\{i\}}.$$

Lemma 4.10. *Then, for any $z \in (\Delta^*)^m$, the integral*

$$g_i(z) := \int_{(\Delta^*)^m} \frac{(\overline{z}_i - \overline{w}_i) f(w)}{\|z-w\|^{2m} \overline{w}_I} dw \wedge d\overline{w}$$

is absolutely convergent.

Proof. Let us fix $z \in (\Delta^*)^m$. Write $f = f_1 + f_2$, where f_1 is a smooth function which is supported in a sufficiently small polydisc $B(z, 2a) := \prod_{j=1}^m B(z_j, 2a)$ around z and f_2 vanishes identically on $\prod_{j=1}^m B(z_j, a)$. Then we have

$$\int_{(\Delta^*)^m} \frac{|\overline{z}_i - \overline{w}_i||f_2(w)|}{\|z-w\|^{2m} |\overline{w}_I|} dw \wedge d\overline{w} \Bigg| \quad \leq \quad C \int_{(\Delta^*)^m} \frac{1}{|w|} dw \wedge d\overline{w} < +\infty,$$

for some constant $C > 0$, where here we denote $|\overline{w}_I| = |w_I| = \prod_{i\in I} |w_i|$ and $|w| = \prod_{i=1}^m |w_i|$. On the other hand, we have

$$\int_{(\Delta^*)^m} \left| \frac{(\overline{z}_i - \overline{w}_i) f_1(w)}{\|z-w\|^{2m}\overline{w}_I} dw \wedge d\overline{w} \right| \quad \leq \quad C' \int_{B(z,2a)} \frac{1}{\|z-w\|^{2m-1}} dw \wedge d\overline{w} < +\infty,$$

for some $C' > 0$ and where the last integral converges from the standard fact that $\int_{B^n(0,1)} \frac{dx}{\|x\|^\alpha}$ converges if $\alpha < n$ ($B^n(0,1) \subseteq \mathbf{R}^n$ is the unit ball). □

Lemma 4.11. *There exists a universal constant $C_{m,N}$, depending only on m and N, such that for any $z \in (\Delta^*)^m$ we have*

$$|g_i(z)| \leq C_{m,N} C_{\eta,N} \frac{|\log|z||^{-N+1}}{|z_{I-\{i\}}|}.$$

Proof. We have

$$|g_i(z)| \leq \int_{(\Delta^*)^m} \frac{|\overline{z}_i - \overline{w}_i||f(w)|}{\|z-w\|^{2m}|\overline{w}_I|} dw \wedge d\overline{w} \leq C_{\eta,N} \int_{(\Delta^*)^m} \frac{|\overline{z}_i - \overline{w}_i||\log|w||^{-N}}{\|z-w\|^{2m}|\overline{w}_I|} dw \wedge d\overline{w}. \quad (7)$$

We will show the inequality by induction on m. The base case $m = 1$ and $I = \{i\}$ is the one treated in [23, Lemme 1] [1] for which we give a proof for the commodity of the reader. There are two cases for the reduction step of the induction, one for those variables j such that $j \in I$ and another one for those variables j not belonging to I. We treat both at the same time using the same dévisage of the integral as the proof of the base case.

The base case : When $m = 1$ and $I = \{i\}$ we have

$$g_1(z) = \int_{\Delta^*} \frac{f(w)}{(z-w)w} dw \wedge d\overline{w}.$$

We let $r := |z|$ and we decompose Δ^* into three regions

$$\Delta^* = D_1 \cup D_2 \cup D_3,$$

where $D_1 = \Delta^*_{r/2}$ is the punctured disc of radius $r/2$, $D_2 = B(z, r/2) \cap \Delta^*$ is the intersection of Δ^* with the (punctured) disc around z of radius $r/2$ and $D_3 = \Delta^* - (D_1 \cup D_2)$. For $w \in D_1$, we have $|z-w| \geq |z|/2$ and $|w| \leq |z|/2$ so that in particular $|\log|w||^{-N} \leq |\log|z||^{-N}$. Hence

$$\int_{D_1} \frac{|\log|w||^{-N}}{|z-w||w|} dw \wedge d\overline{w} \leq 2 \frac{|\log|z||^{-N}}{|z|} \int_{D_1} \frac{dw \wedge d\overline{w}}{|w|} = 2\pi |\log|z||^{-N},$$

where the last integral is calculated using polar coordinates. When $w \in D_2$, we have $|w| \geq |z|/2$ and also $|w| \leq \frac{3}{2}|z|$ so that that $|\log|w||^{-N} \leq |\log(\frac{3}{2}|z|)|^{-N}$. Hence

$$\begin{aligned} \int_{D_2} \frac{|\log|w||^{-N}}{|z-w||w|} dw \wedge d\overline{w} \quad &\leq \quad 2 \frac{|\log(\frac{3}{2}|z|)|^{-N}}{|z|} \int_{B(z,r/2)} \frac{dw \wedge d\overline{w}}{|z-w|} \\ &= \quad 2 \frac{|\log(\frac{3}{2}|z|)|^{-N}}{|z|} \int_{\Delta^*_{r/2}} \frac{dw \wedge d\overline{w}}{|w|} = 2\pi |\log(\frac{3}{2}|z|)|^{-N} \end{aligned}$$

[1] Actually, in [23, Lemme 1] only the case of slowly increasing functions, i.e. when $-N > 0$ in our notation, is treated. The arguments for the present case are commented in [21, Lemma 3.1.1] but there is a misprint in the second term, where it should be $2r$ instead of $r/2$.

Finally, when $w \in D_3$, we use that $|z-w| \geq |w/3|$ and we get, for $N \neq 1$,

$$\begin{aligned}\int_{D_3} \frac{|\log|w||^{-N}}{|z-w||w|} dw \wedge d\overline{w} &\leq 3\int_{D_3} \frac{|\log|w||^{-N}}{|w|^2} dw \wedge d\overline{w} \leq 3\int_{D_3 \cup D_2} \frac{|\log|w||^{-N}}{|w|^2} dw \wedge d\overline{w} \\ &= 6\pi \int_{r/2}^{1} \frac{|\log\rho|^{-N}}{\rho} d\rho = \frac{6\pi}{N-1} |\log(|z|/2)|^{-N+1} \leq \frac{6\pi}{N-1} |\log(|z|)|^{-N+1},\end{aligned}$$

where the last equality follows from the direct calculation $\int_{r/2}^{1} \frac{|\log\rho|^{-N}}{\rho} d\rho = \frac{|\log r/2|^{-N+1}}{N-1}$ for $N \neq 1$. Putting all together we get

$$|g_1(z)| \leq C_{\eta,N} \left(2\pi |\log|z||^{-N} + 2\pi |\log(\frac{3}{2}|z|)|^{-N} + \frac{6\pi}{N-1} |\log(|z|)|^{-N+1} \right).$$

We finally observe that $|\log(\frac{3}{2}|z|)^N \leq 2^N |\log|z||^N$ for all $z \in \Delta^*$. This concludes the proof of the base case.

The induction step : For any $1 \leq j \leq m$, $j \neq i$, let $r_j := |z_j|$ and decompose as before

$$(\Delta^*)^m = D_1^{(j)} \cup D_2^{(j)} \cup D_3^{(j)},$$

where $D_1^{(j)}$ (resp. $D_2^{(j)}$, resp. $D_3^{(j)}$) is obtained by replacing the $j-$th term of $(\Delta^*)^m$ by $D_1 = \Delta^*_{r_j/2}$ (resp. $D_2 = B(z_j, r_j/2) \cap \Delta^*$, resp. $D_3 = \Delta^* - (D_1 \cup D_2)$).

(1) When $w_j \in D_1$, we have $|z_j - w_j| \geq r_j/2 = |z_j|/2$. Letting $z' = (z_1, \cdots, \widehat{z}_j, \cdots, z_m) \in (\Delta^*)^{m-1}$ and idem for w', by the weighted AM-GM inequality we have

$$\|z-w\|^2 = |z_j - w_j|^2 + \|z' - w'\|^2 \geq m \left(|z_j - w_j|^2 \left(\frac{\|z'-w'\|^2}{m-1} \right)^{m-1} \right)^{1/m}, \tag{8}$$

so that $1/\|z-w\|^{2m} \leq \frac{1}{|z_j - w_j|^2} \frac{1}{\|z'-w'\|^{2(m-1)}} \leq 4 \frac{1}{|z_j|^2} \frac{1}{\|z'-w'\|^{2(m-1)}}$.

(a) When $j \in I$, using the above inequality we get

$$\int_{D_1^{(j)}} \frac{|\overline{z}_i - \overline{w}_i| |\log|w||^{-N}}{\|z-w\|^{2m} |\overline{w}_I|} dw \wedge d\overline{w} \leq$$

$$\begin{aligned}&\leq 4 \frac{|\log|z_j||^{-N}}{|z_j|^2} \left(\int_{\Delta^*_{r_j/2}} \frac{dw_j \wedge d\overline{w}_j}{|w_j|} \right) \left(\int_{(\Delta^*)^{m-1}} \frac{|\overline{z}_i - \overline{w}_i| |\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)} |w'_{I'}|} dw' \wedge d\overline{w}' \right) \\ &= 4\pi \frac{|\log|z_j||^{-N}}{|z_j|} \int_{(\Delta^*)^{m-1}} \frac{|\overline{z}_i - \overline{w}_i| |\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)} |w'_{I'}|} dw' \wedge d\overline{w}',\end{aligned}$$

where we have denoted $I' = I - \{j\}$.

(b) Assume now that $j \notin I$. The same calculation shows that

$$\int_{D_1^{(j)}} \frac{|\overline{z}_i - \overline{w}_i| |\log|w||^{-N}}{\|z-w\|^{2m} |\overline{w}_I|} dw \wedge d\overline{w} \leq$$

$$\begin{aligned}&\leq 4 \frac{|\log|z_j||^{-N}}{|z_j|^2} \left(\int_{\Delta^*_{r_j/2}} dw_j \wedge d\overline{w}_j \right) \left(\int_{(\Delta^*)^{m-1}} \frac{|\overline{z}_i - \overline{w}_i| |\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)} |w'_I|} dw' \wedge d\overline{w}' \right) \\ &= 2\pi |\log|z_j||^{-N} \left(\int_{(\Delta^*)^{m-1}} \frac{|\overline{z}_i - \overline{w}_i| |\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)} |w'_I|} dw' \wedge d\overline{w}' \right),\end{aligned}$$

(2)(a) Let now $w_j \in D_2$ with $j \in I$. We have $|w_j| \geq |z_j|/2$ and $|\log|w_j||^{-N} \leq |\log(\frac{3}{2}|z_j|)|^{-N}$ as in the base case. We calculate

$$\int_{D_2^{(j)}} \frac{|\overline{z}_i - \overline{w}_i||\log|w||^{-N}}{\|z-w\|^{2m}|\overline{w}_I|} dw \wedge d\overline{w}$$

$$\leq 2\frac{|\log(\frac{3}{2}|z_j|)|^{-N}}{|z_j|} \int_{(\Delta^*)^{m-1}} \frac{|z_i - w_i||\log|w'||^{-N}}{|w'_{I'}|} \left(\int_{D_2} \frac{dw_j \wedge d\overline{w}_j}{\|z-w\|^{2m}} \right) dw' \wedge d\overline{w}'. \quad (9)$$

Using polar coordinates we calculate

$$\begin{aligned} \int_{D_2} \frac{dw_j \wedge d\overline{w}_j}{\|z-w\|^{2m}} &= 2\pi \int_0^{r_j/2} \frac{\rho_j}{(\rho_j^2 + \|z'-w'\|^2)^m} d\rho_j \\ &= \frac{\pi}{m-1} \left(\frac{1}{\|z'-w'\|^{2(m-1)}} - \frac{1}{(r_j^2/2 + \|z'-w'\|^2)^{m-1}} \right) \\ &\leq \frac{\pi}{m-1} \frac{1}{\|z'-w'\|^{2(m-1)}}. \end{aligned}$$

Hence (9) is majorated by

$$\frac{2\pi}{m-1} \frac{|\log(\frac{3}{2}|z_j|)|^{-N}}{|z_j|} \left(\int_{(\Delta^*)^{m-1}} \frac{|z_i - w_i||\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)}|w'_{I'}|} dw' \wedge d\overline{w}' \right).$$

(2)(b) When $j \notin I$, the first inequality becomes

$$\int_{D_2^{(j)}} \frac{|\overline{z}_i - \overline{w}_i||\log|w||^{-N}}{\|z-w\|^{2m}|\overline{w}_I|} dw \wedge d\overline{w} \leq |\log(\frac{3}{2}|z_j|)|^{-N} \int_{(\Delta^*)^{m-1}} \frac{|z_i - w_i||\log|w'||^{-N}}{|w'_I|} \left(\int_{D_2} \frac{dw_j \wedge d\overline{w}_j}{\|z-w\|^{2m}} \right) dw' \wedge d\overline{w}',$$

and the same estimates explained in (a) give the result.

(3)(a) Let $w_j \in D_3$, with $j \in I$. Using again the AM-GM inequality we get

$$\int_{D_3^{(j)}} \frac{|\overline{z}_i - \overline{w}_i||\log|w||^{-N}}{\|z-w\|^{2m}|\overline{w}_I|} dw \wedge d\overline{w} \leq 4 \left(\int_{D_3} \frac{|\log|w_j||^{-N}}{|z_j - w_j|^2|w_j|} dw_j \wedge d\overline{w}_j \right) \left(\int_{(\Delta^*)^{m-1}} \frac{|z_i - w_i||\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)}|w'_{I'}|} dw' \wedge d\overline{w}' \right).$$

Since on D_3 we have $|z_j - w_j| \geq |w_j|/3$ and we obtain

$$\begin{aligned} \int_{D_3} \frac{|\log|w_j||^{-N}}{|z_j - w_j|^2|w_j|} dw_j \wedge d\overline{w}_j &\leq 3 \int_{D_3 \cup D_2} \frac{|\log|w_j||^{-N}}{|w_j|^3} dw_j \wedge d\overline{w}_j = 6\pi \int_{r_j/2}^{1/2} \frac{|\log \rho_j|^{-N}}{\rho_j^2} d\rho_j, \\ &\leq 12\pi \frac{1}{|z_j|} \int_{r_j/2}^{1} \frac{|\log \rho_j|^{-N}}{\rho_j} d\rho_j \leq \frac{12\pi}{N-1} \frac{|\log|z_j||^{-N+1}}{|z_j|}. \end{aligned}$$

(3)(b) Finally, we let $w_j \in D_3$, with $j \notin I$. Using the same argument we get

$$\int_{D_3^{(j)}} \frac{|\overline{z}_i - \overline{w}_i||\log|w||^{-N}}{\|z-w\|^{2m}|\overline{w}_I|} dw \wedge d\overline{w} \leq 4 \left(\int_{D_3} \frac{|\log|w_j||^{-N}}{|z_j - w_j|^2} dw_j \wedge d\overline{w}_j \right) \left(\int_{(\Delta^*)^{m-1}} \frac{|z_i - w_i||\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)}|w'_I|} dw' \wedge d\overline{w}' \right).$$

Since on D_3 we have $|z_j - w_j| \geq |w_j|/3$, we obtain

$$\int_{D_3} \frac{|\log|w_j||^{-N}}{|z_j - w_j|^2} dw_j \wedge d\overline{w}_j \leq 6\pi \int_{r_j/2}^{1/2} \frac{|\log \rho_j|^{-N}}{\rho_j} d\rho_j \leq 6\pi |\log|z_j||^{-N+1}.$$

Putting (1), (2) and (3) together as in the base case, we deduce that, for $j \in I$,

$$|g_i(z)| \leq C_{\eta,N} C' \frac{|\log|z_j||^{-N+1}}{|z_j|} \left(\int_{(\Delta^*)^{m-1}} \frac{|z_i - w_i||\log|w'||^{-N}}{\|z'-w'\|^{2(m-1)}|w'_{I'}|} dw' \wedge d\overline{w}' \right),$$

for some constant C' depending only on m and N (and the same with no $|z_j|$ on the denominator when $j \notin I$). This concludes the proof of the inductive step and hence the proof of the lemma. □

Lemma 4.12. *For any differential operator D, we have*

$$D(K\eta) = K(D\eta).$$

Proof. By the change of variable $u = z - w$ in the integral defining $D(K\eta)$ we have

$$\begin{aligned} D(K\eta)(z) &= \frac{(m-1)!}{(2\pi i)^m} \sum_{i=1}^{m} D\left(\int \frac{\overline{\Phi_i(u)} \wedge \Phi(z-u)}{\|u\|^{2m}} \wedge \eta(z-u) \right) \\ &= \frac{(m-1)!}{(2\pi i)^m} \sum_{i=1}^{m} \int \frac{\overline{\Phi_i(u)} \wedge \Phi(z-u)}{\|u\|^{2m}} \wedge D\eta(z-u) \\ &= K(D\eta). \end{aligned}$$

In the above, we used differentiation under the integral because of absolute convergence (Lemma 4.10), and the second equality follows from the fact that the coefficient function of $\frac{\overline{\Phi_i(u)} \wedge \Phi(z-u)}{\|u\|^{2m}}$ is independent of z. □

Before stating our main result, we recall the definition of the topology on the space of rapidly decreasing differential forms.

Definition 4.13. *The space of rapidly decreasing functions on $(\Delta^*)^m$ is equipped with the Schwartz topology given be the family of semi-norms*

$$p_{N,D}(f) = \sup_z |\log|z||^N |Df(z)|,$$

where $N \in \mathbf{N}$ and D is as in definition 4.1. The space of rapidly decreasing differential forms on $(\Delta^*)^m$ is a free module over rapidly decreasing functions as is equipped with the induced topology.

We recall the reader that a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of rapidly decreasing functions converges to 0 for the Schwartz topology if and only if, for all N and D, we have $p_{N,D}(\varphi_n) \to 0$ as $n \to +\infty$.

Theorem 4.14. *The Bochner-Martinelli formula induces a well defined continuous operator*

$$K : \mathscr{A}_{rd}^{0,q} \to \mathscr{A}_{rd}^{0,q-1}.$$

Moreover, if satisfies $\overline{\partial} K + K\overline{\partial} = id$.

Proof. To see that the operator is well defined, we need to prove that if $\eta \in \mathscr{A}_{rd}^{0,q}$ is a rapidly decreasing form, then the integral defining $K\eta$ converges and gives a rapidly decreasing differential form. First observe that one can restrict to the local situation and work on $(\Delta^*)^m$ with coordinates $(z_1, \ldots, z_m)$. Let $q > 0$ and let $\eta(z) = \sum_{|I|=q} f_I(z) d\overline{z}_I$ be a rapidly decreasing differential form of type $(0, q)$. We have

$$(K\eta)(z) = \sum_I \sum_{i\in I} \left(\int_{(\Delta^*)^m} \frac{(\overline{z}_i - \overline{w}_i) f(w)}{\|z-w\|^{2m} \overline{w}_I} dw \wedge d\overline{w} \right) d\overline{z}_{I-\{i\}}$$

It suffices to show that each differential form

$$\omega_i(z) = \left(\int_{(\Delta^*)^m} \frac{(\overline{z}_i - \overline{w}_i) f(w)}{\|z-w\|^{2m} \overline{w}_I} dw \wedge d\overline{w} \right) d\overline{z}_{I-\{i\}},$$

is well defined and rapidly decreasing. Lemma 4.10 shows that all these integrals converge absolutely. Lemma 4.11 shows that the first condition of rapidly decreasing (cf. Definition 4.1) is satisfied. Let D denote any differential operator as in Definition 4.1, then by Lemma 4.12 we know that

$$D\omega_I = \left(\int_{(\Delta^*)^m} \frac{(\overline{z}_i - \overline{w}_i) Df(w)}{\|z-w\|^{2m} \overline{w}_I} dw \wedge d\overline{w} \right) d\overline{z}_{I-\{i\}},$$

and hence one can apply Lemma 4.11 to conclude that each ω_i is rapidly decreasing. By Lemma 4.11, the operator K is continuous for every semi-norm $p_{N,D}$ and hence is continuous as an operator on the space of rapidly decreasing differential forms equiped with the Schwartz topology. It follows from the Bochner-Martinelli-Koppelman formula ([32, Theorem 2.3]) that the identity $\overline{\partial} K + K \overline{\partial} = \mathrm{id}$ holds on the dense subspace of compactly supported differential forms, and hence the result follows by continuity of the operator. □

4.4. **Tempered currents.** We now define sheaves of tempered currents by considering continuous linear forms on rapidly decreasing forms and show that they are filtered quasi-isomorphic to the cohomology of X.

Definition 4.15. *For any $0 \leq p, q \leq d$, we define the sheaf $\mathscr{D}^{p,q}$ of tempered currents to be the sheaf on $\overline{X}$ assigning to any open set $U \subseteq \overline{X}$ the module $\mathscr{D}^{p,q}(U)$ of continuous complex linear forms on the compactly supported sections $\Gamma_c(U, \mathscr{A}_{rd}^{d-p,d-q})$. Similarly we denote by $\mathscr{D}^*_{\mathbf{R}}$ as the dual of $\mathscr{A}^*_{si,\mathbf{R}}$.*

Remark 4.16. As in Definition 4.13. The space $\Gamma_c(U, \mathscr{A}^0_{rd})$ of rapidly decreasing functions on $\overline{X}$ is equipped with the usual Schwartz topology given be the family of semi-norms

$$p_{N,D}(f) = \sup_z |\log |z||^N |Df(z)|,$$

where $N \in \mathbf{N}$ and D is as in definition 4.1.

Remark 4.17. Observe that the boundary D might intersect U and hence an element in $\Gamma_c(U, \mathscr{A}^0_{rd})$ is a function on $U \cap X$ with growth conditions along $D \cap U$. In particular, is has compact support in $U \subseteq \overline{X}$ but not necessarily in X.

For any open subset $U \subseteq \overline{X}$, let $T \in \mathscr{D}^{p,q}(U)$ and $\omega \in \mathscr{A}^{p',q'}_{si}(U)$. Since the product of a slowly increasing differential form against a rapidly decreasing differential form is rapidly decreasing, the formula

$$(\omega T)(\eta) = T(\omega \wedge \eta)$$

is a well defined element in $\mathscr{D}^{p+p',q+q'}(U)$. This induces an $\mathscr{A}^0_{si}$-bilinear map and hence a map

$$\mathscr{A}^{p,q}_{si} \otimes_{\mathscr{A}^0_{si}} \mathscr{D}^{p',q'} \to \mathscr{D}^{p+p',q+q'},$$

equipping $\mathscr{D}^{*,*}$ with a bigraded $\mathscr{A}^{*,*}_{si}$-module.

Lemma 4.18. *The map*

$$\mathscr{A}^{p,q}_{si} \otimes_{\mathscr{A}^0_{si}} \mathscr{D}^{p',q'} \cong \mathscr{D}^{p+p',q+q'}$$

is an isomorphism.

Proof. To prove that this map is an isomorphism, we immediately reduce to the case where $p' = q' = 0$. This map is injective since $\mathscr{A}^{p,q}_{si}$ is free over $\mathscr{A}^0_{si}$. Let $\{\frac{dz_I}{z_I} \wedge dz_J \wedge \frac{d\overline{z}_{I'}}{\overline{z}_{I'}} \wedge d\overline{z}_{J'}\}_{I,I',J,J'}$, where $I, I' \subseteq \{1, \ldots, k\}, J, J' \subseteq \{k+1, \ldots, d\}$ are such that $|I| + |J| = p$,

$|I'|+|J'|=q$, be a basis of $\mathscr{A}_{si}^{p,q}(U)$ over $\mathscr{A}_{si}^{0}(U)$, where $U\subseteq\overline{X}$ is a sufficiently small coordinate open subset. If $T\in\mathscr{D}^{p,q}(U)$, then, defining $T_{I,J}\in\mathscr{D}^{0,0}(U)$ by

$$T_{I,J}(\frac{dz_{[1,k]}}{z_{[1,k]}}\wedge dz_{[k+1,d]}\wedge\frac{d\overline{z}_{[1,k]}}{\overline{z}_{[1,k]}}\wedge d\overline{z}_{[k+1,d]})=T(\frac{dz_{I^c}}{z_{I^c}}\wedge dz_{J^c}\wedge\frac{d\overline{z}_{I'^c}}{\overline{z}_{I'^c}}\wedge d\overline{z}_{J'^c}),$$

where $I^c=\{1,\dots,k\}-I$, $J^c=\{k+1,\dots,d\}-J$ and idem for I'^c and J'^c, we have

$$T=\sum_{I,J,I',J'}T_{I,J}\otimes(\frac{dz_I}{z_I}\wedge dz_J\wedge\frac{\overline{z}_{I'}}{\overline{z}_{I'}}\wedge d\overline{z}_{J'}),$$

showing that the multiplication map is surjective. This finishes the proof. □

In fact, the proof of the lemma shows the following.

Corollary 4.19. *The map*

$$\Omega^p_{\overline{X}}(\log\ D)\otimes_{\mathcal{O}_{\overline{X}}}\mathscr{D}^{0,q}\cong\mathscr{D}^{p,q}$$

is an isomorphism.

By Lemma 4.18, for any coordinate open subset $U\subseteq\overline{X}$, we can write any tempered current $T\in\mathscr{D}^{p,q}(U)$ as a differential form with coefficients in tempered distributions:

$$T=\sum_{I,I',J,J'}T_{I,J}\frac{dz_I}{z_I}\wedge dz_J\wedge\frac{\overline{z}_{I'}}{\overline{z}_{I'}}\wedge d\overline{z}_{J'},$$

where $T_{I,J}\in\mathscr{D}^{d,d}(U)$. The exterior derivative of smooth forms induces a differential $d:\mathscr{D}^n\to\mathscr{D}^{n+1}$ defined by

$$dT(\omega)=(-1)^{n+1}T(d\omega).$$

We have $d=\partial+\overline{\partial}$ with $\partial:\mathscr{D}^{p,q}\to\mathscr{D}^{p+1,q}$, $\overline{\partial}:\mathscr{D}^{p,q}\to\mathscr{D}^{p,q+1}$ defined in the analogous way. This defines complexes of sheaves of tempered currents $\mathscr{D}^*$, $\mathscr{D}^{*,*}$ (with differentials $d,\partial,\overline{\partial}$). Since currents are modules over $\mathscr{A}_{si}^0$ all these complexes are complexes of fine sheaves. We will denote by $\mathscr{D}^{*,*}(\overline{X})$ the corresponding complexes of global sections. Moreover $\mathscr{D}^*$ is equipped with a Hodge filtration given by

$$F^p\mathscr{D}^*=\bigoplus_{p'\geq p}\mathscr{D}^{p',q}.$$

For any open subset $U\subseteq\overline{X}$, there is a natural way to associate to any form $\omega\in\mathscr{A}_{si}^{p,q}(U)$ a current $T_\omega\in\mathscr{D}^{p,q}(U)$ given by

$$T_\omega(\eta)=\frac{1}{(2\pi i)^d}\int_U\omega\wedge\eta.$$

Using Lemma 4.18, the current T_ω can also be described as the product of ω with the normalised trace distribution in $\mathscr{D}^{0,0}(U)$ defined by $\eta\in\mathscr{A}_{rd}^{d,d}(U)\mapsto\frac{1}{(2\pi i)^d}\int_U\eta$. By Stokes formula [4] we have $dT_\omega=T_{d\omega}$, $\partial T_\omega=T_{\partial\omega}$, etc., which implies that

$$\mathscr{A}_{si}^*\to\mathscr{D}^*$$

is a filtered morphism of complexes.

Theorem 4.20. *There is a filtered quasi-isomorphism*

$$(\mathscr{A}_{si}^*,F)\to(\mathscr{D}^*,F)$$

compatible with real structures.

Proof. By Proposition 4.7, it suffices to show that there is a long exact sequence

$$0 \to \Omega^p_{\overline{X}}(\log\, D) \to \mathscr{D}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{D}^{p,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathscr{D}^{p,d} \xrightarrow{\overline{\partial}} 0. \tag{10}$$

We first reduce to the case $p = 0$. Assume that we have a long exact sequence

$$0 \to \mathcal{O}_{\overline{X}} \to \mathscr{D}^{0,0} \xrightarrow{\overline{\partial}} \mathscr{D}^{0,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathscr{D}^{0,d} \xrightarrow{\overline{\partial}} 0.$$

Since the sheaf $\Omega^p_{\overline{X}}(\log\, D)$ is a locally free $\mathcal{O}_{\overline{X}}$-module of finite type, it is in particular flat and we get a long exact sequence

$$0 \to \Omega^p_{\overline{X}}(\log\, D) \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}} \to \Omega^p_{\overline{X}}(\log\, D) \otimes_{\mathcal{O}_{\overline{X}}} \mathscr{D}^{0,0} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^p_{\overline{X}}(\log\, D) \otimes_{\mathcal{O}_{\overline{X}}} \mathscr{D}^{0,d} \xrightarrow{\overline{\partial}} 0.$$

Hence the general case follows from Corollary 4.19.

Let us prove the case $p = 0$. By Theorem 4.14, we have a continuous operator

$$K : \mathscr{A}^{0,q}_{rd} \to \mathscr{A}^{0,q-1}_{rd}.$$

The operator K induces an operator

$$K : \mathscr{D}^{0,q} \to \mathscr{D}^{0,q-1}$$

defined via $(KT)(\omega) = T(K\omega)$. This operator satisfies $\overline{\partial}K + K\overline{\partial} = id$, showing that the complex of (10) is exact in the middle.

It remains to show that the kernel of $\overline{\partial} : \mathscr{D}^{0,0} \xrightarrow{\overline{\partial}} \mathscr{D}^{0,1}$ is precisely $\mathcal{O}_{\overline{X}}$. It follows from the regularity of the $\overline{\partial}$-operator (cf. [19, p. 380]) that if $T \in \mathscr{D}^{0,0}$ is a current such that $\overline{\partial}T = 0$, then $T = T_\varphi$ for some holomorphic function φ on X. Since T_φ is a tempered current, this implies that φ is slowly increasing and hence holomorphic on the whole $\overline{X}$ by Lemma 4.6. We illustrate this in the case of dimension one: the function φ is in fact holomorphic on $\overline{X}$. Indeed, arguing by contradiction, write $\varphi(z) = \sum_{n=-\infty}^{+\infty} a_n(\varphi) z^n$ the Laurent expansion of φ and assume without loss of generality that $a_{-1}(\varphi) \neq 0$ and moreover that $\varphi(z) = 1/z$. Let then $\omega = z\frac{dz}{z} \wedge \frac{d\overline{z}}{\overline{z}}$. Then ω is a rapidly decreasing differential form but

$$\int_{\Delta^*} \frac{1}{z}\omega = \int_{\Delta^*} \frac{dz \wedge d\overline{z}}{|z|^2}$$

does not converge, implying that T cannot be evaluated at the form ω. The case of higher dimension reduces to the case of dimension 1. This finishes the proof of the proposition. □

4.5. **Application to DB-cohomology.** We now apply the results obtained in the last section to give some useful descriptions of DB-cohomology. We recall that DB-cohomology is defined as

$$\mathbf{R}(p)_{\mathcal{D}} := \mathrm{cone}(Rj_*\mathbf{R}(p) \oplus F^p\Omega^*_{\overline{X}}(\log D) \to Rj_*\Omega^*_X)[-1]. \tag{11}$$

Let $\mathscr{A}^*_{si}(\overline{X})$, resp. $\mathscr{A}^*_{si,\mathbf{R}(p-1)}(\overline{X})$, denote the global sections of the complex of sheaves $\mathscr{A}^*_{si}$, resp. $\mathscr{A}^*_{si,\mathbf{R}(p-1)}$.

Proposition 4.21. *There is a quasi-isomorphism*

$$\mathbf{R}(p)_{\mathcal{D}} \simeq \mathrm{cone}(F^p\mathscr{A}^*_{si}(\overline{X}) \to \mathscr{A}^*_{si,\mathbf{R}(p-1)}(\overline{X}))[-1],$$

where the arrow is induced by the projection $\pi_{p-1} : \mathbf{C} \to \mathbf{R}(p-1)$ defined by $\pi_{p-1}(z) = \frac{z+(-1)^{p-1}\overline{z}}{2}$. In particular, we have canonical isomorphisms

$$H^n_{\mathcal{D}}(X, \mathbf{R}(p)) \simeq \frac{\{(\phi, \phi') \in F^p\mathscr{A}^n_{si}(\overline{X}) \oplus \mathscr{A}^{n-1}_{si,\mathbf{R}(p-1)}(\overline{X}) \,|\, d\phi = 0, d\phi' = \pi_{p-1}(\phi)\}}{d(\tilde{\phi}, \tilde{\phi}')}, \tag{12}$$

Proof. This follows from Proposition 4.7 and Proposition 4.8. □

Remark 4.22. Let $r_{1,1} : H^1_{\mathcal{M}}(X, \mathbf{Q}(1)) \to H^1_{\mathcal{D}}(X, \mathbf{R}(1))$ be Beilinson's regulator. Recall the canonical isomorphism $\mathcal{O}(X)^\times \otimes \mathbf{Q} \simeq H^1_{\mathcal{M}}(X, \mathbf{Q}(1))$. Then for $u \in \mathcal{O}(X)^\times$, the Deligne cohomology class $r_{1,1}(u \otimes 1)$ is represented by $(d\log(u), \log|u|) \in F^1\mathscr{A}^1_{si}(\overline{X}) \oplus \mathscr{A}^0_{si,\mathbf{R}(0)}(\overline{X})$.

The following result is the key to the calculations of this article.

Theorem 4.23. *We have*

$$\mathbf{R}(p)_{\mathcal{D}} = \operatorname{cone}\left(F^p\mathscr{D}^*(\overline{X}) \to \mathscr{D}^*_{\mathbf{R}(p-1)}(\overline{X})\right)[-1].$$

In particular, we have

$$H^n_{\mathcal{D}}(X, \mathbf{R}(p)) = \frac{\{(S,T) : dS = 0, dT = \pi_{p-1}(S)\}}{d(\widetilde{S}, \widetilde{T})},$$

where $(S,T) \in F^p\mathscr{D}^n(\overline{X}) \oplus \mathscr{D}^{n-1}_{\mathbf{R}(p-1)}(\overline{X})$ *and* $d(\widetilde{S}, \widetilde{T}) = (dS, dT - \pi_{p-1}(S))$.

Proof. This is a consequence of Proposition 4.7 and Theorem 4.20. □

In what follows, for $(S,T) \in F^p\mathscr{D}^n(\overline{X}) \oplus \mathscr{D}^{n-1}_{\mathbf{R}(p-1)}(\overline{X})$ such that $dS = 0$ and $dT = \pi_{p-1}(S)$, we will denote by $[(S,T)] \in H^n_{\mathcal{D}}(X, \mathbf{R}(p))$ the cohomology class of the pair (S,T).

Proposition 4.24. *Let* $x \in H^n_{\mathcal{D}}(X/\mathbf{R}, \mathbf{R}(n))$ *be a Deligne–Beilinson cohomology class which is represented, via the isomorphism of Proposition 4.21, by a pair* (ϕ, ϕ') *of smooth slowly increasing differential forms. Then via the isomorphism of Proposition 4.23, the class* x *is represented by the pair of currents* $(T_\phi, T_{\phi'})$.

Proof. This is immediate from Proposition 4.21 and the quasi-isomorphisms of Theorem 4.20. □

We also need to recall the functoriality of DB-cohomology for proper morphisms. Let $f : X' \to X$ be a proper morphism of pure relative codimension c. Let $\overline{X'}$ denote a smooth compactification of X' such that $Y' = \overline{X'} - X'$ is a simple normal crossing divisor. Assume further that f extends to a morphism $\overline{X'} \to \overline{X}$ that we still denote by f and such that $f^{-1}(Y) = Y'$. By Poincaré duality between DB-cohomology [26, Theorem 1.15] and homology and covariance of DB-homology by proper maps, one has a functorial map

$$f_* : H^n_{\mathcal{D}}(X', \mathbf{R}(p)) \to H^{n+2c}_{\mathcal{D}}(X, \mathbf{R}(p+c)).$$

If T is a tempered current on $\overline{X}'$ of type (p,q) then the formula

$$f_*T(\omega) = T(f^*\omega)$$

defines an element $f_*T \in \mathscr{D}^{p+c,q+c}(\overline{X})$.

Proposition 4.25. *Via the isomorphism of Theorem 4.23, we have*

$$f_*([(T,T')]) = [(f_*T, f_*T')].$$

Proof. The morphism f_* on Betti cohomology coincides by construction with the usual description of the pushforward given by Poincaré duality and covariance of Betti homology for proper morphisms. The result then follows from [26] or [8, §5.5], precisely the discussion after Corollary 5.49 of *loc. cit.*. □

We conclude with the construction of a linear form on DB-cohomology associated to certain rapidly decreasing differential forms.

Proposition 4.26. *Let $\omega \in \mathscr{A}^n_{rd}(\overline{X})$ be a smooth closed rapidly decreasing differential form of Hodge type components inside $\{(a,b) : a < p\}$. Then the assignment $(S,T) \mapsto T(\omega)$ induces a map*

$$\langle -, \omega \rangle : H^{n+1}_{\mathcal{D}}(X, \mathbf{R}(p)) \to \mathbf{C}.$$

Proof. By Theorem 4.23 we have

$$H^{n+1}_{\mathcal{D}}(X, \mathbf{R}(p)) = \{(S,T) \in F^p\mathscr{D}^{n+1}(\overline{X}) \oplus \mathscr{D}^n_{\mathbf{R}(p-1)}(\overline{X})\}/\sim .$$

In order to show that the linear form $(S,T) \mapsto T(\omega)$ is well defined at the level of cohomology, we need to see that it vanishes at any coboundary. Let $(\tilde{S}, \tilde{T}) \in F^p\mathscr{D}^{n+1}(\overline{X}) \oplus \mathscr{D}^n_{\mathbf{R}(p-1)}(\overline{X})$. We have $d(\tilde{S}, \tilde{T}) = (d\tilde{S}, d\tilde{T} - \pi_{p-1}(\tilde{S}))$ and we need to check that $(d\tilde{T} - \pi_{p-1}(\tilde{S}))(\omega) = 0$. We have

$$d\tilde{T}(\omega) = -\tilde{T}(d\omega) = 0$$

since ω is closed. Moreover, $\tilde{S} \in F^p\mathscr{D}^{n+1}(\overline{X})$, which implies that $\tilde{S}$ vanishes on forms of type (a,b) with $a < p$, whence the result. □

5. Archimedean regulators and non-critical values of the Spin L-function

We now apply the general results of last section to the elements constructed in Definition 2.3. In this section, we fix neat levels U and $V = \mathbf{H}(\mathbf{A}_f) \cap U$ and smooth toroidal compactifications $\overline{\mathrm{Sh}_{\mathbf{G}}(U)}$ and $\overline{\mathrm{Sh}_{\mathbf{H}}(V)}$ such that the boundaries are normal crossing divisors. By [20, Proposition 3.4], it is possible to extend ι to a morphism $\overline{\mathrm{Sh}_{\mathbf{H}}(V)} \to \overline{\mathrm{Sh}_{\mathbf{G}}(U)}$.

5.1. Integral expression for the pairing.

Let

$$\mathrm{Eis}_{\mathcal{M}} : \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}})^{V_1} \to H^7_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)\big)$$

denote the morphism defined in Definition 2.3. Recall that it is defined as the composite

$$\begin{aligned} \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}})^{V_1} &\xrightarrow{\ u\ } H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}(V_1), \overline{\mathbf{Q}}(1)) \\ &\xrightarrow{p^*_{1,\mathcal{M}}} H^1_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{H}}(V), \overline{\mathbf{Q}}(1)\big) \\ &\xrightarrow{\iota_{*,\mathcal{M}}} H^7_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)\big). \end{aligned}$$

Let

$$r_{\mathcal{D}} : H^7_{\mathcal{M}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4)\big) \to H^7_{\mathcal{D}}\big(\mathrm{Sh}_{\mathbf{G}}(U), \mathbf{R}(4)\big) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$$

denote Beilinson higher regulator.

Lemma 5.1. *Let $\Phi_f \in \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}})^{V_1}$. Then, via the isomorphisms given by Theorem 4.23, the cohomology class $r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f))$ is represented by the pair of tempered currents*

$$(\iota_* T_{p_1^* \log|u(\Phi_f)|}, \iota_* T_{p_1^* d\log u(\Phi_f)}).$$

Proof. According to [26, §3.7], the regulator maps are morphisms between twisted Poincaré duality theories. As a consequence, we have the commutative diagram

$$\begin{array}{ccccc} H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathrm{GL}_2}(V_1), \overline{\mathbf{Q}}(1)) & \xrightarrow{p^*_{1,\mathcal{M}}} & H^1_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{H}}(V), \overline{\mathbf{Q}}(1)) & \xrightarrow{\iota_{*,\mathcal{M}}} & H^{d+1}_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(t)) \\ {\scriptstyle r_{\mathcal{D}}}\big\downarrow & & {\scriptstyle r_{\mathcal{D}}}\big\downarrow & & {\scriptstyle r_{\mathcal{D}}}\big\downarrow \\ H^1_{\mathcal{D}}(\mathrm{Sh}_{\mathrm{GL}_2}(V_1)/\mathbf{R}, \mathbf{R}(1)) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} & \xrightarrow{p^*_{1,\mathcal{D}}} & H^1_{\mathcal{D}}(\mathrm{Sh}_{\mathbf{H}}(V)/\mathbf{R}, \mathbf{R}(1)) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} & \xrightarrow{\iota_{*,\mathcal{D}}} & H^{d+1}_{\mathcal{D}}(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathbf{R}(t)) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}. \end{array}$$

Via the isomorphism of Proposition 4.21, the morphism $p^*_{1,\mathcal{D}}$ is induced by the pullback of differential forms. The statement of the Lemma follows from Remark 4.22, Proposition 4.24 and Proposition 4.25. □

Let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character such that $\pi_f^U \neq 0$. Writing $\pi_\infty|_{\mathbf{G}_0(\mathbf{R})} \simeq \pi_\infty^1 \oplus \overline{\pi}_\infty^1$, we assume that

$$\mathrm{Hom}_{K_\infty}\left(\bigwedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes_{\mathbf{C}} \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^-, \pi_\infty^1\right) \neq 0.$$

Following Lemma 3.4, we consider a cusp form $\Psi = \Psi_\infty \otimes \Psi_f$ in the space of π such that Ψ_∞ is a highest weight vector of the minimal K_∞-type τ_∞^1 of π_∞^1 and such that Ψ_f is a non-zero vector in π_f^U and we let ω_Ψ be the associated harmonic cuspidal differential form. Analogously we do it for $\overline{\pi}_\infty^1$ at the place of π_∞^1.

Theorem 5.2. *Let $\Phi_f \in \mathcal{S}_0(\mathbf{A}_f, \overline{\mathbf{Q}})^{V_1}$. Then, we have*

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M},n}(\overline{\Phi})), \omega_\Psi\rangle = \int_{\mathrm{Sh}_{\mathbf{H}}(V)} p_1^* \log|u(\Phi_f)| \iota^* \omega_\Psi.$$

Proof. According to Lemma 5.1, the DB-cohomology class $r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M},n}(\overline{\Phi})) \in H^7_{\mathcal{D}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathbf{R}(4)) \otimes \overline{\mathbf{Q}}$ is represented by the pair of tempered currents $(\iota_* T_{p_1^* \log|u(\Phi_f)|}, \iota_* T_{p_1^* d\log u(\Phi_f)})$. It follows from [24, Proposition 2.6.1] that the differential form ω_Ψ is rapidly decreasing. As it is moreover of Hodge type $(3,3)$, by Proposition 4.26 we have

$$\begin{aligned}\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M},n}(\overline{\Phi})), \omega_\Psi\rangle &= \iota_* T_{p_1^* \log|u(\Phi_f)|}(\omega_\Psi) \\ &= T_{p_1^* \log|u(\Phi_f)|}(\iota^* \omega_\Psi) \\ &= \int_{\mathrm{Sh}_{\mathbf{H}}(V)} p_1^* \log|u(\Phi_f)| \iota^* \omega_\Psi.\end{aligned}$$

This finishes the proof. □

5.2. **The adelic integrals.** We now use Kronecker limit formula to rewrite Theorem 5.2 in terms of values of adelic Eisenstein series. Throughout the section, we let π be a cuspidal automorphic representation as in §5.1.

Fix the choice of a measure on $\mathbf{H}(\mathbf{A})$ as follows. For each finite place p of $\mathbf{Q}$, we take the Haar measure dh_p on $\mathbf{H}(\mathbf{Q}_p)$ that assigns volume one to $\mathbf{H}(\mathbf{Z}_p)$. For the archimedean place, we fix $X_0 := (X_{2e_1} \wedge X_{2e_2} \wedge X_{2e_3}) \otimes (X_{-2e_1} \wedge X_{-2e_2} \wedge X_{-2e_3}) \in \bigwedge^6 \mathfrak{h}_{\mathbf{C}}/\mathfrak{k}_{\mathbf{H},\mathbf{C}} = \wedge^3 \mathfrak{p}_{\mathbf{H},\mathbf{C}}^+ \otimes \wedge^3 \mathfrak{p}_{\mathbf{H},\mathbf{C}}^- \subseteq \wedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes \wedge^3 \mathfrak{p}_{\mathbf{C}}^-$ as the basis of the 1-dimensional subspace $\wedge^6 \mathfrak{p}_{\mathbf{H},\mathbf{C}}$ of $\wedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes \wedge^3 \mathfrak{p}_{\mathbf{C}}^-$ [2]. Such a choice of basis induces an equivalence between top differential ω forms on $X_{\mathbf{H}} = \mathbf{H}(\mathbf{R})/K_{\mathbf{H},\infty}$ and invariant measures $d_\omega h_\infty$ on $\mathbf{H}(\mathbf{R})$ assigning measure one to $K_{\mathbf{H},\infty}$ (cf. [22, p. 83] for details). We then define $dh = d_\omega h_\infty \prod_p dh_p$.

Let $A \in \mathcal{U}(\mathfrak{h})$ be the operator such that $\mathrm{pr}_{\tau_{(2,2,-4)}}(X_0) = A.X_{(2,2,-4)}$, where $\mathrm{pr}_{\tau_{(2,2,-4)}} : \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes_{\mathbf{C}} \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^- \to \tau_{(2,2,-4)}$ denotes the projector to the irreducible factor $\tau_{(2,2,-4)}$ (cf. Lemma 3.1). We give an explicit formula for A in Lemma 5.4 below.

After applying Proposition 2.1 and Lemma 5.1, Theorem 5.2 implies the following.

Theorem 5.3. *Let $\Phi_f \in \mathcal{S}_0(\mathbf{A}_f^2, \overline{\mathbf{Q}})^{V_1}$ and let ω_Ψ be as in Theorem 5.2. We have*

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f)), \omega_\Psi\rangle = \frac{h_V}{\mathrm{vol}(V)} \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} E(h_1, \Phi, 0)(A.\Psi)(h) dh, \tag{13}$$

where $h_V = 2^{-2}|\mathbf{Z}_{\mathbf{G}}(\mathbf{Q})\backslash \mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f)/(\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) \cap V)|$.

[2] by a slight abuse of language, we denote by X_0 the vector in $\wedge^6 \mathfrak{p}_{\mathbf{H},\mathbf{C}}$ as well as its image in $\wedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes \wedge^3 \mathfrak{p}_{\mathbf{C}}^-$ by ι, this should cause no confusion

Proof. Recall that Theorem 5.2 gives

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f)), \omega_\Psi\rangle = \int_{\mathrm{Sh}_{\mathbf{H}}(V)} \xi \wedge \iota^*\omega_\Psi,$$

where, by Lemma 5.1 and Proposition 2.1, $\xi = \mathrm{pr}_1^* \log|u(\Phi_f)| = \mathrm{pr}_1^* E(g,\Phi,0)$. Thanks to the equivalence between top differential forms on $X_{\mathbf{H}}$ and invariant measures on $\mathbf{H}(\mathbf{R})$ explained above, we pass from integrating over $\mathrm{Sh}_{\mathbf{H}}(V)$ to integrating over $\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash\mathbf{H}(\mathbf{A})$. More precisely, we have

$$\begin{aligned}
\int_{\mathrm{Sh}_{\mathbf{H}}(V)} \xi \wedge \iota^*\omega_\Psi &= \int_{\mathbf{H}(\mathbf{Q})\backslash\mathbf{H}(\mathbf{A})/\mathbf{Z}_{\mathbf{H}}(\mathbf{R})K_{\mathbf{H},\infty}V} E(h_1,\Phi,0)\omega_\Psi(X_0)(h)dh \\
&= h_V \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash\mathbf{H}(\mathbf{A})/V} E(h_1,\Phi,0)\omega_\Psi(X_0)(h)dh \\
&= \frac{h_V}{\mathrm{vol}(V)} \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash\mathbf{H}(\mathbf{A})} E(h_1,\Phi,0)\omega_\Psi(X_0)(h)dh,
\end{aligned}$$

where we have used that $|\mathbf{Z}_{\mathbf{H}}(\mathbf{R})/(\mathbf{Z}_{\mathbf{G}}\cap\mathbf{H})(\mathbf{R})| = 2^2$. Finally, note that $\omega_\Psi(X_0)(h) = (A.\Psi)(h)$ by definition of ω_Ψ. This proves the desired formula. □

We finish this section by giving an explicit formula for the projector A of the theorem.

Lemma 5.4. *Up to renormalizing X_0 by an explicit non-zero rational factor, the projections of X_0 to $\tau_{(2,2,-4)}$ and $\tau_{(4,-2,-2)}$ are given, respectively, by $A.X_{(2,2,-4)}$ and $A'.X_{(4,-2,-2)}$, where $A = \mathrm{Ad}^2_{X_{e_3-e_2}} \circ \mathrm{Ad}^2_{X_{e_3-e_1}}$, $A' = \mathrm{Ad}^2_{X_{e_2-e_1}} \circ \mathrm{Ad}^2_{X_{e_3-e_1}}$.*

Proof. Recall that X_0 is a weight $(0,0,0)$-vector (with respect to the action of $\mathfrak{h}_{\mathbf{C}}$) in $\wedge^3\mathfrak{p}^+_{\mathbf{C}}\otimes\wedge^3\mathfrak{p}^-_{\mathbf{C}}$. Thus, we may write $X_0 = Y \oplus \alpha x_{(2,2,-4)}$, where Y belongs to the complement of $\tau_{(2,2,-4)}$ in the decomposition of $\wedge^3\mathfrak{p}^+_{\mathbf{C}}\otimes\wedge^3\mathfrak{p}^-_{\mathbf{C}}$ as the sum of its weight subspaces (cf. Example 3.1), the vector $x_{(2,2,-4)}$ is a generator of the one dimensional weight $(0,0,0)$-eigenspace of $\tau_{(2,2,-4)}$, and α is a scalar. We can assume $x_{(2,2,-4)} = \mathrm{Ad}^2_{X_{e_3-e_2}} \circ \mathrm{Ad}^2_{X_{e_3-e_1}}(X_{(2,2,-4)})$, where $X_{(2,2,-4)} = X_{2e_1}\wedge X_{2e_2}\wedge X_{e_1+e_2}\otimes X_{-e_1-e_3}\wedge X_{-e_2-e_3}\wedge X_{-2e_3}$ is a highest weight vector for $\tau_{(2,2,-4)}$. Since the the weight $(2,2,-4)$ has multiplicity one in $\wedge^3\mathfrak{p}^+_{\mathbf{C}}\otimes\wedge^3\mathfrak{p}^-_{\mathbf{C}}$, we have that $\mathrm{Ad}^2_{X_{e_1-e_3}} \circ \mathrm{Ad}^2_{X_{e_2-e_3}}(Y) = 0$ and hence

$$\mathrm{Ad}^2_{X_{e_1-e_3}} \circ \mathrm{Ad}^2_{X_{e_2-e_3}}(X_0) = \alpha\, \mathrm{Ad}^2_{X_{e_1-e_3}} \circ \mathrm{Ad}^2_{X_{e_2-e_3}} \circ \mathrm{Ad}^2_{X_{e_3-e_2}} \circ \mathrm{Ad}^2_{X_{e_3-e_1}}(X_{(2,2,-4)}).$$

A direct computation [3] shows that

- $\mathrm{Ad}^2_{X_{e_1-e_3}} \circ \mathrm{Ad}^2_{X_{e_2-e_3}}(X_0) = 2^6 X_{(2,2,-4)}$,
- $\mathrm{Ad}^2_{X_{e_1-e_3}} \circ \mathrm{Ad}^2_{X_{e_2-e_3}} \circ \mathrm{Ad}^2_{X_{e_3-e_2}} \circ \mathrm{Ad}^2_{X_{e_3-e_1}}(X_{(2,2,-4)}) = 2^{10}3^2 5^2 X_{(2,2,-4)}$.

Therefore, the projection of X_0 to $\tau_{(2,2,-4)}$ is $\frac{1}{3600}x_{(2,2,-4)}$. The other projection follows (with the same coefficient) by applying the action of complex conjugation. This finishes the proof of the Lemma. □

Finally, we use the main result of [42] to write the adelic integral calculating the archimedean regulator in terms of a special value of a Spin L-function for GSp_6.

[3] The authors have found Sage package for Lie groups very useful for these computations.

5.3. **The Spin L-function.** Recall the following.

Definition 5.5. *For a character χ of $\mathbf{Q}_\ell^\times$, define*

$$L(\chi, s) := \begin{cases} (1-\chi(\ell)\ell^{-s})^{-1} & \text{if } \chi_{|_{\mathbf{Z}_\ell^\times}} = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\chi_0, \chi_1, \chi_2, \chi_3$ be smooth characters of $\mathbf{Q}_\ell^\times$. They define an unramified character $\underline{\chi}$ of the Borel $B_\ell = T_\ell U_{B,\ell}$ of $\mathbf{G}(\mathbf{Q}_\ell)$, which is trivial on the unipotent radical $U_{B,\ell}$, and on the diagonal torus T_ℓ is

$$d = diag(a, b, c, \mu a^{-1}, \mu b^{-1}, \mu c^{-1}) \mapsto \underline{\chi}(d) := \chi_1(a)\chi_2(b)\chi_3(c)\chi_0(\mu).$$

The modular character of the Borel subgroup $\delta_{B_\ell} : T_\ell \to \mathbf{C}$ is given by

$$diag(a, b, c, \mu a^{-1}, \mu b^{-1}, \mu c^{-1}) \mapsto \tfrac{|a|^6|b|^4|c|^2}{|\mu|^6}.$$

Definition 5.6. *The (normalized) principal series representation $\pi_\ell(\underline{\chi}) = \pi_\ell(\chi_0, \chi_1, \chi_2, \chi_3)$ is the representation of $\mathbf{G}(\mathbf{Q}_\ell)$ whose underlying vector space is the space of functions $f : \mathbf{G}(\mathbf{Q}_\ell) \to \mathbf{C}$ satisfying*

$$f(dng) = \tfrac{|a|^3|b|^2|c|}{|\mu|^3}\underline{\chi}(d)f(g),$$

for every $d = diag(a, b, c, \mu a^{-1}, \mu b^{-1}, \mu c^{-1})$ and $u \in U_{B,\ell}$, and where the action of $\mathbf{G}(\mathbf{Q}_\ell)$ is given by right-translation.

Definition 5.7. *Let $\pi_\ell = \pi_\ell(\underline{\chi})$ be an irreducible principal series. It's Spin L-factor is defined as*

$$L(\pi_\ell, \mathrm{Spin}, s) := L(\chi_0, s) \prod_{k=1}^{3} \prod_{1 \le i_1 < \cdots < i_k \le 3} L(\chi_0\chi_{i_1} \cdots \chi_{i_k}, s).$$

Now let $\pi = \pi_\infty \otimes \bigotimes'_\ell \pi_\ell$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ and let Σ be a finite set of primes containing all the bad finite primes for π.

Definition 5.8. *The partial Spin L-function of π is defined as*

$$L^\Sigma(\pi, \mathrm{Spin}, s) := \prod_{\ell \notin \Sigma} L(\pi_\ell, \mathrm{Spin}, s).$$

5.4. **The integral representation of the Spin L-function.** Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character. Denote by $I(\Phi, \Psi, s)$ the integral

$$\int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_\mathbf{G}(\mathbf{A})\backslash\mathbf{H}(\mathbf{A})} E(h_1, \Phi, s)\Psi(h)dh,$$

where $\Phi \in \mathcal{S}(\mathbf{A}^2)$ and Ψ is a cusp form in the space of π. Here, we assume that Φ and Ψ are factorizable.

5.4.1. *Fourier coefficients of type* $(4\,2)$. Let $\mathcal{O}$ be the unipotent orbit of $\mathbf{G}$ associated to the partition $(4\,2)$. To such $\mathcal{O}$ one can define a set of Fourier coefficients as follows. Denote by $h_\mathcal{O}$ the one-dimensional torus

$$t \mapsto \mathrm{diag}(t^3, t, t, t^{-3}, t^{-1}, t^{-1})$$

attached to $\mathcal{O}$ (cf. [11, p. 82]). Given any positive root α (for the action of the diagonal torus of $\mathbf{G}$), there is a non-negative integer n such that

$$h_\mathcal{O}(t)x_\alpha(u)h_\mathcal{O}(t)^{-1} = x_\alpha(t^n u),$$

where x_α denotes the one-parameter subgroup associated to α. Let $U_2(\mathcal{O})$ denote the subgroup of the unipotent radical U_B of the standard Borel B of $\mathbf{G}$ generated by the x_α such that $n \geq 2$. If $\alpha \neq e_2' - e_3'$, then $n \geq 2$, thus $U_2(\mathcal{O})$ coincides with the unipotent radical U_P of the standard parabolic P with Levi $\mathrm{GL}_1^2 \times \mathrm{GL}_2$, given by

$$\left\{ \begin{pmatrix} a & & & \\ & g & & \\ & & \mu a^{-1} & \\ & & & \mu^t g^{-1} \end{pmatrix} \ : \ a, \mu \in \mathrm{GL}_1, \ g \in \mathrm{GL}_2 \right\}.$$

Let $L = \mathbf{Q}(\sqrt{D})$ be an étale quadratic extension of $\mathbf{Q}$ and let $\chi : U_P(\mathbf{Q})\backslash U_P(\mathbf{A}) \to \mathbf{C}^\times$ be the non-degenerate unitary character associated to L as in [42, §2.1].

Definition 5.9. *Let Ψ be a cusp form in the space of π. Define the Fourier coefficient*

$$\Psi_{\chi,U_P}(g) := \int_{U_P(\mathbf{Q})\backslash U_P(\mathbf{A})} \chi^{-1}(u)\Psi(ug)du.$$

We say that π supports a Fourier coefficients of type (4 2) if $\Psi_{\chi,U_P} \neq 0$ for some choice of Ψ and L. Our main results will concern automorphic cuspidal representations π supporting a non-zero Fourier coefficient of type (4 2). The following lemma shows that, under some mild assumptions, the cuspidal automorphic representations that we consider in this article always support such a non-zero coefficient.

Lemma 5.10. *Let $\pi = \pi_\infty \otimes \pi_f$ be an automorphic cuspidal representation of $\mathbf{G}(\mathbf{A})$ with trivial central character and such that $\pi_\infty = \pi_\infty^{3,3} \oplus \overline{\pi}_\infty^{3,3}$ (cf. Lemma 3.3). Assume that π does not admit a non-trivial cuspidal theta lift to the split orthogonal group $\mathrm{SO}_{12}(\mathbf{A})$. Then π admits a non-zero Fourier coefficient of type* (4 2).

Proof. We first observe that, by [18, Theorem 2.7], π admits a non-zero Fourier coefficient of type (6), (4 2) or (2 2 2). The former case corresponds to π being generic. We will show that, according to our Hodge type and the assumption of the lemma, π cannot have a Fourier coefficient of either type (6) or (2 2 2).

We first show that, since π_∞ is of Hodge type $(3,3)$, it cannot be generic. Recall that, after [47, Proposition 6.19], one can read the Hodge type of a discrete series representation defined by an element $w \in \mathfrak{W}_{\mathrm{Sp}_6}/\mathfrak{W}_{K_\infty}$ (notations are as in §3.3) as follows. The element w induces a reordering of the roots and the Hodge type of the discrete series representation (and hence of the cuspidal form π) is (p,q), where p (resp. q) is the number of positive (resp. negative) simple non-compact roots. Moreover, it is well known (cf. for instance [36, Proposition 4.1]) that π is generic exactly when $w = w_3, w_6$ in the notation of the proof of Lemma 3.3, which are of Hodge type $(4,2)$ and $(2,4)$. We conclude that π cannot be generic, i.e. it does not support a Fourier coefficient of type (6).

Now, suppose that π admits a non-zero Fourier coefficient of type (2 2 2). By [17, Main Theorem B], π lifts to a non-trivial cuspidal representation on $\mathrm{SO}_{12}(\mathbf{A})$. This contradicts our hypotheses, thus proving the result. □

Remark 5.11. Observe that [17, Main Theorem A] implies that, if π supports a Fourier coefficient of type (4 2), then it admits a non-trivial theta lift to $\mathrm{SO}_{12}(\mathbf{A})$, but this lift will not be cuspidal.

5.4.2. *The unfolding.* Recall the following.

Proposition 5.12 ([16], Proposition 7.1)**.** *The integral $I(\Phi, \Psi, s)$ unfolds to*

$$\int_{U_{B_{\mathbf{H}}}(\mathbf{A})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} f(h_1, \Phi, s)\Psi_{\chi,U_P}(h)dh,$$

where $U_{B_{\mathbf{H}}}$ is the unipotent radical of the Borel $B_{\mathbf{H}}$ of $\mathbf{H}$ and $f(h_1, \Phi, s)$ is the normalised section defined in §2.3.1.

As explained in [42], the Fourier coefficient Ψ_{χ,U_P} might not factorise in general, thus Proposition 5.12 does not imply that $I(\Phi, \Psi, s)$ factors into an Euler product; however, in [42], the authors define and study local integrals corresponding to this unfolded integral, and use them to relate the global integral $I(\Phi, \Psi, s)$ to values of the Spin L-function of π, as we now recall.

5.4.3. *Connection with values of the Spin L-function.* Recall that we have taken $\pi = \pi_\infty \otimes (\otimes_p \pi_p)$ to be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character. We further suppose that π supports a Fourier coefficient of type (4 2), i.e. that there is a cusp form Ψ in the space of π such that Ψ_{χ,U_P} is not identically zero.

We now recall following [42] the definition of the local integrals corresponding to $I(\Phi, \Psi, s)$ and their properties. We start with the following definition.

Definition 5.13. *A (U_P, χ)-model for π_p is a linear functional $\Lambda : \pi_p \to \mathbf{C}$ such that*

$$\Lambda(u \cdot v) = \chi(u)\Lambda(v),$$

for all $v \in \pi_p$ and $u \in U_P$.

For a (U_P, χ)-model Λ for π_p, a vector v in the space of π_p, and $\Phi_p \in \mathcal{S}(\mathbf{Q}_p^2, \mathbf{C})$, define

$$I_p(\Phi_p, v, s) := \int_{U_{B_{\mathbf{H}}}(\mathbf{Q}_p)\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}_p)\backslash \mathbf{H}(\mathbf{Q}_p)} f(g_1, \Phi_p, s)\Lambda(g \cdot v)dg,$$

where $f(g, \Phi_p, s) \in \mathrm{Ind}_{B_{\mathrm{GL}_2}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\delta_{B_{\mathrm{GL}_2}}^s)$ denotes the function

$$|\det(g)|^s \int_{\mathrm{GL}_1(\mathbf{Q}_p)} \Phi_p((0,t)g)|t|^{2s}dt.$$

One has the following.

Proposition 5.14 ([42] Theorem 1.1, Proposition 5.1)**.**

(1) *If p is a finite odd prime and π_p is unramified, let $v_0 \in \pi_p$ be a spherical vector and let $\Phi_p = \mathrm{char}(\mathbf{Z}_p^2)$; then, for any (U_P, χ)-model Λ for π_p, we have*

$$I_p(\Phi_p, v_0, s) = \Lambda(v_0)L(\pi_p, \mathrm{Spin}, s).$$

(2) *If π_p is ramified and v_0 is a vector in π_p, then there exists a vector v in π_p and a function $\Phi_p \in \mathcal{S}_0(\mathbf{Q}_p^2, \overline{\mathbf{Q}})$ such that for all (U_P, χ)-models Λ*

$$I_p(\Phi_p, v, s) = \Lambda(v_0).$$

Remark 5.15. As explained in the proof of [42, Proposition 5.1], in the case of a finite bad place p, one can choose Φ_p to be $\mathrm{char}((0,1) + p^n\mathbf{Z}_p)$, with n a suitable positive integer depending on v_0.

Finally, consider the archimedean integral

$$I_\infty(\Phi_\infty, \Psi, s) := \int_{U_{B_{\mathbf{H}}}(\mathbf{R})\mathbf{Z}_{\mathbf{G}}(\mathbf{R})\backslash \mathbf{H}(\mathbf{R})} f(h_1, \Phi_\infty, s)\Psi_{\chi,U_P}(h)dh. \tag{14}$$

This integral has been studied in [16], where it is shown that it can be made non-zero at arbitrary $s = s_0$ if one has some freedom on the choice of Φ_∞ and Ψ_∞.

5.5. **The regulator computation.** We now state and prove one of the main results of this manuscript.

Let $\pi = \pi_\infty \otimes (\otimes_p' \pi_p)$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character, such that

(DS) π_∞ is a discrete series whose restriction to $\mathrm{Sp}_6(\mathbf{R})$ is $\pi_\infty^{3,3} \oplus \bar{\pi}_\infty^{3,3}$,

(FC) there exists a factorizable cusp form $\Psi = \Psi_\infty \otimes \Psi_f$ in the space of π which admits a non-zero Fourier coefficient of type (4 2) associated to a real étale quadratic extension $L/\mathbf{Q}$, and such that Ψ_∞ is a highest weight vector of the minimal K_∞-type of $\pi_\infty^{3,3}$.

Remark 5.16. In view of Lemma 5.10, the existence of a Fourier coefficient of type (4 2) might be partly replaced by asking that π lifts to a non-cuspidal automorphic representation of the split $\mathbf{SO}_{12}(\mathbf{A})$. Furthermore, notice that if π does not support such a Fourier coefficient, the following results will still be true but all the terms will vanish.

Let us fix a neat open compact subgroup $U = \prod_p U_p$ of $\mathbf{G}(\mathbf{A}_f)$, for which $\Psi_f \in \pi_f^U$, and $\Phi_f = \otimes_p \Phi_p \in \mathcal{S}_0(\mathbf{A}_f^2, \mathbf{Q})^{V_1}$, with $V_1 = \mathrm{pr}_1(U \cap \mathbf{H})$, which satisfy the following. Let Σ be a finite set of primes containing ∞ and all the bad finite primes for π; for every prime $p \notin \Sigma$, we set $U_p = \mathbf{G}(\mathbf{Z}_p)$ and $\Phi_p = \mathrm{char}(\mathbf{Z}_p^2)$. Then, we have the following.

Theorem 5.17. *Assume (DS), (FC) and that Ψ is invariant under $\mathbf{G}(\mathbf{Z}_p)$ for every $p \notin \Sigma$. Then we have*

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f)), \omega_\Psi \rangle = C_V \lim_{s \to 0} \left(I_\Sigma(\Phi_\Sigma, A.\Psi_\Sigma, s) L^\Sigma(\pi, \mathrm{Spin}, s) \right),$$

where $L^\Sigma(\pi, \mathrm{Spin}, s) = \prod_{p \notin \Sigma} L(\pi_p, \mathrm{Spin}, s)$, and

$$I_\Sigma(\Phi, A.\Psi, s) := \int_{U_{B_{\mathbf{H}}}(\mathbf{A}_\Sigma)\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_\Sigma)\backslash \mathbf{H}(\mathbf{A}_\Sigma)} f(h_1, \Phi_\Sigma, s)(A.\Psi)_{\chi, U_P}(h) dh,$$

the operator A is defined in Lemma 5.4 and $C_V = \frac{h_V}{\mathrm{vol}(V)}$ are as in Theorem 5.3.

Proof. This follows from Theorem 5.3 and Proposition 5.14. For the sake of clarity, we give a sketch of its proof, following [40].

Recall that, from Theorem 5.3 and Proposition 5.12, we have

$$\begin{aligned}
\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f)), \omega_\Psi \rangle &= C_V \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} E(h_1, \Phi, 0)\Psi'(h) dh \\
&= C_V \int_{U_{B_{\mathbf{H}}}(\mathbf{A})\mathbf{Z}_{\mathbf{G}}(\mathbf{A})\backslash \mathbf{H}(\mathbf{A})} f(h_1, \Phi, 0)\Psi'_{\chi, U_P}(h) dh \\
&=: C_V I(\Phi, \Psi', 0),
\end{aligned}$$

where we denote $C_V = \frac{h_V}{\mathrm{vol}(V)}$ and $\Psi' = A.\Psi$.

We now study the quantity $I(\Phi, \Psi', 0)$. Given a finite set of primes S containing Σ, define

$$I_S(\Phi, \Psi', s) := \int_{U_{B_{\mathbf{H}}}(\mathbf{A}_S)\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_S)\backslash \mathbf{H}(\mathbf{A}_S)} f(h_1, \Phi_S, s)\Psi'_{\chi, U_P}(h) dh, \tag{15}$$

where $\Phi_S = \prod_{v \in S} \Phi_v$. Then, in the range of convergence

$$I(\Phi, \Psi', s) = \lim_{\Sigma \subseteq S} I_S(\Phi, \Psi', s).$$

Notice that, by [42, Theorem 2.7], if $p \notin S$,

$$I_{S \cup \{p\}}(\Phi, \Psi', s) = L(\pi_p, \mathrm{Spin}, s) I_S(\Phi, \Psi', s).$$

Indeed, $I_{S\cup\{p\}}(\Phi,\Psi',s)$ equals to

$$\int_{U_{B_{\mathbf{H}}}(\mathbf{A}_S)\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_S)\backslash\mathbf{H}(\mathbf{A}_S)} f(h_1,\Phi_S,s)\left(\int_{U_{B_{\mathbf{H}}}(\mathbf{Q}_p)\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}_p)\backslash\mathbf{H}(\mathbf{Q}_p)} f(h_{1,p},\Phi_p,s)\Psi'_{\chi,U_P}(h_ph)dh_p\right)dh.$$

As $p\notin\Sigma$, π_p is unramified at p. Fix a spherical vector v_0 for π_p; as Ψ' is invariant under the action of $\mathbf{G}(\mathbf{Z}_p)$, there is a $\mathbf{G}(\mathbf{Z}_p)$-equivariant map $\varphi_p:\pi_p\to\pi$ sending v_0 to Ψ'. Thus, for a fixed $h\in\mathbf{G}(\mathbf{A}_S)$, the functional $\Lambda_h:\pi_p\to\mathbf{C}$ defined by $\Lambda_h(v):=\varphi_p(v)_{\chi,U_P}(h)$ is clearly a (U_P,χ)-model and $\Lambda_h(h_p\cdot v_0)=\Psi'_{\chi,U_P}(h_ph)$. Proposition 5.14(1) implies then

$$\begin{aligned}\int_{U_{B_{\mathbf{H}}}(\mathbf{Q}_p)\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}_p)\backslash\mathbf{H}(\mathbf{Q}_p)} & f(h_{1,p},\Phi_p,s)\Psi'_{\chi,U_P}(h_ph)dh_p\\ &= \int_{U_{B_{\mathbf{H}}}(\mathbf{Q}_p)\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}_p)\backslash\mathbf{H}(\mathbf{Q}_p)} f(h_{1,p},\Phi_p,s)\Lambda_h(h_p\cdot v_0)dh_p\\ &= L(\pi_p,\mathrm{Spin},s)\Lambda_h(v_0)\\ &= L(\pi_p,\mathrm{Spin},s)\Psi'_{\chi,U_P}(h),\end{aligned}$$

which implies the desired equality.

Taking the limit varying the set $S\supseteq\Sigma$, we get

$$I(\Phi,\Psi',0)=\lim_{s\to 0}\left(I_\Sigma(\Phi,\Psi',s)L^\Sigma(\pi,\mathrm{Spin},s)\right),$$

which implies the result. □

The following Corollary studies further the finite integral $I_\Sigma(\Phi,A.\Psi,s)$. For this, we fix the components at finite bad primes of our Schwartz function Φ and the level. If $p\in\Sigma$ is a finite place, we let $\Phi_p=\mathrm{char}((0,1)+p^n\mathbf{Z}_p^2)$, and

$$U_p=\{g\in\mathbf{G}(\mathbf{Z}_p)\ :\ g\equiv I\ (\bmod p^n)\},$$

with n the positive integer given by Proposition 5.14(2) and Remark 5.15.

Corollary 5.18. *Let Ψ and ω_Ψ be as in Theorem 5.17. Then, there exist $\widetilde{\Psi}=\Psi_\infty\otimes\widetilde{\Psi}_f$ in π, with Ψ_f and $\widetilde{\Psi}_f$ coinciding outside Σ, such that*

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f)),\omega_{\widetilde{\Psi}}\rangle=C_V\lim_{s\to 0}\left(I_\infty(\Phi_\infty,A.\Psi,s)L^\Sigma(\pi,\mathrm{Spin},s)\right),$$

where

$$I_\infty(\Phi_\infty,A.\Psi,s)=\int_{U_{B_{\mathbf{H}}}(\mathbf{R})\mathbf{Z}_{\mathbf{G}}(\mathbf{R})\backslash\mathbf{H}(\mathbf{R})} f(h_1,\Phi_\infty,s)(A.\Psi)_{\chi,U_P}(h)dh.$$

Proof. Let $p\in\Sigma$ be a finite place. As Ψ is factorizable and π_p irreducible, there exists a $\mathbf{G}(\mathbf{Q}_p)$-equivariant map $\varphi_p:\pi_p\to\pi$, which sends $v_p\mapsto A.\Psi$, where the vector $v_p\in\pi_p$ denotes the local component of $A.\Psi$. By the choice of Φ_p, Proposition 5.14 (2) produces a vector $\widetilde{v}_p$ in π_p which depends on v_p, such that for all (U_P,χ)-models Λ of π_p

$$I_p(\Phi_p,\widetilde{v}_p,s)=\Lambda(v_p).$$

For a fixed $h\in\mathbf{G}(\mathbf{A}_{\Sigma\smallsetminus\{p\}})$, define the functional $\Lambda_h:\pi_p\to\mathbf{C}$ as $\Lambda_h(v):=\varphi_p(v)_{\chi,U_P}(h)$. This is a (U_P,χ)-model for π_p and $\Lambda_h(h_p\cdot v)=\varphi_p(v)_{\chi,U_P}(h_ph)$. We have that, for this model Λ_h,

$$\int_{U_{B_{\mathbf{H}}}(\mathbf{Q}_p)\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}_p)\backslash\mathbf{H}(\mathbf{Q}_p)} f(h_{1,p},\Phi_p,s)\Lambda_h(h_p\cdot\widetilde{v}_p)dh_p=\Lambda_h(v_p)=(A.\Psi)_{\chi,U_P}(h).$$

Now one proceeds similarly to the proof of Theorem 5.17 to use this equality to show that

$$I_\Sigma(\Phi, A.\widetilde{\Psi}, s) = I_\infty(\Phi_\infty, A.\Psi, s),$$

where the cusp form $\widetilde{\Psi}$ coincides with Ψ at ∞ and away from Σ, and has component $\widetilde{v}_p$ at each $p \in \Sigma$. Finally, since $\tilde{\Psi}$ and Ψ coincide almost everywhere as well as at ∞, the result follows from Theorem 5.17.

□

We conclude with two remarks on the non-triviality of the limiting value

$$\lim_{s\to 0} \left(I_\infty(\Phi_\infty, A.\Psi, s) L^\Sigma(\pi, \mathrm{Spin}, s) \right).$$

Remark 5.19. The archimedean integral $I_\infty(\Phi_\infty, A.\Psi, s)$ is expected to be equal to the Gamma factor of the completed Spin L-function as predicted by the rule of Serre. This would imply that it has a pole at $s = 0$ of order equal to the order of vanishing of $L^\Sigma(\pi, \mathrm{Spin}, s)$ at $s = 0$. The analogous expectation in the case of GSp_4 was confirmed in the articles [35], [38].

Remark 5.20. It follows from [16, Proposition 12.1] that the archimedean integral can be made non-zero at arbitrary $s = s_0$ if one has some freedom on the choice of Φ_∞ and Ψ_∞. This shows that the archimedean integral does not vanish identically. However their result is not enough for our purposes as Ψ_∞ is fixed to be in the minimal K_∞-type of $\pi_\infty^{3,3}$.

5.6. **A remark on the non-vanishing of the regulator.** We have the following direct consequence of Theorem 5.17 regarding the non-vanishing of motivic cohomology.

Let N denote the positive integer defined as the product of prime numbers ℓ such that π_ℓ is ramified. The fact that π_∞ is cohomological implies that there exists a number field L whose ring of integers $\mathcal{O}_L$ contains the eigenvalues of the spherical Hecke algebra $\mathcal{H}^{sph,N}$ away from N and with coefficients in $\mathbb{Z}$ acting on $\bigotimes'_{\ell \nmid N} \pi_\ell^{\mathbf{G}(\mathbf{Z}_\ell)}$. We let $\theta_\pi : \mathcal{H}^{sph,N} \to L$ denote the Hecke character of π and we let $\mathfrak{m}_\pi := \ker(\theta_\pi)$.

Corollary 5.21. *We assume that there exists an automorphic representation π of $\mathbf{G}(\mathbf{A})$ and a cusp form Ψ in π, which satisfy all the running assumptions of Theorem 5.17 and such that*

$$\lim_{s\to 0} \left(I_\infty(\Phi_\infty, A.\Psi, s) L^\Sigma(\pi, \mathrm{Spin}, s) \right) \neq 0.$$

Then, the class $\mathrm{Eis}_{\mathcal{M}}(\Phi_f)$ is non-trivial and thus the localisation $H^7_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4))_{\mathfrak{m}_\pi}$ is non-zero.

By assuming spin functoriality for π and by using a generalisation of the prime number theorem due to Jacquet and Shalika [25], we can improve the corollary by establishing that the non-vanishing of our pairing relies on the non-vanishing of the archimedean integral, as we now explain.

Let $\mathrm{Spin} : {}^L\mathbf{G} \to {}^L\mathrm{GL}_8$ denote the homomorphism between L-groups induced by the spin representation $\mathrm{Spin} : \mathrm{GSpin}_7 \to \mathrm{GL}_8$. Roughly, Langlands' functoriality would predict the existence of a *spin lift* to GL_8 for π, i.e. the existence of an automorphic representation Π of $\mathrm{GL}_8(\mathbf{A})$ whose L-parameter ϕ_{Π_v} at each place v is obtained from composing the L-parameter ϕ_{π_v} of π_v with Spin, thus implying that

$$L(\pi, \mathrm{Spin}, s) = L(\Pi, s),$$

where the latter denotes the standard L-function associated to Π.

Remark 5.22. In [31], a *potential* version of spin functoriality is discussed and proved. If we further assume that π_∞ is spin-regular (cf. [31]), the result [31, Theorem C], which builds upon [1, Theorem A], produces a cuspidal automorphic representation Π of $\mathrm{GL}_8(\mathbf{A}_F)$, over a finite totally real extension $F/\mathbf{Q}$, with the desired properties. For instance, at each finite place v of F above an odd prime $p \notin \Sigma$ one has $L(\Pi_v, s) = L(\pi_p, \mathrm{Spin}, s)$.

Corollary 5.23. *Let Ψ and ω_Ψ be as in Theorem 5.17. Suppose that there exists a cuspidal spin lift Π to $\mathbf{GL}_8$ of π and that*

$$I_\infty(\widehat{\Phi}_\infty, A.\Psi, 1) \neq 0,$$

where $\widehat{\Phi}$ denotes the Fourier transform of Φ. Then, the class $\mathrm{Eis}_{\mathcal{M}}(\Phi_f)$ is non-zero and thus the localisation $H^7_{\mathcal{M}}(\mathrm{Sh}_{\mathbf{G}}(U), \overline{\mathbf{Q}}(4))_{\mathfrak{m}_\pi}$ is non-zero.

Proof. Recall that the Eisenstein $\mathrm{E}(g, \Phi, s)$ satisfies the functional equation

$$\mathrm{E}(g, \Phi, s) = \mathrm{E}(g, \widehat{\Phi}, 1-s).$$

This implies that

$$I(\Phi, \Psi, s) = I(\widehat{\Phi}, \Psi, 1-s) = I_\Sigma(\widehat{\Phi}, \Psi, 1-s) L^\Sigma(\pi, \mathrm{Spin}, 1-s).$$

Since $L^\Sigma(\pi, \mathrm{Spin}, s) = L^\Sigma(\Pi, s)$, by Corollary 5.18, we get that

$$\langle r_{\mathcal{D}}(\mathrm{Eis}_{\mathcal{M}}(\Phi_f)), \omega_{\widetilde{\Psi}} \rangle =_{\overline{\mathbf{Q}}^\times} \lim_{s \to 1} \left(I_\infty(\widehat{\Phi}_\infty, A.\Psi, s) L^\Sigma(\Pi, s) \right).$$

We now claim that $L^\Sigma(\Pi, 1) \neq 0$. In [25], it is shown that $L(\Pi, s) \neq 0$ for any s with $\mathrm{Re}(s) = 1$. Writing

$$L^\Sigma(\Pi, s) = L(\Pi, s) \prod_{p \in \Sigma} L(\Pi_p, s)^{-1},$$

our claim follows from the fact that each Euler factor $L(\Pi_p, s)$ has no pole in the region $\mathrm{Re}(s) \geq \frac{1}{2}$ (e.g. [44, p. 317]). The result follows. □

References

[1] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math., 179 (2014), pp. 501–609.

[2] A. A. Beĭlinson, *Higher regulators and values of L-functions*, J. of Soviet Math., (1985), pp. 2036–2070.

[3] A. A. Beĭlinson, *Higher regulators of modular curves*, in Applications of algebraic K-theory to algebraic geometry and number theory,, vol. 55 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1986, pp. 1–34.

[4] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4), 7 (1974), pp. 235–272 (1975).

[5] A. Borel and N. R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, vol. 67, American Mathematical Soc., 2013.

[6] F. Brunault and M. Chida, *Regulators for Rankin-Selberg products of modular forms*, Ann. Math. Qué., 40 (2016), pp. 221–249.

[7] J. I. Burgos, *A C^∞ logarithmic Dolbeault complex*, Compositio Math., 92 (1994), pp. 61–86.

[8] J. I. Burgos Gil, J. Kramer, and U. Kühn, *Cohomological arithmetic Chow rings*, Journal of the Institute of Mathematics of Jussieu, 6 (2007), pp. 1–172.

[9] A. Cauchi, F. Lemma, and J. R. Jacinto, *Algebraic cycles and functorial lifts from G_2 to* PGSp_6, arXiv preprint arXiv:2202.09394, (2022).

[10] A. Cauchi and J. Rodrigues Jacinto, *Norm-compatible systems of Galois cohomology classes for* GSp_6, Documenta Mathematica, 25 (2020).

[11] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, CRC Press, 1993.

[12] P. Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math., (1971), pp. 5–57.

[13] C. Deninger, *Higher regulators and Hecke L-series of imaginary quadratic fields. I*, Invent. Math., 96 (1989), pp. 1–69.

[14] ———, *Higher regulators and Hecke L-series of imaginary quadratic fields. II*, Ann. of Math. (2), 132 (1990), pp. 131–158.

[15] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, vol. 22, Springer Science & Business Media, 2013.

[16] W. T. Gan and N. Gurevich, *Cap representations of G_2 and the spin L-function of $PGSp6$*, Israel Journal of Mathematics, 170 (2009), pp. 1–52.

[17] D. Ginzburg and N. Gurevich, *On the first occurrence of cuspidal representations of symplectic groups*, Journal of the Institute of Mathematics of Jussieu, 5 (2006), pp. 35–52.

[18] D. Ginzburg, S. Rallis, and D. Soudry, *On Fourier coefficients of automorphic forms of symplectic groups*, Manuscripta Math., 111 (2003), pp. 1–16.

[19] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, 1994.

[20] M. Harris, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3), 59 (1989), pp. 1–22.

[21] ———, *Automorphic forms of $\overline{\partial}$-cohomology type as coherent cohomology classes*, J. Differential Geom., 32 (1990), pp. 1–63.

[22] ———, *L-functions and periods of polarized regular motives*, Journal fur die Reine und Angewandte Mathematik, (1997), pp. 75–161.

[23] M. Harris and D. H. Phong, *Cohomologie de Dolbeault à croissance logarithmique à l'infini*, C. R. Acad. Sci. Paris Sér. I Math., 302 (1986), pp. 307–310.

[24] M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties. III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory*, Mém. Soc. Math. Fr. (N.S.), (2001), pp. vi+116.

[25] H. Jacquet and J. A. Shalika, *A non-vanishing theorem for zeta functions of* $\mathrm{GL}(n)$, Inventiones mathematicae, 38 (1976), pp. 1–16.

[26] U. Jannsen, *Deligne homology, Hodge-D-conjecture, and motives*, in Beilinson's conjectures on special values of L-functions, M. Rapoport, P. Schneider, and N. Schappacher, eds., vol. 4, Academic Press, 1988.

[27] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque, 295 (2004), pp. 117–290.

[28] K. Kato, T. Matsubara, and C. Nakayama, *Log C^∞-functions and degenerations of Hodge structures*, in Algebraic geometry 2000, Azumino (Hotaka), vol. 36 of Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 2002, pp. 269–320.

[29] G. Kings, *Higher regulators, Hilbert modular surfaces, and special values of L-functions*, Duke Math. J., 92 (1998), pp. 61–127.

[30] A. W. Knapp, *Representation theory of semisimple groups: an overview based on examples*, vol. 36, Princeton university press, 2001.

[31] A. Kret and S. W. Shin, *Galois representations for general symplectic groups*, arXiv preprint arXiv:1609.04223, (2016).

[32] C. Laurent-Thiébaut, *Holomorphic function theory in several variables: An introduction*, Springer Science & Business Media, 2010.

[33] A. Lei, D. Loeffler, and S. L. Zerbes, *Euler systems for Hilbert modular surfaces*, Forum Math. Sigma, 6 (2018), pp. Paper No. e23, 67.

[34] F. Lemma, *On higher regulators of Siegel threefolds II: the connection to the special value*, Compos. Math., 153 (2017), pp. 889–946.

[35] T. Miyazaki, *The Generalized Whittaker Functions for* $\mathrm{Sp}(2,\mathbf{R})$ *and the Gamma Factor of the Andrianov L-function*, Journal of Mathematical Sciences-University of Tokyo, 7 (2000), pp. 241–295.

[36] C. Moeglin and D. Renard, *Sur les paquets d'Arthur de* $\mathrm{Sp}(2n,\mathbf{R})$ *contenant des modules unitaires de plus haut poids, scalaires*, Nagoya Mathematical Journal, 241 (2021), pp. 44–124.

[37] A. Mokrane and J. Tilouine, *Cohomology of Siegel varieties with p-adic integral coefficients and applications*, Astérisque, 280 (2002).

[38] T. Moriyama, *Generalized Whittaker functions on* $\mathrm{GSp}(2,\mathbf{R})$ *associated with indefinite quadratic forms*, Journal of the Mathematical Society of Japan, 63 (2011), pp. 1203–1262.

[39] V. Navarro Aznar, *Sur la théorie de Hodge-Deligne*, Invent. Math., 90 (1987), pp. 11–76.

[40] I. Piatetski-Shapiro and S. Rallis, *A new way to get Euler products*, J. reine angew. Math., 392 (1988), pp. 110–124.

[41] A. Pollack and S. Shah, *A class number formula for Picard modular surfaces*, arXiv preprint arXiv:1801.07383, (2017).

[42] ———, *The spin L-function on* GSp_6 *via a non-unique model*, Amer. J. Math., 140 (2018), pp. 753–788.

[43] D. Ramakrishnan, *Higher regulators on quaternionic Shimura curves and values of L-functions*, in Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), vol. 55 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1986, pp. 377–387.

[44] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Mathematical Journal, 81 (1996), pp. 269–322.

[45] P. Schneider, *Introduction to the Beilinson's conjectures*, in Beilinson's conjectures on special values of L-functions, M. Rapoport, P. Schneider, and N. Schappacher, eds., vol. 4, Academic Press, 1988.

[46] J.-P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Séminaire Delange-Pisot-Poitou. Théorie des nombres, 11 (1969), pp. 1–15.

[47] D. A. Vogan and G. J. Zuckerman, *Unitary representations with non–zero cohomology*, Compositio Mathematica, 53 (1984), pp. 51–90.

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