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ALGEBRAIC CYCLES AND FUNCTORIAL LIFTS FROM G_2 TO PGSp_6

ANTONIO CAUCHI, FRANCESCO LEMMA AND JOAQUÍN RODRIGUES JACINTO

ABSTRACT. We establish instances of Beilinson–Tate conjectures for automorphic representations of PGSp_6 whose Spin L -function has a pole at $s = 1$. Using the exceptional theta correspondence between the split group of type G_2 and PGSp_6 and assuming the non-vanishing of a certain archimedean integral, this allows us to confirm a conjecture of Gross and Savin on rank 7 motives of type G_2 .

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1. INTRODUCTION

In this paper we establish a connection between algebraic cycles in Siegel sixfolds and the residue at $s = 1$ of Spin L -functions of automorphic representations of GSp_6 , as predicted by conjectures of Beilinson and Tate. Moreover, we exploit an exceptional theta correspondence between the split group of type G_2 and PGSp_6 to answer a question of Gross and Savin.

1.1. Motivation. Let $\pi = \pi_\infty \otimes \pi_f$ be a cohomological cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$, let $M(\pi_f)$ denote the Spin Chow motive with coefficients in a number field L conjecturally attached to π and let $L(s, M(\pi_f)(3))$ be its Hasse-Weil L -function. Let

$$r_{\mathcal{H}} : H_{\mathcal{M}}^1(M(\pi_f)(4)) \oplus N(M(\pi_f)(3)) \rightarrow H_{\mathcal{H}}^1(M(\pi_f)(4))$$

denote Beilinson-Deligne regulator. Here $H_{\mathcal{M}}^1(M(\pi_f)(4))$ denotes the first motivic cohomology group of $M(\pi_f)(4)$, the group $N(M(\pi_f)(3))$ denotes algebraic cycles in $M(\pi_f)(3)$ up to homological equivalence and $H_{\mathcal{H}}^1(M(\pi_f)(4))$ denotes the first absolute Hodge cohomology group of $M(\pi_f)(4)$.

Conjecture 1.1. (*Beilinson–Tate*)

(1) *The map $r_{\mathcal{H}}$ induces an isomorphism*

$$(H_{\mathcal{M}}^1(M(\pi_f)(4)) \oplus N(M(\pi_f)(3))) \otimes_{\mathbf{Q}} \mathbf{R} \rightarrow H_{\mathcal{H}}^1(M(\pi_f)(4)),$$

(2) $\mathrm{ord}_{s=0} L(s, M(\pi_f)(3)) = \dim_L H_{\mathcal{M}}^1(M(\pi_f)(4))$,

$$(3) \quad -\text{ord}_{s=1} L(s, M(\pi_f)(3)) = \dim_L N(M(\pi_f)(3))$$

$$(4) \quad \det(\text{Im } r_{\mathcal{H}}) = L^*(1, M(\pi_f)(3)) \mathcal{D}(M(\pi_f)(4)), \text{ where } \mathcal{D}(M(\pi_f)(4)) \text{ denotes the Deligne } L\text{-structure of } \det(H_{\mathcal{H}}^1(M(\pi_f)(4))).$$

In [9], we studied the contribution of the motivic cohomology to this conjecture. This corresponds to the case where $L(s, M(\pi_f)(3))$ does not have a pole at $s = 1$. In this article, we focus on the contribution of algebraic cycles, which corresponds to the case where $L(s, M(\pi_f)(3))$ has a simple pole at $s = 1$.

The ℓ -adic étale realization $M_{\ell}(\pi_f)$ of $M(\pi_f)$ is expected to be a $\text{GL}_8(\overline{\mathbf{Q}}_{\ell})$ -valued Galois representation factoring through the Spin representation $\text{Spin} : \text{Spin}_7(\overline{\mathbf{Q}}_{\ell}) \rightarrow \text{GL}_8(\overline{\mathbf{Q}}_{\ell})$. If $L(s, M(\pi_f)(3))$ has a pole at $s = 1$, Conjecture 1.1 (3) implies the existence of an invariant vector in this 8-dimensional Galois representation. As the stabilizer in $\text{Spin}_7(\overline{\mathbf{Q}}_{\ell})$ of a generic vector in the Spin representation is the exceptional group $G_2(\overline{\mathbf{Q}}_{\ell})$, by Langlands reciprocity principle, π should be a functorial lift from a group G of type G_2 . In fact, we have $\text{Spin}|_{G_2} = \text{Std} \oplus \mathbf{1}$, where Std denotes the standard representation of G_2 and $\mathbf{1}$ denotes the trivial representation. Then, if σ is a cuspidal automorphic representation of $G(\mathbf{A})$ lifting to π , Gross and Savin [27] conjectured that the motive $M(\pi_f)$ decomposes as the direct sum of the rank 7 motive $M(\sigma_f)$ attached to σ and the rank 1 trivial motive generated by the class given in Conjecture 1.1. Moreover, inspired by local calculations, they conjectured that this class should arise from a Hilbert modular threefold.

1.2. Main results. Let F denote a real étale quadratic \mathbf{Q} -algebra, i.e., F is either a quadratic extension of \mathbf{Q} or $\mathbf{Q} \times \mathbf{Q}$. Associated to the totally real étale cubic algebra $E = \mathbf{Q} \times F$ of \mathbf{Q} there is a Hilbert modular threefold $\text{Sh}_{\mathbf{H}}/\mathbf{Q}$, with underlying reductive group $\mathbf{H} = \{g \in \text{Res}_{E/\mathbf{Q}} \text{GL}_{2,E} \mid \det(g) \in \mathbf{G}_m\}$. The group \mathbf{H} embeds naturally into $\mathbf{G} = \text{GSp}_6$ and one has a closed embedding $\iota : \text{Sh}_{\mathbf{H}} \hookrightarrow \text{Sh}_{\mathbf{G}}$ of codimension 3 in the Shimura variety attached to \mathbf{G} , which is the Siegel variety of dimension 6. For any irreducible algebraic representation V of \mathbf{G} such that ι^*V contains the trivial representation of \mathbf{H} , the cycle $\text{Sh}_{\mathbf{H}}$ of $\text{Sh}_{\mathbf{G}}$ induces a class

$$\mathcal{Z}_{\mathbf{H},\mathcal{M}} \in H_{\mathcal{M}}^6(\text{Sh}_{\mathbf{G}}, \mathcal{V}_{\mathcal{M}}(3)),$$

where $\mathcal{V}_{\mathcal{M}}$ is the Chow local system associated to V and $H_{\mathcal{M}}^6(\text{Sh}_{\mathbf{G}}, \mathcal{V}_{\mathcal{M}}(3))$ is the motivic cohomology group of $\text{Sh}_{\mathbf{G}}$ with coefficients in $\mathcal{V}_{\mathcal{M}}(3)$. We denote by $\mathcal{Z}_{\mathbf{H},\mathcal{H}} \in H_{\mathcal{H}}^7(\text{Sh}_{\mathbf{G}}, \mathcal{V}_{\mathcal{H}}(4))$, resp. $\mathcal{Z}_{\mathbf{H},B} \in H_B^6(\text{Sh}_{\mathbf{G}}, \mathcal{V}_B(3))$ the image of $\mathcal{Z}_{\mathbf{H},\mathcal{M}}$ in absolute Hodge cohomology, resp. Betti cohomology (see Definition 3.11 for the definition of $\mathcal{Z}_{\mathbf{H},\mathcal{H}}$). Let π be a cuspidal automorphic representation of $\text{PGSp}_6(\mathbf{A})$ whose archimedean component belongs to the discrete series L -packet of V and has Hodge type $(3, 3)$. For a cusp form Ψ in the space of π , whose archimedean component is in the minimal K -type of π_{∞} , we have a vector valued harmonic differential form ω_{Ψ} whose cohomology class $[\omega_{\Psi}]$ is an element of $H_{dR,c}^6(\text{Sh}_{\mathbf{G}}, \mathcal{V}_{dR})$. Poincaré duality induces maps

$$\begin{aligned} \langle \cdot, [\omega_{\Psi}] \rangle_B &: H_B^6(\text{Sh}_{\mathbf{G}}, \mathcal{V}_B(3)) \rightarrow \mathbf{C}, \\ \langle \cdot, [\omega_{\Psi}] \rangle_{\mathcal{H}} &: H_{\mathcal{H}}^7(\text{Sh}_{\mathbf{G}}, \mathcal{V}_{\mathcal{H}}(4)) \rightarrow \mathbf{C}. \end{aligned}$$

The pairings $\langle \mathcal{Z}_{\mathbf{H},B}, [\omega_{\Psi}] \rangle_B$ and $\langle \mathcal{Z}_{\mathbf{H},\mathcal{H}}, [\omega_{\Psi}] \rangle_{\mathcal{H}}$ are computed in terms of the residue of a certain adelic integral of Rankin-Selberg type considered in [48]. In *loc. cit.* it is shown that, if π supports certain Fourier coefficient associated to F , then the local factors at unramified

places v of this integral represent the degree 8 local spin L -function $L(s, \pi_v, \mathrm{Spin})$ of π_v . The following result gives evidence for Conjecture 1.1 for the motive associated to π .

Theorem 1.2 (Theorem 5.12). *Let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$ such that π_∞ is a discrete series of Hodge type $(3, 3)$ in the discrete series L -packet of V . If the integral $\mathcal{I}_S(\Phi, \Psi, 1)$ defined in Theorem 5.8 is non-zero for some Schwartz-Bruhat function Φ , then*

$$\langle \mathcal{Z}_{\mathbf{H}, B}, [\omega_\Psi] \rangle_B = \langle \mathcal{Z}_{\mathbf{H}, \mathcal{H}}, [\omega_\Psi] \rangle_{\mathcal{H}} = C \cdot \mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}),$$

where C is a non-zero constant.

Remark 1.3. We point out that, according to [17, Proposition 12.1] there exist a Schwartz-Bruhat function Φ and a vector $\Psi \in V_\pi$ such that $\mathcal{I}_S(\Phi, \Psi, 1)$ is non-zero. However we do not know if this holds for Ψ_∞ in the minimal K -type of π_∞ .

As a corollary of this theorem, one can deduce, under the additional assumption that π is the Steinberg representation at a finite place, a weak version of Conjecture 1.1(1) (Theorem 8.4) and Conjecture 1.1(3) (Corollary 5.13).

When $\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin})$ is non-zero then (cf. [48, Theorem 1.3]) π is a weak functorial lift of a cuspidal automorphic representation σ of $G_2(\mathbf{A})$, where G_2 is the split form of the exceptional group of type G_2 . Moreover (cf. Proposition 8.1), we have

$$\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) = L^S(1, \sigma, \mathrm{Std}) \mathrm{Res}_{s=1} \zeta^S(s).$$

Hence the theorem above gives a cohomological formula for the critical value $L^S(1, \sigma, \mathrm{Std})$.

Our second main theorem concerns the program of Gross and Savin on rank seven motives of Galois type G_2 . The first step towards the conjecture of Gross and Savin was done by Kret and Shin in [41], where they more generally constructed $G\mathrm{Spin}$ -valued Galois representations associated to cohomological cuspidal automorphic forms of symplectic groups. Moreover, based on the calculations of [27], they verified ([41, Theorem 11.1]) that, for suitable automorphic representations of PGSp_6 in the image of the exceptional theta correspondence from the compact form G_2^c of type G_2 , the image of their Galois representation lies actually in $G_2(\overline{\mathbf{Q}}_\ell)$. More precisely, let ρ_π be the $\mathrm{Spin}_7(\overline{\mathbf{Q}}_\ell)$ -valued Galois representation attached to π . Assuming that π is a theta lift of σ , we have

$$\mathrm{Spin} \circ \rho_\pi = \mathrm{Std} \circ \rho_\sigma \oplus \mathbf{1} \tag{1}$$

where $\mathrm{Std} \circ \rho_\sigma$ is the standard Galois representation attached to σ . We obtain a similar decomposition using the exceptional theta correspondence from the split form G_2 (cf. Theorem 8.8).

Theorem 1.4 (Theorem 8.11). *Let $\sigma = \bigotimes'_v \sigma_v$ be an irreducible cuspidal automorphic representation of $G_2(\mathbf{A})$ such that:*

- σ_∞ is a quaternionic discrete series,
- σ_p is the Steinberg representation for some prime number p .

Let π be a theta lift of σ to $\mathrm{PGSp}_6(\mathbf{A})$. Assume that the integral $\mathcal{I}_S(\Phi, \Psi, 1)$ is non-zero for some Φ and Ψ as above. Then, the trivial representation $\mathbf{1}$ in (1) is generated by the étale realization of $\mathcal{Z}_{\mathbf{H}, \mathcal{M}}$.

We conclude this introduction explaining a result used as an intermediate step in the proof of Theorem 1.4 which has its own interest. By a result of Gan [15, Theorem 3.1], every

cuspidal automorphic representation σ of $G_2(\mathbf{A})$ supports a Fourier coefficient associated to an étale cubic algebra E .

Theorem 1.5 (Theorem 7.2, Proposition 7.13). *Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$. Assume that*

- σ is not globally generic;
- σ_p is generic at some finite place p .

Then the big theta lift $\Theta(\sigma)$ is cuspidal. Moreover, σ supports a Fourier coefficient associated to $\mathbf{Q} \times F$, for some étale quadratic algebra F , if and only if $\Theta(\sigma)$ supports a rank 2 Fourier coefficient associated to F .

Remark 1.6. The assumptions of Theorem 1.5 are implied by the ones of Theorem 1.4 as the quaternionic discrete series are not generic, while the Steinberg representation is.

1.3. Overview of the proofs. The main difficulty for calculating the pairing of Theorem 1.2 between the motivic class and the cohomology class $[\omega_\Psi]$ resides on the fact that the first class is constructed from the decomposition into irreducible components of the restriction of V to the subgroup \mathbf{H} , while the test vector is constructed from its restriction to the maximal compact subgroup $\mathbf{U}(3)$ of $\mathbf{G}(\mathbf{R})$. One needs to carefully study the relationship between these two different decompositions (Theorem 4.2). As a consequence we get a formula for the pairing in terms of a period integral (Proposition 4.8 and Proposition 4.10). These adelic integrals are in turn related to the residue of the partial Spin L -function of π by means of the work of Pollack and Shah (Proposition 5.10), which allows to conclude the proof. Theorem 1.4 follows basically from Theorem 1.2 and 1.5. The proof of Theorem 1.5 goes as follows. We first prove (Theorem 7.2 and Corollary 7.3) that σ lifts to a cuspidal representation using the tower of exceptional correspondences for G_2 studied in [23], which reduces the problem to the vanishing of certain automorphic period integrals. Finally, we establish (Proposition 7.13) a correspondence between Fourier coefficients of σ and its theta lift, which in particular implies the non-vanishing of the latter.

1.4. Structure of the manuscript. In section 2 we fix notation, conventions, and basic results that will be useful in the body of the article. In particular, we prove that, under some mild assumptions, the localization at a maximal ideal of the Hecke algebra of the cohomology of the Siegel sixfold is cuspidal and concentrated in the middle degree. We also introduce Absolute Hodge cohomology and compute the dimension of its π_f -isotypical component. In section 3 we explain the construction of the motivic class $\mathcal{Z}_{\mathcal{M}}$ and its realizations. In Section 4 we construct the harmonic differential form ω_Ψ associated to a suitable cuspidal form Ψ in the space of π and we prove our first main result concerning the calculation of the pairing between the motivic class and the cohomology class $[\omega_\Psi]$. In Section 5, we use the results of Pollack and Shah to relate the pairing to the residue of the Spin L -function. Sections 6 and 7 are devoted to the study of the exceptional theta correspondence between G_2 and PGSp_6 and contain the proof of Theorem 1.5. Finally, in Section 8 we relate the pairing to a critical value of the standard L -function of G_2 . We also deduce from the work of Kret and Shin the existence of Galois representations attached to certain cuspidal representations of G_2 and we conclude with a proof of Theorem 1.4.

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2. PRELIMINARIES

2.1. Algebraic groups and algebraic representations. Let ψ denote an antisymmetric non-degenerate bilinear form on a finite dimensional \mathbf{Q} -vector space V . The symplectic group $\mathbf{GSp}(V, \psi)$ is the \mathbf{Q} -group scheme defined by

$$\mathbf{GSp}(V, \psi) = \{g \in \mathbf{GL}(V) \mid \forall v, w \in V, \psi(gv, gw) = \nu(g)\psi(v, w), \nu(g) \in \mathbf{G}_m\}$$

Then $\nu : \mathbf{GSp}(V, \psi) \rightarrow \mathbf{G}_m$ is a character. Let I_n denote the identity matrix of size n . When V is the \mathbf{Q} -vector space \mathbf{Q}^{2n} endowed with the bilinear form whose matrix is $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, we let \mathbf{GSp}_{2n} denote $\mathbf{GSp}(\mathbf{Q}^{2n}, J)$ and we let \mathbf{Sp}_{2n} denote $\ker \nu$. In this paper, we are mainly interested in the case $n = 3$. Hence we will denote by \mathbf{G} the group \mathbf{GSp}_6 and by \mathbf{G}_0 the group \mathbf{Sp}_6 . Let $\mathbf{T} \subset \mathbf{G}$ denote the maximal diagonal torus and $\mathbf{B} \subset \mathbf{G}$ denote the standard Borel. We have

$$\mathbf{T} = \{\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu, \alpha_3^{-1}\nu), \alpha_1, \alpha_2, \alpha_3, \nu \in \mathbf{G}_m\}.$$

We associate to any 4-uple $(\lambda_1, \lambda_2, \lambda_3, c) \in \mathbf{Z}^4$ such that $c \equiv \lambda_1 + \lambda_2 + \lambda_3 \pmod{2}$ the algebraic character $\lambda(\lambda_1, \lambda_2, \lambda_3, c)$ of \mathbf{T} defined by

$$\lambda(\lambda_1, \lambda_2, \lambda_3, c) : \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu, \alpha_3^{-1}\nu) \mapsto \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \alpha_3^{\lambda_3} \nu^{\frac{c - \lambda_1 - \lambda_2 - \lambda_3}{2}}.$$

This defines an isomorphism between the group of 4-uples $(\lambda_1, \lambda_2, \lambda_3, c) \in \mathbf{Z}^4$ such that $c \equiv \lambda_1 + \lambda_2 + \lambda_3 \pmod{2}$ and the group of algebraic characters of \mathbf{T} . Let $\rho_1 = \lambda(1, -1, 0, 0)$ and $\rho_2 = \lambda(0, 1, -1, 0)$ denote the short simple roots and let $\rho_3 = \lambda(0, 0, 2, 0)$ denote the long simple root. The set of roots of \mathbf{T} in \mathbf{G} is $R = R^+ \cup R^-$ where

$$R^+ = \{\rho_1, \rho_2, \rho_1 + \rho_2, \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3, \rho_1 + 2\rho_2 + \rho_3, 2\rho_1 + 2\rho_2 + \rho_3, 2\rho_2 + \rho_3, \rho_3\}$$

is the set of positive roots with respect to \mathbf{B} and $R^- = -R^+$. A weight $\lambda = \lambda(\lambda_1, \lambda_2, \lambda_3, c)$ is dominant for \mathbf{B} if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. For any such λ , there exists a unique (up to isomorphism) irreducible algebraic representation V^λ of \mathbf{G} of highest weight λ and every irreducible algebraic representation of \mathbf{G} is obtained in this way (up to isomorphism). Similarly, irreducible algebraic representations of \mathbf{GSp}_4 are classified by their highest weight which is a character of the shape $\lambda(\lambda_1, \lambda_2, c)$ with $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \equiv c \pmod{2}$ (see for example [44, §2.3] for more details). We will also use the classification of irreducible algebraic representations of the groups $\mathbf{G}_0 = \mathbf{Sp}_6$ and \mathbf{Sp}_4 . Hence let us recall that the diagonal maximal torus $\mathbf{T}_0 = \mathbf{T} \cap \mathbf{G}_0$ of \mathbf{G}_0 is

$$\mathbf{T}_0 = \{\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}), \alpha_1, \alpha_2, \alpha_3 \in \mathbf{G}_m\}.$$

and that its group of algebraic characters is isomorphic to \mathbf{Z}^3 via $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda(\lambda_1, \lambda_2, \lambda_3)$ where

$$\lambda(\lambda_1, \lambda_2, \lambda_3) : \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}) \mapsto \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \alpha_3^{\lambda_3}. \quad (2)$$

A weight $\lambda = \lambda(\lambda_1, \lambda_2, \lambda_3)$ is dominant with respect to the standard Borel $\mathbf{B}_0 = \mathbf{B} \cap \mathbf{G}_0$ if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and for any such λ there exists a unique (up to isomorphism) irreducible algebraic representation V^λ of \mathbf{G}_0 of highest weight λ and every irreducible algebraic representation of \mathbf{G}_0 is obtained in this way (up to isomorphism). Similarly, irreducible algebraic representations of \mathbf{Sp}_4 are classified by characters $\lambda(\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2$, with obvious notation.

2.2. Compact Lie groups and representations. Let $U(n) = \{g \in GL_n(\mathbf{C}) \mid {}^t \bar{g}g = I_n\}$ denote the unitary group and let $K_\infty \subset \mathbf{G}_0(\mathbf{R})$ be the subgroup defined as

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid AA^t + BB^t = 1, AB^t = BA^t \right\}.$$

We have an isomorphism $\kappa : U(3) \simeq K_\infty$ defined by $A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. In fact K_∞ is a maximal compact subgroup of $\mathbf{G}_0(\mathbf{R})$. Let $T_\infty \subset K_\infty$ denote $\{\kappa(\text{diag}(z_1, z_2, z_3)), z_1, z_2, z_3 \in U(1)\}$. Then T_∞ is Cartan subgroup of K_∞ . Its group of algebraic characters is isomorphic to \mathbf{Z}^3 via $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda'(\lambda_1, \lambda_2, \lambda_3)$, where

$$\lambda'(\lambda_1, \lambda_2, \lambda_3) : \kappa(\text{diag}(z_1, z_2, z_3)) \mapsto z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3}.$$

An algebraic character is dominant if $\lambda_1 \geq \lambda_2 \geq \lambda_3$. For any dominant integral weight λ' , there exists a unique (up to isomorphism) irreducible representation $\tau_{\lambda'}$ of K_∞ in a finite dimensional \mathbf{C} -vector space and every irreducible representation of K_∞ is obtained in this way (up to isomorphism). In what follows, we will simply denote the irreducible representation of highest weight $\lambda'(\lambda_1, \lambda_2, \lambda_3)$ by $\tau_{(\lambda_1, \lambda_2, \lambda_3)}$. Let us explain the connection between the weights λ of \mathbf{T}_0 defined by equation (2) in the previous section and the weights λ' defined above. Let $J \in \mathbf{G}_0(\mathbf{C})$ denote the matrix $J = \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 & iI_3 \\ iI_3 & I_3 \end{pmatrix}$. Then we have

$$J^{-1} \kappa(\text{diag}(z_1, z_2, z_3)) J = \text{diag}(z_1, z_2, z_3)$$

and so, for any $(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Z}^3$, we have

$$\lambda(\lambda_1, \lambda_2, \lambda_3) (J^{-1} \kappa(z_1, z_2, z_3) J) = \lambda'(\lambda_1, \lambda_2, \lambda_3) (\text{diag}(z_1, z_2, z_3)).$$

In brief, the character $\lambda'(\lambda_1, \lambda_2, \lambda_3)$ of T_∞ is conjugated to the restriction of $\lambda(\lambda_1, \lambda_2, \lambda_3)$ to $U(1)^3 \subset \mathbf{C}^\times \times \mathbf{C}^\times \times \mathbf{C}^\times = \mathbf{T}_0(\mathbf{C})$.

2.3. Lie algebras. Let \mathfrak{g}_0 , resp. \mathfrak{k} , denote the Lie algebra of $\mathbf{G}_0(\mathbf{R})$, resp. K_∞ , and let $\mathfrak{g}_{0, \mathbf{C}}$, resp. $\mathfrak{k}_{\mathbf{C}}$, denote its complexification. Then

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_6(\mathbf{R}) \mid B = B^t, C = C^t, A = -D^t, B = B^t \right\}, \\ \mathfrak{k} &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M_6(\mathbf{R}) \mid A = -A^t, B = B^t \right\}. \end{aligned}$$

The Lie algebra \mathfrak{k} is the 1-eigenspace for the Cartan involution $\theta(X) = -X^t$. The (-1) -eigenspace is $\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in M_6(\mathbf{R}) \mid A = A^t, B = B^t \right\}$. Letting

$$\mathfrak{p}_{\mathbf{C}}^\pm = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \in M_6(\mathbf{C}) \mid A = A^t \right\},$$

we have $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_{\mathbf{C}}^+ \oplus \mathfrak{p}_{\mathbf{C}}^-$ and one has the Cartan decomposition

$$\mathfrak{g}_{0, \mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}^+ \oplus \mathfrak{p}_{\mathbf{C}}^-.$$

For $1 \leq j \leq 3$, let $D_j \in M_3(\mathbf{C})$ be the matrix with entry 1 at position (j, j) and 0 elsewhere. Define $T_j = \begin{pmatrix} 0 & D_j \\ -D_j & 0 \end{pmatrix}$. Then the Lie algebra \mathfrak{h} of T_∞ is $\mathfrak{h} = \mathbf{R} \cdot T_1 \oplus \mathbf{R} \cdot T_2 \oplus \mathbf{R} \cdot T_3$. This is a compact Cartan subalgebra of \mathfrak{g}_0 . Let (e_1, e_2, e_3) denote the basis of $\mathfrak{h}_{\mathbf{C}}^*$ dual to $(-iT_1, -iT_2, -iT_3)$. A system of positive roots for $(\mathfrak{g}_{0, \mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ is then given by

$$\{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, 2e_1, 2e_2, 2e_3\}.$$

The simple roots are $e_1 - e_2, e_2 - e_3$ and $2e_3$. We note that $\mathfrak{p}_{\mathbf{C}}^+$ is spanned by the root spaces corresponding to the positive roots of type $2e_j$ and $e_j + e_k$. We denote $\Delta = \{\pm 2e_j, \pm(e_j \pm e_k)\}$ the set of all roots, $\Delta_{\mathbf{c}} = \{\pm(e_j - e_k)\}$ the set of compact roots and $\Delta_{\text{nc}} = \Delta - \Delta_{\mathbf{c}}$ the non-compact roots. Finally, we note $\Delta^+, \Delta_{\mathbf{c}}^+$ and Δ_{nc}^+ the set of positive, positive compact and positive non-compact roots, respectively.

2.4. Weyl groups. Recall that the Weyl group of \mathbf{G}_0 is given by $\mathfrak{W}_{\mathbf{G}_0} = \{\pm 1\}^3 \rtimes \mathfrak{S}_3$. The reflection σ_j in the hyperplane orthogonal to $2e_j$ simply reverses the sign of e_j while leaving the other e_k fixed. The reflection σ_{jk} in the hyperplane orthogonal to $e_j - e_k$ exchanges e_j and e_k and leaves the remaining e_ℓ fixed. The Weyl group \mathfrak{W}_{K_∞} of $K_\infty \cong U(3)$ is isomorphic to \mathfrak{S}_3 and, via the embedding into \mathbf{G} , identifies with the subgroup of $\mathfrak{W}_{\mathbf{G}_0}$ generated by the σ_{jk} . With the identification $\mathfrak{W}_{\mathbf{G}_0} = N(\mathbf{T}_0)/Z(\mathbf{T}_0)$, an explicit description of $\mathfrak{W}_{\mathbf{G}_0}$ and \mathfrak{W}_{K_∞} is given as follows. The matrices corresponding to the reflections σ_{jk} are $\begin{pmatrix} S_{jk} & 0 \\ 0 & -S_{jk} \end{pmatrix}$, where S_{jk} is the matrix with entry 1 at places (ℓ, ℓ) , $\ell \neq j, k$, (k, j) and (j, k) and zeroes elsewhere. The matrices corresponding to the reflection σ_j in the hyperplane orthogonal to $2e_j$ are of the form $\begin{pmatrix} 0 & U_j \\ -U_j & 0 \end{pmatrix}$, where U_j denotes the diagonal matrix with -1 at the place (j, j) and ones at the other entries of the diagonal. This gives an explicit description of the elements of \mathfrak{W}_{K_∞} and their length:

$$\mathfrak{W}_{K_\infty} = \{1, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{12}\sigma_{13}, \sigma_{12}\sigma_{23}\} \xrightarrow{\ell(\bullet)} \{0, 1, 1, 1, 2, 2\}.$$

2.5. Discrete series. We recall standard facts on discrete series for $\mathbf{G}_0(\mathbf{R}) = \mathbf{Sp}_6(\mathbf{R})$ and for $\mathbf{PGSp}_6(\mathbf{R})$. For any non-singular weight Λ define

$$\Delta^+(\Lambda) := \{\alpha \in \Delta \mid \langle \alpha, \Lambda \rangle > 0\}, \quad \Delta_c^+(\Lambda) = \Delta^+(\Lambda) \cap \Delta_c,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on \mathbf{R}^3 . Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a weight of T_∞ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = (3, 2, 1)$. As $|\mathfrak{W}_{\mathbf{G}_0}/\mathfrak{W}_{K_\infty}| = 8$, by [39, Theorem 9.20], the set of equivalence classes of irreducible discrete series representations of $\mathbf{G}_0(\mathbf{R})$ with Harish-Chandra parameter $\lambda + \rho$ contains 8 elements. More precisely, choose representatives $\{w_1, \dots, w_8\}$ of $\mathfrak{W}_{\mathbf{G}_0}/\mathfrak{W}_{K_\infty}$ of increasing length and such that for any $1 \leq i \leq 8$, then the weight $w_i(\lambda + \rho)$ is dominant for K_∞ . The representatives, defined by their action on ρ , are $w_1(3, 2, 1) = (3, 2, 1)$, $w_2(3, 2, 1) = (3, 2, -1)$, $w_3(3, 2, 1) = (3, 1, -2)$, $w_4(3, 2, 1) = (2, 1, -3)$, $w_5(3, 2, 1) = (3, -1, -2)$, $w_6(3, 2, 1) = (2, -1, -3)$, $w_7(3, 2, 1) = (1, -2, -3)$, $w_8(3, 2, 1) = (-1, -2, -3)$. Then, for any $1 \leq i \leq 8$, there exists an irreducible discrete series π_∞^Λ , where $\Lambda = w_i(\lambda + \rho)$, of Harish-Chandra parameter Λ and containing with multiplicity 1 the minimal K_∞ -type with highest weight $\Lambda + \delta_{\mathbf{G}_0} - 2\delta_{K_\infty}$ where $\delta_{\mathbf{G}_0}$, resp. δ_{K_∞} , is the half-sum of roots, resp. of compact roots, which are positive with respect to the Weyl chamber in which Λ lies, i.e., $2\delta_{\mathbf{G}_0} := \sum_{\alpha \in \Delta^+(\Lambda)} \alpha$, $2\delta_{K_\infty} := \sum_{\alpha \in \Delta_c^+(\Lambda)} \alpha$. Moreover, for $i \neq j$, $\Lambda = w_i(\lambda + \rho)$, $\Lambda = w_j(\lambda + \rho)$, the representations π_∞^Λ and π_∞^Λ are not equivalent and any discrete series of \mathbf{G}_0 is obtained in this way. Let V^λ be the irreducible algebraic representation of \mathbf{G}_0 of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (for \mathbf{T}_0).

Definition 2.1. *The discrete series L-packet $P(V^\lambda)$ associated to λ is the set of isomorphism classes of discrete series of $\mathbf{G}_0(\mathbf{R})$ whose Harish-Chandra parameter is of the form $\Lambda = w_i(\lambda + \rho)$ as i varies.*

By [7, Theorem II.5.3], for each $\pi_\infty^\Lambda \in P(V^\lambda)$, the space

$$\mathrm{Hom}_{K_\infty} \left(\bigwedge^6 \mathfrak{g}_{0, \mathbf{C}} / \mathfrak{k}_{\mathbf{C}} \otimes V^\lambda, \pi_\infty^\Lambda \right)$$

has dimension 1. This is a consequence of the fact (cf. the proof of [7, Theorem II.5.3]) that the minimal K_∞ -type of π_∞^Λ appears uniquely in $\bigwedge^6 \mathfrak{g}_{0, \mathbf{C}} / \mathfrak{k}_{\mathbf{C}} \otimes V^\lambda$. Using the Cartan decomposition, we get

$$\bigwedge^6 \mathfrak{g}_{0, \mathbf{C}} / \mathfrak{k}_{\mathbf{C}} = \bigoplus_{p+q=6} \bigwedge^p \mathfrak{p}_{\mathbf{C}}^+ \otimes_{\mathbf{C}} \bigwedge^q \mathfrak{p}_{\mathbf{C}}^-.$$

Hence, there exists a unique pair (p, q) such that $\mathrm{Hom}_{K_\infty}(\bigwedge^p \mathfrak{p}_\mathbf{C}^+ \otimes \bigwedge^q \mathfrak{p}_\mathbf{C}^- \otimes V^\lambda, \pi_\infty^\Lambda)$ is non-zero and hence of dimension one. We call such a pair (p, q) the Hodge type of π_∞^Λ .

Lemma 2.2. *There exist two elements $\pi_{\infty,1}^{3,3}$ and $\overline{\pi}_{\infty,1}^{3,3}$ in $P(V^\lambda)$ of Hodge type $(3, 3)$. They are characterized by having Harish-Chandra parameters $(\lambda_2 + 2, \lambda_3 + 1, -\lambda_1 - 3)$ and $(\lambda_1 + 3, -\lambda_3 - 1, -\lambda_2 - 2)$ and minimal K_∞ -types $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$ and $\tau_{(\lambda_1+4, -\lambda_3-2, -\lambda_2-2)}$ respectively.*

Proof. The discrete series $\pi_{\infty,1}^{3,3}$ and $\overline{\pi}_{\infty,1}^{3,3}$ correspond to the Weyl representatives w_4 and w_5 . Since $w_4\lambda = (\lambda_2, \lambda_3, -\lambda_1)$ and $w_5\lambda = (\lambda_1, -\lambda_3, -\lambda_2)$, the Harish-Chandra parameters of $\pi_{\infty,1}^{3,3}$ and $\overline{\pi}_{\infty,1}^{3,3}$ are as desired. When $\Lambda = w_4(\lambda + \rho)$ (resp. $\Lambda = w_5(\lambda + \rho)$), observe that $\delta_{\mathbf{G}_0}$ equals to $(2, 1, -3)$ (resp. $(3, -1, -2)$), while $\delta_{K_\infty} = (1, 0, -1)$ in both cases. Hence, using the formula above, the minimal K_∞ -types of $\pi_{\infty,1}^{3,3}$ and $\overline{\pi}_{\infty,1}^{3,3}$ are $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$ and $\tau_{(\lambda_1+4, -\lambda_3-2, -\lambda_2-2)}$ respectively.

Recall that, after [56, Proposition 6.19], the Hodge type of a discrete series representation of Harish-Chandra parameter Λ is (p, q) , where p (resp. q) is the number of positive non-compact roots in $\Delta^+(\Lambda)$ (resp. $\Delta^-(\Lambda)$). Using this, one easily checks that the Hodge type of $\pi_{\infty,1}^{3,3}$ and $\overline{\pi}_{\infty,1}^{3,3}$ is $(3, 3)$. □

The picture for $\mathrm{PGSp}_6(\mathbf{R})$ is similar, but the set of its Harish-Chandra parameters changes slightly. This is due to the fact that, since its maximal compact subgroup has two connected components, the set of parameters has to be considered up to the action of \mathfrak{W}_{K_∞} and of w_8 , as the latter, which is the anti-diagonal matrix with all entries -1 , now belongs to the connected component away from the identity of the maximal compact subgroup. Concretely, any parameter $\mu = (\mu_1, \mu_2, \mu_3)$ has to be identified with $w_8\mu = (-\mu_3, -\mu_2, -\mu_1)$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and $\sum_i \lambda_i \equiv 0 \pmod{2}$, then the irreducible algebraic \mathbf{G} -representation $V^{(\lambda,0)}$ of highest weight $\lambda(\lambda_1, \lambda_2, \lambda_3, 0)$ defines a representation of PGSp_6 . The corresponding discrete series L -packet $P(V^{(\lambda,0)})$ for $\mathrm{PGSp}_6(\mathbf{R})$ has thus four elements. Any element $\pi_\infty \in P(V^{(\lambda,0)})$ of Harish-Chandra parameter μ , viewed as a $\mathbf{G}(\mathbf{R})$ -representation, decomposes when restricted to $\mathbf{G}_0(\mathbf{R})$ as the direct sum of two discrete series in $P(V^\lambda)$ of Harish-Chandra parameters μ and $w_8\mu$. As a consequence, for any such π_∞ , the space

$$H^6(\mathfrak{g}, K_\mathbf{G}; \pi_\infty \otimes V^{(\lambda,0)}) = \mathrm{Hom}_{K_\mathbf{G}} \left(\bigwedge^6 \mathfrak{g}_\mathbf{C} / \mathrm{Lie}(K_\mathbf{G})_\mathbf{C}, \pi_\infty \otimes V^{(\lambda,0)} \right),$$

where $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, $\mathfrak{g}_\mathbf{C}$ is its complexification and $K_\mathbf{G} = \mathbf{R}_+^\times K_\infty$, is 2-dimensional. The discussion above implies the following.

Lemma 2.3. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a dominant weight for \mathbf{G}_0 such that $\sum_i \lambda_i \equiv 0 \pmod{2}$. Then there exists a unique discrete series $\pi_\infty^{3,3} \in P(V^{(\lambda,0)})$ of $\mathrm{PGSp}_6(\mathbf{R})$, with Harish-Chandra parameter $(\lambda_2 + 2, \lambda_3 + 1, -\lambda_1 - 3)$, such that*

$$\pi_\infty^{3,3} |_{\mathbf{G}_0(\mathbf{R})} = \pi_{\infty,1}^{3,3} \oplus \overline{\pi}_{\infty,1}^{3,3}.$$

We will refer to $\pi_\infty^{3,3}$ as the discrete series of $\mathrm{PGSp}_6(\mathbf{R})$ in $P(V^{(\lambda,0)})$ of Hodge type $(3, 3)$.

2.6. Shimura varieties. Let F denote a real étale quadratic \mathbf{Q} -algebra, i.e. F is either a totally real quadratic extension of \mathbf{Q} or $\mathbf{Q} \times \mathbf{Q}$. Denote by $\mathrm{GL}_{2,F}^*/\mathbf{Q}$ the subgroup scheme

of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F}$ sitting in the Cartesian diagram

$$\begin{array}{ccc} \mathrm{GL}_{2,F}^* & \hookrightarrow & \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F} \\ \downarrow & & \downarrow \det \\ \mathbf{G}_m & \hookrightarrow & \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_{m,F}. \end{array}$$

For instance, when $F = \mathbf{Q} \times \mathbf{Q}$, we have

$$\mathrm{GL}_{2,F}^* = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g_1) = \det(g_2)\}.$$

Let \mathbf{H} denote the group

$$\mathbf{H} = \mathrm{GL}_2 \boxtimes \mathrm{GL}_{2,F}^* = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_{2,F}^* \mid \det(g_1) = \det(g_2)\}. \quad (3)$$

We embed \mathbf{H} into \mathbf{G} as follows. Let us consider the $\mathbf{Q} \times F$ -module

$$V := \mathbf{Q}e_1 \oplus Fe_2 \oplus \mathbf{Q}f_1 \oplus Ff_2,$$

where $V_1 := \mathbf{Q}e_1 \oplus \mathbf{Q}f_1$ and $V_2 := Fe_2 \oplus Ff_2$ are respectively the standard representations of GL_2 and $\mathrm{GL}_{2,F}^*$. We equip V with the $\mathbf{Q} \times F$ -valued alternating form $\psi' : V \times V \rightarrow \mathbf{Q} \times F$, such that $\psi'(e_1, f_1) = (1, 0)$, $\psi'(e_2, f_2) = (0, \frac{1}{2})$ and V_1 is orthogonal to V_2 . The group \mathbf{H} acts naturally on V and preserves ψ' up to a scalar. We can regard V as a 6-dimensional \mathbf{Q} -vector space with \mathbf{Q} -valued symplectic form $\psi := \mathrm{tr}_{(\mathbf{Q} \times F)/\mathbf{Q}} \circ \psi'$. Explicitly, we have

$$\psi(\alpha e_1 + \alpha e_2, \beta f_1 + \beta f_2) = \alpha\beta + \frac{1}{2}\mathrm{tr}_{F/\mathbf{Q}}(\alpha\beta).$$

This identification defines an embedding $\mathbf{H} \hookrightarrow \mathrm{GSp}(V, \psi)$. We now identify $\mathrm{GSp}(V, \psi)$ with \mathbf{G} by choosing a suitable \mathbf{Q} -basis of V . Recall that the set of real quadratic \mathbf{Q} -algebras is parametrized by $D \in \mathbf{Q}_{>0}^\times / (\mathbf{Q}_{>0}^\times)^2$, via $D \mapsto F = \mathbf{Q} \oplus \mathbf{Q}\sqrt{D}$. Using the decomposition $F = \mathbf{Q} \oplus \mathbf{Q}\sqrt{D}$, we consider the \mathbf{Q} -basis of V given by

$$\{e_1, e_2, e_3, f_1, f_2, f_3\} := \left\{ e_1, e_2, \sqrt{D}e_2, f_1, f_2, \frac{1}{\sqrt{D}}f_2 \right\}.$$

In this basis, ψ is represented by the matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, thus we obtain an isomorphism $\mathrm{GSp}(V, \psi) \simeq \mathbf{G}$ and the embedding

$$\iota : \mathbf{H} \hookrightarrow \mathbf{G}.$$

Note that the group

$$\mathbf{H}' := \mathrm{GL}_2 \boxtimes \mathrm{GSp}_4 := \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GSp}_4 \mid \det(g_1) = \nu(g_2)\},$$

is also naturally embedded in \mathbf{G} and ι factors through \mathbf{H}' .

Recall from [9, §2.2] that there is a 3-dimensional Shimura variety $\mathrm{Sh}_{\mathbf{H}}$ associated to the $\mathbf{H}(\mathbf{R})$ -conjugacy class of

$$h : \mathbf{S} \longrightarrow \mathbf{H}_{/\mathbf{R}}, \quad x + iy \mapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right),$$

where $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m/\mathbf{C}$ is the Deligne torus. The associated Shimura datum has reflex field is \mathbf{Q} and the Shimura variety $\mathrm{Sh}_{\mathbf{H}}$ can be described as follows. If $V \subseteq \mathbf{H}(\mathbf{A}_f)$ is a fibre product (over the similitude characters) $V_1 \times_{\mathbf{A}_f^\times} V_2$ of sufficiently small subgroups, we have

$$\mathrm{Sh}_{\mathbf{H}}(V) = \mathrm{Sh}_{\mathrm{GL}_2}(V_1) \times_{\mathbf{G}_m} \mathrm{Sh}_{\mathrm{GL}_{2,F}^*}(V_2),$$

where $\times_{\mathbf{G}_m}$ denotes the fibre product over the zero dimensional Shimura variety of level $W = \det(V_1) = \det(V_2)$. The connected components are given by

$$\pi_0(\mathrm{Sh}_{\mathbf{H}}(V)(\mathbf{C})) = \hat{\mathbf{Z}}^\times / W.$$

Hence, $\text{Sh}_{\mathbf{H}}$ can be thought as the fibre product of a modular curve and a Hilbert-Blumenthal modular surface. We also recall that the complex points of $\text{Sh}_{\mathbf{H}}(V)$ are given by

$$\text{Sh}_{\mathbf{H}}(V)(\mathbf{C}) = \mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A}) / \mathbf{Z}_{\mathbf{H}}(\mathbf{R}) K_{\mathbf{H}, \infty} V,$$

where $\mathbf{Z}_{\mathbf{H}}$ denotes the center of \mathbf{H} and $K_{\mathbf{H}, \infty} \subseteq \mathbf{H}(\mathbf{R})$ is the maximal compact defined as the product $U(1) \times U(1) \times U(1)$.

The embedding $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$ induces a Shimura datum for \mathbf{G} whose reflex field is \mathbf{Q} . For any sufficiently small compact open subgroup U of $\mathbf{G}(\mathbf{A}_f)$, denote by $\text{Sh}_{\mathbf{G}}(U)$ the associated Shimura variety of dimension 6. We also write $\iota : \text{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \hookrightarrow \text{Sh}_{\mathbf{G}}(U)$ the closed embedding of codimension 3 induced by the group homomorphism $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$.

2.7. Cohomology of Siegel sixfolds. Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ having non-zero fixed vectors by a neat compact open group $U \subseteq \mathbf{G}(\mathbf{A}_f)$. We assume that π has trivial central character and hence we regard it as a cuspidal automorphic representation of $\text{PGSp}_6(\mathbf{A})$. Our purpose is to establish that, under mild assumptions, suitable localizations at π of cuspidal, L^2 , inner Betti and Betti cohomologies coincide and are concentrated in the middle degree. The assumptions are the following.

(DS) the archimedean component π_{∞} is a discrete series representation of $\text{PGSp}_6(\mathbf{R})$,

(St) at a finite place p the component π_p is the Steinberg representation of $\text{PGSp}_6(\mathbf{Q}_p)$.

Let us fix for the rest of this section $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Z}^3$ satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and $\sum \lambda_i \equiv 0 \pmod{2}$. We will denote by V , without mentioning λ anymore, the irreducible algebraic representation of \mathbf{G} of highest weight $(\lambda, 0)$. As V has trivial central character, it will be considered as an irreducible representation of PGSp_6 . Then π_{∞} belongs to the discrete series L -packet $P(V)$. As a consequence

$$H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_{\infty} \otimes V) = \text{Hom}_{K_{\mathbf{G}}} \left(\bigwedge^6 \mathfrak{g}_{\mathbf{C}} / \text{Lie}(K_{\mathbf{G}})_{\mathbf{C}}; \pi_{\infty} \otimes V \right) \neq 0,$$

where $K_{\mathbf{G}} = \mathbf{R}_+^{\times} K_{\infty}$.

There are natural inclusions of spaces of \mathbf{C} -valued functions

$$\mathcal{C}_{\text{cusp}}^{\infty}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})) \subseteq \mathcal{C}_{\text{rd}}^{\infty}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})) \subseteq \mathcal{C}_{(2)}^{\infty}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})) \subseteq \mathcal{C}^{\infty}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})),$$

where these spaces denote, respectively, the space of cuspidal square-integrable functions, rapidly decreasing functions, square-integrable functions and smooth functions, and

$$\mathcal{C}_{\text{c/center}}^{\infty}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})) \subseteq \mathcal{C}_{\text{rd}}^{\infty}(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})),$$

where the first space is the space of compactly supported modulo the center functions (for the precise definition of these spaces, we refer to [6]). Tensoring by V the inclusions above and applying the $(\mathfrak{g}, K_{\mathbf{G}})$ -cohomology functor, we obtain the natural maps

$$\begin{array}{ccccccc} H_{\text{cusp}}^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) & \rightarrow & H_{\text{rd}}^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) & \rightarrow & H_{(2)}^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) & \rightarrow & H^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}), \\ & & \uparrow & & & & \\ & & H_{\text{c}}^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) & & & & \end{array}$$

where $\mathcal{V}_{\mathbf{C}}$ is the \mathbf{C} -local system associated to V . Let $H_1^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})$ denote the image of $H_{\text{c}}^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})$ in $H^{\bullet}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})$. Let N denote the positive integer defined as the product of prime numbers ℓ such that π_{ℓ} is ramified. The fact that π_{∞} is cohomological implies that there exists a number field L whose ring of integers \mathcal{O}_L contains the eigenvalues of the spherical Hecke algebra $\mathcal{H}^{\text{sph}, N}$ away from N and with coefficients in \mathbf{Z} acting on

$\bigotimes'_{\ell \nmid N} \pi_\ell^{\mathbf{G}(\mathbf{Z}_\ell)}$. Let $\mathcal{H}_L^{\text{sph}, N}$ denote the spherical Hecke algebra away from N with coefficients in L , let $\theta_\pi : \mathcal{H}_L^{\text{sph}, N} \rightarrow L$ denote the Hecke character of π and let $\mathfrak{m}_\pi := \ker(\theta_\pi)$. Considering the localization at \mathfrak{m}_π of the above cohomology groups, we have the following result.

Proposition 2.4. *Let π satisfy the hypothesis (DS) and (St) above. Then*

$$H_{\text{cusp}}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H_{(2)}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H_!^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}.$$

Proof. By [6, Theorem 5.3 & Corollary 5.5], the compositions of the horizontal maps

$$H_{\text{cusp}}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) \hookrightarrow H_*^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}),$$

for $*$ \in $\{\text{rd}, (2), \emptyset\}$, are injections. By [6, Theorem 5.2], one has an isomorphism

$$H_c^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) \cong H_{\text{rd}}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}).$$

Hence, if the equality $H_{\text{cusp}}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H_{(2)}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}$ holds, we have

$$H_{\text{cusp}}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H_{(2)}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H_!^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}.$$

We show the former equality as follows. By [5, §4],

$$H_{(2)}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) = \bigoplus_{\sigma \subset L_d^2} \sigma_f^U \otimes H^\bullet(\mathfrak{g}, K_{\mathbf{G}}; \sigma_\infty \otimes V)^{m(\sigma)}, \quad (4)$$

where σ runs over the set of isomorphism classes of automorphic representations appearing in the discrete spectrum L_d^2 of $L^2(Z(\mathbf{A})\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A}))$. Similarly,

$$H_{\text{cusp}}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}) = \bigoplus_{\sigma \subset L_0^2} \sigma_f^U \otimes H^\bullet(\mathfrak{g}, K_{\mathbf{G}}; \sigma_\infty \otimes V)^{m_0(\sigma)},$$

where σ runs over the set of isomorphism classes of automorphic representations in the cuspidal spectrum $L_0^2 \subset L_d^2$. From (4), we can write

$$H_{(2)}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = \bigoplus_{\sigma = \sigma_\infty \otimes \sigma_f} \sigma_f^U \otimes H^\bullet(\mathfrak{g}, K_{\mathbf{G}}; \sigma_\infty \otimes V)^{m(\sigma)},$$

where $\sigma \in L_d^2$ is such that $\sigma_\ell^{\mathbf{G}(\mathbf{Z}_\ell)} \simeq \pi_\ell^{\mathbf{G}(\mathbf{Z}_\ell)} \neq 0$ at all $\ell \nmid N$. Notice that the latter implies that $\sigma_f^N \simeq \pi_f^N$, where for any automorphic representation τ we have denoted $\tau_f^N = \bigotimes_{\ell \nmid N} \tau_\ell$. By [41, Lemma 8.1(2)], the Steinberg condition implies that the representation π_ℓ is tempered and unitary at each $\ell \nmid N$ (as π has trivial central character). Thus, if σ contributes non-trivially to the above sum, its local component at a finite place $\ell \nmid N$ is tempered. This implies that σ is necessarily cuspidal and thus appears in $H_{\text{cusp}}^\bullet(\text{Sh}_{\mathbf{G}}, \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}$ with multiplicity $m_0(\sigma) = m(\sigma)$. This last statement follows from the fact that any non-cuspidal automorphic representation appearing in L_d^2 is obtained as a residue of an Eisenstein series and in particular it is non-tempered at every place (cf. [42, Proposition 4.5.4]). We are left to show that

$$H_{(2)}^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H^\bullet(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}.$$

Recall that Franke's decreasing filtration on the space of automorphic forms for $\mathbf{G}(\mathbf{A})$ (cf. [57, §4.7]) yields a spectral sequence $E_1^{p,q} \Rightarrow H^{p+q}(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})$, where

$$E_1^{p,q} = \bigoplus_{\substack{(w,P) \in B(p) \\ \ell(w) \leq p+q}} \bigoplus_{\sigma = \sigma_\infty \otimes \sigma_f} (\text{Ind}_{P(\mathbf{A}_f)}^{\mathbf{G}(\mathbf{A}_f)} \sigma_f)^U \otimes H^{p+q-\ell(w)}(\mathfrak{m}, K_M; \sigma_\infty \otimes W^{w(\lambda+\rho)-\rho}),$$

where, for all $p \in \mathbf{Z}_{\geq 0}$, $B(p)$ denotes a certain subset depending on p of elements (w, P) (cf. [57, §4.8]), with $w \in \mathfrak{W}_{\mathbf{G}}$ and $P = M \cdot U_P$ a standard parabolic subgroup of \mathbf{G} , $W^{w(\lambda+\rho)-\rho}$

denotes the irreducible algebraic representation of M of highest weight $w(\lambda + \rho) - \rho$, and σ runs over the set of isomorphism classes of automorphic representations appearing in the discrete spectrum of $L^2(Z_M(\mathbf{A})M(\mathbf{Q})\backslash M(\mathbf{A}))$. By the proof of [41, Lemma 8.1(1)], we have that $E_{1, \mathfrak{m}_\pi}^{p, q}$ are zero unless when $(w, P) = (1, \mathbf{G})$, in which case there exists a unique $p_0 \in \mathbf{Z}_{\geq 0}$, for which

$$E_{1, \mathfrak{m}_\pi}^{p, q} = \begin{cases} H_{(2)}^{p+q}(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} & \text{if } p = p_0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the spectral sequence for the localization degenerates at the first page and gives

$$H_{(2)}^{p_0+\bullet}(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = E_{1, \mathfrak{m}_\pi}^{p_0, \bullet} = H^{p_0+\bullet}(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}.$$

□

Proposition 2.5. *Let π satisfy the hypothesis **(DS)** and **(St)** above. Then, we have*

$$H^\bullet(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} = H^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} \neq 0.$$

Proof. Suppose that τ_f contributes to $H^i(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi}$. As we noted in the proof of Proposition 2.4, this implies that, for every $\ell \nmid N$, $\tau_\ell \simeq \pi_\ell$ is tempered and unitary (cf. [41, Lemma 8.1(2)]). Let us fix $\ell \nmid N$; the action of the Frobenius correspondence on intersection cohomology Frob_ℓ on $IH^i(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\tau_f]$ and thus on $H^i(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\tau_f]$ is pure of weight i , i.e. its eigenvalues all have absolute value $\ell^{i/2}$ (cf. [46, Remark 7.2.5]). On the other hand, by the congruence relation conjectured in [4, §6] and verified in [59], Frob_ℓ is a root of the Hecke polynomial

$$H_\ell(T) := \det(T - \ell^3 \mathrm{spin}(\mathrm{Fr}_\ell \times \hat{g})),$$

which is a polynomial in T whose coefficients are elements in the coordinate ring of the set of Fr_ℓ -conjugacy classes of semisimple elements of $\widehat{\mathbf{G}}(\mathbf{C}) = \mathbf{G}\mathrm{Spin}_7(\mathbf{C})$, for Fr_ℓ a Frobenius element in the Weil group of \mathbf{Q}_ℓ . By the untwisted Satake isomorphism, we can see $H_\ell(T)$ as a polynomial with coefficients in the spherical Hecke algebra $\mathcal{H}(\mathbf{G}(\mathbf{Q}_\ell)/\mathbf{G}(\mathbf{Z}_\ell), \mathbf{Q})$ (cf. [59, (2.2.1) & Corollary (2.8)]) and thus we can denote by $H_\ell(T; \tau_\ell)$ the specialization of $H_\ell(T)$ to τ_ℓ , i.e.

$$H_\ell(T; \tau_\ell) = \det(T - \ell^3 \mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Fr}_\ell))),$$

where ϕ_{τ_ℓ} is the unramified Langlands parameter of τ_ℓ . The congruence relation gives that $H_\ell(\mathrm{Frob}_\ell; \tau_\ell) = 0$ on $IH^\bullet(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\tau_f]$, which implies that the eigenvalues of Frob_ℓ on $IH^\bullet(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\tau_f]$ are a subset of the ones of $\ell^3 \mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Frob}_\ell))$. As τ_ℓ is tempered, all the eigenvalues of $\mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Fr}_\ell))$ have absolute value equal to 1 (cf. [25, §6]). Hence the eigenvalues of $\ell^3 \mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Fr}_\ell))$, and thus of Frob_ℓ , have all absolute value equal to ℓ^3 . In particular, $H^i(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\tau_f]$ is zero unless $i = 6$. Finally, notice that $H^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})_{\mathfrak{m}_\pi} \neq 0$ as the assumption **(DS)** implies $H^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\pi_f] \neq 0$. □

Remark 2.6. The proof of Proposition 2.5 is similar to the one of [41, Proposition 8.2], where the proof is carried on with a trace formula argument.

2.8. Hodge theory. We keep the same notation as §2.7. In particular, $\pi = \pi_\infty \otimes \pi_f$ is a cuspidal automorphic representation of \mathbf{G} with trivial central character which satisfies **(DS)** and **(St)**, with $\pi_\infty \in P(V)$ for some irreducible algebraic representation V of \mathbf{G} as above.

Let \mathcal{V} denote the \mathbf{Q} -local system on $\mathrm{Sh}_{\mathbf{G}}(U)$ attached to V . We can take the π_f -isotypic component $H_{B, *}^6[\pi_f]$ of $H_*^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})$, where $* \in \{\emptyset, !\}$ and where $\mathcal{V}_{\mathbf{C}}$ denotes $\mathcal{V} \otimes_{\mathbf{Q}} \mathbf{C}$. Propositions 2.4 and 2.5 imply

$$H_B^\bullet[\pi_f] = H_{B, !}^\bullet[\pi_f] = H_{B, !}^6[\pi_f] \neq 0. \quad (5)$$

By [4, (2.3.1)] (see also [53, Proposition 2.15]), if L is a sufficiently large number field, $H_B^6[\pi_f]$ appears as a sub-quotient of $H_1^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_L)$, where \mathcal{V}_L denotes $\mathcal{V} \otimes_{\mathbf{Q}} L$. In particular, we have a projection

$$\mathrm{pr}_{\pi} : H^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_L)_{\mathrm{m}\pi}(n) \rightarrow H_B^6[\pi_f](n).$$

Since $H_1^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_L)$ is a pure L -Hodge structure of weight 6, we have

$$H_B^6[\pi_f] = \pi_f^U(L) \otimes M_B(\pi_f),$$

with $\pi_f^U(L)$ a realization of π_f^U over L and $M_B(\pi_f)$ a pure L -Hodge structure of weight 6. Thus we have a decomposition

$$M_B(\pi_f) \otimes \mathbf{C} = \bigoplus_{p+q=6} H^{p,q}(\pi_f).$$

Lemma 2.7. *Under the hypothesis (DS) and (St)*

$$\dim_{\mathbf{C}} H^{p,q}(\pi_f) = \begin{cases} 1 & \text{if } p \neq 3, \\ 2 & \text{if } p = 3. \end{cases}$$

In particular, we have $\dim_L M_B(\pi_f) = 8$.

Proof. Thanks to (5), we have that

$$H_B^6[\pi_f] \otimes \mathbf{C} = H_{B,!}^6[\pi_f] \otimes \mathbf{C} = H_{B,\mathrm{cusp}}^6[\pi_f] \otimes \mathbf{C},$$

hence

$$H_B^6[\pi_f] \otimes \mathbf{C} = \pi_f^U \otimes \bigoplus_{\sigma_{\infty}} H^6(\mathfrak{g}, K_{\mathbf{G}}; \sigma_{\infty} \otimes V)^{m(\sigma)},$$

where σ_{∞} runs over the elements of the discrete series L -packet $P(V)$ of $\mathbf{PGSp}_6(\mathbf{R})$ and $m(\sigma)$ denotes the multiplicity of $\sigma = \sigma_{\infty} \otimes \pi_f$. Notice that $H^6(\mathfrak{g}, K_{\mathbf{G}}; \sigma_{\infty} \otimes V)$ equals

$$\mathrm{Hom}_{K_{\infty}} \left(\bigwedge^6 \mathfrak{g}_0/\mathfrak{k}, \sigma_{\infty}^1 \otimes V \right) \oplus \mathrm{Hom}_{K_{\infty}} \left(\bigwedge^6 \mathfrak{g}_0/\mathfrak{k}, \bar{\sigma}_{\infty}^1 \otimes V \right) \quad (6)$$

where we have denoted $\sigma_{\infty|_{\mathbf{G}_0(\mathbf{R})}} = \sigma_{\infty}^1 \oplus \bar{\sigma}_{\infty}^1$. According to [7, Theorem II.5.3 b)], each space in the decomposition above is 1-dimensional. Moreover there exists a unique pair of integers $(r_{\sigma_{\infty}}, s_{\sigma_{\infty}})$ satisfying $r_{\sigma_{\infty}} + s_{\sigma_{\infty}} = 6$ such that (6) equals

$$\mathrm{Hom}_{K_{\infty}} \left(\bigwedge^{r_{\sigma_{\infty}}} \mathfrak{p}_{\mathbf{C}}^+ \otimes \bigwedge^{s_{\sigma_{\infty}}} \mathfrak{p}_{\mathbf{C}}^-, \sigma_{\infty}^1 \otimes V \right) \oplus \mathrm{Hom}_{K_{\infty}} \left(\bigwedge^{s_{\sigma_{\infty}}} \mathfrak{p}_{\mathbf{C}}^+ \otimes \bigwedge^{r_{\sigma_{\infty}}} \mathfrak{p}_{\mathbf{C}}^-, \bar{\sigma}_{\infty}^1 \otimes V \right).$$

As we remarked in §2.5, the set $P(V)$ has four elements and is in bijection with the set of Hodge types up to conjugation. Since the Hodge structure in $H_{B,\mathrm{cusp}}^6[\pi_f]$ is induced by this splitting, we deduce that

$$\dim_{\mathbf{C}} H^{r_{\sigma_{\infty}}, s_{\sigma_{\infty}}}(\pi_f) = \begin{cases} m(\sigma) & \text{if } r_{\sigma_{\infty}} \neq 3, \\ 2m(\sigma) & \text{if } r_{\sigma_{\infty}} = 3. \end{cases}$$

By [41, Theorem 12.1], the multiplicity of σ is either 0 or 1, while thanks to [41, Corollary 8.4 & Corollary 12.4] the dimension of $M_B(\pi_f)$ equals 8. Hence $m(\sigma) = 1$ for all $\sigma_{\infty} \in P(V)$, which concludes the proof. \square

2.9. Absolute Hodge cohomology. Let us first recall some definitions from [3]. A mixed \mathbf{R} -Hodge structure consists of a finite dimensional \mathbf{R} -vector space $M_{\mathbf{R}}$ equipped with an increasing finite filtration W_* called the weight filtration and a decreasing finite filtration F^* on $M_{\mathbf{C}} = M_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ called the Hodge filtration, such that each pair $(\mathrm{Gr}_n^W M_{\mathbf{R}}, (\mathrm{Gr}_n^W M_{\mathbf{C}}, F^*))$ is a pure \mathbf{R} -Hodge structure of weight n ([12, Définition 2.1.10]). The category of mixed \mathbf{R} -Hodge structures is an abelian category [12, Théorème (2.3.5)] and we denote it by $\mathrm{MHS}_{\mathbf{R}}$.

Definition 2.8. A real mixed \mathbf{R} -Hodge structure is given by a mixed \mathbf{R} -Hodge structure such that $M_{\mathbf{R}}$ is equipped with an involution F_{∞}^* stabilizing the weight filtration and whose \mathbf{C} -antilinear complexification $\overline{F_{\infty}^*} = F_{\infty}^* \otimes c$, where c denotes the complex conjugation, defines an involution on $M_{\mathbf{C}}$ stabilizing the Hodge filtration.

We will refer to F_{∞}^* as the real Frobenius and to $\overline{F_{\infty}^*}$ as the de Rham involution. We denote by $\mathrm{MHS}_{\mathbf{R}}^+$ the abelian category of real mixed Hodge A -structures. For any pair of objects $M, N \in D(\mathrm{MHS}_{\mathbf{R}}^+)$, one has $R\mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}^+}(M, N) = R\mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}}(M, N)^{\overline{F_{\infty}^*}}$, since we are assuming that A is a field and hence taking invariants by $\overline{F_{\infty}^*}$ is an exact functor.

Definition 2.9. If $M = (M_{\mathbf{R}}, F_{\infty}^*) \in C(\mathrm{MHS}_{\mathbf{R}}^+)$ is a complex of real mixed \mathbf{R} -Hodge structure, its absolute Hodge cohomology is defined as

$$R\Gamma_{\mathcal{H}}(M) = R\mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}}(\mathbf{R}(0), M_{\mathbf{R}}).$$

Its real absolute Hodge cohomology is defined as

$$R\Gamma_{\mathcal{H}/\mathbf{R}}(M) := R\mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}^+}(\mathbf{R}(0), M) = R\Gamma_{\mathcal{H}}(M_{\mathbf{R}})^{\overline{F_{\infty}^*}}.$$

The cohomology groups $H_B^i(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}})$, where $\mathcal{V}_{\mathbf{R}} = \mathcal{V} \otimes_{\mathbf{Q}} \mathbf{R}$, are equipped with a real Frobenius F_{∞}^* acting as the complex conjugation on (the complex points) $\mathrm{Sh}_{\mathbf{G}}(U)$ and on $\mathcal{V}_{\mathbf{R}}$, define real mixed \mathbf{R} -Hodge structures. This can be deduced directly from [12] since the cohomology with coefficients is a direct factor of the cohomology of a fiber product of the universal abelian variety of $\mathrm{Sh}_{\mathbf{G}}(U)$, or from the theory of mixed Hodge modules of [51]. We let $M \in C(\mathrm{MHS}_{\mathbf{R}}^+)$ be the complex of real mixed \mathbf{R} -Hodge structures given by $(\bigoplus_{i \in \mathbf{N}} H_B^i(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}})[-i], F_{\infty}^*)$ and we define the absolute real Hodge cohomology $H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathcal{V}_{\mathbf{R}}(4))$ of $\mathrm{Sh}_{\mathbf{G}}(U)$ and coefficients in $\mathcal{V}_{\mathbf{R}}(4)$ to be $H^1(R\Gamma_{\mathcal{H}/\mathbf{R}}(M(4)))$. Then we have the short exact sequence

$$0 \rightarrow \mathrm{Ext}_{\mathrm{MHS}_{\mathbf{R}}^+}^1(\mathbf{R}(0), H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))) \rightarrow$$

$$H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathcal{V}_{\mathbf{R}}(4)) \rightarrow \mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}^+}(\mathbf{R}(0), H_B^7(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))) \rightarrow 0$$

If $\pi = \pi_{\infty} \otimes \pi_f$ is as above, we denote by

$$H_{\mathcal{H}}^1(M(\pi_f)\mathbf{R}(4)) := (H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathcal{V}_{\mathbf{R}}(4)) \otimes L) [\pi_f]$$

the π_f -isotypical component.

Lemma 2.10. Under the hypothesis **(DS)** and **(St)**, we have a canonical short exact sequence of finite rank free $\mathbf{R} \otimes_{\mathbf{Q}} L$ -modules

$$0 \rightarrow F^4 H_{\mathrm{dR}}^6[\pi_f] \rightarrow H_{\mathbf{B}}^6[\pi_f]^{F_{\infty}^* = -1}(3) \rightarrow H_{\mathcal{H}}^1(M(\pi_f)\mathbf{R}(4)) \rightarrow 0.$$

Moreover, we have

$$\dim_{\mathbf{R} \otimes_{\mathbf{Q}} L} H_{\mathcal{H}}^1(M(\pi_f)\mathbf{R}(4)) = \dim_{\mathbf{C}} \pi_f^U.$$

Proof. It follows from the existence of the short exact sequence above and from Proposition 2.5 that we have a canonical isomorphism

$$H_{\mathcal{H}}^1(M(\pi_f)_{\mathbf{R}}(4)) \simeq \text{Ext}_{\text{MHS}_{\mathbf{R}}^+}^1(\mathbf{R}(0), H_B^6(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))[\pi_f]).$$

Hence, the first statement of the Lemma follows as in [44, Lemma 4.11]. In particular, the map $F^4 H_{\text{dR}}^6[\pi_f] \rightarrow H_B^6[\pi_f]^{F_{\infty}^*=-1}(3)$ is defined by the composition of

$$F^4 H_{\text{dR}}^6[\pi_f] \rightarrow H_{\text{dR}}^6[\pi_f] \otimes \mathbf{C} \simeq H_B^6(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\pi_f],$$

of the projection to $H_B^6(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\pi_f]^{\overline{F_{\infty}^*}=1}$, where $\overline{F_{\infty}^*}$ is the complexification $F_{\infty}^* \otimes c$, with c denoting the complex conjugation, and of the projection

$$H_B^6(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}})[\pi_f]^{\overline{F_{\infty}^*}=1} = H_B^6[\pi_f]^{F_{\infty}^*=-1}(3) \oplus H_B^6[\pi_f]^{F_{\infty}^*=1}(4) \rightarrow H_B^6[\pi_f]^{F_{\infty}^*=-1}(3).$$

Finally, by Lemma 2.7, we have that

$$\dim_{\mathbf{R} \otimes_{\mathbf{Q}} L} F^4 H_{\text{dR}}^6[\pi_f] = 3 \dim_L \pi_f^U(L) = 3 \dim_{\mathbf{C}} \pi_f^U.$$

On the other hand,

$$\dim_{\mathbf{R} \otimes_{\mathbf{Q}} L} H_B^6[\pi_f]^{F_{\infty}^*=-1}(3) = (3 + h^{3,+}) \dim_{\mathbf{C}} \pi_f^U,$$

where $h^{3,+}$ is the dimension of the \mathbf{C} -vector space $\{x \in H^{3,3}(\pi_f) : F_{\infty}^*(x) = -x\}$ (cf. [9, §3.4.2]). By the proof of Lemma 2.7, we have $h^{3,+} = 1$, which implies the result. \square

3. CONSTRUCTION OF THE MOTIVIC CLASS

3.1. Cartan product. Before starting, we briefly recall some properties of the Cartan product of irreducible representations that will be needed (cf. [54, §2.5] for more details). Let A denote either a connected compact Lie group or a semisimple algebraic group. Fix a Cartan subgroup of A and an orientation of the roots. Irreducible algebraic representations of A are parametrized by dominant weights. If λ and σ are two dominant weights with corresponding representations V^{λ} and V^{σ} , then the representation $V^{\lambda+\sigma}$ appears in $V^{\lambda} \otimes V^{\sigma}$ with multiplicity one. We denote it by $V^{\lambda} \cdot V^{\sigma}$ and we call it the Cartan component of $V^{\lambda} \otimes V^{\sigma}$. Clearly, the tensor product of two highest weight vectors maps to a corresponding highest weight vector. We denote by $v \otimes w \mapsto v \cdot w$ the projection from $V^{\lambda} \otimes V^{\sigma}$ to its Cartan component $V^{\lambda} \cdot V^{\sigma}$.

Lemma 3.1. [54, Lemma 2.12] *Every non-zero pure tensor in $V^{\lambda} \otimes V^{\sigma}$ projects non trivially to the Cartan component.*

3.2. Branching laws. In what follows, we fix a totally real field F over which \mathbf{H} splits. Since \mathbf{H} is split over F , its finite dimensional irreducible representations are determined by the highest weight theory and we can thus use the branching laws for algebraic representations from \mathbf{G} to \mathbf{H} established in [10].

Lemma 3.2. *The \mathbf{G} -representation V^{λ} over F of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, c)$ contains the trivial \mathbf{H} -representation if and only if $c = 0$ and $\lambda_1 = \lambda_2 + \lambda_3$. When this holds the trivial representation of \mathbf{H} appears in $(V^{\lambda})|_{\mathbf{H}}$ with multiplicity $\lambda_2 - \lambda_3 + 1$.*

Proof. From [10, Lemma 2.10], the sum of all irreducible sub- \mathbf{H} -representations of V^{λ} isomorphic (up to a twist) to $\text{Sym}^{(k,0,0)}$ for some $k \geq 0$ is given by

$$\bigoplus_{\substack{k=|\lambda_1-\lambda_2-\lambda_3| \\ k \equiv |\lambda| \pmod{2}}}^{\lambda_1-\lambda_2+\lambda_3} r \cdot \text{Sym}^{(k,0,0)} \otimes \det^{\frac{|\lambda|-k}{2}},$$

for $r = \lambda_2 - \lambda_3 + 1$. From this we deduce that V^λ contains the trivial \mathbf{H} -representation with multiplicity $r = \lambda_2 - \lambda_3 + 1$ if and only if $\lambda_1 - \lambda_2 - \lambda_3 = 0$. \square

It will be useful to construct explicitly generators of the trivial \mathbf{H} -representations inside V^λ given by the branching laws. We achieve this by constructing some vectors in the representations $V^{(1,1,0,0)}$ and $V^{(2,1,1,0)}$ and then by taking their Cartan product. From now on, all the representations are defined over F . Moreover, since the branching laws are determined by the restriction to the derived subgroups, in the following we work with the groups

$$\mathbf{H}_0 := \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \hookrightarrow \mathbf{H}'_0 := \mathrm{SL}_2 \times \mathrm{Sp}_4 \hookrightarrow \mathbf{G}_0 = \mathrm{Sp}_6.$$

Recall that we associate to any $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2$ such that $\lambda_1 \geq \lambda_2 \geq 0$, the irreducible Sp_4 -representation with highest weight λ . Applying the branching laws [10, Proposition 2.8], we get the following decompositions of representations of \mathbf{H}'_0 :

$$\begin{aligned} V^{(1,1,0)} &= (\mathrm{Sym}^0 \boxtimes V^{(0,0)}) \oplus (\mathrm{Sym}^0 \boxtimes V^{(1,1)}) \oplus (\mathrm{Sym}^1 \boxtimes V^{(1,0)}), \\ V^{(2,1,1)} &= (\mathrm{Sym}^0 \boxtimes V^{(1,1)}) \oplus (\mathrm{Sym}^0 \boxtimes V^{(2,0)}) \oplus (\mathrm{Sym}^1 \boxtimes V^{(1,0)}) \oplus (\mathrm{Sym}^1 \boxtimes V^{(2,1)}) \\ &\quad \oplus (\mathrm{Sym}^2 \boxtimes V^{(1,1)}). \end{aligned}$$

By Lemma 3.2, $V^{(1,1,0)}$ contains two copies of the trivial \mathbf{H}_0 -representation, each of which lies resp. in $\mathrm{Sym}^0 \boxtimes V^{(0,0)}$ and $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$, while $V^{(2,1,1)}$ contains a unique trivial \mathbf{H}_0 -representation appearing in $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$. Using these facts, we can explicitly define generators of these three trivial representations of \mathbf{H}_0 .

Let V be the standard representation of \mathbf{G}_0 with its symplectic basis $\langle e_1, e_2, e_3, f_1, f_2, f_3 \rangle$ given in §2.6. According to our choice of embedding $\mathbf{H}'_0 \hookrightarrow \mathbf{G}_0$, $\langle e_1, f_1 \rangle$, resp. $\langle e_2, e_3, f_2, f_3 \rangle$, defines a basis of the standard representation of SL_2 , resp. Sp_4 . We first recall how one can realize the representations $V^{(1,1,0)}$ and $V^{(1,1,1)}$. As explained in [14, §17.1], $V^{(1,1,0)}$ is realized inside $\bigwedge^2 V$ as the complement of the \mathbf{G}_0 -invariant subspace generated by the vector $e_1 \wedge f_1 + e_2 \wedge f_2 + e_3 \wedge f_3$ corresponding to the symplectic form or, in other words, as the kernel of the map $\bigwedge^2 V \rightarrow V$ sending $v_1 \wedge v_2$ to $\psi(v_1, v_2)$. By [14, Theorem 17.5], the irreducible representation $V^{(1,1,1)}$ is identified with the kernel of the map $\varphi : \bigwedge^3 V \rightarrow V$, $v_1 \wedge v_2 \wedge v_3 \mapsto \sum_{i < j, k \neq i, j} \psi(v_i, v_j) (-1)^{i-j+1} v_k$.

Lemma 3.3. *Let $F(0)$ denote the trivial \mathbf{H}_0 -representation. We have the following*

$$\begin{aligned} v &:= e_2 \wedge f_2 - e_3 \wedge f_3 \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(1,1)} \subseteq V^{(1,1,0)}, \\ w &:= e_2 \wedge f_2 + e_3 \wedge f_3 - 2e_1 \wedge f_1 \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(0,0)} \subseteq V^{(1,1,0)}, \\ z &:= z_1 - z_2 \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(1,1)} \subseteq V^{(2,1,1)}, \end{aligned}$$

where

$$\begin{aligned} z_1 &:= e_1 \cdot (f_1 \wedge e_2 \wedge f_2 - f_1 \wedge e_3 \wedge f_3), \\ z_2 &:= f_1 \cdot (e_1 \wedge e_2 \wedge f_2 - e_1 \wedge e_3 \wedge f_3), \end{aligned}$$

and \cdot denotes the Cartan product.

Proof. The vector v is obtained from the highest weight vector $e_1 \wedge e_2$ in $V^{(1,1,0)}$ by applying the composition $X_{(0,1,-1)} \circ X_{(0,-2,0)} \circ X_{(-1,0,1)}$, where $X_{(-1,0,1)}, X_{(0,-2,0)}, X_{(0,1,-1)} \in \mathfrak{sp}_6$ denote the weight vectors for $\lambda(-1, 0, 1)$, $\lambda(0, -2, 0)$, and $\lambda(0, 1, -1)$ respectively (cf. [14, §16.1] for the precise description). Moreover, the vector $X_{(-1,0,1)}(e_1 \wedge e_2) = -e_2 \wedge e_3$ is of weight $(0, 1, 1)$, which appears only in the component $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$, and $X_{(0,1,-1)}, X_{(0,-2,0)} \in$

$\mathfrak{sp}_4 \subseteq \mathfrak{sp}_6$ so v still lies inside $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$. The vector w is \mathbf{H}'_0 -invariant and therefore it generates the only trivial \mathbf{H}'_0 -representation in $V^{(1,1,0)}$. We now explain the definition of z . Note that $e_1 \in V^{(1,0,0)}$ and $f_1 \wedge e_2 \wedge f_2 - f_1 \wedge e_3 \wedge f_3 \in V^{(1,1,1)}$. Thus, by the properties of the Cartan product

$$V^{(1,0,0)} \otimes V^{(1,1,1)} = V^{(1,1,0)} \oplus V^{(2,1,1)} \rightarrow V^{(2,1,1)}, \quad v_1 \otimes v_2 \mapsto v_1 \cdot v_2,$$

z_1 is a non-zero vector in $V^{(2,1,1)}$ by Lemma 3.1. The vector z_1 is fixed by $\{I_2\} \times \mathrm{SL}_2^2$, but not by $\mathrm{SL}_2 \times \{I_2\} \times \{I_2\}$, however, as it is easy to verify, we have that

$$z = z_1 + h \cdot z_1 = z_1 - z_2 \in F(0) \subset V^{(2,1,1)}, \quad \text{with } h = \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, I_2, I_2 \right),$$

generates the unique trivial \mathbf{H}_0 -representation of $V^{(2,1,1)}$. \square

Lemma 3.4. *Let $\lambda = (\lambda_2 + \lambda_3, \lambda_2, \lambda_3, 0)$ with $\lambda_2 \geq \lambda_3 \geq 0$. For each $\lambda_2 \geq \mu \geq \lambda_3$, the vector*

$$v^{[\lambda, \mu]} := v^{\lambda_2 - \mu} \cdot w^{\mu - \lambda_3} \cdot z^{\lambda_3} \in F(0) \subseteq (V^\lambda)|_{\mathbf{H}}$$

realizes a distinct copy of the trivial representation $F(0)$ of \mathbf{H} inside $(V^\lambda)|_{\mathbf{H}}$.

Proof. For $p, q, r \in \mathbf{N}$, we have

$$v^p \cdot w^q \cdot z^r \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(p+r, p+r)} \subseteq V^{(p+q+2r, p+q+r)}.$$

The vectors v, w, z are \mathbf{H} -highest weight vectors, thus $v^{[\lambda, \mu]}$ is such one too. We are left to show that each of the vectors is different. This follows from the fact that each $v^{[\lambda, \mu]}$ lies in $\mathrm{Sym}^0 \boxtimes V^{(\lambda_2 + \lambda_3 - \mu, \lambda_2 + \lambda_3 - \mu)} \otimes \nu^{\mu - \lambda_2 - \lambda_3}$ and these representations are all different as μ varies. \square

3.3. The motivic class. As in the section above, we fix a totally real field F such that \mathbf{H} splits over F . For a smooth quasi-projective scheme S over a field of characteristic zero, let $\mathrm{CHM}_L(S)$ denote the tensor category of relative Chow motives over S with coefficients in a number field L and denote by $M : \mathrm{Var}/S \rightarrow \mathrm{CHM}_L(S)$ the contravariant functor from the category of smooth projective schemes over S to the category of relative Chow motives over S (cf. [1, §2.1]). By [13, Corollary 3.2], if A/S is an abelian scheme of relative dimension g , there is a decomposition $M(A) = \bigoplus_{i=1}^{2g} h^i(A)$ in $\mathrm{CHM}_L(S)$. Let G temporarily denote one of the groups \mathbf{H} or \mathbf{G} , and denote by $\mathrm{Rep}_F(G)$ the category of finite dimensional algebraic representations of G defined over F . In [1], Ancona constructs an additive functor

$$\mu_U^G : \mathrm{Rep}_F(G) \longrightarrow \mathrm{CHM}_F(\mathrm{Sh}_G(U)),$$

where U is a sufficiently small open compact subgroup of $G(\mathbf{A}_f)$. We recall some of its properties.

Proposition 3.5 ([1, Théorème 8.6]). *The functor μ_U^G respects duals, tensor products and satisfies the following properties.*

- (1) *If V is the standard representation of G , then $\mu_U^G(V) = h^1(\mathcal{A}_G)$, where \mathcal{A}_G is the universal abelian scheme over $\mathrm{Sh}_G(U)$.*
- (2) *If $\nu : G \rightarrow \mathbf{G}_m$ is the multiplier, then $\mu_U^G(\nu)$ is the Lefschetz motive $F(-1)$.*
- (3) *For a G -representation V defined over F , the Betti realization of $\mu_U^G(V)$ is the local system \mathcal{V}_F associated to the vector bundle*

$$G(\mathbf{Q}) \backslash (X_G \times V \times (G(\mathbf{A}_f)/U)) \rightarrow \mathrm{Sh}_G(U)(\mathbf{C}).$$

- (4) For any prime v of F above ℓ and G -representation V , the v -adic étale realization \mathcal{V}_v of $\mu_U^G(V)$ is the étale sheaf associated to $V \otimes_F F_v$, with U acting on the left via $U \hookrightarrow G(\mathbf{A}_f) \rightarrow G(\mathbf{Q}_\ell)$.

Definition 3.6. Let V^λ be the finite dimensional irreducible algebraic representation over \mathbf{Q} of \mathbf{G} of highest weight λ . We denote by \mathcal{V}_F^λ the relative Chow motive associated to $V^\lambda \otimes F$.

Let $U \subset \mathbf{G}(\mathbf{A}_f)$ be a sufficiently small compact open subgroup and let $U' = U \cap \mathbf{H}(\mathbf{A}_f)$. Recall that we have a closed embedding $\iota : \mathrm{Sh}_{\mathbf{H}}(U') \hookrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$ which is of codimension 3. Let V^λ the algebraic representation of \mathbf{G} (over F) of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, c)$ such that $\lambda_1 = \lambda_2 + \lambda_3$ and $c = 0$. Using the branching laws of Lemma 3.2 and [55, Theorem 1.2], we get the following (cf. [10, Proposition 2.17]).

Proposition 3.7. For any $\lambda_2 \geq \mu \geq \lambda_3$, we have a Gysin morphism

$$\iota_*^{[\lambda, \mu]} : H_{\mathcal{M}}^0(\mathrm{Sh}_{\mathbf{H}}(U'), F(0)) \rightarrow H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)),$$

corresponding to the embedding of $F(0) \subset \iota^* V^\lambda$ given by the \mathbf{H} -trivial vector $v^{[\lambda, \mu]}$ of Lemma 3.4.

Definition 3.8. We let $\mathcal{Z}_{\mathbf{H}, \mathcal{M}}^{[\lambda, \mu]} \in H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))$ be the image by $\iota_*^{[\lambda, \mu]}$ of

$$\mathbf{1}_{\mathrm{Sh}_{\mathbf{H}}(U')} \in \mathrm{CH}^0(\mathrm{Sh}_{\mathbf{H}}(U'))_F = H_{\mathcal{M}}^0(\mathrm{Sh}_{\mathbf{H}}(U'), F(0)).$$

3.4. Realizations.

3.4.1. *Étale realization.* Let \mathfrak{l} be a prime of F above ℓ . We have an étale cycle class map

$$\mathrm{cl}_{\mathrm{ét}} : H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)) \rightarrow H_{\mathrm{ét}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathfrak{l}}^\lambda(3)) \rightarrow H_{\mathrm{ét}}^6(\mathrm{Sh}_{\mathbf{G}}(U)_{\overline{\mathbf{Q}}}, \mathcal{V}_{\mathfrak{l}}^\lambda(3))^{\mathrm{G}_{\mathbf{Q}}},$$

where the last arrow is the natural map obtained from the Hochschild-Serre spectral sequence. We define the following.

Definition 3.9. We let $\mathcal{Z}_{\mathbf{H}, \mathrm{ét}}^{[\lambda, \mu]} := \mathrm{cl}_{\mathrm{ét}}(\mathcal{Z}_{\mathbf{H}, \mathcal{M}}^{[\lambda, \mu]}) \in H_{\mathrm{ét}}^6(\mathrm{Sh}_{\mathbf{G}}(U)_{\overline{\mathbf{Q}}}, \mathcal{V}_{\mathfrak{l}}^\lambda(3))^{\mathrm{G}_{\mathbf{Q}}}$.

Remark 3.10.

- Notice that $\mathcal{Z}_{\mathbf{H}, \mathrm{ét}}^{[\lambda, \mu]}$ equals to the image of $\mathbf{1} \in H_{\mathrm{ét}}^0(\mathrm{Sh}_{\mathbf{H}}(U')_{\overline{\mathbf{Q}}}, F_{\mathfrak{l}}(0))$ via the étale Gysin map

$$\iota_{\mathrm{ét}, * }^{[\lambda, \mu]} : H_{\mathrm{ét}}^0(\mathrm{Sh}_{\mathbf{H}}(U')_{\overline{\mathbf{Q}}}, F_{\mathfrak{l}}(0)) \rightarrow H_{\mathrm{ét}}^0(\mathrm{Sh}_{\mathbf{H}}(U')_{\overline{\mathbf{Q}}}, \iota^* \mathcal{V}_{\mathfrak{l}}^\lambda) \rightarrow H_{\mathrm{ét}}^6(\mathrm{Sh}_{\mathbf{G}}(U)_{\overline{\mathbf{Q}}}, \mathcal{V}_{\mathfrak{l}}^\lambda(3)).$$

- As the representation V^λ is self dual, we have a Galois equivariant perfect pairing

$$H_{\mathrm{ét}, c}^6(\mathrm{Sh}_{\mathbf{G}}(U)_{\overline{\mathbf{Q}}}, \mathcal{V}_{\mathfrak{l}}^\lambda(3)) \times H_{\mathrm{ét}}^6(\mathrm{Sh}_{\mathbf{G}}(U)_{\overline{\mathbf{Q}}}, \mathcal{V}_{\mathfrak{l}}^\lambda(3)) \rightarrow F_{\mathfrak{l}}(0).$$

Hence, by duality $\mathcal{Z}_{\mathbf{H}, \mathrm{ét}}^{[\lambda, \mu]}$ determines a map

$$H_{\mathrm{ét}, c}^6(\mathrm{Sh}_{\mathbf{G}}(U)_{\overline{\mathbf{Q}}}, \mathcal{V}_{\mathfrak{l}}^\lambda(3)) \rightarrow F_{\mathfrak{l}}(0).$$

3.4.2. *Betti realizations.* As in the previous subsection, we define the class

$$\mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]} \in H_B^6(\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C}), \mathcal{V}_F^\lambda(3))$$

as the image of $\mathcal{Z}_{\mathbf{H}, \mathcal{M}}^{[\lambda, \mu]}$ via the Betti cycle class map

$$\mathrm{cl}_B : H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)) \rightarrow H_B^6(\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C}), \mathcal{V}_F^\lambda(3)).$$

Note that, as F is totally real, the image

$$\mathrm{Im}(\mathrm{cl}_B) \subset H_B^6(\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C}), \mathcal{V}_{\mathbf{R}}^\lambda(3))^{\mathrm{F}_\infty^* = 1},$$

where F_∞^* denotes the composition of the map induced by complex conjugation on the \mathbf{C} -points of $\mathrm{Sh}_{\mathbf{G}}(U)$ with complex conjugation on the coefficients.

3.4.3. *Absolute Hodge realizations.* Let $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))^0 = \ker(\mathrm{cl}_B)$ denote the subgroup of homologically trivial classes and let $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}}$ denote the quotient $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))/H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))^0$. Note that when $\lambda_2 = \lambda_3 = 0$, i.e. the representation V^λ is the trivial representation, then $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)) = H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), F(3)) = \mathrm{CH}^3(\mathrm{Sh}_{\mathbf{G}}(U))_F$ is the usual Chow group of 3-codimensional cycles modulo rational equivalence and the space $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}} = \mathbf{N}^3(\mathrm{Sh}_{\mathbf{G}}(U))_F$ is the space of 3-codimensional cycles modulo homological equivalence, with coefficients in F . In this section, we define a natural injective map

$$H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}} \rightarrow H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}^\lambda(4)). \quad (7)$$

The definition is similar to the one for smooth projective varieties (see [52, §5]) and we recall it for the convenience of the reader. The cycle class map is an injection

$$\mathrm{cl}_B : H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}} \rightarrow H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F(3))^{F_\infty^*=1} \cap H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}^\lambda(3))^{0,0}$$

where $H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}^\lambda(3))^{0,0}$ denotes the subspace of

$$W_0 H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}^\lambda(3)) = \mathrm{Gr}_0^W H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}^\lambda(3))$$

of vectors which have Hodge type $(0, 0)$. The composite of the inclusions

$$\begin{aligned} H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F(3))^{F_\infty^*=1} \cap H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}^\lambda(3))^{0,0} &\hookrightarrow W_0 H_{dR}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(3)) \\ &\hookrightarrow W_2 H_{dR}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(3)) \\ &= W_0 H_{dR}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4)) \end{aligned}$$

and of the projection

$$W_0 H_{dR}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4)) \rightarrow W_0 H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4)) \setminus W_0 H_{dR}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4)) / F^0 W_0 H_{dR}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))$$

is injective. As the last space above is canonically isomorphic to

$$\mathrm{Ext}_{\mathrm{MHS}_{\mathbf{R}}^+}^1(\mathbf{R}(0), H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))),$$

we obtain a natural injective map

$$H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}} \rightarrow \mathrm{Ext}_{\mathrm{MHS}_{\mathbf{R}}^+}^1(\mathbf{R}(0), H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))).$$

Composing this map with the canonical injection

$$\mathrm{Ext}_{\mathrm{MHS}_{\mathbf{R}}^+}^1(\mathbf{R}(0), H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))) \rightarrow H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}^\lambda(4))$$

we obtain the map (7). We denote by $\overline{\mathrm{cl}}_{\mathcal{H}} : H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)) \rightarrow H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}^\lambda(4))$ the composition of the map (7) with the projection $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)) \rightarrow H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}}$.

Definition 3.11. *We define*

$$\mathcal{Z}_{\mathbf{H}, \mathcal{H}}^{[\lambda, \mu]} := \overline{\mathrm{cl}}_{\mathcal{H}}(\mathcal{Z}_{\mathbf{H}, \mathcal{M}}^{[\lambda, \mu]}) \in H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}^\lambda(4)).$$

Remark 3.12. Let π be a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$ which satisfies the hypotheses of Lemma 2.10 and let S be a finite set of places containing the ramified places of π_f and ∞ . By the conjectures of Beilinson and Tate and the local calculations of Gross and Savin in [27], the family of cycles $\mathcal{Z}_{\mathbf{H}, \mathcal{H}}^{[\lambda, \mu]}$ obtained by varying the cubic étale algebra E/\mathbf{Q} defining \mathbf{H} , and their Hecke translates are expected to generate $H_{\mathcal{H}}^1(M(\pi_f)_{\mathbf{R}}(4))$ when $\mathrm{ord}_{s=1} L^S(s, \pi, \mathrm{Spin}) = -1$. One of our main results will confirm this expectation.

4. CONSTRUCTION OF THE DIFFERENTIAL FORM AND PAIRING WITH THE MOTIVIC CLASS

The purpose of this chapter is to study the Betti and Hodge realizations of the cycle constructed in §3.3 by relating their pairing with a suitable cuspidal harmonic differential form to an automorphic period.

4.1. Test vectors. Recall that the discrete series L -packets for $\mathrm{PGSp}_6(\mathbf{R})$ have four elements, each indexed by a Hodge type (and its conjugate). Let π denote a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$ for which π_∞ is the discrete series of Hodge type $(3,3)$ in the L -packet of V^λ where $\lambda = (\lambda_1, \lambda_2, \lambda_3, 0)$ and $\lambda_1 = \lambda_2 + \lambda_3$. This translates into saying that π is a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character for which

$$H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_\infty \otimes V^\lambda) \neq 0,$$

and such that $\pi_\infty|_{\mathbf{G}_0(\mathbf{R})} = \pi_{\infty,1}^{3,3} \oplus \bar{\pi}_{\infty,1}^{3,3}$ is the direct sum of the discrete series representations of respective Harish Chandra parameters $(\lambda_2 + 2, \lambda_3 + 1, -\lambda_1 - 3)$ and $(\lambda_1 + 3, -\lambda_3 - 1, -\lambda_2 - 2)$. Recall that these discrete series contain with multiplicity one their minimal K_∞ -types $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$ and $\tau_{(\lambda_1+4, -\lambda_3-2, -\lambda_2-2)}$ respectively. On the other hand, as K_∞ -representations we have

$$\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \supseteq \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^- = \bigoplus_i \tau_i \supseteq \tau_{(2,2,-4)} \oplus \tau_{(4,-2,-2)},$$

where the equality expresses the decomposition of the tensor product into irreducible K_∞ -representations. This fact will be useful to construct an element in

$$H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_\infty \otimes V^\lambda) = \mathrm{Hom}_{K_\infty} \left(\bigwedge^6 \mathfrak{p}_{\mathbf{C}}, \pi_\infty \otimes V^\lambda \right) \simeq \mathrm{Hom}_{K_\infty} \left(\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \otimes V^\lambda, \pi_\infty \right),$$

where the last equality follows from the fact that V^λ is self-dual. Before stating the next result, let us fix the following data.

- A highest weight vector Ψ_∞ , resp. $\bar{\Psi}_\infty$, of the minimal K_∞ -type $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$, resp. $\tau_{(\lambda_1+4, -\lambda_3-2, -\lambda_2-2)}$ of $\pi_{\infty,1}^{3,3}$, resp. $\bar{\pi}_{\infty,1}^{3,3}$.
- A highest weight vector $X_{(2,2,-4)}$, resp. $X_{(4,-2,-2)}$, of $\tau_{(2,2,-4)}$, resp. $\tau_{(4,-2,-2)}$.
- A highest weight vector $v^{\lambda'}$, resp. $v^{\bar{\lambda}'}$, of $\tau_{\lambda'} \subseteq V^\lambda$, resp. $\tau_{\bar{\lambda}'} \subseteq V^\lambda$, where $\tau_{\lambda'}$ and $\tau_{\bar{\lambda}'}$ denote the irreducible algebraic K_∞ -representations of highest weight $\lambda' = (\lambda_2, \lambda_3, -\lambda_1)$ and $\bar{\lambda}' = (\lambda_1, -\lambda_3, -\lambda_2)$ respectively.

Lemma 4.1. *The spaces $\mathrm{Hom}_{K_\infty}(\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \otimes V^\lambda, \pi_{\infty,1}^{3,3})$ and $\mathrm{Hom}_{K_\infty}(\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \otimes V^\lambda, \bar{\pi}_{\infty,1}^{3,3})$ are of dimension one and the elements*

$$\omega_{\Psi_\infty} \in \mathrm{Hom}_{K_\infty}(\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \otimes V^\lambda, \pi_{\infty,1}^{3,3}), \omega_{\bar{\Psi}_\infty} \in \mathrm{Hom}_{K_\infty}(\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \otimes V^\lambda, \bar{\pi}_{\infty,1}^{3,3})$$

defined by

$$\omega_{\Psi_\infty}(X_{(2,2,-4)} \otimes v^{\lambda'}) = \Psi_\infty$$

$$\omega_{\bar{\Psi}_\infty}(X_{(4,-2,-2)} \otimes v^{\bar{\lambda}'}) = \bar{\Psi}_\infty$$

are generators of these spaces.

Proof. This is a consequence of [7, Theorem II.5.3 b)] and its proof. □

4.2. Restriction to \mathbf{H} . Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, 0)$, with $\lambda_1 = \lambda_2 + \lambda_3$ and let V^λ be as above. Let \mathfrak{h} , resp. $\mathfrak{k}_{\mathbf{H}}$ denote the Lie algebra of $\mathbf{H}(\mathbf{R})$, resp. the maximal compact modulo the center $K_{\mathbf{H}}$. Observe that via the embedding $\iota : \mathbf{H}(\mathbf{R}) \hookrightarrow \mathbf{G}(\mathbf{R})$, the group $K_{\mathbf{H}}$ is isomorphic to T_∞ . One has a Cartan decomposition $\mathfrak{h}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{H},\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{H},\mathbf{C}}$, where $\mathfrak{p}_{\mathbf{H},\mathbf{C}}$ is 6-dimensional and is spanned by the non-compact root spaces. We fix once and for all a generator X_0 of the 1-dimensional \mathbf{C} -vector space $\bigwedge^6 \mathfrak{p}_{\mathbf{H},\mathbf{C}} \subseteq \bigwedge^6 \mathfrak{p}_{\mathbf{C}}$ as in [9, §4.4]. The main result of this section is the following.

Theorem 4.2. *Let ω_{Ψ_∞} and $\omega_{\overline{\Psi}_\infty}$ be the elements of $\text{Hom}_{K_\infty} \left(\bigwedge^6 \mathfrak{p}_{\mathbf{C}} \otimes V^\lambda, \pi_\infty \right)$ defined in Lemma 4.1. Let X_0 be as above and let v be any \mathbf{H} -invariant vector in V^λ . Then*

$$\omega_{\Psi_\infty}(X_0 \otimes v) \neq 0,$$

$$\omega_{\overline{\Psi}_\infty}(X_0 \otimes v) \neq 0.$$

The proof of Theorem 4.2 will be constructive and occupies the rest of this section. We start by recalling the following result.

Lemma 4.3 ([9, Lemma 4.21]). *Let X_0 be as above. Then the image of X_0 by*

$$\bigwedge^6 \mathfrak{p}_{\mathbf{H},\mathbf{C}} \rightarrow \bigwedge^6 \mathfrak{p}_{\mathbf{C}} \rightarrow \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^+ \otimes \bigwedge^3 \mathfrak{p}_{\mathbf{C}}^- \rightarrow \tau_{(2,2,-4)},$$

where the first map is induced by the embedding $\mathbf{H} \rightarrow \mathbf{G}$ and the second and the third maps are the natural projections, is non zero.

We next study the interaction between the branching laws of V^λ to the subgroup \mathbf{H} of \mathbf{G} and to its maximal compact subgroup. More precisely, we show that the \mathbf{H} -invariant vectors constructed in Lemma 3.4 project non-trivially to $\tau_{\lambda'}$ and $\tau_{\overline{\lambda}'}$ and moreover that their projections form a basis of the corresponding $(0,0,0)$ -weight spaces for the action of T_∞ .

Lemma 4.4. *Let $\tau_{\lambda'}$ and $\tau_{\overline{\lambda}'}$ be the irreducible algebraic sub- K_∞ -representations of V^λ of highest weight $\lambda' = (\lambda_2, \lambda_3, -\lambda_1)$ and $\overline{\lambda}' = (\lambda_1, -\lambda_3, -\lambda_2)$. Then the weight $(0,0,0)$ appears in both $\tau_{\lambda'}$ and $\tau_{\overline{\lambda}'}$ with multiplicity $\lambda_2 - \lambda_3 + 1$.*

Proof. Let $n_0(\lambda')$ denote the multiplicity of the weight $(0,0,0)$ in $\tau_{\lambda'}$. Kostant multiplicity formula reads as

$$n_0(\lambda') = \sum_{w \in \mathfrak{W}_{K_\infty}} (-1)^{\ell(w)} P(w(\lambda' + \rho_{K_\infty}) - \rho_{K_\infty}),$$

where $\rho_{K_\infty} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{g}}^+} \alpha = (1, 0, -1)$ and the function $\mu \mapsto P(\mu)$ calculates the number of ways for which the weight μ can be expressed as a linear combination

$$\alpha(e_1 - e_2) + \beta(e_1 - e_3) + \gamma(e_2 - e_3),$$

with $\alpha, \beta, \gamma \in \mathbf{Z}_{\geq 0}$ (cf. [14]). Using this formula, it is a tedious but straightforward calculation to verify that $n_0(\lambda') = \lambda_2 - \lambda_3 + 1$ and the same for $\overline{\lambda}' = w_8 \lambda'$. □

According to Lemma 4.4, there are $\lambda_2 - \lambda_3 + 1$ linearly independent vectors of weight $(0,0,0)$ in $\tau_{\lambda'}$. We now show that these weight vectors correspond one to one to the \mathbf{H} -invariant vectors of Lemma 3.2.

Lemma 4.5. *Let v, w be the vectors of $V^{(1,1,0)}$ and let z be the vector of $V^{(2,1,1)}$ defined in Lemma 3.3. The irreducible algebraic representation $\tau_{(1,0,-1)}$, resp. $\tau_{(1,1,-2)}$ and $\tau_{(2,-1,-1)}$, appear in the restriction of $V^{(1,1,0)}$, resp. of $V^{(1,1,1)}$, to K_∞ with multiplicity 1. Moreover, we have $v, w \in \tau_{(1,0,-1)} \subseteq V^{(1,1,0)}$, and $z \in \tau_{(1,1,-2)} \oplus \tau_{(2,-1,-1)} \subseteq V^{(1,1,1)}$, with z projecting non-trivially to each factor of this decomposition.*

Proof. First observe that $v, w \in V^{(1,1,0)}$ and $z \in V^{(2,1,1)}$ are vectors of weight $(0,0,0)$ both for the split and the compact tori of $\mathbf{G}_0(\mathbf{R})$. Indeed these vectors are fixed (up to a constant) by the matrix J sending the non-compact torus \mathbf{T}_0 to the compact torus T_∞ defined in §2.2. Using branching laws from $\mathbf{G}_0(\mathbf{R})$ to K_∞ , we have a decomposition of K_∞ -representations

$$V^{(1,1,0)} = \tau_{(1,1,0)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(0,-1,-1)}.$$

The weight $(0,0,0)$ appears only in $\tau_{(1,0,-1)}$ and with multiplicity 2. Since it has also multiplicity 2 in $V^{(1,1,0)}$, we deduce that $\{v, w\}$ forms a basis for the $(0,0,0)$ -eigenspace of $\tau_{(1,0,-1)}$. On the other hand, we have

$$\begin{aligned} V^{(2,1,1)} &= \tau_{(-1,-1,-2)} \oplus \tau_{(1,-1,-2)} \oplus \tau_{(1,1,0)} \oplus \tau_{(1,1,-2)} \\ &\quad \oplus \tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)} \oplus \tau_{(2,1,-1)} \oplus \tau_{(2,1,1)} \oplus \tau_{(0,-1,-1)}. \end{aligned}$$

The weight $(0,0,0)$ only appears in $\tau_{(1,1,-2)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)}$, which implies that

$$z \in \tau_{(1,1,-2)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)}.$$

Notice that the decomposition of the standard representation of \mathbf{G}_0

$$V = \tau_{(1,0,0)} \oplus \tau_{(0,0,-1)}$$

of K_∞ -representations can be realized by picking the basis $\{v_1, v_2, v_3, w_1, w_2, w_3\}$, where $v_r := e_r + if_r$ and $w_r := ie_r + f_r$. The set $\{v_r\}_{1 \leq r \leq 3}$, resp. $\{w_r\}_{1 \leq r \leq 3}$, defines a basis for $\tau_{(1,0,0)}$, resp. $\tau_{(0,0,-1)}$. We now write z in terms of this basis. By using the relations

$$\begin{aligned} e_r &= \frac{1}{2}v_r - \frac{i}{2}w_r, \\ f_r &= \frac{1}{2}w_r - \frac{i}{2}v_r, \end{aligned}$$

we have that

$$e_1 \otimes f_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3) - f_1 \otimes e_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3)$$

equals to

$$\frac{1}{4}(v_1 \otimes w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) - w_1 \otimes v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)).$$

Thus,

$$z = z_1 - z_2 = \frac{1}{4}(v_1 \cdot w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) - w_1 \cdot v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)).$$

Notice that the vector $w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) \in V^{(1,1,1)}$ is of weight $(-1,0,0)$ for T_∞ , while $v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) \in V^{(1,1,1)}$ is of weight $(1,0,0)$ for T_∞ . As

$$V^{(1,1,1)} = \tau_{(1,1,1)} \oplus \tau_{(1,-1,-1)} \oplus \tau_{(1,1,-1)} \oplus \tau_{(-1,-1,-1)},$$

and the fact that the weight $(-1,0,0)$ appears only in $\tau_{(1,-1,-1)}$ and $(1,0,0)$ only in $\tau_{(1,1,-1)}$, we have that

$$\begin{aligned} w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) &\in \tau_{(1,-1,-1)} \\ v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) &\in \tau_{(1,1,-1)} \end{aligned}$$

By the properties of the Cartan product, the vector $s_1 := v_1 \cdot w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)$ is non-zero in $\tau_{(2,-1,-1)}$, while $s_2 := w_1 \cdot v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)$ is non-zero in $\tau_{(1,1,-2)}$. This shows that the vector $z \in V^{(2,1,1)}$ lives in $\tau_{(2,-1,-1)} \oplus \tau_{(1,1,-2)}$, thus finishing the proof. \square

Proposition 4.6. *The set $\{\text{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})\}_\mu$, resp. $\{\text{pr}_{\tau_{\bar{\lambda}'}}(v^{[\lambda,\mu]})\}_\mu$, forms a basis of the weight $(0,0,0)$ -eigenspace of $\tau_{\lambda'} \subset V^\lambda$, resp. $\tau_{\bar{\lambda}'} \subset V^\lambda$.*

Proof. Recall that we have defined

$$v^{[\lambda,\mu]} := v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \cdot z^{\lambda_3} \in F(0) \subseteq (V^\lambda)|_{\mathbf{H}}.$$

By Lemma 4.5, we have that $v, w \in \tau_{(1,0,-1)} \subseteq V^{(1,1,0)}$ so that, for any $\lambda_3 \leq \mu \leq \lambda_2$, we have $v^{\lambda_2-\mu} \otimes w^{\mu-\lambda_3} \in \tau_{(1,0,-1)}^{\otimes \lambda_2-\lambda_3}$ and we deduce that the projection of $v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \in V^{(\lambda_2-\lambda_3, \lambda_2-\lambda_3, 0)}$ to $\tau_{(\lambda_2-\lambda_3, 0, \lambda_3-\lambda_2)}$ coincides with their Cartan product with respect to K_∞ . Moreover, each of these projections is non-zero because of Lemma 3.1. Since the vectors

$$v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \in \tau_{(\lambda_2-\lambda_3, 0, \lambda_3-\lambda_2)}$$

are all different as they live in different \mathbf{H}'_0 sub-representations (cf. the proof of Lemma 3.4), we conclude that they span the $\lambda_2 - \lambda_3 + 1$ -dimensional weight $(0,0,0)$ -eigenspace of $\tau_{(\lambda_2-\lambda_3, 0, \lambda_3-\lambda_2)}$. We now show that z^{λ_3} projects non-trivially to both $\tau_{(2\lambda_3, -\lambda_3, -\lambda_3)}$ and $\tau_{(\lambda_3, \lambda_3, -2\lambda_3)}$. Notice that, as the weights $(2\lambda_3, -\lambda_3, -\lambda_3)$ and $(\lambda_3, \lambda_3, -2\lambda_3)$ are extremal in $V^{(2\lambda_3, \lambda_3, \lambda_3)}$ and appear uniquely, we have a commutative diagram

$$\begin{array}{ccc} (V^{(2,1,1)})^{\otimes \lambda_3} & \xrightarrow{\quad \cdot \quad} & V^{(2\lambda_3, \lambda_3, \lambda_3)} \\ \text{(pr}_1, \text{pr}'_1) \downarrow & & \downarrow \text{pr}_2 \\ (\tau_{(2,-1,-1)})^{\otimes \lambda_3} \oplus (\tau_{(1,1,-2)})^{\otimes \lambda_3} & \xrightarrow{\quad \cdot \quad} & \tau_{(2\lambda_3, -\lambda_3, -\lambda_3)} \oplus \tau_{(\lambda_3, \lambda_3, -2\lambda_3)}, \end{array}$$

where the horizontal arrows are the Cartan projections and the vertical arrows are the natural projections given by the decomposition of $V^{(2r,r,r)}$ as K_∞ -representations. Thanks to the commutativity of the diagram, we know that the vector $z^{\otimes \lambda_3} \in (V^{(2,1,1)})^{\otimes \lambda_3}$ maps to

$$\text{pr}_2(z^{\lambda_3}) = \text{pr}_1(z)^{\lambda_3} + \text{pr}'_1(z)^{\lambda_3} = s_1^{\lambda_3} + s_2^{\lambda_3},$$

where s_1, s_2 are as in Lemma 4.5. This shows, again by Lemma 3.1, that each $v^{[\lambda,\mu]}$ projects non-trivially to both $\tau_{\lambda'}$ and $\tau_{\bar{\lambda}'}$ and that each of these projections are different by Lemma 3.4. Indeed,

$$\begin{aligned} \text{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]}) &= v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \cdot s_1^{\lambda_3}, \\ \text{pr}_{\tau_{\bar{\lambda}'}}(v^{[\lambda,\mu]}) &= v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \cdot s_2^{\lambda_3}. \end{aligned}$$

By Lemma 4.4, this means that $\{\text{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})\}_\mu$ (resp. $\{\text{pr}_{\tau_{\bar{\lambda}'}}(v^{[\lambda,\mu]})\}_\mu$) defines a basis of the weight $(0,0,0)$ -eigenspace of $\tau_{\lambda'}$, resp. $\tau_{\bar{\lambda}'}$. This finishes the proof. \square

We are now in condition of finishing the proof of Theorem 4.2

Proof of Theorem 4.2. By construction, the map ω_{Ψ_∞} factors through $\tau_{\lambda'+(2,2,-4)} \subseteq \tau_{(2,2,-4)} \otimes \tau_{\lambda'}$. Lemma 4.3 shows that the projection of X_0 to $\tau_{(2,2,-4)}$ is non-zero, while Proposition 4.6 shows that $\text{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})$ is non-zero. Since $\tau_{\lambda'+(2,2,-4)}$ is the Cartan product of $\tau_{(2,2,-4)}$ and $\tau_{\lambda'}$, we deduce from Lemma 3.1 that the image of the pure tensor $\text{pr}_{\tau_{(2,2,-4)}}(X_0) \otimes \text{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})$ is non-zero. \square

4.3. The pairing. Let π denote a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$ for which π_∞ is the discrete series of Hodge type (3,3) in the L -packet of V^λ with $\lambda = (\lambda_2 + \lambda_3, \lambda_2, \lambda_3, 0)$. Let $\Psi = \Psi_\infty \otimes \Psi_f$ denote a cusp form in $\pi = \pi_\infty \otimes \pi_f$. We assume that Ψ_∞ is a highest weight vector of the minimal K_∞ -type $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$ of $\pi_\infty|_{\mathbf{G}_0(\mathbf{R})}$. We let $[\omega_{\Psi_\infty}] \in H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_\infty \otimes V^\lambda)$ be the cohomology class of the harmonic differential form ω_{Ψ_∞} defined in Lemma 4.1. We also assume that $\Psi_f \in V_{\pi_f}$ is U -invariant. Then we have $[\omega_\Psi] := [\omega_{\Psi_\infty} \otimes \Psi_f] \in H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi^U \otimes V^\lambda)$.

Lemma 4.7. *There is a Hecke-equivariant inclusion*

$$H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi^U \otimes V^\lambda) \subset H_{dR,c}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{C}}^\lambda).$$

Moreover, if π_w is the Steinberg representation for some finite place w , such inclusion is unique.

Proof. Let $\mathcal{C}_{rd}^\infty(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})/U, V^\lambda)$ denote the space of V^λ -valued \mathcal{C}^∞ -functions on the double quotient $\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})/U$ which, together with all their right $\mathfrak{U}(\mathfrak{g}_{\mathbf{C}})$ -derivatives, are rapidly decreasing in the sense of [30]. As π is cuspidal and cusp forms are rapidly decreasing, we have $H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi_\infty \otimes V_{\mathbf{C}}^\lambda)^{m(\pi)} \otimes \pi_f^U \subset H^6(\mathfrak{g}, K_{\mathbf{G}}; \mathcal{C}_{rd}^\infty(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})/U, V^\lambda))$. Thus the result follows from the fact that, according to [6, Theorem 5.2] (see also [30, Theorem 1.4.1]), there exists a canonical Hecke equivariant isomorphism $H_{dR,c}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}^\lambda) \simeq H^6(\mathfrak{g}, K_{\mathbf{G}}; \mathcal{C}_{rd}^\infty(\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})/U, V^\lambda))$. Finally, if π_w is Steinberg at a finite place w , we have, as in Lemma 2.7, that $m(\pi) = 1$. \square

4.3.1. The pairing in Betti cohomology. Poincaré duality is a perfect pairing

$$\langle \cdot, \cdot \rangle : H_B^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3)) \times H_{B,c}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda) \rightarrow F(-3),$$

which is a morphism of mixed F -Hodge structures. Fix the choice of a measure dh on $\mathbf{H}(\mathbf{A})$ as follows. For each finite place p , we take the Haar measure dh_p on $\mathbf{H}(\mathbf{Q}_p)$ that assigns volume 1 to $\mathbf{H}(\mathbf{Z}_p)$. For the archimedean place, we let $X_0 \in \bigwedge^6 \mathfrak{p}_{\mathbf{H}, \mathbf{C}}$ be the generator fixed at the beginning of section 4.2. The choice of X_0 induces an equivalence between top differential forms on $X_{\mathbf{H}} = \mathbf{H}(\mathbf{R})/K_{\mathbf{H}, \infty}$ and invariant measures dh_∞ on $\mathbf{H}(\mathbf{R})$ assigning measure one to $K_{\mathbf{H}, \infty}$ (cf. [31, p. 83] for details). We let dh_∞ denote the measure associated in this way to the pullback of $\iota^{[\lambda, \mu]*} \omega_\Psi$ to $X_{\mathbf{H}}$ and we then define $dh = dh_\infty \prod_p dh_p$.

Proposition 4.8. *We have*

$$\langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle = \frac{h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \int_{\mathbf{H}(\mathbf{Q}) \backslash \mathbf{Z}_{\mathbf{G}}(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A})} A^{[\lambda, \mu]} \cdot \Psi(h) dh,$$

where $h_{U'} = 4^{-1} |\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}) \backslash \mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) / (\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) \cap U')|$ and $A^{[\lambda, \mu]} \in U(\mathfrak{k}_{\mathbf{C}})$ is an element for which $A^{[\lambda, \mu]} \cdot \Psi_\infty = \omega_{\Psi_\infty}(X_0 \otimes v^{[\lambda, \mu]})$.

Proof. By [6, Corollary 5.5], there exists a \mathcal{V}^λ -valued rapidly decreasing differential form η of degree five on $\mathrm{Sh}_{\mathbf{G}}(U)$ such that $\omega_c := \omega_\Psi + d\eta$ is compactly supported. We have

$$\begin{aligned} \langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle &= \langle \mathrm{cl}_B(\iota_*^{[\lambda, \mu]} \mathbf{1}_{\mathrm{Sh}_{\mathbf{H}}(U')}), [\omega_c] \rangle \\ &= \langle \iota_*^{[\lambda, \mu]} \mathrm{cl}_B(\mathbf{1}_{\mathrm{Sh}_{\mathbf{H}}(U')}), [\omega_c] \rangle \\ &= \langle \mathrm{cl}_B(\mathbf{1}_{\mathrm{Sh}_{\mathbf{H}}(U')}), \iota^{[\lambda, \mu]*} [\omega_c] \rangle \\ &= \frac{1}{(2\pi i)^3} \int_{\mathrm{Sh}_{\mathbf{H}}(U')} \iota^{[\lambda, \mu]*} \omega_c, \end{aligned}$$

where $\iota^{[\lambda,\mu]*} : \iota^*V^\lambda \rightarrow F(0)$ is the \mathbf{H} -equivariant projection dual to the inclusion $\iota^{[\lambda,\mu]} : F(0) \rightarrow \iota^*V^\lambda$ defined by $1 \mapsto v^{[\lambda,\mu]} \in V^\lambda$, where $v^{[\lambda,\mu]}$ is the vector defined in Lemma 3.4. According to [6, §5.6], we have

$$\int_{\mathrm{Sh}_{\mathbf{H}}(U')} \iota^{[\lambda,\mu]*} d\eta = 0.$$

Hence, using Theorem 4.2 we have

$$\begin{aligned} \langle \mathcal{Z}_{\mathbf{H},B}^{[\lambda,\mu]}, [\omega_\Psi] \rangle &= \frac{1}{(2\pi i)^3} \int_{\mathrm{Sh}_{\mathbf{H}}(U')} \iota^{[\lambda,\mu]*} \omega_\Psi \\ &= \frac{1}{(2\pi i)^3} \int_{\mathrm{Sh}_{\mathbf{H}}(U')} \omega_\Psi \left(X_0 \otimes v^{[\lambda,\mu]} \right) (h) dh \\ &= \frac{1}{(2\pi i)^3} \int_{\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A}) / \mathbf{Z}_{\mathbf{H}}(\mathbf{R}) K_{\mathbf{H},\infty} U'} A^{[\lambda,\mu]} \cdot \Psi(h) dh \\ &= \frac{h_{U'}}{(2\pi i)^3} \int_{\mathbf{H}(\mathbf{Q}) \mathbf{Z}_{\mathbf{G}}(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A}) / U'} A^{[\lambda,\mu]} \cdot \Psi(h) dh \\ &= \frac{h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \int_{\mathbf{H}(\mathbf{Q}) \mathbf{Z}_{\mathbf{G}}(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A})} A^{[\lambda,\mu]} \cdot \Psi(h) dh, \end{aligned}$$

where the third equality follows from Theorem 4.2 as $\omega_\Psi(X_0 \otimes v^{[\lambda,\mu]})$ is non-zero and thus it is of the form $A^{[\lambda,\mu]} \cdot \Psi_\infty$, for some $A^{[\lambda,\mu]} \in U(\mathfrak{k}_{\mathbf{C}})$, because Ψ_∞ is the highest weight vector of the minimal K_∞ -type $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$. Moreover, the fourth equality follows from the fact that Ψ is fixed by the center of \mathbf{G} , whence, using that $|\mathbf{Z}_{\mathbf{H}}(\mathbf{R}) / (\mathbf{Z}_{\mathbf{G}} \cap \mathbf{H})(\mathbf{R})| = 4$, the constant $h_{U'}$ is equal to $4^{-1} |\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}) \backslash \mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) / (\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) \cap U')|$. \square

Remark 4.9. In view of Proposition 4.8, we immediately notice that if π is not \mathbf{H} -distinguished, namely

$$\int_{\mathbf{H}(\mathbf{Q}) \mathbf{Z}_{\mathbf{G}}(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A})} \varphi_\pi(h) dh = 0,$$

for any cusp form φ_π in the space of π , we have that $\mathrm{pr}_{\pi \vee} \mathcal{Z}_{\mathbf{H},B}^{[\lambda,\mu]} = 0$. As we discuss later in §8, the \mathbf{H} -distinguishability is related to the property of π being a (functorial) lift from G_2 , which is (conjecturally) equivalent to the fact that the Spin L -function of π has a pole at $s = 1$.

4.3.2. *The pairing in absolute Hodge cohomology.* Let

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathcal{V}_{\mathbf{R}}(4)) \times H_{\mathcal{H},c}^6(\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{R}, \mathcal{V}_{\mathbf{R}}(3)) \rightarrow \mathbf{R}$$

be the natural pairing between absolute Hodge cohomology and compactly supported cohomology as constructed in [3, §4.2]. In order to ease notations, we will denote by $H_B^*(i)$ and $H_{B,c}^*(i)$ the cohomology groups $H_{B,c}^*(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F(i))$ and $H_B^*(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F(i))$, respectively. Recall from §2.9 that absolute Hodge cohomology and compactly supported cohomology live in exact sequences

$$0 \rightarrow \mathrm{Ext}_{\mathrm{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), H_B^6(4)) \rightarrow H_{\mathcal{H}}^7(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4)) \rightarrow \mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}}(\mathbf{R}(0), H_B^7(4)) \rightarrow 0, \quad (8)$$

$$0 \rightarrow \mathrm{Ext}_{\mathrm{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), H_{B,c}^5(3)) \rightarrow H_{\mathcal{H},c}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(3)) \rightarrow \mathrm{Hom}_{\mathrm{MHS}_{\mathbf{R}}}(\mathbf{R}(0), H_{B,c}^6(3)) \rightarrow 0, \quad (9)$$

which are deduced from the description of absolute Hodge cohomology as a cone of a diagram of complexes of Hodge structures. Let $[\omega_\Psi] \in H_{B,c}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}^\lambda(3))$ be the compactly supported cohomology class of the harmonic differential form ω_Ψ . This class is of

Hodge type (3, 3) and hence, since $W_0 H_{B,c}^6(3) = H_{B,c}^6(3)$, it naturally lives in the space $\text{Hom}_{\text{MHS}_{\mathbf{R}}}(\mathbf{R}(0), H_{B,c}^6(3)) = W_0 H_{B,c}(3) \cap F^0 H_{B,c}(3)_{\mathbf{C}}$. Denote by $[\widetilde{\omega}_{\Psi}]$ any lift of $[\omega_{\Psi}]$ in $H_{\mathcal{H},c}^6(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(3))$ via the surjection of the exact sequence (9).

Proposition 4.10. *The pairing $\langle \mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}, [\widetilde{\omega}_{\Psi}] \rangle_{\mathcal{H}}$ depends only on $[\omega_{\Psi}]$ and not on the choice of lift. We denote this value by $\langle \mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}, [\omega_{\Psi}] \rangle_{\mathcal{H}}$. Moreover, the pairing is given by the natural Poincaré duality pairing. In particular, we have*

$$\langle \mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}, [\omega_{\Psi}] \rangle_{\mathcal{H}} = \frac{h_U}{(2\pi i)^3 \cdot \text{vol}(U')} \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A})} A^{[\lambda,\mu]} \cdot \Psi(h) dh.$$

Proof. We give a sketch of the proof and we refer to [3] or to [8, §5.1] for the facts used here. It follows from the description of the pairing between absolute Hodge cohomology and compactly supported cohomology given in [3, §4.2] that, since our cycle class $\mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}$ lives in the subspace $\text{Ext}_{\text{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), H_B^6(4))$ of $H_{\mathcal{H}}^7(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(4))$, the map

$$\langle [\mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}], - \rangle : H_{\mathcal{H},c}^6(\text{Sh}_{\mathbf{G}}(U), \mathcal{V}_{\mathbf{R}}(3)) \rightarrow \mathbf{R}$$

factors through $\text{Hom}_{\text{MHS}_{\mathbf{R}}}(\mathbf{R}(0), H_{B,c}^6(3))$ and coincides with the natural Poincaré duality pairing

$$\begin{aligned} \text{Ext}_{\text{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), H_B^6(4)) \otimes \text{Hom}_{\text{MHS}_{\mathbf{R}}}(\mathbf{R}(0), H_{B,c}^6(3)) &\rightarrow \text{Ext}_{\text{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), H_B^6(4) \otimes H_{B,c}^6(3)) \\ &\xrightarrow{\cup} \text{Ext}_{\text{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), H_{B,c}^{12}(7)) \\ &\xrightarrow{\text{Tr}} \text{Ext}_{\text{MHS}_{\mathbf{R}}}^1(\mathbf{R}(0), \mathbf{R}(1)) = \mathbf{R}, \end{aligned}$$

This shows the first two assertions. The last formula follows from Proposition 4.8. \square

5. THE INTEGRAL REPRESENTATION OF POLLACK-SHAH AND THE RESIDUE OF THE SPIN L -FUNCTION

In this section, we explain the precise connection between the period integral appearing in the statement of Proposition 4.8 and the residue of the spin L -function of π in the case where the cubic totally real étale algebra over \mathbf{Q} defining \mathbf{H} is of the form $\mathbf{Q} \times F$, with F a quadratic real étale algebra over \mathbf{Q} . We start by recalling well known analytic properties of some Eisenstein series for GL_2 .

5.1. Eisenstein series for GL_2 . Let \mathbf{T}_2 denote the maximal diagonal torus of GL_2 and let $\mathbf{B}_2 = \mathbf{T}_2 \mathbf{U}_2$ denote the standard Borel. We denote by δ the character of \mathbf{T}_2 defined by $\text{diag}(t_1, t_2) \mapsto t_1/t_2$ and we regard δ as a character of \mathbf{B}_2 by extending it trivially to the unipotent radical. Let $\Phi \in \mathcal{S}(\mathbf{A}^2)$ be a Schwartz-Bruhat function. Following Jacquet, for any $s \in \mathbf{C}$, we attach to Φ the function $f_{\Phi} \in \text{Ind}_{\mathbf{B}_2(\mathbf{A})}^{\text{GL}_2(\mathbf{A})} \delta^s$ defined by

$$f_{\Phi}(h, s) = |\det(h)|^s \int_{\mathbf{A}^{\times}} \Phi((0, t)h) |t|^{2s} d^{\times} t$$

and the Eisenstein series

$$E_{\Phi}(h, s) = \sum_{\gamma \in \mathbf{B}_2(\mathbf{Q}) \backslash \text{GL}_2(\mathbf{Q})} f_{\Phi}(\gamma h, s).$$

In the statement of the following Lemma, we denote by $\widehat{\Phi}(0) = \int_{\mathbf{A}^2} \Phi(x, y) dx dy$ the value at 0 of the Fourier transform of Φ .

Lemma 5.1.

- (1) [36, Proposition 19.2]. *The Eisenstein series $E_\Phi(h, s)$ is absolutely convergent for $\operatorname{Re}(s)$ big enough and has a meromorphic continuation to \mathbf{C} .*
(2) [38, Lemma (4.2)]. *There exists a constant $c > 0$ such that*

$$E_\Phi(h, s) = \frac{c|\det(h)|^{s-1}\widehat{\Phi}(0)}{s-1} - \frac{c|\det(h)|^s\Phi(0)}{s} + R(h, s)$$

where $R(h, s)$ is an entire function in s for any $h \in \mathbf{GL}_2(\mathbf{A})$.

5.2. Fourier coefficients. Here we discuss the definition and basic properties of some Fourier coefficients for cusp forms for \mathbf{G} , which appear in the integral representation of the Spin L -function of [48].

5.2.1. The Siegel parabolic. We let $Q = L_3U_3$ denote the standard Siegel parabolic subgroup of \mathbf{G} , with Levi $L_3 \simeq \mathbf{GL}_3 \times \mathbf{GL}_1$. Explicitly,

$$L_3 = \left\{ m(g, \mu) = \begin{pmatrix} g & & & \\ & \mu^t g^{-1} & & \\ & & & \\ & & & \end{pmatrix} \mid g \in \mathbf{GL}_3, \mu \in \mathbf{GL}_1 \right\},$$

$$U_3 = \left\{ n(u) = \begin{pmatrix} I_3 & u \\ & I_3 \end{pmatrix}, u \in M_3 \mid u^t = u \right\}.$$

Denote $\operatorname{Sym}(3) = \{\alpha \in M_3 \mid \alpha^t = \alpha\}$. To each $\alpha \in \operatorname{Sym}(3)(\mathbf{Q})$, we associate the unitary character $\psi_\alpha : U_3(\mathbf{Q}) \backslash U_3(\mathbf{A}) \rightarrow \mathbf{C}^\times$ by $n(u) \in U_3(\mathbf{A}) \mapsto e(\operatorname{Tr}(\alpha u))$ where $e : \mathbf{Q} \backslash \mathbf{A} \rightarrow \mathbf{C}^\times$ is the additive character with $e_\infty(x) := e^{2\pi i x}$ for $x \in \mathbf{R}$, and conductor 1 at the finite places. For each $\alpha \in \operatorname{Sym}(3)(\mathbf{Q})$, we define a Fourier coefficient along U_3 for a cuspidal automorphic representation π of $\mathbf{G}(\mathbf{A})$ as follows.

Definition 5.2. *Let Ψ be a cusp form in the space of π . Define*

$$\Psi_{U_3, \psi_\alpha}(g) := \int_{U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})} \psi_\alpha^{-1}(u) \Psi(ug) du.$$

We let $L_3(\mathbf{Q})$ acts on $\operatorname{Sym}(3)(\mathbf{Q})$ via the right action $\alpha \cdot m(g, \mu) = \mu^{-1} g^t \alpha g$.

Lemma 5.3. *Let $\alpha, \beta \in \operatorname{Sym}(3)(\mathbf{Q})$. If there exists $m \in L_3(\mathbf{Q})$ such that $\beta = \alpha \cdot m$, then*

$$\Psi_{U_3, \psi_\beta}(g) = \Psi_{U_3, \psi_\alpha}(mg)$$

Proof. Suppose that $\beta = \alpha \cdot m$ with $m = m(g, \mu)$. The result follows from the equality

$$\psi_\beta(n(u)) = e(\operatorname{Tr}(\mu^{-1} g^t \alpha g u)) = e(\operatorname{Tr}(\alpha g u \mu^{-1} g^t)) = \psi_\alpha(mn(u)m^{-1}).$$

□

In this manuscript, we are interested in Fourier coefficients associated to the set of rank two elements of $\operatorname{Sym}(3)(\mathbf{Q})$, which we denote by $\operatorname{Sym}^{\operatorname{rk}2}(3)(\mathbf{Q})$. Let $D \in \mathbf{Q}^\times$ and let F denote the étale quadratic extension $\mathbf{Q}(\sqrt{D})$ of \mathbf{Q} . If D is not a square then F is a field, else $F = \mathbf{Q} \times \mathbf{Q}$.

Definition 5.4. *We let $\psi_D : U_3(\mathbf{Q}) \backslash U_3(\mathbf{A}) \rightarrow \mathbf{C}^\times$ be the unitary character*

$$\psi_D : n(u) \mapsto e(\operatorname{Tr}(\alpha_D u)) = e(u_{33} - Du_{22})$$

associated to $\alpha_D = \begin{pmatrix} 0 & \\ & -D \\ & & 1 \end{pmatrix} \in \operatorname{Sym}^{\operatorname{rk}2}(3)(\mathbf{Q})$.

We have the following:

Lemma 5.5. *A set of representatives of $\operatorname{Sym}^{\operatorname{rk}2}(3)(\mathbf{Q}) / \sim_{M_3(\mathbf{Q})}$ is given by*

$$\{\alpha_D : D \in \mathbf{Q}^\times / (\mathbf{Q}^\times)^2\}.$$

In view of these two lemmas, the set of Fourier coefficients associated to the Siegel parabolic and a rank 2 symmetric matrix is parametrized by the set of étale quadratic algebras of \mathbf{Q} .

5.2.2. *Fourier coefficients of type (42).* We now turn our attention to Fourier coefficients associated to the unipotent orbit of \mathbf{G} associated to the partition (42). The corresponding unipotent subgroup is the unipotent radical subgroup of the non-maximal standard parabolic $P = L_P \cdot U_P$, which arises as the intersection of the Siegel parabolic Q with the Klingen parabolic. Notice that P has Levi $L_P = \mathrm{GL}_2 \times \mathrm{GL}_1^2$, given by

$$\left\{ \begin{pmatrix} a & & & \\ & g & & \\ & & \mu a^{-1} & \\ & & & \mu^t g^{-1} \end{pmatrix} : a, \mu \in \mathrm{GL}_1, g \in \mathrm{GL}_2 \right\}.$$

Following [48, §2.1], we define a unitary character which we still denote $\psi_D : U_P(\mathbf{Q}) \backslash U_P(\mathbf{A}) \rightarrow \mathbf{C}^\times$ as follows. Every element of $U_P/[U_P, U_P]$ can be expressed as the product of $n(v)\tilde{n}(u)$, where

$$n(v) = \begin{pmatrix} 1 & v_1 & v_2 & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \in \mathbf{G}, \quad \tilde{n}(u) = \begin{pmatrix} 1 & & & & \\ & 1 & u_{22} & u_{23} & \\ & & 1 & u_{23} & u_{33} \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \in U_3.$$

We will denote by N_v (resp. N_u) the set of the $n(v)$'s (resp. $\tilde{n}(u)$'s). If $n \equiv n(v)\tilde{n}(u)$ modulo $[U_P, U_P]$, define

$$\psi_D(n) := e(v_1 + u_{33} - Du_{22}) = e(v_1)\psi_D(n(u)).$$

Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$. We define the following Fourier coefficients.

Definition 5.6. *Let Ψ be a cusp form in the space of π . Define*

$$\Psi_{U_P, \psi_D}(g) := \int_{U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})} \psi_D^{-1}(u) \Psi(ug) du.$$

In the following proposition, we relate these Fourier coefficients to the ones for the Siegel parabolic associated to rank 2 symmetric matrices.

Proposition 5.7. *For a cusp form Ψ in the space of π , the following two conditions are equivalent.*

- (1) $\Psi_{U_P, \psi_D}(g) \not\equiv 0$.
- (2) *There exists $\alpha \in \mathrm{Sym}^{\mathrm{rk}2}(3)(\mathbf{Q})$ with $\alpha \sim_{L(\mathbf{Q})} \alpha_D$ such that $\Psi_{U_3, \alpha}(g) \not\equiv 0$.*

Proof. Fourier expand $\Psi_{U_3, \psi_D}(g)$ over N_v to get

$$\Psi_{U_3, \psi_D}(g) = \int_{(\mathbf{Q} \backslash \mathbf{A})^2} \Psi_{U_3, \psi_D}(n(v)g) dv + \sum_{\gamma \in \mathrm{Stab}_L(\psi_D)(\mathbf{Q}) \backslash L(\mathbf{Q})} \Psi_{U_P, \psi_D}(\gamma g).$$

The term

$$\int_{(\mathbf{Q} \backslash \mathbf{A})^2} \Psi_{U_3, \psi_D}(n(v)g) dv = \int_{N_u(\mathbf{Q}) \backslash N_u(\mathbf{A})} \psi_D^{-1}(\tilde{n}(u)) \int_{U_K(\mathbf{Q}) \backslash U_K(\mathbf{A})} \Psi(n_k \tilde{n}(u)g) dn_k d\tilde{n}(u)$$

and the inner integral vanishes because of cuspidality of Ψ along the unipotent radical U_K of the Klingen parabolic. Thus

$$\Psi_{U_3, \psi_D}(g) = \sum_{\gamma} \Psi_{U_P, \psi_D}(\gamma g).$$

This relation implies the result as follows. If $\Psi_{U_3, \psi_D}(g) \neq 0$, the Fourier coefficient $\Psi_{U_P, \psi_D}(g)$ does not vanish identically. Viceversa, if $\Psi_{U_P, \psi_D}(g) \neq 0$ then there is a character ψ' in the $L(\mathbf{Q})$ -orbit of ψ_D such that $\Psi_{U_3, \psi'}(g) \neq 0$. \square

5.3. The Spin L -function and its residue at $s = 1$. Let π denote any cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with trivial central character. Let S denote a finite set of places of \mathbf{Q} containing the ones where π is ramified and the archimedean place. If $\text{Spin} : \text{Spin}_7(\mathbf{C}) \rightarrow \text{GL}(V_8)$ denotes the 8-dimensional spin representation, the partial Spin L -function of π is defined to be

$$L^S(s, \pi, \text{Spin}) := \prod_{\ell \notin S} \frac{1}{\det(1 - \ell^{-s} \text{Spin}(s_{\pi_\ell}))},$$

where s_{π_ℓ} denotes the Satake parameter of the unramified local component π_ℓ . Let \mathbf{H} be the group (3) associated to the étale cubic algebra $\mathbf{Q} \times F$, where $F = \mathbf{Q}(\sqrt{D})$ with either $D \neq 1 \in \mathbf{Q}_{>0}^\times / (\mathbf{Q}^\times)^2$, in which case F is a real quadratic field, or $D \equiv 1 \pmod{(\mathbf{Q}^\times)^2}$, in which case $F = \mathbf{Q} \times \mathbf{Q}$. For any cusp form $\Psi \in V_\pi$, Pollack-Shah [48] give an integral representation

$$\mathcal{I}(\Phi, \Psi, s) = \int_{\mathbf{Z}(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} E_\Phi(h_1, s) \Psi(h) dh.$$

of $L^S(s, \pi, \text{Spin})$. For any Φ and Ψ , the integral $\mathcal{I}(\Phi, \Psi, s)$ is absolutely convergent for $\text{Re}(s)$ big enough and has a meromorphic continuation to \mathbf{C} . According to [17, Proposition 7.1], for $\text{Re}(s)$ big enough we have the unfolding

$$\mathcal{I}(\Phi, \Psi, s) = \int_{U_{B_{\mathbf{H}}}(\mathbf{A})Z(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A})} f_\Phi(h_1, s) \Psi_{U_P, \psi_D}(h) dh$$

where $U_{B_{\mathbf{H}}}$ is the unipotent radical of the upper triangular Borel subgroup $B_{\mathbf{H}}$ of \mathbf{H} and Ψ_{U_P, ψ_D} is the Fourier coefficient of Definition 5.6.

Theorem 5.8 ([48]). *For a set Σ of places of \mathbf{Q} , denote*

$$\mathcal{I}_\Sigma(\Phi, \Psi, s) = \int_{U_{B_{\mathbf{H}}}(\mathbf{Q}_\Sigma)Z_{\mathbf{G}}(\mathbf{Q}_\Sigma) \backslash \mathbf{H}(\mathbf{Q}_\Sigma)} f(h_1, \Phi_\Sigma, s) \Psi_{U_P, \psi_D}(h) dh.$$

Let Ψ be a cusp form in the space of π . Then, for any factorizable Schwartz-Bruhat function Φ on \mathbf{A}^2 and up to enlarging S , we have

$$\mathcal{I}(\Phi, \Psi, s) = \mathcal{I}_S(\Phi, \Psi, s) L^S(s, \pi, \text{Spin}).$$

Moreover, there exists a cusp form $\tilde{\Psi}$ in the space of π and a factorizable Schwartz-Bruhat function Φ on \mathbf{A}^2 such that

$$\mathcal{I}(\Phi, \tilde{\Psi}, s) = \mathcal{I}_\infty(\Phi, \Psi, s) L^S(s, \pi, \text{Spin}).$$

Note that if π does not support a rank two Fourier coefficient (for the Siegel parabolic Q) and thus, by Proposition 5.7, a Fourier coefficient for P , the integral $\mathcal{I}(\Phi, \Psi, s)$ is identically zero.

Corollary 5.9 ([48]). *Suppose that π supports a rank two Fourier coefficient. Then, the partial Spin L -function $L^S(s, \pi, \text{Spin})$ has meromorphic continuation in s , is holomorphic outside $s = 1$, and has at worst a simple pole at $s = 1$.*

As we explain in the later sections, using results of Gan and Gan-Gurevich, they further prove that when $L^S(s, \pi, \text{Spin})$ has a simple pole at $s = 1$, π lifts to the split G_2 under the exceptional theta correspondence. This observation is based on the following key relation between the residue at $s = 1$ of $L^S(s, \pi, \text{Spin})$ and the automorphic period we have introduced in §4.3.

Proposition 5.10. *For any factorizable Schwartz-Bruhat function Φ on \mathbf{A}^2 , we have*

$$c \cdot \widehat{\Phi}(0) \cdot \int_{\mathbf{Z}(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} \Psi(h) dh = \text{Res}_{s=1} (\mathcal{I}_S(\Phi, \Psi, s) L^S(s, \pi, \text{Spin})),$$

where $c > 0$ is the constant of Lemma 5.1.

Proof. Thanks to Lemma 5.1, the residue at $s = 1$ of $\mathcal{I}(\Phi, \Psi, s)$ equals

$$c \cdot \widehat{\Phi}(0) \cdot \int_{\mathbf{Z}(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} \Psi(h) dh.$$

The result then follows from Theorem 5.8. \square

The following characterization is implicit in the work [48] and has been proved in [17] for certain CAP representations.

Proposition 5.11. *Suppose that π is a cuspidal automorphic representation of $\text{PGSp}_6(\mathbf{A})$ which supports a Fourier coefficient of type (42) associated to a quadratic étale \mathbf{Q} -algebra F . Then $L^S(s, \pi, \text{Spin})$ has a simple pole at $s = 1$ if and only if π is \mathbf{H} -distinguished, with $\mathbf{H} = \text{GL}_2 \boxtimes \text{GL}_{2,F}^*$, i.e. if there exists a cusp form Ψ in π such that*

$$\int_{\mathbf{Z}(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} \Psi(h) dh \neq 0.$$

Proof. By [48, Theorem 2.7] (cf. Theorem 5.8), given a cusp form Ψ in π , there exists a cusp form $\tilde{\Psi}$ and a Schwartz-Bruhat function Φ such that

$$\mathcal{I}(\Phi, \tilde{\Psi}, s) = \mathcal{I}_\infty(\Phi, \Psi, s) L^S(s, \pi, \text{Spin})$$

By Proposition 5.10, taking residues at $s = 1$ on both sides we have

$$c \cdot \widehat{\Phi}(0) \cdot \int_{\mathbf{Z}(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} \tilde{\Psi}(h) dh = \text{Res}_{s=1} (\mathcal{I}_\infty(\Phi, \Psi, s) L^S(s, \pi, \text{Spin})),$$

where $c > 0$ is the constant of Lemma 5.1. We now use [17, Proposition 12.1] to deduce that there exists local data Φ_∞ and Ψ_∞ such that $\mathcal{I}_\infty(\Phi, \Psi, 1) \neq 0$. Hence, up to modifying Ψ and Φ at ∞ , we obtain

$$\widehat{\Phi}(0) \cdot \int_{\mathbf{Z}(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} \tilde{\Psi}(h) dh = C \cdot \text{Res}_{s=1} L^S(s, \pi, \text{Spin}),$$

with C a certain non-zero constant in \mathbf{C} . Note finally that we have the freedom to choose Φ such that $\widehat{\Phi}(0) \neq 0$. This follows from the fact that, given the two non-zero linear maps $l_1 : \mathcal{S}(\mathbf{A}^2) \rightarrow \mathbf{C}, \Phi \mapsto \mathcal{I}_\infty(\Phi, \Psi, 1)$ and $l_2 : \mathcal{S}(\mathbf{A}^2) \rightarrow \mathbf{C}, \Phi \mapsto \widehat{\Phi}(0)$, $\ker(l_1) \cup \ker(l_2) \neq \mathcal{S}(\mathbf{A}^2)$, hence the result follows. \square

We now state our first main result. Let $\mathcal{Z}_{\mathbf{H},B}^{[\lambda,\mu]}$, $\mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}$, and ω_Ψ be as in §3.4 and §4.3.

Theorem 5.12. *If $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1) \neq 0$, then*

$$\langle \mathcal{Z}_{\mathbf{H},\mathcal{H}}^{[\lambda,\mu]}, [\omega_\Psi] \rangle_{\mathcal{H}} = \langle \mathcal{Z}_{\mathbf{H},B}^{[\lambda,\mu]}, [\omega_\Psi] \rangle = C \cdot \text{Res}_{s=1} L^S(s, \pi, \text{Spin}),$$

where C is a non-zero constant.

Proof. By Proposition 4.8 and Proposition 4.10, we have that

$$\langle \mathcal{Z}_{\mathbf{H}, \mathcal{H}}^{[\lambda, \mu]}, [\omega_\Psi] \rangle_{\mathcal{H}} = \langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle = \frac{h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \int_{\mathbf{H}(\mathbf{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{A}) \backslash \mathbf{H}(\mathbf{A})} \Psi^{[\lambda, \mu]}(h) dh,$$

where $U' = U \cap \mathbf{H}(\mathbf{A}_f)$ and $h_{U'} = 4^{-1} |\mathbf{Z}_{\mathbf{G}}(\mathbf{Q}) \backslash \mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) / (\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_f) \cap U')|$. Let Φ denote a factorizable Schwartz-Bruhat function on \mathbf{A}^2 , for which $\widehat{\Phi}(0) \neq 0$. By Proposition 5.10, we have

$$\begin{aligned} \langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle &= \frac{c \cdot \widehat{\Phi}(0) h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \mathrm{Res}_{s=1} \left(\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, s) L^S(s, \pi, \mathrm{Spin}) \right) \\ &= C \cdot \mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) \end{aligned}$$

where

$$C = \frac{c \cdot \widehat{\Phi}(0) h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0.$$

This finishes the proof. \square

Let $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}}$ denote the F -vector space defined in section 3.4.3 and let $H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}}[\pi_f^\vee]$ denote its π_f^\vee -isotypical component. This is a finite dimensional L -vector space, where L is the number field introduced in § 2.8. Tate conjecture for the motive attached to π (see Conjecture 1.1 (3)) predicts the equality

$$-\mathrm{ord}_{s=1} L^S(s, \pi, \mathrm{Spin}) = \dim_L H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}}[\pi_f^\vee].$$

Corollary 5.13. *If $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0$, then we have*

$$-\mathrm{ord}_{s=1} L^S(s, \pi, \mathrm{Spin}) \leq \dim_L H_{\mathcal{M}}^6(\mathrm{Sh}_{\mathbf{G}}(U), \mathcal{V}_F^\lambda(3))_{\mathrm{hom}}[\pi_f^\vee].$$

Proof. If $L^S(s, \pi, \mathrm{Spin})$ does not have a pole at $s = 1$, there is nothing to prove. Else we have $-\mathrm{ord}_{s=1} L^S(s, \pi, \mathrm{Spin}) = 1$ by [48, Theorem 1.3]. As a consequence $\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) \neq 0$. By Theorem 5.12, under the assumption that $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0$ this implies that $\langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle \neq 0$. In particular, the projection of $\mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}$ to the π_f^\vee -isotypical component is non-zero. This implies the statement. \square

6. EXCEPTIONAL THETA LIFTS FROM G_2 TO PGSp_6

In this section, we discuss the exceptional theta correspondence for the dual reductive pair (G_2, PGSp_6) and describe the set of Fourier coefficients associated to the Heisenberg parabolic for cuspidal automorphic forms of $G_2(\mathbf{A})$. It has as its solely purpose to fix notations and to recall some well known results that will be used later so that the knowledgeable reader might skip it.

6.1. Split G_2 and E_7 . In this section we will follow the exposition of the Appendix of [32] by Savin.

6.1.1. The group G_2 . Let \mathbb{H} be the algebra of Hamilton quaternions over \mathbf{Q} with the usual basis $\{1, i, j, k\}$. The conjugate \bar{a} of an element $a = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{H}$ is $\bar{a} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$. The split octonion algebra over \mathbf{Q} is $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with multiplication

$$(a, b) \cdot (c, d) = (ac + d\bar{b}, \bar{a}d + cb).$$

Then \mathbb{O} is a non-commutative, non-associative \mathbf{Q} -algebra. However it is alternative, which means that for any $x, y \in \mathbb{O}$ we have $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ and $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ (see [35]). If $x = (a, b)$, let $\bar{x} = (\bar{a}, -b)$. Then $x \mapsto \bar{x}$ is a \mathbf{Q} -linear involution on \mathbb{O} satisfying

$\overline{x \cdot y} = \overline{y} \cdot \overline{x}$. The norm $N : \mathbb{O} \rightarrow \mathbf{Q}$ is the quadratic form defined by $x \mapsto x \cdot \overline{x} = \overline{x} \cdot x$. The trace $\text{Tr} : \mathbb{O} \rightarrow \mathbf{Q}$ is defined by $x \mapsto x + \overline{x}$. For any $x, y, z \in \mathbb{O}$, the properties

$$\begin{aligned} N(x \cdot y) &= N(x)N(y), \\ \text{Tr}(x \cdot y) &= \text{Tr}(y \cdot x), \\ \text{Tr}(x \cdot (y \cdot z)) &= \text{Tr}((x \cdot y) \cdot z) \end{aligned}$$

are satisfied. For $x, y \in \mathbb{O}$, we write $y \in x^\perp$ if y is orthogonal to x with respect to the bilinear form $(x, y) \mapsto \text{Tr}(x \cdot \overline{y})$, which means that $x \cdot \overline{y} + y \cdot \overline{x} = 0$.

Let $l = (0, 1) \in \mathbb{O}$ so that $\{1, i, j, k, l, li, lj, lk\}$ is a basis of \mathbb{O} . From this, one constructs another useful basis $\{s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4\}$, where

$$\begin{aligned} s_1 &= \frac{1}{2}(i + li), s_2 = \frac{1}{2}(j + lj), s_3 = \frac{1}{2}(k + lk), s_4 = \frac{1}{2}(1 + l), \\ t_1 &= \frac{1}{2}(i - li), t_2 = \frac{1}{2}(j - lj), t_3 = \frac{1}{2}(k - lk), t_4 = \frac{1}{2}(1 - l). \end{aligned}$$

The following multiplication table for this basis is given in Table 1 of the Appendix of [32].

	s_1	s_2	s_3	t_1	t_2	t_3	s_4	t_4
s_1	0	$-t_3$	t_2	s_4	0	0	0	s_1
s_2	t_3	0	$-t_1$	0	s_4	0	0	s_2
s_3	$-t_2$	t_1	0	0	0	s_4	0	s_3
t_1	t_4	0	0	0	s_3	$-s_2$	t_1	0
t_2	0	t_4	0	$-s_3$	0	s_1	t_2	0
t_3	0	0	t_4	s_2	$-s_1$	0	t_3	0
s_4	s_1	s_2	s_3	0	0	0	s_4	0
t_4	0	0	0	t_1	t_2	t_3	0	t_4

We define

$$G_2 := \{g \in \text{GL}(\mathbb{O}) \mid g(x \cdot y) = (gx) \cdot (gy), \forall x, y \in \mathbb{O}\}.$$

to be the group of automorphisms of \mathbb{O} . We note that G_2 acts transitively on non-zero elements of trace zero and norm zero. We will denote the set of trace zero octonions by either \mathbb{O}^0 or V_7 , where the latter notation emphasises that this set defines the standard irreducible 7-dimensional representation of G_2 and induces an embedding

$$G_2 \hookrightarrow \text{SO}_7.$$

6.1.2. *The dual reductive pair.* We consider the Albert algebra J over \mathbf{Q} , which is the set of matrices

$$A = \begin{pmatrix} d & \overline{z} & y \\ z & e & \overline{x} \\ \overline{y} & x & f \end{pmatrix}$$

where $d, e, f \in \mathbf{Q}$ and $x, y, z \in \mathbb{O}$. The algebra J is equipped with a cubic form, called the determinant, which is given by

$$\det(A) = def - dN(x) - eN(y) - fN(z) + \text{Tr}(xyz).$$

The group of isogenies of this form is a group of type E_6 and its orbits on J are classified by the rank. We will need to consider the set Ω of rank 1 elements $A \in J$, i.e. those $A \neq 0$ such that $A^2 = \text{Tr}(A) \cdot A$. This condition means that the entries of A satisfy the equalities

$$\begin{aligned} N(x) &= ef, \quad N(y) = df, \quad N(z) = de, \\ dx &= \bar{y} \cdot \bar{z}, \quad ey = \bar{z} \cdot \bar{x}, \quad fz = \bar{x} \cdot \bar{y}. \end{aligned}$$

Let G denote the split adjoint group of type E_7 , which is constructed from J by the Koecher-Tits construction (see Section 3 of [40]). The group G has a maximal parabolic $P = MN$ and its opposite $\bar{P} = M\bar{N}$, with $N \simeq J$ and such that the action under conjugation of the Levi M on N gives an isomorphism of M and the group of similitudes of the cubic form on J

$$M \cong \{g \in \text{GL}(J) \mid \det(gA) = \lambda \det(A) \text{ for some } \lambda \in \mathbf{G}_m \text{ and } \forall A \in J\}.$$

The group G_2 can be realized as a subgroup of M via its action on J by the rule

$$g \cdot \begin{pmatrix} d & \bar{z} & y \\ z & e & \bar{x} \\ \bar{y} & x & f \end{pmatrix} = \begin{pmatrix} d & g\bar{z} & gy \\ gz & e & g\bar{x} \\ g\bar{y} & gx & f \end{pmatrix}.$$

This action has fixed points J_3 , the Jordan algebra of symmetric 3×3 matrices with entries in \mathbf{Q} . Note that the left action of GL_3 on $J_3 \cong N$ given by

$$g \cdot A = \det(g)^{-1} g A g^t \tag{10}$$

extends to an action on J preserving the determinant form up to scalar, thus defining an embedding of GL_3 into M . Then GL_3 is the centraliser of G_2 in M and $Q = \text{GL}_3 U_3$ (which is the Siegel parabolic of \mathbf{PGSp}_6) is the centralizer of G_2 in P . Similarly, the opposite \bar{Q} is the centralizer of G_2 in \bar{P} . This gives the dual reductive pair (G_2, \mathbf{PGSp}_6) in G .

6.2. Fourier coefficients for G_2 .

6.2.1. *Root system and the Heisenberg parabolic.* Let T be a (rank 2) maximal split torus over \mathbf{Q} in G_2 and let Δ , resp. $\Delta^+ \subset \Delta$, be the set of roots, resp. a subset of positive roots, for G_2 . Let a , resp. b , denote the long, resp. short, simple root in Δ^+ . Then

$$\Delta^+ = \{a, b, a + b, a + 2b, a + 3b, 2a + 3b\}.$$

We let $B = TU$ denote the Borel subgroup of G_2 associated to Δ^+ . Other than B , there are two proper standard parabolic subgroups P_a and P_b of G_2 , such that $P_a \cap P_b = B$. They are characterized by the following. For any $\alpha \in \Delta^+$, denote by $x_\alpha : \mathbf{G}_a \hookrightarrow U$ the one parameter unipotent subgroup associated to α . Then, for each $r \in \{a, b\}$, the Levi L_r of P_r is isomorphic to GL_2 and contains x_r . We fix an isomorphism $\text{GL}_2 \simeq L_r$ such that $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \mapsto x_r(u)$. We denote by $H := P_b$ the so-called Heisenberg parabolic and let $L_H U_H$ denote its Levi decomposition. The unipotent radical U_H is of dimension 5 and admits the filtration

$$1 \subseteq [U_H, U_H] \subseteq U_H,$$

with $U_H/[U_H, U_H]$ being the four dimensional abelian unipotent group generated by

$$\{x_a, x_{a+b}, x_{a+2b}, x_{a+3b}\},$$

while $[U_H, U_H]$ is isomorphic to the one parameter unipotent subgroup x_{2a+3b} . We now give a more detailed description of U_H and L_H which will be useful later. First describe an embedding of SL_3 into G_2 . The group G_2 acts transitively on the set \mathbb{O}^0 of trace zero octonions. In particular, it acts transitively on the set

$$\Gamma_c := \{x \in \mathbb{O}^0 \mid N(x) = -c\}.$$

By [35, Theorem 4], the stabilizer of an element $y_0 \in \Gamma_1$ is isomorphic to SL_3 . Choose y_0 such that the unipotent radical U_{SL_3} of the upper triangular Borel of SL_3 is generated by the one-parameter subgroups

$$\{x_a, x_{a+3b}, x_{2a+3b}\}.$$

In terms of the basis chosen in §6.1.1, this is achieved by choosing $y_0 = s_4 - t_4$. In this case, one shows (cf. [49, Lemma 2]) that the stabilizer of y_0 leaves invariant the subspace $\langle s_1, s_2, s_3 \rangle$ and is identified with $\mathrm{SL}_3 = \mathrm{SL}(\langle s_1, s_2, s_3 \rangle)$.

6.2.2. *The Lie algebra of G_2 .* The multiplication map on \mathbb{O} induces a map $V_7 \otimes V_7 \rightarrow V_7$ given by $x \otimes y \mapsto \frac{xy - yx}{2}$. This map is alternating, hence it induces a G_2 -equivariant map $\wedge^2 V_7 \rightarrow V_7$ which is surjective. Then the Lie algebra \mathfrak{g}_2 of G_2 can be identified with the kernel of this map. Under this identification, one has an explicit description of the action of \mathfrak{g}_2 on V_7 , namely

$$(w \wedge x) \cdot v = \langle x, v \rangle w - \langle w, v \rangle x.$$

We will also need (cf. [14, §22.2]) the decomposition

$$\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus \mathrm{Std}_3 \oplus \mathrm{Std}_3^*, \quad (11)$$

where Std_3 is the standard representation of SL_3 with basis $\{v_1, v_2, v_3\}$ and Std_3^* is its dual with basis $\{\delta_1, \delta_2, \delta_3\}$ and where we denote by E_{ij} , $1 \leq i < j \leq 3$ the standard basis vectors of \mathfrak{sl}_3 . The identification between the two descriptions (cf. [47, §2.2]) of \mathfrak{g}_2 is given by $E_{ij} = t_j \wedge s_i$, $1 \leq i < j \leq 3$, $v_i = (s_4 - t_4) \wedge s_i + t_{i+1} \wedge t_{i+2}$ and $\delta_i = (s_4 - t_4) \wedge t_i + s_{i+1} \wedge s_{i+2}$, $1 \leq i \leq 3$, where indices are taken modulo 3. Moreover, the component \mathfrak{sl}_3 is the Lie algebra of the copy of SL_3 embedded into G_2 as above. In particular, E_{12}, E_{13} and E_{23} are root vectors for the roots $a, 2a + 3b$ and $a + 3b$ respectively. Moreover, the vectors v_1, v_2 and δ_3 are root vectors for the roots $a + b, b$ and $a + 2b$, respectively. Via (11), the Lie algebra \mathfrak{u}_H of U_H is

$$\mathfrak{u}_H = \mathfrak{u}_{\mathrm{SL}_3} \oplus \mathbf{Q}v_1 \oplus \mathbf{Q}\delta_3, \quad (12)$$

Under (11) the Lie algebra \mathfrak{l}_H of the Levi L_H is generated by the Cartan subalgebra and the root vectors v_2, δ_2 .

6.2.3. *Fourier coefficients.* We now describe the Fourier coefficients for G_2 associated to the Heisenberg parabolic. We closely follow [47] and refer to it for more details. In order to describe the Fourier coefficients associated to H , we need to study the L_H -representation $V_H := U_H/[U_H, U_H]$. As a GL_2 -representation, V_H is isomorphic to $\mathrm{Sym}^3(\mathrm{Std}_2) \otimes \det^{-1}(\mathrm{Std}_2)$, where Std_2 denotes the standard representation of GL_2 . Under the identification of (12), (a representative of) an element of $V_H(\mathbf{Q})$ can be written as

$$x_a(\lambda_1)x_{a+b}(\lambda_2/3)x_{a+2b}(\lambda_3/3)x_{a+3b}(\lambda_4), \quad \text{with } \lambda_i \in \mathbf{Q},$$

which corresponds to the binary cubic polynomial

$$p(x, y) = \lambda_1 x^3 + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3$$

where x, y form a basis of Std_2 . Associated to p , there is the cubic \mathbf{Q} -algebra R with basis $\{1, i, j\}$ with multiplicative table

$$\begin{aligned} ij &= -ad \\ i^2 &= -ac + bi - aj \\ j^2 &= -bd + di - cj. \end{aligned}$$

Example 6.1.

- (1) ([26, 3.2]) If $p(x, y) = x^2y - xy^2$ then the associated \mathbf{Q} -algebra R is isomorphic to \mathbf{Q}^3 .
- (2) ([26, 3.3]) If $p(x, y) = x^3 - Dxy^2$ (or equivalently $p(x, y) = -Dx^2y + y^3$ using the action of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) then the associated \mathbf{Q} -algebra R is isomorphic to $\mathbf{Q} \oplus \mathbf{Q}(\sqrt{D})$.

There is an action of $\mathrm{GL}_2(\mathbf{Q})$ on the set of bases $\{1, i, j\}$ of a given cubic algebra R , which makes the association $p(X, Y) \mapsto (R, \{1, i, j\})$ $\mathrm{GL}_2(\mathbf{Q})$ -equivariant. Since any cubic algebra admits a basis of this shape, we have the following.

Proposition 6.2 ([26, Proposition 2.1]). *There is a bijection between the $\mathrm{GL}_2(\mathbf{Q})$ -orbits on $V_H(\mathbf{Q})$ and the set of isomorphism classes of cubic \mathbf{Q} -algebras. Moreover, each orbit has a well-defined discriminant in $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2$.*

Let $e : \mathbf{Q} \setminus \mathbf{A} \rightarrow \mathbf{C}^\times$ be the additive character introduced in §5.2.1. Let $\langle \cdot, \cdot \rangle$ denote the symplectic pairing on V_H defined as follows. If $v, v' \in V_H$ correspond to $p(x, y)$ and $p'(x, y)$ respectively, then

$$\langle v, v' \rangle = \lambda_1 \lambda'_4 - \frac{1}{3} \lambda_2 \lambda'_3 + \frac{1}{3} \lambda_3 \lambda'_2 - \lambda_4 \lambda'_1.$$

Any character $\psi : U_H(\mathbf{Q}) \setminus U_H(\mathbf{A}) \rightarrow \mathbf{C}^\times$ factors through $V_H(\mathbf{A})$, hence we consider the projection \bar{n} of $n \in U_H(\mathbf{A})$ to $V_H(\mathbf{A})$, which, by (12), can be written as

$$\bar{n} = x_a(\lambda'_1)x_{a+b}(\lambda'_2/3)x_{a+2b}(\lambda'_3/3)x_{a+3b}(\lambda'_4).$$

If $v \in V_H(\mathbf{Q})$ corresponds to $p(x, y)$, we then define $\psi_v : U_H(\mathbf{Q}) \setminus U_H(\mathbf{A}) \rightarrow \mathbf{C}^\times$ by

$$n \mapsto e(\langle v, \bar{n} \rangle) = e(\lambda_1 \lambda'_4 - \frac{1}{3} \lambda_2 \lambda'_3 + \frac{1}{3} \lambda_3 \lambda'_2 - \lambda_4 \lambda'_1).$$

The character ψ_v is non-degenerate if and only if v corresponds to an étale cubic algebra over \mathbf{Q} . In this manuscript, we are interested in étale cubic algebras of the form $\mathbf{Q} \times F$, with F of either the form $\mathbf{Q}(\sqrt{D})$ (with $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2 \ni D \not\equiv 1$) or $\mathbf{Q} \times \mathbf{Q}$ (with $D \equiv 1 \pmod{(\mathbf{Q}^\times)^2}$).

Definition 6.3. Let $\psi_{H,D} : U_H(\mathbf{Q}) \setminus U_H(\mathbf{A}) \rightarrow \mathbf{C}^\times$ denote the character associated to $\mathbf{Q} \times F$. Given a cusp form φ for $G_2(\mathbf{A})$, define

$$\varphi_{U_H, \psi_{H,D}}(g) := \int_{U_H(\mathbf{Q}) \setminus U_H(\mathbf{A})} \psi_{H,D}^{-1}(n) \varphi(ng) dn.$$

6.3. The theta lift from G_2 to \mathbf{PGSp}_6 . Let $\Pi = \bigotimes'_v \Pi_v$ denote the restricted tensor product of the minimal representations Π_v of $E_7(\mathbf{Q}_v)$ over all places v of \mathbf{Q} . A unitary model of the minimal representation is given by $L^2(\Omega)$, where recall that Ω denotes the subset of rank 1 elements in J . There is a unique up to a non-zero scalar embedding

$$\theta : \Pi \rightarrow \mathcal{A}(E_7(\mathbf{Q}) \setminus E_7(\mathbf{A}))$$

of Π in the space $\mathcal{A}(E_7(\mathbf{Q}) \setminus E_7(\mathbf{A}))$ of automorphic forms of E_7 (see [22], [40]). For $f \in \Pi$ and $\varphi \in \mathcal{A}(G_2(\mathbf{Q}) \setminus G_2(\mathbf{A}))$, we define a function $\Theta(f, \varphi)$ on $\mathbf{PGSp}_6(\mathbf{A})$ by

$$\Theta(f, \varphi)(g) = \int_{G_2(\mathbf{Q}) \setminus G_2(\mathbf{A})} \theta(f)(g'g) \varphi(g') dg'.$$

Definition 6.4. Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$.

- (1) Define $\Theta(\sigma)$ to be the span of the functions $\Theta(f, \varphi)$, where $f \in \Pi$ and φ runs through the cusp forms in the contragredient σ^\vee of σ .
- (2) We say that a cuspidal automorphic representation π of $\mathbf{PGSp}_6(\mathbf{A})$ is a Θ -lift of σ if it appears as an irreducible subquotient of $\Theta(\sigma)$.

If a Θ -lift of σ exists, then its local constituents are compatible with the local Theta correspondence between G_2 and PGSp_6 .

Proposition 6.5. *Let π be a Θ -lift of σ , then π_v is an irreducible subquotient of $\Theta(\sigma_v)$.*

Proof. See [32, Theorem 1.7 (i)]. □

After imposing certain local conditions on σ , in the next section we use one of the main results of [23] to show that $\Theta(\sigma)$ is non-zero and cuspidal, thus proving the existence of a non-trivial Θ -lift of σ . Before doing so, we first recall the properties of the local theta correspondence needed later.

6.3.1. *Discrete series and a conjecture of Gross.* Let T_c denote a compact torus in $G_2(\mathbf{R})$, which is contained in the maximal compact subgroup $K_{G_2} \simeq (\mathrm{SU}_2 \times \mathrm{SU}_2)/\mu_2$ of $G_2(\mathbf{R})$. We abuse notation denoting again by a, b the simple positive roots for T_c (with the short root b which we assume to be compact) and Δ^+ the resulting set of positive roots. Then, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = 3a + 5b$. The set of positive compact roots is given by

$$\Delta_c^+ = \{b, 2a + 3b\},$$

which, in the notation of [45], is $\{2\varepsilon_2, 2\varepsilon_1\}$. The Weyl group \mathfrak{W}_{G_2} is isomorphic to the dihedral group D_6 of 12 elements and it is generated by w_a and w_b , where w_α denotes the reflections around the line orthogonal to α . The Weyl group $\mathfrak{W}_{K_{G_2}} \simeq (\mathbf{Z}/2\mathbf{Z})^2$ is generated by w_b and $w_{2a+3b} = w_a w_b w_a w_b w_a$.

Let γ be a dominant weight for G_2 with respect to T_c . The set of equivalence classes of irreducible discrete series of $G_2(\mathbf{R})$ associated to γ has cardinality equal to $|\mathfrak{W}_{G_2}/\mathfrak{W}_{K_{G_2}}| = 3$. Choose representatives $\{w_1, w_2, w_3\}$ of $\mathfrak{W}_{G_2}/\mathfrak{W}_{K_{G_2}}$ such that $w_i\rho$ is dominant for K_{G_2} . Then, for any $1 \leq i \leq 3$, there exists an irreducible discrete series σ_∞^Γ of Harish-Chandra parameter $\Gamma = w_i(\gamma + \rho)$ and minimal K_{G_2} -type $\Gamma + \delta_{G_2} - 2\delta_{K_{G_2}}$, where δ_{G_2} , resp. $\delta_{K_{G_2}}$, is the half-sum of roots, resp. compact roots, which are positive with respect to the Weyl chamber in which Γ lies. Precisely, if we let $w_1 = \mathrm{id}$, $w_2 = w_a$, and $w_3 = w_b w_a$, then

$$\begin{aligned} w_1\rho &= \rho = 3\varepsilon_1 + \varepsilon_2, \\ w_2\rho &= 2a + 5b = 2\varepsilon_1 + 4\varepsilon_2, \\ w_3\rho &= a + 4b = \varepsilon_1 + 5\varepsilon_2. \end{aligned}$$

We let $\mathcal{D}_{3,1}$, $\mathcal{D}_{2,4}$, and $\mathcal{D}_{1,5}$ denote the sets of discrete series of $G_2(\mathbf{R})$ whose Harish-Chandra parameter lies in the Weyl chamber corresponding to $w_1\rho$, $w_2\rho$, and $w_3\rho$ respectively. Elements of $\mathcal{D}_{3,1}$ are the quaternionic discrete series, while elements of $\mathcal{D}_{2,4}$ are the generic discrete series.

Gross has given a precise conjectural description of the entire discrete spectrum of the dual pair (G_2, PGSp_6) (cf. [45, Conjecture 1.2]). Recall that there are four families of discrete series for $\mathrm{PGSp}_6(\mathbf{R})$, indexed by the set of Hodge types up to conjugation. In particular, the discrete series of $\mathrm{PGSp}_6(\mathbf{R})$ of Hodge type $(4, 2)$, resp. $(6, 0)$, are the generic, resp. holomorphic discrete series.

Conjecture 6.6 (Gross). *Let Π_∞ be the minimal representation of $E_7(\mathbf{R})$. The discrete spectrum of the restriction of Π_∞ to the dual pair $G_2(\mathbf{R}) \times \mathrm{PGSp}_6(\mathbf{R})$ is the direct sum of all tensor products $\sigma_\infty \otimes \theta(\sigma_\infty)$, where σ_∞ belongs to the discrete series of G_2 . If σ_∞ has infinitesimal character $\gamma + \rho = r\varepsilon_1 + s\varepsilon_2$ and belongs to either $\mathcal{D}_{3,1}$, $\mathcal{D}_{2,4}$, or $\mathcal{D}_{1,5}$, then*

$\theta(\sigma_\infty)$ is the discrete series of $\mathbf{PGSp}_6(\mathbf{R})$ with infinitesimal character $(r, \frac{1}{2}(r+s), \frac{1}{2}(r-s))$ and Hodge type $(3, 3)$, $(4, 2)$, or $(5, 1)$ respectively.

The conjecture of Gross has been verified for discrete series in $\mathcal{D}_{3,1}$ by Li (cf. §6.3.2 below). Moreover, it has been partially confirmed for the generic family $\mathcal{D}_{2,4}$ by Harris-Khare-Thorne in [32, Theorem 1.5 and Theorem 1.7(ii)] using the main result of Savin's appendix to [32] and the non-vanishing of the global theta lift given by [23, Corollary 4.2], while Li verified it for a proper subset $\mathcal{D}'_{1,5}$ of $\mathcal{D}_{1,5}$ in [45, Theorem 4.3]. We also note that the remaining equivalence class of holomorphic discrete series of $\mathbf{PGSp}_6(\mathbf{R})$ (of Hodge type $(6, 0)$) is realized in an exceptional theta correspondence studied by Gross-Savin between the compact real form $G_2^c(\mathbf{R})$ of G_2 and $\mathbf{PGSp}_6(\mathbf{R})$ and moreover this is the only Hodge type that appears in that correspondence (cf. [27, Theorem 3.5]).

6.3.2. Quaternionic discrete series and their theta lift. We describe the main result of [45], which verifies the conjecture of Gross for quaternionic discrete series. We first notice that a discrete series $\sigma_\infty^{x,y}$ of Harish-Chandra parameter $x\varepsilon_1 + y\varepsilon_2$ lies in the set of quaternionic discrete series $\mathcal{D}_{3,1}$ if x, y are two non-negative integers such that $x - 3 \geq y - 1 \geq 0$ and $x - y$ is even. The minimal K_{G_2} -type of $\sigma_\infty^{x,y} \in \mathcal{D}_{3,1}$ is given by

$$\mathrm{Sym}^{x+1}(\mathrm{Std}_{\varepsilon_1}) \boxtimes \mathrm{Sym}^{y-1}(\mathrm{Std}_{\varepsilon_2}),$$

where $\mathrm{Std}_{\varepsilon_1}$ (resp. $\mathrm{Std}_{\varepsilon_2}$) is the standard representation of the SU_2 corresponding to the long root ε_1 (resp. the short root ε_2).

Proposition 6.7. *Let Π_∞ denote the minimal representation of $E_7(\mathbf{R})$. We have*

$$\Pi_\infty|_{G_2(\mathbf{R}) \times \mathbf{PGSp}_6(\mathbf{R})} \supseteq \bigoplus_{\sigma_\infty^{x,y} \in \mathcal{D}_{3,1}} \sigma_\infty^{x,y} \otimes \theta(\sigma_\infty^{x,y})$$

where $\theta(\sigma_\infty^{x,y}) \in P(V^\lambda)$, with $\lambda = (x - 3, \frac{1}{2}(x + y) - 2, \frac{1}{2}(x - y) - 1, 0)$, is the discrete series $\pi_\infty^{3,3}$ of Hodge type $(3, 3)$ and Harish-Chandra parameter $(\frac{1}{2}(x + y), \frac{1}{2}(x - y), -x)$.

Proof. See [45, Theorem 1.1] and [33, Theorem 5.4]. \square

The set $\mathcal{D}_{3,1}$ contains an important family of discrete series, which were studied by Gross and Wallach in [28] and [29].

Definition 6.8. *For every $n \geq 2$, the quaternionic discrete series σ_n is the element of $\mathcal{D}_{3,1}$ of Harish-Chandra parameter $(2n - 1)\varepsilon_1 + \varepsilon_2$ and minimal K_{G_2} -type*

$$\mathrm{Sym}^{2n}(\mathrm{Std}_{\varepsilon_1}) \boxtimes \mathbf{1}.$$

A fundamental property of the members of this family is that they admit (unique) models with respect to the unipotent radical of the Heisenberg parabolic and non-degenerate characters corresponding to totally real étale cubic algebras. Recall, as in §6.2.3, that a non-degenerate character $\psi : U_H(\mathbf{R}) \rightarrow \mathbf{C}^\times$ corresponds to a cubic algebra, whose discriminant is either positive or negative. The first type corresponds to the $\mathrm{GL}_2(\mathbf{R})$ -orbit on $V_H(\mathbf{R})$ given by \mathbf{R}^3 , while the second to the $\mathrm{GL}_2(\mathbf{R})$ -orbit of $\mathbf{R} \times \mathbf{C}$. A representative $\psi : U_H(\mathbf{R}) \rightarrow \mathbf{C}^\times$ of the totally real orbit is given by $e^{2\pi i f}$, where $f : U_H(\mathbf{R}) \rightarrow \mathbf{R}$ is non-zero on the one parameter unipotent subgroups x_{a+b} and x_{a+2b} and trivial on x_a and x_{a+3b} (cf. [16, §6]). A special case of the main result of [58] gives the following.

Proposition 6.9. *Let ψ be a non-degenerate character of $V_H(\mathbf{R})$. There is (at most) a one dimensional space of ψ -equivariant linear functionals on σ_n . Moreover,*

$$\dim \mathrm{Hom}_{U_H(\mathbf{R})}(\sigma_n, \psi) = 1,$$

exactly when ψ corresponds to a totally real cubic algebra.

6.3.3. The non-archimedean theta correspondence. We describe the properties of the non-archimedean theta correspondence which will be later needed to study the global theta correspondence. Let σ be an irreducible admissible representation of $G_2(\mathbf{Q}_p)$.

Definition 6.10. Let $\Theta(\sigma)$ be the equivalence class of irreducible admissible representations π of $\mathrm{PGSp}_6(\mathbf{Q}_p)$ such that $\sigma \otimes \pi$ is a quotient of the minimal representation Π_p of $E_7(\mathbf{Q}_p)$. Denote by $\theta(\sigma)$ the maximal semisimple quotient of $\Theta(\sigma)$.

Proposition 6.11. For an irreducible admissible representation σ of $G_2(\mathbf{Q}_p)$, $\Theta(\sigma)$ has finite length with unique irreducible quotient (if non-zero) $\theta(\sigma)$. Moreover, one has the following.

- (1) Let σ be an unramified generic representation of $G_2(\mathbf{Q}_p)$ with Satake parameter s , then $\pi = \theta(\sigma)$ is the unramified representation of $\mathrm{PGSp}_6(\mathbf{Q}_p)$ whose Satake parameter is $\varphi \circ s$, where $\varphi : G_2 \hookrightarrow \mathrm{Spin}_7$ is the map of L -groups.
- (2) Let St_{G_2} (resp. $\mathrm{St}_{\mathrm{PGSp}_6}$) be the Steinberg representation of $G_2(\mathbf{Q}_p)$ (resp. $\mathrm{PGSp}_6(\mathbf{Q}_p)$); then $\theta(\mathrm{St}_{G_2}) = \mathrm{St}_{\mathrm{PGSp}_6}$.

Proof. See [19, Theorem 1.2, Theorem 15.3(v)] and [27, Proposition 3.1]. \square

7. CUSPIDALITY AND FOURIER COEFFICIENTS OF THE GLOBAL THETA LIFT

In this section, based on the works [23], [27], and the appendix of Savin in [32], we give a criterion on the cuspidality of representations in the image of the exceptional theta lift and on their possession of Fourier coefficients of type (42).

7.1. Cuspidality of the global lift. Let V denote the unipotent subgroup of SL_3 (embedded into G_2 as in §6.2.1) generated by the roots $a + 3b$ and $2a + 3b$. We further consider the subgroup SL_2 embedded into G_2 via the Levi of the “long root” parabolic P_a and denote, for any cusp form φ for $G_2(\mathbf{A})$,

$$\varphi^{\mathrm{SL}_2 V}(g) := \int_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \int_{V(\mathbf{Q}) \backslash V(\mathbf{A})} \varphi(vmg) dv dm.$$

We are thankful to David Ginzburg, who kindly shared with us a proof that the non-vanishing of the period appearing in [21, Theorem 3.7(3)] implies that σ is globally generic, i.e. that it has non-zero Whittaker coefficients. We crucially use this fact to prove the following.

Lemma 7.1. Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$, which is not globally generic. For any cusp form $\varphi \in V_\sigma$ and $g \in G_2(\mathbf{A})$, we have $\varphi^{\mathrm{SL}_2 V}(g) = 0$.

Proof. Let Z denote the unipotent subgroup of G_2 generated by the roots $a + 2b$, $a + 3b$, and $2a + 3b$. If we Fourier expand the period $\varphi^{\mathrm{SL}_2 V}(g)$ along the 1-dimensional unipotent subgroup $x_{a+2b}(r)$ of G_2 , we get

$$\varphi^{\mathrm{SL}_2 V}(g) = \varphi^{\mathrm{SL}_2 Z}(g) + \sum_{\psi_Z} \varphi^{\mathrm{SL}_2, \psi_Z}(g),$$

where the sum runs over non-trivial additive characters $\psi_Z : Z(\mathbf{Q}) \backslash Z(\mathbf{A}) \rightarrow \mathbf{C}^\times$ supported on the root $a + 2b$, $\varphi^{\mathrm{SL}_2 Z}(g)$ is the period of φ over $[\mathrm{SL}_2 Z]$, and

$$\varphi^{\mathrm{SL}_2, \psi_Z}(g) := \int_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \int_{(\mathbf{Q} \backslash \mathbf{A})^3} \varphi(x_{a+2b}(r_1) x_{a+3b}(r_2) x_{2a+3b}(r_3) mg) \psi_Z(r_1) dr_i dm.$$

By [23, Lemma 2.1], $\varphi^{\mathrm{SL}_2 Z}(g) = 0$ for all φ in σ and $g \in G_2(\mathbf{A})$. Hence, $\varphi^{\mathrm{SL}_2 V}(g) = 0$ if and only if $\varphi^{\mathrm{SL}_2, \psi_Z}(g) = 0$ for all choices of data. We argue by contradiction. Suppose that $\varphi^{\mathrm{SL}_2, \psi_Z}(g) \neq 0$ for a certain choice of data. We now show that this implies that σ supports Whittaker Fourier coefficients, thus contradicting our hypothesis. Let V_a be the unipotent radical of P_a . Since V is normal in V_a , we can consider the quotient $V_0 = V_a/V$, which is isomorphic to the Heisenberg group in three variables (cf. [20, §1.1]). Note that it is generated by the roots $b, a+b$, and $a+2b$. We can then apply the results of [34] for the Jacobi group given by $\mathrm{SL}_2 V_0$. In particular, [34, Proposition 1.3] implies that $\varphi^{\mathrm{SL}_2, \psi_Z}(1)$ is zero for all choices of data if and only if

$$I_1 := \int_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \int_{(\mathbf{Q} \backslash \mathbf{A})^5} \varphi(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)x_{a+3b}(r_2)x_{2a+3b}(r_3)m) \cdot \bar{\theta}_{\mathrm{SL}_2}^{\phi_1}(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)m)\theta_{\mathrm{SL}_2}^{\phi_2}(m)dv_i dr_i dm$$

is zero for all choices of data. Here $\theta_{\mathrm{SL}_2}^{\bullet}(g)$ is the theta function on the double cover of the Jacobi group $\mathrm{SL}_2 V_0$ (cf. [34, p. 620]). The period I_1 appears as the residue of the global integral studied in [20]: since $\theta_{\mathrm{SL}_2}^{\phi_2}(m)$ realizes the residue at $s = 3/4$ of the normalized Eisenstein series $\mathrm{Eis}_{\tilde{\mathrm{SL}}_2}(m, s)$ on the metaplectic cover of SL_2 , by [20, Theorem 4], the integral I_1 is the residue of

$$I_2(s) := \int_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \int_{(\mathbf{Q} \backslash \mathbf{A})^5} \varphi(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)x_{a+3b}(r_2)x_{2a+3b}(r_3)m) \cdot \bar{\theta}_{\mathrm{SL}_2}^{\phi_1}(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)m)\mathrm{Eis}_{\tilde{\mathrm{SL}}_2}(m, s)dv_i dr_i dm.$$

If I_1 is not zero, then, for $\mathrm{Re}(s)$ large enough, the integral $I_2(s)$ is not zero. By [20, Theorem 1], $I_2(s)$ contains the Whittaker coefficient for φ as an inner integration. This shows that if $\varphi^{\mathrm{SL}_2, \psi_Z}(1) \neq 0$ for some choice of data, $I_2(s) \neq 0$ and, in particular, σ is globally generic. This finishes the proof. \square

Theorem 7.2. *Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$. Assume that*

- (1) σ is not globally generic;
- (2) there exists a finite place p such that σ_p is generic.

Then the big theta lift $\Theta(\sigma)$ of σ to PGSp_6 is cuspidal.

Proof. We show the result by using the tower of theta lifts from G_2 and its properties studied in [23]. If σ lifts trivially to PGSp_6 then there is nothing to prove, so suppose that σ has a non-zero theta lift π to PGSp_6 . Then, by [23, Theorem A] π is cuspidal if and only if the lifts of σ to PGSp_4 and PGL_3 are both zero. By [23, Theorem 4.1 (3)], the lift to PGSp_4 is zero if and only if

$$\varphi^{\mathrm{SL}_3}(g) = \int_{[\mathrm{SL}_3]} \varphi(xg)dx = 0 \text{ and } \varphi^{\mathrm{SU}(2,1)}(g) = \int_{[\mathrm{SU}(2,1)]} \varphi(xg)dx = 0$$

for any $g \in G_2(\mathbf{A})$, any $\varphi \in V_{\sigma^\vee}$. Here, SL_3 embeds into G_2 as the stabilizer of a norm -1 vector (cf. §6.2.1), while $\mathrm{SU}(2,1)$ is realized as the stabilizer of a norm $-c$ vector, with c not a square in \mathbf{Q} . We argue by contradiction. Suppose that σ^\vee has a non-trivial $\mathrm{SU}(2,1)$ -functional. This implies that, at every finite v , σ_v admits one. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{SU}(2,1)}(\sigma_v^\vee, \mathbf{C}) = \mathrm{Hom}_{G_2}(\mathfrak{c}\text{-Ind}_{\mathrm{SU}(2,1)}^{G_2}(\mathbf{C}), \sigma_v)$$

and hence, since σ_v is irreducible, one deduces that each local component σ_v of σ is a quotient of $C_c^\infty(G_2(\mathbf{Q}_v)/\mathrm{SU}(2,1)(\mathbf{Q}_v))$. In particular, σ_p is identified with such a quotient. This is a contradiction as, by hypothesis, σ_p is generic but, by [27, Lemma 4.10],

$C_c^\infty(G_2(\mathbf{Q}_p)/\mathrm{SU}(2,1)(\mathbf{Q}_p))$ does not admit a Whittaker functional. The same argument also shows the vanishing of φ^{SL_3} . We claim finally that the theta lift of σ to PGL_3 also vanishes. Since σ is not globally generic, Lemma 7.1 shows that, for all $\varphi \in \sigma$, $\varphi^{\mathrm{SL}_2 V}(g) = 0$. We can then apply [23, Theorem 4.1(4)] to deduce that the theta lift of σ to PGL_3 is zero and conclude the proof. \square

Corollary 7.3. *Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$. Assume that*

- (1) σ_∞ is a discrete series;
- (2) there exists a finite place p such that σ_p is Steinberg.

Then $\Theta(\sigma)$ is cuspidal.

Proof. We distinguish two cases. We first suppose that σ is globally generic. Then we apply [32, Theorem 1.7(ii)] to deduce that its theta lift is cuspidal. If, instead, σ is not globally generic, the result follows from Theorem 7.2 as the Steinberg representation $\sigma_p = \mathrm{St}_{G_2}$ is generic. \square

7.2. Calculation of orbits. This preparatory section presents an elementary but crucial calculation needed in the proof of Proposition 7.7.

Let $e : \mathbf{Q} \backslash \mathbf{A} \rightarrow \mathbf{C}^\times$ be the standard non-trivial character introduced in §5.2 and let $A \in J(\mathbf{Q})$. We define the character $\psi_A : N(\mathbf{Q}) \backslash N(\mathbf{A}) \rightarrow \mathbf{C}^\times$ by $\psi_A(X) = e(\mathrm{Tr}(A \circ X))$ where $A \circ X = \frac{1}{2}(AX + XA)$ is the Jordan product. Recall from §5.2 that, for any $B \in J_3(\mathbf{Q})$, we define a character $\psi_B : U_3(\mathbf{Q}) \backslash U_3(\mathbf{A}) \rightarrow \mathbf{C}^\times$ by $\psi_B(n(X)) = e(\mathrm{Tr}(BX))$. In particular, we have denoted ψ_D the character associated to

$$\alpha_D = \begin{pmatrix} 0 & & \\ & -D & \\ & & 1 \end{pmatrix} \in J_3(\mathbf{Q}).$$

Define

$$\omega(\mathbf{Q}) := \{A \in \Omega(\mathbf{Q}) \mid \psi_A|_{U_3(\mathbf{A})} = \psi_D\},$$

i.e. the set of rank 1 matrices in $J(\mathbf{Q})$ inducing the same character as α_D on the unipotent radical of the Siegel parabolic. In the following, we will always see $\omega(\mathbf{Q})$ inside $\overline{N}(\mathbf{Q})$. In particular, if $g \in \mathrm{GL}_3(\mathbf{Q}) \subseteq M(\mathbf{Q})$, its action on A is the dual action to (10), namely

$$g \cdot A = \det(g)(g^t)^{-1}Ag^{-1}. \text{ Finally, denote by } A(x, y, z) \text{ the matrix } \begin{pmatrix} 0 & \bar{z} & y \\ z & -D & \bar{x} \\ \bar{y} & x & 1 \end{pmatrix} \in J.$$

Lemma 7.4. *We have*

$$\omega(\mathbf{Q}) = \left\{ A(x, y, z) : \mathrm{Tr}(x) = \mathrm{Tr}(z) = 0, N(x) = -D, N(z) = 0, z \in x^\perp, y = -D^{-1}zx \right\}.$$

Proof. Let

$$A = \begin{pmatrix} d & \bar{z} & y \\ z & e & \bar{x} \\ \bar{y} & x & f \end{pmatrix} \in J$$

Similarly to the proof of [27, Lemma 3.4], the condition $\psi_A|_{U_3(\mathbf{A})} = \psi_D$ is equivalent to

$$\begin{aligned} d &= 0, & e &= -D, & f &= 1, \\ \bar{x} &= -x, & \bar{y} &= -y, & \bar{z} &= -z \end{aligned}$$

The condition that A has rank 1 gives

$$N(x) = -D, \quad N(y) = N(z) = 0,$$

$$yz = 0, \quad zx = -Dy, \quad xy = z.$$

We claim that these conditions imply that $z \in x^\perp$, which means that $z\bar{x} + x\bar{z} = 0$, or equivalently $zx = -xz$. Indeed, multiplying $z = xy$ on the left by x and using alternativity, we obtain

$$xz = x(xy) = (xx)y = Dy = -zx.$$

Finally, as $N(x) = -D$ and $\text{Tr}(x) = 0$, we have $x^2 = D$ and hence $x^{-1} = D^{-1}x$, which implies that

$$y = x^{-1}z = D^{-1}xz.$$

This shows one inclusion of the statement.

In the other direction let $x, z \in \mathbb{O}$ be as in the right hand side of the statement. We have to show that $y := -D^{-1}zx$ has norm and trace equal to zero and that $xy = z$. We have

$$N(y) = (-D)^{-2}N(z)N(x) = 0$$

and

$$\text{Tr}(y) = -D^{-1}\text{Tr}(zx) = -D^{-1}\text{Tr}(xz) = D^{-1}\text{Tr}(zx) = -\text{Tr}(y)$$

because $xz = -zx$. Hence $\text{Tr}(y) = 0$. Moreover

$$xy = -D^{-1}x(zx) = D^{-1}x(xz) = D^{-1}(xx)z = z.$$

This shows that $A \in \omega(\mathbf{Q})$ and concludes the proof of the lemma. \square

As for any $A(x, y, z) \in \omega(\mathbf{Q})$, the octonion $y = -D^{-1}zx$ is determined by x and z and we will often denote $A(x, y, z)$ by $A(x, z)$. Note that there is an action of $G_2(\mathbf{Q})$ on the set $\omega(\mathbf{Q})$ given by the action on the coefficients. The following proposition describing the orbits of this action will be essential.

Proposition 7.5. *The group $G_2(\mathbf{Q})$ acts on $\omega(\mathbf{Q})$ with a finite number of orbits. Moreover, representatives of the orbits and their respective stabilizers are given as follows.*

- (1) *If D is a square in \mathbf{Q}^\times :*
 - (a) $A_3 = A(x, 0)$, where $x = (s_4 - t_4)\sqrt{D}$ and $\text{Stab}_{G_2(\mathbf{Q})}(A(x, 0)) \cong \text{SL}_3$, where SL_3 is embedded into G_2 as §6.2.1.
 - (b) $A_2 = A(x, t_3)$ with $\text{Stab}_{G_2}(A_2) = \text{SL}_2V \subset \text{SL}_3$, where SL_2 and V embed into SL_3 as in §7.1.
 - (c) $A_1 = A(x, s_3)$ with $\text{Stab}_{G_2}(A_1) = \text{SL}_2\bar{V} \subset \text{SL}_3$, where SL_2 is as in (1)(b) and \bar{V} is the opposite unipotent subgroup to V .
 - (d) $A_0 = A(x, s_1 + t_3)$ with $\text{Stab}_{G_2}(A_0) = U_D$, where U_D denotes the unipotent radical of the upper-triangular Borel of SL_3 (denoted by U_{SL_3} in §7.1).
- (2) *If D is not square in \mathbf{Q}^\times :*
 - (a) $A_1 = A(x, 0) \in \omega(\mathbf{Q})$, for any $x \neq 0$ for which $N(x) = -D$, with

$$\text{Stab}_{G_2(\mathbf{Q})}(A(x, 0)) \cong \text{SU}(2, 1).$$

To emphasize its dependence on D , we denote this stabilizer by $\text{SU}_D(2, 1)$.

- (b) $A_0 = A(x, z)$, for any norm zero z in x^\perp , with $\text{Stab}_{G_2}(A_0) \simeq U_D$, where U_D denotes the unipotent radical of the upper-triangular Borel of $\text{SU}_D(2, 1)$.

Proof. Step 1. By [49, Theorem 1], the group G_2 acts transitively on the set of trace zero elements of norm $-D$ and hence on the sets $A(x, 0)$. The description of the stabilizer in (1)(a) follows from [35, Theorem 4] or [49, Lemma 2]. The description of the stabilizer in (2)(a) follows from [35, Theorem 3] or [49, Lemma 3]. More precisely, according to [49, Lemma 3] the space x^\perp has the structure of a 3-dimensional $\mathbf{Q}(\sqrt{D})$ -vector space and the action of $\text{Stab}_{G_2}(x)$ on x^\perp induces an isomorphism $\text{Stab}_{G_2}(x) \simeq \text{SU}_D(2, 1)$.

Step 2. We now study the remaining G_2 -orbits when D is a square in \mathbf{Q} . Again, we can assume that $D = 1$. Recall from §6.2.1 that SL_3 embeds into G_2 as the stabilizer of $s_4 - t_4$. This identification is explicitly given as follows (cf. [49, Lemma 2]). An element of $g \in \text{SL}_3$ induces an action on \mathbb{O}^0 fixing $s_4 - t_4$ and given by the left multiplication by g on $\langle s_1, s_2, s_3 \rangle$ and by $(g^t)^{-1}$ on $\langle t_1, t_2, t_3 \rangle$. One verifies that this actions respects multiplication and hence defines an element in G_2 . Assume $z \neq 0$ is such that $A(x, z) \in \omega(\mathbf{Q})$. Since z is trace zero and orthogonal to $x = s_4 - t_4$ we can write $z = z_1 + z_2$ with $z_1 = \sum_i \alpha_i s_i$ and $z_2 = \sum_i \beta_i t_i$. Since the group SL_3 acts transitively on the non-zero elements of $\langle s_1, s_2, s_3 \rangle$ and $\langle t_1, t_2, t_3 \rangle$, then the cases where $z_1 = 0$ or $z_2 = 0$ give rise to exactly two orbits. When $z_1 = 0$, taking $z_2 = t_3$ as a generator of this orbit, the corresponding stabilizer is

$$\left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \text{SL}_3,$$

which coincides with $\text{SL}_2 V$ as in (1)(b). Similarly, when $z_2 = 0$, taking $z_1 = s_3$ as the generator of the orbit, then the stabilizer is

$$\left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \right\} \subset \text{SL}_3.$$

This is nothing but $\text{SL}_2 \bar{V}$, with SL_2 which again embeds in the Levi of the long root parabolic P_a and \bar{V} is the opposite unipotent subgroup to V generated by the negative roots $-a - 3b$, $-2a - 3b$. Finally we treat the case $z_1, z_2 \neq 0$. Write

$$z = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 + \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3.$$

The condition $N(z) = 0$ translates then in

$$\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 = 0. \tag{13}$$

We can assume that $z_2 = t_3$. Then $\alpha_3 = 0$ by (13) and, using the action of the stabilizer of t_3 , we can assume that $z_1 = s_1$. It is then immediate to check that the stabilizer of $A(s_4 - t_4, s_1 + t_3)$ is as in (1)(d). This concludes the proof of (1).

Step 3. We finally deal with the case where D is not a square in \mathbf{Q} . By Witt's theorem, the group $\text{SU}(2, 1)$ acts transitively on the isotropic vectors of the three dimensional $\mathbf{Q}(\sqrt{D})$ vector space x^\perp . We thus have two orbits for $G_2(\mathbf{Q})$ on $\omega(\mathbf{Q})$, generated by $A(x, 0)$ and $A(x, z)$, where z is any non-zero vector in x^\perp with zero norm. We are now left with calculating the stabilizer of the latter orbit. Recall that the action of $\text{SU}(2, 1)$ on x^\perp is given by its natural action on $\mathbf{Q}(\sqrt{D})^3$. More precisely, after extending scalars to $\mathbf{Q}(\sqrt{D})$, we can decompose

$$x^\perp \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{D}) = \mathbf{Q}(\sqrt{D}) \langle s_1, s_2, s_3 \rangle \oplus \mathbf{Q}(\sqrt{D}) \langle t_1, t_2, t_3 \rangle.$$

The projection to the first component induces an isomorphism of $\mathbf{Q}(\sqrt{D})$ -vector spaces $x^\perp \simeq \mathbf{Q}(\sqrt{D}) \langle s_1, s_2, s_3 \rangle$ (cf. [49, Lemma 3]), with $\text{SU}(2, 1)$ acting naturally on the basis

$\{s_1, s_2, s_3\}$. Here, we choose the Hermitian form defining $\mathrm{SU}(2, 1)$ given by

$$\begin{pmatrix} & & \sqrt{D}^{-1} \\ & 1 & \\ -\sqrt{D}^{-1} & & \end{pmatrix} \in \mathrm{GL}_3(\mathbf{Q}(\sqrt{D})).$$

We can then suppose that z is sent to s_1 and the corresponding stabilizer is given by

$$\left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap \mathrm{SU}(2, 1) = U_D.$$

□

7.3. Non-vanishing of Fourier coefficients I. Recall that we have denoted by $\Pi = \bigotimes'_v \Pi_v$ the minimal representation of the group E_7 . Moreover, in §6.3, for $f \in \Pi$ and $\varphi \in \mathcal{A}(G_2(\mathbf{Q}) \backslash G_2(\mathbf{A}))$, we have defined the function $\Theta(f, \varphi)$ on $\mathrm{PGSp}_6(\mathbf{A})$ by

$$\Theta(f, \varphi)(g) = \int_{G_2(\mathbf{Q}) \backslash G_2(\mathbf{A})} \theta(f)(g'g)\varphi(g')dg'. \quad (14)$$

For any $A \in J(\mathbf{Q})$ and $f \in \Pi$, consider the Fourier coefficient

$$\theta(f)_A(g) = \int_{N(\mathbf{Q}) \backslash N(\mathbf{A})} \theta(f)(ng)\psi_A^{-1}(n)dn.$$

We then have the Fourier expansion (cf. [32, §A.3])

$$\theta(f)(g) = \theta(f)_0(g) + \sum_{A \in \Omega(\mathbf{Q})} \theta(f)_A(g), \quad (15)$$

where $\Omega(\mathbf{Q}) \subset J(\mathbf{Q})$ is the subset of rank 1 elements.

The following Lemma will be used in the proof of Proposition 7.7. Its proof is similar to the one of [27, Lemma 4.6] but we give details for the convenience of the reader. Let A_0 be the representative of the open G_2 -orbit on $\omega(\mathbf{Q})$ given in Proposition 7.5. Note that there is no harm in conjugating $A_0 \in J(\mathbf{Q})$ by an element of the Levi $\mathrm{GL}_3(\mathbf{Q})$ of the Siegel parabolic of PGSp_6 . Thus, conjugating by $\mathrm{diag}(n, n, n)$, A_0 gets multiplied by n^2 and so we can assume that the entries x, y, z of A_0 are in $\mathbb{O}(\mathbf{Z})$.

Lemma 7.6. *Let S denote a finite number of places containing 2 and ∞ , and let $f = \bigotimes'_v f_v \in \Pi$ be such that, for $v \notin S$, we have $f_v = f_v^0$ where f_v^0 denotes the spherical vector normalized such that $f_v^0(A_0) = 1$. Let $\mathbf{Q}_S = \prod_{v \in S} \mathbf{Q}_v$. If $g \in G_2(\mathbf{A})$, we write $g = g_S g^S$ where $g_S \in G_2(\mathbf{Q}_S)$ and $g^S \in \prod_{v \notin S} G_2(\mathbf{Q}_v)$. Then there exists a non-zero constant c_{A_0} such that for every $g \in G_2(\mathbf{A})$ we have*

$$\theta(f)_{A_0}(g) = c_{A_0} f_S(g_S^{-1} A_0) \prod_{v \notin S} \chi_v(g_v)$$

where $f_S = \bigotimes_{v \in S} f_v$ and χ_v is the characteristic function of $U_D(\mathbf{Z}_v) \backslash G_2(\mathbf{Z}_v)$.

Proof. By uniqueness of local functionals ([32, Theorem A.4]), there exists a non-zero scalar c_{A_0} such that for any $g \in E_7(\mathbf{A})$, we have $\theta(f)_{A_0}(g) = c_{A_0} (\Pi(g)f)(A_0)$. For $g \in G_2(\mathbf{A})$ we have $(\Pi(g)f)(A_0) = f(g^{-1}A_0)$ where $g^{-1}A_0$ is the result of the natural action of g^{-1} on the off diagonal entries of A_0 . Hence $\theta(f)_{A_0}(g) = c_{A_0} f(g^{-1}A_0) = c_{A_0} \prod_v f_v(g_v^{-1}A_0)$ for $g \in G_2(\mathbf{A})$. Let us prove that for any $p \notin S$, we have $f_p(g_p^{-1}A_0) = \chi_p(g_p)$. So let $g_p \in G_2(\mathbf{Q}_p)$ be such that $f_p^0(g_p^{-1}A_0) \neq 0$ and let x', y', z' denote the off diagonal entries of $g_p^{-1}A_0$. According to [32, Theorem A.5] the spherical vector f_p^0 is supported in $J(\mathbf{Z}_p)$. Hence $x', y', z' \in \mathbb{O}(\mathbf{Z}_p)$. Consider $\mathbb{O}(\mathbf{F}_p)$ the split octonion algebra over \mathbf{F}_p . The projections of

(x, y, z) and (x', y', z') to $\mathbb{O}(\mathbf{F}_p)$ are $G_2(\mathbf{F}_p)$ -conjugated by the proof of Step 1 in Proposition 7.5, which is still valid over the base field \mathbf{F}_p as long as $p \neq 2$. It follows from Hensel lemma that (x, y, z) and (x', y', z') are $G_2(\mathbf{Z}_p)$ -conjugated. Therefore the function $g_p \mapsto f_p^0(g_p^{-1}A_0)$ is supported in $U_D(\mathbf{Z}_p) \backslash G_2(\mathbf{Z}_p) \subset U_D(\mathbf{Q}_p) \backslash G_2(\mathbf{Q}_p)$. Since f_p^0 is $G_2(\mathbf{Z}_p)$ -invariant, for $g_p \in G_2(\mathbf{Z}_p)$ we have $f_p^0(g_p^{-1}A_0) = f_p^0(A_0) = 1$. This completes the proof. \square

Proposition 7.7. *Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$ as in Theorem 7.2 and let $\varphi \in \sigma^\vee$ be a cuspidal form. Then, the following conditions are equivalent*

- (1) $\Theta(f, \varphi)_{U_P, \psi_D}(1) \neq 0$ for some choice of f .
- (2) $\varphi^{U_D}(g) \neq 0$ for some $g \in G_2(\mathbf{A})$.

In particular, if any of the conditions holds then $\Theta(\sigma)$ is non-zero.

Proof. Recall first that, according to Proposition 5.7, we have $\Theta(f, \varphi)_{U_P, \psi_D} \neq 0$ if and only if $\Theta(f, \varphi)_{U_3, \alpha} \neq 0$ for some $\alpha \in \text{Sym}^{\text{rk}^2}(3)(\mathbf{Q})$ with $\alpha \sim_{L(\mathbf{Q})} \alpha_D$. We write

$$\begin{aligned} \Theta(f, \varphi)_{U_3, \psi_D}(1) &= \int_{U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})} \Theta(f, \varphi)(u) \psi_D^{-1}(u) du \\ &= \int_{G_2(\mathbf{Q}) \backslash G_2(\mathbf{A})} \int_{U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})} \sum_{A \in \Omega(\mathbf{Q})} \theta(f)_A(ug) \varphi(g) \psi_D^{-1}(u) dudg, \end{aligned}$$

where in the second equality we used the definition (14) of $\Theta(f, \varphi)$ and the Fourier expansion (15) of $\theta(f)$. Since $U_3 \subseteq N$, we have that $\theta(f)_A(ug) = \psi_A(u) \theta(f)_A(g)$ and

$$\int_{U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})} \psi_A(u) \psi_D^{-1}(u) = \begin{cases} \text{vol}(U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})) & \text{if } \psi_D = \psi_A|_{U_3(\mathbf{A})} \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have

$$\Theta(f, \varphi)_{U_3, \psi_D}(1) = \text{vol}(U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})) \int_{G_2(\mathbf{Q}) \backslash G_2(\mathbf{A})} \sum_{A \in \omega(\mathbf{Q})} \theta(f)_A(g) \varphi(g) dg. \quad (16)$$

Let $(A_i)_i$ be the finite representatives of the orbits of the action of $G_2(\mathbf{Q})$ on $\omega(\mathbf{Q})$ as given by Proposition 7.5, and write Stab_{A_i} for the stabilizers of A_i in G_2 . The integral on the right hand side of (16) becomes

$$\sum_i \int_{G_2(\mathbf{Q}) \backslash G_2(\mathbf{A})} \sum_{g' \in \text{Stab}_{A_i}(\mathbf{Q}) \backslash G_2(\mathbf{Q})} \theta(f)_{A_i}(g'g) \varphi(g) dg = \sum_i \int_{\text{Stab}_{A_i}(\mathbf{Q}) \backslash G_2(\mathbf{A})} \theta(f)_{A_i}(g) \varphi(g) dg.$$

Observe now that, by [32, Theorem A.4] we have $\theta(f)_{A_i}(g) = c_{A_i} f(g^{-1}A_i)$ for any $g \in G_2$. Hence, since $\text{Stab}_{A_i}(\mathbf{A})$ fixes the matrix A_i , we deduce that the function $g \mapsto \theta(f)_{A_i}(g)$ is left $\text{Stab}_{A_i}(\mathbf{A})$ -invariant. Making an inner integration over $\text{Stab}_{A_i}(\mathbf{Q}) \backslash \text{Stab}_{A_i}(\mathbf{A})$ in each term of the outer sum, we deduce that the above equals

$$\sum_i \int_{\text{Stab}_{A_i}(\mathbf{A}) \backslash G_2(\mathbf{A})} \theta(f)_{A_i}(g) \varphi^{\text{Stab}_{A_i}}(g) dg,$$

where $\varphi^{\text{Stab}_{A_i}}(g)$ denotes the period of φ over $\text{Stab}_{A_i}(\mathbf{Q}) \backslash \text{Stab}_{A_i}(\mathbf{A})$. We now analyse two different possibilities. If D is not a square in \mathbf{Q} , then, by (2) of Proposition 7.5, $G_2(\mathbf{Q})$ acts on $\omega(\mathbf{Q})$ with two orbits, one closed and one open. Let A_0, A_1 denote representatives of these two orbits with stabilizers $\text{Stab}_{A_0} = U_D$ and $\text{Stab}_{A_1} = \text{SU}_D(2, 1)$ in G_2 . By the proof of Theorem 7.2, $\varphi^{\text{SU}_D(2, 1)}(g) = 0$, and hence the only surviving term is the one corresponding to the orbit represented by A_0 . If D is a square in \mathbf{Q} , then by (1) of Proposition 7.5, $G_2(\mathbf{Q})$ acts on the set $\omega(\mathbf{Q})$ with four orbits, three closed and one open. Let $A_i, 0 \leq i \leq 3$

denote representatives of those orbits, with A_0 representing the open one. The corresponding stabilizers are U_D , SL_3 , SL_2V and its conjugate $\mathrm{SL}_2\bar{V}$. By the proof of Theorem 7.2, we have $\varphi^{\mathrm{SL}_3}(g) = 0$. By hypothesis σ (and σ^\vee) is not globally generic, hence Lemma 7.1 implies that $\varphi^{\mathrm{SL}_2V}(g) = \varphi^{\mathrm{SL}_2\bar{V}}(g) = 0$. From this, we deduce that, for any D ,

$$\Theta(f, \varphi)_{U_3, \psi_D}(1) = \int_{U_D(\mathbf{A}) \backslash G_2(\mathbf{A})} \theta(f)_{A_0}(g) \varphi^{U_D}(g) dg, \quad (17)$$

where $\varphi^{U_D}(g)$ is the constant term of φ along U_D . This shows that if $\Theta(f, \varphi)_{U_3, \psi_D}(1) \neq 0$ then $\varphi^{U_D} \neq 0$ since the period appears as an inner integral of the Fourier coefficient.

We now show the converse, i.e. that if $\varphi^{U_D} \neq 0$ then, for some choice of $f \in \Pi$, the Fourier coefficient $\Theta(f, \varphi)_{U_3, \psi_D}$ does not vanish. Let S be as in Lemma 7.6. By enlarging S if necessary, we can assume that the cusp form φ is $G_2(\mathbf{Z}_v)$ -invariant for all $v \notin S$. By Lemma 7.6, the integral of (17) equals

$$c_{A_0} \cdot \left(\int_{U_D(\mathbf{Q}_S) \backslash G_2(\mathbf{Q}_S)} f_S(g^{-1}A_0) \varphi^{U_D}(g) dg \right) \cdot \prod_{v \notin S} \mathrm{vol}(U_D(\mathbf{Z}_v) \backslash G_2(\mathbf{Z}_v), dg_v).$$

It remains to show that, when $\varphi^{U_D} \neq 0$ then for a good choice of f at the places in S , the integral

$$\int_{U_D(\mathbf{Q}_S) \backslash G_2(\mathbf{Q}_S)} f_S(g^{-1}A_0) \varphi^{U_D}(g) dg \neq 0.$$

It follows from [32, Theorem A.4] that f_S can be any smooth compactly supported function on $\Omega(\mathbf{Q}_S)$. Let $g_0 \in G_2(\mathbf{Q}_S)$ be such that $\varphi^{U_D}(g_0) \neq 0$. We can take a non-negative f supported in a sufficiently small neighborhood of g_0 to ensure the non-vanishing of the integral. This finishes the proof of the proposition. \square

7.4. Non-vanishing of Fourier coefficients II. The purpose of this section is to prove the following result.

Theorem 7.8. *Let F denote a quadratic étale algebra and $\sigma = \sigma_\infty \otimes \sigma_f$ be a cuspidal automorphic representation of $G_2(\mathbf{A})$ such that*

- σ_∞ is a discrete series in $\mathcal{D}_{3,1}$.
- there exists a finite prime p such that σ_p is Steinberg.
- The representation σ supports Fourier coefficient associated to the cubic algebra $\mathbf{Q} \times F$.

The theta lift $\Theta(\sigma) = \otimes'_v \Theta(\sigma_v)$ is a non-zero cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$. Let π be any non-zero irreducible subquotient of $\Theta(\sigma)$. Then

- π_∞ is a discrete series of Hodge type $(3, 3)$.
- π_p is Steinberg.
- The representation π supports a non-trivial Fourier coefficient of type (42) associated to F .

Remark 7.9. By [15, Theorem 3.1], every irreducible non-trivial automorphic representation of $G_2(\mathbf{A})$ has a non-zero Fourier coefficient for some étale cubic algebra R . If R does not split as $\mathbf{Q} \times F$, then, one can show that the theta lift of σ supports a Fourier coefficient for the Siegel parabolic associated to a symmetric matrix of rank three. Together with [32, Theorem A.9], which deals with the generic case, this gives a complete correspondence between Fourier coefficients of σ and $\Theta(\sigma)$. A detailed proof of this statement will appear in a forthcoming work of ours.

Let us first fix some notations first. Recall from §6.1.2 that the centralizer of G_2 in M is GL_3 and let

$$U_0 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

be the unipotent radical of its Borel subgroup of upper triangular matrices. Note that the unipotent subgroup U_0U_3 is the unipotent radical of the parabolic subgroup P of PGSp_6 of Levi $\mathrm{GL}_2 \times \mathrm{GL}_1^2$ appearing in §5.2.2.

Definition 7.10. Define the character $\psi_0 : U_0(\mathbf{Q}) \backslash U_0(\mathbf{A}) \rightarrow \mathbf{C}^\times$ by sending

$$\psi_0(u) = e(a).$$

Note that $\psi_0\psi_D$ is the character (simply denoted by ψ_D) on $U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})$ introduced in §5.2.2.

As explained in §7.2, we view the space Ω of rank 1 elements in J inside \overline{N} so that U_0 acts on Ω via the natural right action of $\mathrm{GL}_3 \subseteq M$ on \overline{N} . Then, we let U_0 act on the left on ω and hence on the triples (x, y, z) of off-diagonal terms by the rule

$$u^{-1} \cdot (x, y, z) = (x + ay + bz, y, z). \quad (18)$$

7.4.1. *The relation between U_P and U_H .* In what follows, we relate the unipotent subgroup U_0 to the unipotent radical U_H of the Heisenberg parabolic. Such a relation will be employed in Proposition 7.13 to establish a relation between Fourier coefficients for the Heisenberg parabolic of G_2 -cusp forms and Fourier coefficients of type (42) of their theta lifts.

Before stating our result, we make the following comments on the choice of representatives of the open orbits in Proposition 7.5. First, suppose that $D = d^2$, with $d \in \mathbf{Q}^\times$. There is no harm in assuming $d \in \mathbf{Z}$. Recall that the stabilizer in G_2 of the vector $s_4 - t_4$ can be identified with $\mathrm{SL}_3 = \mathrm{SL}(\langle s_1, s_2, s_3 \rangle)$. Since the Heisenberg parabolic $H = L_H \cdot U_H$ is the stabilizer of the flag $\langle s_1, t_3 \rangle$, its unipotent radical U_H contains $U_D = \mathrm{Stab}_{G_2}(A_0)$, where

$$A_0 = A(d(s_4 - t_4), s_1 - t_3, d(s_1 + t_3)) \in J(\mathbf{Z})$$

is the representative of the open orbit of the action of G_2 on $\omega(\mathbf{Q})$ as in Proposition 7.5. Moreover, U_H/U_D is 2-dimensional and supported on the roots $a + b$ and $a + 2b$. Let us now suppose that D is not a square in \mathbf{Q}^\times . The vector $x = s_2 + Dt_2$ is a trace zero octonion of norm D and orthogonal to t_3 . We choose the representative of the open orbit to be

$$A_0 = A(s_2 + Dt_2, s_1, t_3) \in J(\mathbf{Z}).$$

Lemma 7.11. *There is a natural surjection $p : U_H \rightarrow U_0$ inducing an isomorphism*

$$U_H/U_D \rightarrow U_0.$$

Proof. By the description of the action in (18) and the linear independence of the coordinates (x, y, z) of the representative of the open orbit, one sees that U_0 acts freely on it. Hence, the result follows from showing that any element in U_H acts on the triple (x, y, z) as an element of U_0 and vice versa. *Case 1.* We start with the case where D is a square in \mathbf{Q}^\times . The action of U_0 is given by

$$u^{-1} \cdot (d(s_4 - t_4), s_1 - t_3, d(s_1 + t_3)) = (d(s_4 - t_4) + (a + db)s_1 + (db - a)t_3, s_1 - t_3, d(s_1 + t_3)). \quad (19)$$

Since any element of U_H fixes s_1 and t_3 , it suffices to show that U_H acts on $(s_4 - t_4)$ as an element of U_0 . We verify this by studying the action of the Lie algebra. By (12), we know that the Lie algebra of U_H is generated by the Lie algebra of the unipotent upper-triangular

subgroup U_D in SL_3 and by the vectors v_1 and δ_3 . Using the explicit action of the action of the Lie algebra given in §6.2.2, one checks that

$$\begin{aligned} E_{ij} \cdot (s_4 - t_4) &= 0, \\ v_1 \cdot (s_4 - t_4) &= s_1, \\ \delta_3 \cdot (s_4 - t_4) &= t_3. \end{aligned}$$

The above equations show that, for $u_1 = x_{a+b}(\lambda_1)$ and $u_2 = x_{a+2b}(\lambda_2)$ for some scalars λ_1, λ_2 , we have

$$\begin{aligned} u_1 \cdot (d(s_4 - t_4)) &= d(s_4 - t_4 + \lambda_1 s_1), \\ u_2 \cdot (d(s_4 - t_4)) &= d(s_4 - t_4 + \lambda_2 t_3). \end{aligned}$$

This gives the desired isomorphism: if $u \in U_H/U_D$ is identified with the product of $x_{a+b}(\lambda_1)x_{a+2b}(\lambda_2)$, then, from Equation (19), we see that it gets sent to the element

$$\begin{pmatrix} 1 & d\frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U_0.$$

Case 2. We now suppose that D is not a square in \mathbf{Q}^\times . Similarly to Case 1, it suffices to calculate $u \cdot (s_2 + Dt_2)$ for any $u \in U_H$. As above, one checks that

$$\begin{aligned} E_{12} \cdot (s_2 + Dt_2) &= s_1, \\ E_{23} \cdot (s_2 + Dt_2) &= Dt_3, \\ E_{13} \cdot (s_2 + Dt_2) &= 0, \\ v_1 \cdot (s_2 + Dt_2) &= t_3, \\ \delta_3 \cdot (s_2 + Dt_2) &= -Ds_1. \end{aligned}$$

This implies that if $u \in V_H = U_H/[U_H, U_H]$ equals to $x_a(\lambda_1)x_{a+b}(\lambda_2)x_{a+2b}(\lambda_3)x_{a+3b}(\lambda_4)$, then

$$u \cdot (s_2 + Dt_2) = s_2 + Dt_2 + (\lambda_1 - \lambda_3 D)s_1 + (\lambda_2 + D\lambda_4)t_3.$$

In particular, U_D embeds into U_H as the subgroup of matrices with $\lambda_1 = \lambda_3 D$ and $\lambda_2 = -\lambda_4 D$, and the map $p : U_H/U_D \rightarrow U_0$ sends u to the element

$$\begin{pmatrix} 1 & \lambda_1 - \lambda_3 D & \lambda_2 + \lambda_4 D \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U_0.$$

□

Corollary 7.12. *Under the isomorphism $p : U_H/U_D \rightarrow U_0$, we have*

$$\psi_{H,D} = \psi_0 \circ p,$$

where $\psi_{H,D} : U_H(\mathbf{Q}) \backslash U_H(\mathbf{A}) \rightarrow \mathbf{C}^\times$ is the character corresponding to the étale cubic algebra $\mathbf{Q} \times \mathbf{Q}(\sqrt{D})$.

Proof. We start with the case where D is a square in \mathbf{Q}^\times . For simplicity, we can (and do) assume that $D = 1$. From Lemma 7.11, if $n \in U_H/U_D$ is identified with the product of $x_{a+b}(\lambda_1)x_{a+2b}(\lambda_2)$, it is sent via p to

$$\begin{pmatrix} 1 & \frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U_0.$$

Hence, the character $\psi_0 \circ p : U_H(\mathbf{Q}) \backslash U_H(\mathbf{A}) \rightarrow \mathbf{C}^\times$ sends $n \mapsto e(\frac{\lambda_1 - \lambda_2}{2})$. We now show that this corresponds to the character $\psi_{H,D}$ associated to $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ as in §6.2.3. Recall that each character on $U_H(\mathbf{Q}) \backslash U_H(\mathbf{A})$ is of the form $n \mapsto e(\langle w, \bar{n} \rangle)$, where \bar{n} denotes the projection of n to $U_H/[U_H, U_H]$ and $w \in U_H(\mathbf{Q})/[U_H(\mathbf{Q}), U_H(\mathbf{Q})]$ corresponds to a binary cubic form

$$f_w(x, y) = \lambda_1 x^3 + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3,$$

with $\lambda_i \in \mathbf{Q}$. Furthermore, as $\bar{n} = x_a(\lambda'_1)x_{a+b}(\lambda'_2/3)x_{a+2b}(\lambda'_3/3)x_{a+3b}(\lambda'_4)$ corresponds to $f'(x, y) = \lambda'_1 x^3 + \lambda'_2 x^2 y + \lambda'_3 x y^2 + \lambda'_4 y^3$, the pairing is

$$\langle w, \bar{n} \rangle = \lambda_1 \lambda'_4 - \frac{\lambda_2 \lambda'_3}{3} + \frac{\lambda_3 \lambda'_2}{3} - \lambda_4 \lambda'_1.$$

Then, the character $\psi_0 \circ p$ corresponds to an element w_D for which $\lambda_1, \lambda_4 = 0$ and $\lambda_2, \lambda_3 = 1/2$, namely the binary cubic polynomial $f_D(x, y) = \frac{1}{2}(x^2 y + x y^2)$. The latter is in the $L_H(\mathbf{Q})$ -orbit corresponding to the cubic algebra \mathbf{Q}^3 . Indeed, if we let $g = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix} \in L_H(\mathbf{Q})$ act on f_D , we get

$$g \cdot f_D(x, y) = -\frac{1}{4} f_D(2x, -2y) = \frac{1}{8}(8x^2 y - 8x y^2) = x^2 y - x y^2,$$

which corresponds to \mathbf{Q}^3 by Example 6.1(1).

We now suppose that D is not a square in \mathbf{Q}^\times . Then, by Lemma 7.11, if

$$n \equiv x_a(\lambda_1)x_{a+b}(\lambda_2)x_{a+2b}(\lambda_3)x_{a+3b}(\lambda_4) \pmod{[U_H, U_H]},$$

the character $\psi_0 \circ p : U_H(\mathbf{Q}) \backslash U_H(\mathbf{A}) \rightarrow \mathbf{C}^\times$ sends $n \mapsto e(\lambda_1 - \lambda_3 D)$. This character is associated to the binary cubic polynomial $f_D(x, y) = D x^2 y - y^3$, which corresponds to $\mathbf{Q} \times \mathbf{Q}(\sqrt{D})$ by Example 6.1(2). \square

7.4.2. Comparison of Fourier coefficients. The following proposition can be paired with Proposition 7.7 to give three equivalent ways of proving that the theta lift of an automorphic representation of G_2 does not vanish.

Proposition 7.13. *Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$ as in Theorem 7.2 and let $\varphi \in \sigma^\vee$ be a cuspidal form. The following conditions are equivalent*

- (1) $\Theta(f, \varphi)_{U_P, \psi_D}(1) \neq 0$ for some choice of $f \in \Pi$,
- (2) $\varphi_{U_H, \psi_{H,D}}(g) \neq 0$ for some $g \in G_2(\mathbf{A})$.

In particular, if any of the conditions holds then $\Theta(\sigma)$ is non-zero.

Proof. Decomposing $U_P = U_0 U_3$, we have

$$\Theta(f, \varphi)_{U_P, \psi_D}(1) = \int_{U_0(\mathbf{Q}) \backslash U_0(\mathbf{A})} \int_{U_3(\mathbf{Q}) \backslash U_3(\mathbf{A})} \Theta(f, \varphi)(uu') \psi_D^{-1}(u') \psi_{U_0}^{-1}(u) du' du.$$

As in the proof of Proposition 7.7, this equals

$$\int_{U_0(\mathbf{Q}) \backslash U_0(\mathbf{A})} \int_{U_D(\mathbf{A}) \backslash G_2(\mathbf{A})} \theta(f)_{A_0}(ug) \varphi^{U_D}(g) \psi_{U_0}^{-1}(u) dg du.$$

Exchanging integrals and making an inner integration over $U_D(\mathbf{A}) \backslash U_H(\mathbf{A})$, we get

$$\int_{U_H(\mathbf{A}) \backslash G_2(\mathbf{A})} \int_{U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \left(\int_{U_0(\mathbf{Q}) \backslash U_0(\mathbf{A})} \theta(f)_{A_0}(uu'g) \psi_{U_0}^{-1}(u) du \right) \varphi^{U_D}(u'g) du' dg.$$

The isomorphism $p : U_H/U_D \cong U_0$ of Lemma 7.11 induces

$$U_0(\mathbf{Q}) \backslash U_0(\mathbf{A}) \cong U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})$$

such that $\psi_{H,D} = \psi_0 \circ p$ (cf. Corollary 7.12). Thus, we can write the integral as

$$\int_{U_H(\mathbf{A}) \backslash G_2(\mathbf{A})} \int_{U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \left(\int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \theta(f)_{A_0}(uu'g) \psi_{H,D}^{-1}(u) du \right) \varphi^{U_D}(u'g) du' dg.$$

Exchanging integrals, we have

$$\begin{aligned} & \int_{U_H(\mathbf{A}) \backslash G_2(\mathbf{A})} \int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \left(\int_{U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \theta(f)_{A_0}(uu'g) \varphi^{U_D}(u'g) du' \right) \psi_{H,D}^{-1}(u) dudg \\ &= \int_{U_H \backslash G_2(\mathbf{A})} \int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \left(\int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \sum_{\gamma \in U_D \backslash U_H(\mathbf{Q})} \theta(f)_{A_0}(u\gamma u'g) \varphi^{U_D}(\gamma u'g) du' \right) \psi_{H,D}^{-1}(u) dudg \\ &= \int_{U_H \backslash G_2(\mathbf{A})} \int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \sum_{\gamma} \left(\int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \theta(f)_{A_0}(\gamma uu'g) \varphi^{U_D}(\gamma u'g) du' \right) \psi_{H,D}^{-1}(u) dudg \end{aligned}$$

Changing variable $u' \mapsto u'' = \gamma uu' = u\gamma u''$ in the inner integral, the above becomes

$$\int_{U_H(\mathbf{A}) \backslash G_2(\mathbf{A})} \int_{U_H(\mathbf{Q}) U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \left(\int_{U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \theta(f)_{A_0}(u''g) \varphi^{U_D}(u^{-1}u''g) du'' \right) \psi_{H,D}^{-1}(u) dudg$$

which, after rearranging the integrals, equals to

$$\begin{aligned} & \int_{U_H(\mathbf{A}) \backslash G_2(\mathbf{A})} \int_{U_D(\mathbf{A}) \backslash U_H(\mathbf{A})} \theta(f)_{A_0}(u''g) \varphi_{U_H, \psi_{H,D}}(u''g) du'' dg \\ &= \int_{U_D(\mathbf{A}) \backslash G_2(\mathbf{A})} \theta(f)_{A_0}(g) \varphi_{U_H, \psi_{H,D}}(g) dg. \end{aligned}$$

This shows that (1) implies (2). The proof of the converse is identical as the one given in Proposition 7.7. \square

We are now ready to give a proof of Theorem 7.8.

Proof of Theorem 7.8. Let σ be a cuspidal automorphic representation satisfying the hypotheses of the Theorem. We first apply Corollary 7.3 to deduce that $\Theta(\sigma)$ is cuspidal. Moreover, by Proposition 7.13, the theta lift supports a Fourier coefficient of type (42) and, in particular, it is non-zero. Finally, by the compatibility between the global and local correspondences (Proposition 6.5), the result of Li (Proposition 6.7) implies that the archimedean component of π is a discrete series of Hodge type (3, 3), while its component at p is the Steinberg representation by Proposition 6.11(2). This finishes the proof. \square

8. THE CYCLE CLASS FORMULA AND THE STANDARD MOTIVE FOR G_2

We conclude this article with the arithmetic applications described in the introduction.

8.1. The relation between L -functions of G_2 and PGSp_6 . The dual group of G_2 is $G_2(\mathbf{C})$, which can be realized as the intersection $\mathrm{SO}_7(\mathbf{C}) \cap \mathrm{Spin}_7(\mathbf{C})$. More precisely, we have the commutative diagram

$$\begin{array}{ccccc}
 & & \text{Std} & & \\
 & \swarrow & \text{---} & \searrow & \\
 G_2(\mathbf{C}) & \hookrightarrow & \mathrm{SO}_7(\mathbf{C}) & \hookrightarrow & \mathrm{GL}_7(\mathbf{C}) \\
 \downarrow \zeta & & \downarrow & & \downarrow \\
 \mathrm{Spin}_7(\mathbf{C}) & \hookrightarrow & \mathrm{SO}_8(\mathbf{C}) & \hookrightarrow & \mathrm{GL}_8(\mathbf{C}), \\
 & \swarrow & \text{Spin} & \searrow &
 \end{array} \tag{20}$$

where $\mathrm{Std} : G_2(\mathbf{C}) \rightarrow \mathrm{GL}(V_7)$ is the standard representation given by trace zero octonions, $\mathrm{Spin} : \mathrm{Spin}_7(\mathbf{C}) \rightarrow \mathrm{GL}(V_8)$ is the 8-dimensional spin representation, while the embedding ζ is defined from the fact that the stabilizer in $\mathrm{Spin}_7(\mathbf{C})$ of a generic vector of V_8 is isomorphic to $G_2(\mathbf{C})$. From the commutative diagram, one immediately sees that

$$V_{8|G_2} = V_7 \oplus \mathbf{1}.$$

In particular, if π_ℓ is an unramified smooth representation of $\mathrm{PGSp}_6(\mathbf{Q}_\ell)$ with Satake parameter s_{π_ℓ} belonging to $\zeta(G_2(\mathbf{C}))$, then

$$L(s, \pi_\ell, \mathrm{Spin}) = L(s, \pi_\ell, \mathrm{Std})\zeta_\ell(s),$$

where

$$L(s, \pi_\ell, \mathrm{Std}) := \frac{1}{\det(1 - \ell^{-s}\mathrm{Std}(s_{\pi_\ell}))}$$

denotes the Euler factor at ℓ of the 7-dimensional standard L -function for G_2 .

Let now π be a cuspidal automorphic representation of PGSp_6 , which is unramified outside a finite set of places S containing the Archimedean place. As a special case of Langlands functoriality, one then expects that if $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$, π is a functorial lift from either G_2 or G_2^c , where recall that G_2^c denotes the form of G_2 which is compact at ∞ and split at all finite places of \mathbf{Q} . We invite the reader to consult [21], [17], [48], and [18] for results in this direction. In particular, using results of [11], [18] and [41], one deduces the following.

Proposition 8.1. *Suppose that π satisfies the hypotheses (DS) and (St) of §2.7. Then π is tempered and the following are equivalent:*

- (1) *For almost all ℓ , the Satake parameter $s_{\pi_\ell} \in \zeta(G_2(\mathbf{C}))$,*
- (2) *There exists a cuspidal automorphic representation σ of either G_2 or G_2^c such that π is a weak functorial lift of σ .*

Moreover, if these conditions hold, the partial L -function $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) = L^S(1, \sigma, \mathrm{Std}) \prod_{\ell \in S} (1 - \ell^{-1}) \neq 0.$$

Proof. Since π is cohomological and it is Steinberg at a finite place, we can apply [41, Lemma 2.7] to deduce that π is essentially tempered at all places. As π has trivial central character, this is equivalent to being tempered. The first part of the statement then follows from [18,

Theorem 1.1]. We now discuss the formula for the residue of the Spin L -function. If either (hence both) condition (1) or (2) holds, the commutative diagram above implies that

$$L^S(s, \pi, \text{Spin}) = L^S(s, \pi, \text{Std})\zeta^S(s),$$

where $L^S(s, \pi, \text{Std})$ is the partial L -function of π associated to the standard 7-dimensional representation of Spin_7 . By [43, Theorem 1.1.1], the restriction to $\text{Sp}_6(\mathbf{A})$ of π contains a cuspidal automorphic representation π^b , such that (up to possibly enlarge S)

$$L^S(s, \pi, \text{Std}) = L^S(s, \pi^b, \text{Std}).$$

By [41, Corollary 2.2 & Lemma 2.3], there exists a cuspidal automorphic representation π^\sharp of $\text{GL}_7(\mathbf{A})$ such that

$$L^S(s, \pi^b, \text{Std}) = L^S(s, \pi^\sharp),$$

where $L^S(s, \pi^\sharp)$ denotes the standard L -function of π^\sharp . We claim that $L^S(1, \pi^\sharp) \neq 0$. By [37, Theorem (1.3)], $L(s, \pi^\sharp) \neq 0$ for any s with $\text{Re}(s) = 1$. If we write

$$L^S(s, \pi^\sharp) = L(s, \pi^\sharp) \prod_{\ell \in S} L(s, \pi_\ell^\sharp)^{-1},$$

then our claim follows from the fact that each $L(s, \pi_\ell^\sharp)$ has no pole at $s = 1$ (cf. [50, p. 317]). This implies that (up to possibly enlarge S)

$$L^S(1, \pi, \text{Std}) = L^S(1, \sigma, \text{Std}) \neq 0,$$

where the first equality is a consequence of the fact that the Satake parameters of σ and π agree almost everywhere. Thus, $L^S(s, \pi, \text{Spin})$ has a simple pole at $s = 1$ with residue

$$\text{Res}_{s=1} L^S(s, \pi, \text{Spin}) = L^S(1, \sigma, \text{Std}) \text{Res}_{s=1} \zeta^S(s) \neq 0.$$

□

The hypotheses of Proposition 8.1 can be weakened at the cost of working with an integral representation of the Spin L -function. The following result uses the integral $\mathcal{I}(\Psi, \Psi, s)$ of Theorem 5.8 and its corollary.

Proposition 8.2 ([48, Theorem 1.3]). *Let π be a cuspidal automorphic representation of $\mathbf{PGSp}_6(\mathbf{A})$ supporting a Fourier coefficient of type (4.2). If $L^S(s, \pi, \text{Spin})$ has a simple pole at $s = 1$, then there exists a cuspidal automorphic representation σ of G_2 such that π is a weak functorial lift of σ , and*

$$\text{Res}_{s=1} L^S(s, \pi, \text{Spin}) = L^S(1, \sigma, \text{Std}) \prod_{\ell \in S} (1 - \ell^{-1}).$$

8.2. The cycle class formula. Let π be a cuspidal automorphic representation of $\mathbf{PGSp}_6(\mathbf{A})$ such that π_∞ is the discrete series of Hodge type (3,3) in the L -packet of V^λ with $\lambda = (\lambda_2 + \lambda_3, \lambda_2, \lambda_3, 0)$. Let $\Psi = \Psi_\infty \otimes \Psi_f$ denote a cusp form in π such that Ψ_∞ is a highest weight vector of the minimal K_∞ -type $\tau_{(\lambda_2+2, \lambda_3+2, -\lambda_1-4)}$ of $\pi_{\infty,1}^{3,3} \subseteq \pi_\infty|_{\text{Sp}_6(\mathbf{R})}$. We assume that $\Psi_f \in \pi_f$ is U -invariant, so that we have (cf. §4.3)

$$[\omega_\Psi] = [\omega_{\Psi_\infty} \otimes \Psi_f] \in H^6(\mathfrak{g}, K_{\mathbf{G}}; \pi^U \otimes V^\lambda).$$

For any $\lambda_2 \geq \mu \geq \lambda_3$, we denote $\Psi^{[\lambda, \mu]} := A^{[\lambda, \mu]} \cdot \Psi_\infty \otimes \Psi_f$, where $A^{[\lambda, \mu]}$ is the operator that appeared in Proposition 4.8. Finally, we denote by pr_{π^\vee} the projection to the π_f^\vee -isotypic component in Betti cohomology of $\text{Sh}_{\mathbf{G}}(U)$. In Proposition 4.8 and Theorem 5.12, we related the value $\langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle$ to the residue at $s = 1$ of the partial Spin L -function of π . Combining these results with Proposition 8.2, we have the following.

Theorem 8.3. *Suppose $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0$. Assume that $L^S(s, \pi, \text{Spin})$ has a simple pole at $s = 1$ and let σ be the weak functorial lift of π given by Proposition 8.2. Then the class $\text{pr}_{\pi^\vee}(\mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}) \neq 0$ and*

$$\langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle = C' \cdot L^S(1, \sigma, \text{Std}),$$

with C' a non-zero constant.

Proof. By Theorem 5.12, we have that

$$\langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle = C \cdot \text{Res}_{s=1} L^S(s, \pi, \text{Spin}),$$

where

$$C = \frac{c \cdot \widehat{\Phi}(0) h_{U'}}{(2\pi i)^3 \cdot \text{vol}(U')} \mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1).$$

By hypothesis, the integral $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, s)$ is not identical zero, which implies that π supports a Fourier coefficient of type (4.2). Thus, by Proposition 8.2 we get

$$\text{Res}_{s=1} L^S(s, \pi, \text{Spin}) = L^S(1, \sigma, \text{Std}) \prod_{\ell \in S} (1 - \ell^{-1}).$$

This finishes the proof by setting $C' := C \cdot \prod_{\ell \in S} (1 - \ell^{-1})$. \square

Our result slightly improves if we assume that π is Steinberg at a finite place. When this holds, by Lemma 2.10, we have

$$\dim_{\mathbf{R} \otimes_{\mathbf{Q}} L} H_{\mathcal{H}}^1(M(\pi_f^\vee)_{\mathbf{R}}(4)) = \dim_{\mathbf{C}} (\pi_f^\vee)^U.$$

Combining Theorem 5.12, Proposition 8.1, and Proposition 4.10, we obtain the following.

Theorem 8.4. *Suppose that π satisfies the following hypotheses:*

- π_p is Steinberg at a prime $p \in S$;
- $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0$ for some μ ;
- for all $\ell \notin S$, the Satake parameter $s_{\pi_\ell} \in \zeta(G_2(\mathbf{C}))$.

Then there exists a cuspidal automorphic representation σ of G_2 such that

$$\langle \mathcal{Z}_{\mathbf{H}, B}^{[\lambda, \mu]}, [\omega_\Psi] \rangle = \langle \mathcal{Z}_{\mathbf{H}, \mathcal{H}}^{[\lambda, \mu]}, [\omega_\Psi] \rangle_{\mathcal{H}} = C' \cdot L^S(1, \sigma, \text{Std}) \neq 0,$$

with C' the constant of Theorem 8.3. In particular $\text{pr}_{\pi^\vee} \mathcal{Z}_{\mathbf{H}, \mathcal{H}}^{[\lambda, \mu]}$ and its Hecke translates generate $H_{\mathcal{H}}^1(M(\pi_f^\vee)_{\mathbf{R}}(4))$.

Remark 8.5. This verifies a weaker form of Conjecture 1.1(3) for the motive $M(\pi_f^\vee)(3)$ at the cost of supposing that $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1)$ is non-zero for some μ .

Remark 8.6. By Proposition 5.11, an automorphic representation π which satisfies the hypotheses of Theorem 8.4 is \mathbf{H} -distinguished, namely the map $\mathcal{P}_{\mathbf{H}} \in \text{Hom}_{\mathbf{H}(\mathbf{A})}(\pi, \mathbf{1})$ defined by

$$\Psi \mapsto \mathcal{P}_{\mathbf{H}}(\Psi) := \int_{Z(\mathbf{A})\mathbf{H}(\mathbf{Q}) \backslash \mathbf{H}(\mathbf{A})} \Psi(h) dh$$

is not identically zero. Then, asking $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0$ for some μ is equivalent to asking that the map obtained as the composition of $\mathcal{P}_{\mathbf{H}}$ with an $\mathbf{H}(\mathbf{R})$ -equivariant embedding $\pi_\infty \rightarrow \pi$ restricts non-trivially to the minimal K_∞ -type of π_∞ .

Remark 8.7. Let σ be a cuspidal automorphic representation of $G_2(\mathbf{A})$, unramified outside a finite set S of primes, such that σ_∞ is the quaternionic discrete series σ_n for $n \geq 2$ of Definition 6.8, and σ_p is Steinberg at a finite prime $p \in S$. Then [15, Theorem 3.1] and Proposition 6.9 imply that there always exists a cubic totally real étale algebra R such that σ supports a Fourier coefficient associated to R . Assume that such algebra splits as $\mathbf{Q} \times F$. By Theorem 7.8, $\Theta(\sigma)$ is a non-zero and cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A})$. If we let π be an irreducible constituent of $\Theta(\sigma)$, then π satisfies all the running assumptions of Theorem 8.4 up to asking that $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1) \neq 0$, where now $\lambda = (2n - 4, n - 2, n - 2, 0)$ and $\mu = n - 2$. By [48, Proposition 5.1], there exists a cusp form $\tilde{\Psi}^{[\lambda, \mu]}$ in π , with Ψ_f and $\tilde{\Psi}_f$ equal away from S , and a Schwartz-Bruhat function Φ' such that

$$\mathcal{I}_S(\Phi', \tilde{\Psi}^{[\lambda, \mu]}, s) = \mathcal{I}_\infty(\Phi, \Psi^{[\lambda, \mu]}, s).$$

Hence, up to modifying the components of Ψ at the finite places in S , our hypothesis reduces to the non-triviality of $\mathcal{I}_\infty(\Phi, \Psi^{[\lambda, \mu]}, 1)$. The explicit formulae for the Fourier coefficient $\Psi_{U_P, \psi_D}^{[\lambda, \mu]}$ given in Proposition 7.13 allow us to write $\mathcal{I}_\infty(\Phi, \Psi^{[\lambda, \mu]}, s)$ as an archimedean integral involving the generalized G_2 -Whittaker functions studied in [47] and a function f_∞ in the minimal representation Π_∞ of $E_7(\mathbf{R})$, whose projection to π_∞ equals $A^{[\lambda, \mu]} \cdot \Psi_\infty$. The uniqueness of the former proved in *loc.cit.* seems to suggest the possibility of calculating $\mathcal{I}_\infty(\Phi, \Psi^{[\lambda, \mu]}, s)$ up to determining $f_\infty \in \Pi_\infty$ explicitly. We hope to come back to this problem in the future.

8.3. Galois representations of G_2 -type and a question of Gross and Savin. By the work of [41], under some mild conditions, we can associate to the cuspidal automorphic representation σ of $G_2(\mathbf{A})$ a G_2 -valued Galois representation ρ_σ which satisfies the expected properties. Using the exceptional theta correspondence, this representation is realized in the middle degree cohomology $H_{\text{ét}}^6(\mathrm{Sh}_{\mathbf{G}, \overline{\mathbf{Q}}}, \mathcal{V}_1^\lambda(3))$ of the Siegel sixfold $\mathrm{Sh}_{\mathbf{G}}$, where we have denoted with $H_{\text{ét}}^6(\mathrm{Sh}_{\mathbf{G}, \overline{\mathbf{Q}}}, \mathcal{V}_1^\lambda(3))$ the direct limit of the cohomology at level U as U varies. This direct limit is a smooth admissible representation of $\mathbf{G}(\mathbf{A}_f)$, endowed with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ commuting with the one of $\mathbf{G}(\mathbf{A}_f)$. The theorem below for the compact form of G_2 is shown in [41, Theorem 11.1 and Corollary 11.3]. The same proof works for the split form of G_2 as long as one has some information on its lift to PGSp_6 , and we only sketch it for the convenience of the reader.

Theorem 8.8. *Let σ be an automorphic cuspidal representation of $G_2(\mathbf{A})$ such that*

- σ_∞ is a quaternionic discrete series;
- σ_p is the Steinberg representation at some finite prime p ;
- σ supports a Fourier coefficient to a cubic algebra of the form $\mathbf{Q} \times F$.

Then, for each prime ℓ and $\iota : \mathbf{C} \cong \overline{\mathbf{Q}}_\ell$, there exists a Galois representation $\rho_\sigma = \rho_{\sigma, \iota} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow G_2(\overline{\mathbf{Q}}_\ell)$ such that

- *For every finite place $v \neq \ell$ where σ is unramified, ρ_σ is unramified at v . Moreover, the semisimple part of $\rho_\sigma(\mathrm{Frob}_v)$ is conjugate to the Satake parameter $\iota(s_{\sigma_v})$ in $G_2(\overline{\mathbf{Q}}_\ell)$.*
- *ρ_{σ_ℓ} is de Rham, and it is crystalline if σ is unramified at ℓ .*
- *$\zeta \circ \rho_\sigma = \rho_\pi$, where π is a theta lift of σ , and $\zeta : G_2(\mathbf{C}) \rightarrow \mathrm{Spin}_7(\mathbf{C})$ is the embedding appearing in (20).*
- *the Zariski closure of the image of ρ_σ maps onto either the image of a principal SL_2 in G_2 or onto G_2 .*

Proof. By Theorem 7.8, the big theta lift $\Theta(\sigma)$ is non-zero and cuspidal. Moreover, if we let π denote an irreducible subquotient of $\Theta(\sigma)$, its archimedean component π_∞ is a discrete series of Hodge type $(3, 3)$, and π_p is the Steinberg representation. By [41, Theorem A], there exists a representation $\rho_\pi : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{Spin}_7(\overline{\mathbf{Q}}_\ell)$ attached to π . By the proof of [41, Theorem 11.1], one has that the image of ρ_π is contained in $G_2(\overline{\mathbf{Q}}_\ell)$, and thus we have ρ_σ such that $\zeta \circ \rho_\sigma = \rho_\pi$ for a suitable choice of embedding $\zeta : G_2(\mathbf{C}) \rightarrow \text{Spin}_7(\mathbf{C})$ fitting in the diagram (20). Hence, by [41, Theorem A] and Proposition 6.11(1), the representation $\rho_\sigma : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow G_2(\overline{\mathbf{Q}}_\ell)$ satisfies the desired first three properties. Finally, by [41, Theorem A (v)], the Zariski closure of ρ_π must map onto either a principal SL_2 in $\text{SO}_7 \cap G_2$, or G_2 . \square

Remark 8.9. The hypotheses of the Theorem above can be substantially weakened to asking that σ_∞ is a discrete series and σ_p is Steinberg. This follows from establishing a complete correspondence of Fourier coefficients between σ and its theta lift and by the results of [41]. A proof of this fact is beyond the scope of this article and it will appear in forthcoming work.

By construction, the composition of the Galois representation ρ_π (and thus ρ_σ) with the Spin representation appears in the middle degree cohomology of $\text{Sh}_{\mathbf{G}}$. If σ_∞ has Harish-Chandra parameter $x\varepsilon_1 + y\varepsilon_2$, with $x - 3 \geq y - 1 \geq 0$ and $x - y$ even, the archimedean component π_∞ is a discrete series in the L -packet of V^λ , with $\lambda = (x - 3, \frac{x+y-4}{2}, \frac{x-y-2}{2}, 0)$. By Lemma 2.7, the π_f^\vee -isotypic component of $H_{\text{ét},1}^6(\text{Sh}_{\mathbf{G},\overline{\mathbf{Q}}}, \mathcal{V}_1^\lambda(3))$ is 8-dimensional, and we have

$$H_{\text{ét},1}^6(\text{Sh}_{\mathbf{G},\overline{\mathbf{Q}}}, \mathcal{V}_1^\lambda(3))[\pi_f^\vee] \otimes \overline{\mathbf{Q}}_\ell = V_{\text{Spin} \circ \rho_\pi} \otimes \pi_f^\vee = V_{\text{Spin} \circ \zeta \circ \rho_\sigma} \otimes \pi_f^\vee.$$

If the image of ρ_σ is Zariski dense in $G_2(\overline{\mathbf{Q}}_\ell)$, we have $\text{Spin} \circ \zeta \circ \rho_\sigma = \text{Std} \circ \rho_\sigma \oplus \mathbf{1}$, where $\text{Std} \circ \rho_\sigma$ is the irreducible “standard” Galois representation attached to σ . If not, by Theorem 8.8, the image of ρ_σ is Zariski dense onto a principal $\xi : \text{SL}_2(\overline{\mathbf{Q}}_\ell) \rightarrow G_2(\overline{\mathbf{Q}}_\ell)$. Then, the branching law of [24, (7.1)] gives that $\text{Spin} \circ \zeta \circ \rho_\sigma = \text{Sym}^6 \circ \rho_\sigma \oplus \mathbf{1}$, where $\text{Sym}^6 \circ \rho_\sigma$ is the irreducible symmetric sixth power Galois representation attached to σ . Denote by $M_\ell(\pi_f)$ the Galois representation $V_{\text{Spin} \circ \rho_\pi}$ and let $M_\ell(\sigma_f)$ be either the Galois representation $V_{\text{Std} \circ \rho_\sigma}$ or $V_{\text{Sym}^6 \circ \rho_\sigma}$. Then, we have that $M_\ell(\sigma_f)^{G_{\mathbf{Q}}} = 0$ and $M_\ell(\pi_f)$ decomposes as the direct sum of $M_\ell(\sigma_f)$ and the trivial representation $\mathbf{1}$ with multiplicity 1.

Remark 8.10. In the case where ρ_σ is not Zariski dense in $G_2(\overline{\mathbf{Q}}_\ell)$, the Satake parameter $s_{\sigma_p} \in \xi(\text{SL}_2(\mathbf{C}))$ for any unramified prime p . By Langlands reciprocity principle, σ should be the functorial lift of a cuspidal automorphic representation τ of $\text{PGL}_2(\mathbf{A})$, while $V_{\text{Sym}^6 \circ \rho_\sigma}$ should be a geometric realization of the motive of the symmetric sixth power of τ .

Tate conjecture predicts the existence of a cycle which gives rise to the trivial representation appearing in the decomposition above. In [27], Gross and Savin, inspired by local computations, conjectured that this cycle should come from a Hilbert modular 3-fold inside $\text{Sh}_{\mathbf{G}}$. The Theorem below confirms this expectation for cuspidal automorphic representations σ of G_2 which satisfy the hypotheses of Theorem 8.8. In particular, we suppose that σ supports a Fourier coefficient for a (necessarily totally real) cubic étale algebra $\mathbf{Q} \times F$. Let $\mathcal{Z}_{\mathbf{H},\text{ét}}^{[\lambda,\mu]} := \text{cl}_{\text{ét}}(\mathcal{Z}_{\mathbf{H},\mathcal{M}}^{[\lambda,\mu]}) \in H_{\text{ét}}^6(\text{Sh}_{\mathbf{G}}(\mathbf{U})_{\overline{\mathbf{Q}}}, \mathcal{V}_1^\lambda(3))^{G_{\mathbf{Q}}}$ be the étale realization of the motivic class $\mathcal{Z}_{\mathbf{H},\mathcal{M}}^{[\lambda,\mu]}$ (see Definition 3.9), where $\mathbf{H} = \text{GL}_2 \boxtimes \text{GL}_{2,F}^*$. Since the construction of the class is compatible with pull-backs, the cycles at finite level are matched into a class $\mathcal{Z}_{\mathbf{H},\text{ét}}^{[\lambda,\mu]} \in H_{\text{ét}}^6(\text{Sh}_{\mathbf{G},\overline{\mathbf{Q}}}, \mathcal{V}_1^\lambda(3))^{G_{\mathbf{Q}}}$. By composing the projection to the π_f^\vee -isotypic component together with the projection given by the vector $\Psi_f \in \pi_f$, we get

$$\mathcal{Z}_{\mathbf{H},\text{ét}}^\sigma := \Psi_f(\text{pr}_{\pi^\vee}(\mathcal{Z}_{\mathbf{H},\text{ét}}^{[\lambda,\mu]})) \in (M_\ell(\sigma_f) \oplus \mathbf{1})^{G_{\mathbf{Q}}} = \mathbf{1}.$$

Theorem 8.11. *Let σ be an automorphic cuspidal representation of $G_2(\mathbf{A})$ such that*

- σ_∞ is a quaternionic discrete series;
- σ_p is the Steinberg representation at some finite prime p ;
- σ supports a Fourier coefficient to a cubic algebra of the form $\mathbf{Q} \times F$.

Assume that the integral $\mathcal{I}_S(\Phi, \Psi^{[\lambda, \mu]}, 1)$ is non-zero for some Schwartz-Bruhat function Φ and $\Psi^{[\lambda, \mu]}$ as in §8.2. Then the class $\mathcal{Z}_{\mathbf{H}, \acute{e}t}^\sigma$ generates the trivial sub-representation $\mathbf{1}$ of $M_\ell(\pi_f)$.

Proof. By the comparison theorem between étale and Betti cohomology [2, Exposé XI, Theorem 4.4 (iii)] and Theorem 8.4, we know that the projection $\text{pr}_{\pi_f^\vee} \mathcal{Z}_{\mathbf{H}, \acute{e}t}^{[\lambda, \mu]}$ to $M_\ell(\pi_f) \otimes \pi_f^\vee$ generates a one dimensional subspace, which is trivial for the action of the Galois group. As we have explained above, the image of ρ_σ is either dense in $G_2(\overline{\mathbf{Q}}_\ell)$ or in $SL_2(\overline{\mathbf{Q}}_\ell) \rightarrow PGL_2(\overline{\mathbf{Q}}_\ell) \hookrightarrow G_2(\overline{\mathbf{Q}}_\ell)$. In either case, the representation $M_\ell(\sigma_f)$ is irreducible and the trivial factor $\mathbf{1}$ in $M_\ell(\pi_f)$ is hence generated by the image of $\mathcal{Z}_{\mathbf{H}, \acute{e}t}^\sigma$. \square

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