



# Topological proofs of categorical coherence

Pierre-Louis Curien, Guillaume Laplante-Anfossi

## ► To cite this version:

Pierre-Louis Curien, Guillaume Laplante-Anfossi. Topological proofs of categorical coherence. 2023. hal-03991861

**HAL Id: hal-03991861**

**<https://hal.science/hal-03991861>**

Preprint submitted on 16 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# TOPOLOGICAL PROOFS OF CATEGORICAL COHERENCE

PIERRE-LOUIS CURIEN AND GUILLAUME LAPLANTE-ANFOSSI

*"We shall construct  $KP_n$ , as a CW-complex, in Section 2 and show that it is an  $(n - 1)$ -ball. This gives an instant one-step proof of MacLane's theorem in full generality."*

– Mikhail M. Kapranov

ABSTRACT. In this note, we give a short topological proof of coherence for categorified non-symmetric operads by using the fact that the diagrams involved are the 1-skeleton of simply connected CW complexes. In particular, we obtain a "one-step" topological proof of MacLane's coherence theorem, as suggested by Kapranov in 1993. In addition, we use Morse theory to give a second topological proof which is very close to MacLane's original argument. We use the same methods to deduce other categorical coherence results and discuss possible generalisations to higher categories.

## INTRODUCTION

The  $n$ -dimensional permutoassociahedron, a CW-complex whose faces are in bijection with parenthesized permutations of  $n + 1$  letters, was first introduced by M. Kapranov in his study of higher dimensional Yang–Baxter equations, through the moduli spaces of curves  $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$  and the solutions of the Knizhnik–Zamolodchikov equation [Kap93]. It was later realized as a convex polytope by V. Reiner and G. M. Ziegler [RZ94], and more recently as a simple polytope in [BIP19] and through the nested braid fan in [CL21].

The present note stems from a desire to understand the epigraph, taken from the introduction of [Kap93]: what is the precise relationship between the permutoassociahedron and MacLane's coherence theorem for symmetric monoidal categories? We show that the *simple connectedness* of the former implies the latter, thereby refining and proving Kapranov's claim (see Theorem 2.8).

This is done through a general "topological coherence theorem" which applies to any simply connected, regular CW complex (Theorem 1.1). We use it to prove coherence result for categorified non-symmetric operads, simplifying the proof of [DP15]. These objects, introduced in [DP15], are an operadic generalization of non-symmetric monoidal categories.

We also investigate a topological incarnation of MacLane's original argument, in the spirit of rewriting theory. Using Morse theory on affine cell complexes [BB97], we obtain a general topological theorem which applies to a certain family of simply connected polyhedral complexes; the ones that admit a terminating and confluent rewriting system on their 1-skeleton

---

*Date:* February 15, 2023.

*2010 Mathematics Subject Classification.* Primary 18N20, Secondary 52B11.

*Key words and phrases.* Categorified operads, categorical coherence, Seifert–Van Kampen theorem, polytopes, MacLane coherence theorem, rewriting theory, Morse theory.

(Theorem 1.4). In particular, this second theorem can be applied to polytopes, allowing us to give a second, rewriting-theoretic proof of both previously mentioned coherence results.

These two general topological coherence theorems can be used to prove other categorical results where polytopes appear, such as coherence for monoidal functors between monoidal categories [Eps66], which we present in Section 2.4. This all points towards further investigation of the relationship between  $n$ -categorical coherence and  $n$ -connectedness of appropriate spaces. As a motivation in this direction, the present results could well be the strict,  $n = 1$  case of recent theorems of S. Barkan in the  $\infty$ -categorical setting [Bar22], see Section 3.1.

## 1. TOPOLOGICAL COHERENCE

**1.1. Coherence à la Van Kampen.** Let  $X$  be a regular CW complex, and let  $X_k$ ,  $k \geq 0$  denote its  $k$ -skeleton. Let  $\mathcal{F}(X)$  be the groupoid with set of objects  $X_0$  and morphisms spanned by the following set: for each  $x \in X_0$ , one identity morphism  $\text{id}_x : x \rightarrow x$ ; and for each 1-cell  $\alpha \in X_1$ , one morphism  $\alpha : x \rightarrow y$  oriented according to its attaching map, and one inverse morphism  $\alpha^{-1} : y \rightarrow x$  in the opposite direction. In other words,  $\mathcal{F}(X)$  is the free groupoid generated by the morphisms  $\alpha$ . A *combinatorial path* on  $X$  is a morphism in  $\mathcal{F}(X)$ , that is, a composable sequence of  $\alpha$  and  $\alpha^{-1}$  morphisms (a *word* in  $\alpha$  and  $\alpha^{-1}$ ). Two combinatorial paths  $\gamma, \gamma' \in \mathcal{F}(X)(x, y)$  with the same endpoints are said to be *parallel*.

The attaching map of a 2-cell  $A$  of  $X$  defines a morphism  $\gamma_A \in \mathcal{F}(X)(x, x)$  for a certain  $x \in A_0$ , given by the sequence of 1-cells in its image. Two parallel combinatorial paths  $\gamma, \gamma'$  are said to be *elementary combinatorially homotopic* if they differ exactly by a relation of the form  $\gamma_A = \text{id}_x$ , for some 2-cell  $A$ . That is, one can rewrite  $\gamma$  into  $\gamma'$  or  $\gamma'$  into  $\gamma$  by replacing some (possibly empty) subword of  $\gamma$  with an equivalent subword using the relation  $\gamma_A = \text{id}_x$ . More generally, two parallel combinatorial paths are *combinatorially homotopic* if they are related by a sequence of elementary combinatorial homotopies.

**Theorem 1.1.** *Any two parallel combinatorial paths on  $X$  are combinatorially homotopic if and only if every path component of  $X$  is simply connected.*

*Proof.* Let  $\Pi(X)$  denote the fundamental groupoid of  $X$ , that is the groupoid with objects the vertices of  $X$  and morphisms the homotopy classes of paths between them. Let  $\mathcal{C}(X)$  denote the quotient of the groupoid  $\mathcal{F}(X)$  by the relation “being combinatorially homotopic”. Then, we have an isomorphism of groupoids

$$\Pi(X) \cong \mathcal{C}(X) .$$

To show this, one proceeds in three steps. First, one shows that the fundamental groupoid  $\Pi(X_1)$  of the 1-skeleton of  $X$  is free on the homotopy classes of maps generated by the attaching maps of the 1-cells, that is, free on the  $\alpha$ -morphisms [Bro06, 9.1.5]. Thus, one gets  $\Pi(X_1) \cong \mathcal{F}(X)$ . Second, one shows that the fundamental groupoid  $\Pi(X_2)$  of the 2-skeleton of  $X$  is the free groupoid  $\Pi(X_1)$  modulo the relations  $\gamma_A = 1$ , for  $A$  a 2-cell of  $X$  [Bro06, 9.1.6]. This is done through repeated application of the Seifert–Van Kampen theorem; one then has  $\Pi(X_2) \cong \mathcal{C}(X)$ . Third, one shows that the inclusion of  $X_2$  in  $X$  induces an isomorphism of fundamental groupoids  $\Pi(X_2) \cong \Pi(X)$  [Bro06, 9.1.7], which concludes the proof of the isomorphism  $\Pi(X) \cong \mathcal{C}(X)$ . The theorem then follows, since

every path component of  $X$  is simply connected if and only if its fundamental groupoid  $\Pi(X)$  is trivial.  $\square$

**1.2. Coherence à la Morse.** Let  $X \subset \mathbb{R}^n$  be a polyhedral complex, and let  $\vec{v} \in \mathbb{R}^n$  be *generic* on the edges of  $X$ , meaning that for any pair of vertices  $x, y \in X$  belonging to the same edge of  $X$ , we have  $\langle \vec{v}, x \rangle \neq \langle \vec{v}, y \rangle$ . Such a generic vector  $\vec{v}$  induces a natural orientation on the edges of  $X$ , directed from the source vertex where the functional  $\langle \vec{v}, - \rangle$  is minimal to the target vertex where it is maximal.

In general, for any face  $F \subset X$  of  $X$ , there is a unique *source* vertex  $\text{sc}(F)$  such that all its adjacent edges  $e \subset F$  are outgoing, and a unique *sink* vertex  $\text{sk}(F)$  whose adjacent edges are all incoming. When the complex  $X$  has a unique *global sink*, a vertex whose adjacent edges  $e \subset X$  are all incoming, we will denote it by  $\text{sk}(X)$ .

Let  $H := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle = 0\}$  be the linear hyperplane orthogonal to  $\vec{v}$ . For every vertex  $x \in X$ , choose  $\varepsilon > 0$  such that the interval between  $\langle \vec{v}, x \rangle$  and  $\langle \vec{v}, x \rangle + \varepsilon$  does not contain the image of any other vertex under the “height” function  $\langle \vec{v}, - \rangle$ .

**Definition 1.2.** The *outgoing link* of a vertex  $x \in X$  is the intersection  $\mathcal{F} \cap (H + x + \varepsilon \vec{v})$  of the family of faces  $\mathcal{F} := \{F \subset X \mid \text{sc}(F) = x\}$  with the affine hyperplane  $H + x + \varepsilon \vec{v}$ .

A combinatorial path  $\gamma$  on  $X$  is *oriented* if for any pair  $(e, f)$  of consecutive edges in  $\gamma$ , we have that  $\text{sk}(e) = \text{sc}(f)$ . When no ambiguity arises, we will omit the adjective “combinatorial” and say only “oriented path”. Two parallel oriented paths are said to be *elementary combinatorially homotopic* if they are as non-oriented paths. They are *combinatorially homotopic* if they are related by a sequence of elementary combinatorial homotopies between oriented paths.

The following Lemma and its consequence Theorem 1.4 translate into topological terms the proof of [ML63, Theorem 3.1].

**Lemma 1.3.** *Let  $X$  be a polyhedral complex, and let  $\vec{v}$  be generic on the edges of  $X$ . If there is a unique global sink  $\text{sk}(X)$ , then the outgoing link of every vertex is connected if and only if any two parallel oriented paths on  $X$  are combinatorially homotopic.*

*Proof.* We prove the first implication. Suppose that the outgoing link of every vertex is connected. Let  $\gamma$  and  $\gamma'$  be two parallel oriented paths between two vertices  $x$  and  $y$ . We prove that they are combinatorially homotopic. We proceed by induction on the maximal length  $m$  of an oriented path between  $x$  and  $y$  in  $X$ . Without loss of generality, we can suppose that  $y = \text{sk}(X)$ , since if  $y \neq \text{sk}(X)$  we can always find an oriented path between  $y$  and  $\text{sk}(X)$ . The cases when  $m = 0$  and  $m = 1$  are trivial. Suppose that the hypothesis holds up to  $m = k - 1, k \geq 2$ , and consider two paths  $\gamma$  and  $\gamma'$  for which  $m = k$ . Let  $e$  and  $e'$  denote the edges of  $\gamma$  and  $\gamma'$  that are adjacent to  $x$ . We examine three cases.

- (1) If  $e = e'$ , we can apply the induction hypothesis to  $\gamma \setminus e$  and  $\gamma' \setminus e'$ .
- (2) If  $e \neq e'$  and both edges are on the same 2-face  $F$  of  $X$ , then using the induction hypothesis we have that  $\gamma$  and  $\gamma'$  are respectively combinatorially homotopic to the paths  $\delta$  and  $\delta'$  defined as follows: they go from  $x = \text{sc}(F)$  to  $\text{sk}(F)$  by the unique path containing  $e$  and  $e'$ , respectively, and then from  $\text{sk}(F)$  to  $y$  along the same arbitrary oriented path. Since  $\delta$  and  $\delta'$  are combinatorially homotopic by definition, the

conclusion follows from the transitivity of the combinatorial homotopy equivalence relation.

- (3) Suppose that  $e \neq e'$ , and that  $e$  and  $e'$  are *not* on the same 2-face of  $X$ . Since the outgoing link of  $x$  is connected, there exists a path  $\theta$  between  $e$  and  $e'$  in this link. For every edge  $e_i$  of  $X$  in the path  $\theta$ , choose an oriented path  $\gamma_i$  in  $X$  from  $x$  to  $y = \text{sk}(X)$  going through  $e_i$ . Now apply Point (2) above to every pair of parallel oriented paths  $(\gamma_i, \gamma_{i+1})$  with  $e_i$  and  $e_{i+1}$  consecutive in  $\theta$ , and conclude again by transitivity of the combinatorial homotopy equivalence relation.

In the other direction, suppose that every pair of parallel oriented combinatorial paths are combinatorially homotopic. We show that for any vertex  $x$ , its outgoing link is connected. Indeed, take two edges  $e, e'$  of  $X$  with source  $x$ , and consider their extensions to oriented paths  $\gamma, \gamma'$  from  $x$  to  $\text{sk}(X)$ . By hypothesis, these two paths are combinatorially homotopic, that is, there is a sequence of parallel oriented paths from  $\gamma$  to  $\gamma'$ . The collection of first edges in each of these paths defines a path between  $e$  and  $e'$  in the outgoing link of  $x$ . Thus, this link is connected.  $\square$

**Theorem 1.4.** *Let  $X$  be a polyhedral complex, and let  $\vec{v}$  be generic on the edges of  $X$ . Suppose that*

- i) there is a unique sink  $\text{sk}(X)$ , and*
- ii) the outgoing link of every vertex is connected.*

*Then, any two parallel combinatorial paths on  $X$  are combinatorially homotopic.*

*Proof.* By Lemma 1.3, the conclusion holds for *oriented* paths. Let us show that this implies the non-oriented version. Let  $\gamma$  be a (non-oriented) combinatorial path on  $X$  between  $x$  and  $y$ . For every vertex  $z$  along  $\gamma$ , one can choose an oriented path  $\delta_z$  from  $z$  to  $\text{sk}(X)$ . We observe that for any edge  $e : z \rightarrow z'$  of  $\gamma$ , the oriented paths  $\delta_z$  and  $\delta_{z'}e$  are combinatorially homotopic by hypothesis. Going from  $x$  to  $y$  inductively one edge at a time and using transitivity of the homotopy equivalence relation, one obtains that  $\gamma$  is combinatorially homotopic to  $\delta_y^{-1}\delta_x$ . Taking another combinatorial path  $\gamma'$  parallel to  $\gamma$ , the same argument shows that  $\gamma'$  is combinatorial homotopic to  $\delta_y^{-1}\delta_x$ . Thus  $\gamma$  and  $\gamma'$  are combinatorially homotopic, which completes the proof.  $\square$

*Remark 1.5.* One can consider the abstract rewriting system defined by  $\vec{v}$  on the vertices of  $X$ , saying that  $x$  can be rewritten into  $y$  if and only if there is an oriented path from  $x$  to  $y$  in  $X$ . The hypotheses of Theorem 1.4 impose that this rewriting system is terminating and confluent [BN98, Definition 2.1.3].

The class of polyhedral complexes to which Theorem 1.4 applies is a strict subclass of simply connected complexes, as the following proposition shows.

**Proposition 1.6.** *Let  $X$  be a polyhedral complex. If there is a generic vector  $\vec{v} \in \mathbb{R}^n$  such that the outgoing link of every vertex is connected, then every path component of  $X$  is simply connected.*

*Proof.* Let  $\vec{v} \in \mathbb{R}^n$  be generic with respect to  $X$ , and suppose that the outgoing link of every vertex is connected. Since  $\vec{v}$  is generic on edges, it defines a Morse function  $\langle \vec{v}, - \rangle$  on  $X$ ,

in the sense of [BB97, Definition 2.2]. As in classical Morse theory, one can determine the homotopy type of  $X$  by considering its successive level sets. For  $t \in \mathbb{R}$  denote by  $X_t$  the closed subspace of  $X$  containing points  $x$  such that  $\langle x, \vec{v} \rangle$  is at least  $t$ . Let  $x$  be a vertex of  $X$  of height  $h = \langle x, \vec{v} \rangle$ . Observe first that  $X_{h+\epsilon}$ , for some small  $\epsilon > 0$ , is homotopy equivalent to  $X_{h'}$  where  $h' > h$  is the next greater height at which there is a vertex. That is, the homotopy type of  $X$  can only change at vertices [BB97, Lemma 2.3]. Then, one proves that  $X_h$  is homotopy equivalent to the pushout of  $X_{h+\epsilon}$  with the cone over the outgoing link of  $x$  along the outgoing link of  $x$  [BB97, Lemma 2.5]. By our assumption, the outgoing link of  $x$  is connected, and thus the cone over it is simply connected. Since the pushout of simply connected spaces over a connected space is always simply connected (this is an application of the Seifert–Van Kampen theorem), we obtain by induction that every path component of  $X$  is simply connected [BB97, Point (3) of Corollary 2.6].  $\square$

The converse of Proposition 1.6 is not true in general: many simply connected polyhedral complexes, as the one represented in Figure 1, have disconnected outgoing links, for many (sometimes for all) choices of generic orientation vectors.

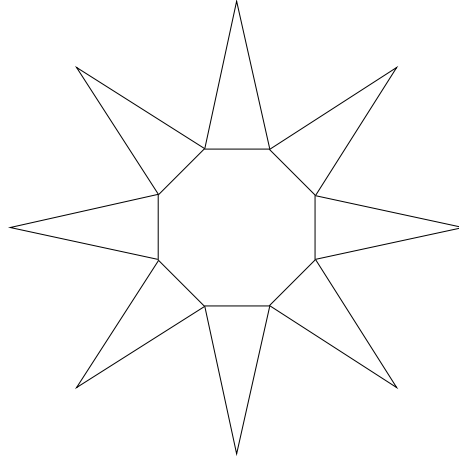


FIGURE 1. A simply connected polyhedral complex which admits disconnected outgoing links for every choice of generic vector.

This implies that the converse of Theorem 1.4 does not hold, and thus that MacLane’s original proof is far from reaching the full generality of Theorem 1.1. However, it will be sufficient for our purposes, presented in the next Section, since it applies to any polytope.

**Proposition 1.7.** *Let  $P \subset \mathbb{R}^n$  be a polytope, and let  $\vec{v} \in \mathbb{R}^n$  be generic with respect to  $P$ . Then,  $P$  admits a unique sink  $\text{sk}(P)$ , and the outgoing link of every vertex is connected.*

*Proof.* The existence and uniqueness of a sink is one of the basic, very useful facts about polytopes, see [Zie95, Theorem 3.7]. For the second part, we first observe that the *link* of a vertex  $x$  in a polytope, called the *vertex figure* and denoted  $P/x$ , is itself a polytope of dimension  $\dim P - 1$ , whose  $(k - 1)$ -faces are in bijection with the  $k$ -faces of  $P$  that contain  $x$

[Zie95, Proposition 2.4]. Define the linear hyperplane  $H := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle = 0\}$ , and consider the two half-spaces  $H^- := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle < 0\}$  and  $H^+ := \{y \in \mathbb{R}^n \mid \langle \vec{v}, y \rangle > 0\}$ . Since  $\vec{v}$  is not perpendicular to any edge of  $P$ , it defines a partition of the vertices of  $P/x$  into two connected components: the vertices that lie in  $H^-$ , which correspond to incoming edges of  $P$  at  $x$ , and the vertices that lie in  $H^+$ , which correspond to outgoing edges of  $P$  at  $x$ . Thus, the outgoing link of  $x$  is connected, and the proof is complete.  $\square$

## 2. CATEGORICAL COHERENCE

**2.1. Basic definitions.** Throughout this section we consider structures without units. Unless otherwise stated, the adjective “non-unital” will be implicitly assumed.

**Definition 2.1.** A *categorified non-symmetric operad*  $\mathcal{P}$  is a collection  $\{\mathcal{P}(n)\}_{n \in \mathbb{N}}$  of small categories equipped with bifunctors

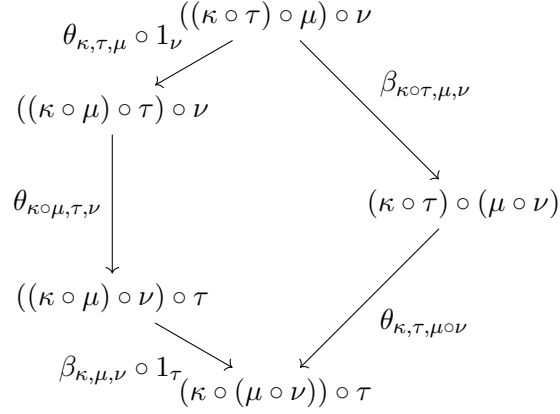
$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(k) \longrightarrow \mathcal{P}(n+k-1), \quad \text{for } 1 \leq i \leq n,$$

and for each  $\kappa \in \mathcal{P}(m)$ ,  $\mu \in \mathcal{P}(n)$ ,  $\nu \in \mathcal{P}(k)$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$  natural isomorphisms

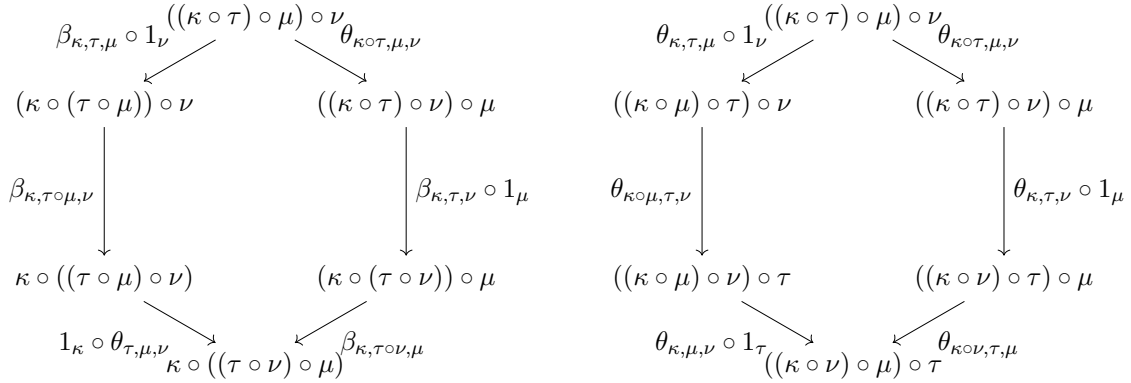
$$\begin{aligned} \beta_{\kappa, \mu, \nu} &: (\kappa \circ_i \mu) \circ_{j+i-1} \nu \xrightarrow{\cong} \kappa \circ_i (\mu \circ_j \nu), \\ \theta_{\kappa, \nu, \mu} &: (\kappa \circ_i \nu) \circ_{j+k-1} \mu \xrightarrow{\cong} (\kappa \circ_j \mu) \circ_i \nu, \quad \text{when } i < j, \end{aligned}$$

such that the following diagrams commute: the pentagonal

$$\begin{array}{ccc} & ((\kappa \circ \tau) \circ \mu) \circ \nu & \\ \beta_{\kappa, \tau, \mu} \circ 1_\nu \swarrow & & \searrow \beta_{\kappa \circ \tau, \mu, \nu} \\ (\kappa \circ (\tau \circ \mu) \circ \nu) & & (\kappa \circ \tau) \circ (\mu \circ \nu) \\ \downarrow \beta_{\kappa, \tau \circ \mu, \nu} & & \downarrow \beta_{\kappa, \tau, \mu \circ \nu} \\ \kappa \circ ((\tau \circ \mu) \circ \nu) & & (\kappa \circ \tau) \circ (\mu \circ \nu) \\ \searrow 1_\kappa \circ \beta_{\tau, \mu, \nu} & & \swarrow \beta_{\kappa, \tau, \mu \circ \nu} \\ & \kappa \circ (\tau \circ (\mu \circ \nu)) & \end{array} \quad \begin{array}{ccc} & ((\kappa \circ \tau) \circ \mu) \circ \nu & \\ \theta_{\kappa \circ \tau, \mu, \nu} \swarrow & & \searrow \beta_{\kappa, \tau, \mu} \circ 1_\nu \\ ((\kappa \circ \tau) \circ \nu) \circ \mu & & (\kappa \circ (\tau \circ \mu) \circ \nu) \\ \downarrow \theta_{\kappa, \tau, \nu} \circ 1_\mu & & \downarrow \theta_{\kappa, \tau \circ \mu, \nu} \\ ((\kappa \circ \nu) \circ \tau) \circ \mu & & (\kappa \circ \nu) \circ (\tau \circ \mu) \\ \searrow \beta_{\kappa \circ \nu, \tau, \mu} & & \swarrow \theta_{\kappa, \tau \circ \mu, \nu} \\ & (\kappa \circ \nu) \circ (\tau \circ \mu) & \end{array}$$



and hexagonal identities



The diagrams above hold for all instances of composable  $\beta$  and  $\theta$ ; these depend on the indices  $i, j, k$ , which are omitted for the sake of readability. Observe that a categorified non-symmetric operad concentrated in arity 1 is a non-symmetric monoidal category.

One can picture an object  $\mu \in \mathcal{P}(n)$  as a planar tree with one vertex decorated by  $\mu$ ,  $n$  leaves and one root (a *corolla*). The  $\circ_i$  bifunctors then correspond to the operation of *grafting* a corolla on top of another. Iterated applications of the  $\circ_i$  can be visualized as fully nested planar trees [Lap22, Definition 2.2], with vertices decorated by objects of  $\mathcal{P}$ , see Figure 2.

The  $\beta$  and  $\theta$  arrows correspond to the sequential and parallel axioms of an operad, and relate the two possible ways of nesting a tree with 3 vertices, see Figure 3. Moreover, there is then one coherence diagram (pentagon or hexagon) for every planar tree with 4 vertices, see Figure 4.

*Remark 2.2.* K. Došen and Z. Petrić introduced in [DP15, Section 12] the notion of weak Cat-operad. Despite looking different at first sight, the two notions are in fact equivalent.



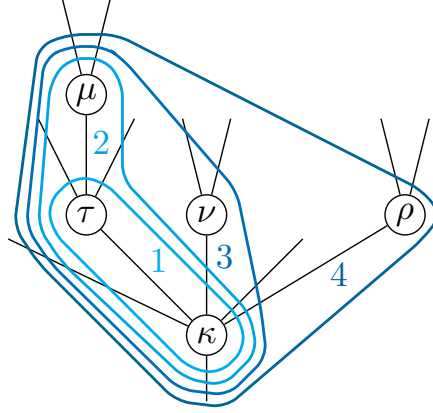


FIGURE 2. A fully nested planar tree.

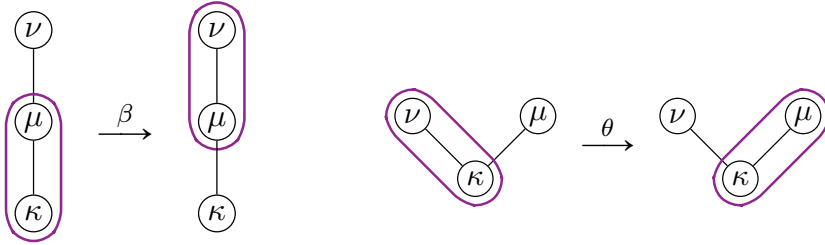
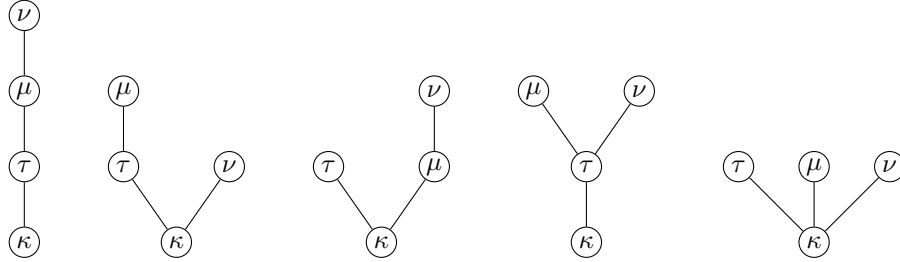
FIGURE 3. The  $\beta$  and  $\theta$  isomorphisms defining a categorified non-symmetric operad.

FIGURE 4. The five planar trees with four vertices, giving rise to the pentagonal (first three) and hexagonal (last two) identities.

The crucial observation is the following: the  $\theta$ -isomorphisms of Došen–Petric comprise both the isomorphisms  $\theta$  in Definition 2.1 and their inverses  $\theta^{-1}$ . Therefore, there are only two pentagonal coherence diagrams in the definition of a weak Cat-operad, the equations  $(\beta \text{ pent}_e)$  and  $(\beta \theta 2_e)$  of [DP15, Section 9]. The set of diagrams of the form  $(\beta \text{ pent}_e)$  is the same as the set of diagrams which arises from the first pentagon in Definition 2.1, while the

set of diagrams of the form  $(\beta\theta 2_e)$  is partitioned into the sets of diagrams which arise from the second and third pentagons in Definition 2.1.

We will give in Theorem 2.6 two topological proofs of coherence for categorified non-symmetric operads. A benefit of our presentation is that, adopting the oriented approach (see the second proof of Theorem 2.6), we get a proof of coherence where  $\beta$  and  $\theta$  are both treated as rewriting rules, in contrast with the proof in [DP15], which proceeds in two stages: first get rid of  $\beta$  (rewriting), then deal with  $\theta$ . See also Remark 2.9.

**Definition 2.3.** A *strong morphism of categorified non-symmetric operads*  $F : \mathcal{P} \rightarrow \mathcal{Q}$  is a collection of functors  $F_n : \mathcal{P}(n) \rightarrow \mathcal{Q}(n)$  together with natural isomorphisms

$$\gamma_{\kappa,\mu} : F_{m-1+n}(\kappa \circ_i \mu) \xrightarrow{\cong} F_m(\kappa) \circ_i F_n(\mu)$$

such that the following diagrams commute:

$$\begin{array}{ccc} & F((\kappa \circ \mu) \circ \nu) & \\ \gamma_{\kappa \circ \mu, \nu} \swarrow & & \searrow F(\beta_{\kappa, \mu, \nu}) \\ F(\kappa \circ \mu) \circ F(\nu) & & F(\kappa \circ (\mu \circ \nu)) \\ \downarrow \gamma_{\kappa \circ \mu} \circ 1_{F(\nu)} & & \downarrow \gamma_{\kappa, \mu \circ \nu} \\ (F(\kappa) \circ F(\mu)) \circ F(\nu) & & F(\kappa) \circ F(\mu \circ \nu) \\ \searrow \beta_{F(\kappa), F(\mu), F(\nu)} & & \swarrow 1_{F(\kappa)} \circ \gamma_{\mu, \nu} \\ & F(\kappa) \circ (F(\mu) \circ F(\nu)) & \end{array} \quad \begin{array}{ccc} & F((\kappa \circ \nu) \circ \mu) & \\ \gamma_{\kappa \circ \nu, \mu} \swarrow & & \searrow F(\theta_{\kappa, \nu, \mu}) \\ F(\kappa \circ \nu) \circ F(\mu) & & F((\kappa \circ \mu) \circ \nu) \\ \downarrow \gamma_{\kappa, \nu} \circ 1_{F(\mu)} & & \downarrow \gamma_{\kappa \circ \mu, \nu} \\ (F(\kappa) \circ F(\nu)) \circ F(\mu) & & F(\kappa \circ \mu) \circ F(\nu) \\ \searrow \theta_{F(\kappa), F(\nu), F(\mu)} & & \swarrow \gamma_{\kappa, \nu} \circ 1_{F(\mu)} \\ & (F(\kappa) \circ F(\mu)) \circ F(\nu) & \end{array}$$

It is said to be *strict* if the natural isomorphisms are identities.

Once again, the diagrams above hold for all instances of  $\beta$  and  $\theta$  arrows, and we have omitted the  $(i, j, k)$ -indices for readability. Observe that a strong (resp. strict) morphism between categorified non-symmetric operads concentrated in arity 1 is a strong (resp. strict) monoidal functor between non-symmetric monoidal categories.

**2.2. Coherence theorem.** We now aim at the coherence theorem for categorified non-symmetric operads. In order to state the theorem, we construct the free non-symmetric categorified operad on a family of sets  $S = \{S_n\}_{n \geq 1}$ . We define a family of categories  $\mathcal{S} = \{\mathcal{S}_n\}_{n \geq 1}$  whose objects are given by the following rules:

- (1) if  $\mu \in S_n$ , then  $\mu$  is an object of  $\mathcal{S}_n$ ;
- (2) if  $\mu \in \mathcal{S}_m$  and  $\nu \in S_n$ , then  $\mu \circ_i \nu$  is an object of  $\mathcal{S}_{m-1+n}$ , for any  $1 \leq i \leq m$ .

If an object  $\mu$  is in  $\mathcal{S}_n$ , we say that  $\mu$  has *arity*  $n$ . Now we define a set  $M$  of basic morphisms  $\beta : (\kappa \circ_i \mu) \circ_{j+i-1} \nu \leftrightarrow \kappa \circ_i (\mu \circ_j \nu) : \beta^{-1}$  for every  $\kappa \in \mathcal{S}_m, \mu \in \mathcal{S}_n, \nu \in \mathcal{S}_k, 1 \leq i \leq m$  and  $1 \leq j \leq n$ , and  $\theta : (\kappa \circ_i \nu) \circ_{j-1+k} \mu \leftrightarrow (\kappa \circ_j \mu) \circ_i \nu : \theta^{-1}$  whenever  $i < j$ . We then define the generating morphisms of the family  $\mathcal{S}$  by the following rules:

- (1) if  $\phi \in M$ , then  $\phi$  is a generating morphism of  $\mathcal{S}$ ;
- (2) if  $\phi : t_1 \rightarrow t_2$  is a generating morphism in  $\mathcal{S}$ , and  $t_3 \in \mathcal{S}$ , then  $\phi \circ_i \text{id} : t_1 \circ_i t_3 \rightarrow t_2 \circ_i t_3$  and  $\text{id} \circ_j \phi : t_3 \circ_j t_1 \rightarrow t_3 \circ_j t_2$  are morphisms, for any  $i$  (resp.  $j$ ) between 1 and the arity of  $t_1$  (resp.  $t_3$ ).

Note that by construction, for every morphism  $\phi : t_1 \rightarrow t_2$ , the objects  $t_1$  and  $t_2$  have the same arity, and we say that  $\phi$  has this *arity*. We then define  $\mathcal{S}_n$  as the free category over all generating morphisms of arity  $n$ . This finishes the construction of our family  $\mathcal{S}$  of categories.

One can think about the objects of  $\mathcal{S}$  as the set of fully nested planar trees whose vertices are decorated by objects of  $S$ , subject to the requirement that if a vertex has  $n$  incoming edges, it must be decorated by an element of  $S_n$ . The nests in the tree represent the order in which the objects decorating the vertices should be formally composed, as represented in Figure 2. Morphisms in  $\mathcal{S}$  are sequences of applications of the associativity rules  $\beta$  and  $\theta$ , moving one nest at a time to a neighbouring pair of vertices as in Figure 3.

**Definition 2.4.** We denote by  $\mathcal{F}(S)$  the quotient of the family of categories  $\mathcal{S}$  by localization (inverting the  $\beta$  and  $\theta$  morphisms), the axioms of bifunctors, and the coherence diagrams defining a categorified non-symmetric operad.

*Remark 2.5.* In the tree representation for  $\mathcal{S}$ , there is a coherence diagram for every decorated planar tree with 4 vertices, see Figure 4. In the rewriting system formed by the fully nested planar trees and the  $\beta$  and  $\theta$  arrows, these correspond precisely to the critical pairs [BN98, Section 6.2].

We obtain that  $\mathcal{F}(S)$  is the free categorified non-symmetric operad on  $S$ . That is, for any categorified non-symmetric operad  $\mathcal{P}$ , and for any family of functions  $\rho_n : S_n \rightarrow \text{Ob}(\mathcal{P}(n))$ , there is a unique *strict* morphism of non-symmetric categorified operads  $\mathcal{F}(S) \rightarrow \mathcal{P}$  which extends  $\rho = \{\rho_n\}_{n \geq 1}$ . By precomposing it with the quotient map  $\mathcal{S} \rightarrow \mathcal{F}(S)$ , we get a levelwise functor  $[-] : \mathcal{S} \rightarrow \mathcal{P}$ .

**Theorem 2.6** (Coherence theorem). *For any categorified non-symmetric operad  $\mathcal{P}$ , for any family of functions  $\rho : S \rightarrow \text{Ob}(\mathcal{P})$ , and for any two parallel morphisms  $\phi_1, \phi_2 : t_1 \rightarrow t_2$  in  $\mathcal{S}$ , we have  $[\phi_1] = [\phi_2]$ .*

*Proof.* The morphisms of  $\mathcal{S}$  are in bijection with combinatorial paths on a family of polytopes called operahedra [Lap22, Section 2.1], see also [DP15, Section 13] and [COI19], whose faces are in bijection with the set of all nestings on a planar tree. There is one operahedron for each planar tree  $t$ , and its vertices are in bijection with the maximal (full) nestings of  $t$  and edges between them are in bijection with the generating morphisms between the corresponding terms. Two parallel morphisms in  $\mathcal{S}$  thus define two parallel combinatorial paths on some operahedron. Since an operahedron is simply connected, Theorem 1.1 implies that these two combinatorial paths are combinatorially homotopic. The result then follows from the fact that a 2-face of an operahedron is exactly either a square (witnessing naturality), a pentagon or an hexagon (witnessing a coherence condition) as in Definition 2.1 above.  $\square$

*Second proof.* Alternatively, since the operahedra are polytopes, one can use Proposition 1.7 and Theorem 1.4. Choosing a generic vector  $\vec{v}$  which has strictly decreasing coordinates

for the Loday realizations [Lap22, Section 2.2] gives the orientations of the diagrams given in Definition 2.1 on every 2-face [Lap22, Proposition 3.11]. The posets/rewriting system obtained generalize the Tamari lattice on fully nested linear trees [Lap22, Definition 2.8]. One then obtains a topological proof of coherence which is almost word for word the original proof of MacLane [ML63, Theorem 3.1], suitably generalized to categorified operads.  $\square$

Following Remark 2.2, we have that Theorem 2.6 gives an alternative, more economical proof of coherence for weak Cat-operads [DP15, Proposition 14.2]. Incidentally, it gives an alternative input to the proof of coherence for cyclic symmetric categorified operads [CO20].

Restricting the theorem above to non-symmetric operads concentrated in arity 1, the category  $\mathcal{F}(S)$  becomes the free non-symmetric monoidal category on  $S$ , and we get the following corollary.

**Corollary 2.7** (MacLane’s coherence theorem for non-symmetric monoidal categories). *For any non-symmetric monoidal category  $\mathcal{C}$ , for any function  $\rho : S \rightarrow \text{Ob}(\mathcal{C})$ , and for any two parallel morphisms  $\phi_1, \phi_2 : t_1 \rightarrow t_2$  in  $\mathcal{S}$ , we have  $[\phi_1] = [\phi_2]$ .*

**2.3. Symmetric monoidal categories.** One can use the same ideas to prove MacLane’s coherence theorem for *symmetric* monoidal categories, using the family of permutoassociahedra [Kap93, RZ94, BIP19, CL21]. Here we simply formulate the theorem, the proof is really the same as the one of Theorem 2.6.

Let  $W_n$  denote the set of non-commutative, fully parenthesized words on  $n$  distinct letters. Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. Each  $w \in W_n$  defines in an obvious way a functor  $[w] : \mathcal{C}^{\times n} \rightarrow \mathcal{C}$ . For instance, if  $w = ((ab)c)(ed)$ , then the associated functor is defined on objects and morphisms by the formula

$$\begin{aligned} [w] & : \mathcal{C}^{\times n} \rightarrow \mathcal{C} \\ (a, b, c, d, e) & \mapsto ((a \otimes b) \otimes c) \otimes (e \otimes d) . \end{aligned}$$

We consider the free category  $\mathcal{W}_n$  which has objects the elements of  $W_n$ , and morphisms generated by the ones of the form  $\phi : w_1 \rightarrow w_2$ , where the word  $w_2$  is obtained from  $w_1$  by applying either  $\alpha : ((ww')w'') \rightarrow (w(w'w''))$ ,  $\alpha^{-1}$ , or  $\tau : ww' \rightarrow w'w$  to a subword of  $w_1$ . To any such  $\phi$  one can associate a natural transformation  $[\phi] : [w_1] \rightarrow [w_2]$  in  $\mathcal{C}$  in the obvious way.

**Theorem 2.8** (MacLane’s coherence theorem). *For any symmetric monoidal category  $\mathcal{C}$ , and for any pair of parallel morphisms  $\phi_1, \phi_2 : w_1 \rightarrow w_2$  in  $\mathcal{W} = \{\mathcal{W}_n\}_{n \geq 1}$ , we have  $[\phi_1] = [\phi_2]$ .*

*Remark 2.9.* MacLane’s original proof [ML63] proceeds in two stages, (anachronically) much like the one of Došen–Petrić for weak Cat-operads (Remark 2.2). But the second proof of Theorem 2.6 suggests a one stage proof: indeed, fixing a total order on the letters, allowing  $\tau : ab \rightarrow ba$  only if the maximal letter is in  $a$  (and adding  $\tau^{-1}$ ), we get a terminating and confluent rewriting system.

**2.4. Further applications.** One can also use the same strategy to prove coherence for *unital* non-symmetric monoidal categories, using the unital associahedra of F. Muro and A. Tonks [MT14].

It is natural to ask if the construction of unital associahedra could be extended to the permutoassociahedra, in such a way as to provide a topological proof of coherence for unital symmetric monoidal categories. The question of the existence of these constructions at the operadic level (i.e. does there exist unital operahedra, symmetric operahedra, and unital symmetric operahedra?) is, to our knowledge, still open as well.

Another immediate application of Theorem 1.1 is the coherence of strong non-symmetric monoidal functors between non-symmetric monoidal categories [Eps66]. The corresponding topological objects are in this case the family of multiplihedra [Sta70, For08]. The generalization to strong morphisms between non-symmetric categorified operads also goes through, involving this time the family of multiploperahedra described at the end of the introduction in [LAM22].

In the same spirit as in Theorem 2.6, one could obtain coherence results for categorifications of many operad-like structures, for instance the ones described in [BMO20]: categorified modular operads, wheeled properads, and permutads (shuffle algebras), among others. In order to treat cyclic and symmetric structures, one could take inspiration from the reduction process followed in [CO20] for the case of cyclic symmetric categorified operads.

### 3. PERSPECTIVES

**3.1. Higher categories.** Theorem 1.1 demonstrates that, in the case of monoidal categories, coherence is equivalent to the vanishing of the first homotopy groups of the associahedra. Since the associahedra are contractible, and therefore all their homotopy groups vanish, one could hope for a topological proof of higher dimensional coherence theorems. Seeing a monoidal category as a bicategory with one object, one can ask about a coherence theorem for tricategories with one object. A first look at the diagrams in the beginning of [GPS95, Section 2] suggests that such a theorem should be at least related to the vanishing of the second homotopy groups of the associahedra. To formulate higher dimensional statements, one needs a good structure of pasting scheme on each associahedron, which is the subject of ongoing work [LAMM].

Recent results of S. Barkan provide evidence for these higher dimensional statements, in the context of  $\infty$ -operads [Bar22]. In this vein, it seems likely that the present results could be interpreted as a strict version and a special case of [Bar22, Theorem B]. It would be interesting to see how the permutoassociahedra arise in the strictification process, and how they are related to operadic partition complexes.

*Acknowledgements.* We would like to thank all the participants of the Roberta Seminar held on September 30th, 2020 in Paris for planting the seeds of the present note. We would like to thank Andrea Bianchi for enlightening discussions, for pointing out mistakes in a first draft, and for reminding us of the Seifert–Van Kampen theorem. Finally, we are grateful to Zoran Petrić for precious discussions on weak Cat-operads, and to Marcy Robertson for helpful comments. We would like to thank the Copenhagen Center for Geometry and Topology as well as the Max Planck Institute for Mathematics in Bonn where part of this work was carried out. The second author was supported by the Australian Research Council Future Fellowship FT210100256.

# REFERENCES

- [Bar22] Shaul Barkan, *Arity Approximation of  $\infty$ -Operads*, ArXiv e-prints (2022), [arXiv:2207.07200](#).
- [BB97] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, *Inventiones mathematicae* **129** (1997), no. 3, 445–470.
- [BIP19] Djordje Baralić, Jelena Ivanović, and Zoran Petrić, *A simple permutoassociahedron*, *Discrete Mathematics* **342** (2019), no. 12, 111591.
- [BMO20] Michael Batanin, Martin Markl, and Jovana Obradović, *Minimal models for graphs-related operadic algebras*, ArXiv e-prints (2020), [arXiv:2002.06640](#).
- [BN98] Franz Baader and Tobias Nipkow, *Term Rewriting and All That*, Cambridge University Press, Cambridge, 1998.
- [Bro06] Ronald Brown, *Topology and groupoids*, Booksurge Publishing, 2006.
- [CL21] Federico Castillo and Fu Liu, *The permuto-associahedron revisited*, ArXiv e-prints (2021), [arXiv:2109.08658](#).
- [CO20] Pierre-Louis Curien and Jovana Obradović, *Categorified Cyclic Operads*, *Applied Categorical Structures* **28** (2020), no. 1, 59–112.
- [COI19] Pierre-Louis Curien, Jovana Obradović, and Jelena Ivanović, *Syntactic aspects of hypergraph polytopes*, *Journal of Homotopy and Related Structures* **14** (2019), no. 1, 235–279.
- [DP15] Kosta Došen and Zoran Petrić, *Weak cat-operads*, *Logical Methods in Computer Science* **11** (2015), no. 1, 1–23.
- [Eps66] D. B. A. Epstein, *Functors between tensored categories*, *Inventiones mathematicae* **1** (1966), no. 3, 221–228.
- [For08] Stefan Forcey, *Convex hull realizations of the multiplihedra*, *Topology Appl.* **156** (2008), no. 2, 326–347.
- [GPS95] R. Gordon, A. J. Power, and Ross Street, *Coherence for tricategories*, vol. 117, American Mathematical Society, Providence, RI, September 1995.
- [Kap93] Mikhail M. Kapranov, *The permutoassociahedron, Mac Lane’s coherence theorem and asymptotic zones for the KZ equation*, *Journal of Pure and Applied Algebra* **85** (1993), no. 2, 119–142.
- [LAM22] Guillaume Laplante-Anfossi and Thibaut Mazuir, *The diagonal of the multiplihedra and the tensor product of  $A$ -infinity morphisms*, ArXiv e-prints (2022), [arXiv:2206.05566](#).
- [LMM] Guillaume Laplante-Anfossi and Anibal M. Medina-Mardones, *Higher categories, fiber polytopes and the Steenrod construction*, in preparation.
- [Lap22] Guillaume Laplante-Anfossi, *The diagonal of the operahedra*, *Advances in Mathematics* **405** (2022), 108494.
- [ML63] Saunders Mac Lane, *Natural associativity and commutativity*, *Rice Univ. Studies* **49** (1963), no. 4, 28–46.
- [MT14] Fernando Muro and Andrew Tonks, *Unital associahedra*, *Forum Mathematicum* **26** (2014), no. 2, 593–620.
- [RZ94] Victor Reiner and Günter M. Ziegler, *Coxeter-associahedra*, *Mathematika* **41** (1994), no. 2, 364–393.
- [Sta70] James D. Stasheff, *H-spaces from a homotopy point of view*, *Lecture Notes in Mathematics*, Vol. 161, Springer-Verlag, Berlin, 1970.
- [Zie95] Günter M. Ziegler, *Lectures on polytopes*, *Graduate Texts in Mathematics*, vol. 152, Springer-Verlag, New York, 1995.

IRIF, UNIVERSITÉ PARIS DIDEROT AND  $\pi r^2$  TEAM, INRIA, FRANCE.

*E-mail address:* [curien@irif.fr](mailto:curien@irif.fr)

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, VICTORIA, AUSTRALIA.

*E-mail address:* [guillaume.laplanteanfossi@unimelb.edu.au](mailto:guillaume.laplanteanfossi@unimelb.edu.au)