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AN INEQUALITY FOR THE SOLUTIONS TO AN EQUATION WITH BOUNDARY CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give a uniform estimate and an inequality for solutions of an equation with Dirichlet boundary condition.

1. INTRODUCTION AND MAIN RESULT

We set $\Delta = -\nabla^i \nabla_i$ the Laplace-Beltrami operator on a connected compact Riemannian manifold with boundary (M, g) of dimension $n \geq 3$ with metric g .

We consider the following equation:

$$(P) \begin{cases} \Delta u + hu = Vu^{N-1}, & u > 0 & \text{in } M, \\ u = 0 & & \text{on } \partial M. \end{cases}$$

Here, we assume the solutions in the sense of distributions and here also in $C^{2,\alpha}(\bar{M})$, $\alpha > 0$ and the operator $\Delta + h$ is coercive in $H_0^1(M)$ with h a smooth function and $0 \leq V \leq b$ with $V \not\equiv 0$ a smooth function and $\|V\|_{C^\alpha} \leq A$ and $N = \frac{2n}{n-2}$ the critical Sobolev exponent.

In [1-25] we have various estimates and inequalities of type $\sup + \inf$ and $\sup \times \inf$. Here we look to the case of lower bounds for these inequalities as in [2], [4], [5], [7] and [21].

Our main results are:

Theorem 1.1. *For all $b, A, M_0 > 0$ and all compact K of M , there is a positive constant $c = c(b, A, \alpha, K, M, g, n, M_0) > 0$ such that:*

$$\inf_K u \geq c \text{ if } \sup_M u \leq M_0.$$

for all solution $u > 0$ of (P) relative to $V \geq 0$ with bound b .

As a consequence of the previous theorem, we have $\inf_K u \geq c = c(\sup_M u)$ with $c > 0$. We write this as: the following implicit Harnack type inequality:

Theorem 1.2. *For all $b, A > 0$ and all compact K of M , there is a positive constant $c = c(b, A, \alpha, K, M, g, n, \sup_M u) > 0$ such that:*

$$\inf_K u \geq c(b, A, \alpha, K, M, g, n, \sup_M u).$$

for all solution $u > 0$ of (P) relative to $V \geq 0$ with bound b .

Remarks:

- 1) The function $M_0 \mapsto c$ is non-increasing function of M_0 .
- 2) We need uniform C^α regularity for V , because we will use the strong maximum principle which require C^2 regularity of the solutions.
- 3) We can replace the exponent N by $q \geq 2 + \epsilon_0$ with $\epsilon_0 > 0$.
- 4) These results can be applied to $h = \frac{n-2}{4(n-1)} S_g$ with S_g the scalar curvature, in the non-negative case. We obtain the prescribed scalar curvature equation and the Yamabe equation on manifolds with boundary, with Dirichlet boundary conditions. For example $S_g \geq 0$.

2. PROOFS OF THE THEOREMS

Proofs of Theorems 1 and 2:

We argue by contradiction, suppose that for a K compact of M we have sequences of solutions (u_i, V_i) of (P) such that: $\inf_K u_i \rightarrow 0$ and $\sup_M u_i \leq M_0$. By the elliptic estimates, see [17], we have $u_i \rightarrow u \geq 0$ uniformly in $C^{2,\alpha}(\bar{M})$.

Now we use the condition $\inf_K u_i \rightarrow 0$ to have for the solution $u \geq 0$, $u(x_0) = 0$ with $x_0 \in K \subset\subset M$.

Then we use the maximum principle applied to $-u$ to have $u \equiv 0$ on \bar{M} . Thus,

$$u_i \rightarrow 0 \text{ in } C^{2,\alpha}(\bar{M}).$$

But if we multiply the equation by u_i and use integration by part, the coercitivity of the operator and the Sobolev embedding we obtain for some constant $C_2 > 0$ independant from i :

$$C_2 \|u_i\|_N^2 \leq C_1 \|u_i\|_{H_0^1(M)}^2 \leq \int_M |\nabla u_i|^2 + h u_i^2 = \int_M (\Delta u_i + h u_i) u_i = \int_M V_i u_i^N \leq b \|u_i\|_N^N,$$

thus,

$$\|u_i\|_N^{N-2} \geq \frac{C_2}{b} > 0,$$

with uniform bound.

And this contradict: $u_i \rightarrow 0$ in $C^0(\bar{M})$.

Thus, for all $b, A > 0$, $M_0 > 0$, for all compact K of M , there is a positive constant $c = c(b, A, \alpha, K, M, g, n, M_0)$ such that:

$$\sup_M u \leq M_0 \Rightarrow \inf_K u \geq c > 0,$$

with by definition,

$$c = c(M_0) = \sup\{c' > 0, \inf_K u \geq c', \text{ if } \sup_M u \leq M_0, M_0 > 0\}.$$

With this definition, by taking twice the supremum in $c' > 0$, we see that if $0 < \sup_M u \leq M_0 \leq K_0 \Rightarrow \inf_K u \geq c(K_0) \Rightarrow \inf_K u \geq c(M_0) \geq c(K_0) \Rightarrow M_0 \mapsto c$ is non-increasing function of M_0 .

Take $u > 0$ a regular solution of (P) relative to V and set $M_0 = \sup_M u > 0$, then for all $v > 0$ solution of (P) relative to $W \geq 0$ with bound, b , we have the previous inequality.

But $v = u > 0$ is a particular solution, thus it satisfies the inequality and we have:

$$\inf_K u \geq c = c(b, A, \alpha, K, M, g, n, \sup_M u) > 0.$$

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