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# AN INEQUALITY FOR THE SOLUTIONS TO AN EQUATION WITH BOUNDARY CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give a uniform estimate and an inequality for solutions of an equation with Dirichlet boundary condition.

## 1. INTRODUCTION AND MAIN RESULT

We set  $\Delta = -\nabla^i \nabla_i$  the Laplace-Beltrami operator on a connected compact Riemannian manifold with boundary  $(M, g)$  of dimension  $n \geq 3$  with metric  $g$ .

We consider the following equation:

$$(P) \begin{cases} \Delta u + hu = Vu^{N-1}, & u > 0 & \text{in } M, \\ u = 0 & & \text{on } \partial M. \end{cases}$$

Here, we assume the solutions in the sense of distributions and here also in  $C^{2,\alpha}(\bar{M})$ ,  $\alpha > 0$  and the operator  $\Delta + h$  is coercive in  $H_0^1(M)$  with  $h$  a smooth function and  $0 \leq V \leq b$  with  $V \not\equiv 0$  a smooth function and  $\|V\|_{C^\alpha} \leq A$  and  $N = \frac{2n}{n-2}$  the critical Sobolev exponent.

In [1-25] we have various estimates and inequalities of type  $\sup + \inf$  and  $\sup \times \inf$ . Here we look to the case of lower bounds for these inequalities as in [2], [4], [5], [7] and [21].

Our main results are:

**Theorem 1.1.** *For all  $b, A, M_0 > 0$  and all compact  $K$  of  $M$ , there is a positive constant  $c = c(b, A, \alpha, K, M, g, n, M_0) > 0$  such that:*

$$\inf_K u \geq c \text{ if } \sup_M u \leq M_0.$$

for all solution  $u > 0$  of (P) relative to  $V \geq 0$  with bound  $b$ .

As a consequence of the previous theorem, we have  $\inf_K u \geq c = c(\sup_M u)$  with  $c > 0$ . We write this as: the following implicit Harnack type inequality:

**Theorem 1.2.** *For all  $b, A > 0$  and all compact  $K$  of  $M$ , there is a positive constant  $c = c(b, A, \alpha, K, M, g, n, \sup_M u) > 0$  such that:*

$$\inf_K u \geq c(b, A, \alpha, K, M, g, n, \sup_M u).$$

for all solution  $u > 0$  of (P) relative to  $V \geq 0$  with bound  $b$ .

### Remarks:

- 1) The function  $M_0 \mapsto c$  is non-increasing function of  $M_0$ .
- 2) We need uniform  $C^\alpha$  regularity for  $V$ , because we will use the strong maximum principle which require  $C^2$  regularity of the solutions.
- 3) We can replace the exponent  $N$  by  $q \geq 2 + \epsilon_0$  with  $\epsilon_0 > 0$ .
- 4) These results can be applied to  $h = \frac{n-2}{4(n-1)} S_g$  with  $S_g$  the scalar curvature, in the non-negative case. We obtain the prescribed scalar curvature equation and the Yamabe equation on manifolds with boundary, with Dirichlet boundary conditions. For example  $S_g \geq 0$ .

## 2. PROOFS OF THE THEOREMS

Proofs of Theorems 1 and 2:

We argue by contradiction, suppose that for a  $K$  compact of  $M$  we have sequences of solutions  $(u_i, V_i)$  of  $(P)$  such that:  $\inf_K u_i \rightarrow 0$  and  $\sup_M u_i \leq M_0$ . By the elliptic estimates, see [17], we have  $u_i \rightarrow u \geq 0$  uniformly in  $C^{2,\alpha}(\bar{M})$ .

Now we use the condition  $\inf_K u_i \rightarrow 0$  to have for the solution  $u \geq 0$ ,  $u(x_0) = 0$  with  $x_0 \in K \subset\subset M$ .

Then we use the maximum principle applied to  $-u$  to have  $u \equiv 0$  on  $\bar{M}$ . Thus,

$$u_i \rightarrow 0 \text{ in } C^{2,\alpha}(\bar{M}).$$

But if we multiply the equation by  $u_i$  and use integration by part, the coercitivity of the operator and the Sobolev embedding we obtain for some constant  $C_2 > 0$  independant from  $i$ :

$$C_2 \|u_i\|_N^2 \leq C_1 \|u_i\|_{H_0^1(M)}^2 \leq \int_M |\nabla u_i|^2 + h u_i^2 = \int_M (\Delta u_i + h u_i) u_i = \int_M V_i u_i^N \leq b \|u_i\|_N^N,$$

thus,

$$\|u_i\|_N^{N-2} \geq \frac{C_2}{b} > 0,$$

with uniform bound.

And this contradict:  $u_i \rightarrow 0$  in  $C^0(\bar{M})$ .

Thus, for all  $b, A > 0$ ,  $M_0 > 0$ , for all compact  $K$  of  $M$ , there is a positive constant  $c = c(b, A, \alpha, K, M, g, n, M_0)$  such that:

$$\sup_M u \leq M_0 \Rightarrow \inf_K u \geq c > 0,$$

with by definition,

$$c = c(M_0) = \sup\{c' > 0, \inf_K u \geq c', \text{ if } \sup_M u \leq M_0, M_0 > 0\}.$$

With this definition, by taking twice the supremum in  $c' > 0$ , we see that if  $0 < \sup_M u \leq M_0 \leq K_0 \Rightarrow \inf_K u \geq c(K_0) \Rightarrow \inf_K u \geq c(M_0) \geq c(K_0) \Rightarrow M_0 \mapsto c$  is non-increasing function of  $M_0$ .

Take  $u > 0$  a regular solution of  $(P)$  relative to  $V$  and set  $M_0 = \sup_M u > 0$ , then for all  $v > 0$  solution of  $(P)$  relative to  $W \geq 0$  with bound,  $b$ , we have the previous inequality.

But  $v = u > 0$  is a particular solution, thus it satisfies the inequality and we have:

$$\inf_K u \geq c = c(b, A, \alpha, K, M, g, n, \sup_M u) > 0.$$

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